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Abstract

In this paper we prove a Carleman estimate with second large parameter for a second order hyperbolic operator in a Riemannian manifold $\mathcal{M}$. Our Carleman estimate holds in the whole cylindrical domain $\mathcal{M}\times(0,T)$ independently of the level set generated by a weight function if functions under consideration vanish on boundary $\partial(\mathcal{M}\times(0,T))$. The proof is direct by using calculus of tensor fields in a Riemannian manifold.

1 Introduction

Since [6] where the unique continuation for an elliptic equation with non-analytical coefficients is proved, the theory of Carleman estimates have been comprehensively developed and we refer for example to Hörmander [10], Isakov [14], Lavtent'ev, Romanov and Shishat·skiĭ[21], Tataru [22] and the references therein. In particular, for Carleman estimates for hyperbolic operators, see [10], [21], Bellassoued and Yamamoto [4], Imanuvilov [11]. A Carleman estimate is an L^2-weight estimate with weight function $e^{2s\varphi}$ which is valid uniformly for all large parameter $s>0$.

Carleman estimates are important tools not only for the unique continuation but also for the observability inequality (see e.g., [20]) and inverse problems (see e.g., Bukhgeim and Klibanov [5], Bellassoued and Yamamoto [3], Imanuvilov and Yamamoto [13], Isakov [14], Klibanov [19], Klibanov and Timonov [20], Yamamoto [23]). In usual Carleman estimates, only one large parameter s is involved. However, in establishing the unique continuation and the observability inequality, and solving inverse problems for some systems in the mathematical physics such as the thermoelasticity system, we need a Carleman estimate wth second large parameter γ, where we set $\varphi = e^{\gamma\psi}$. Such Carleman estimates are proved in Isakov and Kim [15], [16] and also see Eller [7], Eller and Isakov [8] where functions under consideration are assumed to have compact supports.

In this paper, considering a second order hyperbolic operator in a Riemannian manifold, we prove a Carleman estimate with second large parameter γ for functions not having compact supports and vanishing on the boundary. The proof is direct mainly by means of integration by parts and the concept is similar to the proof of a Carleman estimate for a parabolic equation (e.g., [23]).

We formulate our Carleman estimate. Let $(\mathcal{M},\ \mathrm{g})$ be a compact Riemannian manifold with boundary $\partial\mathcal{M}$. All manifolds will be assumed smooth (which means $\mathcal{C}^\infty$) and oriented. We denote by Δ_{g} the Laplace-Beltrami operator associated to the metric g. In local coordinates, $\mathrm{g}(x) = (\mathrm{g}_{jk})$, Δ_{g} is given by

$$\Delta_{\mathrm{g}} = \frac{1}{\sqrt{\det \mathrm{g}}} \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} \left(\sqrt{\det \mathrm{g}}\, \mathrm{g}^{jk} \frac{\partial}{\partial x_k} \right). \tag{1.1}$$

Here (g^{jk}) is the inverse of the metric g and $\det \mathrm{g} = \det(\mathrm{g}_{jk})$.

Let us consider the following second order hyperbolic operator of second order

$$P = \partial_t^2 - \Delta_{\mathrm{g}} + P_1 \tag{1.2}$$

where P_1 is a first order partial operator with coefficients in $L^\infty(\mathbb{R} \times \mathcal{M})$.

Throughout this paper we use the following notations:

$$a(x,\xi) = \sum_{j,k=1}^{n} \mathrm{g}^{jk}(x)\xi_j\xi_k. \tag{1.3}$$

Given two symbols p and q we define their Poisson bracket as

$$\{p,q\}(x,\xi) = \frac{\partial p}{\partial \xi} \cdot \frac{\partial q}{\partial x} - \frac{\partial p}{\partial x} \cdot \frac{\partial q}{\partial \xi} = \sum_{j=1}^{n} \left(\frac{\partial p}{\partial \xi_j}\frac{\partial q}{\partial x_j} - \frac{\partial p}{\partial x_j}\frac{\partial q}{\partial \xi_j} \right). \tag{1.4}$$

The theory about differential calculus of tensor fields on Riemannian manifold can be found in [17]. Let $(\mathcal{M}, \mathrm{g})$ be an n-dimensional, $n \geq 2$, compact Riemannian manifold, with smooth boundary and smooth metric g. Fix a coordinate system $x = [x_1, \dots, x_n]$ and let $\left[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right]$ be the coordinate vector fields. For each $x \in \mathcal{M}$, define the inner product and the norm on the tangent space $T_x\mathcal{M}$ by

$$\mathrm{g}(X,Y) = \langle X,Y\rangle_{\mathrm{g}} = \sum_{j,k=1}^{n} \mathrm{g}_{jk}\alpha_j\beta_k,$$

$$|X|_{\mathrm{g}} = \langle X,X\rangle_{\mathrm{g}}^{1/2}, \qquad \forall X = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^{n} \beta_i \frac{\partial}{\partial x_i}.$$

Here and henceforth we identify $X = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}$ with $(\alpha_1, ..., \alpha_n) \in \mathbb{R}^n$. Moreover $(X)_j$ denotes the j-th coordinate of X. For $\mathcal{C}^1$-function f on $\mathcal{M}$, we define the gradient of f is the vector field $\nabla_{\mathrm{g}} f$ such that

$$X(f) = \langle \nabla_{\mathrm{g}} f, X\rangle_{\mathrm{g}}$$

for all vector fields X on $\mathcal{M}$. Then, with the above notation, we have

$$\nabla_{\mathrm{g}} f = \sum_{i,j=1}^{n} \mathrm{g}^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}. \tag{1.5}$$

We note that with the above identification, we see $(\nabla_{\mathrm{g}} f)_j = \sum_{i=1}^{n} \mathrm{g}^{ij} \frac{\partial f}{\partial x_i}$.

The metric tensor g induce the Riemannian volume $dv_{\mathrm{g}} = (\det \mathrm{g})^{1/2}\, dx_1 \wedge \cdots \wedge dx_n$. We denote by $L^2(\mathcal{M})$ the completion of $\mathcal{C}^\infty(\mathcal{M})$ with the usual inner product

$$\langle f_1, f_2 \rangle = \int_{\mathcal{M}} f_1(x) f_2(x) dv_{\mathrm{g}}, \qquad \forall f_1, f_2 \in \mathcal{C}^\infty(\mathcal{M}).$$

The Sobolev space $H^1(\mathcal{M})$ is the completion of $\mathcal{C}^\infty(\mathcal{M})$ with respect to the norm $\|\cdot\|_{H^1(\mathcal{M})}$,

$$\|f\|^2_{H^1(\mathcal{M})} = \|f\|^2_{L^2(\mathcal{M})} + \|\nabla f\|^2_{L^2(\mathcal{M})}.$$

Recalling the co-normal derivative defined below, we have

$$\partial_\nu u := \nabla_{\mathrm{g}} u \cdot \nu = \sum_{j,k=1}^{n} \mathrm{g}^{jk} \nu_j \frac{\partial u}{\partial x_k} \tag{1.6}$$

where ν is the outward vector field to $\partial\mathcal{M}$.
Moreover, using covariant derivatives (see [9]), it is possible to define coordinate invariant norm in $H^k(\mathcal{M})$, $k \geq 0$, and let

$$H^1_0(\mathcal{M}) = \left\{ v \in H^1(\mathcal{M}), \quad v = 0 \text{ on } \partial\mathcal{M} \right\}. \tag{1.7}$$

In order to state our Carleman estimate we need to introduce the following assumptions.
Assumption (A.1): We assume that there exists a positive function $\vartheta : \overline{\mathcal{M}} \longrightarrow \mathbb{R}$ of class $\mathcal{C}^2$ such that

$$\{a, \{a, \vartheta\}\}(x, \xi) > 0, \quad x \in \overline{\mathcal{M}}, \quad \xi \in T_x\mathcal{M} \backslash \{0\}. \tag{1.8}$$

Since $\overline{\mathcal{M}}$ is compact and $a(x,\xi)$ is a homogenous function with respect ξ, it follows from (1.8) that there exists a positive constant $\varrho > 0$ such that

$$\frac{1}{4} \{a, \{a, \vartheta\}\}(x, \xi) \geq 2\varrho |\, \widetilde{\xi}\, |^2_{\mathrm{g}}, \quad x \in \overline{\mathcal{M}}, \quad \xi \in T_x\mathcal{M} \backslash \{0\}. \tag{1.9}$$

Here $\widetilde{\xi}_j = \displaystyle\sum_{i=1}^{n} \mathrm{g}^{ij} \xi_i$.
Assumption (A.2): Moreover we assume that $\vartheta(x)$ has no critical points on $\overline{\mathcal{M}}$:

$$\min_{x \in \overline{\mathcal{M}}} |\nabla_{\mathrm{g}} \vartheta(x)|^2_{\mathrm{g}} > 0. \tag{1.10}$$

Assumption (A.3): Under assumption (A.1)-(A.2), let a subboundary Γ_0 satisfy

$$\{x \in \Gamma;\ \nabla_{\mathrm{g}} \vartheta \cdot \nu(x) \geq 0\} \subset \Gamma_0.$$

Let us define

$$Q = \mathcal{M} \times (0,T), \quad \Sigma_0 = \Gamma_0 \times (0,T)$$

and

$$\psi(x,t) = \vartheta(x) - \beta\,(t - t_0)^2, \quad 0 < \beta < \varrho, \quad 0 < t_0 < T \tag{1.11}$$

where the constant ϱ is given in (1.9). We define the weight function $\varphi : \mathcal{M} \times \mathbb{R} \longrightarrow \mathbb{R}$ by $\varphi(x,t) = e^{\gamma\psi(x,t)}$, where $\gamma > 0$ is a large parameter and let

$$\sigma = s\gamma\varphi,$$

where s is real numbers. Let us introduce the following notation

$$\mathcal{H}^1_0(Q) = \left\{ u \in H^1(0,T; L^2(\mathcal{M})) \cap L^2(0,T; H^1_0(\mathcal{M})),\ \partial_t^j u(\tau, \cdot) = 0,\ \tau \in \{0, T\},\ j \in \{0,1\} \right\}. \tag{1.12}$$

The following Carleman estimate is our main result:

Theorem 1 *Assume that (A.1), (A.2) and (A.3) hold. Then there exist constants $C > 0$ and $\gamma_* > 0$ such that for any $\gamma > \gamma_*$ there exist $s_* = s_*(\gamma)$ such that for all $s \geq s_*$ the following Carleman estimate holds*

$$C \int_Q e^{2s\varphi} \sigma \left(|\nabla_{\mathrm{g}} v|^2_{\mathrm{g}} + |\partial_t v|^2 + \sigma^2 |v|^2 \right) d\mathrm{v}_{\mathrm{g}} dt \leq \int_Q e^{2s\varphi} |Pv|^2 \, d\mathrm{v}_{\mathrm{g}} dt + \int_{\Sigma_0} \sigma e^{2s\varphi} |\partial_\nu v|^2 \, d\omega_{\mathrm{g}} dt \tag{1.13}$$

whenever $v \in \mathcal{H}^1_0(Q)$, and the right hand side is finite. Here $d\omega_{\mathrm{g}}$ is the volum form of $\partial\mathcal{M}$.

Remark 1 We show a simple example of metric g and ϑ satisfying (A.1) and (A.2). Let $\mathcal{M} \subset \mathbb{R}^n$ be a bounded domain with smooth boundary.
We take

$$\mathrm{g}^{jk}(x) = \mu(x)\delta_{jk}, \quad \mu(x) \geq \mu_0 > 0, \quad \forall x \in \mathcal{M}.$$

Then

$$P = \partial_t^2 - \mu(x)\Delta + P_1.$$

Let us consider $x_0 \in \mathbb{R}^n \backslash \overline{\mathcal{M}}$. Put $\vartheta(x) = |x - x_0|^2$. In this case an elementary calculation shows that

$$\frac{1}{4} \{a, \{a, \vartheta\}\}(x, \xi) = 2\mu(x) \left(1 - \frac{\nabla\mu \cdot (x - x_0)}{2\mu} \right) |\,\widetilde{\xi}\,|^2_{\mathrm{g}} + 2\mu(\nabla\mu \cdot \xi)(\xi \cdot (x - x_0)). \tag{1.14}$$

We assume that there exists $\varrho \in (0, \mu_0)$ such that

$$\frac{3}{2} |\nabla (\log \mu)| \, |x - x_0| \leq 1 - \frac{\varrho}{\mu_0}, \quad x \in \overline{\mathcal{M}}, \tag{1.15}$$

then we obtain

$$\frac{1}{4} \{a, \{a, \vartheta\}\}(x, \xi) \geq 2\varrho |\,\widetilde{\xi}\,|^2_{\mathrm{g}}. \tag{1.16}$$

Then (A.1) is satisfied. Since $\nabla_{\mathrm{g}}\vartheta(x) \neq 0$ for all $x \in \overline{\mathcal{M}}$ then (A.2) is also satisfied.
Moreover for $\mu(x)$ satisfying (1.15), also $g^{jk}(x) = \mu(x)\delta_{jk} + \varepsilon b_{jk}(x)$ satisfies (A.1) and (A.2) if $\varepsilon > 0$ is sufficiently small and $b_{jk} \in C^\infty(Q)$ (e.g., [4]).

As for other conditions admitting Carleman estimates, see also Amirov and Yamamoto [1] and Imanuvilov, Isakov and Yamamoto [12].

Theorem 1 is a Carleman estimate which holds over the whole domain Q, not only in level sets of a weight function, for functions which vanish on ∂Q. We need not assume that the functions under consideration have compact supports and so ours is different from the Carleman estimates presented in [15], [16] and [7], [8].

2 Preliminaries

In this section we collect some formulas to be involved in the sequel. Our interset is focused on Riemannian manifolds which are manifolds equipped with metric structure. Precisely a Riemannian manifold $(\mathcal{M}, \mathrm{g})$ is a manifold $\mathcal{M}$ with a positive definite 2-covariant tensor field g called the metric tensor. In local coordonates, g is given by a smooth, positive definite, symetric matrix function $\mathrm{g} = (\mathrm{g}_{jk})$.
We denote by $\mathrm{div}(X)$ the divergence of a vector field $X \in H^1(\mathcal{M})$ on $\mathcal{M}$, that is, in local coordinates,

$$\mathrm{div}(X) = \frac{1}{\sqrt{\det \mathrm{g}}} \sum_{i=1}^{n} \partial_i \left(\sqrt{\det \mathrm{g}}\, \alpha_i \right), \quad X = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}. \tag{2.1}$$

For $X \in H^1(\mathcal{M})$ we have the following divergence formula

$$\int_{\mathcal{M}} \mathrm{div}(X) d\mathrm{v}_{\mathrm{g}} = \int_{\partial\mathcal{M}} \langle X, \nu \rangle \, d\omega_{\mathrm{g}} \tag{2.2}$$

where $d\omega_{\mathrm{g}}$ is the volume form of $\partial\mathcal{M}$, and for $f \in H^1(\mathcal{M})$ we have following Green formula

$$\int_{\mathcal{M}} \mathrm{div}(X)\, f d\mathrm{v}_{\mathrm{g}} = -\int_{\mathcal{M}} \langle X, \nabla_{\mathrm{g}} f \rangle_{\mathrm{g}} \, d\mathrm{v}_{\mathrm{g}} + \int_{\partial\mathcal{M}} X \cdot \nu f d\omega_{\mathrm{g}}. \tag{2.3}$$

Then if $f \in H^1(\mathcal{M})$ and $w \in H^2(\mathcal{M})$, the following identity holds

$$\int_{\mathcal{M}} \Delta_{\mathrm{g}} w f d\mathrm{v}_{\mathrm{g}} = -\int_{\mathcal{M}} \langle \nabla_{\mathrm{g}} w, \nabla_{\mathrm{g}} f \rangle_{\mathrm{g}} \, d\mathrm{v}_{\mathrm{g}} + \int_{\partial\mathcal{M}} \partial_\nu w f d\omega_{\mathrm{g}}. \tag{2.4}$$

For $\vartheta \in \mathcal{C}^2(\mathcal{M})$, the Hessian of ϑ with respect to the metric g is defined by

$$\mathbb{D}^2\vartheta(X, X)(x) = \sum_{i,j=1}^{n} \alpha_i \left(\sum_{l=1}^{n} \frac{\partial \vartheta_l}{\partial x_i} \mathrm{g}_{lj} + \sum_{k,l=1}^{n} \vartheta_k \mathrm{g}_{lj} \Gamma_{ik}^{l} \right) \alpha_j, \quad \forall X = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}, \tag{2.5}$$

where we recall that $\vartheta_l(x) = (\nabla_{\mathrm{g}} \vartheta(x))_l$ is the $l-$th coordinate of $\nabla_{\mathrm{g}} \vartheta(x)$ and

$$(\nabla_{\mathrm{g}} \vartheta(x))_l = \vartheta_l(x) = \sum_{j=1}^{n} \mathrm{g}^{jl}(x) \frac{\partial \vartheta}{\partial x_j}(x), \quad l = 1, ..., n \tag{2.6}$$

and Γ_{ik}^{l} is the connection coefficient (Cristoffel symbol) of the Levi-Civita connection $\mathbb{D}$ to the metric g, that is,

$$\Gamma_{ik}^{l}(x) = \frac{1}{2} \sum_{p=1}^{n} \mathrm{g}^{lp}(x) \left(\frac{\partial \mathrm{g}_{kp}}{\partial x_i} + \frac{\partial \mathrm{g}_{ip}}{\partial x_k} - \frac{\partial \mathrm{g}_{ik}}{\partial x_p} \right). \tag{2.7}$$

Let X and Y be vector fields with components α_p and β_q. Then the l-th component of the covariant derivative of Y with respect to X is given by

$$(\mathbb{D}_X Y)_l = \sum_{p,q=1}^{n} \alpha_p \left(\frac{\partial \beta_l}{\partial x_p} + \Gamma_{pq}^{l} \beta_q \right). \tag{2.8}$$

We list a few formulad that will be used for the proof (see [17], p. 140 and [18], p.41). For any functions $f_1, f_2 \in \mathcal{C}^2(\mathcal{M})$ and any vectors fields X, Y and Z, we have

$$\begin{array}{rcl} Z(\langle X, Y\rangle_{\mathrm{g}}) & = & \langle \mathbb{D}_Z X, Y\rangle_{\mathrm{g}} + \langle X, \mathbb{D}_Z Y\rangle_{\mathrm{g}}\,, \\ \langle \nabla_{\mathrm{g}} f_1, Z\rangle & = & Z(f_1), \\ \langle \mathbb{D}_X(\nabla_{\mathrm{g}} f_1), Y\rangle_{\mathrm{g}} & = & \mathbb{D}^2 f_1(X,Y), \\ \nabla_{\mathrm{g}}(f_1 f_2) & = & f_2 \nabla_{\mathrm{g}} f_1 + f_1 \nabla_{\mathrm{g}} f_2, \\ \mathrm{div}(f_1 X) & = & f_1 \mathrm{div}(X) + \langle X, \nabla_{\mathrm{g}} f_1\rangle_{\mathrm{g}}\,. \end{array} \tag{2.9}$$

The following technical lemma holds true, which is proved in Appendix A of [2].

Lemma 2.1 *Let ϑ be a $\mathcal{C}^2$ function. Then we have the following identity:*

$$\{a, \{a, \vartheta\}\}(x,\xi) = 4\mathbb{D}^2\vartheta(\widetilde{\xi}, \widetilde{\xi}), \quad \forall\, x \in \mathcal{M},\, \xi \in T_x\mathcal{M}\backslash\,\{0\}\,. \tag{2.10}$$

Here $\widetilde{\xi}_j = \sum_{i=1}^n \mathrm{g}^{ij}(x)\xi_i$.

By assumption (A.1) we derive

$$\mathbb{D}^2\vartheta(X,X) \geq 2\varrho\,|X|_{\mathrm{g}}^2\,, \quad \forall X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}. \tag{2.11}$$

3 Proof of Theorem 1

In this section we complete the proof of Theorem 1. We will divide the proof in three steps. Henceforth we recall that $Q = \mathcal{M} \times (0,T)$ and $\Sigma_0 = \Gamma_0 \times (0,T)$, and we set $(f,g) = \int_Q fg d\mathrm{v}_{\mathrm{g}} dt$ and $\Sigma = \partial\mathcal{M} \times (0,T)$.

3.1 Change of variables

In this step, we set the differential equation satisfied by a new function z, which will be u up to weight function. That is, let us introduce the new functions $z = e^{s\varphi}u$ and $G = e^{s\varphi}f$, where $f = (\partial_t^2 - \Delta_{\mathrm{g}})u$. We easily obtain that

$$M_1 z + M_2 z = G_{s,\gamma} \tag{3.1}$$

where

$$\begin{array}{rcl} M_1 z & = & \partial_t^2 z - \Delta_{\mathrm{g}} z + \sigma^2\left(|\partial_t\psi|^2 - |\nabla_{\mathrm{g}}\psi|_{\mathrm{g}}^2\right) z\,, \\ M_2 z & = & -2\sigma\left(\partial_t z \partial_t\psi - \langle \nabla_{\mathrm{g}} z, \nabla_{\mathrm{g}}\psi\rangle_{\mathrm{g}}\right) - \gamma\sigma\left(|\partial_t\psi|^2 - |\nabla_{\mathrm{g}}\psi|_{\mathrm{g}}^2\right) z \end{array} \tag{3.2}$$

and

$$G_{s,\gamma} = G + \sigma\left(\partial_t^2\psi - \Delta_{\mathrm{g}}\psi\right) z. \tag{3.3}$$

With the previous notations, we have

$$\|M_1 z\|^2 + \|M_2 z\|^2 + 2\,(M_1 z, M_2 z) = \|G_{s,\gamma}\|^2\,. \tag{3.4}$$

Now, we will make the computation of $2\left(M_1z, M_2z\right)$. For this, we will develop the six terms appearing in $\left(M_1z, M_2z\right)$ and integrate by parts several times with respect to the space and time variables.

We have

$$
\begin{aligned}
\left(M_1z, M_2z\right) &= -2\int_Q \sigma\partial_t^2 z\left(\partial_t z\partial_t\psi - \left\langle\nabla_{\mathrm{g}} z, \nabla_g\psi\right\rangle_{\mathrm{g}}\right) d\mathrm{v}_{\mathrm{g}}dt \\
&\quad -\gamma\int_Q \sigma\partial_t^2 z\left(\left|\partial_t\psi\right|^2 - \left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2\right) z d\mathrm{v}_{\mathrm{g}}dt \\
&\quad +2\int_Q \sigma\Delta_{\mathrm{g}} z\left(\partial_t z\partial_t\psi - \left\langle\nabla_{\mathrm{g}} z, \nabla_{\mathrm{g}}\psi\right\rangle_{\mathrm{g}}\right) d\mathrm{v}_{\mathrm{g}}dt \\
&\quad +\gamma\int_Q \sigma\Delta_{\mathrm{g}} z\left(\left|\partial_t\psi\right|^2 - \left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2\right) z d\mathrm{v}_{\mathrm{g}}dt \\
&\quad -2\int_Q \sigma^3\left(\left|\partial_t\psi\right|^2 - \left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2\right) z\left(\partial_t z\partial_t\psi - \left\langle\nabla_{\mathrm{g}} z, \nabla_{\mathrm{g}}\psi\right\rangle_{\mathrm{g}}\right) d\mathrm{v}_{\mathrm{g}}dt \\
&\quad -\gamma\int_Q \sigma^3\left(\left|\partial_t\psi\right|^2 - \left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2\right)^2 |z|^2\, d\mathrm{v}_{\mathrm{g}}dt \\
&= \sum_{j=1}^{6}\mathcal{I}_j. \qquad (3.5)
\end{aligned}
$$

First we have

$$
\begin{aligned}
\mathcal{I}_1 &= -\int_Q \sigma\partial_t\psi\frac{\partial}{\partial t}\left(\left|\partial_t z\right|^2\right) d\mathrm{v}_{\mathrm{g}}dt - \int_Q \sigma\left\langle\nabla_{\mathrm{g}}\left(\left|\partial_t z\right|^2\right), \nabla_{\mathrm{g}}\psi\right\rangle_{\mathrm{g}} d\mathrm{v}_{\mathrm{g}}dt \\
&\quad -2\gamma\int_Q \sigma\left(\partial_t\psi\partial_t z\right)\left\langle\nabla_{\mathrm{g}}\psi, \nabla_{\mathrm{g}} z\right\rangle_{\mathrm{g}} d\mathrm{v}_{\mathrm{g}}dt \\
&= \gamma\int_Q \sigma\left|\partial_t\psi\right|^2\left|\partial_t z\right|^2 d\mathrm{v}_{\mathrm{g}}dt + \int_Q \sigma\partial_t^2\psi\left|\partial_t z\right|^2 d\mathrm{v}_{\mathrm{g}}dt \\
&\quad +\gamma\int_Q \sigma\left|\partial_t z\right|^2\left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2 d\mathrm{v}_{\mathrm{g}}dt + \int_Q \sigma\left|\partial_t z\right|^2\Delta_{\mathrm{g}}\psi d\mathrm{v}_{\mathrm{g}}dt \\
&\quad -2\gamma\int_Q \sigma\left(\partial_t\psi\partial_t z\right)\left\langle\nabla_{\mathrm{g}}\psi, \nabla_{\mathrm{g}} z\right\rangle_{\mathrm{g}} d\mathrm{v}_{\mathrm{g}}dt. \qquad (3.6)
\end{aligned}
$$

We also have

$$
\begin{aligned}
\mathcal{I}_2 &= -\gamma\int_Q \sigma\partial_t^2 z\left(\left|\partial_t\psi\right|^2 - \left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2\right) z\, d\mathrm{v}_{\mathrm{g}}dt \\
&= \gamma\int_Q \sigma\left|\partial_t z\right|^2\left(\left|\partial_t\psi\right|^2 - \left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2\right) d\mathrm{v}_{\mathrm{g}}dt \\
&\quad +\gamma^2\int_Q \sigma\left(\partial_t\psi\partial_t z\right)\left(\left|\partial_t\psi\right|^2 - \left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2\right) z\, d\mathrm{v}_{\mathrm{g}}dt + 2\gamma\int_Q \sigma\left(\partial_t z z\right)\left(\partial_t\psi\partial_t^2\psi\right) d\mathrm{v}_{\mathrm{g}}dt \\
&= \gamma\int_Q \sigma\left|\partial_t z\right|^2\left(\left|\partial_t\psi\right|^2 - \left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2\right) d\mathrm{v}_{\mathrm{g}}dt - \frac{\gamma^3}{2}\int_Q \sigma\left|\partial_t\psi\right|^2\left(\left|\partial_t\psi\right|^2 - \left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2\right)|z|^2\, d\mathrm{v}_{\mathrm{g}}dt \\
&\quad -\frac{5}{2}\gamma^2\int_Q \sigma|z|^2\left|\partial_t\psi\right|^2\partial_t^2\psi d\mathrm{v}_{\mathrm{g}}dt + \frac{\gamma^2}{2}\int_Q \sigma|z|^2\partial_t^2\psi\left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2 d\mathrm{v}_{\mathrm{g}}dt \\
&\quad -\gamma\int_Q \sigma|z|^2\left|\partial_t^2\psi\right|^2 d\mathrm{v}_{\mathrm{g}}dt. \qquad (3.7)
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\mathcal{I}_3 &= 2\int_Q \sigma\Delta_{\rm g} z\left(\partial_t z\partial_t\psi - \langle\nabla_{\rm g} z,\nabla_{\rm g}\psi\rangle_{\rm g}\right)d{\rm v}_{\rm g}dt\\
&= -2\gamma\int_Q \sigma\langle\nabla_{\rm g}\psi,\nabla_{\rm g} z\rangle_{\rm g}\left(\partial_t z\partial_t\psi - \langle\nabla_{\rm g} z,\nabla_{\rm g}\psi\rangle_{\rm g}\right)d{\rm v}_{\rm g}dt\\
&\quad -2\int_Q \sigma\langle\nabla_{\rm g} z,(\partial_t\nabla_{\rm g} z\partial_t\psi)\rangle_{\rm g}\,d{\rm v}_{\rm g}dt + 2\int_Q\sigma\left\langle\nabla_{\rm g} z,\nabla_{\rm g}\left(\langle\nabla_{\rm g} z,\nabla_{\rm g}\psi\rangle_{\rm g}\right)\right\rangle_{\rm g}d{\rm v}_{\rm g}dt\\
&\quad -2\left[\int_\Sigma \sigma\left|\nabla_{\rm g} z\cdot\nu\right|^2\nabla_{\rm g}\psi\cdot\nu d\omega_{\rm g}dt\right]\\
&= -2\gamma\int_Q\sigma\langle\nabla_{\rm g} z,\nabla_{\rm g}\psi\rangle_{\rm g}(\partial_t\psi\partial_t z)\,d{\rm v}_{\rm g}dt + 2\gamma\int_Q\sigma\left|\langle\nabla_{\rm g}\psi,\nabla_{\rm g} z\rangle_{\rm g}\right|^2d{\rm v}_{\rm g}dt\\
&\quad -\int_Q\sigma\frac{\partial}{\partial t}\left(|\nabla_{\rm g} z|_{\rm g}^2\right)\partial_t\psi d{\rm v}_{\rm g}dt + 2\int_Q\sigma\left\langle\nabla_{\rm g} z,\nabla_{\rm g}\left(\langle\nabla_{\rm g} z,\nabla_{\rm g}\psi\rangle_{\rm g}\right)\right\rangle_{\rm g}d{\rm v}_{\rm g}dt\\
&\quad -2\left[\int_\Sigma\sigma\left|\nabla_{\rm g} z\cdot\nu\right|^2\nabla_{\rm g}\psi\cdot\nu d\omega_{\rm g}dt\right].
\end{aligned}
\tag{3.8}
$$

Applying (2.9) with $Z = \nabla_{\rm g} z$, we obtain

$$
\begin{aligned}
\left\langle\nabla_{\rm g} z,\nabla_{\rm g}\left(\langle\nabla_{\rm g} z,\nabla_{\rm g}\psi\rangle_{\rm g}\right)\right\rangle_{\rm g} &= \nabla_{\rm g} z\left(\langle\nabla_{\rm g} z,\nabla_g\psi\rangle_{\rm g}\right)\\
&= \left\langle\mathbb{D}_{\nabla_{\rm g} z}\nabla_{\rm g} z,\nabla_g\psi\right\rangle_{\rm g} + \left\langle\nabla_{\rm g} z,\mathbb{D}_{\nabla_{\rm g} z}\nabla_g\psi\right\rangle_{\rm g}\\
&= \mathbb{D}^2\psi\left(\nabla_{\rm g} z,\nabla_{\rm g} z\right) + \mathbb{D}^2 z\left(\nabla_{\rm g} z,\nabla_{\rm g}\psi\right)
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
\left\langle\nabla_{\rm g}\psi,\nabla_{\rm g}\left(|\nabla_{\rm g} z|_{\rm g}^2\right)\right\rangle_{\rm g} &= \nabla_{\rm g}\psi\left(\langle\nabla_{\rm g} z,\nabla_{\rm g} z\rangle_{\rm g}\right)\\
&= \left\langle\mathbb{D}_{\nabla_{\rm g}\psi}\nabla_{\rm g} z,\nabla_g z\right\rangle_{\rm g} + \left\langle\nabla_{\rm g} z,\mathbb{D}_{\nabla_{\rm g}\psi}\nabla_{\rm g} z\right\rangle_{\rm g}\\
&= 2\mathbb{D}^2 z\left(\nabla_{\rm g} z,\nabla_{\rm g}\psi\right)
\end{aligned}
\tag{3.10}
$$

we deduce that

$$
\begin{aligned}
\mathcal{I}_3 &= -2\gamma\int_Q\sigma\langle\nabla_{\rm g}\psi,\nabla_{\rm g} z\rangle_{\rm g}(\partial_t\psi\partial_t z)\,d{\rm v}_{\rm g}dt + 2\gamma\int_Q\sigma\left|\langle\nabla_{\rm g}\psi,\nabla_{\rm g} z\rangle_{\rm g}\right|^2d{\rm v}_{\rm g}dt\\
&\quad +\gamma\int_Q\sigma|\nabla_{\rm g} z|_{\rm g}^2|\partial_t\psi|^2d{\rm v}_{\rm g}dt + \int_Q\sigma|\nabla_{\rm g} z|_{\rm g}^2\,\partial_t^2\psi\,d{\rm v}_{\rm g}dt\\
&\quad +2\int_Q\sigma\mathbb{D}^2\psi(\nabla_{\rm g} z,\nabla_{\rm g} z)d{\rm v}_{\rm g}dt - \gamma\int_Q\sigma|\nabla_{\rm g} z|_{\rm g}^2|\nabla_{\rm g}\psi|_{\rm g}^2d{\rm v}_{\rm g}dt\\
&\quad -\int_Q\sigma|\nabla_{\rm g} z|_{\rm g}^2\Delta_{\rm g}\psi d{\rm v}_{\rm g}dt\\
&\quad -\left[\int_\Sigma\sigma|\partial_\nu z|^2\nabla_{\rm g}\psi\cdot\nu d\omega_{\rm g}dt\right].
\end{aligned}
\tag{3.11}
$$

On the other hand, we have

$$
\begin{aligned}
\mathcal{I}_4 &= \gamma\int_Q\sigma\Delta_{\rm g} z\left(|\partial_t\psi|^2 - |\nabla_{\rm g}\psi|_{\rm g}^2\right)z d{\rm v}_{\rm g}dt\\
&= -\gamma^2\int_Q\sigma\langle\nabla_{\rm g}\psi,\nabla_{\rm g} z\rangle_{\rm g}\left(|\partial_t\psi|^2 - |\nabla_{\rm g}\psi|_{\rm g}^2\right)z d{\rm v}_{\rm g}dt
\end{aligned}
$$

$$
\begin{aligned}
& +\gamma\int_Q \sigma\left\langle\nabla_{\mathrm g} z, \nabla_{\mathrm g}\left(|\nabla_{\mathrm g}\psi|_{\mathrm g}^2\right)\right\rangle_{\mathrm g} z d\mathrm{v}_{\mathrm g}dt-\gamma\int_Q\sigma\,|\nabla_{\mathrm g} z|^2\left(|\partial_t\psi|^2-|\nabla_{\mathrm g}\psi|_{\mathrm g}^2\right)d\mathrm{v}_{\mathrm g}dt\\
= & -\frac{\gamma^2}{2}\int_Q\sigma\left\langle\nabla_{\mathrm g}\psi,\nabla_{\mathrm g}\left(|z|^2\right)\right\rangle_{\mathrm g}\left(|\partial_t\psi|^2-|\nabla_{\mathrm g} z|_{\mathrm g}^2\right)d\mathrm{v}_{\mathrm g}dt\\
& +\frac{\gamma}{2}\int_Q\sigma\left\langle\nabla_{\mathrm g}\left(|z|^2\right),\nabla_{\mathrm g}\left(|\nabla_{\mathrm g}\psi|_{\mathrm g}^2\right)\right\rangle_{\mathrm g}\left(|\partial_t\psi|^2-|\nabla_{\mathrm g}\psi|_{\mathrm g}^2\right)d\mathrm{v}_{\mathrm g}dt\\
& -\gamma\int_Q\sigma\,|\nabla_{\mathrm g} z|_{\mathrm g}^2\left(|\partial_t\psi|^2-|\nabla_{\mathrm g}\psi|_{\mathrm g}^2\right)d\mathrm{v}_{\mathrm g}dt\\
= & \frac{\gamma^2}{2}\int_Q\sigma\,|z|^2\left(\Delta_{\mathrm g}\psi+\gamma\,|\nabla_{\mathrm g}\psi|^2\right)_{\mathrm g}\left(|\partial_t\psi|^2-|\nabla_{\mathrm g}\psi|_{\mathrm g}^2\right)d\mathrm{v}_{\mathrm g}dt\\
& -\gamma^2\int_Q\sigma\,|z|^2\left\langle\nabla_{\mathrm g}\psi,\nabla_{\mathrm g}\left(|\nabla_{\mathrm g}\psi|_{\mathrm g}^2\right)\right\rangle_{\mathrm g}d\mathrm{v}_{\mathrm g}dt-\frac{\gamma}{2}\int_Q\sigma\,|z|^2\,\Delta_{\mathrm g}\left(|\nabla_{\mathrm g}\psi|_{\mathrm g}^2\right)d\mathrm{v}_{\mathrm g}dt\\
& -\gamma\int_Q\sigma\,|\nabla_{\mathrm g} z|_{\mathrm g}^2\left(|\partial_t\psi|^2-|\nabla_{\mathrm g}\psi|_{\mathrm g}^2\right)d\mathrm{v}_{\mathrm g}dt. \qquad (3.12)
\end{aligned}
$$

We also have

$$
\begin{aligned}
\mathcal{I}_5 = & -2\int_Q\sigma^3\left(|\partial_t\psi|^2-|\nabla_{\mathrm g}\psi|_{\mathrm g}^2\right)z\left(\partial_t\psi\partial_t z-\langle\nabla_{\mathrm g}\psi,\nabla_{\mathrm g} z\rangle_{\mathrm g}\right)d\mathrm{v}_{\mathrm g}dt\\
= & -\int_Q\sigma^3\frac{\partial}{\partial t}\left(|z|^2\right)\partial_t\psi\left(|\partial_t\psi|^2-|\nabla_{\mathrm g}\psi|_{\mathrm g}^2\right)d\mathrm{v}_{\mathrm g}dt\\
& +\int_Q\sigma^3\left\langle\nabla\left(|z|^2\right),\nabla_{\mathrm g}\psi\right\rangle_{\mathrm g}\left(|\partial_t\psi|^2-|\nabla_{\mathrm g}\psi|_{\mathrm g}^2\right)d\mathrm{v}_{\mathrm g}dt\\
= & \;3\gamma\int_Q\sigma^3\left(|\partial_t\psi|^2-|\nabla_{\mathrm g}\psi|_{\mathrm g}^2\right)^2|z|^2\,d\mathrm{v}_{\mathrm g}dt\\
& +\int_Q\sigma^3\,|z|^2\left(\partial_t^2\psi-\Delta_{\mathrm g}\psi\right)\left(|\partial_t\psi|^2-|\nabla_{\mathrm g}\psi|_{\mathrm g}^2\right)d\mathrm{v}_{\mathrm g}dt\\
& +2\int_Q\sigma^3\,|z|^2\,|\partial_t\psi|^2\,\partial_t^2\psi\,d\mathrm{v}_{\mathrm g}dt+\int_Q\sigma^3\,|z|^2\left\langle\nabla_{\mathrm g}\psi,\nabla_{\mathrm g}\left(|\nabla_{\mathrm g}\psi|_{\mathrm g}^2\right)\right\rangle_{\mathrm g}d\mathrm{v}_{\mathrm g}dt. \qquad (3.13)
\end{aligned}
$$

Furthermore

$$
\mathcal{I}_6=-\gamma\int_Q\sigma^3\left(|\partial_t\psi|^2-|\nabla_{\mathrm g}\psi|_{\mathrm g}^2\right)^2|z|^2\,d\mathrm{v}_{\mathrm g}dt. \qquad (3.14)
$$

Since

$$
\left\langle\nabla_{\mathrm g}\psi,\nabla_{\mathrm g}\left(|\nabla_{\mathrm g}\psi|_{\mathrm g}^2\right)\right\rangle_{\mathrm g}=\mathbb{D}_{\nabla_{\mathrm g}\psi}\left(\langle\nabla_{\mathrm g}\psi,\nabla_{\mathrm g}\psi\rangle_{\mathrm g}\right)=2\mathbb{D}^2\psi\left(\nabla_{\mathrm g}\psi,\nabla_{\mathrm g}\psi\right), \qquad (3.15)
$$

we obtain

$$
\begin{aligned}
(M_1 z, M_2 z) = & \Bigg(\gamma\int_Q\sigma\,|\partial_t z|^2\,|\partial_t\psi|^2\,d\mathrm{v}_{\mathrm g}dt+\int_Q\sigma\,|\partial_t z|^2\,\partial_t^2\psi d\mathrm{v}_{\mathrm g}dt+\gamma\int_Q\sigma\,|\partial_t z|^2\,|\nabla_{\mathrm g}\psi|_{\mathrm g}^2\,d\mathrm{v}_{\mathrm g}dt\\
& +\int_Q\sigma\,|\partial_t z|^2\,\Delta_{\mathrm g}\psi\,d\mathrm{v}_{\mathrm g}dt-4\gamma\int_Q\sigma\left(\partial_t\psi\partial_t z\right)\langle\nabla_{\mathrm g} z,\nabla_{\mathrm g}\psi\rangle_{\mathrm g}\,d\mathrm{v}_{\mathrm g}dt\\
& +\gamma\int_Q\sigma\,|\partial_t z|^2\left(|\partial_t\psi|^2-|\nabla_{\mathrm g}\psi|_{\mathrm g}^2\right)d\mathrm{v}_{\mathrm g}dt+2\gamma\int_Q\sigma\left|\langle\nabla_{\mathrm g}\psi,\nabla_{\mathrm g} z\rangle_{\mathrm g}\right|^2d\mathrm{v}_{\mathrm g}dt\Bigg)\\
& +\Bigg(\gamma\int_Q\sigma\,|\nabla_{\mathrm g} z|_{\mathrm g}^2\,|\partial_t\psi|^2\,d\mathrm{v}_{\mathrm g}dt+\int_Q\sigma\,|\nabla_{\mathrm g} z|_{\mathrm g}^2\,\partial_t^2\psi d\mathrm{v}_{\mathrm g}dt
\end{aligned}
$$

$$
\begin{aligned}
& +2\int_Q \sigma\mathbb{D}^2\psi(\nabla_{\rm g}z,\nabla_{\rm g}z)d{\rm v}_{\rm g}dt-\gamma\int_Q \sigma\left|\nabla_{\rm g}z\right|_{\rm g}^2\left|\nabla_{\rm g}\psi\right|_{\rm g}^2 d{\rm v}_{\rm g}dt \\
& -\int_Q \sigma\left|\nabla_{\rm g}z\right|_{\rm g}^2\Delta_{\rm g}\psi d{\rm v}_{\rm g}dt-\gamma\int_Q \sigma\left|\nabla_{\rm g}z\right|_{\rm g}^2\left(\left|\partial_t\psi\right|^2-\left|\nabla_{\rm g}\psi\right|_{\rm g}^2\right)d{\rm v}_{\rm g}dt\Bigg) \\
& +\Bigg(2\gamma\int_Q \sigma^3\left(\left|\partial_t\psi\right|^2-\left|\nabla_{\rm g}\psi\right|^2\right)_{\rm g}^2\left|z\right|^2 d{\rm v}_{\rm g}dt \\
& +\int_Q \sigma^3\left|z\right|^2\left(\partial_t^2\psi-\Delta_{\rm g}\psi\right)\left(\left|\partial_t\psi\right|^2-\left|\nabla_{\rm g}\psi\right|_{\rm g}^2\right)d{\rm v}_{\rm g}dt \\
& +2\int_Q \sigma^3\left|z\right|^2\left|\partial_t\psi\right|^2\partial_t^2\psi\, d{\rm v}_{\rm g}dt \\
& +2\int_Q \sigma^3\left|z\right|^2\mathbb{D}^2\psi(\nabla_{\rm g}\psi,\nabla_{\rm g}\psi)d{\rm v}_{\rm g}dt\Bigg)-\left[\int_\Sigma \sigma\left|\partial_\nu z\right|^2\nabla_{\rm g}\psi\cdot\nu d\omega_{\rm g}dt\right]+\mathcal{R}_1 \\
\equiv\ & \mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3+\mathcal{B}_0+\mathcal{R}_1
\end{aligned} \tag{3.16}
$$

where the terms $\mathcal{B}_0$ and $\mathcal{R}_1$ satisfy

$$
\left|\mathcal{R}_1\right|\leq C\gamma\int_Q \sigma^2\left|z\right|^2 d{\rm v}_{\rm g}dt \tag{3.17}
$$

and

$$
\mathcal{B}_0=-\left[\int_\Sigma \sigma\left|\partial_\nu z\right|^2\nabla_{\rm g}\psi\cdot\nu d\omega_{\rm g}dt\right] \tag{3.18}
$$

3.2 Interior estimate

The terms $\mathcal{J}_1,\mathcal{J}_2$ and $\mathcal{J}_3$ are given by:

$$
\begin{aligned}
\mathcal{J}_1 &= 2\gamma\int_Q \sigma\left(\partial_t\psi\partial_t z-\left\langle\nabla_{\rm g}\psi,\nabla_{\rm g}z\right\rangle_{\rm g}\right)^2 d{\rm v}_{\rm g}dt+\int_Q \sigma\left|\partial_t z\right|^2\left(\partial_t^2\psi+\Delta_{\rm g}\psi\right)d{\rm v}_{\rm g}dt \\
&\geq \int_Q \sigma\left|\partial_t z\right|^2\left(\partial_t^2\psi+\Delta_{\rm g}\psi\right)d{\rm v}_{\rm g}dt.
\end{aligned} \tag{3.19}
$$

$$
\mathcal{J}_2=2\int_Q \sigma\mathbb{D}^2\psi\left(\nabla_{\rm g}z,\nabla_{\rm g}z\right)d{\rm v}_{\rm g}dt+\int_Q \sigma\left|\nabla_{\rm g}z\right|^2\left(\partial_t^2\psi-\Delta_{\rm g}\psi\right)d{\rm v}_{\rm g}dt. \tag{3.20}
$$

$$
\begin{aligned}
\mathcal{J}_3 &= 2\gamma\int_Q \sigma^3\left(\left|\partial_t\psi\right|^2-\left|\nabla_{\rm g}\psi\right|_{\rm g}^2\right)^2\left|z\right|^2 d{\rm v}_{\rm g}dt \\
&+\int_Q \sigma^3\left|z\right|^2\left(\partial_t^2\psi-\Delta_{\rm g}\psi\right)\left(\left|\partial_t\psi\right|^2-\left|\nabla_{\rm g}\psi\right|_{\rm g}^2\right)d{\rm v}_{\rm g}dt+2\int_Q \sigma^3\left|z\right|^2\left|\partial_t\psi\right|^2\partial_t^2\psi\, d{\rm v}_{\rm g}dt \\
&+2\int_Q \sigma^3\left|z\right|^2\mathbb{D}^2\psi(\nabla_{\rm g}\psi,\nabla_{\rm g}\psi)d{\rm v}_{\rm g}dt.
\end{aligned} \tag{3.21}
$$

On the other hand, we have

$$
\begin{aligned}
\int_Q M_1 z(\sigma z\Delta_{\rm g}\psi)d{\rm v}_{\rm g}dt &= \int_Q \partial_t^2 z(\sigma z\Delta_{\rm g}\psi)d{\rm v}_{\rm g}dt-\int_Q \Delta_{\rm g}z(z\sigma\Delta_{\rm g}\psi)d{\rm v}_{\rm g}dt \\
&+\int_Q \sigma^3\left(\left|\partial_t\psi\right|^2-\left|\nabla_{\rm g}\right|_{\rm g}^2\right)\Delta_{\rm g}\psi\left|z\right|^2 d{\rm v}_{\rm g}dt
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_Q \sigma \left|\partial_t z\right|^2 \Delta_{\mathrm{g}}\psi d\mathrm{v}_{\mathrm{g}} dt - \gamma \int_Q \sigma(\partial_t \psi)(\partial_t z) z \Delta_{\mathrm{g}}\psi d\mathrm{v}_{\mathrm{g}} dt \\
&\quad + \int_Q \sigma \left|\nabla_{\mathrm{g}} z\right|_{\mathrm{g}}^2 \Delta_{\mathrm{g}}\psi d\mathrm{v}_{\mathrm{g}} dt + \gamma \int_Q \sigma \left\langle \nabla_{\mathrm{g}}\psi, \nabla_{\mathrm{g}} z\right\rangle_{\mathrm{g}} \Delta_{\mathrm{g}}\psi z d\mathrm{v}_{\mathrm{g}} dt \\
&\quad + \int_Q \sigma \left\langle \nabla_{\mathrm{g}} z, \nabla_{\mathrm{g}}(\Delta_{\mathrm{g}}\psi) z\right\rangle_{\mathrm{g}} d\mathrm{v}_{\mathrm{g}} dt \\
&\quad + \int_Q \sigma^3 \left(\left|\partial_t \psi\right|^2 - \left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2\right) \Delta_{\mathrm{g}}\psi \left|z\right|^2 d\mathrm{v}_{\mathrm{g}} dt. \qquad (3.22)
\end{aligned}
$$

Then for $\varepsilon > 0$, we choose a constant $C > 0$ such that

$$
\begin{aligned}
\left|\int_Q \sigma \left(\left|\nabla_{\mathrm{g}} z\right|_{\mathrm{g}}^2 - \left|\partial_t z\right|^2\right) \Delta_{\mathrm{g}}\psi d\mathrm{v}_{\mathrm{g}} dt\right| &\leq C \int_Q \sigma^3 \left|z\right|^2 \left|\left|\partial_t \psi\right|^2 - \left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2\right| d\mathrm{v}_{\mathrm{g}} dt \\
&\quad + \varepsilon \left\|M_1 z\right\|^2 + \left|\mathcal{R}_2\right| \qquad (3.23)
\end{aligned}
$$

where the term $\mathcal{R}_2$ satisfies

$$
\left|\mathcal{R}_2\right| \leq C_\varepsilon \left(\gamma \int_Q \left|\nabla_{\mathrm{g}} z\right|_{\mathrm{g}}^2 d\mathrm{v}_{\mathrm{g}} dt + \gamma \int_Q \sigma^2 \left|z\right|^2 d\mathrm{v}_{\mathrm{g}} dt\right). \qquad (3.24)
$$

Therefore, we find that

$$
\begin{aligned}
\mathcal{J}_1 + \mathcal{J}_2 &\geq 2\int_Q \sigma \mathbb{D}^2\psi(\nabla_{\mathrm{g}} z, \nabla_{\mathrm{g}} z) d\mathrm{v}_{\mathrm{g}} dt - 2\beta \int_Q \sigma(\left|\nabla_{\mathrm{g}} z\right|_{\mathrm{g}}^2 + \left|\partial_t z\right|^2) d\mathrm{v}_{\mathrm{g}} dt \\
&\quad + \int_Q \sigma \left(\left|\nabla_{\mathrm{g}} z\right|_{\mathrm{g}}^2 - \left|\partial_t z\right|^2\right) \Delta_{\mathrm{g}}\psi d\mathrm{v}_{\mathrm{g}} dt \\
&\geq 2\int_Q \sigma \mathbb{D}^2\psi(\nabla_{\mathrm{g}} z, \nabla_{\mathrm{g}} z) - 2\beta \int_Q \sigma(\left|\nabla_{\mathrm{g}} z\right|_{\mathrm{g}}^2 + \left|\partial_t z\right|^2) d\mathrm{v}_{\mathrm{g}} dt \\
&\quad - C\int_Q \sigma^3 \left|z\right|^2 \left|\left|\partial_t \psi\right|^2 - \left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2\right| d\mathrm{v}_{\mathrm{g}} dt - \varepsilon \left\|M_1 z\right\|^2 - \left|\mathcal{R}_2\right|. \qquad (3.25)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\int_Q M_1 z(\sigma z) d\mathrm{v}_{\mathrm{g}} dt &= -\int_Q \sigma \left|\partial_t z\right|^2 d\mathrm{v}_{\mathrm{g}} dt - \gamma \int_Q \sigma \partial_t \psi \partial_t z z d\mathrm{v}_{\mathrm{g}} dt \\
&\quad + \int_Q \sigma \left|\nabla_{\mathrm{g}} z\right|_{\mathrm{g}}^2 d\mathrm{v}_{\mathrm{g}} dt + \gamma \int_Q \sigma \left\langle \nabla_{\mathrm{g}}\psi, \nabla_{\mathrm{g}} z\right\rangle_{\mathrm{g}} z d\mathrm{v}_{\mathrm{g}} dt \\
&\quad + \int_Q \sigma^3 \left(\left|\partial_t \psi\right|^2 - \left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2\right) \left|z\right|^2 d\mathrm{v}_{\mathrm{g}} dt. \qquad (3.26)
\end{aligned}
$$

We deduce that

$$
\begin{aligned}
\int_Q \sigma \left|\partial_t z\right|^2 d\mathrm{v}_{\mathrm{g}} dt &\leq C\int_Q \sigma^3 \left|z\right|^2 \left|\left|\partial_t \psi\right|^2 - \left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2\right| d\mathrm{v}_{\mathrm{g}} dt \\
&\quad + \varepsilon \left\|M_1 z\right\|^2 + \left|\mathcal{R}_3\right| + \int_Q \sigma \left|\nabla_{\mathrm{g}} z\right|_{\mathrm{g}}^2 d\mathrm{v}_{\mathrm{g}} dt \qquad (3.27)
\end{aligned}
$$

where

$$
\left|\mathcal{R}_3\right| \leq C_\varepsilon \left(\gamma \int_Q \left(\left|\nabla_{\mathrm{g}} z\right|_{\mathrm{g}}^2 + \left|\partial_t z\right|^2\right) d\mathrm{v}_{\mathrm{g}} dt + \gamma \int_Q \sigma^2 \left|z\right|^2 d\mathrm{v}_{\mathrm{g}} dt\right). \qquad (3.28)
$$

Combining (3.28), (3.25) and using (2.11), we obtain

$$\begin{aligned}
\mathcal{J}_1+\mathcal{J}_2 &\geq 2\int_Q \sigma\mathbb{D}^2\psi(\nabla_{\mathrm{g}} z,\nabla_{\mathrm{g}} z)d\mathrm{v}_{\mathrm{g}}dt-4\beta\int_Q \sigma\left|\nabla_{\mathrm{g}} z\right|_{\mathrm{g}}^2 d\mathrm{v}_{\mathrm{g}}dt \\
&\quad -C\int_Q \sigma^3\left|z\right|^2\left|\left|\partial_t\psi\right|^2-\left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2\right|d\mathrm{v}_{\mathrm{g}}dt-\varepsilon\left\|M_1 z\right\|^2-\left|\mathcal{R}_2\right|-\left|\mathcal{R}_3\right| \\
&\geq 4(\varrho-\beta)\int_Q \sigma\left|\nabla_{\mathrm{g}} z\right|_{\mathrm{g}}^2 d\mathrm{v}_{\mathrm{g}}dt-C\int_Q \sigma^3\left|z\right|^2\left|\left|\partial_t\psi\right|^2-\left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2\right|d\mathrm{v}_{\mathrm{g}}dt \\
&\quad -\varepsilon\left\|M_1 z\right\|^2-\left|\mathcal{R}_2\right|-\left|\mathcal{R}_3\right| \\
&\geq (\varrho-\beta)\int_Q \sigma\left(\left|\nabla_{\mathrm{g}} z\right|_{\mathrm{g}}^2+\left|\partial_t z\right|^2\right)d\mathrm{v}_{\mathrm{g}}dt-C\Bigg(\int_Q \sigma^3\left|z\right|^2\left|\left|\partial_t\psi\right|^2-\left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2\right|d\mathrm{v}_{\mathrm{g}}dt \\
&\quad +\varepsilon\left\|M_1 z\right\|^2+\left|\mathcal{R}_2\right|+\left|\mathcal{R}_3\right|\Bigg).
\end{aligned} \tag{3.29}$$

We also see that

$$\begin{aligned}
\mathcal{J}_3 &= 2\gamma\int_Q \sigma^3\left(\left|\partial_t\psi\right|^2-\left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2\right)^2\left|z\right|^2 d\mathrm{v}_{\mathrm{g}}dt \\
&\quad +\int_Q \sigma^3\left|z\right|^2\left(\partial_t^2\psi-\Delta_{\mathrm{g}}\psi\right)\left(\left|\partial_t\psi\right|^2-\left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2\right)d\mathrm{v}_{\mathrm{g}}dt-4\beta\int_Q \sigma^3\left|z\right|^2\left|\partial_t\psi\right|^2\, d\mathrm{v}_{\mathrm{g}}dt \\
&\quad +2\int_Q \sigma^3\left|z\right|^2\mathbb{D}^2\psi(\nabla_{\mathrm{g}}\psi,\nabla_{\mathrm{g}}\psi)d\mathrm{v}_{\mathrm{g}}dt \\
&\geq 2\gamma\int_Q \sigma^3\left(\left|\partial_t\psi\right|^2-\left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2\right)^2\left|z\right|^2 d\mathrm{v}_{\mathrm{g}}dt-C\int_Q \sigma^3\left|z\right|^2\left|\left|\partial_t\psi\right|^2-\left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2\right|d\mathrm{v}_{\mathrm{g}}dt \\
&\quad +4\int_Q \sigma^3\left(\varrho\left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2-\beta\left|\partial_t\psi\right|^2\right)\left|z\right|^2 d\mathrm{v}_{\mathrm{g}}dt.
\end{aligned} \tag{3.30}$$

Additionally, we find that

$$\begin{aligned}
\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3 &\geq 2(\varrho-\beta)\int_Q \sigma\left(\left|\nabla_{\mathrm{g}} z\right|_{\mathrm{g}}^2+\left|\partial_t z\right|^2\right)d\mathrm{v}_{\mathrm{g}}dt+2\gamma\int_Q \sigma^3\left(\left|\partial_t\psi\right|^2-\left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2\right)^2\left|z\right|^2 d\mathrm{v}_{\mathrm{g}}dt \\
&\quad +4\int_Q \sigma^3\left(\varrho\left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2-\beta\left|\partial_t\psi\right|^2\right)\left|z\right|^2 d\mathrm{v}_{\mathrm{g}}dt \\
&\quad -C\Big(\int_Q \sigma^3\left|z\right|^2\left|\left|\partial_t\psi\right|^2-\left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2\right|d\mathrm{v}_{\mathrm{g}}dt+\varepsilon\left\|M_1 z\right\|^2+\left|\mathcal{R}_2\right|+\left|\mathcal{R}_3\right|\Big).
\end{aligned} \tag{3.31}$$

3.3 Conclusion

Let $\eta>0$ be small such that $\beta(1+\eta)<\varrho$. Let us consider

$$Q^\eta=\left\{(x,t)\in Q,\ \left|\left|\partial_t\psi\right|^2-\left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2\right|\leq\eta\left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2\right\}.$$

Then

$$\begin{aligned}
\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3 &\geq 2(\varrho-\beta)\int_Q \sigma\left(\left|\nabla_{\mathrm{g}} z\right|_{\mathrm{g}}^2+\left|\partial_t z\right|^2\right)d\mathrm{v}_{\mathrm{g}}dt \\
&\quad +2\gamma\int_{Q\setminus Q^\eta}\sigma^3\left(\left|\partial_t\psi\right|^2-\left|\nabla_{\mathrm{g}}\psi\right|_{\mathrm{g}}^2\right)^2\left|z\right|^2 d\mathrm{v}_{\mathrm{g}}dt
\end{aligned}$$

$$
\begin{aligned}
& -C\eta\int_{Q^\eta}\sigma^3\,|z|^2\,d\mathrm{v_g}dt - C\int_{Q\backslash Q^\eta}\sigma^3\,|z|^2\,d\mathrm{v_g}dt \\
& +4(\varrho-\beta(1+\eta))\int_{Q^\eta}\sigma^3\,|z|^2\,|\nabla_{\mathrm{g}}\psi|_{\mathrm{g}}^2\,d\mathrm{v_g}dt \\
& -C\left(\varepsilon\,\|M_1z\|^2+|\mathcal{R}_2|+|\mathcal{R}_3|\right) \\
\geq\; & \delta\int_Q\sigma\left(|\nabla_{\mathrm{g}}z|_{\mathrm{g}}^2+|\partial_tz|^2\right)d\mathrm{v_g}dt+2\gamma\eta^2C_1\int_{Q\backslash Q^\eta}\sigma^3\,|z|^2\,d\mathrm{v_g}dt \\
& -C_2\eta\int_{Q^\eta}\sigma^3\,|z|^2\,d\mathrm{v_g}dt - C_3\int_{Q\backslash Q^\eta}\sigma^3\,|z|^2\,d\mathrm{v_g}dt \\
& +C_4(\varrho-\beta(1+\eta))\int_{Q^\eta}\sigma^3\,|z|^2\,d\mathrm{v_g}dt - C\left(\varepsilon\,\|M_1z\|^2+|\mathcal{R}_2|+|\mathcal{R}_3|\right) \\
\geq\; & \delta\int_Q\sigma\left(|\nabla_{\mathrm{g}}z|_{\mathrm{g}}^2+|\partial_tz|^2\right)d\mathrm{v_g}dt+(2\gamma\eta^2C_1-C_3)\int_{Q\backslash Q^\eta}\sigma^3\,|z|^2\,d\mathrm{v_g}dt \\
& +(C_4-\eta C_2)\int_{Q^\eta}\sigma^3\,|z|^2\,d\mathrm{v_g}dt - C\left(\varepsilon\,\|M_1z\|^2+|\mathcal{R}_2|+|\mathcal{R}_3|\right). \qquad (3.32)
\end{aligned}
$$

Then for η small and γ large, we obtain

$$
\begin{aligned}
\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3 \;\geq\; & \delta\int_Q\sigma\left(|\nabla_{\mathrm{g}}z|_{\mathrm{g}}^2+|\partial_tz|^2\right)d\mathrm{v_g}dt + C\int_Q\sigma^3\,|z|^2\,d\mathrm{v_g}dt \\
& -C\left(\varepsilon\,\|M_1z\|^2+|\mathcal{R}_2|+|\mathcal{R}_3|\right). \qquad (3.33)
\end{aligned}
$$

By (3.16) we find

$$
\begin{aligned}
2\,(M_1z,M_2z)-\mathcal{B}_0 \;\geq\; & C\int_Q\sigma\left(|\nabla_{\mathrm{g}}z|_{\mathrm{g}}^2+|\partial_tz|^2\right)+\sigma^3\,|z|^2\,d\mathrm{v_g}dt \\
& -C\left(\varepsilon\,\|M_1z\|^2+|\mathcal{R}_1|+|\mathcal{R}_2|+|\mathcal{R}_3|\right). \qquad (3.34)
\end{aligned}
$$

Then there exists $s_*(\gamma)$ such that for any $s\geq s_*$ and ε small, we have

$$
\|G\|^2-\mathcal{B}_0\geq C\int_Q\sigma\left(\left(|\nabla_{\mathrm{g}}z|_{\mathrm{g}}^2+|\partial_tz|^2\right)+\sigma^2\,|z|^2\right)d\mathrm{v_g}dt. \qquad (3.35)
$$

The proof is completed.

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