UTMS 2011–10 April 28, 2011

Uniform estimate for distributions of the sum of i.i.d. random variables with fat tail: Infinite variance case

by

Kenji Nakahara

UNIVERSITY OF TOKYO
GRADUATE SCHOOL OF MATHEMATICAL SCIENCES
KOMABA, TOKYO, JAPAN

Uniform Estimate for Distributions of the Sum of i.i.d. Random Variables with Fat Tail: Infinite variance case

Kenji NAKAHARA

Abstract

In the previous preprint [14], We showed uniform estimates of distributions of the sum of i.i.d. random variables with finite variance in the threshold case. In this preprint, we show a uniform estimate without variance condition in the threshold case.

1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X_n, n = 1, 2, \ldots,$ be independent identically distributed random variables whose probability law is μ. Let $F : \mathbb{R} \to [0, 1]$ and $\bar{F} : \mathbb{R} \to [0, 1]$ be given by $F(x) = \mu((-\infty, x])$ and $\bar{F}(x) = \mu((x, \infty)), \quad x \in \mathbb{R}$. We assume the following.

(A1) $\bar{F}(x)$ is a regularly varying function of index $-\alpha$ for some $\alpha \geq 2$, as $x \to \infty$,i.e., if we let

$$L(x) = x^{\alpha} \bar{F}(x) \ , \ x \geq 1,$$

then $L(x) > 0$ for any $x \geq 1$, and for any $a > 0$

$$\frac{L(ax)}{L(x)} \to 1 \ , \ x \to \infty.$$

(A2) $\int_{-\infty}^{0} |x|^{2+\delta_0} \mu(dx) < \infty$ for some $\delta_0 \in (0, 1)$ and $\int_{\mathbb{R}} x \mu(dx) = 0$.

(A3) The probability law μ is absolutely continuous and has a density function $\rho : \mathbb{R} \to [0, \infty)$ which is right continuous and has a finite total variation.

Let us define $\Phi_k : \mathbb{R} \to \mathbb{R}, k = 0, 1, 2, 3$ by

$$\Phi_0(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp(-\frac{y^2}{2}) dy, \qquad x \in \mathbb{R},$$

$$\Phi_1(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) = -\frac{d}{dx}\Phi_0(x),$$

and

$$\Phi_k(x) = (-1)^{k-1}\frac{d^{k-1}}{dx^{k-1}}\Phi_1(x), \qquad k = 2, 3.$$

Let $t_n = \sup\{t > 0; n\int_{-\infty}^{t} x^2\mu(dx) > t^2\}$. Then from (A1), (A2) we can see that

$$P(\sum_{k=1}^{n} X_k > t_n s) \to \Phi_0(s), \qquad n \to \infty, \quad s \geq 1.$$

Let $v_n = \int_{-\infty}^{t_n} x^2\mu(dx)$ for $n \geq 1$. We also define $H : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} H(n, s) = \Phi_0(s) \quad &+ \quad n\int_{-\infty}^{s} \bar{F}(t_n(s - x))\Phi_1(x)dx \\ &- \quad \left(v_n^{-1/2} n^{1/2}\Phi_1(s)\int_0^{\infty} x\mu(dx) + \frac{\Phi_2(s)}{2} v_n^{-1}\int_0^{t_n} x^2\mu(dx)\right). \end{aligned}$$

In this paper, we show the following theorem, which is corresponding to Theorem 2 in the previous preprint.

Theorem 1. *Assume* (A1) *for* $\alpha = 2$*,* (A2) *and* (A3)*. Then for any* $\delta \in (0, 1)$*, there is a* $C > 0$ *such that*

$$\sup_{s\in[1,\infty)} |\frac{P(\sum_{k=1}^{n} X_k > t_n s)}{H(n, s)} - 1| \leq C(n\bar{F}(t_n))^{1-\delta}, \qquad n \geq 1. \tag{1}$$

In particular,

$$\sup_{s\in[1,\infty)} |\frac{P(\sum_{k=1}^{n} X_k > t_n s)}{\Phi_0(s) + n\bar{F}(t_n s)} - 1| \to 0, \qquad n \to \infty.$$

We also prove the following to obtain Theorem 1.

Theorem 2. *Assume* (A1) *for* $\alpha = 2$*,* (A2) *and* (A3)*. Then for any* $\delta \in (0, 1)$*, there is a* $C > 0$ *such that*

$$|P(\sum_{k=1}^{n} X_k > t_n s) - H(n, s)| \leq C(n\bar{F}(t_n))^{2-\delta}, \qquad s \geq 1.$$

Theorem 2 is corresponding to Theorem 4 in [14]. Throughout this paper we assume (A1) for $\alpha = 2$, (A2) and (A3). Then we see that $t_n = n^{1/2}v_n^{1/2}$ and $n\bar{F}(t_n) = \dfrac{L(t_n)}{v_n} \to 0, n \to \infty$ (see Equation (2)). See section 2 in [14] for the properties of regularly varying functions needed in this paper.

2 Estimate for moments and characteristic functions

Let

$$\eta_k(t) = \int_{-\infty}^{t} x^k \mu(dx), \qquad t > 0,\ k = 1, 2,$$

and

$$\eta_3(t) = \int_{1}^{t} x^3 \mu(dx), \qquad t > 1.$$

Then we see that

$$-\eta_1(t) = \int_{t}^{\infty} x\mu(dx) = \int_{t}^{\infty} \bar{F}(x)dx + t\bar{F}(t), \qquad t > 0,$$

$$\eta_3(t) = \bar{F}(1) - t^3\bar{F}(t) + 3\int_{1}^{t} x^2\bar{F}(x)dx \qquad t > 1.$$

and $\eta_2(t)$ is slowly varying.

Let $t_n = \sup\{t > 0; n\eta_2(t) > t^2\}$ and $v_n = \eta_2(t_n) = \int_{-\infty}^{1} x^2\mu(dx) - L(t_n) + L(1) + 2\int_1^{t_n} x^{-1}L(x)dx$.

Note that $t_n = n^{1/2}\eta_2(t_n)^{1/2} \geq n^{1/2}\eta_2(0) \to \infty, \quad n \to \infty$.

Let $a_n = n\bar{F}(t_n)$. Then for any $t_0 > 0$, we see that for $t > t_0$,

$$\frac{1}{L(t)}\int_{1}^{t} x^{-1}L(x)dx = \int_{1/t}^{1} \frac{L(tx)}{L(t)}\frac{dz}{z} \geq \int_{1/t_0}^{1} \frac{L(tx)}{L(t)}\frac{dz}{z} \to \int_{1/t_0}^{1} \frac{dz}{z} = \log t_0.$$

Since t_0 is arbitrary, we see that

$$a_n = \frac{L(t_n)}{v_n} \to 0, \quad n \to \infty. \tag{2}$$

Proposition 1. *There is a $C > 0$ such that*

$$-n\frac{\eta_1(t_n)}{t_n} \leq Ca_n, \tag{3}$$

$$n\frac{\eta_3(t_n)}{t_n^3} \leq Ca_n. \tag{4}$$

for any $n \geq 1$.

Proof. Similarly to Proposition 8 in [14], we can prove Proposition 1. □

Proposition 2. *There is $c_1 > 0$ such that for any integer n, m with $n \geq m$ and $\xi \in \mathbb{R}$ with $|\xi| \geq a_n^{-\delta}$,*

$$|\varphi(t_n^{-1}\xi, \mu(t_n))|^n \leq (1 + \frac{c_1\eta_2(t_n|\xi|^{-1})}{mv_n}|\xi|^2)^{-m/4}.$$

In particular, there is $c_2 > 0$ such that for any integer n, m with $n \geq m$ and $\xi \in \mathbb{R}$ with $|\xi| \in (a_n^{-\delta}, t_n)$,

$$|\varphi(t_n^{-1}\xi, \mu(t_n))|^n \leq (1 + \frac{c_2}{m}|\xi|)^{-m/4}.$$

Proof. Let $t > 2$. We see that for $\xi \in (-t^{-1}, t^{-1})$,

$$\begin{aligned}
& |\varphi(\xi, \mu(t))|^2 \\
= \ & (1 - \bar{F}(t))^2 \int_{\mathbb{R}} \int_{\mathbb{R}} \exp(i\xi(x-y))\rho(x)1_{(-\infty,t)}(x)\rho(y)1_{(-\infty,t)}(y)dxdy \\
\leq \ & 1 - \int_{\mathbb{R}} \int_{\mathbb{R}} (1 - \cos(\xi(x-y)))\rho(x)1_{(-t,t)}(x)\rho(y)1_{(-t,t)}(y)dxdy \\
\leq \ & 1 - \frac{|\xi|^2}{4} \int_{\mathbb{R}} \int_{\mathbb{R}} (x-y)^2\rho(x)1_{(-t,t)}(x)\rho(y)1_{(-t,t)}(y)dxdy.
\end{aligned}$$

Similarly we have for $\xi \in \mathbb{R}$ with $|\xi| > t^{-1}$,

$$\begin{aligned}
& |\varphi(\xi, \mu(t))|^2 \\
\leq \ & 1 - \frac{|\xi|^2}{4} \int_{\mathbb{R}} \int_{\mathbb{R}} (x-y)^2\rho(x)1_{(-|\xi|^{-1},|\xi|^{-1})}(x)\rho(y)1_{(-|\xi|^{-1},|\xi|^{-1})}(y)dxdy.
\end{aligned}$$

We can easily see that

$$\eta_2(t)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} (x-y)^2\rho(x)1_{(-t,t)}(x)\rho(y)1_{(-t,t)}(y)dxdy \to 2, \qquad t \to \infty.$$

Hence we see that there is a $c_1 > 0$ such that for any $n \geq 2$ and $\xi \in \mathbb{R}$ with $|\xi| \geq a_n^{-\delta}$,

$$|\varphi(t_n^{-1}\xi, \mu(t_n))| \leq (1 - \frac{c_1\eta_2(t_n|\xi|^{-1})}{nv_n}|\xi|^2)^{1/2} \leq (1 + \frac{c_1\eta_2(t_n|\xi|^{-1})}{nv_n}|\xi|^2)^{-1/4}.$$

It is easy to check that $(1 + x/\beta)^\beta \geq 1 + x$ for any $\beta \geq 1$ and $x \geq 0$. Therefore if $n \geq m$, we have

$$\left(1 + \frac{c_1\eta_2(t_n|\xi|^{-1})}{nv_n}|\xi|^2\right)^{n/m} \geq 1 + \frac{c_1\eta_2(t_n|\xi|^{-1})}{mv_n}|\xi|^2.$$

Since $\eta_2(t)$ is slowly varying, we see that for $\xi \in \mathbb{R}$ with $t_n \geq |\xi| \geq a_n^{-\delta}$,

$$\frac{\eta_2(t_n|\xi|^{-1})}{v_n} = \frac{\eta_2(t_n|\xi|^{-1})}{\eta_2(t_n|\xi|^{-1}|\xi|)} \geq M(1)^{-1}|\xi|^{-1}.$$

Therefore we have our assertion. □

3 Asymptotic expansion of characteristic functions

Remind that $t_n = n^{1/2} v_n^{1/2}$ and $a_n = n\bar{F}(t_n) = v_n^{-1} L(t_n)$.

In this section, we prove the following Lemma.

Lemma 1. *Let*

$$\begin{aligned} R_{n,0}(\xi) &= \exp(\frac{\xi^2}{2})\varphi(n^{-1/2}\xi;\mu(t_n))^n - (1 + n(\varphi(n^{-1/2}\xi;\mu(t_n)) - 1) + \frac{\xi^2}{2}), \\ R_{n,1}(\xi) &= \exp(\frac{\xi^2}{2})\varphi(n^{-1/2}\xi;\mu(t_n))^n - 1, \\ R_{n,2}(\xi) &= \exp(\frac{\xi^2}{2})\varphi(n^{-1/2}\xi;\mu(t_n))^{n-1} - 1. \end{aligned}$$

Then there is a $C > 0$ such that

$$|R_{n,0}(\xi)| \le C a_n^{2-5\delta} |\xi| \tag{5}$$

and

$$|R_{n,1}(\xi)| + |R_{n,2}(\xi)| \le C a_n^{1-2\delta} |\xi|, \tag{6}$$

for any $n \ge 8$ and $\xi \in \mathbb{R}$ with $|\xi| \le a_n^{-\delta}$.

As a corollary to Lemma 1, we have the following.

Corollary 1. *Let*

$$\tilde{R}_0(n,s) = \mu(t_n)^{*n}((t_n s, \infty)) - \Phi_0(s) - \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-is\xi}}{i\xi}\left(n(\varphi(n^{-1/2}\xi;\mu(t_n)) - 1) + \frac{\xi^2}{2}\right) e^{-\xi^2/2} d\xi,$$

$$\tilde{R}_{1,k}(n,s) = \mu(t_n)^{*(n-k)}((t_n s, \infty)) - \Phi_0(s), \qquad k = 0, 1,$$

and

$$\tilde{R}_2(n,s) = \frac{1}{2\pi}\int_{\mathbb{R}} \left| \varphi(n^{-1/2}\xi;\mu(t_n))^{n-1} - e^{-\frac{\xi^2}{2}} \right| d\xi.$$

Then there is a $C > 0$ such that for any $n \ge 1$ and $s \in \mathbb{R}$, we have

$$|\tilde{R}_0(n,s)| \le C a_n^{2-6\delta} \tag{7}$$

and

$$|\tilde{R}_{1,0}(n,s)| + |\tilde{R}_{1,0}(n,s)| + |\tilde{R}_2(n,s)| \le C a_n^{1-4\delta}. \tag{8}$$

Proof. From Proposition 7 in [14], we see that

$$\begin{aligned}
&\tilde{R}_0(n,s)\\
=\ & \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-is\xi}}{i\xi}\left(\varphi(n^{-1/2}\xi;\mu(t_n))^n - e^{-\frac{\xi^2}{2}} - \left(n(\varphi(n^{-1/2}\xi;\mu(t_n))-1)+\frac{\xi^2}{2}\right)e^{-\frac{\xi^2}{2}}\right)d\xi\\
=\ & \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-is\xi}}{i\xi}R_{n,0}(\xi)e^{-\xi^2/2}d\xi.
\end{aligned}$$

By Lemma 1, there is a $C_0 > 0$ such that

$$\int_{|\xi|\le a_n^{-\delta}} \frac{|R_{n,0}(\xi)|}{|\xi|}d\xi \le C_0 a_n^{2-6\delta}.$$

It is easy to see that

$$n|\varphi(n^{-1/2}\xi;\mu(t_n))-1| \le \frac{nt_n^{-1}|\eta_1(t_n)||\xi|}{1-\bar{F}(t_n)} + \frac{|\xi|^2}{2\eta_2(t_n)(1-\bar{F}(t_n))}, \qquad \xi\in\mathbb{R}.$$

From the above inequality and Proposition 6 in [14] and 2, we see that for any $m \ge 2/\delta$, there is a $C_1 > 0$ such that for any $n \ge 4m$

$$|\varphi(n^{-1/2}\xi;\mu(t_n))|^n + \left|n(\varphi(n^{-1/2}\xi;\mu(t_n))-1)+1+\frac{\xi^2}{2}\right|e^{-\frac{\xi^2}{2}} \le C_1|\xi|^{-m}, for\ |\xi|\in(a_n^{-\delta}, v_n^{1/2}a_n^{-\delta})$$

and

$$|\varphi(n^{-1/2}\xi;\mu(t_n))|^n + \left|n(\varphi(n^{-1/2}\xi;\mu(t_n))-1)+1+\frac{\xi^2}{2}\right|e^{-\frac{\xi^2}{2}} \le C_1\left(\frac{|\xi|}{v_n^{1/2}}\right)^{-m},$$

for $|\xi| \ge v_n^{1/2}a_n^{-\delta}$. Hence we have

$$\begin{aligned}
&\int_{|\xi|>a_n^{-\delta}} |\xi|^{-1}\left|\varphi(n^{-1/2}\xi;\mu(t_n))^n - e^{-\frac{\xi^2}{2}} - \left(n(\varphi(n^{-1/2}\xi;\mu(t_n))-1)+\frac{\xi^2}{2}\right)e^{-\frac{\xi^2}{2}}\right|d\xi\\
\le\ & 2C_1\int_{a_n^{-\delta}}^{v_n^{1/2}a_n^{-\delta}} |\xi|^{-m-1}v_n^{1/2}d\xi + 2C_1\int_{v_n^{1/2}a_n^{-\delta}}^{\infty}\left(\frac{|\xi|}{v_n^{1/2}}\right)^{-m-1}d\xi\\
=\ & \frac{4C_1}{m}a_n^{m\delta} \le \frac{4C_1}{m}a_n^2.
\end{aligned}$$

Therefore we have Equation (7).

We also see that

$$\begin{aligned}
\tilde{R}_{1,k}(n,s) &= \frac{1}{2\pi}\int_{\mathbb{R}}\frac{e^{-is\xi}}{i\xi}\left(\varphi(n^{-1/2}\xi;\mu(t_n))^n - e^{-\frac{\xi^2}{2}}\right)d\xi\\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\frac{e^{-is\xi}}{i\xi}R_{n,1+k}(\xi)e^{-\xi^2/2}d\xi,\\
\tilde{R}_2(n,s) &= \frac{1}{2\pi}\int_{\mathbb{R}}|R_{n,2}(\xi)|e^{-\xi^2/2}d\xi.
\end{aligned}$$

Similarly to Equation (7), we have Equation (8). $\square$

We make some preparations to prove Lemma 1.

Let

$$R_0(n,\xi) = \varphi(t_n^{-1}\xi, \mu(t_n)) - (1 - \frac{\xi^2}{2n}).$$

First we prove the following.

Proposition 3. *There is a constant $C > 0$ such that for any $n \geq 1$, and $\xi \in \mathbb{R}$ with $|\xi| \leq a_n^{-\delta}$,*

$$|nR_0(n,\xi)| \leq Ca_n^{1-2\delta}|\xi|$$

and

$$n|\varphi(n^{-1/2}\xi; \mu(t_n)) - 1| \quad \leq \quad Ca_n^{-\delta}|\xi|.$$

In particular

$$\sup\{|nR_0(n,\xi)|; |\xi| \leq a_n^{-\delta}\} \to 0, \qquad n \to \infty. \tag{9}$$

Proof. Similarly to Proposition 9 in [14], we can prove Proposition 3. □

Let

$$R_{1,k}(n,\xi) = (n-k)\log\varphi(t_n^{-1}\xi; \mu(t_n)) - n(\varphi(n^{-1/2}\xi; \mu(t_n)) - 1), \quad k = 0, 1.$$

Proposition 4. *There is a $C > 0$, such that for any $\xi \in \mathbb{R}$ with $|\xi| \leq a_n^{-\delta}$,*

$$|R_{1,k}(n,\xi)| \leq Cn^{-1}a_n^{-3\delta}|\xi|.$$

In particular

$$\sup\{|R_{1,k}(n,\xi)|; |\xi| \leq a_n^{-\delta}\} \to 0, \qquad n \to \infty. \tag{10}$$

Proof. Similarly to Proposition 10 in [14], we can prove Proposition 4. □

Let us prove Lemma 1. Note that for $k = 0, 1$

$$\log(e^{\xi^2/2}\varphi(n^{-1/2}\xi; \mu(t_n))^{n-k}) = nR_0(n,\xi) + R_{1,k}(n,\xi).$$

We see that

$$e^{\xi^2/2}\varphi(n^{-1/2}\xi; \mu(t_n))^{n-k} = \exp(nR_0(n,\xi) + R_{1,k}(n,\xi)).$$

Hence we see that

$$\begin{aligned} R_{n,0}(\xi) &= e^{\xi^2/2}\varphi(n^{-1/2}\xi;\mu(t_n))^n - (1+nR_0(n,\xi)) \\ &= \exp(nR_0(n,\xi)) - (1+nR_0(n,\xi)) + \exp(nR_0(n,\xi))(\exp(R_{1,0}(n,\xi))-1) \end{aligned}$$

From Equation (9), we see that there is a $C>0$ such that

$$|R_{n,0}(\xi)| \leq C\left(|nR_0(n,\xi)|^2 + |R_{1,0}(n,\xi)|\right).$$

Therefore we have Equation (5) from Proposition 3 and 4. Proof of Equation (6) is similar to Equation (5).

4 Proof of Theorem 2

Note that

$$P(\sum_{l=1}^n X_l > t_n s) = \sum_{k=0}^n I_k(n,s),$$

where

$$I_k(n,s) = P(\sum_{l=1}^n X_l > t_n s, \sum_{l=1}^n 1_{\{X_l > t_n\}} = k), \qquad k=0,1,\ldots,n.$$

Then we have

$$I_k(n,s) = \binom{n}{k} P(\sum_{l=1}^n X_l > t_n s, X_i > t_n, i=1,\ldots,k, X_j \leq t_n, j=k+1,\ldots,n),$$

for $k=0,1,\ldots,n$.

Let $\bar{F}_{n,0}(x) = P(X_1 > t_n x, X_1 \leq t_n) = (1-\bar{F}(t_n))\mu(t_n)((t_n^{-1}x,\infty))$ and $\bar{F}_{n,1}(x) = P(X_1 > t_n x, X_1 > t_n)$. Note that $\bar{F}_{n,0}(x) + \bar{F}_{n,1}(x) = \bar{F}(t_n x)$.

We show estimations on $I_0(n,s)$ and $I_1(n,s)$. Since the proofs of the estimates are same as Proposition 11, 12 in [14], we omit the proofs.

Proposition 5. *There is a $C>0$ such that*

$$\begin{aligned} & |I_0(n,s) - (1-n)\Phi_0(s) - \frac{1}{2}\Phi_2(s) - n\int_{\mathbb{R}} \bar{F}_{n,0}(s-x)\Phi_1(x)dx| \\ \leq\ & Ca_n^{2-5\delta}, \qquad n\geq 1, s\geq 1. \end{aligned}$$

Proposition 6. *There is a $C > 0$ such that*

$$|I_1(n,s) - n\int_{\mathbb{R}} \bar{F}_{n,1}(s-x)\Phi_1(x)dx| \le Ca_n^{2-5\delta}, \qquad n \ge 1, s \ge 1.$$

Now, let us prove Theorem 2.

From Proposition 5 and 6, we see that there is a $C > 0$ such that

$$\begin{aligned} & |I_0(n,s) + I_1(n,s) - (1-n)\Phi_0(s) - \frac{1}{2}\Phi_2(s) - n\int_{\mathbb{R}} \bar{F}(t_n(s-x))\Phi_1(x)dx| \\ \le\ & Ca_n^{2-5\delta}. \end{aligned}$$

Note that

$$\begin{aligned} & \int_{\mathbb{R}} \bar{F}(t_n(s-x))\Phi_1(x)dx - \Phi_0(s) \\ =\ & \int_{-\infty}^{s} \bar{F}(t_n(s-x))\Phi_1(x)dx + \int_s^{\infty} (\bar{F}(t_n(s-x)) - 1_{\{x>s\}})\Phi_1(x)dx \\ =\ & \int_{-\infty}^{s} \bar{F}(t_n(s-x))\Phi_1(x)dx - \int_s^{\infty} F(t_n(s-x))\Phi_1(x)dx \end{aligned}$$

and

$$n\int_s^{\infty} F((t_n(s-x))\Phi_1(x)dx = nt_n^{-1}\int_{-\infty}^{0} F(y)\Phi_1(s - t_n^{-1}y)dy.$$

Let $R(s,y) = \Phi_1(s-y) - \Phi_1(s) - \Phi_2(s)y$, for $s > 0$ and $y \le 0$, then we see that there is a $C_1 > 0$ such that

$$|R(s,y)| \le C_1|y|^{1+\delta_0}.$$

Hence we have

$$\begin{aligned} & n|\int_s^{\infty} F(t_n(s-x))\Phi_1(x)dx - \sum_{k=1}^{2} t_n^{-k}\Phi_k(s)\int_{-\infty}^{0} y^{k-1}F(y)dy| \\ =\ & nt_n^{-1}|\int_{-\infty}^{0} R(s,t_n^{-1}y)F(y)dy| \\ \le\ & C_1 n^{-\delta_0/2}\eta_2(t_n)^{-(1+\delta_0)/2}\int_{-\infty}^{0} y^{1+\delta_0}F(y)dy \\ \le\ & Cn^{-\delta_0/2}, \end{aligned}$$

where $C = C_1\eta_2(0)^{-(1+\delta_0)/2}\int_{-\infty}^{0} y^{1+\delta_0}F(y)dy < \infty$.

Since

$$\int_{-\infty}^{0} F(y)dy = \int_{-\infty}^{0} y\mu(dy) = -\int_{0}^{\infty} y\mu(dy)$$

and

$$\int_{-\infty}^{0} yF(y)dy = \frac{1}{2}\int_{-\infty}^{0} y^2\mu(dy) = \frac{\eta_2(t_n)}{2} - \frac{1}{2}\int_{0}^{t_n} y^2\mu(dy),$$

we see that

$$\frac{1}{2}\Phi_2(s) - nt_n^{-2}\Phi_2(s)\int_{-\infty}^{0} yF(y)dy = \frac{\Phi_2(s)}{2}\eta_2(t_n)^{-1}\int_{0}^{t_n} y^2\mu(dy).$$

Therefore we have

$$\begin{aligned} & |(1-n)\Phi_0(s) + \frac{1}{2}\Phi_2(s) + n\int_{\mathbb{R}} \bar{F}(t_n(s-x))\Phi_1(x)dx - H(n,s)| \\ \leq \; & Cn^{-\delta_0/2}. \end{aligned}$$

We also see that

$$\sum_{k=2}^{n} I_k(n,s) \leq \sum_{k=2}^{n} \frac{n(n-1)}{k(k-1)}\binom{n-2}{k-2}\bar{F}(t_n)^k(1-\bar{F}(t_n))^{n-k} \leq \frac{n(n-1)}{2}\bar{F}(t_n)^2 = a_n^2.$$

This completes the proof of Theorem 2.

5 Proof of Theorem 1

Recall that $t_n = \sup\{t > 0; n\eta_2(t) > t^2\}$,$v_n = \eta_2(t_n)$ and $a_n = \frac{L(t_n)}{v_n}$. Let $v_n(t) = \eta_2(t_n t)$ for $t > 0$.

Let

$$\begin{aligned} \hat{F}_n(s) &= \int_{-\infty}^{s} \bar{F}(t_n(s-x))\Phi_1(x)dx, \\ A(n,s) &= n\hat{F}_n(s) - v_n^{-1/2}n^{1/2}\Phi_1(s)\int_{0}^{\infty} x\mu(dx) - \frac{v_n^{-1}}{2}\Phi_2(s)\int_{0}^{t_n} x^2\mu(dx), \\ &= n\hat{F}_n(s) - v_n^{-1/2}n^{1/2}\Phi_1(s)\int_{0}^{\infty} \bar{F}(x)dx - v_n^{-1}\Phi_2(s)\left(\int_{0}^{t_n} x\bar{F}(x)dx - \frac{L(t_n)}{2}\right), \\ H(n,s) &= \Phi_0(s) + A(n,s), \end{aligned}$$

and

$$H_0(n,s) = \Phi_0(s) + n\bar{F}(t_n s).$$

Similarly to Lemma 2 in [14], we can prove the following.

Lemma 2.

$$\sup_{s\in[1,\infty)}\left|\frac{H(n,s)}{H_0(n,s)}-1\right|\to 0,\qquad n\to\infty.$$

We also prove the following.

Lemma 3. *For any $\beta>0$ and $\delta\in(0,1)$, there is a $C>0$ such that we have*

$$\sup_{s>a_n^{-\beta}}\left|\frac{P(\sum_{k=1}^n X_k>t_n s)}{H(n,s)}-1\right|\le Ca_n^{1-\delta}.$$

We make some preparations to prove Lemma 3. Similarly to Proposition 26 in [7], we can prove the following.

Proposition 7. (1) *For any $t,s>0$, and $n\ge 2$,*

$$P(\sum_{k=2}^n X_k 1_{\{X_k\le tn^{1/2}\}}>sn^{\frac12})\le \exp(\frac{3}{t^2}E[X_1^2 1_{\{X_1\le tn^{1/2}\}}]-\frac{s}{t}).$$

(2) *For any $s,t>0$, $\varepsilon\in(0,1)$ with $t<(1-\varepsilon)s$,*

$$|P(\sum_{k=1}^n X_k>sn^{\frac12})-nP(X_1+\sum_{k=2}^n X_k1_{\{X_k\le tn^{1/2}\}}>sn^{\frac12},\sum_{k=2}^n X_k 1_{\{X_k\le tn^{1/2}\}}\le\varepsilon sn^{\frac12})|$$
$$\le 2n(n-1)\bar F(tn^{\frac12})^2+\exp(\frac{3}{t^2}E[X_1^2 1_{\{X_1\le tn^{1/2}\}}]-\frac{s}{t})+n\bar F(tn^{\frac12})\exp(\frac{3}{t^2}E[X_1^21_{\{X_1\le tn^{1/2}\}}]-\frac{\varepsilon s}{2t}).$$

Proof. We prove this proposition briefly. We see that

$$\begin{aligned}P(\sum_{k=2}^n X_k1_{\{X_k\le tn^{1/2}\}}>sn^{1/2}) &\le \exp(-\frac{s}{t})E[\exp(\frac{1}{tn^{1/2}}\sum_{k=2}^n X_k1_{\{X_k\le tn^{1/2}\}})]\\ &\le \exp(-\frac{s}{t})E[\exp(\frac{1}{tn^{1/2}}X_11_{\{X_1\le tn^{1/2}\}})]^{n-1}.\end{aligned}$$

It is easy to see that $e^x\le 1+x+x^2(1\vee e^x)$ for any $x\in\mathbb{R}$. So we have

$$\begin{aligned}E[\exp(\frac{1}{tn^{1/2}}X_11_{\{X_1\le tn^{1/2}\}})] &\le 1+\frac{1}{tn^{1/2}}E[X_11_{\{X_1\le tn^{1/2}\}}]+\frac{1}{t^2n}E[X_1^21_{\{X_1\le tn^{1/2}\}}]\exp(1)\\ &\le 1-\frac{1}{tn^{1/2}}E[X_11_{\{X_1>tn^{1/2}\}}]+\frac{3}{t^2n}E[X_1^21_{\{X_1\le tn^{1/2}\}}]\\ &\le 1+\frac{3}{t^2n}E[X_1^21_{\{X_1\le tn^{1/2}\}}].\end{aligned}$$

Since $\log(1+x) \le x$ for $x > 0$, we see that

$$(n-1)\log E[\exp(\frac{1}{tn^{1/2}} X_1 1_{\{X_1 \le tn^{1/2}\}})] \le \frac{3}{t^2} E[X_1^2 1_{\{X_1 \le tn^{1/2}\}}].$$

Hence we have the assertion (1).

Note that

$$P(\sum_{k=1}^n X_k > sn^{1/2}) = \sum_{m=0}^n I_m,$$

where

$$I_m = P(\sum_{k=1}^n X_k > sn^{1/2}, \sum_{k=1}^n 1_{\{X_k > tn^{1/2}\}} = m), \qquad m = 0, 1, \dots, n.$$

Then we have

$$I_m = \binom{n}{m} P(\sum_{k=1}^n X_k > sn^{1/2}, X_i > tn^{1/2}, i = 1, \dots, m,\ X_j \le tn^{1/2}, j = m+1, \dots, n),$$

for $m = 0, 1, \dots, n$. We can easily see that

$$\sum_{m=2}^n I_m \le \frac{n(n-1)}{2} \bar{F}(tn^{1/2})^2. \tag{11}$$

From (1), we have

$$I_0 \le \exp(\frac{3}{t^2} E[X_1^2 1_{\{X_1 \le tn^{1/2}\}}] - \frac{s}{t}). \tag{12}$$

Let $A_1 = \{X_1 > tn^{1/2}\}$, $A_2 = \{X_k \le tn^{1/2}, k = 2, 3, \dots, n\}$,
$B_1 = \{X_1 + \sum_{k=2}^n X_k 1_{\{X_k \le tn^{1/2}\}} > sn^{1/2}\}$ and $B_2 = \{\sum_{k=2}^n X_k 1_{\{X_k \le tn^{1/2}\}} \le \varepsilon sn^{1/2}\}$.

Note that $B_1 \cap B_2 \subset A_1$, since $t < (1-\varepsilon)s$. So we see that

$$\begin{aligned} & |P(B_1 \cap A_1 \cap A_2) - P(B_1 \cap B_2)| \\ \le\ & P(B_1 \cap B_2^c \cap A_1 \cap A_2) + P(B_1 \cap B_2 \cap A_1 \cap A_2^c) \\ \le\ & P(A_1)P(B_2^c) + P(A_1)P(A_2^c). \end{aligned} \tag{13}$$

Note that

$$P(A_2^c) \le \sum_{k=2}^n P(X_k > tn^{1/2}) = (n-1)\bar{F}(tn^{1/2}).$$

Also, by the assertion (1) we have

$$P(B_2^c) \le \exp(\frac{3}{t^2} E[X_1^2 1_{\{X_1 \le tn^{1/2}\}}] - \frac{\varepsilon s}{2t}).$$

Since $I_1 = nP(B_1 \cap A_1 \cap A_2)$, we have the assertion (2) from Equations (11), (12) and (13). This completes the proof. $\square$

We apply for Proposition 7 with $v_n^{1/2}s$, $v_n^{1/2}t$. Then we have

$$P(\sum_{k=2}^{n} X_k 1_{\{X_k \le t_n t\}} > st_n) \le 2\exp(\frac{3v_n(t)}{t^2 v_n} - \frac{s}{t}) \tag{14}$$

and

$$\begin{aligned} & |P(\sum_{k=1}^{n} X_k > st_n) - nP(X_1 + \sum_{k=2}^{n} X_k 1_{\{X_k \le t_n t\}} > st_n, \sum_{k=2}^{n} X_k 1_{\{X_k \le t_n t\}} \le \varepsilon t_n s)| \\ \le \quad & 2n(n-1)\bar{F}(t_n t)^2 + \exp(\frac{3\eta_2(t_n t)}{t^2 \eta_2(t_n)} - \frac{s}{t}) + 2n\bar{F}(t_n t)\exp(\frac{6\eta_2(t_n t)}{t^2\eta_2(t_n)} - \frac{\varepsilon s}{2t}). \end{aligned} \tag{15}$$

Since $\eta_2(t)$ is slowly varying, we see that there is a $C > 0$ such that $\eta_2(t_n t)/\eta_2(t_n) \le Ct$ for $t \ge 1$. So we have

$$\begin{aligned} |P(\sum_{k=1}^{n} X_k > st_n) - nP(X_1 + \sum_{k=2}^{n} X_k 1_{\{X_k \le t_n t\}} > st_n, \sum_{k=2}^{n} X_k 1_{\{X_k \le t_n t\}} \le \varepsilon t_n s)| \\ \le 2n(n-1)\bar{F}(t_n t)^2 + \exp(\frac{3C}{t} - \frac{s}{t}) \\ +2n\bar{F}(t_n t)\exp(\frac{3C}{t} - \frac{\varepsilon s}{2t}). \end{aligned} \tag{16}$$

Also we prove the following for the proof of Lemma 3.

Proposition 8. *For any γ, δ, $\varepsilon \in (0,1)$ and $\beta > 0$, there is a $C > 0$ such that*

$$\begin{aligned} & |P(X_1 + \sum_{k=2}^{n} X_k 1_{\{X_k \le s^\gamma t_n\}} > st_n, \sum_{k=2}^{n} X_k 1_{\{X_k \le s^\gamma t_n\}} \le \varepsilon st_n) \\ & - \int_{-\infty}^{\varepsilon s} \bar{F}(t_n(s-x))\Phi_1(x)dx| \\ & \le C\bar{F}((1-\varepsilon)n^{1/2}s)a_n^{1-4\delta}, \qquad for\ s > a_n^{-\beta}. \end{aligned}$$

Proof. We can prove Proposition 8 similarly to Proposition 20 in [14]. □

Now let us prove Lemma 3. Since

$$\begin{aligned}
& H(n,s) - n\int_{-\infty}^{\varepsilon s} \bar{F}(t_n(s-x))\Phi_1(x)dx \\
= \;& \Phi_0(s) - n\int_{\varepsilon s}^{s} \bar{F}(t_n(s-x))\Phi_1(x)dx \\
& + v_n^{-1/2} n^{1/2}\Phi_1(s)\int_0^\infty x\mu(dx) + v_n^{-1}\frac{\Phi_2(s)}{2}\int_0^{t_n} x^2\mu(dx) \\
= \;& \Phi_0(s) - v_n^{-1/2}n^{1/2}\eta_1((1-\varepsilon)t_n s)\Phi_1(s) + v_n^{-1}\frac{\Phi_2(s)}{2}\int_{(1-\varepsilon)t_n s}^{t_n} x^2\mu(dx) \\
& - v_n^{-1/2}n^{1/2}\Big(\int_0^{(1-\varepsilon)t_n s} \bar{F}(z)(\Phi_1(s - t_n^{-1}z) - \Phi_1(s) - t_n^{-1}z\Phi_2(s))dz\Big) \\
& - v_n^{-1}\frac{L((1-\varepsilon)t_n s)}{(1-\varepsilon)s}\Phi_1(s) - v_n^{-1}L((1-\varepsilon)t_n s)\Phi_2(s) \\
= \;& \Phi_0(s) - v_n^{-1/2}n^{1/2}\eta_1((1-\varepsilon)t_n s)\Phi_1(s) + v_n^{-1}\frac{\Phi_2(s)}{2}\int_{(1-\varepsilon)t_n s}^{t_n} x^2\mu(dx) \\
& - v_n^{-1/2}n^{1/2}\Big(\int_0^{(1-\varepsilon)t_n s} \bar{F}(z)(\Phi_1(s - t_n^{-1}z) - \Phi_1(s) - t_n^{-1}z\Phi_2(s))dz\Big) \\
& - \frac{\eta_2((1-\varepsilon)st_n)}{(1-\varepsilon)s\eta_2(t_n)}\frac{L((1-\varepsilon)st_n)}{\eta_2((1-\varepsilon)st_n)}\Phi_1(s) - \frac{\eta_2((1-\varepsilon)st_n)}{\eta_2(t_n)}\frac{L((1-\varepsilon)st_n)}{\eta_2((1-\varepsilon)st_n)}\Phi_2(s),
\end{aligned}$$

it is easy to see that there is a $C_1 > 0$ such that for $s \geq 1$

$$|H(n,s) - n\int_{-\infty}^{\varepsilon s} \bar{F}(t_n(s-x))\Phi_1(x)dx| \leq C_1 s^3 \Phi_1(\varepsilon s).$$

Combining Equation (13) and Proposition 8, we see that there is a $C_1, C_2 > 0$ such that

$$\begin{aligned}
& |P(\sum_{k=1}^n X_k > st_n) - n\int_{-\infty}^{\varepsilon s} \bar{F}(t_n(s-x))\Phi_1(x)dx| \\
\leq \;& 2n(n-1)\bar{F}(s^\gamma t_n)^2 + \exp(\frac{3C_1}{s^\gamma} - \frac{s}{s^\gamma}) + n\bar{F}(s^\gamma t_n)\exp(\frac{3C_1}{s^\gamma} - \frac{\varepsilon s}{2s^\gamma}) \\
& + C_2 n\bar{F}((1-\varepsilon)t_n s)a_n^{1-\delta}.
\end{aligned}$$

Hence we see that there is a $C > 0$ such that

$$\sup_{s > a_n^{-\beta}} (n\bar{F}(t_n s))^{-1}|P(\sum_{k=1}^n X_k > st_n) - H(n,s)| \leq C a_n^{1-\delta}.$$

Therefore by Lamma 2, we have our assertion.

Now let us prove Theorem 1. From Theorem 2, we see that there is a $C_1 > 0$ such that

$$|P(\sum_{k=1}^n X_k > st_n) - H(n,s)| \leq C_1 a_n^{2-\delta/2}, \qquad s \geq 1.$$

Note that for any $\varepsilon > 0$, there is a $C_2 > 0$ such that $n\bar{F}(t_n s) \geq C_2^{-1} s^{-3} a_n \geq C_2^{-1} a_n^{1+\delta/2}$ for $s \leq a_n^{-\delta/6}$. Hence by Lemma 2, we see that there is a $C_3 > 0$ such that

$$H(n,s)^{-1} \leq C_3 (n\bar{F}(t_n s))^{-1} \leq C_2 C_3 a_n^{-1-\delta/2}, \qquad s \leq a_n^{-\delta/6}.$$

So we have

$$\sup_{s \leq a_n^{-\delta/6}} \left| \frac{P(\sum_{k=1}^n X_k > s t_n)}{H(n,s)} - 1 \right| \leq C_1 C_2 C_3 a_n^{1-\delta}.$$

From this inequality and Lemma 3, we have Equation (1). The latter assertion is obvious from Equation (1) and Lemma 2.

Reference

[1] Borovkov, A., Large deviations for random walks with semiexponential distributions. *Siberian Math. J.*, **41** (2000), 1061-1093

[2] Borovkov, A. and K Borovkov, *Asymptotic Analysis of Random Walks: Heavy Tailed Distributions*, Cambridge University Press (2008), Cambridge.

[3] Cline, D.B.H. and Hsing, T. Large deviation probabilities for sums and maxima of random variables with heavy or subexponetial tails. preprint, Texas A&M University (1991).

[4] Cramér, Sur un nouveau théorème limite de la théorie des probabilités. *Actualités Sci. Indust.* **736** Paris (1939).

[5] Embrechts, P., Klüppelberg, C. and T. Milkosch, *Modelling Extremal Events*, Springer, Berlin Heidelberg, (1997).

[6] Feller, *An Introduction to Probability Theory and Its Applications II*, Wiley, New York (1971).

[7] Fushiya, H., and S. Kusuoka, Uniform Estimate for Distributions of the Sum of i.i.d. Random Variables with Fat Tail, *J. Math. Sci. Univ. Tokyo* **17** (2010), 79-121.

[8] Heyde, C.C., A contribution to the theory of large deviations for sums of independent random variables, *Z. Wahrscheinlichleitstheorie verw. Gebiete* **7** (1967), 303-308.

[9] Linnik, Yu. V., On the probability of large deviation for sums of independent variables, Proc. 4th Berkley Symp. Math. Statist. Probability 2, 289-306, (1961), Univ. Cal. Press.

[10] Nagaev, A.V., Limit theorems for large deviations where Cramér's conditions are violated (in Russian). *Izv. Akad. Nauk UzSSR. Ser. Fiz.-Mat. Nauk.* **6** (1969) 17-22.

[11] Nagaev, A.V., Integral limit theorems taking large deviations into account when Cramer's condition does not hold. I,II. *Theory Probab. Appl.* **14** (1969), 51-64; 193-208.

[12] Nagaev, S.V., Some limit theorems for large deviations. *Theory Probab. Appl*, **41** (1965), 214-235.

[13] Nagaev, S.V., Large deviations of sums of independent random variables, *Ann. Probab.* **7** (1979), 745-789.

[14] K. Nakahara, Uniform Estimate for Distributions of the Sum of i.i.d. Random Variables with Fat Tail: Threshold case, preprint. Univ. of Tokyo. submitted to J. Math. Sci. Univ. Tokyo.

[15] Petrov, V.V., *Sums of independent random variables*. Springer, New York etc (1975).

[16] L.V. Rozovskii, Probabilities of large deviations of sums of independent random variables with common distribution function in the domain of attraction of the normal law, *Theory Probab Appl.* **34** (1989), 625-644.

[17] L.V. Rozovskii, Probabilities of large deviations on the whole axis, *Theory Probab Appl.* **38** (1994), 53-79.

UTMS

2011–1 Qing Liu: *Fattening and comparison principle for level-set equation of mean curvature type.*

2011–2 Oleg Yu. Imanuvilov, Gunther Uhlmann, and Masahiro Yamamoto: *Global uniqueness in determining the potential for the two dimensional Schrödinger equation from cauchy data measured on disjoint subsets of the boundary.*

2011–3 Junjiro Noguchi: *Connections and the second main theorem for holomorphic curves.*

2011–4 Toshio Oshima and Nobukazu Shimeno: *Boundary value problems on Riemannian symmetric spaces of the noncompact type.*

2011–5 Toshio Oshima: *Fractional calculus of Weyl algebra and Fuchsian differential equations.*

2011–6 Junjiro Noguchi and Jörg Winkelmann: *Order of meromorphic maps and rationality of the image space.*

2011–7 Mourad Choulli, Oleg Yu. Imanuvilov, Jean-Pierre Puel and Masahiro Yamamoto: *Inverse source problem for the lineraized Navier-Stokes equations with interior data in arbitrary sub-domain.*

2011–8 Toshiyuki Kobayashi and Yoshiki Oshima: *Classification of discretely decomposable $A_{\mathfrak{q}}(\lambda)$ with respect to reductive symmetric pairs.*

2011–9 Shigeo Kusuoka and Song Liang: *A classical mechanical model of Brownian motion with one particle coupled to a random wave field.*

2011–10 Kenji Nakahara: *Uniform estimate for distributions of the sum of i.i.d. random variables with fat tail: Infinite variance case.*

The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012