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Inverse heat problem of determining time-dependent source parameter in reproducing kernel space

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Abstract A method by the reproducing kernel Hilbert space is applied to an inverse heat problem of determining a time-dependent source parameter. The problem is reduced to a system of linear equations. The exact and approximate solutions are both obtained in a reproducing kernel space. The approximate solution and its partial derivatives are proved to converge to the exact solution and its partial derivatives, respectively. The proposed method improves the previous method. Our numerical results show that the method is of high precision.

Keywords exact solution; inverse problem; source control parameter; parabolic partial differential equation; reproducing kernel

2000 MR Subject Classification 65M32; 47B32; 35K20; 35R30

1 Introduction

The literature on the numerical approximation of solutions to inverse problems for parabolic partial differential equations is large and still growing rapidly. Many methods based on the finite difference, the finite element, the spectral, the finite volume, the boundary element and the meshless methods have been proposed to approximate solutions and we can refer for example to [1-8] and the references therein.

We consider the following inverse problem of simultaneously finding unknown coefficients $p(t)$ and $w(x,t)$ from the following parabolic equation (e.g. [2,5])

$$w_t = w_{xx} + qw_x + p(t)w + k(x,t), \quad (x,t) \in D = [0,1] \times [0,1], \tag{1.1}$$

$$w(x,0) = g(x), \quad x \in [0,1], \tag{1.2}$$

$$w(0,t) = h_1(t), \quad w(1,t) = h_2(t), \quad t \in [0,1], \tag{1.3}$$

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subject to the pointwise observation data at x^*

$$w(x^*, t) = E(t), \quad x^* \in (0, 1), \quad t \in [0, 1]. \tag{1.4}$$

Throughout this paper, we assume that $k(x, t)$, $g(x)$, $h_1(t)$, $h_2(t)$ and $E(t)$ are known and sufficiently smooth functions, q is a known constant and x^* is a fixed prescribed interior point in $(0, 1)$. Equation (1.1) models a heat process with heat source whose intensity is proportional to the temperature with coefficient $p(t)$ (e.g. [2]). Equation (1.4) represents the temperature at a given point x^* in a spatial domain at time t. Thus the purpose of solving this inverse problem is to identify the source parameter $p(t)$ that produces a desired temperature profile at each time t and a given point x^* in a spatial domain (see e.g. [2]).

The existence and uniqueness of the solutions to this problem and also some more applications are discussed in [6-12]. Various numerical methods [13-21] are developed for this inverse problem and related inverse parabolic problems. As for other types of inverse parabolic problems, see e.g. [2,22]. The approach in the current paper is by the reproducing kernel Hilbert space and is different for example from the finite difference method [5,21].

In recent years, there is much interest in the use of reproducing kernel for the solution of nonlinear physical and engineering problems [23-28]. The reproducing kernel has been applied successfully to wavelet transforms [29], stochastic processes [30], signal processing [31], machine learning [32], ill-posed Cauchy problems for elliptic equations [33], inverse problems [34,35], etc. Those papers indicate that the reproducing kernel method (RKM) [36-38] possesses many outstanding advantages, which can handle the nonlinear and ill-posed problems. Also the numerical solutions can be obtained by this method for the practical problems that cannot be solved efficiently before [39-45].

The most important advantages of RKM are as follows:

(i) The approximate solution converges uniformly to the exact solution, also partial derivatives of the approximate solution converge uniformly to partial derivatives of the exact solution.

(ii) The structure of numerical programming is simple and the calculations are very fast.

(iii) The accuracy of approximate solution is very high.

In this paper, we give the representation of exact solution to problem (1.1)-(1.4) in the reproducing kernel space, and improve the existing methods as follows: first, we obtain the reproducing kernel spaces by redefining the inner products, which are simpler than the former [27], and so it can decrease the accumulative errors to improve the precision and decrease the runtime; second, this approach reduces problem (1.1)-(1.4) to a system of linear equations, and avoids the Gram-Schmidt orthogonalization process [28]. Numerical calculations indicate that this method improves the precision and decreases the runtime, especially when the number of knots is large.

Before applying our method to problem (1.1)-(1.4), we apply two different procedures for the problem, which are inspired by [5,6,15,21]. Henceforth $\cdot'$ denotes the derivative in the variable under consideration.

Procedure I

Here we assume

$$E(t) - h_1(t)(1-x^*) - h_2(t)x^* \neq 0, \quad t \in [0,1]. \tag{1.5}$$

By the maximum principle for the parabolic equation, condition (1.5) is satisfied for example, if

$$h_1(t) = h_2(t) = 0\,, \quad k(x,t) \geq 0, \quad g(x) \geq 0 (\not\equiv 0), \qquad x,t \in [0,1]. \tag{1.6}$$

Then we set

$$v(x,t) = w(x,t)r(t), \qquad r(t) = \exp\biggl(-\int_0^t p(s)ds\biggr). \tag{1.7}$$

Then the direct calculations yield

$$v_t = v_{xx} + qv_x + r(t)k(x,t), \quad (x,t) \in D,$$

$$v(x,0) = g(x), \quad x \in [0,1],$$

$$v(0,t) = r(t)h_1(t), \quad t \in [0,1],$$

$$v(1,t) = r(t)h_2(t), \quad t \in [0,1],$$

$$v(x^*,t) = r(t)E(t), \quad t \in [0,1]. \tag{1.8}$$

Again we set

$$u(x,t) = v(x,t) - (v(0,t) - g(0))\,(1-x) - (v(1,t) - g(1))\,x - g(x).$$

Then we can further transform the original problem to

$$Lu(x,t) = f(x,t), \quad (x,t) \in D \tag{1.9}$$

$$u(x,0) = 0, \quad x \in [0,1], \tag{1.10}$$

$$u(0,t) = 0, \quad t \in [0,1], \tag{1.11}$$

$$u(1,t) = 0, \quad t \in [0,1]. \tag{1.12}$$

Here we set

$$Lu = u_t - u_{xx} - qu_x - F(x,t)\frac{u(x^*,t)}{M(t)} + H(x,t)\biggl(\frac{u(x^*,t)}{M(t)}\biggr)',$$

$$
\begin{aligned}
f(x,t) = \quad & (g(x^*) - g(0)(1-x^*) - g(1)x^*)\left(\frac{F(x,t)}{M(t)} + \frac{H(x,t)M'(t)}{M^2(t)}\right) \\
& + g''(x) + q(g'(x) + g(0) - g(1)),
\end{aligned}
$$

where

$$
F(x,t) = q\Big(h_2(t) - h_1(t)\Big) - \partial_t H(x,t) + k(x,t),
$$

$$
H(x,t) = (1-x)h_1(t) + xh_2(t)
$$

and

$$
M(t) = E(t) - h_1(t)(1-x^*) - h_2(t)x^*.
$$

Procedure II

We assume

$$
E(t) - \Big(h_1(t)(1-x^*) + h_2(t)\exp\left(\frac{q}{2}\right)x^*\Big)\exp\left(-\frac{q}{2}x^*\right) \neq 0, \; t \in [0,1]. \tag{1.13}
$$

The condition (1.13) is satisfied for example under condition (1.6). Then we set

$$
v(x,t) = w(x,t)r(t)\exp\left(\frac{q}{2}x\right), \quad r(t) = \exp\left(-\int_0^t (p(s) - q^2/4)ds\right). \tag{1.14}
$$

Then the problem becomes

$$
\begin{gathered}
v_t = v_{xx} + r(t)\exp\left(\frac{q}{2}x\right)k(x,t), \quad (x,t) \in D, \\
v(x,0) = g(x)\exp\left(\frac{q}{2}x\right), \quad x \in [0,1], \\
v(0,t) = h_1(t)r(t), \quad t \in [0,1], \\
v(1,t) = h_2(t)r(t)\exp\left(\frac{q}{2}\right), \quad t \in [0,1], \\
v(x^*,t) = r(t)E(t)\exp\left(\frac{q}{2}x^*\right), \quad t \in [0,1].
\end{gathered} \tag{1.15}
$$

Thus by direct calculations:

$$
u(x,t) = v(x,t) - (v(0,t) - g(0))(1-x) - \Big(v(1,t) - g(1)\exp\left(\frac{q}{2}\right)\Big)x - v(x,0).
$$

The original problem (1.1)-(1.4) can further be reduced to (1.9)-(1.12). Here we set

$$
Lu = u_t - u_{xx} + \exp\left(-\frac{q}{2}x^*\right)\left(F(x,t)\frac{u(x^*,t)}{M(t)} + H(x,t)\left(\frac{u(x^*,t)}{M(t)}\right)'\right),
$$

$$
\begin{aligned}
f(x,t) = \quad & \Big(g(x^*) - \Big(g(0)(1-x^*) + g(1)\exp\left(\frac{q}{2}\right)x^*\Big)\exp\left(-\frac{q}{2}x^*\right)\Big) \\
& \times\left(-\frac{F(x,t)}{M(t)} + \frac{H(x,t)M'(t)}{M^2(t)}\right) + \Big(g''(x) + qg'(x) + \frac{q^2}{4}g(x)\Big)\exp\left(\frac{q}{2}x\right),
\end{aligned}
$$

where

$$F(x,t) = \partial_t H(x,t) - k(x,t)\exp\left(\frac{q}{2}x\right),$$

$$H(x,t) = (1-x)h_1(t) + xh_2(t)\exp\left(\frac{q}{2}\right)$$

and

$$M(t) = E(t) - \left(h_1(t)(1-x^*) + h_2(t)\exp\left(\frac{q}{2}\right)x^*\right)\exp\left(-\frac{q}{2}x^*\right).$$

Our method is composed of

(i) Solve (1.9)-(1.12) which is an initial-boundary value problem for a non-classical heat equation.

(ii) Find $p(t)$ by (1.7)-(1.8) in Procedure I or (1.14)-(1.15) in Procedure II.

We apply our numerical method to problem (1.9)-(1.12) for each procedure.

This paper is organized as follows: in Section 2, we construct reproducing kernel spaces according to (1.9)-(1.12). Section 3 gives the exact and approximate solutions in the reproducing kernel space. The convergence analysis is presented in Section 4. The numerical example is studied in Section 5. Finally a conclusion is given in Section 6.

2 Several reproducing kernel spaces

In this section, we construct the reproducing kernel spaces according to (1.9)-(1.12) by redefining the inner products, which are simpler than the former [27]. Therefore it can decrease the accumulative errors and thereby improve the precision and decrease consumedly the runtime.

2.1 The reproducing kernel space $W_3[0,1]$

The inner product space $W_3[0,1]$ is defined as $W_3[0,1] = \{u \mid u, u', u''$ are absolutely continuous real value functions, $u, u', u'', u^{(3)} \in L^2[0,1], u(0) = 0, u(1) = 0\}$. The inner product and the norm in $W_3[0,1]$ are given respectively by

$$< u, v >_{W_3} = \sum_{i=0}^{2} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{(3)}(x)v^{(3)}(x)dx,$$

$$\| u \|_{W_3} = \sqrt{< u, u >_{W_3}},$$

where $u, v \in W_3[0,1]$.

Theorem 2.1: *The space $W_3[0,1]$ is a reproducing kernel space. That is, for every fixed $x \in [0,1]$, there is a function $R_x(y) \in W_3[0,1]$, $y \in [0,1]$, such that for every $u \in W_3[0,1]$, $< u, R_x >_{W_3} = u(x)$. The reproducing* *kernel $R_x(y)$ is given by*

$$R_x(y) = \begin{cases} -\dfrac{(x-1)y}{18720}\Big(156y^4 + 12x(360 - 300y - 100y^2 - 15y^3 + 3y^4) \\ \quad + x^2(6 - 4x + x^2)(120 + 30y + 10y^2 - 5y^3 + y^4)\Big), & y \leq x, \\ -\dfrac{x(y-1)}{18720}\Big(10(3+x)xy(-120 + 6y - 4y^2 + y^3) \\ \quad + 5y(24 - x^3)(36 + 6y - 4y^2 + y^3) \\ \quad + x^4(156 + 36y + 6y^2 - 4y^3 + y^4)\Big), & y > x. \end{cases} \tag{2.1}$$

The proof of Theorem 2.1 is similar to [46, Theorem 2.1].

2.2 The reproducing kernel space $W_2[0,1]$

The inner product space $W_2[0,1]$ is defined by $W_2[0,1] = \{u \mid u, u'$ are absolutely continuous real value functions, $u, u', u'' \in L^2[0,1], u(0) = 0\}$. The inner product and norm in $W_2[0,1]$ are given respectively by

$$< u, v >_{W_2} = \sum_{i=0}^{1} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{(2)}(x)v^{(2)}(x)dx,$$

$$\| u \|_{W_2} = \sqrt{< u, u >_{W_2}},$$

where $u, v \in W_2[0,1]$. Similarly, we can prove that $W_2[0,1]$ is a complete reproducing kernel space and its reproducing kernel is

$$R_x^{\{2\}}(y) = \begin{cases} -\frac{1}{6}y\big(y^2 - 3x(2+y)\big), & y \leq x, \\ -\frac{1}{6}x\big(x^2 - 3y(2+x)\big), & y > x. \end{cases} \tag{2.2}$$

2.3 The reproducing kernel space $W_1[0,1]$

The inner product space $W_1[0,1]$ is defined by $W_1[0,1] = \{u \mid u$ is absolutely continuous real value function, $u, u' \in L^2[0,1]\}$. The inner product and norm in $W_1[0,1]$ are given respectively by

$$< u, v >_{W_1} = u(0)v(0) + \int_0^1 u'(x)v'(x)dx,$$

$$\| u \|_{W_1} = \sqrt{< u, u >_{W_1}},$$

where $u, v \in W_1[0,1]$. Similarly, we can prove that $W_1[0,1]$ is a reproducing kernel space and its reproducing kernel is

$$R_x^{\{1\}}(y) = \begin{cases} 1 + y, & y \leq x, \\ 1 + x, & y > x. \end{cases} \tag{2.3}$$

2.4 The reproducing kernel space $W_{(3,2)}(D)$ and $W_{(1,1)}(D)$

Assume that $\{p_i(x)\}_{i=1}^{\infty}$ is an orthonormal basis of $W_3[0,1]$ and $\{q_i(t)\}_{i=1}^{\infty}$ is an orthonormal basis of $W_2[0,1]$. Now we define $W_{(3,2)}(D)$ by

$$W_{(3,2)}(D) = \{u|u(x,t) = \sum_{i,j=1}^{\infty} c_{ij}p_i(x)q_j(t), \quad \sum_{i,j=1}^{\infty} \mid c_{ij} \mid^2 < \infty, \quad c_{ij} \in \mathbb{R}\}.$$

The inner product of $W_{(3,2)}(D)$ is defined dy

$$< u_1, u_2 >_{W_{(3,2)}} = \sum_{i,j=1}^{\infty} c_{ij}d_{ij},$$

where $u_1 = \sum\limits_{i,j=1}^{\infty} c_{ij}p_i(x)q_j(t)$ and $u_2 = \sum\limits_{i,j=1}^{\infty} d_{ij}p_i(x)q_j(t)$. The norm is denoted by

$$\|u\|^2_{W_{(3,2)}} =< u, u >_{W_{(3,2)}} .$$

By [36], it is easy to prove the following Propositions 2.1 and 2.2.

Proposition 2.1. *If $u(x,t) = u_1(x)u_2(t)$ and $v(x,t) = v_1(x)v_2(t) \in W_{(3,2)}(D)$, then*

$$< u,\, v >_{W_{(3,2)}} =< u_1,\, v_1 >_{W_3} < u_2,\, v_2 >_{W_2} .$$

Proposition 2.2. *$W_{(3,2)}(D)$ is a reproducing kernel space and the reproducing kernel is*

$$K_{(\xi,\eta)}(x,t) = R_\xi(x)R_\eta^{\{2\}}(t),$$

where $R_\xi(x)$, $R_\eta^{\{2\}}(t)$ are given by (2.1) and (2.2) respectively.

Similar to the definition of $W_{(3,2)}(D)$, we can define $W_{(1,1)}(D)$. $W_{(1,1)}(D)$ is also a reproducing kernel space, and the reproducing kernel is

$$\overline{K}_{(\xi,\eta)}(x,t) = R_\xi^{\{1\}}(x)R_\eta^{\{1\}}(t),$$

where $R_\xi^{\{1\}}(x)$, $R_\eta^{\{1\}}(t)$ are given by (2.3).

3 The solution of Eqs.(1.9)-(1.12)

In this section, the exact solution of problem (1.9)-(1.12) is given in the reproducing kernel space $W_{(3,2)}(D)$.

In Eqs.(1.9)-(1.12), since $k(x,t)$, $g(x)$, $h_1(t)$, $h_2(t)$ and $E(t)$ are sufficiently smooth, $L : W_{(3,2)}(D) \to W_{(1,1)}(D)$ is a bounded linear operator. Put $M = (x,t), M_i = (x_i,t_i)$, $\varphi_i(M) = \overline{K}_{M_i}(M)$, and $\psi_i(M) = L^*\varphi_i(M)$, where $\overline{K}$ is the reproducing kernel of $W_{(1,1)}(D)$ and L^* is the adjoint operator of L. The orthonormal system $\{\overline{\psi}_i(M)\}_{i=1}^{\infty}$ of $W_{(3,2)}(D)$ can be derived from the Gram-Schmidt orthogonalization process of $\{\psi_i(M)\}_{i=1}^{\infty}$,

$$\overline{\psi}_i(M) = \sum_{k=1}^{i} \beta_{ik}\psi_k(M), \quad (\beta_{ii} > 0, i = 1,2,...).$$

Theorems 3.1 and 3.2 can be proved similarly to Lemma 3.2 and Theorem 3.1 respectively in [28] where a forward problem for a heat equation with non-local boundary condition is discussed.

Theorem 3.1. *For Eqs.*(1.9)-(1.12)*, if* $\{M_i\}_{i=1}^{\infty}$ *is dense in* D*, then* $\{\psi_i(M)\}_{i=1}^{\infty}$ *is the complete system of* $W_{(3,2)}(D)$ *and* $\psi_i(M) = L_N K_M(N)|_{N=M_i}$.

The subscript N of L_N indicates that the operator L applies to the function of N.

Theorem 3.2. *If* $\{M_i\}_{i=1}^{\infty}$ *is dense in* D *and the solution of Eqs.*(1.9)-(1.12) *is unique, then the solution of Eqs.*(1.9)-(1.12) *satisfies the form*

$$u(M) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(M_k) \overline{\psi}_i(M). \tag{3.1}$$

Now, the approximate solution $u_n(M)$ can be obtained by the n-term interception of the exact solution $u(M)$ and

$$u_n(M) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} f(M_k) \overline{\psi}_i(M). \tag{3.2}$$

Theorem 3.3. *If* $u(M)$ *is the solution of Eqs.*(1.9)-(1.12) *represented in the form of* (3.1)*,* $u_n = P_n u(M)$*, where* P_n *is an orthogonal projector from* $W_{(3,2)}$ *to Span*$\{\overline{\psi}_i(M)\}_{i=1}^{n}$*, then* $Lu_n(M_i) = f(M_i)$*,* $i = 1, 2, ..., n$.

Proof.

$$\begin{aligned} Lu_n(M_i) &=< Lu_n(M), \varphi_i(M) >=< u_n(M), L^* \varphi_i(M) > \\ &=< P_n u(M), \psi_i(M) >=< u(M), P_n \psi_i(M) > \\ &=< u(M), \psi_i(M) >=< Lu(M), \varphi_i(M) > \\ &= Lu(M_i) = f(M_i), \quad i = 1, 2, ..., n. \end{aligned}$$

□

Now

$$\begin{aligned} u_n(M) &= \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} f(M_k) \overline{\psi}_i(M) \\ &= \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} f(M_k) \sum_{l=1}^{i} \beta_{il} \psi_l(M) \\ &= \sum_{i=1}^{n} C_i \psi_i(M), \end{aligned} \tag{3.3}$$

where $C_i = \sum_{k=1}^{i} \beta_{ik} f(M_k) \sum_{l=1}^{i} \beta_{il}$, then from Theorem 3.3 we have

$$Lu_n(M_j) = \sum_{i=1}^{n} C_i L\psi_i(M_j) = f(M_j), \;\; j = 1, 2, \cdots, n. \tag{3.4}$$

Thus, from Eq.(3.4), we obtain $C_i, i = 1, 2, \cdots, n$. Taking them into Eq.(3.3), we get the approximate solution $u_n(M)$ of Eqs.(1.9)-(1.12). Then we may obtain the approximation (w_n, p_n) of the original inverse problem from (1.7) and (1.8) for procedure I, and from (1.14) and (1.15) for procedure II.

Using Eqs.(3.3) and (3.4) to solve Eqs.(1.9)-(1.12), we can avoid the Gram-Schmidt orthogonalization process [28] of $\{\psi_i(M)\}_{i=1}^{\infty}$, and so we can improve the precision and decrease consumedly the runtime when the number of knots is the same, especially when the number of knots is large. It is efficiently applied to solving some model problems, and is of high precision.

4 Convergence analysis

We assume that $\{M_i\}_{i=1}^{\infty}$ is dense in D. We discuss the convergence of the approximate solutions constructed in Section 3. Let $u(M)$ be the exact solution of Eqs.(1.9)-(1.12), $u_n(M)$ be the n-term approximation solution of Eqs.(1.9)-(1.12). We set $\| u \|_C \triangleq \max\limits_{M\in D} |u(M)|$. Then arguing similarly to [28], we have

Theorem 4.1. *(i)* $\| u - u_n \|_{W_{(3,2)}(D)} \to 0$, $n \to \infty$. *Moreover a sequence* $\|u - u_n\|_{W_{3,2}(D)}$ *is monotonically decreasing in* n.

(ii)
$$\left\| \frac{\partial^{i+j} u}{\partial x^i \partial t^j} - \frac{\partial^{i+j} u_n}{\partial x^i \partial t^j} \right\|_C \to 0,\ \ n \to \infty;\ \ i = 0, 1, 2;\ \ j = 0, 1;\ \ i + j = 0, 1, 2.$$

5 Numerical example

In this section, some numerical examples are studied to demonstrate that our method is effective and the accuracy of approximate solution is high.

The domain D is divided into an $N \times M$ mesh with the spatial step size $h = 1/N$ in x direction and the time step size $k = 1/M$, respectively, in which N and M are integers.

Example 1

Consider problem (1.1)-(1.4) with the following conditions:

$$\begin{cases} w(x,0) = \sin\left(\dfrac{\pi}{2}x\right), \\ w(0,t) = 0, \qquad w(1,t) = \exp(t), \\ k(x,t) = \left(\left(\dfrac{\pi^2}{4} - t\right)\sin\left(\dfrac{\pi}{2}x\right) - \pi\cos\left(\dfrac{\pi}{2}x\right)\right)\exp(t), \\ q = 2, \qquad x^* = \dfrac{1}{2}, \\ E(t) = \dfrac{\sqrt{2}}{2}\exp(t). \end{cases}$$

The exact solution is $w(x,t) = \sin\left(\dfrac{\pi}{2}x\right)\exp(t)$ and $p(t) = 1 + t$.

With our method and finite difference method (FDM) [21], the root mean square (RMS) errors of the $w(x,t)$ and $p(t)$, and CPU time are presented in Tables 1-3 for procedure I.

From the above results, we can see that our method uses shorter CPU time, and obtains good results.

Another solution examples have been done to control the sensitivity of procedure I to errors. Artificial errors were introduced into the additional condition data by defining functions $E(t) = E(t)(1+d)$ where d represents the level of noise in the corresponding piece of data. Results with grid $N \times M = 8 \times 5$ and noise $d = 0.01$, $d = 0.025$ and $d = 0.033$ are given in Figures 1 and 2 for procedure I.

As seen from the figures that errors results are worsening.

Example 2

Consider problem (1.1)-(1.4) with the following conditions:

$$\begin{cases} w(x,0) = x, \\ w(0,t) = 0, \qquad w(1,t) = \exp(t), \\ k(x,t) = -(2 + xt^2)\exp(t), \\ q = 2, \qquad x^* = \dfrac{1}{2}, \\ E(t) = \dfrac{1}{2}\exp(t). \end{cases}$$

The exact solution is $w(x,t) = x\exp(t)$ and $p(t) = 1 + t^2$.

With our method and finite difference method (FDM) [5], the root mean square (RMS) errors of the $w(x,t)$ and $p(t)$, and CPU time are presented in Tables 4-6 for procedure II.

From the above results, we can see that our method uses shorter CPU time, and obtains good results.

Another solution examples have been done to control the sensitivity of procedure II to errors. Results with grid $N \times M = 8 \times 9$ and noise $d = 0.01$, $d = 0.025$ and $d = 0.033$ are given in Figures 3 and 4 for procedure II.

As seen from the figures that errors results are worsening.

6 Conclusions

In this article, our method has been successfully applied to an inverse problem of determining a t-dependent function of the source term which is proportional to the temperature. Our method is based on the reproducing kernel Hilbert space, and the approximate solution and its partial derivatives are proved to converge to the exact solution and its partial derivatives, respectively. Our method improves the previous method. The computational results confirmed the efficiency, reliability and accuracy of our method, and our method is applicable to more general inverse source problem for parabolic equations.

References

[1] R. Pourgholi, M. Rostamian, and M. Emamjome, *A numerical method for solving a nonlinear inverse parabolic problem*, Inverse Probl. Sci. Eng. 18 (2010), pp. 1151-1164.

[2] M. Dehghan, *Parameter determination in a partial differential equation from the overspecified data*, Math. Comput. Model. 41 (2005), pp. 197-213.

[3] M. Dehghan, *An inverse problem of finding a source parameter in a semilinear parabolic equation*, Appl. Math. Model. 25 (2001), pp. 743-754.

[4] J.R. Cannon and H.M. Yin, *On a class of nonlinear parabolic equations with nonlinear trace type functionals inverse problems*, Inverse Probl. 7 (1991), pp. 149-161.

[5] E.C. Baran, *Numerical procedures for determining of an unknown parameter in parabolic equation*, Appl. Math. Comput. 162 (2005), pp. 1219-1226.

[6] J.R. Cannon, Y. Lin, and S. Xu, *Numerical procedures for the determination of an unknown coefficient in semi-linear parabolic differential equations*, Inverse Probl. 10 (1994), pp. 227-243.

[7] J.R. Cannon and Y. Lin, *An inverse problem of finding a parameter in a semi-linear heat equation*, J. Math. Anal. Appl. 145 (1990), pp. 470-484.

[8] J.R. Cannon, Y. Lin, and S. Wang, *Determination of source parameter in parabolic equations*, Meccanica 27 (1992), pp. 85-94.

[9] A.I. Prilepko and D.G. Orlovskii, *Determination of the parameter of an evolution equation and inverse problems of mathematical physics. I*, Differentsial′nye Uravneniya 21 (1985), pp. 119-129.

[10] A.I. Prilepko and V.V. Soloev, *Solvability of the inverse boundary value problem of finding a coefficient of a lower order term in a parabolic equation*, J. Diff. Eq. 23 (1987), pp. 136-143.

[11] J.A. MacBain and J.B. Bednar, *Existence and uniqueness properties for one-dimensional magnetotellurics inversion problem*, J. Math. Phys. 27 (1986), pp. 645-649.

[12] J.A. MacBain, *Inversion theory for a parametrized diffusion problem*, SIAM. J. Appl. Math. 47 (1987), pp. 1386-1391.

[13] S. Wang and Y. Lin, *A finite difference solution to an inverse problem determining a control function in a parabolic partial differential equation*, Inverse Probl. 5 (1989), pp. 631-640.

[14] A. Grebennikov, *Local spline-approximation method of solving some coefficient inverse problems for differential equation of the parabolic type*, Inverse Probl. Sci. Eng. 9 (2001), pp. 455-469 .

[15] A.G. Fatullayev, *Numerical procedure for the determination of an unknown parameter in a parabolic equation*, Int. J. Comp. Math. 78 (2001), pp. 103-109.

[16] R. Yang and Y. Li, *Numerical method for the simultaneous determination of unknown parameters in a parabolic equation*, Int. J. Comp. Math. 87 (2010), pp. 150-157.

[17] R. Cheng, *Determination of a control parameter in a one-dimensional parabolic equation using the moving least-square approximation*, Int. J. Comp. Math. 85 (2008), pp. 1363-1373.

[18] B.T. Johansson and D. Lesnic, *A procedure for determining a spacewise dependent heat source and the initial temperature*, Appl. Anal. 87 (2008), pp. 265-276.

[19] B.T. Johansson and M. Pricop, *A method for identifying a spacewise-dependent heat source under stochastic noise interference*, Inverse Probl. Sci. Eng. 18 (2010), pp. 51-63.

[20] S.A. Yousefi and M. Dehghan, *Legendre multiscaling functions for solving the one-dimensional parabolic inverse problem*, Numer. Meth. Part. Differ. Equ. 25 (2009), pp. 1502-1510.

[21] A. Fatullayev and E. Can, *Numerical procedures for determining unknown source parameter in parabolic equations*, Math. Comput. Simul. 54 (2000), pp. 159-167.

[22] M. Dehghan, *Identification of a time-dependent coefficient in a partial differential equation subject to an extra measurement*, Numer. Meth. Part. Differ. Equ. 21 (2005), pp. 611-622.

[23] A. Shaw, K.N. Kaushik, and D. Roy, *Mesh-free approximations via the error reproducing kernel method and applications to nonlinear systems developing shocks*, Int. J. Non-Linear Mech. 44 (2009), pp. 417-431.

[24] Y. Lin, M. Cui, and Y. Zheng, *Representation of the exact solution for infinite system of linear equations*, Appl. Math. Comput. 168 (2005), pp. 636-650.

[25] K.M. Liew, T.Y. Ng, and Y.C. Wu, *Meshfree method for large deformation analysis-a reproducing kernel particle approach*, Eng. Struct. 24 (2002), pp. 543-551.

[26] A. Shaw, D. Roy, S.R. Reid, and M. Aleyaasin, *Reproducing kernel collocation method applied to the non-linear dynamics of pipe whip in a plane*, Int. J. Impact. Eng. 34 (2007), pp. 1637-1654.

[27] M. Cui and F. Geng, *A computational method for solving one-dimensional variable-coefficient Burgers equation*, Appl. Math. Comput. 188 (2007), pp. 1389-1401.

[28] Y. Zhou, M. Cui, and Y. Lin, *Numerical algorithm for parabolic problems with non-classical conditions*, J. Comput. Appl. Math. 230 (2009), pp. 770-780.

[29] I. Antoniou and K. Gustafson, *Wavelets and stochastic processes*, Math. Comput. Simul. 49 (1999), pp. 81-104.

[30] H. Hult, *Approximating some Volterra type stochastic integrals with applications to parameter estimation*, Stoch. Proc. Appl. 105 (2003), pp. 1-32.

[31] J. Xian, S.P. Luo, and W. Lin, *Weighted sampling and signal reconstruction in spline subspaces*, Signal Process 86 (2006), pp. 331-340.

[32] M. Pontil, *A note on different covering numbers in learning theory*, J. Complexity 19 (2003), pp. 665-671.

[33] T. Takeuchi and M. Yamamoto, *Tikhonov regularization by a reproducing kernel Hilbert space for the Cauchy problem for an elliptic equation*, SIAM. J. Sci. Comput. 31 (2008), pp. 112-142.

[34] Y.C. Hon and T. Takeuchi, *Discretized Tikhonov regularization by reproducing kernel Hilbert space for backward heat conduction problem*, Adv. Comput. Math. 34 (2011), pp. 167-183.

[35] S. Saitoh, *Applications of the reproducing kernel theory to inverse problems*, Comm. Korean Math. Soc. 16 (2001), pp. 371-383.

[36] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. 68 (1950), pp. 1-50.

[37] S. Saitoh, *Theory of reproducing kernels and its applications*, Longman Scientific & Technical, Harlow; copublished in the United States with JohnWiley & Sons, Inc., New York, 1988.

[38] M. Cui and Y. Lin, *Nonlinear Numerical Analysis in the Reproducing Kernel Space*, Nova Science Publishers, Inc., New York, 2009.

[39] D. Alpay and C. Dubi, *Carathéodory Fejér interpolation in the ball with mixed derivatives*, Linear Algebra Appl. 382 (2004), pp. 117-133.

[40] D. Alpay, V. Bolotnikov, and H.T. Kaptanoğlu, *The Schur algorithm and reproducing kernel Hilbert spaces in the ball*, Linear Algebra Appl. 342 (2002), pp. 163-186.

[41] G.W. Wasilkowski and H. Woźniakowski, *Finite-order weights imply tractability of linear multivariate problems*, J. Approx. Theory 130 (2004), pp. 57-77.

[42] Y. Xu, *Representation of reproducing kernels and the Lebesgue constants on the ball*, J. Approx. Theory 112 (2001), pp. 295-310.

[43] G.E. Fasshauer, *Dual bases and discrete reproducing kernels: a unified framework for RBF and MLS approximation*, Eng. Anal. Bound Elem. 29 (2005), pp. 313-325.

[44] Z. Luo and J. Levesley, *Error estimates and convergence rates for variational Hermite interpolation*, J. Approx. Theory 95 (1998), pp. 264-279.

[45] O. Christensen and T. Strohmer, *The finite section method and problems in frame theory*, J. Approx. Theory 133 (2005), pp. 221-237.

[46] W. Wang, M. Cui, and B. Han, *A new method for solving a class of singular two-point boundary value problems*, Appl. Math. Comput. 206 (2008), pp. 721-727.

Table 1: RMS errors of $w(x,t)$ with our method and FDM [21] for Example 1.

Our method	$h=0.1$	$h=0.05$	$h=0.02$
$k=0.05$	1.75E-4	3.36E-5	3.84E-6
$k=0.02$	2.09E-4	4.15E-5	5.00E-6
FDM [21]	$h=0.1$	$h=0.05$	$h=0.02$
$k=0.05$	2.81E-3	2.59E-3	2.52E-3
$k=0.02$	1.21E-3	1.00E-3	9.48E-4

Table 2: RMS errors of $p(t)$ with our method and FDM [21] for Example 1.

Our method	$h=0.1$	$h=0.05$	$h=0.02$
$k=0.05$	5.14E-3	1.47E-3	2.55E-4
$k=0.02$	4.97E-3	1.45E-3	2.54E-4
FDM [21]	$h=0.1$	$h=0.05$	$h=0.02$
$k=0.05$	9.56E-2	9.05E-2	8.90E-2
$k=0.02$	3.99E-2	3.52E-2	3.39E-2

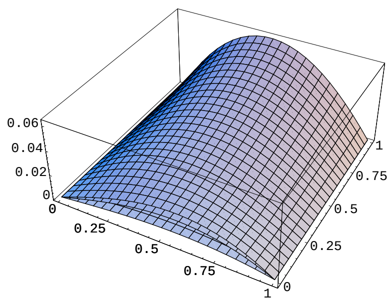

Figure 1: Error $|w - w_{40}|$ with our method for Example 1: (down) $d = 0.01$, (middle) $d = 0.025$, (top) $d = 0.033$.

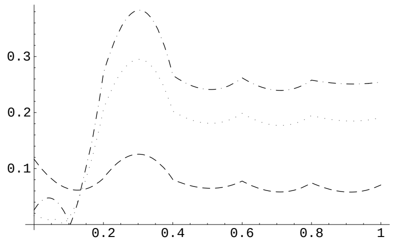

Figure 2: Error $|p-p_5|$ with our method for Example 1: $(---)$ $d=0.01$, $(\cdots)$ $d=0.025$, $(-\cdot-)$ $d=0.033$.

Table 3: CPU time with our method and FDM [21] for Example 1.

Method	$N \times M$	RMS errors of $w(x,t)$	RMS errors of $p(t)$	CPU time (s)
Our method	4×4	1.3×10^{-3}	1.6×10^{-2}	0.733
	5×5	8.3×10^{-4}	1.4×10^{-2}	1.794
	6×6	5.3×10^{-4}	1.0×10^{-2}	3.822
	7×7	3.8×10^{-4}	8.7×10^{-3}	7.113
	8×8	2.7×10^{-4}	7.0×10^{-3}	12.184
FDM [21]	30×30	1.6×10^{-3}	5.8×10^{-2}	1.029
	40×40	1.2×10^{-3}	4.3×10^{-2}	2.433
	50×50	9.5×10^{-4}	3.4×10^{-2}	5.367
	56×56	8.4×10^{-4}	3.0×10^{-2}	8.237
	66×66	7.1×10^{-4}	2.5×10^{-2}	15.459

Table 4: RMS errors of $w(x,t)$ with our method and FDM [5] for Example 2.

Our method	$h = 0.1$	$h = 0.05$	$h = 0.02$
$k = 0.05$	1.14E-4	3.48E-5	6.84E-6
$k = 0.02$	1.07E-4	3.36E-5	5.91E-6
FDM [5]	$h = 0.1$	$h = 0.05$	$h = 0.02$
$k = 0.05$	1.45E-3	1.40E-3	1.38E-3
$k = 0.02$	5.49E-4	5.02E-4	4.89E-4

Table 5: RMS errors of $p(t)$ with our method and FDM [5] for Example 2.

Our method	$h = 0.1$	$h = 0.05$	$h = 0.02$
$k = 0.05$	5.48E-3	1.72E-3	3.06E-4
$k = 0.02$	5.23E-3	1.64E-3	2.86E-4
FDM [5]	$h = 0.1$	$h = 0.05$	$h = 0.02$
$k = 0.05$	2.02E-1	1.95E-1	1.93E-1
$k = 0.02$	7.43E-2	6.88E-2	6.73E-2

Table 6: CPU time with our method and FDM [5] for Example 2.

Method	$N \times M$	RMS errors of $w(x,t)$	RMS errors of $p(t)$	CPU time (s)
Our method	3×3	8.3×10^{-4}	4.8×10^{-2}	0.125
	4×4	2.9×10^{-4}	1.7×10^{-2}	0.405
	5×5	1.9×10^{-4}	1.0×10^{-2}	0.998
	7×7	1.6×10^{-4}	8.0×10^{-3}	3.853
	8×8	1.4×10^{-4}	7.0×10^{-3}	6.568
FDM [5]	20×20	1.4×10^{-3}	1.9×10^{-1}	0.219
	26×26	1.0×10^{-3}	1.4×10^{-1}	0.578
	34×34	7.5×10^{-4}	1.0×10^{-1}	1.388
	50×50	4.9×10^{-4}	6.7×10^{-2}	5.274
	56×56	4.3×10^{-4}	6.0×10^{-2}	8.035

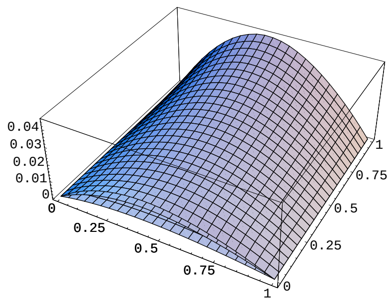

Figure 3: Error $|w - w_{72}|$ with our method for Example 2: (down) $d = 0.01$, (middle) $d = 0.025$, (top) $d = 0.033$.

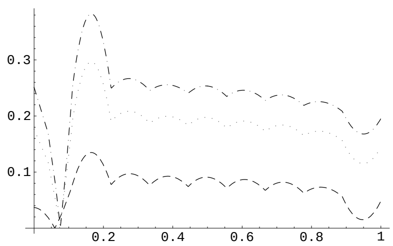

Figure 4: Error $|p - p_9|$ with our method for Example 2: $(---)$ $d = 0.01$, $(\cdots)$ $d = 0.025$, $(-\cdot-)$ $d = 0.033$.

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