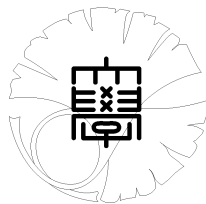# Hybridized discontinuous Galerkin
method with lifting operator

by

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Hybridized Discontinuous Galerkin
Method with Lifting Operator

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Abstract

In this paper, we propose a new hybridized discontinuous Galerkin method for the Poisson equation with homogeneous Dirichlet boundary condition. Our method has the advantage that the stability is better than the previous hybridized method. We derive L^2 and H^1 error estimates of optimal order. Some numerical results are presented to verify our analysis.

Keywords. discontinuous Galerkin method, hybridized method, error analysis

1   Introduction

The discontinuous Galerkin finite-element methods (DGFEMs) is one of the active research fields of numerical analysis in the last decade. They allow us to use discontinuous approximate functions across the element boundaries and have the robustness to variation of element geometry. That is, we can utilize many kind of polynomials as approximate functions on elements and many kind of polyhedral domains as elements simultaneously. Consequently, DGFEM fits adaptive computations so that mathematical analysis as well as actual applications has been developed for various problems. For more details, we refer to [2, 3, 4]. However, the size and band-widths of the resulting matrices can be much larger than those of the conventional FEM, which is a disadvantage from the viewpoint of computational cost. To surmount this obstacle, recently new class of DGFEM, which is called hybridized DGFEMs, is proposed and analyzed by B. Cockburn and his colleagues; for example, see [9]. Thus, we introduce new unknown function $\hat{U}_h$ on inter-element edges and characterize it as the weak solution of a target PDE. We then obtain the discrete system for $\hat{U}_h$ and the size of the system becomes smaller. On the other hand, it should be kept in mind that DGFEM has another origin. Some class of nonconforming and hybrid FEM's, which are called hybrid displacement method, use discontinuous functions as approximate field functions;

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see for example [5, 6]. In [10] and [11], F. Kikuchi and Y. Ando developed a variant of the hybrid displacement one, and applied it to plate problems. Their approach enables one to use conventional element matrices and vectors. It, however, suffered from numerical instability and was not fully successful. Recently, the author and his colleagues proposed a new DGFEM that is based on the hybrid displacement approach by stabilizing their old method and applied it to linear elasticity problems in [7]. A key point of our method is to introduce penalty terms in order to ensure the stability. We, then, carried out theoretical analysis by using the 2D Poisson equation as a model problem, and gave some concrete finite element models with numerical results and observations in [8]. However, an issue still remains. The stability is guaranteed only when the penalty parameters are taken from a certain interval, and we know only the existence of such an interval and do not know concrete information about it.

The purpose of this paper is to propose a new hybridized DGFEM that is stable for arbitrary penalty parameters. Our strategy is to introduce the lifting operator and define the penalty term in terms of the lifting operator. In order to state our idea as clearly as possible, we consider the Poisson equation with homogeneous Dirichlet condition:

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1}$$

where Ω is a convex polygonal domain and $f \in L^2(\Omega)$.

This paper is composed of six sections. In Section 2, we introduce the triangulation and finite element spaces, and then describe the lifting operator. Section 3 is devoted to the formulation of our proposed hybridized DGFEM, and mathematical analysis including error estimates is given in Section 4. In Section 5, we report some results of numerical computations and confirm our theoretical results. Finally, we conclude this paper in Section 6.

2 Preliminaries

2.1 Notation

Let $\Omega \subset \mathbb{R}^n$, for an integer $n \geq 2$, be a convex polygonal domain. We introduce a triangulation $\mathcal{T}_h = \{K\}$ of Ω in the sense [8], where $h = \max_{K \in \mathcal{T}_h} h_K$ and h_K stands for the diameter of K. That is each $K \in \mathcal{T}_h$ is an m-polygonal domain, where m is an integer and can differ with K. We assume that m is bounded from above independently of a family of triangulations $\{\mathcal{T}_h\}_h$, and ∂K does not

intersect with itself. Let $\mathcal{E}_h = \{e \subset \partial K : K \in \mathcal{T}_h\}$ be the set of all edges of elements, and let $\Gamma_h = \bigcup_{K \in \mathcal{T}_h} \partial K$. We define the so-called broken Sobolev space for $k \geq 0$,

$$H^k(\mathcal{T}_h) = \{v \in L^2(\Omega) : v|_K \in H^k(K) \quad \forall K \in \mathcal{T}_h\}.$$

Let $L_0^2(\Gamma_h) = \{\hat{v} \in L^2(\Gamma_h) : \hat{v}|_{\partial\Omega} = 0\}$. We introduce the inner products

$$(u, v)_K = \int_K uv dx \quad \text{for } K \in \mathcal{T}_h,$$

$$\langle \hat{u}, \hat{v} \rangle_e = \int_e \hat{u}\hat{v} ds \quad \text{for } e \in \mathcal{E}_h.$$

The usual m-th order Sobolev seminorm and norm on K are denoted by $|u|_{m,K}$ and $\|u\|_{m,K}$, respectively. We use finite element spaces:

$$U_h \subset H^2(\mathcal{T}_h), \quad \hat{U}_h \subset L_0^2(\Gamma_h).$$

In addition, we set $V_h = U_h \times \hat{U}_h$, and $V(h) = H^2(\mathcal{T}_h) \times L_0^2(\Gamma_h)$.

2.2 Lifting operators

We state the definition of the lifting operator which plays a crucial role in our formulation and analysis. To this end, we fix $K \in \mathcal{T}_h$ and $e \subset \partial K$ for the time being, and set

$$U_h(K) = \{w_h|_K : w_h \in U_h, \ \hat{U}_h(e) = \{\hat{w}_h|_e : \hat{w}_h \in \hat{U}_h\}.$$

Then, for any $\hat{v} \in L^2(e)$, there exists a unique $\mathbf{u}_h \in U_h(K)^n$ such that

$$(\mathbf{u}_h, \mathbf{w}_h)_K = \langle \hat{v}, \mathbf{w}_h \cdot \mathbf{n}_K \rangle_e, \ \forall \mathbf{w}_h \in U_h(K)^n, \tag{2}$$

where $\mathbf{n}_K$ is the unit outward normal vector to ∂K. The lifting operator $\mathbf{L}_{e,K} : L^2(e) \to U_h(K)^n$ is defined as $\mathbf{L}_{e,K}(\hat{v}) = \mathbf{u}_h$. Thus,

$$(\mathbf{L}_{e,K}(\hat{v}), \mathbf{w}_h)_K = \langle \hat{v}, \mathbf{w}_h \cdot \mathbf{n}_K \rangle_e, \ \forall \mathbf{w}_h \in U_h(K)^n. \tag{3}$$

Furthermore, we define $\mathbf{L}_{\partial K} = \sum_{e \subset \partial K} \mathbf{L}_{e,K}$.

3 New hybridized DG scheme

This section is devoted to the presentation of our proposed hybridized DGFEM. Before doing so, we convert the Poisson problem (1) into a suitable weak form (7). A key idea is to introduce unknown functions on inter-element edges. First, multiplying both the sides of (1) by a test function $v \in U_h$ and integrating over each $K \in \mathcal{T}_h$, we have by the integration by parts

$$\sum_{K \in \mathcal{T}_h} [(\nabla u, \nabla v)_K - \langle \mathbf{n}_K \cdot \nabla u, v \rangle_{\partial K}] = (f, v). \tag{4}$$

From the continuity of the flux, we have

$$\sum_{K \in \mathcal{T}_h} \langle \mathbf{n}_K \cdot \nabla u, \hat{v} \rangle = 0 \quad \forall \hat{v} \in L_0^2(\Gamma_h). \tag{5}$$

This, together with (4), implies

$$\sum_{K \in \mathcal{T}_h} [(\nabla u, \nabla v)_K - \langle \mathbf{n}_K \cdot \nabla u, v - \hat{v} \rangle_{\partial K}] = (f, v). \tag{6}$$

Here we set, for $\mathbf{u} = (u, \hat{u})$ and $\mathbf{v} = (v, \hat{v}) \in V(h)$,

$$a_h(\mathbf{u}, \mathbf{v}) = \sum_{K \in \mathcal{T}_h} (\nabla u, \nabla v)_K,$$

$$b_h(\mathbf{u}, \mathbf{v}) = - \sum_{K \in \mathcal{T}_h} \langle \mathbf{n}_K \cdot \nabla u, v - \hat{v} \rangle_{\partial K}.$$

Then, (6) is rewritten as

$$a_h(\mathbf{u}, \mathbf{v}) + b_h(\mathbf{u}, \mathbf{v}) = (f, v). \tag{7}$$

Now we can state our hybridized DGFEM: find $\mathbf{u}_h \in V_h$ such that

$$B_h^L(\mathbf{u}_h, \mathbf{v}_h) := a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, \mathbf{u}_h) + j_h(\mathbf{u}_h, \mathbf{v}_h)$$
$$= (f, v_h) \quad \forall \mathbf{v}_h = (v_h, \hat{v}_h) \in V_h. \tag{8}$$

Here, the third term $b_h(\mathbf{v}_h, \mathbf{u}_h)$ of B_h^L is added in order to symmetrize the scheme and the penalty term $j_h(\mathbf{u}_h, \mathbf{v}_h)$ is defined by

$$j_h(\mathbf{u}, \mathbf{v}) = \sum_{K \in \mathcal{T}_h} (\mathbf{L}_{\partial K}(u - \hat{u}), \mathbf{L}_{\partial K}(v - \hat{v}))_K$$
$$+ \sum_{K \in \mathcal{T}_h} \sum_{e \subset \partial K} \int_e \eta_e h_e^{-1}(u - \hat{u})(v - \hat{v}) ds,$$

with the penalty parameters $\eta_e > 0$, where h_e is the diameter of e.

4 Error estimates

In this section, we give a mathematical analysis of our hybridized DGFEM. To this end, we introduce

$$\||\mathbf{v}\||^2 = \sum_{K \in \mathcal{T}_h} \left(\|\nabla v - \mathbf{L}_{\partial K}(v - \hat{v})\|_{0,K}^2 + \sum_{e \subset \partial K} \frac{\eta_e}{h_e} \|v - \hat{v}\|_{0,e}^2 \right),$$

$$\||\mathbf{v}\||_h^2 = \sum_{K \in \mathcal{T}_h} \left(|v|_{1,K}^2 + \sum_{e \subset \partial K} \frac{\eta_e}{h_e} \|v - \hat{v}\|_{0,e}^2 \right),$$

where η_e is a positive parameter for each $e \in \mathcal{E}_h$.

Theorem 1. *The bilinear form B_h^L satisfies the following three properties.*

(Consistency) *Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ be the exact solution. For $\mathbf{u} = (u, u|_{\Gamma_h})$, we have*
$$B_h^L(\mathbf{u}, \mathbf{v}) = (f, v) \quad \forall \mathbf{v} \in V(h).$$

(Boundedness)
$$|B_h^L(\mathbf{v}, \mathbf{w})| \leq \||\mathbf{v}\|| \||\mathbf{w}\|| \quad \forall \mathbf{v}, \mathbf{w} \in V(h).$$

(Coercivity)
$$B_h^L(\mathbf{v}_h, \mathbf{v}_h) \geq \||\mathbf{v}_h\||^2 \quad \forall \mathbf{v}_h \in V_h.$$

Furthermore, the scheme (8) *admits a unique solution $\mathbf{u}_h \in V_h$ for any $f \in L^2(\Omega)$ and $\{\eta_e\}_e$.*

Proof. The consistency is trivial since $u - u|_{\Gamma_h} = 0$ on Γ_h. The coercivity is a direct consequence of the expression

$$b_h(\mathbf{v}, \mathbf{w}) = -\sum_K (\nabla v, \mathbf{L}_{\partial K}(w - \hat{w}))_K.$$

Combining this with the Schwarz inequality, we immediately deduce the boundedness. Finally, the coercivity implies the uniqueness of (8) and, hence, the system of linear equations (8) admits a unique solution. $\square$

As results of those three properties, we obtain the following a priori error estimates in terms of $\|| \cdot \||$.

Theorem 2. *Let* $\mathbf{u} = (u, u|_{\Gamma_h}) \in V(h)$ *with the exact solution* $u \in H^2(\Omega) \cap H_0^1(\Omega)$ *of the Poisson problem* (1). *Suppose that* $\{\mathcal{T}_h\}_h$ *satisfies*

$$\tau \leq \frac{h_e}{h_K} \quad \forall K \in \mathcal{T}_h, \forall e \subset \partial K \tag{9}$$

with some positive constant τ. *Let* $\mathbf{u}_h = (u_h, \hat{u}_h) \in V_h$ *be the solution of our HDG scheme* (8) *for an arbitrary* $\{\eta_e\}_e$, $\eta_e > 0$. *Then, we have the error estimates*

$$|\!|\!|\mathbf{u} - \mathbf{u}_h|\!|\!| \leq 2 \inf_{v_h \in V_h} |\!|\!|\mathbf{u} - \mathbf{v}_h|\!|\!|. \tag{10}$$

Proof. Let $\mathbf{v}_h \in V_h$ be arbitrary. By Theorem 1, we have

$$
\begin{aligned}
|\!|\!|\mathbf{u}_h - \mathbf{v}_h|\!|\!|^2 &\leq B_h^L(\mathbf{u}_h - \mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h) && \text{(Coercivity)} \\
&= B_h^L(\mathbf{u} - \mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h) && \text{(Consistency)} \\
&\leq |\!|\!|\mathbf{u} - \mathbf{v}_h|\!|\!| \, |\!|\!|\mathbf{u}_h - \mathbf{v}_h|\!|\!|, && \text{(Boundedness)}
\end{aligned}
$$

which implies that

$$|\!|\!|\mathbf{u}_h - \mathbf{v}_h|\!|\!| \leq |\!|\!|\mathbf{u} - \mathbf{v}_h|\!|\!| \quad \forall \mathbf{v}_h \in V_h. \tag{11}$$

Using the triangle inequality, we have

$$
\begin{aligned}
|\!|\!|\mathbf{u} - \mathbf{u}_h|\!|\!| &\leq |\!|\!|\mathbf{u} - \mathbf{v}_h|\!|\!| + |\!|\!|\mathbf{u}_h - \mathbf{v}_h|\!|\!| \\
&\leq 2|\!|\!|\mathbf{u} - \mathbf{v}_h|\!|\!|.
\end{aligned}
$$

From the above, it follows that

$$|\!|\!|\mathbf{u} - \mathbf{u}_h|\!|\!| \leq 2 \inf_{\mathbf{v}_h \in V_h} |\!|\!|\mathbf{u} - \mathbf{v}_h|\!|\!|, \tag{12}$$

which implies that the error of the approximate solution is optimal in the norm $|\!|\!| \cdot |\!|\!|$. $\qquad\square$

As is stated in [8], we assume that the following approximate properties: for $v \in H^{k+1}(K)$ there exist positive constants $C_{k,s}^{\mathrm{e}}$ and $C_{k,s}^{\mathrm{f}}$ such that

$$\inf_{v_h \in U_h} |v - v_h|_{s,K} \leq C_{k,s}^{\mathrm{e}} h_K^{k+1-s} |v|_{k+1,K}, \tag{13}$$

$$\inf_{\hat{v}_h \in \hat{U}_h} |v - \hat{v}_h|_{s,e} \leq C_{k,s}^{\mathrm{f}} h_K^{k+\frac{1}{2}-s} |v|_{k+1,K}. \tag{14}$$

Then we have the error estimates in Theorem 2 are actually of optimal order.

6

Theorem 3. *Under the assumptions in Theorem 2 and the approximate properties* (13) *and* (14)*, we have, if $u \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$,*

$$\||\mathbf{u} - \mathbf{u}_h|\| \leq Ch^k |u|_{k+1,\Omega}, \tag{15}$$

$$\|u - u_h\|_{0,\Omega} \leq Ch^{k+1}|u|_{k+1,\Omega}. \tag{16}$$

In order to prove Theorem 3, we need the following auxiliary result.

Proposition 4. *Let $K \in \mathcal{T}_h$ and $e \subset \partial K$. Then we have*

$$\|\mathbf{L}_{e,K}(\hat{v})\|_{0,K} \leq C_1 h_e^{-1/2} \|\hat{v}\|_{0,e} \ \forall \hat{v} \in L^2(e). \tag{17}$$

Proof. In (3), taking $\mathbf{w}_h = \mathbf{L}_{e,K}(\hat{v})$ yields

$$
\begin{aligned}
\|\mathbf{L}_{e,K}(\hat{v})\|_{0,K}^2 &= (\mathbf{L}_{e,K}(\hat{v}), \mathbf{L}_{e,K}(\hat{v}))_K \\
&= \langle \hat{v}, \mathbf{L}_{e,K}(\hat{v}) \rangle_e \\
&\leq \|\hat{v}\|_{0,e} \|\mathbf{L}_{e,K}(\hat{v})\|_{0,e}.
\end{aligned} \tag{18}
$$

By the trace theorem, there exists C_1 such that

$$\|\mathbf{L}_{e,K}(\hat{v})\|_{0,e} \leq C_1 h_e^{-1/2} \|\mathbf{L}_{e,K}(\hat{v})\|_{0,K}. \tag{19}$$

Here C_1 depends on $U_h(K)$ and $\hat{U}_h(e)$. Combining (18) with (19), we obtain (17). $\qquad\square$

Proof of Theorem 3. As a consequence of Proposition 4, it can be proved that there exists a constant C_2 such that

$$\||\mathbf{v}|\| \leq C_2 \||\mathbf{v}|\|_h \quad \forall \mathbf{v} \in V(h). \tag{20}$$

From (13) and (14), we have

$$\inf_{\mathbf{v}_h \in V_h} \||\mathbf{u} - \mathbf{v}_h|\|_h \leq Ch^k |u|_{k+1,\Omega}. \tag{21}$$

Combining this with (20), we obtain (15). Next, we prove (16). Here we define $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ as the solution of the adjoint problem

$$-\Delta \psi = u - u_h \text{ in } \Omega, \quad \psi = 0 \text{ on } \partial\Omega. \tag{22}$$

Let $\boldsymbol{\psi} = (\psi, \psi|_{\Gamma_h})$. Then, since B_h^L is symmetric, we have

$$B_h^L(\mathbf{v}, \boldsymbol{\psi}) = (u - u_h, v) \quad \forall \mathbf{v} = (v, \hat{v}) \in V(h). \tag{23}$$

7

In particular, taking $\mathbf{v} = \mathbf{u} - \mathbf{u}_h$, we have for any $\boldsymbol{\psi}_h \in V_h$,

$$
\begin{aligned}
\|u - u_h\|_{0,\Omega}^2 &\leq B_h^L(\mathbf{u} - \mathbf{u}_h, \boldsymbol{\psi}) \\
&= B_h^L(\mathbf{u} - \mathbf{u}_h, \boldsymbol{\psi} - \boldsymbol{\psi}_h) \\
&\leq \|\|\mathbf{u} - \mathbf{u}_h\|\| \|\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|\| \\
&\leq C_2 \|\|\mathbf{u} - \mathbf{u}_h\|\| \|\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|\|_h.
\end{aligned}
$$

From (13) and (14), it follows that

$$
\|\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|\|_h \leq Ch|\psi|_{2,\Omega}. \tag{24}
$$

By the regularity of the adjoint problem, we have

$$
|\psi|_{2,\Omega} \leq C\|u - u_h\|_{0,\Omega}. \tag{25}
$$

Thus we obtain (16). $\qquad\square$

Remark 5. In contrast to our previous results of [8], error estimates in Theorem 2 are valid for any positive parameters η_e. This is one of the advantages of our hybridized DGFEM.

5 Numerical results

We now present the numerical results of our method for the following Poisson equation:

$$
\begin{cases}
-\Delta u = 2\pi^2 \sin(\pi x)\sin(\pi y) \text{ in } \Omega, \\
u = 0 \text{ on } \partial\Omega,
\end{cases} \tag{26}
$$

where Ω is a unit square. We use uniform rectangular meshes and P_k–P_k elements ($k = 1, 2, 3$). We computed the approximate solutions for various mesh size $h = 1/N$, see Table 1. We take the unity as the penalty parameters for each $e \in \mathcal{E}_h$. We see from Table 1 that the H^1 and L^2 convergence rate of the approximate solutions are h^k and h^{k+1}, respectively. Figure1 and Figure2 show the approximate solution u_h and $\hat{u}_h$ in the case $k = 1$ and $N = 8$, respectively.

6 Conclusions

We have presented a new hybridized DGFEM by using the lifting operator and examined the stability for arbitrary penalty parameters. Convergence results of

optimal order have been proved and confirmed by numerical experiments. As a model problem, we have considered only the Dirichlet boundary value problem for the Poisson equation. We are interested in application to other problems, for example, Neumann boundary value problem, convection-diffusion equations, Stokes system, and time-dependent problems. They are left here as future study.

7 Acknowledgement

Table 1: L^2 and H^1 errors.

k	N	L^2 error	L^2 rate	H^1 error	H^1 rate
1	4	3.23E-02	1.96	7.15E-01	1.01
	8	8.29E-03	1.96	3.55E-01	1.00
	16	2.14E-03	1.99	1.78E-01	1.00
	32	5.39E-04		8.90E-02	
2	4	4.56E-03	3.18	1.46E-01	2.07
	8	5.04E-04	3.05	3.47E-02	2.02
	16	6.08E-05	3.01	8.58E-03	2.00
	32	7.53E-06		2.14E-03	
3	4	4.48E-04	4.21	2.00E-02	3.12
	8	2.43E-05	4.07	2.30E-03	3.03
	16	1.45E-06	4.02	2.81E-04	3.01
	32	8.94E-08		3.49E-05	

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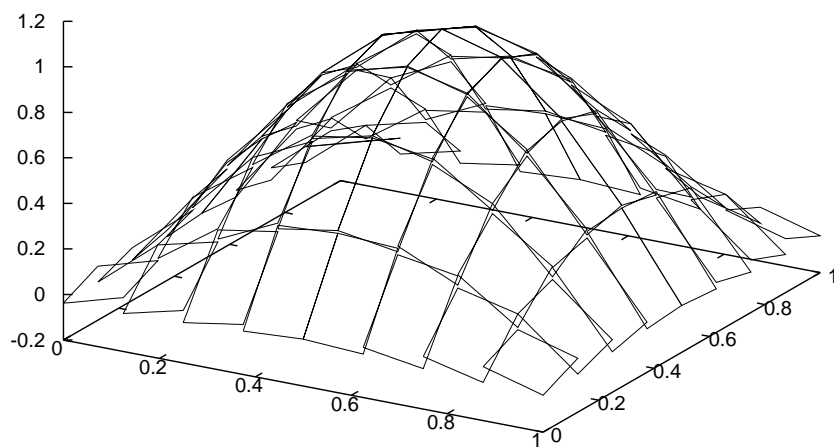
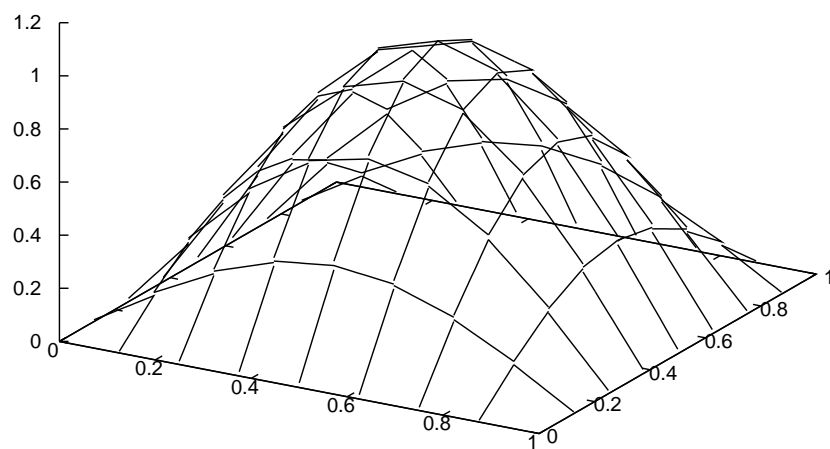Figure 1: The approximate solution u_h in the case $k = 1$ and $N = 8$.

Figure 2: The approximate solution $\hat{u}_h$ in the case $k = 1$ and $N = 8$.

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