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The matrix coefficients with minimal K-types of the spherical and non-spherical principal series representations of $SL(3, \mathbf{R})$

by

Keiju-Sono

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

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ABSTRACT: We compute the holonomic system of rank 6 for the radial part of the matrix coefficients of class one and non-spherical principal series representations of $SL(3,\mathbf{R})$. We give explicit formulas of the coefficients of six power series solutions, and express the matrix coefficients by linear combinations of these power series. Among others, the c-functions of non-spherical principal series are obtained.

1 Introduction

It is a classical result to have the matrix coefficient of the class one principal series of a semisimple real Lie group as a linear combination of asymptotic power series solutions [1]. But for non-spherical case, there seems to be few references.

In the rather recent literature, Masatoshi Iida [4] studied the systems of the differential equations satisfied by the spherical functions of the principal series representations of $Sp(2,\mathbf{R})$ with 1-or 2-dimensional K-types, and found a new integral formula for the radial part of the spherical functions . And Iida and Takayuki Oda [3] investigated the differential equations satisfied by the leading terms of these functions and determined the exact power series expansions of the matrix coefficients of certain generalized principal series representations of $Sp(2,\mathbf{R})$.

In this paper, we handle the case of the group $SL(3,\mathbf{R})$. In this case, the Dirac-Schmid equation together with the Casimir equation gives the holonomic system of rank 6 on the split Cartan subgroup of $SL(3,\mathbf{R})$ for the radial part of the matrix coefficients of the non-spherical principal series belonging to the minimal K-type of dimension three.

We take $K = SO(3,\mathbf{R})$ as a maximal compact subgroup of G and $(\eta, V_\eta), (\tau, V_\tau)$ in $\hat{K}$. We define the space of spherical functions by $C^\infty_{\eta,\tau}(K\backslash G/K) := \{\phi : G \to V_\eta \otimes V_\tau | \phi(k_L g k_R^{-1}) = \eta(k_L) \otimes \tau(k_R)\phi(g), k_L, k_R \in K, g \in G\}$ and studied the case of $\eta = \tau = \mathbf{1}$ (the trivial representation of K) or the case $\eta = \tau = \tau_2$ (the three dimensional tautological representation of K).

In the former case, the representation is a class one principal series representation. Let $\mathfrak{g}$ be the Lie algebra of G and $U(\mathfrak{g})$ its universal enveloping algebra. The spherical function associated to class one principal series is completely determined by the actions of the Capelli elements Cp_2, Cp_3, which are the generators of the center of $U(\mathfrak{g})$ together with the regularity at the identity of G.

Meanwhile, the case of non-spherical principal series which has three dimensional K-types, we constructed two kinds of equations: 1.The equations obtained from the action of Casimir element of degree two 2.The equations obtained from the action of gradient operator, i.e. the Dirac-Schmid operator. We compute the eigenvalues of these operators and construct the equations by combining these results. In both cases, we obtain six power series solutions corresponding to the six characteristic roots.

The key point in this paper is as follows. We have three different non-spherical principal series with the same infinitesimal characters $Z(\mathfrak{g}) \to \mathbf{C}$. We cannot distinguish them only by the elements of $Z(\mathfrak{g})$. This is the reason we need the Dirac-Schmid operator which has distinct eigenvalues for different non-spherical principal series.

In the last section, we determine the exact power series expansions of matrix coefficients of spherical and non-spherical principal series representations by using the formula of hypergeometric functions. The coefficients appearing in the linear combination of power series are called c-functions, and firstly evaluated explicitly by G.Schiffmann [7] only for the case of spherical representations (cf. also the book [8], chap 9 of G.Warner). However, this inductive argument does not work for non-spherical case. This seems to be the reason why there are little results for non-spherical case.

Our method is classical. We investigate a part of the monodromy data of our holonomic system to have the unique solutions invariant under the fundamental group of the regular part of the split Cartan subgroup in $SL(3, \mathbf{R})$.

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2 Preliminaries

2.1 Notation

Let $G = SL(3, \mathbf{R})$ and fix $K = SO(3, \mathbf{R})$ as a maximal compact subgroup of G, and set $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{sl}(3, \mathbf{R}), \mathfrak{k} = \mathrm{Lie}(K) = \mathfrak{so}(3)$. Put

$$A := \{\mathrm{diag}(a_1, a_2, a_3) \in G | \prod_{i=1}^{3} a_i = 1, a_i \in \mathbf{R}_{>0}\}$$

and set $\mathfrak{a} = \mathrm{Lie}(A)$.
The Cartan involution $\theta : G \to G$ is defined by $g \mapsto ({}^t g)^{-1}$ $(g \in G)$, and its Lie

algebra version is $\theta : \mathfrak{g} \to \mathfrak{g}, X \mapsto -{}^tX$.
Then

$$K = G^\theta = \{g \in G |\, \theta(g) = g\}$$

and

$$\mathfrak{k} = \mathfrak{g}^\theta = \{X \in \mathfrak{g} |\, \theta(X) = X\}.$$

Put

$$\mathfrak{p} = \mathfrak{g}^{-\theta} = \{X \in \mathfrak{g} |\, \theta(X) = -X\}.$$

Then we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, called the Cartan decomposition.
Let $E_{i,j}$ $(1 \leq i, j \leq 3)$ be the matrix unit with 1 at the (i, j)-th entry and 0 at other entries. Put

$$H_{i,j} := E_{i,i} - E_{j,j} \in \mathfrak{a}\,(i \neq j).$$

Put $X_{i,j} = E_{i,j} + E_{j,i}\,(i \neq j) \in \mathfrak{p}$ and $K_{i,j} = E_{i,j} - E_{j,i}\,(i \neq j) \in \mathfrak{k}$.

2.2 The principal series representations

Let P_0 be a minimal parabolic subgroup of G given by the upper triangular matrices in G, and $P_0 = MAN$ be the Langlands decomposition of P_0 with $M = K \cap \{\text{diagonals in}\, G\}$, and

$$N = \left\{ \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} \in G | x_i \in \mathbf{R}, i = 1, 2, 3 \right\}.$$

To define a principal series representation with respect to the minimal parabolic subgroup P_0 of G, we firstly fix a character σ of M and a linear form $\nu \in \mathfrak{a}^* \otimes_{\mathbf{R}} \mathbf{C} = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}, \mathbf{C})$. We write $\nu(\mathrm{diag}(t_1, t_2, t_3)) = \nu_1 t_1 + \nu_2 t_2$. Then we can define a representation $\sigma \otimes a^\nu$ of MA, and extend this to P_0 by the identification $P_0/N \simeq MA$. Then we set

$$\pi_{\sigma,\nu} = C^\infty \mathrm{Ind}_{P_0}^G (\sigma \otimes a^{\nu+\rho} \otimes 1_N).$$

Here ρ is the half sum of positive roots of $(\mathfrak{g}, \mathfrak{a})$ given by $a^\rho = a_1^2 a_2$, for $a = \mathrm{diag}(a_1, a_2, a_3) \in A$.
The representation space is

$$C^\infty_{(M,\sigma)}(K) = \{f \in C^\infty(K) | f(mk) = \sigma(m) f(k), m \in M, k \in K\}$$

and the action of G is defined by

$$(\pi(x)f)(k) = a(kx)^{\nu+\rho} f(\kappa(kx))\,(x \in G, k \in K).$$

Here, for $g \in G$, $g = n(g)a(g)\kappa(g)$ $(n(g) \in N, a(g) \in A, \kappa(g) \in K)$ is the Iwasawa decomposition. Next, we difine characters σ_j $(j = 0, 1, 2, 3)$ of M as follows. The group M consisting of four elements is a finite abelian group of

(2,2)-type, and its elements except for the unity are given by

$$m_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, m_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, m_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since M is commutative, all the irreducible unitary representations of M is 1-dimensional. For any $\sigma \in \widehat{M}$, we have $\sigma^2 = 1$. Therefore, the set $\widehat{M}$ consisting of 4 characters $\{\sigma_j | j = 0, 1, 2, 3\}$, where each σ_j, except for the trivial character σ_0, is specified by the following table of values at the elements m_i $(i = 1, 2, 3)$.

	m_1	m_2	m_3
σ_1	1	-1	-1
σ_2	-1	1	-1
σ_3	-1	-1	1

The correspondence of a character of M and the minimal K-type of $\pi_{\sigma,\nu}$ is as follows ([6]).

Proposition 2.1. *1) If σ is the trivial character of M, the representation $\pi_{\sigma,\nu}$ is spherical or class one. That is, it has a unique K-invariant vecter in $H_{\sigma,\nu}$.*
2)If σ is not trivial, the minimal K-type of the restriction $\pi_{\sigma,\nu}|_K$ to K is a 3-dimensional representation of K, which is isomorphic to the unique standard one (τ_2, V_2). The multiplicity of this minimal K-type is one:

$$\dim_{\mathbf{C}} \mathrm{Hom}_K(\tau_2, H_{\sigma,\nu}) = 1.$$

2.3 The definition of spherical functions

Let (π, H_π) be the principal series representation of $G = SL(3, \mathbf{R})$. We want to study the matrix coefficient

$$\Phi_{w,v} : G \to \mathbf{C}, \quad g \mapsto \langle w, \pi(g)v \rangle \quad (w \in H_\pi^*, v \in H_\pi).$$

Let (τ_L, V_L) be the K-type of H_π^* and (τ_R, V_R) be the K-type of H_π. And let $\iota : \tau_L \boxtimes \tau_R \to \pi^* \boxtimes \pi$ be the $K \times K$ embedding. The bilinear form $(w, v) \mapsto \Phi_{w,v}$ is the element of $\mathrm{Hom}_{G\times G}(H_\pi^* \otimes H_\pi, C^\infty(G))$. We define a homomorphism $\mathrm{Hom}_{G\times G}(H_\pi^* \otimes H_\pi, C^\infty(G)) \to \mathrm{Hom}_{K\times K}(V_L \otimes V_R, C^\infty(G))$ by $\Phi \mapsto \Phi \circ \iota$. The space $\mathrm{Hom}_{K\times K}(V_L \otimes V_R, C^\infty(G))$ is identified with a space
$C^\infty_{\tau_L^*,\tau_R^*}(K\backslash G/K)$
$:= \{F : G \to V_L^* \otimes V_R^* | F(k_1 g k_2) = (\tau_L^* \boxtimes \tau_R^*)(k_1, k_2^{-1})F(g),\ k_1, k_2 \in K, g \in G\}$
by the correspondence

$$\langle F_\phi(g), v_1 \otimes v_2 \rangle = \phi(v_1 \otimes v_2)(g) \quad (\forall (v_1, v_2) \in V_L \times V_R)$$

for $\phi \in \mathrm{Hom}_{K\times K}(V_L \otimes V_R, C^\infty(G))$, $F_\phi \in C^\infty_{\tau_L^*,\tau_R^*}(K\backslash G/K)$. The element of $C^\infty_{\tau_L^*,\tau_R^*}(K\backslash G/K)$ is called a spherical function. Because of the Cartan double coset decomposition $G = KAK$, spherical functions are determined by its restriction to A.

3 The double coset Cartan decomposition

Because G has the double coset decomposition $G = KAK$, we consider the decomposition of the standard elements in $\mathfrak{p}$ with respect to the double coset decomposition:

$$\mathfrak{g} = \mathrm{Ad}(a^{-1})\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{k}.$$

Here $a \in A$ is a regular element in A. For $x \in \mathbf{R}_{>0}$, put $sh(x) = \frac{1}{2}(x - \frac{1}{x})$, $ch(x) = \frac{1}{2}(x + \frac{1}{x})$. We have the following decomposition:

Lemma 3.1. *We have*

$$X_{i,j} = -\frac{1}{sh(\frac{a_i}{a_j})}\mathrm{Ad}(a^{-1})K_{i,j} + 0 + \frac{ch(\frac{a_i}{a_j})}{sh(\frac{a_i}{a_j})}K_{i,j} \quad ;$$

$$H_{i,j} = 0 + H_{i,j} + 0$$

with respect to the decomposition $\mathfrak{g} = \mathrm{Ad}(a^{-1})\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{k}$.

4 The $(\mathfrak{g}, K)$-modules of principal series representations

4.1 The Capelli elements

The center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ has two independent generators, and they are obtained as Capelli elements because $\mathfrak{g} = \mathfrak{sl}_3$ is of type A_2 (see [2]). For $i = 1, 2, 3$, we put

$$E'_{i,i} = E_{i,i} - \frac{1}{3}\left(\sum_{k=1}^{3} E_{k,k}\right).$$

The following proposition gives the explicit description of the independent generators of $Z(\mathfrak{g})$.

Proposition 4.1. *The independent generators* $\{Cp_2, Cp_3\}$ *of* $Z(\mathfrak{g})$ *are given as follows:*

$$\begin{aligned} Cp_2 =& (E'_{1,1} - 1)E'_{2,2} + E'_{2,2}(E'_{3,3} + 1) + (E'_{1,1} - 1)(E'_{3,3} + 1) \\ & - E_{2,3}E_{3,2} - E_{1,3}E_{3,1} - E_{1,2}E_{2,1} \\ Cp_3 =& (E'_{1,1} - 1)E'_{2,2}(E'_{3,3} + 1) + E_{1,2}E_{2,3}E_{3,1} + E_{1,3}E_{2,1}E_{3,2} \\ & - (E'_{1,1} - 1)E_{2,3}E_{3,2} - E_{1,3}E'_{2,2}E_{3,1} - E_{1,2}E_{2,1}(E'_{3,3} + 1). \end{aligned}$$

4.2 Reduction of Capelli elements

To compute the action of Capelli elements on class one spherical functions, we may regard the above two elements as elements in $Z(\mathfrak{g})\,(\mathrm{mod} U(\mathfrak{g})\,\mathfrak{k})$, because class one sperical functions are annihilated by the right action of $\mathfrak{k}$. After simple computations, we have the following lemma.

Lemma 4.2. *The Capelli elements Cp_2,Cp_3 satisfy the next congruences:*

$$
\begin{aligned}
Cp_2 \equiv &(E'_{1,1}-1)E'_{2,2} + E'_{2,2}(E'_{3,3}+1) + (E'_{1,1}-1)(E'_{3,3}+1) \\
&- E^2_{2,3} - E^2_{1,3} - E^2_{1,2} \quad (\mathrm{mod}\, U(\mathfrak{g})\mathfrak{k}), \\
Cp_3 \equiv &(E'_{1,1}-1)E'_{2,2}(E'_{3,3}+1) + E_{1,2}E_{2,3}E_{3,1} + E_{1,3}E_{2,1}E_{3,2} \\
&- E^2_{2,3}(E'_{1,1}-1) - E^2_{1,3}E'_{2,2} - E^2_{1,2}(E'_{3,3}+1) \\
&(\mathrm{mod}\, U(\mathfrak{g})\mathfrak{k}).
\end{aligned}
$$

4.3 Eigenvalues of Cp_2, Cp_3

In order to construct the partial differential equations satisfied by spherical functions of class one case, we have to compute the eigenvalues of the actions of the Capelli elements Cp_2, Cp_3. For the class one principal series, $\sigma = \sigma_0$ is the trivial character of M. Let f_0 be the generator of the minimal K-type in $H_{\sigma_0,\nu}$ normalized such that $f_0|K \equiv 1$. The actions of Cp_2, Cp_3 on f_0 are computed in [6], and the result is as follows:

Proposition 4.3. *The Capelli elements Cp_2, Cp_3 act on f_0 by scalar multiples, and the eigenvalues are given as follows:*

$$
Cp_2 f_0 = S_2\left(\frac{1}{3}(2\nu_1-\nu_2), \frac{1}{3}(-\nu_1+2\nu_2), -\frac{1}{3}(\nu_1+\nu_2)\right) f_0,
$$

$$
Cp_3 f_0 = S_3\left(\frac{1}{3}(2\nu_1-\nu_2), \frac{1}{3}(-\nu_1+2\nu_2), -\frac{1}{3}(\nu_1+\nu_2)\right) f_0.
$$

Here, $S_2(a,b,c) = ab+bc+ca$, $S_3(a,b,c) = abc$.

5 The partial differential equations satisfied by class one spherical functions

5.1 Construction of the differential equations

We put

$$
y_1 = y_1(a) := a_1/a_2,\; y_2 = y_2(a) := a_2/a_3
$$

for $a = \mathrm{diag}(a_1, a_2, a_3) \in A$.
By the definition of the action of Lie algebra, we have the following formula.

Lemma 5.1. *For $f(y_1, y_2) = f(a) \in C^\infty(A)$, we have*

$$H_{1,2}f = \left(2y_1\frac{\partial}{\partial y_1} - y_2\frac{\partial}{\partial y_2}\right) f,\ H_{2,3}f = \left(-y_1\frac{\partial}{\partial y_1} + 2y_2\frac{\partial}{\partial y_2}\right) f.$$

Now we want to construct the partial differential equations of class one spherical functions. We define differential operators ∂_1, ∂_2 by

$$\partial_i := y_i\frac{\partial}{\partial y_i} \quad (i = 1, 2).$$

By direct computations, we have the following two lemmas.

Lemma 5.2. *For $1 \leq i, j \leq 3$ such that $i \neq j$, we have*

$$[K_{i,j}, \mathrm{Ad}(a^{-1})K_{i,j}] = -2sh(\frac{a_i}{a_j})H_{i,j}.$$

Lemma 5.3. *For $i, j, k \in \{1, 2, 3\}$ such that $i \neq j, j \neq k, k \neq i$, we have*

$$[K_{i,j}, \mathrm{Ad}(a^{-1})K_{j,k}] = \frac{sh(\frac{a_j}{a_k})}{sh(\frac{a_i}{a_k})}\mathrm{Ad}(a^{-1})K_{i,k} + \frac{sh(\frac{a_i}{a_j})}{sh(\frac{a_i}{a_k})}K_{i,k},$$

$$[K_{i,j}, \mathrm{Ad}(a^{-1})K_{k,i}] = \frac{sh(\frac{a_i}{a_k})}{sh(\frac{a_j}{a_k})}\mathrm{Ad}(a^{-1})K_{j,k} - \frac{sh(\frac{a_i}{a_j})}{sh(\frac{a_j}{a_k})}K_{j,k}.$$

By combining Lemma 5.1, Lemma 5.2, and Lemma 5.3, the actions of Cp_2, Cp_3 in Lemma 4.2 on f_0 are obtained by direct computations. The eigenvalues are obtained in Proposition 4.3. Thus we have the following differential equations:

Theorem 5.4. *Let $F \in C^\infty(K\backslash G/K)$ be a class one spherical function of $G = SL(3, \mathbf{R})$, its restriction to A; $F|_A = F(y_1, y_2)$ satisfies two partial differential equations:*

$$\begin{aligned}
&2(\partial_1^2 - \partial_1\partial_2 + \partial_2^2)F + \left(-\frac{y_2^2+1}{y_2^2-1} + \frac{y_1^2y_2^2+1}{y_1^2y_2^2-1} + 2\frac{y_1^2+1}{y_1^2-1}\right)\partial_1 F \\
&+ \left(2\frac{y_2^2+1}{y_2^2-1} + \frac{y_1^2y_2^2+1}{y_1^2y_2^2-1} - \frac{y_1^2+1}{y_1^2-1}\right)\partial_2 F \\
&+ \left\{-\frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2) + 2\right\} F = 0,
\end{aligned} \tag{5.1}$$

$$
\begin{aligned}
&\partial_1^2\partial_2 F - \partial_1\partial_2^2 F + \left(-1 + \frac{y_2^2}{y_2^2-1} + \frac{y_1^2y_2^2}{y_1^2y_2^2-1}\right)\partial_1^2 F \\
&+ \left(-\frac{2y_2^2}{y_2^2-1} + \frac{2y_1^2}{y_1^2-1}\right)\partial_1\partial_2 F + \left(1 - \frac{y_1^2}{y_1^2-1} - \frac{y_1^2y_2^2}{y_1^2y_2^2-1}\right)\partial_2^2 F \\
&+ \left(1 - \frac{y_1^2y_2^2}{(y_2^2-1)(y_1^2y_2^2-1)} + \frac{3y_1^2y_2^2}{(y_1^2-1)(y_2^2-1)} + \frac{y_1^2y_2^2}{(y_1^2-1)(y_1^2y_2^2-1)}\right. \\
&\left.-\frac{y_2^2}{y_2^2-1} - \frac{2y_1^2}{y_1^2-1}\right)\partial_1 F \\
&+ \left(-1 - \frac{y_1^2y_2^2}{(y_2^2-1)(y_1^2y_2^2-1)} - \frac{3y_1^2y_2^2}{(y_1^2-1)(y_2^2-1)} + \frac{y_1^2y_2^2}{(y_1^2-1)(y_1^2y_2^2-1)}\right. \\
&\left.+\frac{2y_2^2}{y_2^2-1} + \frac{y_1^2}{y_1^2-1}\right)\partial_2 F \\
&+ \frac{1}{27}(2\nu_1-\nu_2)(2\nu_2-\nu_1)(\nu_1+\nu_2)F = 0.
\end{aligned} \tag{5.2}
$$

5.2 Power series solution around $y_1, y_2 = 0$

For the class one spherical function F, we want to find its series expansion at the origin $y_1 = 0$, $y_2 = 0$ by solving (5.1) and (5.2). Firstly, we put

$$
F(y_1, y_2) = \sum_{n,m=0}^{\infty} a_{n,m} y_1^{n+\mu_1} y_2^{m+\mu_2} \quad (a_{0,0} \neq 0). \tag{5.3}
$$

The first task is to compute the characteristic roots (μ_1, μ_2). By substituting (5.3) for F into the equation (5.1) , and picking up the coefficient of $y_1^{n+\mu_1} y_2^{m+\mu_2}$, we have the next equation satisfied by $\{a_{n,m}\}$.

Proposition 5.5. *The coefficients $\{a_{n,m}\}$ satisfy the following recurrence relation:*

$$
\begin{aligned}
&\{2(n'-4)^2 - 2(n'-4)(m'-4) + 2(m'-4)^2 \\
&+ 2(n'-4) + 2(m'-4) + \lambda\}a_{n-4,m-4} \\
&+ \{-2(n'-4)^2 + 2(n'-4)(m'-2) - 2(m'-2)^2 \\
&- 4(n'-4) + 2(m'-2) - \lambda\}a_{n-4,m-2} \\
&+ \{-2(n'-2)^2 + 2(n'-2)(m'-4) - 2(m'-4)^2 \\
&+ 2(n'-2) - 4(m'-4) - \lambda\}a_{n-2,m-4} \\
&+ \{2(n'-2)^2 - 2(n'-2)m' + 2m'^2 + 2(n'-2) - 4m' + \lambda\}a_{n-2,m} \\
&+ \{2n'^2 - 2n'(m'-2) + 2(m'-2)^2 - 4n' + 2(m'-2) + \lambda\}a_{n,m-2} \\
&+ (-2n'^2 + 2n'm' - 2m'^2 + 2n' + 2m' - \lambda)a_{n,m} = 0.
\end{aligned} \tag{5.4}
$$

Here, $\lambda := -\frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2) + 2,\ n^{'} := n + \mu_1,\ m^{'} = m + \mu_2,$ *and* $a_{i,j} = 0$ $(if \quad i < 0\, or\, j < 0)$.

Proposition 5.6. *The characteristic roots take six values:*

$$\begin{aligned}
&(\mu_1, \mu_2)\\
&= \left(\frac{1}{3}(2\nu_1 - \nu_2) + 1, -\frac{1}{3}(2\nu_2 - \nu_1) + 1\right),\\
&\left(\frac{1}{3}(2\nu_2 - \nu_1) + 1, -\frac{1}{3}(2\nu_1 - \nu_2) + 1\right),\\
&\left(\frac{1}{3}(2\nu_2 - \nu_1) + 1, \frac{1}{3}(\nu_1 + \nu_2) + 1\right),\\
&\left(-\frac{1}{3}(\nu_1 + \nu_2) + 1, -\frac{1}{3}(2\nu_2 - \nu_1) + 1\right),\\
&\left(\frac{1}{3}(2\nu_1 - \nu_2) + 1, \frac{1}{3}(\nu_1 + \nu_2) + 1\right),\\
&\left(-\frac{1}{3}(\nu_1 + \nu_2) + 1, -\frac{1}{3}(2\nu_1 - \nu_2) + 1\right).
\end{aligned} \tag{5.5}$$

Proof. Because $a_{i,j} = 0\ (if\, i < 0\, or\, j < 0)$, and $a_{0,0} \neq 0$, by substituting $n = m = 0$ in (5.4), we have

$$-2(\mu_1^2 - \mu_1\mu_2 + \mu_2^2 - \mu_1 - \mu_2) - \lambda = 0.$$

This equation is equivalent to

$$(\mu_1 - 1)^2 - (\mu_1 - 1)(\mu_2 - 1) + (\mu_2 - 1)^2 = \frac{1}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2). \tag{5.6}$$

Next, by computing the recurrence equation given by equation (5.2), and substituting $n = m = 0$ in the coefficient of $a_{n,m}$, we have

$$\mu_1^2\mu_2 - \mu_1\mu_2^2 - \mu_1^2 + \mu_2^2 + \mu_1 - \mu_2 = -\frac{1}{27}(2\nu_1 - \nu_2)(2\nu_2 - \nu_1)(\nu_1 + \nu_2).$$

This equation is equivalent to

$$\{(\mu_1 - 1) - (\mu_2 - 1)\}(\mu_1 - 1)(\mu_2 - 1) = -\frac{1}{27}(2\nu_1 - \nu_2)(2\nu_2 - \nu_1)(\nu_1 + \nu_2). \tag{5.7}$$

By combining (5.6) and (5.7), we have the result. □

Next, we put

$$F(y_1, y_2) = sh(y_1)^{-\frac{1}{2}} sh(y_2)^{-\frac{1}{2}} sh(y_1 y_2)^{-\frac{1}{2}} G(y_1, y_2)\ (0 < y_1, y_2 << 1)$$

and compute the power series of G at the origin $y_1 = y_2 = 0$. We put

$$G(y_1, y_2) = \sum_{n,m=0}^{\infty} \tilde{a}_{n,m} y_1^{n+\tilde{\mu_1}} y_2^{m+\tilde{\mu_2}}\ (\tilde{a}_{0,0} \neq 0). \tag{5.8}$$

Proposition 5.7. *The characteristic roots take six values:*

$$
\begin{aligned}
&(\tilde{\mu}_1, \tilde{\mu}_2) \\
&= \left(\frac{1}{3}(2\nu_1 - \nu_2), -\frac{1}{3}(2\nu_2 - \nu_1)\right), \left(\frac{1}{3}(2\nu_2 - \nu_1), -\frac{1}{3}(2\nu_1 - \nu_2)\right), \\
&\left(\frac{1}{3}(2\nu_2 - \nu_1), \frac{1}{3}(\nu_1 + \nu_2)\right), \left(-\frac{1}{3}(\nu_1 + \nu_2), -\frac{1}{3}(2\nu_2 - \nu_1)\right), \\
&\left(\frac{1}{3}(2\nu_1 - \nu_2), \frac{1}{3}(\nu_1 + \nu_2)\right), \left(-\frac{1}{3}(\nu_1 + \nu_2), -\frac{1}{3}(2\nu_1 - \nu_2)\right).
\end{aligned}
\tag{5.9}
$$

Proof. We have

$$sh(y_1)^{\frac{1}{2}} sh(y_2)^{\frac{1}{2}} sh(y_1 y_2)^{\frac{1}{2}} = y_1^{-1} y_2^{-1} (1 + O(y_1, y_2))$$

when $0 < y_1, y_2 << 1$. By combining this with Proposition 5.6, we have the result. $\square$

By substituting (5.8) into (5.1) , we have the following differential equation:

Proposition 5.8. *For the class one spherical function F, the function*

$$G(y_1, y_2) = sh(y_1)^{\frac{1}{2}} sh(y_2)^{\frac{1}{2}} sh(y_1 y_2)^{\frac{1}{2}} F(y_1, y_2)\,(0 < y_1, y_2 << 1)$$

satisfies

$$2\left(\partial_1^2 - \partial_1 \partial_2 + \partial_2^2\right) G + \left(\lambda^{'} + \frac{1}{2sh(y_1)^2} + \frac{1}{2sh(y_2)^2} + \frac{1}{2sh(y_1 y_2)^2}\right) G = 0. \tag{5.10}$$

Here, $\lambda^{'} = -\frac{2}{3}(\nu_1^2 - \nu_1 \nu_2 + \nu_2^2)$.

When $0 < y < 1$, we have

$$\frac{1}{2sh(y)^2} = 2 \sum_{k=1}^{\infty} k y^{2k}.$$

Hence the equation becomes

$$2\left(\partial_1^2 - \partial_1 \partial_2 + \partial_2^2\right) G + \left(\lambda^{'} + 2\sum_{k=1}^{\infty} k y_1^{2k} + 2\sum_{k=1}^{\infty} k y_2^{2k} + 2\sum_{k=1}^{\infty} k y_1^{2k} y_2^{2k}\right) G = 0. \tag{5.11}$$

By substituting (5.8) into (5.11), and picking up the coefficient of $y_1^{n+\tilde{\mu}_1} y_2^{m+\tilde{\mu}_2}$, we have the recurrence relation of $\{\tilde{a}_{n,m}\}$:

Proposition 5.9. *The coefficients $\{\tilde{a}_{n,m}\}$ satisfy*

$$
\begin{aligned}
&(2n^{'2} - 2n^{'}m^{'} + 2m^{'2} + \lambda^{'})\tilde{a}_{n,m} \\
&+ 2\sum_{k=1}^{\infty} k\tilde{a}_{n-2k,m} + 2\sum_{k=1}^{\infty} k\tilde{a}_{n,m-2k} + 2\sum_{k=1}^{\infty} k\tilde{a}_{n-2k,m-2k} = 0.
\end{aligned}
\tag{5.12}
$$

Here, $n^{'} = n + \tilde{\mu}_1$, $m^{'} = m + \tilde{\mu}_2$.

Note that because $\tilde{a}_{i,j} = 0$ if $i < 0$ or $j < 0$, the summations above are all finite sum. By substituting $n = m = 0$, we have

$$2\tilde{\mu}_1^2 - 2\tilde{\mu}_1\tilde{\mu}_2 + 2\tilde{\mu}_2^2 + \lambda' = 0.$$

Hence we have

$$\begin{aligned} &2n'^2 - 2n'm' + 2m'^2 + \lambda' \\ &= 2\{n^2 - nm + m^2 + (2\tilde{\mu}_1 - \tilde{\mu_2})n + (2\tilde{\mu}_2 - \tilde{\mu_1})m\}. \end{aligned}$$

By substituting this into (5.12), we have

$$\begin{aligned} &\{n^2 - nm + m^2 + (2\tilde{\mu}_1 - \tilde{\mu_2})n + (2\tilde{\mu}_2 - \tilde{\mu_1})m\}\tilde{a}_{n,m} \\ &+ \sum_{k=1}^{\infty} k\tilde{a}_{n-2k,m} + \sum_{k=1}^{\infty} k\tilde{a}_{n,m-2k} + \sum_{k=1}^{\infty} k\tilde{a}_{n-2k,m-2k} = 0. \end{aligned} \tag{5.13}$$

From this equation, easily we have

$$\tilde{a}_{n,m} = 0 \quad (\text{if}\, n \,\text{or}\, m \,\text{is odd}).$$

We put

$$p(n,m) = n^2 - nm + m^2 + (2\tilde{\mu}_1 - \tilde{\mu}_2)n + (2\tilde{\mu}_2 - \tilde{\mu}_1)m.$$

The following theorem is one of the main theorems of this paper, which gives the explicit expression of $\tilde{a}_{2n,2m}$.

Theorem 5.10. *Suppose that $\tilde{a}_{0,0} = 1$ and $p(n,m) \neq 0$ if $(n,m) \neq (0,0)$. Let $\mathbf{P}_{n,m}$ be the family of all sets $\{p(2n_k, 2m_k), \cdots, p(2n_0, 2m_0)\}$ such that*

$$n_k = n, m_k = m, n_0 = m_0 = 0$$

and

$$\begin{gathered} (n_{i+1}, m_{i+1}) = (n_i + l_i, m_i) \textit{ or } (n_i, m_i + l_i) \textit{ or } (n_i + l_i, m_i + l_i) \\ (\exists l_i \in \mathbf{Z}_{>0}), \quad (i = 0, \cdots, k-1). \end{gathered}$$

(Here, k depends on each set).
For $\{p(2n_k, 2m_k), \cdots, p(2n_0, 2m_0)\} \in \mathbf{P}_{n,m}$ and $0 \leq i \leq k-1$, we define $d_i \in \mathbf{Z}$ by $d_i = -l_i$. And we put

$$C_{(n_1,\cdots n_k;m_1,\cdots m_k)} = \prod_{i=0}^{k-1} d_i. \tag{5.14}$$

Then we have

$$\tilde{a}_{2n,2m} = \sum_{\{p(2n_k,2m_k),\cdots,p(2n_0,2m_0)\}\in\mathbf{P}_{n,m}} \frac{C_{(n_1,\cdots n_k;m_1,\cdots m_k)}}{p(2n_k,2m_k)\cdots p(2n_1,2m_1)}. \tag{5.15}$$

for $(n,m) \neq (0,0)$.

Proof. We prove this statement by induction with respect to m. First, we consider the case $m = 0$. Suppose that for $0 \le N \le n$,

$$\tilde{a}_{2N,0} = \sum_{\{p(2n_k,0),\cdots,p(2n_0,0)\}\in \mathbf{P}_{N,0}} \frac{C_{(n_1,\cdots n_k;0,\cdots 0)}}{p(2n_k,0)\cdots p(2n_1,0)}. \tag{5.16}$$

Now, $\{\tilde{a}_{n,0}\}$ satisfies

$$p(2n,0)\tilde{a}_{2n,0} + \sum_{l=1}^{n} l\tilde{a}_{2n-2l,0} = 0.$$

Thus we have

$$\begin{aligned}\tilde{a}_{2n+2,0} &= -\frac{1}{p(2n+2,0)}\sum_{l=1}^{n+1} l\tilde{a}_{2n+2-2l,0}\\ &= -\frac{1}{p(2n+2,0)}\\ &\cdot \sum_{l=1}^{n+1} l \sum_{\{p(2n_k,0),\cdots,p(2n_0,0)\}\in \mathbf{P}_{2n+2-2l,0}} \frac{C_{(n_1,\cdots n_k;0,\cdots 0)}}{p(2n_k,0)\cdots p(2n_1,0)}.\end{aligned}$$

For $\{p(2n_k,0),\cdots,p(2n_0,0)\} \in \mathbf{P}_{2n+2-2l}$, the coefficient of

$$1/p(2n+2,0)p(2n_k,0)\cdots p(2n_1,0)$$

is $-lC_{(n_1,\cdots n_k;0,\cdots 0)} = -l\prod_{i=0}^{k-1} d_i$, and by definition, $-l = d_k$.
Thus, in this case, the coefficient is just as (5.14). Hence (5.15) holds when $m = 0$.
Next, suppose that (5.15) holds when $0 \le M \le m$. Let ι be the translation of parameters $\tilde{\mu}_1 \mapsto \tilde{\mu}_2$, $\tilde{\mu}_2 \mapsto \tilde{\mu}_1$. Then

$$\begin{aligned}\tilde{a}_{0,2m+2} &= \iota(\tilde{a}_{2m+2,0})\\ &= \iota\left(\sum_{\mathbf{P}_{2m+2,0}} \frac{C_{(m_1,\cdots,m_k;0,\cdots,0)}}{p(2m_k,0)\cdots p(2m_1,0)}\right)\\ &= \sum_{\mathbf{P}_{0,2m+2}} \frac{C_{(0,\cdots,0;m_1,\cdots,m_k)}}{p(0,2m_k)\cdots p(0,2m_1)}.\end{aligned}$$

Thus (5.15) holds for $\tilde{a}_{0,2m+2}$.
Next we suppose that (5.15) holds for $\tilde{a}_{0,2m+2}, \tilde{a}_{2,2m+2}, \cdots, \tilde{a}_{2n,2m+2}$. Since $\tilde{a}_{n,m}$ satisfies

$$p(2n+2,2m+2)\tilde{a}_{2n+2,2m+2} + \sum_{l=1}^{n+1} l\tilde{a}_{2n+2-2l,2m+2} + \sum_{l=1}^{m+1} l\tilde{a}_{2n+2,2m+2-2l}$$

$$+ \sum_{l=1}^{\min\{n+1,m+1\}} l\tilde{a}_{2n+2-2l,2m+2-2l} = 0,$$

we have

$$\tilde{a}_{2n+2,2m+2} = -\frac{1}{p(2n+2,2m+2)} \left(\sum_{l=1}^{n+1} l\tilde{a}_{2n+2-2l,2m+2} + \sum_{l=1}^{m+1} l\tilde{a}_{2n+2,2m+2-2l} \right. \\ \left. + \sum_{l=1}^{\min\{n+1,m+1\}} l\tilde{a}_{2n+2-2l,2m+2-2l} \right).$$

Here, by assumption,

$$\tilde{a}_{2n+2-2l,2m+2} = \sum_{\mathbf{P}_{n+1-l,m+1}} \frac{C_{(n_1,\cdots,n_k;m_1,\cdots,m_k)}}{p(n_k,m_k)\cdots p(n_1,m_1)}.$$

For $\{p(n_k,m_k),\cdots p(n_0,m_0)\} \in \mathbf{P}_{n+1-l,m+1}$, we have $\{p(2n+2,2m+2),$ $p(n_k,m_k),\cdots p(n_0,m_0)\} \in \mathbf{P}_{n+1,m+1}$, and the coefficient of $1/p(2n+2,2m+2)p(n_k,m_k)\cdots p(n_1,m_1)$ is $-l\prod_{i=0}^{k-1} d_i$, and by definition, $-l = d_k$. So the coefficient is just as (5.14). Similarly, the coefficients of the terms appearing the rest two summations are just as (5.14).
Each expansion of $\tilde{a}_{2n+2-2l_1,2m+2}, \tilde{a}_{2n+2,2m+2-2l_2}, \tilde{a}_{2n+2-2l_3,2m+2-2l_4}$ $(l_1,\cdots,$ $l_4 \in \mathbf{N})$ has $p(2n+2-2l_1,2m+2)$, $p(2n+2,2m+2-2l_2), p(2n+2-2l_3,2m+2-2l_4)$ respectively, and each of them doesn't appear in the expansions of the rest two kinds of $\tilde{a}_{n,m}$. So there is no term which appears more than two times in the summation above. The fact that all of the elements in $\mathbf{P}_{n+1,m+1}$ appear follows from the assumption of induction and the summation. Hence $\tilde{a}_{2n+2,2m+2}$ is just as (5.15). Thus the induction is completed and we have proved the theorem. □

6 The case of the 3-dimensional tautological representation

Let $\tau_2 : K = SO(3) \hookrightarrow GL(3,\mathbf{R})$ be the tautological representation. Then we say that

$$\{s_1 = {}^t(1,0,0), s_2 = {}^t(0,1,0), s_3 = {}^t(0,0,1)\}$$

is the natural basis of this representation τ_2. We consider a spherical function $\Psi \in C^\infty_{\tau_2,\tau_2}(K\backslash G/K)$. Ψ can be written in terms of the basis $\{s_i | i = 1,2,3\}$:

$$\Psi(g) = \sum_{i=1}^{3}\sum_{j=1}^{3} d_{ij}(g) s_i^L \otimes s_j^R.$$

Lemma 6.1. *For $a \in A$, we have $d_{ij}(a) = 0$ if $i \neq j$.*

Proof. A subgroup M of G is defined by

$$\begin{aligned}M &= Z_K(A) = \{k \in K | ak = ka \quad (\forall a \in A)\} \\ &= \{\mathrm{diag}(\epsilon_1, \epsilon_2, \epsilon_3) | \epsilon_i \in \{\pm 1\}, \epsilon_1\epsilon_2\epsilon_3 = 1\}.\end{aligned}$$

Then for $m \in M$, $a \in A$, we have

$$\tau_L(m)\Psi(a) = \Psi(ma) = \Psi(am) = \tau_R(m^{-1})\Psi(a).$$

Therefore, for example, for $m_3 = \mathrm{diag}(-1, -1, 1) \in M$, we have $\tau_2(m_3) \otimes \Psi(a) = 1 \otimes \tau_2(m_3^{-1})\Psi(a)$. So we have

$$\begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} (d_{ij}(a)) = (d_{ij}(a)) \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}.$$

From this, we have $d_{13}(a) = d_{31}(a) = d_{23}(a) = d_{32}(a) = 0$. Similarly, the actions of the other elements of M show that $d_{ij}(a) = 0$ if $i \neq j$. □

6.1 The action of Casimir operator

We use the same coordinate $y_1 = \frac{a_1}{a_2}, y_2 = \frac{a_2}{a_3}$ ($a = \mathrm{diag}(a_1, a_2, a_3) \in A$) as in the class one case. The Casimir operator C of $SL(3, \mathbf{R})$ is decomposed into two parts with respect to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$:

$$C = C(\mathfrak{p}) + C(\mathfrak{k}).$$

Here,

$$C(\mathfrak{p}) = \frac{2}{3}(H_{1,2}^2 + H_{1,2}H_{2,3} + H_{2,3}^2) + \frac{1}{2}\sum_{i<j} X_{i,j}^2,$$

$$C(\mathfrak{k}) = -\frac{1}{2}\sum_{i<j} K_{i,j}^2.$$

First, we consider the action of $C(\mathfrak{p})$.

$$\begin{aligned}&\left(\text{The action of } \quad \frac{2}{3}(H_{1,2}^2 + H_{1,2}H_{2,3} + H_{2,3}^2)\right) \\ &= \frac{2}{3}\{(2\partial_1 - \partial_2)^2 + (2\partial_1 - \partial_2)(-\partial_1 + 2\partial_2) + (-\partial_1 + 2\partial_2)^2\} \\ &= 2(\partial_1^2 - \partial_1\partial_2 + \partial_2^2).\end{aligned}$$

Next,

$$X_{i,j}^2 = \left\{-\frac{1}{sh(\frac{a_i}{a_j})}\mathrm{Ad}(a^{-1})K_{i,j} + \frac{ch(\frac{a_i}{a_j})}{sh(\frac{a_i}{a_j})}K_{i,j}\right\}^2$$

$$
\begin{aligned}
&= \frac{1}{sh(\frac{a_i}{a_j})^2}(\mathrm{Ad}(a^{-1})K_{i,j})^2 - 2\frac{ch(\frac{a_i}{a_j})}{sh(\frac{a_i}{a_j})^2}\mathrm{Ad}(a^{-1})K_{i,j}\cdot K_{i,j} \\
&+ \frac{ch(\frac{a_i}{a_j})^2}{sh(\frac{a_i}{a_j})^2}K_{i,j}^2 - \frac{ch(\frac{a_i}{a_j})}{sh(\frac{a_i}{a_j})^2}[K_{i,j}, \mathrm{Ad}(a^{-1})K_{i,j}].
\end{aligned}
$$

A direct computation shows that the above bracket product is given as follows:

Lemma 6.2. *For $i \neq j$, we have*
$[K_{i,j}, \mathrm{Ad}(a^{-1})K_{i,j}] = -2sh(\frac{a_i}{a_j})H_{i,j}$.

Therefore, we have

$$
\begin{aligned}
\frac{1}{2}\sum_{i<j} X_{i,j}^2 = & \frac{1}{2}\frac{1}{sh(y_1)^2}(\mathrm{Ad}(a^{-1})K_{1,2})^2 - \frac{ch(y_1)}{sh(y_1)^2}\mathrm{Ad}(a^{-1})K_{1,2}\cdot K_{1,2} \\
&+ \frac{1}{2}\frac{ch(y_1)^2}{sh(y_1)^2}K_{1,2}^2 + \frac{ch(y_1)}{sh(y_1)}H_{1,2} \\
&+ \frac{1}{2}\frac{1}{sh(y_1y_2)^2}(\mathrm{Ad}(a^{-1})K_{1,3})^2 - \frac{ch(y_1y_2)}{sh(y_1y_2)^2}\mathrm{Ad}(a^{-1})K_{1,3}\cdot K_{1,3} \\
&+ \frac{1}{2}\frac{ch(y_1y_2)^2}{sh(y_1y_2)^2}K_{1,3}^2 + \frac{ch(y_1y_2)}{sh(y_1y_2)}H_{1,3} \\
&+ \frac{1}{2}\frac{1}{sh(y_2)^2}(\mathrm{Ad}(a^{-1})K_{2,3})^2 - \frac{ch(y_2)}{sh(y_2)^2}\mathrm{Ad}(a^{-1})K_{2,3}\cdot K_{2,3} \\
&+ \frac{1}{2}\frac{ch(y_2)^2}{sh(y_2)^2}K_{2,3}^2 + \frac{ch(y_2)}{sh(y_2)}H_{2,3}.
\end{aligned}
$$

The actions of $(\mathrm{Ad}(a^{-1})K_{i,j})^2$, $(\mathrm{Ad}(a^{-1})K_{i,j})K_{i,j}$, $K_{i,j}^2$ on $\Psi(g) = \sum_i \sum_j d_{i,j}(g)s_i^L \otimes s_j^R$ are given by

$$
\begin{aligned}
&(\mathrm{Ad}(a^{-1})K_{i,j})^2\Psi(a) = -d_{ii}(a)s_{ii}^{LR} - d_{jj}(a)s_{jj}^{LR}, \\
&(\mathrm{Ad}(a^{-1})K_{i,j})K_{i,j}\Psi(a) = d_{jj}(a)s_{ii}^{LR} + d_{ii}(a)s_{jj}^{LR}, \\
&K_{i,j}^2\Psi(a) = -d_{ii}(a)s_{ii}^{LR} - d_{jj}(a)s_{jj}^{LR}
\end{aligned}
$$

on A. Here, we put $s_{ij}^{LR} := s_i^L \otimes s_j^R$.
Therefore, we have

$$
\begin{aligned}
&C(\mathfrak{p})\Psi(a) \\
&= 2(\partial_1^2 - \partial_1\partial_2 + \partial_2^2)\Psi(a) \\
&+ \left(2\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} - \frac{ch(y_2)}{sh(y_2)}\right)\partial_1\Psi(a)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(-\frac{ch(y_1)}{sh(y_1)}+\frac{ch(y_1y_2)}{sh(y_1y_2)}+2\frac{ch(y_2)}{sh(y_2)}\right)\partial_2\Psi(a)\\
&-\frac{1}{2}\frac{1}{sh(y_1)^2}\{d_{11}(a)s_{11}^{LR}+d_{22}(a)s_{22}^{LR}\}-\frac{1}{2}\frac{1}{sh(y_1y_2)^2}\{d_{11}(a)s_{11}^{LR}+d_{33}(a)s_{33}^{LR}\}\\
&-\frac{1}{2}\frac{1}{sh(y_2)^2}\{d_{22}(a)s_{22}^{LR}+d_{33}(a)s_{33}^{LR}\}\\
&+\frac{ch(y_1)}{sh(y_1)^2}\{d_{22}(a)s_{11}^{LR}+d_{11}(a)s_{22}^{LR}\}+\frac{ch(y_1y_2)}{sh(y_1y_2)^2}\{d_{33}(a)s_{11}^{LR}+d_{11}(a)s_{33}^{LR}\}\\
&+\frac{ch(y_2)}{sh(y_2)^2}\{d_{33}(a)s_{22}^{LR}+d_{22}(a)s_{33}^{LR}\}\\
&-\frac{1}{2}\frac{ch(y_1)^2}{sh(y_1)^2}\{d_{11}(a)s_{11}^{LR}+d_{22}(a)s_{22}^{LR}\}-\frac{1}{2}\frac{ch(y_1y_2)^2}{sh(y_1y_2)^2}\{d_{11}(a)s_{11}^{LR}+d_{33}(a)s_{33}^{LR}\}\\
&-\frac{1}{2}\frac{ch(y_2)^2}{sh(y_2)^2}\{d_{22}(a)s_{22}^{LR}+d_{33}(a)s_{33}^{LR}\}.
\end{aligned}
$$

Next, the action of $C(\mathfrak{k})=-\frac{1}{2}\sum_{i<j}K_{i,j}^2$ is given as follows:

$$
\begin{aligned}
&(\text{ The action of } \quad C(\mathfrak{k}))\\
&=\frac{1}{2}\{d_{11}(a)s_{11}^{LR}+d_{22}(a)s_{22}^{LR}\}+\frac{1}{2}\{d_{22}(a)s_{22}^{LR}+d_{33}(a)s_{33}^{LR}\}\\
&+\frac{1}{2}\{d_{11}(a)s_{11}^{LR}+d_{33}(a)s_{33}^{LR}\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&(C\Psi)(a)\\
&=2(\partial_1^2-\partial_1\partial_2+\partial_2^2)\Psi(a)\\
&+\left(2\frac{ch(y_1)}{sh(y_1)}+\frac{ch(y_1y_2)}{sh(y_1y_2)}-\frac{ch(y_2)}{sh(y_2)}\right)\partial_1\Psi(a)\\
&+\left(-\frac{ch(y_1)}{sh(y_1)}+\frac{ch(y_1y_2)}{sh(y_1y_2)}+2\frac{ch(y_2)}{sh(y_2)}\right)\partial_2\Psi(a)\\
&-\frac{1}{sh(y_1)^2}\{d_{11}(a)s_{11}^{LR}+d_{22}(a)s_{22}^{LR}\}-\frac{1}{sh(y_1y_2)^2}\{d_{11}(a)s_{11}^{LR}+d_{33}(a)s_{33}^{LR}\}\\
&-\frac{1}{sh(y_2)^2}\{d_{22}(a)s_{22}^{LR}+d_{33}(a)s_{33}^{LR}\}\\
&+\frac{ch(y_1)}{sh(y_1)^2}\{d_{22}(a)s_{11}^{LR}+d_{11}(a)s_{22}^{LR}\}+\frac{ch(y_1y_2)}{sh(y_1y_2)^2}\{d_{33}(a)s_{11}^{LR}+d_{11}(a)s_{33}^{LR}\}\\
&+\frac{ch(y_2)}{sh(y_2)^2}\{d_{33}(a)s_{22}^{LR}+d_{22}(a)s_{33}^{LR}\}.
\end{aligned}
$$

The next step is to compute the eigenvalue λ of the Casimir operator C. We compute the action on $f \in H_{\sigma_i,\nu}$ such that $f(e) = 1$. First,

$$
\begin{aligned}
&\left(\text{The action of } \quad \frac{2}{3}(H_{1,2}^2 + H_{1,2}H_{2,3} + H_{2,3}^2)\right) \\
&= \frac{2}{3}\{(\nu_1 - \nu_2 + 1)^2 + (\nu_1 - \nu_2 + 1)(\nu_2 + 1) + (\nu_2 + 1)^2\} \\
&= \frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2 + 3\nu_1 + 3).
\end{aligned}
$$

Next,

$$
\begin{aligned}
\frac{1}{2}\sum_{i<j} X_{i,j}^2 - \frac{1}{2}\sum_{i<J} K_{i,j}^2 &= \frac{1}{2}\sum_{i<j}(E_{i,j} + E_{j,i})^2 - \frac{1}{2}\sum_{i<J}(E_{i,j} - E_{j,i})^2 \\
&= \sum_{i<j}(E_{i,j}E_{j,i} + E_{j,i}E_{i,j}).
\end{aligned}
$$

Since $Xf(e) = 0$ for $X \in \mathfrak{n}$, the action of $E_{i,j}E_{j,i}$ is 0. On the other hand, since

$$[E_{i,j}, E_{j,i}] = H_{i,j},$$

we have

$$E_{j,i}E_{i,j} = E_{i,j}E_{j,i} - H_{i,j}.$$

Thus

$$
\begin{aligned}
&(\text{The action of } \quad \frac{1}{2}\sum_{i<j} X_{i,j}^2 - \frac{1}{2}\sum_{i<J} K_{i,j}^2) \\
&= (\text{The action of } -\sum_{i<j} H_{i,j}) \\
&= (\text{The action of } -2H_{1,3}) = -2(\nu_1 + 2).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\lambda &= \frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2 + 3\nu_1 + 3) - 2(\nu_1 + 2) \\
&= \frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2) - 2.
\end{aligned}
$$

For $\Psi(g) = \sum_{i=1}^3 \sum_{j=1}^3 d_{ij}(g) s_i^L \otimes s_j^R \in C_{\tau_2,\tau_2}^\infty(K\backslash G/K)$, we put

$$d_{11}(a) = F(a) = F(y_1, y_2),$$

$$d_{22}(a) = G(a) = G(y_1, y_2),$$
$$d_{33}(a) = H(a) = H(y_1, y_2).$$

Then, by comparing the coefficients of s_{ii}^{LR} in both sides of the equation $C\Psi = \lambda\Psi$, we have the following three equations:

Theorem 6.3. *F,G,H satisfy the relations:*

$$\begin{aligned}
&2(\partial_1^2 - \partial_1\partial_2 + \partial_2^2)F(y_1, y_2) \\
&+ \left(2\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} - \frac{ch(y_2)}{sh(y_2)}\right)\partial_1 F(y_1, y_2) \\
&+ \left(-\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} + 2\frac{ch(y_2)}{sh(y_2)}\right)\partial_2 F(y_1, y_2) \\
&- \left(\frac{1}{sh(y_1)^2} + \frac{1}{sh(y_1y_2)^2}\right)F(y_1, y_2) \\
&+ \frac{ch(y_1)}{sh(y_1)^2}G(y_1, y_2) + \frac{ch(y_1y_2)}{sh(y_1y_2)^2}H(y_1, y_2) \\
&= \lambda F(y_1, y_2),
\end{aligned} \tag{6.1}$$

$$\begin{aligned}
&2(\partial_1^2 - \partial_1\partial_2 + \partial_2^2)G(y_1, y_2) \\
&+ \left(2\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} - \frac{ch(y_2)}{sh(y_2)}\right)\partial_1 G(y_1, y_2) \\
&+ \left(-\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} + 2\frac{ch(y_2)}{sh(y_2)}\right)\partial_2 G(y_1, y_2) \\
&- \left(\frac{1}{sh(y_1)^2} + \frac{1}{sh(y_2)^2}\right)G(y_1, y_2) \\
&+ \frac{ch(y_1)}{sh(y_1)^2}F(y_1, y_2) + \frac{ch(y_2)}{sh(y_2)^2}H(y_1, y_2) \\
&= \lambda G(y_1, y_2),
\end{aligned} \tag{6.2}$$

$$\begin{aligned}
&2(\partial_1^2 - \partial_1\partial_2 + \partial_2^2)H(y_1, y_2) \\
&+ \left(2\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} - \frac{ch(y_2)}{sh(y_2)}\right)\partial_1 H(y_1, y_2) \\
&+ \left(-\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} + 2\frac{ch(y_2)}{sh(y_2)}\right)\partial_2 H(y_1, y_2) \\
&- \left(\frac{1}{sh(y_2)^2} + \frac{1}{sh(y_1y_2)^2}\right)H(y_1, y_2) \\
&+ \frac{ch(y_1y_2)}{sh(y_1y_2)^2}F(y_1, y_2) + \frac{ch(y_2)}{sh(y_2)^2}G(y_1, y_2) \\
&= \lambda H(y_1, y_2).
\end{aligned} \tag{6.3}$$

Here, $\lambda = \frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2) - 2.$

6.2 The gradient operator

For the spherical function $\Psi(g) \in C^{\infty}_{\tau_2,\tau_2}(K\backslash G/K)$, we define the right gradient operator ∇^R as follows:

Definition 6.4. For the orthonormal basis $\{X_i\}_{i=1}^5$ of $\mathfrak{p}$, the right gradient operator ∇^R is defined by

$$\nabla^R \Psi(g) := \sum_{i=1}^{5} R_{X_i}\Psi \otimes X_i^*.$$

Here, X_i^* is the dual basis of X_i with respect to the inner product $(X,Y) \in \mathfrak{p} \times \mathfrak{p} \to \mathrm{Tr}(XY) \in \mathbf{C}$.

If we take $\{H_{1,2}, H_{2,3}, X_{1,2}, X_{2,3}, X_{1,3}\}$ as a basis of $\mathfrak{p}$, the dual basis is $\{\frac{1}{3}(2H_{1,2}+H_{2,3}), \frac{1}{3}(H_{1,2}+2H_{2,3}), \frac{1}{2}X_{1,2}, \frac{1}{2}X_{2,3}, \frac{1}{2}X_{1,3}\}$. Therefore,

$$\begin{aligned}\nabla^R\Psi(g) =& \frac{1}{3}R_{H_{1,2}}\Psi \otimes (2H_{1,2}+H_{2,3}) + \frac{1}{3}R_{H_{2,3}}\Psi \otimes (H_{1,2}+2H_{2,3}) \\ &+ \frac{1}{2}\sum_{i<j} R_{X_{i,j}}\Psi \otimes X_{i,j}.\end{aligned}$$

Claim 1. We define $\{w_i | 0 \le i \le 4\} \subset \mathfrak{p}_{\mathbf{C}} = \mathfrak{p} \otimes_{\mathbf{R}} \mathbf{C}$ by

$$\begin{aligned} w_0 &:= -2(H_{2,3} - \sqrt{-1}X_{2,3}) \\ w_4 &:= -2(H_{2,3} + \sqrt{-1}X_{2,3}) \\ w_2 &:= \frac{2}{3}(2H_{1,2}+H_{2,3}) \\ w_1 &:= X_{1,3} + \sqrt{-1}X_{1,2} \\ w_3 &:= -X_{1,3} + \sqrt{-1}X_{1,2}. \end{aligned}$$

Then $\{w_i | 0 \le i \le 4\}$ becomes the basis of $\mathfrak{p}_{\mathbf{C}}$.

With this basis, the gradient operator ∇^R is written as

$$\begin{aligned}\nabla^R\Psi =& \frac{1}{16}R_{w_4}\Psi \otimes w_0 + \frac{1}{16}R_{w_0}\Psi \otimes w_4 - \frac{1}{4}R_{w_3}\Psi \otimes w_1 \\ &- \frac{1}{4}R_{w_1}\Psi \otimes w_3 + \frac{3}{8}R_{w_2}\Psi \otimes w_2 \\ &= \frac{1}{4}\Bigg(\frac{1}{4}R_{w_4}\Psi \otimes w_0 + \frac{1}{4}R_{w_0}\Psi \otimes w_4 - R_{w_3}\Psi \otimes w_1 \\ &-R_{w_1}\Psi \otimes w_3 + \frac{3}{2}R_{w_2}\Psi \otimes w_2\Bigg).\end{aligned}$$

K acts on $\mathfrak{p}_{\mathbf{C}}$ by adjoint action. We denote this representation by (τ_4, W_4). By the Clebsh-Gordan theorem, $\tau_2 \otimes \tau_4$ has the irreducible decomposition

$$\tau_2 \otimes \tau_4 \cong \tau_2 \oplus \tau_4 \oplus \tau_6.$$

In this decomposition, the projector of K-modules

$$pr_2 : \tau_2 \otimes \tau_4 \to \tau_2$$

is described as in the following table:

Table 1: Table of $pr_2(s_j \otimes w_k)$

	w_0	w_1	w_2	w_3	w_4
s_1	0	$-\frac{1}{4}(s_3 + \sqrt{-1}s_2)$	$-\frac{1}{3}s_1$	$\frac{1}{4}(s_3 - \sqrt{-1}s_2)$	0
s_2	$\frac{1}{2}(s_2 - \sqrt{-1}s_3)$	$-\frac{\sqrt{-1}}{4}s_1$	$\frac{1}{6}s_2$	$-\frac{\sqrt{-1}}{4}s_1$	$\frac{1}{2}(s_2 + \sqrt{-1}s_3)$
s_3	$-\frac{1}{2}(s_3 + \sqrt{-1}s_2)$	$-\frac{1}{4}s_1$	$\frac{1}{6}s_3$	$\frac{1}{4}s_1$	$\frac{1}{2}(-s_3 + \sqrt{-1}s_2)$

$\nabla^R \Psi$ is a $\tau_2 \otimes (\tau_2 \otimes \mathfrak{p}_{\mathbf{C}})$-valued function. Then, by mapping $s_i^L \otimes s_j^R \otimes w_k$ to $s_i^L \otimes s_j^R w_k$ (here, $s_j^R w_k := pr_2(s_j^R \otimes w_k)$) , we have a K-homomorphism

$$\tilde{pr}_2(\nabla^R) : C^\infty_{\tau_2,\tau_2}(K \backslash G/K) \to C^\infty_{\tau_2,\tau_2}(K \backslash G/K).$$

Since minimal K-type τ_2 is multiplicity one, $\tilde{pr}_2(\nabla^R)$ is a map of constant multiple.
We compute $4\tilde{pr}_2(\nabla^R \Psi)(a)$ for $\Psi(g) = \sum_i \sum_j d_{ij}(g) s_i^L \otimes s_j^R$, $a \in A$.

1)

$$\begin{aligned}
\frac{1}{4}\tilde{pr}_2(R_{w_4}\Psi \otimes w_0) &= \frac{1}{4}\tilde{pr}_2(R_{-2(H_{2,3}+\sqrt{-1}X_{2,3})}\Psi \otimes w_0) \\
&= -\frac{1}{2}\tilde{pr}_2(R_{H_{2,3}}\Psi \otimes w_0) - \frac{\sqrt{-1}}{2}\tilde{pr}_2(R_{X_{2,3}}\Psi \otimes w_0)
\end{aligned}$$

First,

$$\begin{aligned}
&-\frac{1}{2}\tilde{pr}_2(R_{H_{2,3}}\Psi \otimes w_0) \\
&= -\frac{1}{2}(-\partial_1 + 2\partial_2)\sum_{i=1}^{3} d_{ii}(a) s_i^L \otimes s_i^R w_0 \\
&= -\frac{1}{2}(-\partial_1 + 2\partial_2) d_{22}(a) s_2^L \otimes \frac{1}{2}(s_2^R - \sqrt{-1}s_3^R) \\
&-\frac{1}{2}(-\partial_1 + 2\partial_2) d_{33}(a) s_3^L \otimes -\frac{1}{2}(s_3^R + \sqrt{-1}s_2^R) \\
&= -\frac{1}{4}(-\partial_1 + 2\partial_2) d_{22}(a) s_{22}^{LR} + \frac{\sqrt{-1}}{4}(-\partial_1 + 2\partial_2) d_{22}(a) s_{23}^{LR}
\end{aligned}$$

$$+\frac{\sqrt{-1}}{4}(-\partial_1+2\partial_2)d_{33}(a)s^{LR}_{32}+\frac{1}{4}(-\partial_1+2\partial_2)d_{33}(a)s^{LR}_{33}.$$

Next, since

$$X_{2,3}=-\frac{1}{sh(y_2)}\mathrm{Ad}(a^{-1})K_{2,3}+\frac{ch(y_2)}{sh(y_2)}K_{2,3},$$

we have

$$\begin{aligned}
&-\frac{\sqrt{-1}}{2}p\tilde{r}_2(R_{X_{2,3}}\Psi\otimes w_0)\\
&=-\frac{\sqrt{-1}}{2}p\tilde{r}_2\left(-\frac{1}{sh(y_2)}R_{\mathrm{Ad}(a^{-1})K_{2,3}}\Psi\otimes w_0+\frac{ch(y_2)}{sh(y_2)}R_{K_{2,3}}\Psi\otimes w_0\right)\\
&=-\frac{\sqrt{-1}}{2sh(y_2)}d_{22}(a)s^L_3\otimes\frac{1}{2}(s^R_2-\sqrt{-1}s^R_3)\\
&+\frac{\sqrt{-1}}{2sh(y_2)}d_{33}(a)s^L_2\otimes-\frac{1}{2}(s^R_3+\sqrt{-1}s^R_2)\\
&-\frac{\sqrt{-1}ch(y_2)}{2sh(y_2)}d_{22}(a)s^L_2\otimes-\frac{1}{2}(s^R_3+\sqrt{-1}s^R_2)\\
&+\frac{\sqrt{-1}ch(y_2)}{2sh(y_2)}d_{33}(a)s^L_3\otimes\frac{1}{2}(s^R_2-\sqrt{-1}s^R_3)\\
&=\left(\frac{1}{4sh(y_2)}d_{33}(a)-\frac{ch(y_2)}{4sh(y_2)}d_{22}(a)\right)s^{LR}_{22}\\
&+\left(-\frac{\sqrt{-1}}{4sh(y_2)}d_{33}(a)+\frac{\sqrt{-1}ch(y_2)}{4sh(y_2)}d_{22}(a)\right)s^{LR}_{23}\\
&+\left(-\frac{\sqrt{-1}}{4sh(y_2)}d_{22}(a)+\frac{\sqrt{-1}ch(y_2)}{4sh(y_2)}d_{33}(a)\right)s^{LR}_{32}\\
&+\left(-\frac{1}{4sh(y_2)}d_{22}(a)+\frac{ch(y_2)}{4sh(y_2)}d_{33}(a)\right)s^{LR}_{33}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\frac{1}{4}p\tilde{r}_2(R_{w_4}\Psi\otimes w_0)\\
&=-\frac{1}{4}(-\partial_1+2\partial_2)d_{22}(a)s^{LR}_{22}+\frac{\sqrt{-1}}{4}(-\partial_1+2\partial_2)d_{22}(a)s^{LR}_{23}\\
&+\frac{\sqrt{-1}}{4}(-\partial_1+2\partial_2)d_{33}(a)s^{LR}_{32}+\frac{1}{4}(-\partial_1+2\partial_2)d_{33}(a)s^{LR}_{33}\\
&+\left(\frac{1}{4sh(y_2)}d_{33}(a)-\frac{ch(y_2)}{4sh(y_2)}d_{22}(a)\right)s^{LR}_{22}\\
&+\left(-\frac{\sqrt{-1}}{4sh(y_2)}d_{33}(a)+\frac{\sqrt{-1}ch(y_2)}{4sh(y_2)}d_{22}(a)\right)s^{LR}_{23}
\end{aligned}$$

$$
\begin{aligned}
&+\left(-\frac{\sqrt{-1}}{4sh(y_2)}d_{22}(a)+\frac{\sqrt{-1}ch(y_2)}{4sh(y_2)}d_{33}(a)\right)s_{32}^{LR}\\
&+\left(-\frac{1}{4sh(y_2)}d_{22}(a)+\frac{ch(y_2)}{4sh(y_2)}d_{33}(a)\right)s_{33}^{LR}.
\end{aligned}
$$

2) Similarly, we have

$$
\begin{aligned}
&\frac{1}{4}p\tilde{r}_2(R_{w_0}\Psi\otimes w_4)\\
&=-\frac{1}{4}(-\partial_1+2\partial_2)d_{22}(a)s_{22}^{LR}-\frac{\sqrt{-1}}{4}(-\partial_1+2\partial_2)d_{22}(a)s_{23}^{LR}\\
&-\frac{\sqrt{-1}}{4}(-\partial_1+2\partial_2)d_{33}(a)s_{32}^{LR}+\frac{1}{4}(-\partial_1+2\partial_2)d_{33}(a)s_{33}^{LR}\\
&+\left(\frac{1}{4sh(y_2)}d_{33}(a)-\frac{ch(y_2)}{4sh(y_2)}d_{22}(a)\right)s_{22}^{LR}\\
&+\left(\frac{\sqrt{-1}}{4sh(y_2)}d_{33}(a)-\frac{\sqrt{-1}ch(y_2)}{4sh(y_2)}d_{22}(a)\right)s_{23}^{LR}\\
&+\left(\frac{\sqrt{-1}}{4sh(y_2)}d_{22}(a)-\frac{\sqrt{-1}ch(y_2)}{4sh(y_2)}d_{33}(a)\right)s_{32}^{LR}\\
&+\left(-\frac{1}{4sh(y_2)}d_{22}(a)+\frac{ch(y_2)}{4sh(y_2)}d_{33}(a)\right)s_{33}^{LR}.
\end{aligned}
$$

3)

$$
\begin{aligned}
&-p\tilde{r}_2(R_{w_3}\Psi\otimes w_1)\\
&=p\tilde{r}_2(R_{X_{1,3}}\Psi\otimes w_1)-\sqrt{-1}p\tilde{r}_2(R_{X_{1,2}}\otimes w_1).
\end{aligned}
$$

First,

$$
\begin{aligned}
&p\tilde{r}_2(R_{X_{1,3}}\Psi\otimes w_1)\\
&=-\frac{1}{sh(y_1y_2)}R_{\mathrm{Ad}(a^{-1})K_{1,3}}\left(\sum_{i=1}^{3}d_{ii}(a)s_i^L\otimes s_i^R\right)w_1\\
&+\frac{ch(y_1y_2)}{sh(y_1y_2)}R_{K_{1,3}}\left(\sum_{i=1}^{3}d_{ii}(a)s_i^L\otimes s_i^R\right)w_1\\
&=-\frac{1}{sh(y_1y_2)}(-d_{11}(a)s_3^L\otimes s_1^Rw_1+d_{33}(a)s_1^L\otimes s_3^Rw_1)\\
&-\frac{ch(y_1y_2)}{sh(y_1y_2)}(-d_{11}(a)s_1^L\otimes s_3^Rw_1+d_{33}(a)s_3^L\otimes s_1^Rw_1)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{sh(y_1y_2)} d_{11}(a) s_3^L \otimes -\frac{1}{4}(s_3^R + \sqrt{-1}s_2^R) \\
&- \frac{1}{sh(y_1y_2)} d_{33}(a) s_1^L \otimes -\frac{1}{4} s_1^R \\
&+ \frac{ch(y_1y_2)}{sh(y_1y_2)} d_{11}(a) s_1^L \otimes -\frac{1}{4} s_1^R \\
&- \frac{ch(y_1y_2)}{sh(y_1y_2)} d_{33}(a) s_3^L \otimes -\frac{1}{4}(s_3^R + \sqrt{-1}s_2^R) \\
&= \left(\frac{1}{4sh(y_1y_2)} d_{33}(a) - \frac{ch(y_1y_2)}{4sh(y_1y_2)} d_{11}(a) \right) s_{11}^{LR} \\
&+ \left(-\frac{\sqrt{-1}}{4sh(y_1y_2)} d_{11}(a) + \frac{\sqrt{-1}ch(y_1y_2)}{4sh(y_1y_2)} d_{33}(a) \right) s_{32}^{LR} \\
&+ \left(-\frac{1}{4sh(y_1y_2)} d_{11}(a) + \frac{ch(y_1y_2)}{4sh(y_1y_2)} d_{33}(a) \right) s_{33}^{LR}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&- \sqrt{-1} p\tilde{r}_2 (R_{X_{1,2}} \Psi \otimes w_1) \\
&= \frac{\sqrt{-1}}{sh(y_1)} R_{\mathrm{Ad}(a^{-1})K_{1,2}} \left(\sum_{i=1}^{3} d_{ii}(a) s_i^L \otimes s_i^R \right) w_1 \\
&- \frac{\sqrt{-1}ch(y_1)}{sh(y_1)} R_{K_{1,2}} \left(\sum_{i=1}^{3} d_{ii}(a) s_i^L \otimes s_i^R \right) w_1 \\
&= \frac{\sqrt{-1}}{sh(y_1)} (-d_{11}(a) s_2^L \otimes s_1^R w_1 + d_{22}(a) s_1^L \otimes s_2^R w_1) \\
&+ \frac{\sqrt{-1}ch(y_1)}{sh(y_1)} (-d_{11}(a) s_1^L \otimes s_2^R w_1 + d_{22}(a) s_2^L \otimes s_1^R w_1) \\
&= -\frac{\sqrt{-1}}{sh(y_1)} d_{11}(a) s_2^L \otimes -\frac{1}{4}(s_3^R + \sqrt{-1}s_2^R) \\
&+ \frac{\sqrt{-1}}{sh(y_1)} d_{22}(a) s_1^L \otimes -\frac{\sqrt{-1}}{4} s_1^R \\
&- \frac{\sqrt{-1}ch(y_1)}{sh(y_1)} d_{11}(a) s_1^L \otimes -\frac{\sqrt{-1}}{4} s_1^R \\
&+ \frac{\sqrt{-1}ch(y_1)}{sh(y_1)} d_{22}(a) s_2^L \otimes -\frac{1}{4}(s_3^R + \sqrt{-1}s_2^R) \\
&= \left(\frac{1}{4sh(y_1)} d_{22}(a) - \frac{ch(y_1)}{4sh(y_1)} d_{11}(a) \right) s_{11}^{LR} \\
&+ \left(\frac{\sqrt{-1}}{4sh(y_1)} d_{11}(a) - \frac{\sqrt{-1}ch(y_1)}{4sh(y_1)} d_{22}(a) \right) s_{23}^{LR}
\end{aligned}
$$

$$+\left(-\frac{1}{4sh(y_1)}d_{11}(a)+\frac{ch(y_1)}{4sh(y_1)}d_{22}(a)\right)s^{LR}_{22}.$$

Therefore, we have

$$\begin{aligned}
&-\tilde{pr}_2(R_{w_3}\Psi\otimes w_1)\\
&=\left(\frac{1}{4sh(y_1)}d_{22}(a)+\frac{1}{4sh(y_1y_2)}d_{33}(a)\right.\\
&\left.-\frac{ch(y_1y_2)}{4sh(y_1y_2)}d_{11}(a)-\frac{ch(y_1)}{4sh(y_1)}d_{11}(a)\right)s^{LR}_{11}\\
&+\left(\frac{\sqrt{-1}}{4sh(y_1)}d_{11}(a)-\frac{\sqrt{-1}ch(y_1)}{4sh(y_1)}d_{22}(a)\right)s^{LR}_{23}\\
&+\left(-\frac{\sqrt{-1}}{4sh(y_1y_2)}d_{11}(a)+\frac{\sqrt{-1}ch(y_1y_2)}{4sh(y_1y_2)}d_{33}(a)\right)s^{LR}_{32}\\
&+\left(-\frac{1}{4sh(y_1)}d_{11}(a)+\frac{ch(y_1)}{4sh(y_1)}d_{22}(a)\right)s^{LR}_{22}\\
&+\left(-\frac{1}{4sh(y_1y_2)}d_{11}(a)+\frac{ch(y_1y_2)}{4sh(y_1y_2)}d_{33}(a)\right)s^{LR}_{33}.
\end{aligned}$$

4) Similarly, we have

$$\begin{aligned}
&-\tilde{pr}_2(R_{w_1}\Psi\otimes w_3)\\
&=\left(\frac{1}{4sh(y_1y_2)}d_{33}(a)-\frac{ch(y_1y_2)}{4sh(y_1y_2)}d_{11}(a)\right.\\
&\left.+\frac{1}{4sh(y_1)}d_{22}(a)-\frac{ch(y_1)}{4sh(y_1)}d_{11}(a)\right)s^{LR}_{11}\\
&+\left(-\frac{\sqrt{-1}}{4sh(y_1)}d_{11}(a)+\frac{\sqrt{-1}ch(y_1)}{4sh(y_1)}d_{22}(a)\right)s^{LR}_{23}\\
&+\left(\frac{\sqrt{-1}}{4sh(y_1y_2)}d_{11}(a)-\frac{\sqrt{-1}ch(y_1y_2)}{4sh(y_1y_2)}d_{33}(a)\right)s^{LR}_{32}\\
&+\left(-\frac{1}{4sh(y_1)}d_{11}(a)+\frac{ch(y_1)}{4sh(y_1)}d_{22}(a)\right)s^{LR}_{22}\\
&+\left(-\frac{1}{4sh(y_1y_2)}d_{11}(a)+\frac{ch(y_1y_2)}{4sh(y_1y_2)}d_{33}(a)\right)s^{LR}_{33}.
\end{aligned}$$

5)Finally,

$$\frac{3}{2}\tilde{pr}_2(R_{w_2}\Psi\otimes w_2)$$

$$= \frac{3}{2} \tilde{pr}_2 (R_{\frac{2}{3}(2H_{1,2}+H_{2,3})} \Psi \otimes w_2)$$
$$= R_{2H_{1,2}+H_{2,3}} \left(\sum_{i=1}^{3} d_{ii}(a) s_i^L \otimes s_i^R \right) w_2$$
$$= -\partial_1 d_{11}(a) s_{11}^{LR} + \frac{1}{2} \partial_1 d_{22}(a) s_{22}^{LR} + \frac{1}{2} \partial_1 d_{33}(a) s_{33}^{LR}.$$

By summing these results, we have

$$\begin{aligned}
&4\tilde{pr}_2(\nabla_R \Psi)(a) \\
&= \left\{ -\left(\partial_1 + \frac{ch(y_1)}{2sh(y_1)} + \frac{ch(y_1y_2)}{2sh(y_1y_2)} \right) d_{11}(a) \right. \\
&\left. + \frac{1}{2sh(y_1)} d_{22}(a) + \frac{1}{2sh(y_1y_2)} d_{33}(a) \right\} s_{11}^{LR} \\
&+ \left\{ -\frac{1}{2sh(y_1)} d_{11}(a) + \left(\partial_1 - \partial_2 + \frac{ch(y_1)}{2sh(y_1)} - \frac{ch(y_2)}{2sh(y_2)} \right) d_{22}(a) \right. \\
&\left. + \frac{1}{2sh(y_2)} d_{33}(a) \right\} s_{22}^{LR} \\
&+ \left\{ -\frac{1}{2sh(y_1y_2)} d_{11}(a) - \frac{1}{2sh(y_2)} d_{22}(a) \right. \\
&\left. + \left(\partial_2 + \frac{ch(y_2)}{2sh(y_2)} + \frac{ch(y_1y_2)}{2sh(y_1y_2)} \right) d_{33}(a) \right\} s_{33}^{LR}.
\end{aligned}$$

This equals $\lambda_i \sum_{i=1}^{3} d_{ii}(a) s_{ii}^{LR}$, where λ_i $(i = 1, 2, 3)$ are some constants depending on the choice of $\sigma = \sigma_i$ of the principal series representation.
Next, we compute λ_i $(i = 1, 2, 3)$.
Let $K \hookrightarrow GL(3, \mathbf{C})$, $k \mapsto (s_{ij}(k))_{i,j=1}^3$ be the tautological representation of K. It is easy to check that $\{s_{i1}, s_{i2}, s_{i3}\}$ is the generator of the minimal K-type of $C^\infty_{M,\sigma_i}(K)$ $(i = 1, 2, 3)$, and we may identify s_{ij} with s_j $(j = 1, 2, 3)$ via K-isomorphism. The dual s_{ij}^* of s_{ij} is s_{ij} itself $(j = 1, 2, 3)$. Fix $\Phi \in \mathrm{Hom}_{K\times K}(\tau_2^* \boxtimes \tau_2^*, H_\pi^* \boxtimes H_\pi)$. To find the value λ_1, it is sufficient to compute the action of $\tilde{pr}_2 \circ \nabla^R$ on

$$\tilde{\Psi} = \sum_{i,j=1}^{3} (s_{1i} \otimes s_{1j}) \otimes \Phi(s_{1i}^* \otimes s_{1j}^*)(g_1, g_2) \in \{(V_{\tau_2^*}^* \boxtimes V_{\tau_2^*}^*) \otimes (H_\pi^* \boxtimes H_\pi)\}^{K\times K}.$$

Since τ_2^* is multiplicity one in H_π, $\tilde{pr}_2 \circ \nabla^R$ acts on this function by constant multiple, and the matrix coefficient of $\pi_{\sigma_1,\nu}$ inherits this value, because the matrix coefficient can be written $\sum_{i,j=1}^{3} (s_{1i} \otimes s_{1j}) \otimes \Lambda \circ \tilde{\Psi}(s_{1i}^* \otimes s_{1j}^*)$. Here, $\Lambda : H_\pi^* \boxtimes H_\pi \to C^\infty(G)$, $f^* \otimes f \mapsto \langle f^*, \pi(\cdot) f \rangle$ is a $G \times G$-homomorphism.
Let $e \in G$ be the unit element of G.

1)

$$\begin{aligned}\frac{1}{4}p\tilde{r}_2(R_{w_4}\tilde{\Psi}\otimes w_0)(e,e) =& -\frac{1}{2}p\tilde{r}_2(R_{H_{2,3}}\tilde{\Psi}\otimes w_0)(e,e)\\ &-\frac{\sqrt{-1}}{2}p\tilde{r}_2(R_{X_{2,3}}\tilde{\Psi}\otimes w_0)(e,e).\end{aligned}$$

First,

$$\begin{aligned}&-\frac{1}{2}p\tilde{r}_2(R_{H_{2,3}}\tilde{\Psi}\otimes w_0)(e,e)\\ &=-\frac{1}{2}(\nu_2+1)s_{11}\otimes s_{11}w_0=0.\end{aligned}$$

Next, since the action of $X_{2,3}$ is the same as that of $-K_{2,3}$, we have

$$-\frac{\sqrt{-1}}{2}p\tilde{r}_2(R_{X_{2,3}}\tilde{\Psi}\otimes w_0)(e,e)=0.$$

Therefore, we have

$$\frac{1}{4}p\tilde{r}_2(R_{w_4}\tilde{\Psi}\otimes w_0)(e,e)=0.$$

2) Similarly, we have

$$\frac{1}{4}p\tilde{r}_2(R_{w_0}\tilde{\Psi}\otimes w_4)(e,e)=0.$$

3)

$$\begin{aligned}-p\tilde{r}_2(R_{w_3}\tilde{\Psi}\otimes w_1)(e,e) =& p\tilde{r}_2(R_{X_{1,3}}\tilde{\Psi}\otimes w_1)(e,e)\\ &-\sqrt{-1}p\tilde{r}_2(R_{X_{1,2}}\tilde{\Psi}\otimes w_1)(e,e).\end{aligned}$$

First,

$$p\tilde{r}_2(R_{X_{1,3}}\tilde{\Psi}\otimes w_1)(e)=-s_{11}\otimes s_{13}w_1=\frac{1}{4}s_{11}\otimes s_{11}.$$

Next,

$$-\sqrt{-1}p\tilde{r}_2(R_{X_{1,2}}\tilde{\Psi}\otimes w_1)(e,e)=\sqrt{-1}s_{11}\otimes s_{12}w_1=\frac{1}{4}s_{11}\otimes s_{11}.$$

Therefore, we have

$$-p\tilde{r}_2(R_{w_3}\tilde{\Psi}\otimes w_1)(e,e)=\frac{1}{2}s_{11}\otimes s_{11}.$$

4) Similarly, we have

$$-p\tilde{r}_2(R_{w_1}\tilde{\Psi}\otimes w_3)(e,e)=\frac{1}{2}s_{11}\otimes s_{11}.$$

5) Finally,

$$\begin{aligned}
&\frac{3}{2}p\tilde{r}_2(R_{w_2}\tilde{\Psi}\otimes w_2)(e,e)\\
&=2p\tilde{r}_2(R_{H_{1,2}}\tilde{\Psi}\otimes w_2)(e,e)+p\tilde{r}_2(R_{H_{2,3}}\tilde{\Psi}\otimes w_2)(e,e)\\
&=-\frac{2}{3}(\nu_1-\nu_2+1)s_{11}\otimes s_{11}-\frac{1}{3}(\nu_2+1)s_{11}\otimes s_{11}\\
&=\left\{-\frac{1}{3}(2\nu_1-\nu_2)-1\right\}s_{11}\otimes s_{11}.
\end{aligned}$$

By summing these results, we have

$$\begin{aligned}
4p\tilde{r}_2(\nabla_R\tilde{\Psi})(e,e)&=-\frac{1}{3}(2\nu_1-\nu_2)s_{11}\otimes s_{11}\\
&=-\frac{1}{3}(2\nu_1-\nu_2)\tilde{\Psi}(e,e).
\end{aligned}$$

In these computations, we used $s_{11}(e)=1, s_{12}(e)=0, s_{13}(e)=0$. Therefore, we conclude that $\lambda_1=-\frac{1}{3}(2\nu_1-\nu_2)$.
The computation of λ_2,λ_3 is the same as that of λ_1, and the values are $\lambda_2=\frac{1}{3}(\nu_1-2\nu_2)$, $\lambda_3=\frac{1}{3}(\nu_1+\nu_2)$.
Summing up, we have the next result:

Theorem 6.5. *Let $\Psi(g)=\sum_{i=1}^{3}\sum_{j=1}^{3}d_{ij}(g)s_i^L\otimes s_j^R\in C^\infty_{\tau_2,\tau_2}(K\backslash G/K)$ be the matrix coefficient obtained from the non-spherical principal series representation whose character of M is $\sigma=\sigma_i$ $(i=1,2,3)$. Put*

$$d_{11}(a)=F(y_1,y_2), d_{22}(a)=G(y_1,y_2), d_{33}(a)=H(y_1,y_2).$$

Then F, G, H satisfy the following relations:

$$\begin{aligned}
&-\left(\partial_1+\frac{ch(y_1)}{2sh(y_1)}+\frac{ch(y_1y_2)}{2sh(y_1y_2)}\right)F(y_1,y_2)+\frac{1}{2sh(y_1)}G(y_1,y_2)\\
&+\frac{1}{2sh(y_1y_2)}H(y_1,y_2)=\lambda_iF(y_1,y_2)
\end{aligned}\tag{6.4}$$

$$\begin{aligned}
&-\frac{1}{2sh(y_1)}F(y_1,y_2)+\left(\partial_1-\partial_2+\frac{ch(y_1)}{2sh(y_1)}-\frac{ch(y_2)}{2sh(y_2)}\right)G(y_1,y_2)\\
&+\frac{1}{2sh(y_2)}H(y_1,y_2)=\lambda_iG(y_1,y_2)
\end{aligned}\tag{6.5}$$

$$
\begin{aligned}
&-\frac{1}{2sh(y_1y_2)}F(y_1,y_2)-\frac{1}{2sh(y_2)}G(y_1,y_2)\\
&+\left(\partial_2+\frac{ch(y_2)}{2sh(y_2)}+\frac{ch(y_1y_2)}{2sh(y_1y_2)}\right)H(y_1,y_2)=\lambda_i H(y_1,y_2).
\end{aligned}
\tag{6.6}
$$

Here, $\lambda_1=-\frac{1}{3}(2\nu_1-\nu_2),\ \lambda_2=\frac{1}{3}(\nu_1-2\nu_2),\ \lambda_3=\frac{1}{3}(\nu_1+\nu_2).$

6.3 Power series solutions

We give the power series solution of the equations obtained in Theorem 6.3 and Theorem 6.5. Firstly, we modify F,G,H by

$$
\begin{aligned}
\tilde{F}(y_1,y_2)&=sh(y_1)^{\frac{1}{2}}sh(y_2)^{\frac{1}{2}}sh(y_1y_2)^{\frac{1}{2}}F(y_1,y_2)\\
\tilde{G}(y_1,y_2)&=sh(y_1)^{\frac{1}{2}}sh(y_2)^{\frac{1}{2}}sh(y_1y_2)^{\frac{1}{2}}G(y_1,y_2)\\
\tilde{H}(y_1,y_2)&=sh(y_1)^{\frac{1}{2}}sh(y_2)^{\frac{1}{2}}sh(y_1y_2)^{\frac{1}{2}}H(y_1,y_2).
\end{aligned}
$$

Then the six equations are rewritten as follows:

$$
\begin{aligned}
&\left(2\partial_1^2-2\partial_1\partial_2+2\partial_2^2\right)\tilde{F}(y_1,y_2)\\
&+\left(-2-\lambda-\frac{2y_1^2}{(y_1^2-1)^2}+\frac{2y_2^2}{(y_2^2-1)^2}-\frac{2y_1^2y_2^2}{(y_1^2y_2^2-1)^2}\right)\tilde{F}(y_1,y_2)\\
&+2\frac{y_1^3+y_1}{(y_1^2-1)^2}\tilde{G}(y_1,y_2)+2\frac{y_1^3y_2^3+y_1y_2}{(y_1^2y_2^2-1)^2}\tilde{H}(y_1,y_2)=0
\end{aligned}
\tag{6.7}
$$

$$
\begin{aligned}
&\left(2\partial_1^2-2\partial_1\partial_2+2\partial_2^2\right)\tilde{G}(y_1,y_2)\\
&+\left(-2-\lambda-\frac{2y_1^2}{(y_1^2-1)^2}-\frac{2y_2^2}{(y_2^2-1)^2}+\frac{2y_1^2y_2^2}{(y_1^2y_2^2-1)^2}\right)\tilde{G}(y_1,y_2)\\
&+2\frac{y_1^3+y_1}{(y_1^2-1)^2}\tilde{F}(y_1,y_2)+2\frac{y_2^3+y_2}{(y_2^2-1)^2}\tilde{H}(y_1,y_2)=0
\end{aligned}
\tag{6.8}
$$

$$
\begin{aligned}
&\left(2\partial_1^2-2\partial_1\partial_2+2\partial_2^2\right)\tilde{H}(y_1,y_2)\\
&+\left(-2-\lambda+\frac{2y_1^2}{(y_1^2-1)^2}-\frac{2y_2^2}{(y_2^2-1)^2}-\frac{2y_1^2y_2^2}{(y_1^2y_2^2-1)^2}\right)\tilde{H}(y_1,y_2)\\
&+2\frac{y_1^3y_2^3+y_1y_2}{(y_1^2y_2^2-1)^2}\tilde{F}(y_1,y_2)+2\frac{y_2^3+y_2}{(y_2^2-1)^2}\tilde{G}(y_1,y_2)=0
\end{aligned}
\tag{6.9}
$$

$$
\begin{aligned}
&-\partial_1\tilde{F}(y_1,y_2)+\frac{1}{2sh(y_1)}\tilde{G}(y_1,y_2)+\frac{1}{2sh(y_1y_2)}\tilde{H}(y_1,y_2)\\
&=\lambda_i\tilde{F}(y_1,y_2)
\end{aligned}
\tag{6.10}
$$

$$
\begin{aligned}
&-\frac{1}{2sh(y_1)}\tilde{F}(y_1,y_2)+(\partial_1-\partial_2)\,\tilde{G}(y_1,y_2)+\frac{1}{2sh(y_2)}\tilde{H}(y_1,y_2)\\
&=\lambda_i\tilde{G}(y_1,y_2)
\end{aligned}
\tag{6.11}
$$

$$
\begin{aligned}
&-\frac{1}{2sh(y_1y_2)}\tilde{F}(y_1,y_2)-\frac{1}{2sh(y_2)}\tilde{G}(y_1,y_2)+\partial_2\tilde{H}(y_1,y_2)\\
&=\lambda_i\tilde{H}(y_1,y_2).
\end{aligned}
\tag{6.12}
$$

Here, λ is in Theorem 6.3 and λ_i is in Theorem 6.5.
We put

$$\tilde{F}(y_1,y_2)=\sum_{n,m=0}^{\infty}a_{n,m}y_1^{n+\mu_1}y_2^{m+\mu_2} \tag{6.13}$$

$$\tilde{G}(y_1,y_2)=\sum_{n,m=0}^{\infty}b_{n,m}y_1^{n+\mu_1}y_2^{m+\mu_2} \tag{6.14}$$

$$\tilde{H}(y_1,y_2)=\sum_{n,m=0}^{\infty}c_{n,m}y_1^{n+\mu_1}y_2^{m+\mu_2} \tag{6.15}$$

$$(a_{0,0},b_{0,0},c_{0,0})\neq(0,0,0)$$

and compute the characteristic roots (μ_1,μ_2) and the coefficients $(a_{n,m}),(b_{n,m}),(c_{n,m})$. Hereafter we compute under the assumption that $1,\nu_1,\nu_2$ are linearly independent over $\mathbf{Q}$. Since

$$\frac{y^2}{(y^2-1)^2}=\sum_{k=1}^{\infty}ky^{2k},\quad \frac{y^3+y}{(y^2-1)^2}=\sum_{k=1}^{\infty}(2k-1)y^{2k-1}\quad(0<y<1),$$

by substituting these relations into equations (6.7), (6.8), (6.9) and picking up the coefficients of $y_1^{n+\mu_1}y_2^{m+\mu_2}$, we have

$$
\begin{aligned}
&(2n'^2-2n'm'+2m'^2+\lambda^{'})a_{n,m}\\
&-2\sum_{k=1}^{\infty}ka_{n-2k,m}+2\sum_{k=1}^{\infty}ka_{n,m-2k}-2\sum_{k=1}^{\infty}ka_{n-2k,m-2k}\\
&+2\sum_{k=1}^{\infty}(2k-1)b_{n-2k+1,m}+2\sum_{k=1}^{\infty}(2k-1)c_{n-2k+1,m-2k+1}=0
\end{aligned}
\tag{6.16}
$$

$$
\begin{aligned}
&(2n'^2-2n'm'+2m'^2+\lambda^{'})b_{n,m}\\
&-2\sum_{k=1}^{\infty}kb_{n-2k,m}-2\sum_{k=1}^{\infty}kb_{n,m-2k}+2\sum_{k=1}^{\infty}kb_{n-2k,m-2k}\\
&+2\sum_{k=1}^{\infty}(2k-1)a_{n-2k+1,m}+2\sum_{k=1}^{\infty}(2k-1)c_{n,m-2k+1}=0
\end{aligned}
\tag{6.17}
$$

$$
\begin{aligned}
&(2n'^2 - 2n'm' + 2m'^2 + \lambda^{'})c_{n,m} \\
&+ 2\sum_{k=1}^{\infty} kc_{n-2k,m} - 2\sum_{k=1}^{\infty} kc_{n,m-2k} - 2\sum_{k=1}^{\infty} kc_{n-2k,m-2k} \\
&+ 2\sum_{k=1}^{\infty}(2k-1)a_{n-2k+1,m-2k+1} + 2\sum_{k=1}^{\infty}(2k-1)b_{n,m-2k+1} = 0.
\end{aligned} \tag{6.18}
$$

Here, $n' = n + \mu_1$, $m' = m + \mu_2$, $\lambda' = -2 - \lambda$ and $a_{n,m}, b_{n,m}, c_{n,m} = 0$ if $n < 0$ or $m < 0$. By substituting $n = m = 0$ into these equations, since $(a_{0,0}, b_{0,0}, c_{0,0}) \neq (0, 0, 0)$, we have

$$
2\mu_1^2 - 2\mu_1\mu_2 + 2\mu_2^2 + \lambda^{'} = 0. \tag{6.19}
$$

Therefore,

$$
\begin{aligned}
&2n'^2 - 2n'm' + 2m'^2 + \lambda^{'} \\
&= 2(n + \mu_1)^2 - 2(n + \mu_1)(m + \mu_2) + 2(m + \mu_2)^2 + \lambda^{'} \\
&= 2\{n^2 - nm + m^2 + (2\mu_1 - \mu_2)n + (2\mu_2 - \mu_1)m\}.
\end{aligned}
$$

Now, we put

$$
p(n, m) = q(n, m) = r(n, m) = n^2 - nm + m^2 + (2\mu_1 - \mu_2)n + (2\mu_2 - \mu_1)m
$$

and subsutitute $2p(n, m)$, $2q(n, m)$, $2r(n, m)$ for $2n'^2 - 2n'm' + 2m'^2 + \lambda^{'}$ in the equations (6.16), (6.17), (6.18) respectively. Then we have

$$
\begin{aligned}
&p(n, m)a_{n,m} \\
&- \sum_{k=1}^{\infty} ka_{n-2k,m} + \sum_{k=1}^{\infty} ka_{n,m-2k} - \sum_{k=1}^{\infty} ka_{n-2k,m-2k} \\
&+ \sum_{k=1}^{\infty}(2k-1)b_{n-2k+1,m} + \sum_{k=1}^{\infty}(2k-1)c_{n-2k+1,m-2k+1} = 0
\end{aligned} \tag{6.20}
$$

$$
\begin{aligned}
&q(n, m)b_{n,m} \\
&- \sum_{k=1}^{\infty} kb_{n-2k,m} - \sum_{k=1}^{\infty} kb_{n,m-2k} + \sum_{k=1}^{\infty} kb_{n-2k,m-2k} \\
&+ \sum_{k=1}^{\infty}(2k-1)a_{n-2k+1,m} + \sum_{k=1}^{\infty}(2k-1)c_{n,m-2k+1} = 0
\end{aligned} \tag{6.21}
$$

$$
\begin{aligned}
&r(n, m)c_{n,m} \\
&+ \sum_{k=1}^{\infty} kc_{n-2k,m} - \sum_{k=1}^{\infty} kc_{n,m-2k} - \sum_{k=1}^{\infty} kc_{n-2k,m-2k} \\
&+ \sum_{k=1}^{\infty}(2k-1)a_{n-2k+1,m-2k+1} + \sum_{k=1}^{\infty}(2k-1)b_{n,m-2k+1} = 0.
\end{aligned} \tag{6.22}
$$

(Though $p(n,m)$,$q(n,m)$,$r(n,m)$ are the same polynomials, but we use the different symbols. By doing so, the expressions of the coefficients $a_{n,m}$, $b_{n,m}$, $c_{n,m}$ become a little easier.)
Before computing $(a_{n,m})$, $(b_{n,m})$, $(c_{n,m})$, we compute the characteristic roots (μ_1, μ_2). By substituting (6.13), (6.14), (6.15) into (6.10), (6.11), (6.12) and subsutituting $n = m = 0$, we have

$$(-\mu_1 - \lambda_i)a_{0,0} = 0$$

$$(\mu_1 - \mu_2 - \lambda_i)b_{0,0} = 0$$

$$(\mu_2 - \lambda_i)c_{0,0} = 0.$$

Since $(a_{0,0}, b_{0,0}, c_{0,0}) \neq (0,0,0)$, at least one of $-\mu_1 - \lambda_i$, $\mu_1 - \mu_2 - \lambda_i$, $\mu_2 - \lambda_i$ is 0. By combining this with the equation (6.19), we can compute the values of (μ_1, μ_2). (Because of the assumption of the linearly independence of $1, \nu_1, \nu_2$, we know that just one of $-\mu_1 - \lambda_i$, $\mu_1 - \mu_2 - \lambda_i$, $\mu_2 - \lambda_i$ is 0, and the other two are not 0.)

Lemma 6.6. *1) In case of $\sigma = \sigma_1$, $\lambda_i = \lambda_1 = -\frac{1}{3}(2\nu_1 - \nu_2)$.*
a) If $-\mu_1 - \lambda_1 = 0$,

$$(\mu_1, \mu_2) = \left(\frac{1}{3}(2\nu_1 - \nu_2), -\frac{1}{3}(2\nu_2 - \nu_1)\right), \left(\frac{1}{3}(2\nu_1 - \nu_2), \frac{1}{3}(\nu_1 + \nu_2)\right)$$

and $a_{0,0} \neq 0, b_{0,0} = 0, c_{0,0} = 0$.
b) If $\mu_1 - \mu_2 - \lambda_1 = 0$,

$$(\mu_1, \mu_2) = \left(\frac{1}{3}(2\nu_2 - \nu_1), \frac{1}{3}(\nu_1 + \nu_2)\right), \left(-\frac{1}{3}(\nu_1 + \nu_2), -\frac{1}{3}(2\nu_2 - \nu_1)\right)$$

and $a_{0,0} = 0, b_{0,0} \neq 0, c_{0,0} = 0$.
c) If $\mu_2 - \lambda_1 = 0$,

$$(\mu_1, \mu_2) = \left(\frac{1}{3}(2\nu_2 - \nu_1), -\frac{1}{3}(2\nu_1 - \nu_2)\right), \left(-\frac{1}{3}(\nu_1 + \nu_2), -\frac{1}{3}(2\nu_1 - \nu_2)\right)$$

and $a_{0,0} = 0, b_{0,0} = 0, c_{0,0} \neq 0$.

2) In case of $\sigma = \sigma_2$, $\lambda_i = \lambda_2 = \frac{1}{3}(\nu_1 - 2\nu_2)$.
a) If $-\mu_1 - \lambda_2 = 0$,

$$(\mu_1, \mu_2) = \left(\frac{1}{3}(2\nu_2 - \nu_1), \frac{1}{3}(\nu_1 + \nu_2)\right), \left(\frac{1}{3}(2\nu_2 - \nu_1), -\frac{1}{3}(2\nu_1 - \nu_2)\right)$$

and $a_{0,0} \neq 0, b_{0,0} = 0, c_{0,0} = 0$.
b) If $\mu_1 - \mu_2 - \lambda_2 = 0$,

$$(\mu_1, \mu_2) = \left(\frac{1}{3}(2\nu_1 - \nu_2), \frac{1}{3}(\nu_1 + \nu_2)\right), \left(-\frac{1}{3}(\nu_1 + \nu_2), -\frac{1}{3}(2\nu_1 - \nu_2)\right)$$

and $a_{0,0} = 0, b_{0,0} \neq 0, c_{0,0} = 0.$
c) If $\mu_2 - \lambda_2 = 0,$

$$(\mu_1, \mu_2) = \left(\frac{1}{3}(2\nu_1 - \nu_2), -\frac{1}{3}(2\nu_2 - \nu_1)\right), \left(-\frac{1}{3}(\nu_1 + \nu_2), -\frac{1}{3}(2\nu_2 - \nu_1)\right)$$

and $a_{0,0} = 0, b_{0,0} = 0, c_{0,0} \neq 0.$
3) In case of $\sigma = \sigma_3$, $\lambda_i = \lambda_3 = \frac{1}{3}(\nu_1 + \nu_2).$
a) If $-\mu_1 - \lambda_3 = 0,$

$$(\mu_1, \mu_2) = \left(-\frac{1}{3}(\nu_1 + \nu_2), -\frac{1}{3}(2\nu_2 - \nu_1)\right), \left(-\frac{1}{3}(\nu_1 + \nu_2), -\frac{1}{3}(2\nu_1 - \nu_2)\right)$$

and $a_{0,0} \neq 0, b_{0,0} = 0, c_{0,0} = 0.$
b) If $\mu_1 - \mu_2 - \lambda_3 = 0,$

$$(\mu_1, \mu_2) = \left(\frac{1}{3}(2\nu_1 - \nu_2), -\frac{1}{3}(2\nu_2 - \nu_1)\right), \left(\frac{1}{3}(2\nu_2 - \nu_1), -\frac{1}{3}(2\nu_1 - \nu_2)\right)$$

and $a_{0,0} = 0, b_{0,0} \neq 0, c_{0,0} = 0.$
c) If $\mu_2 - \lambda_3 = 0,$

$$(\mu_1, \mu_2) = \left(\frac{1}{3}(2\nu_1 - \nu_2), \frac{1}{3}(\nu_1 + \nu_2)\right), \left(\frac{1}{3}(2\nu_2 - \nu_1), \frac{1}{3}(\nu_1 + \nu_2)\right)$$

and $a_{0,0} = 0, b_{0,0} = 0, c_{0,0} \neq 0.$

The following theorem gives the explicit expressions of the coefficients $(a_{n,m})$, $(b_{n,m})$, $(c_{n,m})$.

Theorem 6.7. *Let* $\mathbf{P}_{n,m}$ *be the family of all sets* $\{\alpha_k(n_k, m_k), \cdots, \alpha_0(n_0, m_0)\}$ *satisfying the next rules;*
A) $\alpha_i = p$ *or* q *or* r $(i = 0, \cdots, k)$, $(n_k, m_k) = (n, m)$, $(n_0, m_0) = (0, 0),$

B) $\alpha_i(n_i, m_i) = p(n_i, m_i) \Rightarrow \alpha_{i-1}(n_{i-1}, m_{i-1}) = p(n_i - 2l_i, m_i)\,; l_i$ *or* $p(n_i, m_i - 2l_i)$; $-l_i$ *or* $p(n_i - 2l_i, m_i - 2l_i)\,; l_i$ *or* $q(n_i - 2l_i + 1, m_i)\,; -(2l_i - 1)$ *or* $r(n_i - 2l_i + 1, m_i - 2l_i + 1)\,; -(2l_i - 1).$

$\alpha_i(n_i, m_i) = q(n_i, m_i) \Rightarrow \alpha_{i-1}(n_{i-1}, m_{i-1}) = q(n_i - 2l_i, m_i)\,; l_i$ *or* $q(n_i, m_i - 2l_i)$; l_i *or* $q(n_i - 2l_i, m_i - 2l_i)\,; -l_i$ *or* $p(n_i - 2l_i + 1, m_i)\,; -(2l_i - 1)$ *or* $r(n_i, m_i - 2l_i + 1)\,; -(2l_i - 1).$

$\alpha_i(n_i, m_i) = r(n_i, m_i) \Rightarrow \alpha_{i-1}(n_{i-1}, m_{i-1}) = r(n_i - 2l_i, m_i)\,; -l_i$ *or* $r(n_i, m_i - 2l_i)$; l_i *or* $r(n_i - 2l_i, m_i - 2l_i)\,; l_i$ *or* $p(n_i - 2l_i + 1, m_i - 2l_i + 1)\,; -(2l_i - 1)$ *or* $q(n_i, m_i - 2l_i + 1)\,; -(2l_i - 1).$

$$(\exists l_i \in \mathbf{Z}_{>0}), \quad (i = 1, \cdots, k).$$

(k *depends on each set*).

For each i, *we express the number after ; of each correspondence as* d_i. *We put*

$$(\delta^a_{n,m}, \delta^b_{n,m}, \delta^c_{n,m}) = \begin{cases} (a_{0,0}, b_{0,0}, c_{0,0}) & (n; even, m; even) \\ (a_{0,0}, c_{0,0}, b_{0,0}) & (n; even, m; odd) \\ (b_{0,0}, a_{0,0}, c_{0,0}) & (n; odd, m; even) \\ (c_{0,0}, b_{0,0}, a_{0,0}) & (n; odd, m; odd) \end{cases}$$

And we put
$\mathbf{P}^p_{n,m} := \mathbf{P}_{n,m} \cap \{\alpha_k = p\}$, $\mathbf{P}^q_{n,m} := \mathbf{P}_{n,m} \cap \{\alpha_k = q\}$, $\mathbf{P}^r_{n,m} := \mathbf{P}_{n,m} \cap \{\alpha_k = r\}$. *Then we have*

$$a_{n,m} = \sum_{\{\alpha_k(n_k,m_k),\cdots,\alpha_0(n_0,m_0)\} \in \mathbf{P}^p_{n,m}} \frac{\left(\prod_{i=1}^k d_i\right) \delta^a_{n,m}}{\alpha_k(n_k, m_k) \cdots \alpha_1(n_1, m_1)}, \tag{6.23}$$

$$b_{n,m} = \sum_{\{\alpha_k(n_k,m_k),\cdots,\alpha_0(n_0,m_0)\} \in \mathbf{P}^q_{n,m}} \frac{\left(\prod_{i=1}^k d_i\right) \delta^b_{n,m}}{\alpha_k(n_k, m_k) \cdots \alpha_1(n_1, m_1)}, \tag{6.24}$$

$$c_{n,m} = \sum_{\{\alpha_k(n_k,m_k),\cdots,\alpha_0(n_0,m_0)\} \in \mathbf{P}^r_{n,m}} \frac{\left(\prod_{i=1}^k d_i\right) \delta^c_{n,m}}{\alpha_k(n_k, m_k) \cdots \alpha_1(n_1, m_1)}. \tag{6.25}$$

for $(n,m) \neq (0,0)$. *Here,* $a_{n,m} = 0$ (*resp.* $b_{n,m} = 0, c_{n,m} = 0$) *if* $\mathbf{P}^p_{n,m} = \emptyset$ (*resp.* $\mathbf{P}^q_{n,m} = \emptyset$, $\mathbf{P}^r_{n,m} = \emptyset$).

Proof. We prove this theorem by the induction with respect to m.
1) Firstly, if $m = 0$, by the equations (6.20), (6.21), (6.22) , we have

$$p(n,0)a_{n,0} - \sum_{l=1}^{\infty} l a_{n-2l,0} + \sum_{l=1}^{\infty} (2l-1) b_{n-2l+1,0} = 0 \tag{6.26}$$

$$q(n,0)b_{n,0} - \sum_{l=1}^{\infty} l b_{n-2l,0} + \sum_{l=1}^{\infty} (2l-1) a_{n-2l+1,0} = 0 \tag{6.27}$$

$$r(n,0)c_{n,0} + \sum_{l=1}^{\infty} l c_{n-2l,0} = 0. \tag{6.28}$$

Suppose that equations (6.20), (6.21), (6.22) hold for $m = 0$, $0 \leq n \leq N$. From the equation (6.26), we have

$$a_{N+1,0} = \frac{1}{p(N+1,0)} \sum_{l=1}^{\infty} l a_{N+1-2l,0} - \frac{1}{p(N+1,0)} \sum_{l=1}^{\infty} (2l-1) b_{N+2-2l,0}. \tag{6.29}$$

Since $\delta^a_{N+1-2l,0} = \delta^b_{N+2-2l,0} = \delta^a_{N+1,0}$, if we expand $a_{N+1-2l,0}$ and $b_{N+2-2l,0}$ as (6.23), (6.24), $\delta^a_{N+1,0}$ appears in the numerator of the right hand side of the equation (6.29).
We add $p(N+1,0)$ in front of each element of $\mathbf{P}^p_{N+1-2l,0}$ and express this family by $\tilde{\mathbf{P}}^p_{N+1-2l,0}$. Next, we add $p(N+1,0)$ in front of each element of $\mathbf{P}^q_{N+2-2l,0}$ and express this family by $\tilde{\mathbf{P}}^q_{N+2-2l,0}$. Then , by difinitions, easily we have

$$\mathbf{P}^p_{N+1,0} = \left(\bigcup_l \tilde{\mathbf{P}}^p_{N+1-2l,0}\right) \cup \left(\bigcup_l \tilde{\mathbf{P}}^q_{N+2-2l,0}\right),$$
$$\tilde{\mathbf{P}}^p_{N+1-2l,0} \cap \tilde{\mathbf{P}}^p_{N+1-2l',0} = \emptyset \quad (\forall\, l, l'\, s.t.\, l \neq l'),$$
$$\tilde{\mathbf{P}}^q_{N+2-2l,0} \cap \tilde{\mathbf{P}}^q_{N+2-2l',0} = \emptyset \quad (\forall\, l, l'\, s.t.\, l \neq l'),$$
$$\tilde{\mathbf{P}}^p_{N+1-2l,0} \cap \tilde{\mathbf{P}}^q_{N+2-2l',0} = \emptyset \quad (\forall\, l, l').$$

Therefore, the set appearing in the expansion of $a_{N+1,0}$ is just as (6.23) for $n = N+1, m = 0$. The rule of products directly follows from the assumption and the equation (6.29). So $a_{N+1,0}$ is just as (6.23) for $n = N+1, m = 0$. Therefore, we have proved (6.23) for $\forall n$ and $m = 0$. The proof of $b_{n,0}$ and $c_{n,0}$ is the same as $a_{n,0}$.

2) Assume that for $0 \leq \forall m \leq M$, the equations (6.23),(6.24),(6.25) hold for all n. We put $m = M+1$. Then, the equations satisfied by $(a_{n,M+1})$, $(b_{n,M+1})$, $(c_{n,M+1})$ are;

$$\begin{aligned} &p(n, M+1)a_{n,M+1} \\ &- \sum_{l=1}^{\infty} l a_{n-2l,M+1} + \sum_{l=1}^{\infty} l a_{n,M+1-2l} - \sum_{l=1}^{\infty} l a_{n-2l,M+1-2l} \\ &+ \sum_{l=1}^{\infty}(2l-1)b_{n-2l+1,M+1} + \sum_{l=1}^{\infty}(2l-1)c_{n-2l+1,M-2l+2} = 0 \end{aligned} \tag{6.30}$$

$$\begin{aligned} &q(n, M+1)b_{n,M+1} \\ &- \sum_{l=1}^{\infty} l b_{n-2l,M+1} - \sum_{l=1}^{\infty} l b_{n,M+1-2l} + \sum_{l=1}^{\infty} l b_{n-2l,M+1-2l} \\ &+ \sum_{l=1}^{\infty}(2l-1)a_{n-2l+1,M+1} + \sum_{l=1}^{\infty}(2l-1)c_{n,M-2l+2} = 0 \end{aligned} \tag{6.31}$$

$$\begin{aligned} &r(n, M+1)c_{n,M+1} \\ &+ \sum_{l=1}^{\infty} l c_{n-2l,M+1} - \sum_{l=1}^{\infty} l c_{n,M+1-2l} - \sum_{l=1}^{\infty} l c_{n-2l,M+1-2l} \\ &+ \sum_{l=1}^{\infty}(2l-1)a_{n-2l+1,M-2l+2} + \sum_{l=1}^{\infty}(2l-1)b_{n,M-2l+2} = 0. \end{aligned} \tag{6.32}$$

Firstly, we put $n = 0$. Then (6.30) becomes

$$a_{0,M+1} = -\frac{1}{p(0, M+1)} \sum_{l=1}^{\infty} l a_{0,M+1-2l}. \tag{6.33}$$

For each l, we add $p(0, M+1)$ in front of each element of $\mathbf{P}^p_{0,M+1-2l}$ and denote this family by $\tilde{\mathbf{P}}^p_{0,M+1-2l}$. Then, since

$$\tilde{\mathbf{P}}^p_{0,M+1-2l} \cap \tilde{\mathbf{P}}^p_{0,M+1-2l'} = \emptyset \quad (l \neq l')$$

and

$$\mathbf{P}^p_{0,M+1} = \bigcup_l \tilde{\mathbf{P}}^p_{0,M+1-2l},$$

the set appearing in the expansion of $a_{0,M+1}$ is just $\mathbf{P}^p_{0,M+1}$. And since $\delta^a_{0,M+1-2l} = \delta^a_{0,M+1}$, $\delta^a_{0,M+1}$ appears in the numerators of $a_{0,M+1}$. Furthermore, the rule of products in the numerators also holds because of the assumption and (6.33). Therefore, for $m = M+1, n = 0$, equation (6.23) holds. The proof of (6.24), (6.25) for $m = M+1, n = 0$ is the same.
Next, suppose that for $0 \leq \forall n \leq N$ and $0 \leq \forall m \leq M+1$, the equations (6.23), (6.24), (6.25) hold.
For $n = N+1$, we have

$$\begin{aligned}
&a_{N+1,M+1} \\
&= \frac{1}{p(N+1, M+1)} \sum_{l=1}^{\infty} l a_{N+1-2l,M+1} \\
&- \frac{1}{p(N+1, M+1)} \sum_{l=1}^{\infty} l a_{N+1,M+1-2l} \\
&+ \frac{1}{p(N+1, M+1)} \sum_{l=1}^{\infty} l a_{N+1-2l,M+1-2l} \\
&- \frac{1}{p(N+1, M+1)} \sum_{l=1}^{\infty} (2l-1) b_{N+2-2l,M+1} \\
&- \frac{1}{p(N+1, M+1)} \sum_{l=1}^{\infty} (2l-1) c_{N+2-2l,M+2-2l}.
\end{aligned}$$

For each l, we add $p(N+1, M+1)$ in front of each element of $\mathbf{P}^p_{N+1-2l,M+1}$ (resp. $\mathbf{P}^p_{N+1,M+1-2l}$, $\mathbf{P}^p_{N+1-2l,M+1-2l}$, $\mathbf{P}^q_{N+2-2l,M+1}$, $\mathbf{P}^r_{N+2-2l,M+1}$) and express this by $\tilde{\mathbf{P}}^l_1$ (resp. $\tilde{\mathbf{P}}^l_2$, $\tilde{\mathbf{P}}^l_3$, $\tilde{\mathbf{P}}^l_4$, $\tilde{\mathbf{P}}^l_5$). Then , since

$$\tilde{\mathbf{P}}^l_i \cap \tilde{\mathbf{P}}^{l'}_i = \emptyset \quad (l \neq l') \quad (i = 1, \cdots 5),$$

$$\tilde{\mathbf{P}}^l_i \cap \tilde{\mathbf{P}}^l_j = \emptyset \quad (i \neq j),$$

$$\mathbf{P}^p_{N+1,M+1} = \bigcup_{i=1}^{5} \bigcup_{l} \mathbf{P}^l_i.$$

the set appearing in the expansion of $a_{N+1,M+1}$ is just $\mathbf{P}^p_{N+1,M+1}$. The assertion about δ^a is the same as above. From (6.34) and the assumption, the rule of numerators also holds. Therefore, for $m = M + 1$ and $n = N + 1$, (6.23) holds. The proof of $b_{N+1,M+1}$, $c_{N+1,M+1}$ is the same. Thus we have completed the proof of this theorem. □

7 The expansion of the matrix coefficients in terms of the power series around $y_1 = y_2 = 0$

In the previous two sections, we obtained power series solutions for differential equations of class one case and three dimensional case. Our purpose in this section is to express the matrix coefficients by their linear combinations. In other words, we want to determine the coefficients of $\psi_{\alpha,\beta}$ in the expressions of matrix coefficients. Here, (α, β) is the characteristic root and $\psi_{\alpha,\beta}$ is the power series solution corresponding to (α, β).

7.1 Class one case

By solving the equations (5.1) and (5.2), we obtained six power series solutions corresponding to the six characteristic roots. For a characteristic root (α, β), we express the power seires solution corresponding to (α, β) by $\psi_{\alpha,\beta}$. We assume that the constant term of $\psi_{\alpha,\beta}$ is 1. As β, we have

$$\beta_1 = \frac{1}{3}(\nu_1 + \nu_2) + 1, \beta_2 = -\frac{1}{3}(2\nu_1 - \nu_2) + 1, \beta_3 = -\frac{1}{3}(2\nu_2 - \nu_1) + 1.$$

And for each β_i, we have two power series solutions.
Therefore, we can write matrix coefficient F by

$$F(y_1, y_2) = \sum_{i=1}^{3} c_i a_i(y_1, y_2) y_2^{\beta_i}.$$

Here, c_i $(i = 1, 2, 3)$ are some constants and $a_i(y_1, y_2)$ $(i = 1, 2, 3)$ are some analytic functions . By substituting $a_i(y_1, y_2)y_2^{\beta_i}$ into the equation (5.1), we have

$$\begin{aligned}
& y_2^{\beta_i}\Big\{2(\partial_1^2 a_i(y_1, y_2) - \partial_1\partial_2 a_i(y_1, y_2) - \beta_i\partial_1 a_i(y_1, y_2) \\
& + \partial_2^2 a_i(y_1, y_2) + 2\beta_i\partial_2 a_i(y_1, y_2) + \beta_i^2 a_i(y_1, y_2)) \\
& + \left(-\frac{y_2^2+1}{y_2^2-1} + \frac{y_1^2y_2^2+1}{y_1^2y_2^2-1} + 2\frac{y_1^2+1}{y_1^2-1}\right)\partial_1 a_i(y_1, y_2)
\end{aligned}$$

$$
\begin{aligned}
&+\left(2\frac{y_2^2+1}{y_2^2-1}+\frac{y_1^2y_2^2+1}{y_1^2y_2^2-1}-\frac{y_1^2+1}{y_1^2-1}\right)(\partial_2 a_i(y_1,y_2)+\beta_i a_i(y_1,y_2))\\
&+\left(-\frac{2}{3}(\nu_1^2-\nu_1\nu_2+\nu_2^2)+2\right)a_i(y_1,y_2)\Bigg\}=0.
\end{aligned}
$$

Dividing both sides by $y_2^{\beta_i}$ and taking the limit $y_2 \to 0$, then we obtain

$$
\begin{aligned}
&2\partial_1^2 a_i(y_1,0)+\left(2\frac{y_1^2+1}{y_1^2-1}-2\beta_i\right)\partial_1 a_i(y_1,0)\\
&+\left(2\beta_i^2-\frac{y_1^2+1}{y_1^2-1}\beta_i-3\beta_i-\frac{2}{3}(\nu_1^2-\nu_1\nu_2+\nu_2^2)+2\right)a_i(y_1,0)=0.
\end{aligned}
\tag{7.1}
$$

We put $y_1^2 = u$, $f_i(u) = a_i(y_1, 0)$. Then the equation (7.1) becomes

$$
\begin{aligned}
&8u^2\frac{d^2 f_i}{du^2}+\left(4\frac{u+1}{u-1}-4\beta_i+8\right)u\frac{df_i}{du}\\
&+\left(2\beta_i^2-\frac{u+1}{u-1}\beta_i-3\beta_i-\frac{2}{3}(\nu_1^2-\nu_1\nu_2+\nu_2^2)+2\right)f_i=0.
\end{aligned}
\tag{7.2}
$$

Next, we put $f_i(u) = u^x g_i(u)\,(x \in \mathbf{C})$ and substitute this into (7.2). Then we have

$$
\begin{aligned}
&8u^2\frac{d^2 g_i}{du^2}+\left(4\frac{u+1}{u-1}-4\beta_i+8+16x\right)u\frac{dg_i}{du}\\
&+\Bigg(8x^2+(4-4\beta_i)x+2\beta_i^2-4\beta_i-\frac{2}{3}(\nu_1^2-\nu_1\nu_2+\nu_2^2)\\
&+2+\frac{8x-2\beta_i}{u-1}\Bigg)g_i=0.
\end{aligned}
\tag{7.3}
$$

Now, we choose x_i satisfying

$$
8x_i^2+(4-4\beta_i)x_i+2\beta_i^2-4\beta_i-\frac{2}{3}(\nu_1^2-\nu_1\nu_2+\nu_2^2)+2=0
$$

and substitute $x = x_i$ into (7.3). Then we have

$$
8u^2\frac{d^2 g_i}{du^2}+\left(4\frac{u+1}{u-1}-4\beta_i+8+16x_i\right)u\frac{dg_i}{du}+\frac{8x_i-2\beta_i}{u-1}g_i=0. \tag{7.4}
$$

Finally, we put $u = \frac{1}{\zeta}$ and substitute this into (7.4). Then we have

$$
\begin{aligned}
&\zeta(\zeta-1)\frac{d^2 g_i}{d\zeta^2}+\left(-\frac{1}{2}(\beta_i-4x_i+1)+\frac{1}{2}(\beta_i-4x_i+3)\zeta\right)\frac{dg_i}{d\zeta}\\
&+(-x_i+\frac{\beta_i}{4})g_i=0.
\end{aligned}
\tag{7.5}
$$

(7.5) is a Gaussian hypergeometric differential equation, and if we define p_i, q_i as the complex numbers satisfying

$$1 + p_i + q_i = \frac{1}{2}(\beta_i - 4x_i + 3),$$

$$p_i q_i = -x_i + \frac{\beta_i}{4}$$

and define r_i by

$$r_i = \frac{1}{2}(\beta_i - 4x_i + 1),$$

then the solution is

$$P\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & p_i & ;\zeta \\ 1-r_i & r_i - p_i - q_i & q_i & \end{array}\right\} = P\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & p_i & ;1-\zeta \\ r_i - p_i - q_i & 1-r_i & q_i & \end{array}\right\}.$$

The regular solution is,

$$\begin{aligned} g_i(y_1) &= {}_2F_1\left(p_i, q_i; 1 - r_i + p_i + q_i; 1 - \zeta\right) \\ &= {}_2F_1\left(p_i, q_i; 1 - r_i + p_i + q_i; 1 - \frac{1}{y_1^2}\right). \end{aligned}$$

(See [9]). Since ${}_2F_1$ satisfies a formula ([5])

$$\begin{aligned} {}_2F_1(a,b;c;z) =& (1-z)^{-a}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}{}_2F_1\left(a, c-b; 1+a-b; \frac{1}{1-z}\right) \\ &+ (1-z)^{-b}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}{}_2F_1\left(b, c-a; 1+b-a; \frac{1}{1-z}\right), \end{aligned} \tag{7.6}$$

we have

$$\begin{aligned} g_i(y_1) =& y_1^{2p_i}\frac{\Gamma(1-r_i+p_i+q_i)\Gamma(q_i-p_i)}{\Gamma(1-r_i+q_i)\Gamma(q_i)}{}_2F_1(p_i, 1-r_i+p_i; 1+p_i-q_i; y_1^2) \\ &+ y_1^{2q_i}\frac{\Gamma(1-r_i+p_i+q_i)\Gamma(p_i-q_i)}{\Gamma(1-r_i+p_i)\Gamma(p_i)}{}_2F_1(q_i, 1-r_i+q_i; 1+q_i-p_i; y_1^2). \end{aligned}$$

Therefore, we have

$$\begin{aligned} & a_i(y_1, 0) \\ &= u^{x_i} g_i(y_1) \\ &= y_1^{2x_i} g_i(y_1) \\ &= y_1^{2(p_i+x_i)}\frac{\Gamma(1-r_i+p_i+q_i)\Gamma(q_i-p_i)}{\Gamma(1-r_i+q_i)\Gamma(q_i)}{}_2F_1(p_i, 1-r_i+p_i; 1+p_i-q_i; y_1^2) \\ &+ y_1^{2(q_i+x_i)}\frac{\Gamma(1-r_i+p_i+q_i)\Gamma(p_i-q_i)}{\Gamma(1-r_i+p_i)\Gamma(p_i)}{}_2F_1(q_i, 1-r_i+q_i; 1+q_i-p_i; y_1^2). \end{aligned} \tag{7.7}$$

Next, for $i = 1, 2, 3$, we compute (x_i, p_i, q_i, r_i). (The final form doesn't depend on the choice of x_i).

A) In case of $\beta_i = \beta_1 = \frac{1}{3}(\nu_1+\nu_2)+1$, $x_1 = \frac{1}{3}\nu_1 - \frac{1}{6}\nu_2$, $p_1 = -\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2}$, $q_1 = \frac{1}{2}$, $r_1 = -\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1$.

B) In case of $\beta_i = \beta_2 = -\frac{1}{3}(2\nu_1 - \nu_2) + 1$, $x_2 = -\frac{1}{6}\nu_1 + \frac{1}{3}\nu_2$, $p_2 = -\frac{1}{2}\nu_2 + \frac{1}{2}$, $q_2 = \frac{1}{2}$, $r_2 = -\frac{1}{2}\nu_2 + 1$.

C) In case of $\beta_i = \beta_3 = -\frac{1}{3}(2\nu_2 - \nu_1) + 1$, $x_3 = \frac{1}{3}\nu_1 - \frac{1}{6}\nu_2$, $p_3 = -\frac{1}{2}\nu_1 + \frac{1}{2}$, $q_3 = \frac{1}{2}$, $r_3 = -\frac{1}{2}\nu_1 + 1$.

By substituting these results into equation (7.7), we have

$$\begin{aligned}
&a_1(y_1, 0)\\
&= y_1^{\frac{1}{3}(2\nu_2-\nu_1)+1} \frac{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})}\\
&\cdot {}_2F_1(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2}, \frac{1}{2}; -\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1; y_1^2)\\
&+ y_1^{\frac{1}{3}(2\nu_1-\nu_2)+1} \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})}\\
&\cdot {}_2F_1(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2}, \frac{1}{2}; \frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + 1; y_1^2),
\end{aligned}$$

$$\begin{aligned}
&a_2(y_1, 0)\\
&= y_1^{-\frac{1}{3}(\nu_1+\nu_2)+1} \frac{\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})} {}_2F_1(-\frac{1}{2}\nu_2 + \frac{1}{2}, \frac{1}{2}; -\frac{1}{2}\nu_2 + 1; y_1^2)\\
&+ y_1^{\frac{1}{3}(2\nu_2-\nu_1)+1} \frac{\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})} {}_2F_1(\frac{1}{2}\nu_2 + \frac{1}{2}, \frac{1}{2}; \frac{1}{2}\nu_2 + 1; y_1^2),
\end{aligned}$$

$$\begin{aligned}
&a_3(y_1, 0)\\
&= y_1^{-\frac{1}{3}(\nu_1+\nu_2)+1} \frac{\Gamma(\frac{1}{2}\nu_1)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})} {}_2F_1(-\frac{1}{2}\nu_1 + \frac{1}{2}, \frac{1}{2}; -\frac{1}{2}\nu_1 + 1; y_1^2)\\
&+ y_1^{\frac{1}{3}(2\nu_1-\nu_2)+1} \frac{\Gamma(-\frac{1}{2}\nu_1)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})} {}_2F_1(\frac{1}{2}\nu_1 + \frac{1}{2}, \frac{1}{2}; \frac{1}{2}\nu_1 + 1; y_1^2).
\end{aligned}$$

Therefore, by compairing the leading terms, we have

$$\begin{aligned}&\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,\frac{1}{3}(\nu_1+\nu_2)+1}(y_1,y_2)\\&=y_1^{\frac{1}{3}(2\nu_2-\nu_1)+1}y_2^{\frac{1}{3}(\nu_1+\nu_2)+1}{}_2F_1(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{1}{2},\frac{1}{2};-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+1;y_1^2)\\&+(\text{higher order terms}),\end{aligned}$$

$$\begin{aligned}&\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,\frac{1}{3}(\nu_1+\nu_2)+1}(y_1,y_2)\\&=y_1^{\frac{1}{3}(2\nu_1-\nu_2)+1}y_2^{\frac{1}{3}(\nu_1+\nu_2)+1}{}_2F_1(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+\frac{1}{2},\frac{1}{2};\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+1;y_1^2)\\&+(\text{higher order terms}),\end{aligned}$$

$$\begin{aligned}&\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1}(y_1,y_2)\\&=y_1^{-\frac{1}{3}(\nu_1+\nu_2)+1}y_2^{-\frac{1}{3}(2\nu_1-\nu_2)+1}{}_2F_1(-\frac{1}{2}\nu_2+\frac{1}{2},\frac{1}{2};-\frac{1}{2}\nu_2+1;y_1^2)\\&+(\text{higher order terms}),\end{aligned}$$

$$\begin{aligned}&\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1}(y_1,y_2)\\&=y_1^{\frac{1}{3}(2\nu_2-\nu_1)+1}y_2^{-\frac{1}{3}(2\nu_1-\nu_2)+1}{}_2F_1(\frac{1}{2}\nu_2+\frac{1}{2},\frac{1}{2};\frac{1}{2}\nu_2+1;y_1^2)\\&+(\text{higher order terms}),\end{aligned}$$

$$\begin{aligned}&\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1}(y_1,y_2)\\&=y_1^{-\frac{1}{3}(\nu_1+\nu_2)+1}y_2^{-\frac{1}{3}(2\nu_2-\nu_1)+1}{}_2F_1(-\frac{1}{2}\nu_1+\frac{1}{2},\frac{1}{2};-\frac{1}{2}\nu_1+1;y_1^2)\\&+(\text{higher order terms}),\end{aligned}$$

$$\begin{aligned}&\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1}(y_1,y_2)\\&=y_1^{\frac{1}{3}(2\nu_1-\nu_2)+1}y_2^{-\frac{1}{3}(2\nu_2-\nu_1)+1}{}_2F_1(\frac{1}{2}\nu_1+\frac{1}{2},\frac{1}{2};\frac{1}{2}\nu_1+1;y_1^2)\\&+(\text{higher order terms}),\end{aligned}$$

and

$$
\begin{aligned}
&F(y_1, y_2)\\
&= c_1 \left\{ \frac{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, \frac{1}{3}(\nu_1+\nu_2)+1} \right.\\
&\left. + \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{\frac{1}{3}(2\nu_1-\nu_2)+1, \frac{1}{3}(\nu_1+\nu_2)+1} \right\}\\
&+ c_2 \left\{ \frac{\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{-\frac{1}{3}(\nu_1+\nu_2)+1, -\frac{1}{3}(2\nu_1-\nu_2)+1} \right.\\
&\left. + \frac{\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, -\frac{1}{3}(2\nu_1-\nu_2)+1} \right\}\\
&+ c_3 \left\{ \frac{\Gamma(\frac{1}{2}\nu_1)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})} \psi_{-\frac{1}{3}(\nu_1+\nu_2)+1, -\frac{1}{3}(2\nu_2-\nu_1)+1} \right.\\
&\left. + \frac{\Gamma(-\frac{1}{2}\nu_1)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})} \psi_{\frac{1}{3}(2\nu_1-\nu_2)+1, -\frac{1}{3}(2\nu_2-\nu_1)+1} \right\}.
\end{aligned}
\tag{7.8}
$$

The next work is to determine c_1, c_2, c_3. To do this, we apply the same method to y_2-part. That is, for $\alpha_1 = -\frac{1}{3}(\nu_1+\nu_2)+1, \alpha_2 = \frac{1}{3}(2\nu_1-\nu_2)+1, \alpha_3 = \frac{1}{3}(2\nu_2-\nu_1)+1$, we can write

$$
F(y_1, y_2) = \sum_{i=1}^{3} d_i b_i(y_1, y_2) y_1^{\alpha_i}
$$

and by investigating the differential equations satisfied by $b_i(0, y_2)$ $(i = 1, 2, 3)$ and comparing the leading terms with respect to y_2, we have

$$
\begin{aligned}
&F(y_1, y_2)\\
&= d_1 \left\{ \frac{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{-\frac{1}{3}(\nu_1+\nu_2)+1, -\frac{1}{3}(2\nu_1-\nu_2)+1} \right.\\
&\left. + \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{-\frac{1}{3}(\nu_1+\nu_2)+1, -\frac{1}{3}(2\nu_2-\nu_1)+1} \right\}\\
&+ d_2 \left\{ \frac{\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{\frac{1}{3}(2\nu_1-\nu_2)+1, -\frac{1}{3}(2\nu_2-\nu_1)+1} \right.\\
&\left. + \frac{\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{\frac{1}{3}(2\nu_1-\nu_2)+1, \frac{1}{3}(\nu_1+\nu_2)+1} \right\}\\
&+ d_3 \left\{ \frac{\Gamma(\frac{1}{2}\nu_1)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})} \psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, -\frac{1}{3}(2\nu_1-\nu_2)+1} \right.\\
&\left. + \frac{\Gamma(-\frac{1}{2}\nu_1)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})} \psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, \frac{1}{3}(\nu_1+\nu_2)+1} \right\}.
\end{aligned}
\tag{7.9}
$$

By compairing the coefficients of $\psi_{\alpha,\beta}$ in the equation (7.8) and (7.9), we can determine c_i, d_i $(i = 1, 2, 3)$ up to constant multiples. In particular,

$$c_1 = \frac{\Gamma(-\frac{1}{2}\nu_1)\Gamma(-\frac{1}{2}\nu_2)}{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})},$$
$$c_2 = \frac{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_1)}{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})},$$
$$c_3 = \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_2)}{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})}.$$

Thus, we completely determined the six coefficients. We have

$$\begin{aligned}
&F(y_1, y_2) \\
&= \frac{\Gamma(-\frac{1}{2}\nu_1)\Gamma(-\frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})}\psi_{\frac{1}{3}(2\nu_2 - \nu_1)+1, \frac{1}{3}(\nu_1+\nu_2)+1} \\
&+ \frac{\Gamma(-\frac{1}{2}\nu_1)\Gamma(-\frac{1}{2}\nu_2)\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})}\psi_{\frac{1}{3}(2\nu_1 - \nu_2)+1, \frac{1}{3}(\nu_1+\nu_2)+1} \\
&+ \frac{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_1)\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1, -\frac{1}{3}(2\nu_1 - \nu_2)+1} \\
&+ \frac{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_1)\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})}\psi_{\frac{1}{3}(2\nu_2 - \nu_1)+1, -\frac{1}{3}(2\nu_1 - \nu_2)+1} \\
&+ \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_1)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1, -\frac{1}{3}(2\nu_2 - \nu_1)+1} \\
&+ \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_2)\Gamma(-\frac{1}{2}\nu_1)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})}\psi_{\frac{1}{3}(2\nu_1 - \nu_2)+1, -\frac{1}{3}(2\nu_2 - \nu_1)+1}.
\end{aligned}$$

7.2 Three dimensional case

Firstly, we take $\sigma = \sigma_1$ for the character of M. Let $\Psi = {}^t(F, G, H)$ be the matrix coefficient with K-type of three dimensional tautological representation, and $\psi_{\alpha,\beta} = {}^t(f_{\alpha,\beta}, g_{\alpha,\beta}, h_{\alpha,\beta})$ be the power series solution around $y_1 = y_2 = 0$ corresponding to the characteristic root (α, β) whose constant term is ${}^t(1, 0, 0)$ or ${}^t(0, 1, 0)$ or ${}^t(0, 0, 1)$.
As α, we have

$$\alpha_1 = -\frac{1}{3}(\nu_1 + \nu_2) + 1, \alpha_2 = \frac{1}{3}(2\nu_1 - \nu_2) + 1, \alpha_3 = \frac{1}{3}(2\nu_2 - \nu_1) + 1.$$

Therefore, we can write

$$F(y_1, y_2) = \sum_{i=1}^{3} d_i b_i^{(1)}(y_1, y_2) y_1^{\alpha_i}$$

$$G(y_1, y_2) = \sum_{i=1}^{3} d_i b_i^{(2)}(y_1, y_2) y_1^{\alpha_i}$$
$$H(y_1, y_2) = \sum_{i=1}^{3} d_i b_i^{(3)}(y_1, y_2) y_1^{\alpha_i}.$$

Here, $d_i (i = 1, 2, 3)$ are some constants and $b_i^{(j)}(y_1, y_2)$ are analytic functions for $0 < y_2 < 1$ and $0 < y_1 << 1$.
By inserting $F = b_2^{(1)}(y_1, y_2) y_1^{\alpha_2}, G = b_2^{(2)}(y_1, y_2) y_1^{\alpha_2}, H = b_2^{(3)}(y_1, y_2) y_1^{\alpha_2}$ into the equation (6.1), we have

$$\begin{aligned}
& y_1^{\alpha_2} \Big\{ 2(\partial_1^2 b_2^{(1)}(y_1, y_2) + 2\alpha_2 \partial_1 b_2^{(1)}(y_1, y_2) + \alpha_2^2 b_2^{(1)}(y_1, y_2) \\
& - \partial_1 \partial_2 b_2^{(1)}(y_1, y_2) - \alpha_2 \partial_2 b_2^{(1)}(y_1, y_2) + \partial_2^2 b_2^{(1)}(y_1, y_2)) \\
& + \left(2\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1 y_2)}{sh(y_1 y_2)} - \frac{ch(y_2)}{sh(y_2)} \right) (\alpha_2 b_2^{(1)}(y_1, y_2) + \partial_1 b_2^{(1)}(y_1, y_2)) \\
& + \left(-\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1 y_2)}{sh(y_1 y_2)} + 2\frac{ch(y_2)}{sh(y_2)} \right) \partial_2 b_2^{(1)}(y_1, y_2) \\
& - \left(\frac{1}{sh(y_1)^2} + \frac{1}{sh(y_1 y_2)^2} \right) b_2^{(1)}(y_1, y_2) \\
& + \frac{ch(y_1)}{sh(y_1)^2} b_2^{(2)}(y_1, y_2) + \frac{ch(y_1 y_2)}{sh(y_1 y_2)^2} b_2^{(3)}(y_1, y_2) \\
& - \lambda b_2^{(1)}(y_1, y_2) \Big\} = 0.
\end{aligned}$$

By dividing both sides by $y_1^{\alpha_2}$ and taking the limit $y_1 \to 0$, we have

$$\begin{aligned}
& 2\partial_2^2 b_2^{(1)}(0, y_2) + \left(2\frac{y_2^2 + 1}{y_2^2 - 1} - 2\alpha_2 \right) \partial_2 b_2^{(1)}(0, y_2) \\
& + \left(2\alpha_2^2 - \frac{y_2^2 + 1}{y_2^2 - 1}\alpha_2 - 3\alpha_2 - \lambda \right) b_2^{(1)}(0, y_2) = 0.
\end{aligned}$$

(Note that because of Lemma 6.6, since $a_{0,0} \neq 0$ if $\alpha = \alpha_2$, $b_2^{(1)}(0, y_2)$ is not identically 0. Hereafter the same statement holds for all the functions we compute.) This equation is the same type of equation as (7.1). By solving this, we have

$$b_2^{(1)}(0, y_2) = y_2^{-\frac{1}{3}(2\nu_2 - \nu_1) + 1} \frac{\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})} {}_2F_1(-\frac{1}{2}\nu_2 + \frac{1}{2}, \frac{1}{2}; -\frac{1}{2}\nu_2 + 1; y_2^2)$$

$$+ y_2^{\frac{1}{3}(\nu_1+\nu_1)+1}\frac{\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_2+\frac{1}{2})}{}_2F_1(\frac{1}{2}\nu_2+\frac{1}{2},\frac{1}{2};\frac{1}{2}\nu_2+1;y_2^2)$$

and by comparing the leading terms, we have

$$\begin{aligned}
&f_{\frac{1}{3}(2\nu_1-\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1}(y_1,y_2) \\
&= y_1^{\frac{1}{3}(2\nu_1-\nu_2)+1}y_2^{-\frac{1}{3}(2\nu_2-\nu_1)+1}{}_2F_1(-\frac{1}{2}\nu_2+\frac{1}{2},\frac{1}{2};-\frac{1}{2}\nu_2+1;y_2^2) \\
&+ (\text{higher order terms}),
\end{aligned}$$

$$\begin{aligned}
&f_{\frac{1}{3}(2\nu_1-\nu_2)+1,\frac{1}{3}(\nu_1+\nu_2)+1}(y_1,y_2) \\
&= y_1^{\frac{1}{3}(2\nu_1-\nu_2)+1}y_2^{\frac{1}{3}(\nu_1+\nu_2)+1}{}_2F_1(\frac{1}{2}\nu_2+\frac{1}{2},\frac{1}{2};\frac{1}{2}\nu_2+1;y_2^2) \\
&+ (\text{higher order terms})
\end{aligned}$$

and

$$\begin{aligned}
\Psi(y_1,y_2) = d_2\Bigg\{&\frac{\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_2+\frac{1}{2})}\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1} \\
&+\frac{\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_2+\frac{1}{2})}\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,\frac{1}{3}(\nu_1+\nu_2)+1}\Bigg\} \\
&+ (\text{linear combination of the other four solutions}).
\end{aligned} \tag{7.10}$$

Next, as β, we have

$$\beta_1 = \frac{1}{3}(\nu_1+\nu_2)+1, \beta_2 = -\frac{1}{3}(2\nu_1-\nu_2)+1, \beta_3 = -\frac{1}{3}(2\nu_2-\nu_1)+1.$$

Therefore, we can write

$$\begin{aligned}
F(y_1,y_2) &= \sum_{i=1}^{3} c_i a_i^{(1)}(y_1,y_2)y_2^{\beta_i}, \\
G(y_1,y_2) &= \sum_{i=1}^{3} c_i a_i^{(2)}(y_1,y_2)y_2^{\beta_i}, \\
H(y_1,y_2) &= \sum_{i=1}^{3} c_i a_i^{(3)}(y_1,y_2)y_2^{\beta_i}.
\end{aligned}$$

By inserting $F = a_2^{(1)}(y_1,y_2)y_2^{\beta_2}$, $G = a_2^{(2)}(y_1,y_2)y_2^{\beta_2}$, $H = a_2^{(3)}(y_1,y_2)y_2^{\beta_2}$ into the equation (6.3) and applying the same method, we have

$$\Psi(y_1,y_2) = c_2\Bigg\{\frac{\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_2+\frac{1}{2})}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1} + \frac{\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_2+\frac{1}{2})}\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1}\Bigg\} + (\text{linear combination of the other four solutions}). \tag{7.11}$$

Next, we take $i=1$ or 3.
By inserting $F = a_i^{(1)}(y_1,y_2)y_2^{\beta_i}, G = a_i^{(2)}(y_1,y_2)y_2^{\beta_i}, H = a_i^{(3)}(y_1,y_2)y_2^{\beta_i}$ into the equation (6.1), we have

$$\begin{aligned}
&y_2^{\beta_i}\Bigg\{2(\partial_1^2 a_i^{(1)}(y_1,y_2) - \partial_1\partial_2 a_i^{(1)}(y_1,y_2) - \beta_i\partial_1 a_i^{(1)}(y_1,y_2)\\
&+\partial_2^2 a_i^{(1)}(y_1,y_2) + 2\beta_i a_i^{(1)}(y_1,y_2) + \beta_i^2 a_i^{(1)}(y_1,y_2))\\
&+\left(2\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} - \frac{ch(y_2)}{sh(y_2)}\right)\partial_1 a_i^{(1)}(y_1,y_2)\\
&+\left(-\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} + 2\frac{ch(y_2)}{sh(y_2)}\right)(\beta_i a_i^{(1)}(y_1,y_2) + \partial_2 a_i^{(1)}(y_1,y_2))\\
&-\left(\frac{1}{sh(y_1)^2} + \frac{1}{sh(y_1y_2)^2}\right)a_i^{(1)}(y_1,y_2)\\
&+\frac{ch(y_1)}{sh(y_1)^2}a_i^{(2)}(y_1,y_2) + \frac{ch(y_1y_2)}{sh(y_1y_2)^2}a_i^{(3)}(y_1,y_2)\\
&-\lambda a_i^{(1)}(y_1,y_2)\Bigg\} = 0.
\end{aligned}$$

By dividing both sides by $y_2^{\beta_i}$ and taking the limit $y_2 \to 0$, we have

$$\begin{aligned}
&2\partial_1^2 a_i^{(1)}(y_1,0) + \left(2\frac{y_1^2+1}{y_1^2-1} - 2\beta_i\right)\partial_1 a_i^{(1)}(y_1,0)\\
&+\left(2\beta_i^2 - \frac{y_1^2+1}{y_1^2-1}\beta_i - 3\beta_i - \frac{4y_1^2}{(y_1^2-1)^2} - \lambda\right)a_i^{(1)}(y_1,0)\\
&+\frac{2(y_1^3+y_1)}{(y_1^2-1)^2}a_i^{(2)}(y_1,0) = 0.
\end{aligned} \tag{7.12}$$

Next, by inserting $F = a_i^{(1)}(y_1,y_2)y_2^{\beta_i}, G = a_i^{(2)}(y_1,y_2)y_2^{\beta_i}, H = a_i^{(3)}(y_1,y_2)y_2^{\beta_i}$ into the equation (6.4), we have

$$\begin{aligned}
&y_2^{\beta_i}\Bigg\{-\partial_1 a_i^{(1)}(y_1,y_2) - \beta_i a_i^{(1)}(y_1,y_2) - \frac{ch(y_1)}{2sh(y_1)}a_i^{(1)}(y_1,y_2)\\
&-\frac{ch(y_1y_2)}{2sh(y_1y_2)}a_i^{(1)}(y_1,y_2) + \frac{1}{2sh(y_1)}a_i^{(2)}(y_1,y_2)\\
&+\frac{1}{2sh(y_1y_2)}a_i^{(3)}(y_1,y_2) - \lambda_1 a_i^{(1)}(y_1,y_2)\Bigg\} = 0.
\end{aligned}$$

By dividing both sides by $y_2^{\beta_i}$ and taking the limit $y_2 \to 0$, we have

$$\frac{y_1}{y_1^2-1} a_i^{(2)}(y_1,0) = \partial_1 a_i^{(1)}(y_1,0) + \left(\frac{y_1^2+1}{2(y_1^2-1)} + \lambda_1 - \frac{1}{2}\right) a_i^{(1)}(y_1,0). \quad (7.13)$$

By combining equations (7.12) and (7.13) to eliminate $a_i^{(2)}(y_1,0)$, we have

$$\begin{aligned} &2\partial_1^2 a_i^{(1)}(y_1,0) + \left(4\frac{y_1^2+1}{y_1^2-1} - 2\beta_i\right)\partial_1 a_i^{(1)}(y_1,0) \\ &+ \left(\frac{y_1^2+1}{y_1^2-1}(2\lambda_1 - \beta_i - 1) + 2\beta_i^2 - 3\beta_i - \lambda + 1\right) a_i^{(1)}(y_1,0) = 0. \end{aligned}$$

We put $y_1^2 = u$, $f_i(u) := a_i^{(1)}(y_1,0)$ and define a differential operator $\tilde{\partial}_1$ by $\tilde{\partial}_1 := u\frac{d}{du}$. Then the equation becomes

$$\begin{aligned} &8{\tilde{\partial}_1}^2 f_i(u) + \left(8 - 4\beta_i + \frac{16}{u-1}\right)\tilde{\partial}_1 f_i(u) \\ &+ \left(2\beta_i^2 - 4\beta_i + 2\lambda_1 - \lambda + \frac{4\lambda_1 - 2\beta_i - 2}{u-1}\right) f_i(u) = 0. \end{aligned}$$

Next, we put $f_i(u) = u^x g_i(u)$ $(x \in \mathbf{C})$. Then the equation becomes

$$\begin{aligned} &8{\tilde{\partial}_1}^2 g_i(u) + \left(8 - 4\beta_i + \frac{16}{u-1} + 16x\right)\tilde{\partial}_1 g_i(u) \\ &+ \left(8x^2 + (8-4\beta_i)x + 2\beta_i^2 - 4\beta_i + 2\lambda_1 - \lambda + \frac{16x + 4\lambda_1 - 2\beta_i - 2}{u-1}\right) g_i(u) = 0. \end{aligned}$$

We take $x = x_i$ as the number satisfying

$$8x_i^2 + (8-4\beta_i)x_i + 2\beta_i^2 - 4\beta_i + 2\lambda_1 - \lambda = 0.$$

Then we have

$$\begin{aligned} &8u^2\frac{d^2 g_i}{du^2} + \left(16 - 4\beta_i + 16x_i + \frac{16}{u-1}\right) u\frac{dg_i}{du} \\ &+ \frac{16x_i + 4\lambda_1 - 2\beta_i - 2}{u-1} g_i(u) = 0. \end{aligned}$$

Finally, we put $u = \frac{1}{\zeta}$. Then the equation becomes

$$\begin{aligned} &\zeta(\zeta-1)\frac{d^2 g_i}{d\zeta^2} \\ &+ \left(\left(\frac{1}{2}\beta_i - 2x_i + 2\right)\zeta - \frac{1}{2}\beta_i + 2x_i\right)\frac{dg_i}{d\zeta} \\ &+ \left(-2x_i - \frac{1}{2}\lambda_1 + \frac{1}{4}\beta_i + \frac{1}{4}\right) g_i = 0. \end{aligned} \quad (7.14)$$

(7.14) is the Gaussian hypergeometric differential equation, and if we define p_i,q_i by complex numbers satisfying

$$1 + p_i + q_i = \frac{1}{2}\beta_i - 2x_i + 2$$

$$p_i q_i = -2x_i - \frac{1}{2}\lambda_1 + \frac{1}{4}\beta_i + \frac{1}{4}$$

and r_i by

$$r_i = \frac{1}{2}\beta_i - 2x_i,$$

then the general solution is

$$P\left\{\begin{array}{ccc} 0 & 1 & \infty \\ 0 & 0 & p_i \ ;\zeta \\ 1-r_i & r_i-p_i-q_i & q_i \end{array}\right\} = P\left\{\begin{array}{ccc} 0 & 1 & \infty \\ 0 & 0 & p_i \ ;1-\zeta \\ r_i-p_i-q_i & 1-r_i & q_i \end{array}\right\}.$$

The regular solution is,

$$\begin{aligned} g_i(y_1) &= {}_2F_1\left(p_i, q_i; 1-r_i+p_i+q_i; 1-\zeta\right) \\ &= {}_2F_1\left(p_i, q_i; 1-r_i+p_i+q_i; 1-\frac{1}{y_1^2}\right). \end{aligned}$$

Since ${}_2F_1$ satisfies a formula (7.6), we have

$$\begin{aligned} g_i(y_1) =& y_1^{2p_i}\frac{\Gamma(1-r_i+p_i+q_i)\Gamma(q_i-p_i)}{\Gamma(1-r_i+q_i)\Gamma(q_i)}{}_2F_1(p_i, 1-r_i+p_i; 1+p_i-q_i; y_1^2) \\ &+ y_1^{2q_i}\frac{\Gamma(1-r_i+p_i+q_i)\Gamma(p_i-q_i)}{\Gamma(1-r_i+p_i)\Gamma(p_i)}{}_2F_1(q_i, 1-r_i+q_i; 1+q_i-p_i; y_1^2). \end{aligned}$$

Therefore, we have

$$\begin{aligned} &a_i^{(1)}(y_1, 0) = u^{x_i}g_i(y_1) = y_1^{2x_i}g_i(y_1) \\ &= y_1^{2(p_i+x_i)}\frac{\Gamma(1-r_i+p_i+q_i)\Gamma(q_i-p_i)}{\Gamma(1-r_i+q_i)\Gamma(q_i)}{}_2F_1(p_i, 1-r_i+p_i; 1+p_i-q_i; y_1^2) \\ &+ y_1^{2(q_i+x_i)}\frac{\Gamma(1-r_i+p_i+q_i)\Gamma(p_i-q_i)}{\Gamma(1-r_i+p_i)\Gamma(p_i)}{}_2F_1(q_i, 1-r_i+q_i; 1+q_i-p_i; y_1^2) \end{aligned} \tag{7.15}$$

for $i = 1, 3$. The values (x_i, p_i, q_i, r_i) $(i = 1, 3)$ are as follows;
A) When $i = 1$, we have

$$\begin{aligned} &x_1 = \frac{1}{3}\nu_1 - \frac{1}{6}\nu_2, \quad p_1 = -\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1, \\ &q_1 = \frac{1}{2}, \quad r_1 = -\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2}. \end{aligned}$$

B) When $i=3$, we have

$$x_3 = \frac{1}{3}\nu_1 - \frac{1}{6}\nu_2, \quad p_3 = -\frac{1}{2}\nu_1 + 1,$$
$$q_3 = \frac{1}{2}, \quad r_3 = -\frac{1}{2}\nu_1 + \frac{1}{2}.$$

By inserting these results into (7.15), we have

$$\begin{aligned}
&a_1^{(1)}(y_1, 0) \\
&= y_1^{\frac{1}{3}(2\nu_2-\nu_1)+2} \frac{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 - \frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + 1)} \\
&\cdot {}_2F_1(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1, \frac{3}{2}; -\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{3}{2}; y_1^2) \\
&+ y_1^{\frac{1}{3}(2\nu_1-\nu_2)+1} \frac{2\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1)} \\
&\cdot {}_2F_1(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + 1, \frac{1}{2}; \frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2}; y_1^2), \\
&a_3^{(1)}(y_1, 0) \\
&= y_1^{-\frac{1}{3}(\nu_1+\nu_2)+2} \frac{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 + 1)} {}_2F_1(-\frac{1}{2}\nu_1 + 1, \frac{3}{2}; -\frac{1}{2}\nu_1 + \frac{3}{2}; y_1^2) \\
&+ y_1^{\frac{1}{3}(2\nu_1-\nu_2)+1} \frac{2\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + 1)} {}_2F_1(\frac{1}{2}\nu_1 + 1, \frac{1}{2}; \frac{1}{2}\nu_1 + \frac{1}{2}; y_1^2).
\end{aligned}$$

Here, we used $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$.
From the equation of $a_1^{(1)}(y_1, 0)$, we have

$$\begin{aligned}
&f_{\frac{1}{3}(2\nu_1-\nu_2)+1, \frac{1}{3}(\nu_1+\nu_2)+1}(y_1, y_2) \\
&= y_1^{\frac{1}{3}(2\nu_1-\nu_2)+1} y_2^{\frac{1}{3}(\nu_1+\nu_2)+1} {}_2F_1(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + 1, \frac{1}{2}; \frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2}; y_1^2) \\
&+ (\text{higher order terms}),
\end{aligned}$$

$$\begin{aligned}
&f_{\frac{1}{3}(2\nu_2-\nu_1)+1, \frac{1}{3}(\nu_1+\nu_2)+1}(y_1, y_2) \\
&= y_1^{\frac{1}{3}(2\nu_2-\nu_1)+2} y_2^{\frac{1}{3}(\nu_1+\nu_2)+1} {}_2F_1(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1, \frac{3}{2}; -\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{3}{2}; y_1^2) \\
&+ (\text{higher order terms}).
\end{aligned}$$

And since the coefficient of $y_1^{\frac{1}{3}(2\nu_2-\nu_1)+2} y_2^{\frac{1}{3}(\nu_1+\nu_2)+1}$ of $f_{\frac{1}{3}(2\nu_2-\nu_1)+1, \frac{1}{3}(\nu_1+\nu_2)+1}$ in $\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, \frac{1}{3}(\nu_1+\nu_2)+1}$ is $\frac{1}{\nu_1-\nu_2-1}$, if the coefficient of $\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1, \frac{1}{3}(\nu_1+\nu_2)+1}$

is $\frac{2\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+1)}$, the coefficient of $\Psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,\frac{1}{3}(\nu_1+\nu_2)+1}$ is

$$\frac{\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2-\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+1)}\times(\nu_1-\nu_2-1)=\frac{2\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+1)}.$$

Therefore, we have

$$\begin{aligned}\Psi =&c_1\left\{\frac{\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+1)}\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,\frac{1}{3}(\nu_1+\nu_2)+1}\right.\\ &\left.+\frac{\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+1)}\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,\frac{1}{3}(\nu_1+\nu_2)+1}\right\}\\ &+(\text{linear combination of the other four solutions}).\end{aligned}\tag{7.16}$$

Similarly, from the equation of $a_3^{(1)}(y_1,0)$, we have

$$\begin{aligned}\Psi =&c_3\left\{\frac{\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+1)}\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1}\right.\\ &\left.+\frac{\Gamma(\frac{1}{2}\nu_1+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1+1)}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1}\right\}\\ &+(\text{linear combination of the other four solutions}).\end{aligned}\tag{7.17}$$

Next, we insert $F=b_i^{(1)}(y_1,y_2)y_1^{\alpha_i}$, $G=b_i^{(2)}(y_1,y_2)y_1^{\alpha_i}$, $H=b_i^{(3)}(y_1,y_2)y_1^{\alpha_i}$ $(i=1,3)$ into the equation (6.3), (6.6). By applying the same method as we used above to $b_i^{(j)}(y_1,y_2)$ $(i=1,3,j=1,2,3)$ (eliminate $b_i^{(2)}(0,y_2)$ and construct the differential equation with respect to $b_i^{(3)}(0,y_2)$), we have

$$\begin{aligned}\Psi =&d_1\left\{\frac{\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+1)}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1}\right.\\ &\left.+\frac{\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+1)}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1}\right\}\\ &+(\text{linear combination of the other four solutions})\end{aligned}\tag{7.18}$$

and

$$\begin{aligned}\Psi =&d_3\left\{\frac{\Gamma(\frac{1}{2}\nu_1+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1+1)}\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1}\right.\\ &\left.+\frac{\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+1)}\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,\frac{1}{3}(\nu_1+\nu_2)+1}\right\}\\ &+(\text{linear combination of the other four solutions}).\end{aligned}\tag{7.19}$$

Now we have six equations with respect to the matrix coefficient Ψ (i.e. (7.16), (7.11), (7.17), (7.18), (7.10), (7.19)). By combining these equations, we obtain two different expressions of Ψ. That is;

$$\Psi =c_1\left\{\frac{\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+1)}\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,\frac{1}{3}(\nu_1+\nu_2)+1}\right.$$

$$
\begin{aligned}
&+\frac{\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+1)}\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,\frac{1}{3}(\nu_1+\nu_2)+1}\Bigg\}\\
&+c_2\Bigg\{\frac{\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_2+\frac{1}{2})}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1}\\
&+\frac{\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_2+\frac{1}{2})}\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1}\Bigg\}\\
&+c_3\Bigg\{\frac{\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+1)}\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1}\\
&+\frac{\Gamma(\frac{1}{2}\nu_1+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1+1)}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1}\Bigg\}\\
&=d_1\Bigg\{\frac{\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+1)}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1}\\
&+\frac{\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+1)}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1}\Bigg\}\\
&+d_2\Bigg\{\frac{\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_2+\frac{1}{2})}\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1}\\
&+\frac{\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_2+\frac{1}{2})}\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,\frac{1}{3}(\nu_1+\nu_2)+1}\Bigg\}\\
&+d_3\Bigg\{\frac{\Gamma(\frac{1}{2}\nu_1+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1+1)}\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1}\\
&+\frac{\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+1)}\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,\frac{1}{3}(\nu_1+\nu_2)+1}\Bigg\}.
\end{aligned}
$$

By compairing the coefficients of $\psi_{\alpha,\beta}$, we have

$$
\begin{aligned}
c_1 &= \frac{\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2})\Gamma(-\frac{1}{2}\nu_2)}{\Gamma(-\frac{1}{2}\nu_1+1)\Gamma(-\frac{1}{2}\nu_2+\frac{1}{2})}\\
c_2 &= \frac{\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+\frac{1}{2})\Gamma(\frac{1}{2}\nu_1+\frac{1}{2})}{\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+1)\Gamma(\frac{1}{2}\nu_1+1)}\\
c_3 &= \frac{\Gamma(\frac{1}{2}\nu_2)\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{1}{2})}{\Gamma(\frac{1}{2}\nu_2+\frac{1}{2})\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+1)}
\end{aligned}
$$

up to constant multiples. Thus we obtained the expression of Ψ in case of $\sigma=\sigma_1$. Note that since the transform $\nu_1\mapsto\nu_2,\nu_2\mapsto\nu_1$ does not change the eigenvalue of Casimir operator λ and change the eigenvalue of gradient operator λ_1 to λ_2, this transform gives the expression of Ψ in case of $\sigma=\sigma_2$. Similarly, the transform $\nu_1\mapsto-\nu_1,\nu_2\mapsto-\nu_1+\nu_2$ gives the expression in case of $\sigma=\sigma_3$. Trerefore, we obtained the following theorem.

Theorem 7.1. *Let the character of M be $\sigma = \sigma_1$. Let $\Psi = {}^t(F, G, H)$ be the matrix coefficient with K-type of three dimensional tautological representation, and $\psi_{\alpha,\beta} = {}^t(f_{\alpha,\beta}, g_{\alpha,\beta}, h_{\alpha,\beta})$ be the power series solution around $y_1 = y_2 = 0$ corresponding to the characteristic root (α, β) whose constant term is ${}^t(1, 0, 0)$ or ${}^t(0, 1, 0)$ or ${}^t(0, 0, 1)$. Then we have*

$$\begin{aligned}
&\Psi(y_1, y_2) \\
&= \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})\Gamma(-\frac{1}{2}\nu_2)\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + 1)\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1)}\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1, \frac{1}{3}(\nu_1+\nu_2)+1} \\
&+ \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})\Gamma(-\frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + 1)\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + 1)}\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, \frac{1}{3}(\nu_1+\nu_2)+1} \\
&+ \frac{\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 + 1)\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + 1)}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1, -\frac{1}{3}(2\nu_1-\nu_2)+1} \\
&+ \frac{\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})\Gamma(-\frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 + 1)\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + 1)}\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, -\frac{1}{3}(2\nu_1-\nu_2)+1} \\
&+ \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_2)\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + 1)\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1)}\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1, -\frac{1}{3}(2\nu_2-\nu_1)+1} \\
&+ \frac{\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_2)\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 + 1)\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1)}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1, -\frac{1}{3}(2\nu_2-\nu_1)+1}.
\end{aligned} \tag{7.20}$$

The transform $\nu_1 \mapsto \nu_2, \nu_2 \mapsto \nu_1$ in (7.20) gives the expression of Ψ in case of $\sigma = \sigma_2$ and the transform $\nu_1 \mapsto -\nu_1, \nu_2 \mapsto -\nu_1 + \nu_2$ in (7.20) gives the expression of Ψ in case of $\sigma = \sigma_3$.

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ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012