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$Pin^{-}(2)$-monopole equations and intersection forms with local coefficients of 4-manifolds

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$\mathrm{Pin}^{-}(2)$-MONOPOLE EQUATIONS AND INTERSECTION FORMS WITH LOCAL COEFFICIENTS OF 4-MANIFOLDS

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ABSTRACT. We introduce a variant of the Seiberg-Witten equations, $\mathrm{Pin}^{-}(2)$-monopole equations, and give its applications to intersection forms with local coefficients of 4-manifolds. The first application is an analogue of Froyshov's results on 4-manifolds which have definite forms with local coefficients. The second is a local coefficient version of Furuta's 10/8-inequality. As a corollary, we construct nonsmoothable spin 4-manifolds satisfying Rohlin's theorem and the 10/8-inequality.

1. Introduction

K. Froyshov[11] recently proved theorems on intersection forms with local coefficients of 4-manifolds which can be considered as a local coefficient analogue of Donaldson's theorem for definite 4-manifolds[7, 8]. To prove his results, he analyzes the moduli space of SO(3)-instantons, and effectively make use of the existence of a kind of reducibles, *twisted reducibles*, whose stabilizers are $\mathbb{Z}/2$, in order to extract the information on local coefficient cohomology.

The first part of this paper proves an analogue of Froyshov's results by Seiberg-Witten theory. In fact, we prove that, if a closed smooth 4-manifold has a definite intersection form with local coefficient, it should be the standard form.

To state the precise statement, we give some preliminaries. Let X be a closed, connected, oriented smooth 4-manifold. Suppose a double covering $\tilde{X}$ of X is given. Let $l = \tilde{X} \times_{\{\pm 1\}} \mathbb{Z}$ and $\lambda = \tilde{X} \times_{\{\pm 1\}} \mathbb{R}$ be its associated bundles with fiber $\mathbb{Z}$ and $\mathbb{R}$. We can consider the cohomology $H^*(X; l)$ with l as bundle of coefficients. Since $l \otimes l = \mathbb{Z}$, we have a homomorphism by the cup product,

$$H^2(X; l) \otimes H^2(X; l) \to H^4(X; \mathbb{Z}) = \mathbb{Z}.$$

This induces a unimodular quadratic form $Q_{X,l}$ on $H^2(X; l)/\text{torsion}$. Let $b_q(X; l)$ be the l-coefficient q-th Betti number, i.e., $b_q(X; l) = \operatorname{rank} H^q(X; l)/\text{torsion}$. The ordinary $\mathbb{Z}$-coefficient Betti numbers are denoted by $b_q(X)$. Now, our first theorem is as follows:

Theorem 1.1. *Let X be a closed, connected, oriented smooth 4-manifold. Suppose that a nontrivial $\mathbb{Z}$-bundle $l \to X$ satisfies the following:*

(1) *The intersection form $Q_{X,l}$ is definite.*
(2) *$w_1(\lambda)^2 = 0$, where $\lambda = l \otimes \mathbb{R}$.*

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Then $Q_{X,l}$ is isomorphic to the diagonal form.

The proof of Theorem 1.1 is outlined as follows. For the double covering $\tilde{X}$ for l, let $\iota\colon \tilde{X} \to \tilde{X}$ be the covering transformation. We consider a Spin^c-structure c on $\tilde{X}$ such that the pullback Spin^c-structure ι^*c is isomorphic to the complex conjugation of c. In fact, if we starts from a Spin^{c-}-structure on X, a $\mathrm{Pin}^-(2)$-variant of Spin^c-structure introduced in §3, we obtain an antilinear involution I covering ι on the spinor bundles and the determinant line bundle of c. Then, I acts on the Seiberg-Witten moduli space $\mathcal{M}$ of c, and we analyze its I-invariant part $\mathcal{M}^I$. The rest of the argument is analogous to the argument in the alternative proof of Donaldson's theorem by the Seiberg-Witten theory (see e.g. [18, 21]). That is, under the assumptions of Theorem 1.1, we prove the virtual dimension of $\mathcal{M}^I$ cannot be greater than $b_1(X;l)$, and obtain an inequality on the characteristic elements of $Q_{X,l}$. Finally, we invoke a theorem of Elkies[9] to prove the form should be the standard form.

The second part of the paper introduces a variant of Seiberg-Witten equations, $\mathrm{Pin}^-(2)$-*monopole equations* we call, which are defined on Spin^{c-}-structures. It turns out that the I-invariant moduli space of the double covering $\tilde{X}$ as above can be identified with the moduli space of solutions of $\mathrm{Pin}^-(2)$-monopole equations. In particular, the moduli space of $\mathrm{Pin}^-(2)$-monopoles is compact. Then we can consider the finite dimensional approximation of the $\mathrm{Pin}^-(2)$-*monopole map* as in [13, 3], which enable us to prove a 10/8-type inequality for intersection forms with local coefficients:

Theorem 1.2. *Let X be a closed connected oriented smooth 4-manifold. For any non-trivial $\mathbb{Z}$-bundle l over X which satisfies $w_1(\lambda)^2 = w_2(X)$, the inequality $b_+(X;l) \geq -\operatorname{sign}(X)/8$ holds.*

Remark 1.3. (1)Since $\alpha \cup \alpha = Sq^1(\alpha)$ for $\alpha \in H^1(X;\mathbb{Z}/2)$, and Sq^1 is the Bockstein homomorphism, $w_2(X) = \alpha \cup \alpha$ holds for some α if $w_2(X)$ has an integral lift of order 2.
(2)We will give an alternative proof of Theorem 1.1 by using the same technique used in the proof of Theorem 1.2.

As an application of Theorem 1.1 and Theorem 1.2, we construct nonsmoothable 4-manifolds satisfying known constraints on smooth 4-manifolds.

Let us consider the spin cases. For smooth spin 4-manifolds, we know two fundamental theorems, Rohlin's theorem(see e.g.[16]) and Furuta's theorem[13]. Rohlin's theorem tells us that the signature of every closed spin 4-manifold is divisible by 16. On the other hand, Furuta's theorem[13] tells us that every closed smooth spin 4-manifold X with indefinite form satisfies the so-called "10/8-inequality"

$$b_2(X) \geq \frac{5}{4}|\operatorname{sign}(X)| + 2.$$

This inequality is improved by M. Furuta and Y. Kametani [14] in the case when $b_1(X) > 0$. We call the improved inequality in [14] the strong 10/8-inequality.

Theorem 1.4. *There exist nonsmoothable spin 4-manifolds which have signatures divisible by 16 and satisfy the strong 10/8-inequality.*

The idea of the construction of such nonsmoothable examples is as follows. Let V be any simply-connected 4-manifold with even definite form Q_V of rank $16k$, and let X be a connected sum of V with sufficiently many $T^2 \times S^2$'s or T^4's so that the 10/8-inequality is satisfied. Since $b_2(M;l) = 0$ and $w_1(\lambda)^2 = 0$ for a non-trivial $\mathbb{Z}$-bundle l on $M = T^2 \times S^2$ or T^4, we can show that X is nonsmoothable by Theorem 1.1. We can also construct similar examples by using Theorem 1.2.

C. Bohr [4] and Lee-Li [17] proved 10/8-type inequalities for non-spin 4-manifolds with even forms. We also construct nonsmoothable non-spin 4-manifolds with even forms satisfying their inequalities.

Theorem 1.5. *There exist nonsmoothable non-spin 4-manifolds X with even indefinite forms satisfying*

$$b_2(X) \geq \frac{5}{4}|\operatorname{sign}(X)|.$$

Remark 1.6. One of the results of Bohr [4] and Lee-Li [17] is that the inequality $b_2(X) \geq 5/4|\operatorname{sign}(X)|$ holds for non-spin 4-manifolds X with even indefinite forms whose 2-primary torsion part of $H_1(X;\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2^k$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. We consturct our examples so that the 2-primary torsion part of $H_1(X;\mathbb{Z})$ is $\mathbb{Z}/2$.

The organization of the paper is as follows. In Section 2, we prove Theorem 1.4 and Theorem 1.5 assuming Theorem 1.1 and Theorem 1.2. In Section 3, we introduce the notion of Spin^{c-}-structures which is a $\mathrm{Pin}^-(2)$-variant of Spin^c-structures. It is also explained that, if a Spin^{c-}-structure on X is given, then a Spin^c-structure on the double covering $\tilde{X}$ is induced, and the covering transformation of $\tilde{X}$ is covered by anti-linear involutions I on the spinor bundles and the determinant line bundle. In Section 4, we study the Seiberg-Witten theory on $\tilde{X}$ with the I-action, especially, analyze the I-invariant part of the moduli spaces. In Section 5, we prove Theorem 1.1. In Section 6, we introduce $\mathrm{Pin}^-(2)$-monopole equations, and show that the moduli space of solutions of $\mathrm{Pin}^-(2)$-monopole equations can be identified with the I-invariant Seiberg-Witten moduli space on the double covering $\tilde{X}$. The Bauer-Furuta theory[13, 3] of $\mathrm{Pin}^-(2)$-monopole map is also considered. In Section 7, the proof of Theorem 1.2 is given by using the equivariant K-theory as in [13, 5]. We also give an alternative proof of Theorem 1.1 by the same technique.

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2. **Applications**

In this section, we prove Theorem 1.4 and Theorem 1.5 assuming Theorem 1.1 and Theorem 1.2. First, we prove the following. (*Cf.* [11], Corollary 1.1.)

Theorem 2.1. *Let V be any closed oriented topological 4-manifold which satisfies either of the following*:

(1) *the intersection form Q_V on $H^2(V;\mathbb{Z})$ is non-standard definite, or*

(2) *there exists an element* $\alpha \in H^1(V;\mathbb{Z}/2)$ *so that* $\alpha \cup \alpha = w_2(X)$*, and* Q_{V,l_α} *satisfies* $b_+(V;l_\alpha) < -\operatorname{sign}(V)/8$*, where* l_α *is the* $\mathbb{Z}$*-bundle corresponding to* α*. (If* $w_2(X) = 0$*, then* α *may be* 0*.)*

Let M *be a closed oriented 4-manifold which admits a nontrivial* $\mathbb{Z}$*-bundle* $l' \to M$ *such that* $b_2(M;l') = 0$ *and* $w_1(\lambda')^2 = 0$*, where* $\lambda' = l' \otimes \mathbb{R}$*. Then the connected sum* $X = V\#M$ *does not admit any smooth structure.*

Before proving Theorem 2.1, we will discuss how to construct V and M as in the theorem. One can construct simply-connected examples of V satisfying (1) by Freedman's theory[10]. Examples of V satisfying (2) can be constructed as follows. Let $|E_8|$ be the simply-connected topological 4-manifold whose form is $-E_8$. (This can be also constructed by Freedman's theory.) Then $V = m|E_8|\#n(S^2 \times S^2)$ with $m > n$ are spin manifolds satisfying (2) with $\alpha = 0$.

As shown in Hambleton-Kreck's paper[15](Proof of Theorem 3), there exist non-spin rational homology 4-spheres Σ_0 and Σ_1 with $\pi_1 = \mathbb{Z}/2$ and Kirby-Siebenmann obstructions $\operatorname{ks}(\Sigma_0) = 0$ and $\operatorname{ks}(\Sigma_1) \neq 0$. For instance, an Enriques surface is topologically decomposed into $|E_8|\#(S^2 \times S^2)\#\Sigma_1$. Then $V = m|E_8|\#n(S^2 \times S^2)\#\Sigma_i$ with $m > n+1$ are non-spin manifolds satisfying (2) with non-zero class $\alpha \in H^1(V;\mathbb{Z}/2) \cong H^1(\Sigma_i;\mathbb{Z}/2) \cong \mathbb{Z}/2$. Note that $b_+(V;l_\alpha) = b_+(V) + 1$ in this case. In fact, for any $\mathbb{Z}$-bundle l over a manifold X, let $\tilde{X}$ be the double covering corresponding to l, and let $\underline{\lambda} = l \otimes \mathbb{R}$ considered as a bundle with discrete fibers. Then, we have in general,

$$b_0(X) - b_1(X) + b_+(X) = b_0(X;l) - b_1(X;l) + b_+(X;l),$$
$$H^*(\tilde{X};\mathbb{R}) = H^*(X;\mathbb{R}) \oplus H^*(X;\underline{\lambda}).$$

Note also that $\operatorname{ks}(V) = 0$ if and only if $m + i \equiv 0 \bmod 2$.

As examples of M, we can take $M = T^2 \times S^2$ or T^4 or their arbitrary connected sum. In fact, $b_2(M;l') = 0$ and $w_1(\lambda')^2 = 0$ for any nontrivial $\mathbb{Z}$-bundle l' over $M = T^2 \times S^2$ or T^4. When M is a connected sum of several $T^2 \times S^2$ or T^4, take l' which is nontrivial on each $T^2 \times S^2$ or T^4 summand.

Proof of Theorem 2.1. Suppose V satisfies (1) and X is smoothable. Take l' as in the assumption, and let $l \to V\#M$ be the connected sum of a trivial $\mathbb{Z}$-bundle on V and l'. Then, $H^2(X;l) = H^2(V;\mathbb{Z}) \oplus H^2(M;l')$ and $Q_{X,l} = Q_V$. Note that $w_1(\lambda)^2 = w_1(\lambda')^2 = 0$. By Theorem 1.1, $Q_{X,l}$ should be standard. This is a contradiction. If V satisfies (2), then consider $l = l_\alpha\#l'$ and use Theorem 1.2. □

Proof of Theorem 1.4. Let V be any simply-connected 4-manifold with even form Q_V of rank $16k$ which satisfies either of the following:

(1) Q_V is definite, or
(2) $Q_V \cong m(-E_8) \oplus nH$ and $m > n$, where H is the hyperbolic form.

Then, take a connected sum of V with sufficiently many $T^2 \times S^2$'s or T^4's so that the 10/8-inequality is satisfied. By Theorem 2.1, it is nonsmoothable. □

Proof of Theorem 1.5. Let $V = \#m|E_8|\#n(S^2 \times S^2)\#\Sigma_i$ with $m > n+1$, and take a connected sum of V with sufficiently many $T^2 \times S^2$'s or T^4's. □

3. Spin^{c-}-structures

In this section, we introduce a variant of Spin^c-structure, Spin^{c-}*-structure* we call. The notion of Spin^{c-}*-structure* was introduced to the author by M. Furuta, and a large part of this section is due to him.

3(i). Spin^{c-}**-groups.** Let $\mathrm{Pin}^-(2)$ be a subgroup of $\mathrm{Sp}(1)$ generated by $\mathrm{U}(1)$ and j, that is, $\mathrm{Pin}^-(2) = \mathrm{U}(1) \sqcup j\,\mathrm{U}(1)$. There is a two-to-one homomorphism $\varphi_0\colon \mathrm{Pin}^-(2) \to \mathrm{O}(2)$, which sends $z \in \mathrm{U}(1)$ in $\mathrm{Pin}^-(2)$ to $z^2 \in \mathrm{U}(1) \subset \mathrm{O}(2)$, and j to the reflection

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let us define $\mathrm{Spin}^{c-}(n) = \mathrm{Spin}(n) \times_{\{\pm 1\}} \mathrm{Pin}^-(2)$. There is an exact sequence

$$1 \to \{\pm 1\} \to \mathrm{Spin}^{c-}(n) \to \mathrm{SO}(n) \times \mathrm{O}(2) \to 1.$$

3(ii). Spin^{c-}**-structures.** Let X be a n-dimensional oriented smooth manifold. Fix a Riemannian metric on X, and let $F(X)$ be its $\mathrm{SO}(n)$-frame bundle. Suppose an $\mathrm{O}(2)$-bundle E over X is given.

Definition 3.1. A Spin^{c-}-structure on $F(X) \times_X E$ is a lift of the principal $\mathrm{SO}(n) \times \mathrm{O}(2)$-bundle $F(X) \times_X E$ to a principal $\mathrm{Spin}^{c-}(n)$-bundle. This is given by the data (P, ϕ) where P is a $\mathrm{Spin}^{c-}(n)$-bundle and φ is a bundle isomorphism $P/\{\pm 1\} \to F(X) \times_X E$.

Remark 3.2. Let G_0 be the identity component of $\mathrm{Spin}^{c-}(n)$. Then G_0 is isomorphic to $\mathrm{Spin}^c(n)$, and $\tilde{X} = P/G_0 \to X$ is a double covering. Note that the determinant line bundle $\det E$ of E is isomorphic to $\tilde{X} \times_{\{\pm 1\}} \mathbb{R}$, where $\{\pm 1\}$ acts on $\mathbb{R}$ by multiplication.

Proposition 3.3. *There exists a* Spin^{c-}*-structure on* $F(X) \times_X E$ *if and only if* $w_2(TX) = w_2(E) + w_1(E)^2$.

Proof. Note that the image of $\mathrm{Pin}^-(2) \subset \mathrm{Sp}(1) = \mathrm{Spin}(3)$ by the canonical homomorphism $\mathrm{Spin}(3) \to \mathrm{SO}(3)$ is an $\mathrm{O}(2)$ in $\mathrm{SO}(3)$. This embedding $\mathrm{O}(2) \subset \mathrm{SO}(3)$ is given by $A \mapsto A \oplus \det A$. By using this embedding, embed $\mathrm{SO}(n) \times \mathrm{O}(2)$ in $\mathrm{SO}(n+3)$. Let $\varphi\colon \mathrm{Spin}(n+3) \to \mathrm{SO}(n+3)$ be the canonical homomorphism. Then $\mathrm{Spin}^{c-}(n) = \varphi^{-1}(\mathrm{SO}(n) \times \mathrm{O}(2))$. Therefore, $w_2(X) = w_2(E \oplus \det E) = w_2(E) + w_1(E)^2$ is the required condition. □

Remark 3.4. Let $l \to X$ be a $\mathbb{Z}$-bundle over X, and $\lambda = l \otimes \mathbb{R}$. The isomorphism classes of $\mathrm{O}(2)$-bundles E whose determinant line bundles $\det E$ are isomorphic to λ are classified by their twisted first Chern classes $\tilde{c}_1(E) \in H^2(X; l)$. See [11], Proposition 2.2.

We concentrate on the case when $n = 4$ below. Let $\mathbb{H}_T$ be a $\mathrm{Spin}^{c-}(4)$-module which is a copy of $\mathbb{H}$ as a vector space, such that the action of $[q_+, q_-, u] \in \mathrm{Spin}^{c-}(4) = (\mathrm{Sp}(1) \times \mathrm{Sp}(1)) \times_{\{\pm 1\}} \mathrm{Pin}(2)$ on $v \in \mathbb{H}_T$ is given by $q_+ v q_-^{-1}$. Then, the associated bundle $P \times_{\mathrm{Spin}^{c-}(4)} \mathbb{H}_T$ is identified with the tangent bundle TX.

Similarly, let $\varphi\colon \mathrm{Spin}^{c-}(4) \to \mathrm{O}(2)$ be the homomorphism defined from $\varphi_0\colon \mathrm{Pin}^-(2) \to \mathrm{O}(2)$. Then the associated bundle $P \times_\varphi \mathrm{O}(2)$ is identified with E.

Let us consider $\mathrm{Spin}^{c-}(4)$-modules $\mathbb{H}_+$ and $\mathbb{H}_-$ which are copies of $\mathbb{H}$ as vector spaces, such that the action of $[q_+, q_-, u] \in \mathrm{Spin}^{c-}(4)$ on $\phi \in \mathbb{H}_\pm$ is given by $q_\pm \phi u^{-1}$. Then, one can obtain the associated bundles $S^\pm = P \times_{\mathrm{Spin}^{c-}(4)} \mathbb{H}_\pm$. These are *positive* and *negative spinor bundles* for the Spin^{c-}-structure.

The Clifford multiplication $\rho_\mathbb{R}\colon \Omega^1(X) \times \Gamma(S^+) \to \Gamma(S^-)$ is defined via $\mathrm{Spin}^{c-}(4)$-equivariant map $\mathbb{H}_T \times \mathbb{H}_+ \to \mathbb{H}_-$ defined by $(v, \phi) \mapsto \bar{v}\phi$. Later we will need a *twisted complex* version of the Clifford multiplication defined as follows. Let G_0 be the identity component of $\mathrm{Spin}^{c-}(4)$. Then G_0 is isomorphic to $\mathrm{Spin}^c(4)$, and $\mathrm{Spin}^{c-}(4)/G_0 \cong \mathbb{Z}/2$. Let $\varepsilon\colon \mathrm{Spin}^{c-}(4) \to \mathrm{Spin}^{c-}(4)/G_0$ be the projection, and let $\mathrm{Spin}^{c-}(4)/G_0 \cong \mathbb{Z}/2$ act on $\mathbb{C}$ by complex conjugation. Then $\mathrm{Spin}^{c-}(4)$ acts on $\mathbb{C}$ via ε and complex conjugation. Define $\rho_0\colon \mathbb{H}_T \otimes_\mathbb{R} \mathbb{C} \times \mathbb{H}_+ \to \mathbb{H}_-$ by $\rho_0(v \otimes a, \phi) = \bar{v}\phi\bar{a}$. This ρ_0 is $\mathrm{Spin}^{c-}(4)$-equivariant. Let us define the bundle K over X by $K = \tilde{X} \times_{\{\pm 1\}} \mathbb{C}$ where $\{\pm 1\}$ acts on $\mathbb{C}$ by complex conjugation. Then we can define via ρ_0 the Clifford multiplication

$$\rho\colon \Omega^1(X; K) \times \Gamma(S^+) \to \Gamma(S^-). \tag{3.5}$$

Note that $K = \underline{\mathbb{R}} \oplus i\lambda$, where $\underline{\mathbb{R}}$ is a trivial $\mathbb{R}$-bundle. By restricting ρ to $\underline{\mathbb{R}}$, $\rho_\mathbb{R}$ is recovered. By restricting ρ to $i\lambda$, we obtain,

$$\rho\colon \Omega^1(X; i\lambda) \times \Gamma(S^+) \to \Gamma(S^-).$$

3(iii). **Double coverings.** In this subsection, we write $\mathrm{Spin}^{c-}(4)$ as G. Note that G has two connected components G_0 and G_1, and the identity component G_0 is $\mathrm{Spin}^c(4)$. If a Spin^{c-}-structure (P, ϕ) on a 4-manifold X is given, then $\tilde{X} = P/G_0$ gives a double covering $\pi\colon \tilde{X} \to X$. Then, we have a G_0-bundle $P \to P/G_0 = \tilde{X}$. The pull-back bundle π^*E has a $\mathrm{SO}(2)$-reduction L, and a bundle isomorphism $\tilde{\phi}\colon P/\{\pm 1\} \to F(\tilde{X}) \times_{\tilde{X}} L$ is induced from ϕ, where $F(\tilde{X}) = \pi^* F(X)$, which can be considered as the frame bundle over $\tilde{X}$ for the pull-back metric. The G_0-bundle P over $\tilde{X}$ and $\tilde{\phi}$ define an ordinary Spin^c-structure c on $\tilde{X}$.

Let $\iota\colon \tilde{X} \to \tilde{X}$ be the covering transformation, and J be $[1, 1, j] \in G = (\mathrm{Sp}(1) \times \mathrm{Sp}(1)) \times_{\{\pm 1\}} \mathrm{Pin}^-(2)$. Then the right J-action on $P \to \tilde{X}$ covers ι. Although the J-action is not a G_0-bundle automorphism of $P \to \tilde{X}$, it can be considered as a kind of automorphism of $P \to \tilde{X}$ which reverses the complex structure as follows. (This can be considered as a natural isomorphism between the pull-back bundle $\iota^* c$ and the complex conjugation of c.)

Let us consider the pull-back G-bundle $\pi^* P \to \tilde{X}$. Then

$$\pi^* P = P \times_G (G/G_0 \times G) = P \times_G (\{\pm 1\} \times G) = P \times_{G_0} G.$$

The bundle $P \times_{G_0} G$ has two components: $P \times_{G_0} G = P_0 \sqcup P_1$, where $P_i = P \times_{G_0} G_i$ for $i = 0, 1$. Note that $P_0 = P \times_{G_0} G_0 = P$ over $\tilde{X}$. Then $\iota\colon \tilde{X} \to \tilde{X}$ has a natural lift $\tilde{\iota}\colon P \times_{G_0} G \to P \times_{G_0} G$ given by $\tilde{\iota}([p, g]) = [pJ, J^{-1}g]$. Note that $\tilde{\iota}$ exchanges the components P_0 and P_1.

On the other hand, we have a natural isomorphism of the fiber bundles $\alpha\colon P_0 \to P_1$ given by $\alpha([p,g]) = [p, gJ]$. Then we have an automorphism I_0 of P_0 given by $I_0 = \alpha \circ \tilde{\iota}$ which reverses the complex structure. Note that I_0 has order 4.

The J-action also induces antilinear automorphisms, denoted by I, on the spinor bundles $S^{\pm} = P \times_{G_0} \mathbb{H}_{\pm}$ given by $I([p,\phi]) = [pJ, J^{-1}\cdot\phi] = [pJ, \phi j]$. Since $J^2 \in G_0$, $I^2([p,\phi]) = [pJ^2, J^{-2}\cdot\phi] = [p,\phi]$. Therefore I is an antilinear involution on each of spinor bundles.

Similarly, the J-action induces an antilinear involution of the determinant line bundle, also denoted by I. This can be seen from the construction above, or noticing the following. Note that $\lambda = \tilde{X} \times_{\{\pm 1\}} \mathbb{R} \to X$ is isomorphic to the determinant $\mathbb{R}$-bundle of E. Let $E_0 \to X$ be the $\mathbb{R}^2$-bundle associated to E. Then, the determinant $\mathbb{C}$-bundle L_0 of c can be identified with the pull-back $\pi^* E_0$ as *real* vector bundles, and the involution ι lifts to $L_0 \cong \pi^* E_0$ as an antilinear bundle automorphism.

Remark 3.6. By using $\mathrm{Pin}^+(2)$ (instead of $\mathrm{Pin}^-(2)$), which is isomorphic to $\mathrm{O}(2)$, but considered as a double covering of $\mathrm{O}(2)$, we can define analogous objects, Spin^{c+}*-structures*. In this case also, one can construct a Spin^c-structure c associated to it on a double covering $\tilde{X}$ of X. But the covering transformation ι lifts on the spinor bundles as a $\mathbb{Z}/4$-action.

4. I-invariant Seiberg-Witten moduli spaces

In the previous section, we introduced the notion of Spin^{c-}-structures, and saw that an involution I is induced on the spinor bundles and the determinant line bundle of the associated Spin^c-structure c on the double covering $\tilde{X}$. In this section, we study the Seiberg-Witten theory on $(\tilde{X}, c)$ with the I-action. In particular, we will analyze the I-invariant part of the moduli space.

Remark 4.1. As mentioned before, we will later introduce $\mathrm{Pin}^-(2)$-monopole equations on Spin^{c-}-structures, and see that the I-invariant Seiberg-Witten moduli space of $(\tilde{X}, c)$ can be identified with the moduli space of $\mathrm{Pin}^-(2)$-monopole solutions.

4(i). **I-invariant Seiberg-Witten moduli spaces.** Let X be a 4-manifold satisfying the assumptions of Theorem 1.1, and $E \to X$ be an $\mathrm{O}(2)$-bundle satisfying $w_2(E) + w_1(E)^2 = w_2(X)$ and $\det E = \lambda$. Then, by Proposition 3.3, there is a Spin^{c-}-structure whose associated $\mathrm{O}(2)$-bundle is E. Furthermore, we have a Spin^c-structure c on the double covering $\tilde{X}$ with the I-action as above. Let $\tilde{S}^+$ be the positive spinor bundle of c, and $\mathcal{A}$ be the space of $\mathrm{U}(1)$-connections on $L = \det \tilde{S}^+$. In this situation, the I-action on $\mathcal{C} := \mathcal{A} \times \Gamma(\tilde{S}^+)$ is induced from the I-action on $\tilde{S}^{\pm}$ and L, and the Seiberg-Witten equations on c is I-equivariant equations for $(A, \Phi) \in \mathcal{C} = \mathcal{A} \times \Gamma(\tilde{S}^+)$. The gauge transformation group $\mathcal{G} = \mathrm{Map}(\tilde{X}, \tilde{S}^1)$ acts on $\mathcal{C}$, and the I-action on $\mathcal{G}$ is given by $Iu = \overline{\iota^* u}$ for $u \in \mathcal{G}$, where $\bar{\cdot}$ means the complex conjugation. (Strictly speaking, we need to take completions of $\mathcal{C}$ and $\mathcal{G}$ by suitable Sobolev norms, e.g., L^2_k-norms for sufficiently large k. Since these things are already standard, the detail is omitted. See [19].)

Now, we concentrate on the I-invariant part of the whole theory. Below, for any object Z on which I acts, the fixed point set is denoted as Z^I. Let $\mathcal{B}^I = \mathcal{C}^I/\mathcal{G}^I$. Note that $\mathcal{G}^I$ can be identified with the space of sections of the fiber bundle over X defined by $\tilde{X} \times_{\{\pm 1\}} S^1 \to X$, where the $\{\pm 1\}$-action on $S^1 = \mathrm{U}(1)$ is given by complex conjugation. Let $\mathcal{M}^I \subset \mathcal{B}^I$ be the I-invariant moduli space which is defined as the space of I-invariant solutions to the Seiberg-Witten equations modulo $\mathcal{G}^I$. The reducible solutions are the solutions of the form $(A, 0)$, and their stabilizers are $\{\pm 1\}$, because the constant maps in $\mathcal{G}^I$ are only ± 1. As noted in Remark 3.4 of [20] or [12], $\mathcal{B}^I$ is embedded continuously in $\mathcal{B}$, and therefore $\mathcal{M}^I$ is a closed subspace of the ordinary Seiberg-Witten moduli space. Hence, $\mathcal{M}^I$ is a compact space.

4(ii). **The deformation complex.** When (A, Φ) is an I-invariant solution, the deformation complex for $\mathcal{M}^I$ at (A, Φ) is given by the restriction of the ordinary deformation complex to its I-invariant part:

$$0 \to \Omega^0(\tilde{X}; i\mathbb{R})^I \to (\Omega^1(\tilde{X}; i\mathbb{R}) \oplus \Gamma(\tilde{S}^+))^I \to (\Omega^+(\tilde{X}; i\mathbb{R}) \oplus \Gamma(\tilde{S}^-))^I \to 0, \tag{4.2}$$

where the I-action on forms is given by the composition of the pullback by ι and the complex conjugation. For calculation of the index of (4.2), 0-th order terms can be neglected, and therefore, the complex (4.2) can be assumed to be a direct sum of the de Rham part and the Dirac part. (See [19], 4.6.) The de Rham part can be identified with the twisted de Rham complex:

$$0 \to \Omega^0(X; i\lambda) \xrightarrow{d} \Omega^1(X; i\lambda) \xrightarrow{d^+} \Omega^+(X; i\lambda) \to 0.$$

The index of the Dirac part is calculated by the Lefschetz formula. More precisely, since the I-action is not complex linear, complexify the operator first, and then apply the Lefschetz formula[2]. Then the index of the Dirac part above is half of the index of the Dirac operator associated to A because the ι-action on $\tilde{X}$ is free. Thus we have,

Proposition 4.3. *The virtual dimension d of $\mathcal{M}^I$ is given by*

$$d = \frac{1}{4}(\tilde{c}_1(E)^2 - \mathrm{sign}(X)) - (b_0(X; l) - b_1(X; l) + b_+(X; l)), \tag{4.4}$$

where $\tilde{c}_1(E) \in H^2(X; l)$ is the twisted Chern class in [11].

Remark 4.5. Note that $b_0(X; l) = 0$ if X is connected and l is nontrivial. The class $\tilde{c}_1(E)$ is the Euler class of E considered in $H^2(X; l)$.

4(iii). **The structure of $\mathcal{M}^I$.** Suppose $b_1(X; l) = 0$ and $b_+(X; l) = 0$. Then, there is a unique reducible class ρ_0 in $\mathcal{M}^I$. Suppose further that $d > 0$. By perturbing the Seiberg-Witten equations by adding an I-invariant self-dual 2-form to the equation for curvature, and if necessary, perturbing the Dirac operator near ρ_0, the moduli space $\mathcal{M}^I$ become a d-dimensional manifold outside ρ_0. Let us fix a small neighborhood $N(\rho_0)$ of ρ_0, and $\overline{\mathcal{M}^I}$ be the closure of $\mathcal{M}^I \setminus N(\rho_0)$. Then, $\overline{\mathcal{M}^I}$ is a compact d-manifold whose boundary is a real projective space $\mathbb{RP}^{d-1}$.

If $b_1(X; l) > 0$ and $b_+(X; l) = 0$, then the space of I-invariant reducible classes forms a $b_1(X; l)$-dimensional torus $T^{b_1(X;l)}$. Let $\overline{\mathcal{M}^I}$ be the closure of the complement of a small neighborhood of the reducible torus $T^{b_1(X;l)}$. If $d' = d - b_1(X;l) > 0$, then, after perturbation, $\overline{\mathcal{M}^I}$ is a compact d-manifold whose boundary is a fiber bundle over $T^{b_1(X;l)}$ with fiber $\mathbb{RP}^{d'-1}$. We will show that the assumption $d' > 0$ causes a contradiction.

Remark 4.6. The manifold $\overline{\mathcal{M}^I}$ could be non-orientable.

4(iv). **The topology of $\mathcal{B}^I$.** We will evaluate the fundamental class of the boundary of $\overline{\mathcal{M}^I}$ by a cohomology class of $\mathcal{B}^I$. Let $(\mathcal{B}^*)^I$ be the space of I-invariant irreducibles, i.e., $(\mathcal{B}^*)^I = \left(\mathcal{A} \times (\Gamma(\tilde{S}^+) \setminus 0)\right)^I / \mathcal{G}^I$.

Proposition 4.7. *The space $(\mathcal{B}^*)^I$ has the same homotopy type with $\mathbb{RP}^\infty \times T^{b_1(X;\lambda)}$.*

Before proving Proposition 4.7, we show the following lemma which will be a key point of our argument.

Lemma 4.8. *If $b_+(X;l) = 0$, then $d' = d - b_1(X;l) \leq 0$.*

Proof. As explained above, if $d' > 0$, then $\overline{\mathcal{M}^I}$ is a compact d-manifold whose boundary is a fiber bundle over $T^{b_1(X;l)}$ with fiber $\mathbb{RP}^{d'-1}$. Note that $(\mathcal{B}^*)^I$ can be considered as a fiber bundle over $T^{b_1(X;\lambda)}$. If we cut down the moduli $\overline{\mathcal{M}^I}$ by a generic fiber F_b over $b \in T^{b_1(X;\lambda)}$, then the cut-down moduli space $(\overline{\mathcal{M}^I})_b = \overline{\mathcal{M}^I} \cap F_b$ is a compact d'-dimensional manifold whose boundary is an $\mathbb{RP}^{d'-1}$. Then there exists a cohomology class $A \in H^{d'-1}(F_b;\mathbb{Z}/2) = H^{d'-1}(\mathbb{RP}^\infty;\mathbb{Z}/2)$ so that $\langle A, [\partial(\overline{\mathcal{M}^I})_b]\rangle \neq 0$. This is a contradiction. $\square$

Let us begin the proof of Proposition 4.7. The proof is divided into several steps.

Lemma 4.9. *The space $(\mathcal{C}^*)^I = \left(\mathcal{A} \times (\Gamma(\tilde{S}^+) \setminus 0)\right)^I$ is contractible.*

Proof. Note that $(\mathcal{C}^*)^I = \mathcal{C}^I \setminus (\mathcal{A} \times 0)^I$. Since the I-action on $\mathcal{C}$ is linear, $(\mathcal{C}^*)^I$ is the complement of a linear subspace with infinite codimension. Therefore $(\mathcal{C}^*)^I$ has the homotopy type of an infinite dimensional sphere, and is contractible. $\square$

Since $\mathcal{G}^I$ acts on $(\mathcal{C}^*)^I$ freely, Lemma 4.9 implies that $(\mathcal{B}^*)^I$ has the homotopy type of the classifying space $B\mathcal{G}^I$. Hence, Proposition 4.7 follows from the next lemma.

Lemma 4.10. $\mathcal{G}^I \simeq (\mathbb{Z}/2) \times \mathbb{Z}^{b_1(X,l)}$.

Proof. We have the following split exact sequence:

$$1 \to \mathcal{G}_0 \to \mathcal{G} \xrightarrow{h} [\tilde{X}, S^1] \cong H^1(\tilde{X};\mathbb{Z}) \to 0, \tag{4.11}$$

where $\mathcal{G}_0$ is the identity component of $\mathcal{G}$, and h sends $u \in \mathcal{G}$ to its homotopy class. Since $\mathcal{G}_0$ has the homotopy type of S^1, we see $\mathcal{G} \simeq S^1 \times \mathbb{Z}^{b_1(\tilde{X})}$. By restricting (4.11) to $\mathcal{G}^I$, we have,

$$1 \to \mathcal{G}_0 \cap \mathcal{G}^I \to \mathcal{G}^I \to h(\mathcal{G}^I) \to 1.$$

Then the image $h(\mathcal{G}^I)$ can be identified with $\pi^* H^1(X;l) \cong \mathbb{Z}^{b_1(X;l)}$. Therefore, it suffices to prove that $\mathcal{G}_0 \cap \mathcal{G}^I \simeq \mathbb{Z}/2$ via the homotopy *in* $\mathcal{G}^I$. Let us introduce another sequence:

$$1 \to \mathrm{Map}(\tilde{X}, \mathbb{Z}) \to \mathrm{Map}(\tilde{X}, \mathbb{R}) \xrightarrow{e} \mathrm{Map}(\tilde{X}, \mathbb{R}/\mathbb{Z}),$$

where e is defined by $e(f) = \exp(2\pi\sqrt{-1}f)$ for $f\colon \tilde{X} \to \mathbb{R}$. Then the image of e is just $\mathcal{G}_0$. Every I-invariant element $u \in \mathcal{G}_0 \cap \mathcal{G}^I$ is represented by f in $\mathrm{Map}(\tilde{X};\mathbb{R})$ satisfying $\iota^* f = m - f$ for some integer m. Since I acts on $\mathrm{Map}(\tilde{X},\mathbb{Z})$ trivially, we may assume $u \in \mathcal{G}_0 \cap \mathcal{G}^I$ is represented by $f \in \mathrm{Map}(\tilde{X};\mathbb{R})$ such that $\iota^* f = -f$ or $\iota^* f = 1 - f$. If $\iota^* f = -f$, then f is deformed I-equivariantly to the constant map 0. If $\iota^* f = 1 - f$, then f is deformed I-equivariantly to the constant map 1/2. Thus, $\mathcal{G}_0 \cap \mathcal{G}^I \simeq \{\pm 1\}$. □

5. Proof of Theorem 1.1

In this section, we complete the proof of Theorem 1.1. Let X be a closed oriented smooth 4-manifold, and $l \to X$ be a $\mathbb{Z}$-bundle over X. The short exact sequence of bundles,

$$0 \to l \xrightarrow{\cdot 2} l \to \mathbb{Z}/2 \to 0,$$

induces a long exact sequence,

$$\cdots \to H^q(X;l) \xrightarrow{\cdot 2} H^q(X;l) \to H^q(X;\mathbb{Z}/2) \to H^{q+1}(X;l) \to \cdots .$$

Lemma 5.1. *The second Stiefel-Whitney class $w_2(X)$ has a lift in $H^2(X;l)$. Moreover, if $c \in H^2(X;l)$ is an characteristic element of $Q_{X,l}$, there exists a torsion class $\delta \in H^2(X;l)$ such that $c + \delta$ is a lift of $w_2(X)$.*

Remark 5.2. An element w in a lattice L is called characteristic if $w \cdot v \equiv v \cdot v \bmod 2$ for any $v \in L$.

Proof of Lemma 5.1. Note that $l^* = l$ and $\mathrm{Hom}(l;\mathbb{Z}/2) = \mathbb{Z}/2$. By the universal coefficient theorem, we have a commutative diagram,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}(H_1(X;l),\mathbb{Z}) & \longrightarrow & H^2(X;l) & \xrightarrow{h_1} & \mathrm{Hom}(H_2(X;l),\mathbb{Z}) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle \rho_1} & & \big\downarrow{\scriptstyle \rho_2} & & \\
0 & \longrightarrow & \mathrm{Ext}(H_1(X;l),\mathbb{Z}/2) & \longrightarrow & H^2(X;\mathbb{Z}/2) & \xrightarrow{h_1} & \mathrm{Hom}(H_2(X;l),\mathbb{Z}/2) & \longrightarrow & 0.
\end{array}$$

As in [1], we can see that $w_2(X)$ has a lift in $H^2(X;l)$ by using Wu's formula. The second statement is also proved by using the diagram. □

Theorem 5.3. *Let X be a closed, connected, oriented smooth 4-manifold. Suppose we have a nontrivial $\mathbb{Z}$-bundle $l \to X$ satisfying $b_+(X;l) = 0$. Let $\lambda = l \otimes \mathbb{R}$. Then, for every cohomology class $C \in H^2(X;l)$ which satisfies $[C]_2 + w_1(\lambda)^2 = w_2(X)$, where $[C]_2$ is the mod 2 reduction of C, the inequality $|C^2| \geq b_2(X;l)$ holds.*

Proof. For $C \in H^2(X;l)$ satisfying the assumption, there is a Spin^{c-}-structure on X whose $\mathrm{O}(2)$-bundle E has $\tilde{c}_1(E) = C$ by Proposition 3.3. Then, we can consider the I-invariant moduli $\mathcal{M}^I$ on the double covering $\tilde{X}$ and the Spin^c-structure corresponding to the Spin^{c-}-structure above. If $b_+(X;l) = 0$, then $C^2 \leq 0$ and $\mathrm{sign}(X) = -b_2(X;l)$, and Lemma 4.8 implies that $d' = 1/4(C^2 - \mathrm{sign}(X)) = 1/4(C^2 + b_2(X;l)) \leq 0$. Thus, $|C^2| \geq b_2(X;l)$ holds. □

To complete the proof of Theorem 1.1, we invoke the following theorem due to Elkies.

Theorem 5.4 (Elkies[9]). *Let L be a lattice over $\mathbb{Z}$. If every characteristic element $w \in L$ satisfies $|w^2| \geq \operatorname{rank} L$, then L is isomorphic to the standard form.*

Proof of Theorem 1.1. We can assume that $b_+(X;l) = 0$ by reversing the orientation if necessary. Under the assumptions of Theorem 1.1, Wu's formula, Lemma 5.1 and Theorem 5.3 imply that every characteristic element C of $Q_{X,l}$ satisfies $|C^2| \geq \operatorname{rank} Q_{X,l}$. Then, by Elkies' theorem, $Q_{X,l}$ should be the standard form. □

6. $\mathrm{Pin}^-(2)$-monopole equations

In this section, we introduce $\mathrm{Pin}^-(2)$-monopole equations, and develop the $\mathrm{Pin}^-(2)$-monopole gauge theory. The whole story is almost parallel to the ordinary Seiberg-Witten case.

6(i). $\mathrm{Pin}^-(2)$**-monopole equations.** Let X be a closed oriented smooth 4-manifold, E be a $\mathrm{O}(2)$-bundle over X, and $\lambda = \det E$. We suppose λ is a nontrivial bundle throughout the rest of the paper. Fix a Riemannian metric on X. Suppose a Spin^{c-}-structure (P, φ) on (X, E) is given. If an $\mathrm{O}(2)$-connection A on E is given, then A and the Levi-Civita connection induces a $\mathrm{Spin}^{c-}(4)$-connection on P, and we can define the Dirac operator via the Clifford multiplication ρ of (3.5) as

$$D_A \colon \Gamma(S^+) \to \Gamma(S^-).$$

If A' is another $\mathrm{O}(2)$-connection on E, then $a = A - A'$ is in $\Omega^1(X; i\lambda)$, and the relation of Dirac operators of A and $A' = A + a$ is given via ρ by

$$D_{A+a}\phi = D_A\phi + \rho(a)\phi.$$

The curvature F_A of A is an element of $\Omega^2(X; i\lambda)$. The space of $i\lambda$-valued self-dual forms, $\Omega^+(X; i\lambda)$, is also associated to the $\mathrm{Spin}^{c-}(4)$-bundle P as follows. Let $\varepsilon\colon \mathrm{Pin}^-(2) \to \mathrm{Pin}^-(2)/\mathrm{U}(1) \cong \{\pm 1\}$ be the projection, and let $\mathrm{Spin}^{c-}(4)$ act on $\operatorname{im}\mathbb{H}$ by $v \in \operatorname{im}\mathbb{H} \to \varepsilon(u) q_+ v q_+^{-1}$ for $[q_+, q_-, u] \in \mathrm{Spin}^{c-}(4)$. Then the space of sections of the associated bundle $P \times_{\mathrm{Spin}^{c-}(4)} \operatorname{im}\mathbb{H}$ is isomorphic to $\Omega^+(X; i\lambda)$. For $\phi \in \mathbb{H}_+$, $\phi i \bar\phi \in \operatorname{im}\mathbb{H}$, and $\mathrm{Spin}^{c-}(4)$ acts on it similarly. Thus, one can define a quadratic map

$$q\colon \Gamma(S^+) \to \Omega^+(X; i\lambda).$$

Let $\mathcal{A}(E)$ be the space of $\mathrm{O}(2)$-connections on E. Then $\mathrm{Pin}^-(2)$-monopole equations for $(A, \phi) \in \mathcal{A}(E) \times \Gamma(S^+)$ are defined by

$$\text{(6.1)} \qquad \begin{cases} D_A\phi = 0, \\ F_A^+ = q(\phi), \end{cases}$$

where F_A^+ is the self-dual part of the curvature F_A.

As in the case of the ordinary Seiberg-Witten equations, it is convenient to work in Sobolev spaces. Fix $k \geq 2$, and take L^2_k-completion of $\mathcal{A}(E) \times \Gamma(S^+)$. The $\mathrm{Pin}^-(2)$-monopole equations (6.1) are assumed as equations for L^2_k-connections/spinors.

6(ii). **Gauge transformations.** The gauge transformation group $\mathcal{G}$ is defined as the space of $\mathrm{Spin}^{c-}(4)$-equivariant diffeomorphisms of P covering the identity map of $P/\mathrm{Pin}^-(2)$. Then, $\mathcal{G}$ can be identified with $\Gamma(P\times_{\mathrm{ad}}\mathrm{Pin}^-(2))$, where ad means the adjoint representation on $\mathrm{Pin}^-(2)$ by the $\mathrm{Pin}^-(2)$-component of $\mathrm{Spin}^{c-}(4)$. Note that $\mathrm{Lie}\,\mathcal{G}\cong\Gamma(P\times_{\mathrm{ad}}\mathbb{R}i)\cong\Omega^0(X;i\lambda)$. We take L^2_{k+1}-completion of $\mathcal{G}$.

Let us look at $\mathcal{G}$ more closely. Recall that $\mathrm{Pin}^-(2)=\mathrm{U}(1)\cup j\,\mathrm{U}(1)$. For $u,z\in\mathrm{U}(1)$, note that $\mathrm{ad}_z(u)=zu\bar{z}=u$, $\mathrm{ad}_{jz}(u)=jzu\bar{z}(-j)=\bar{u}$, $\mathrm{ad}_z(ju)=z^2ju$ and $\mathrm{ad}_{jz}(ju)=\bar{z}^2j\bar{u}$. Then $\mathcal{G}$ can be decomposed into $\mathcal{G}=\mathcal{G}_0\cup\mathcal{G}_1$, where $\mathcal{G}_0=\Gamma(P\times_{\mathrm{ad}}\mathrm{U}(1))$ and $\mathcal{G}_1=\Gamma(P\times_{\mathrm{ad}}j\,\mathrm{U}(1))$. Note that $\mathcal{G}_0\cong\Gamma(\tilde{X}\times_{\{\pm1\}}\mathrm{U}(1))$, where $\{\pm1\}$ acts on $\mathrm{U}(1)$ by complex conjugation. For $\mathcal{G}_1$, the following holds:

Proposition 6.2. *$\mathcal{G}_1=\emptyset$ if and only if $\tilde{c}_1(E)\neq 0$.*

Proof. Note that $P\times_{\mathrm{ad}}j\,\mathrm{U}(1)$ is isomorphic to the bundle of unit vectors of E, and $\tilde{c}_1(E)=0$ if and only if E is isomorphic to $\underline{\mathbb{R}}\oplus i\lambda$. (Recall Remark 3.4.) $\square$

6(iii). **Moduli spaces.** The moduli space $\mathcal{M}$ of $\mathrm{Pin}^-(2)$-monopoles is defined as the space of solutions to (6.1) divided by $\mathcal{G}$.

Proposition 6.3. *The moduli space $\mathcal{M}$ is compact.*

For Dirac operators of Spin^{c-}-structures, one can readily prove the Weitzenböck formula (see [19]),

$$D_A^2\phi=\nabla_A^*\nabla_A\phi+\frac{\kappa}{4}\phi+\frac{\rho(F_A)}{2}\phi, \tag{6.4}$$

where κ is the scalar curvature of the metric on X. With this understood, the proof of Proposition 6.3 is almost parallel to the case of the ordinary Seiberg-Witten theory. The compactness of $\mathcal{M}$ can be seen also from the relation with the Seiberg-Witten theory on the double covering as in the next subsection.

6(iv). **The relation with the Seiberg-Witten theory on the double covering.** Let $\mathcal{A}(E)$ be the space of $\mathrm{O}(2)$-connections on E. As explained in §3(iii), for a Spin^{c-}-structure on (X,E), it is induced a Spin^c-structure c on the double covering $\tilde{X}$ associated to $\lambda=\det E$. Let $\pi\colon\tilde{X}\to X$ be the projection. Let $\tilde{S}^\pm$ be the spinor bundles of c, L be the determinant line bundle of c, and $\mathcal{A}(L)$ be the space of $\mathrm{U}(1)$-connections on L. Then, by construction, we can see that $\pi^*S^\pm\cong\tilde{S}^\pm$, and

$$\Gamma(S^\pm)\cong\Gamma(\tilde{S}^\pm)^I.$$

The relation of $\mathcal{A}(E)$ and $\mathcal{A}(L)$ is given as follows. For an $\mathrm{O}(2)$-connection A on E, A with the Levi-Civita connection determines a $\mathrm{Spin}^{c-}(4)$-connection $\mathbb{A}$ on P. Let us consider the pull-back $\mathrm{Spin}^{c-}(4)$-connection $\pi^*\mathbb{A}$ on $\pi^*P\to\tilde{X}$. Since $\pi^*P=P_0\cup P_1$ (see §3(iii)), the $\mathrm{Spin}^{c-}(4)$-connection $\pi^*\mathbb{A}$ has a $\mathrm{Spin}^c(4)$-reduction $\tilde{\mathbb{A}}$ on the $\mathrm{Spin}^c(4)$-bundle P_0. Then we obtain a $\mathrm{U}(1)$-connection $\tilde{A}$ on L from $\tilde{\mathbb{A}}$, and we can see that

$$\mathcal{A}(E)\cong\mathcal{A}(L)^I.$$

If we write $\tilde{\mathcal{G}} = \mathrm{Map}(\tilde{X}, S^1)$, then $\tilde{\mathcal{G}}^I \cong \Gamma(\tilde{X} \times_{\{\pm 1\}} \mathrm{U}(1)) \cong \mathcal{G}_0$ (see §4(i)). Let $\mathcal{C} = \mathcal{A}(E) \times \Gamma(S^+)$ and $\tilde{\mathcal{C}} = \mathcal{A}(L) \times \Gamma(\tilde{S}^+)$. Then we have

Proposition 6.5. $\mathcal{C}/\mathcal{G}_0 \cong \tilde{\mathcal{C}}^I/\tilde{\mathcal{G}}^I$.

Furthermore, it can be seen that the I-invariant Seiberg-Witten moduli space $\tilde{\mathcal{M}}^I$ of $(\tilde{X}, c)$ can be identified with the $\mathrm{Pin}^-(2)$-monopole moduli space $\mathcal{M}$:

Proposition 6.6. *If $\tilde{c}_1(E) \neq 0$, then $\mathcal{M} \cong \tilde{\mathcal{M}}^I$. If $\tilde{c}_1(E) = 0$, let $\mathcal{M}'$ be the space of* $\mathrm{Pin}^-(2)$*-monopole solutions divided by $\mathcal{G}_0$. Then $\mathcal{M}' \cong \tilde{\mathcal{M}}^I$.*

Since the ordinary Seiberg-Witten moduli space is compact, the compactness of $\mathcal{M}$ (Proposition 6.3) follows from Proposition 6.6, too. Proposition 6.6 also implies the following,

Proposition 6.7. *The virtual dimension d of the* $\mathrm{Pin}^-(2)$*-monopole moduli space $\mathcal{M}$ is given by* (4.4).

Of course, Proposition 6.7 can be also proved by the index theorem.

6(v). **The $\mathrm{Pin}^-(2)$-monopole map.** Let $\mathcal{A} = \mathcal{A}(E)$. The $\mathrm{Pin}^-(2)$-monopole map $\tilde{\mu}$ is defined as follows (*Cf.* [3], p.11):

$$\tilde{\mu}\colon \mathcal{A} \times (\Gamma(S^+) \oplus \Omega^1(X; i\lambda)) \to \mathcal{A} \times (\Gamma(S^-) \oplus \Omega^+(X; i\lambda) \oplus \Omega^0(X; i\lambda) \oplus H^1(X; i\lambda),$$
$$(A, \phi, a) \mapsto (A, D_{A+a}\phi, F_A^+ + d^+a - q(\phi), d^*a, a_{harm}),$$

where a_{harm} is the harmonic part of a. When $\tilde{c}_1(E) \neq 0$, let $\mathcal{G} = \mathcal{G}_0$ act trivially on forms. When $\tilde{c}_1(E) = 0$, let $\mathcal{G}$ act on forms by multiplication of ± 1 via the projection $\mathcal{G} \to \mathcal{G}/\mathcal{G}_0 \cong \{\pm 1\}$. Then the monopole map $\tilde{\mu}$ is $\mathcal{G}$-equivariant.

Choose a base point $*$ on X, and let $\mathcal{G}_e \subset \mathcal{G}_0$ be the based gauge group consisting of $u \in \mathcal{G}$ so that $u(*) = 1$. Let us choose a reference connection A. The subspace $A + \ker d \subset \mathcal{A}$ is preserved by the action of $\mathcal{G}_e$, and the $\mathcal{G}_e$-action is free. The quotient space is isomorphic to the "λ-coefficient Picard torus" $T^{b_1(X;l)} = H^1(X;\lambda)/H^1(X;l)$. Let $\mathcal{V}$ and $\mathcal{W}$ be the quotient spaces,

$$\mathcal{V} = (A + \ker d) \times (\Gamma(S^+) \oplus \Omega^1(X; i\lambda))/\mathcal{G}_e,$$
$$\mathcal{W} = (A + \ker d) \times (\Gamma(S^-) \oplus \Omega^+(X; i\lambda) \oplus \Omega^0(X; i\lambda) \oplus H^1(X; i\lambda))/\mathcal{G}_e.$$

Then $\mathcal{V}$ and $\mathcal{W}$ are bundles over $T^{b_1(X;l)}$. Dividing $\tilde{\mu}$ by $\mathcal{G}_e$, we obtain a fiber preserving map

$$\mu = \tilde{\mu}/\mathcal{G}_e \colon \mathcal{V} \to \mathcal{W}.$$

Constant gauge transformations $\{\pm 1\} \subset \mathcal{G}_0$ still act on $\mathcal{V}$ and $\mathcal{W}$, and μ is a $\mathbb{Z}/2$-equivariant map in general. If $\tilde{c}_1(E) = 0$, then fix an isomorphism $E \cong \underline{\mathbb{R}} \oplus \lambda$, and take a connection on E which is the direct sum of the trivial flat connection on $\underline{\mathbb{R}}$ and a connection on $i\lambda$ as a reference connection A. Then μ is a $\mathbb{Z}/4$-equivariant map.

For a fixed $k > 4$, we take the fiberwise L^p_k-completion of $\mathcal{V}$ and the fiberwise L^p_{k-1}-completion of $\mathcal{W}$. Then we can prove the map μ is a Fredholm proper map as in [3]. In fact, we can readily prove the following by using the Weitzenböck formula (6.4).

Proposition 6.8 ([3])**.** *Preimages $\mu^{-1}(B) \subset \mathcal{V}$ of bounded disk bundles $B \subset \mathcal{W}$ are contained in bounded disk bundles.*

With this understood, we can construct a finite dimensional approximation $f\colon V \to W$ of μ between some finite rank vector bundles over $T^{b_1(X;l)}$ as in [3]. The map f is also a $\mathbb{Z}/2$(or $\mathbb{Z}/4$)-equivariant proper map.

Remark 6.9. We can further develop $\mathrm{Pin}^-(2)$-monopole gauge theory. Many things in the Seiberg-Witten theory could also be considered in the $\mathrm{Pin}^-(2)$-monopole theory. Especially, we can define $\mathrm{Pin}^-(2)$-monopole invariants and their cohomotopy refinements. It would also be interesting to consider gluing formulas, Floer theory, and so on. All of these issues are left to future researches.

7. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 by using equivariant K-theory as in Bryan's paper[5]. We also give an alternative proof of Theorem 1.1 by the same technique.

7(i). **Equivariant K-theory.** We review several facts on equivariant K-theory, especially, the equivariant Thom isomorphism and tom Dieck's character formula for the K-theoretic degree. We refer to the readers §3.3 of [5] and tom Dieck's book [6], pp.254–255.

Let V and W be complex Γ-representations for some compact Lie group Γ. Let BV and BW be Γ-invariant balls in V and W and let $f\colon BV \to BW$ be a Γ-map preserving the boundaries SV and SW. The K-group $K_\Gamma(V)$ is defined as $K_\Gamma(BV, SV)$, and the equivariant Thom isomorphism theorem says that $K_\Gamma(V)$ is a free $R(\Gamma)$-module with the Bott class $\lambda(V)$ as generator, where $R(\Gamma)$ is the complex representation ring of Γ. The map f induces a homomorphism $f^*\colon K_\Gamma(W) \to K_\Gamma(V)$. The K-theoretic degree $\alpha_f \in R(\Gamma)$ is uniquely determined by the relation $f^*(\lambda(W)) = \alpha_f \cdot \lambda(V)$.

For $g \in \Gamma$, let V_g and W_g be the subspaces of V and W fixed by g, and let $V_g^\perp$ and $W_g^\perp$ be their orthogonal complements. Let $f^g\colon V_g \to W_g$ be the restriction of f, and let $d(f^g)$ be the ordinary topological degree of f^g. (Note that $d(f^g) = 0$ if $\dim V_g \neq \dim W_g$.) For $\beta \in R(\Gamma)$, let $\Lambda_{-1}\beta$ be the alternating sum $\sum(-1)^i\Lambda^i\beta$ of exterior powers.

Then tom Dieck's character formula[6] is,

$$\mathrm{tr}_g(\alpha_f) = d(f^g)\,\mathrm{tr}_g(\Lambda_{-1}(W_g^\perp - V_g^\perp)), \tag{7.1}$$

where tr_g is the trace of the g-action.

7(ii). **Proof of Theorem 1.2.** Suppose X and a $\mathbb{Z}$-bundle l satisfy the assumptions of Theorem 1.2. Let $\lambda = l \otimes \mathbb{R}$ and $E = \underline{\mathbb{R}} \oplus \lambda$. By the assumptions, a Spin^{c-}-structure (P, ϕ) for (X, E) exists by Proposition 3.3, and we obtain a finite dimensional approximation $f\colon V \to W$ of the $\mathrm{Pin}^-(2)$-monopole map on (P, ϕ). Since $\tilde{c}_1(E) = 0$, f is a $\Gamma = \mathbb{Z}/4$-equivariant proper map. If $b_1(X; l) > 0$, by restricting f to the fiber over the origin of $T^{b_1(X;l)}$ which is represented by the fixed reference connection A, f can be assumed to be a Γ-map between (real) Γ-representation V and W. In fact, f can be considered as a map of the following form,

$$f\colon \tilde{\mathbb{R}}^m \oplus \mathbb{C}_1^{n+k} \to \tilde{\mathbb{R}}^{m+b} \oplus \mathbb{C}_1^n,$$

where $\Gamma = \mathbb{Z}/4$ acts on $\tilde{\mathbb{R}}$ by multiplication of ± 1 via the surjection $\mathbb{Z}/4 \to \{\pm 1\}$, and on $\mathbb{C}_k$ by multiplication of $g = \exp 2\pi\sqrt{-1}k/4$ for some fixed generator g of Γ, m, n are some positive integers, $b = b_+(X; l)$ and

$$k = \frac{1}{2}\operatorname{ind}_{\mathbb{R}} D_A = \frac{1}{8}(\tilde{c}_1(E)^2 - \operatorname{sign}(X)) = -\frac{1}{8}\operatorname{sign}(X).$$

As in [13], take the complexification of f as $f(u \otimes 1 + v \otimes i) = f(u) \otimes 1 + f(v) \otimes i$. Now the complexified f is of the form,

$$f\colon \tilde{\mathbb{C}}^m \oplus (\mathbb{C}_1 \oplus \mathbb{C}_{-1})^{n+k} \to \tilde{\mathbb{C}}^{m+b} \oplus (\mathbb{C}_1 \oplus \mathbb{C}_{-1})^n,$$

where $\tilde{\mathbb{C}} = \tilde{\mathbb{R}} \otimes \mathbb{C}$. Let us apply tom Dieck's formula (7.1). Since $V_g = W_g = 0$, $d(f^g) = 1$. Then we have,

$$\operatorname{tr}_g(\alpha_f) = \operatorname{tr}_g(\Lambda_{-1}(\tilde{\mathbb{C}}^b - (\mathbb{C}_1 \oplus \mathbb{C}_{-1})^k) = \operatorname{tr}_g((\mathbb{C} - \tilde{\mathbb{C}})^b(2\mathbb{C} - \mathbb{C}_1 \oplus \mathbb{C}_{-1})^{-k}) = 2^{b-k}.$$

Since $\operatorname{tr}_g(\alpha_f)$ is an integer, we have $b - k \geq 0$. Thus, Theorem 1.2 is proved.

Remark 7.2. In the proof of Theorem 1.2, we restrict the finite dimensional approximation f to a fiber, and take the complexification of it. Due to such modifications of f, the inequality we obtained might be somewhat weaker than expected. One could improve the inequality by using the technique of [14].

7(iii). **An alternative proof of Theorem 1.1.** In this subsection, we give an alternative proof of Theorem 1.1 by giving an alternative proof of Lemma 4.8. Suppose X and l satisfy the assumption of Theorem 1.1. We may assume $b_+(X; l) = 0$ by reversing the orientation of X if necessary. Suppose an $O(2)$-bundle E such that $\det E = \lambda$, and a Spin^{c-}-structure are given. Then we have a $\Gamma = \mathbb{Z}/2$-equivariant finite dimensional approximation $f\colon V \to W$ of the $\mathrm{Pin}^-(2)$-monopole map. By restricting f to a fiber if $b_1(X; l) > 0$, we may assume f has the form of

$$f\colon \mathbb{R}^m \oplus \tilde{\mathbb{C}}^n \to \mathbb{R}^m \oplus \tilde{\mathbb{C}}^{n+k},$$

where $\Gamma \cong \{\pm 1\}$ acts on $\mathbb{R}$ trivially, and on $\tilde{\mathbb{C}}$ by multiplication of ± 1, and m, n are some positive integers, and

$$k = -\frac{1}{2}\operatorname{ind}_{\mathbb{R}} D_A = -\frac{1}{8}(\tilde{c}_1(E)^2 - \operatorname{sign}(X)).$$

Take the complexification of f and apply tom Dieck's formula (7.1) for $g = -1$. Then,

$$\operatorname{tr}_g(\alpha_f) = \operatorname{tr}_g((\mathbb{C} - \tilde{\mathbb{C}})^{2k}) = 2^{2k}.$$

Therefore $k \geq 0$, and Lemma 4.8 is proved.

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