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The second main theorem of hypersurfaces in the projective space

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THE SECOND MAIN THEOREM OF HYPERSURFACES IN THE PROJECTIVE SPACE

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Contents

1. Introduction

In this paper, we deal with a holomorphic map from the complex plane $\mathbb{C}$ to the n-dimensional complex projective space $\mathbb{P}^n(\mathbb{C})$ and prove the Nevanlinna Second Main Theorem for some families of non-linear hypersurfaces in $\mathbb{P}^n(\mathbb{C})$. These hypersurfaces may be singular. In the Nevanlinna theory, it has been a fundamental problem to prove the Second Main Theorem for a holomorphic map from $\mathbb{C}$ to $\mathbb{P}^n(\mathbb{C})$. In 1933, H. Cartan [2] proved the Second Main Theorem for hyperplanes. The case of non-linear hypersurfaces had been studied by many authors. For examples, J. Noguchi [9], B. Shiffman [14], Eremenko and Sodin [6], Y. -T. Siu [16], and M. Ru [12]. In these results, the degree of hypersurfaces does not appear in the defect relation. In A. Biancofiore [1], the degree of hypersurfaces concerns the defect relation for special holomorphic mappings. In this paper, the degree of the hypersurfaces appears in our defect relation.

To prove the Second Main Theorem, we use J.-P. Demailly's meromorphic partial projective connection. A meromorphic partial projective connection, which is defined in J.-P. Demailly [3], is a family of locally defined meromorphic connections such that they work as an entirely defined meromorphic connection under the Wronskian operator. We recall the definition and some basic properties of a partial projective connection in §3. The reference of this section is §11 of J.-P. Demailly [3]. In the Nevanlinna theory, the idea of using meromorphic connection is due to Y. -T. Siu [15]. Later, by using meromorphic connection, A. Nadel [7] constructed Kobayashi hyperbolic hypersurfaces in $\mathbb{P}^3(\mathbb{C})$ explicitly. J. El Goul [5] also constructed Kobayashi hyperbolic hypersurfaces in $\mathbb{P}^3(\mathbb{C})$ by simplifying Nadel's method. J.-P. Demailly [3] developed a new general concept called meromorphic partial projective connections. The Nevanlinna theory however was not used in A. Nadel [7], J. El Goul [5], J.-P. Demailly [3], or J.-P. Demailly and J. El Goul [4]. These papers mainly dealt with holomorphic curves into non-linear hypersurfaces of $\mathbb{P}^n(\mathbb{C})$ by using a negative curvature method.

In §7, we prove the Nevanlinna Second Main Theorem for singular hypersurfaces by using the pull back of a meromorphic partial projective connection. The Second Main Theorem for singular divisors was dealt with in B. Shiffman [13]. In B. Shiffman [13], the singular divisor is reduced to the smooth one by resolving the singularity. In this paper, we also resolve the singularity of divisors. By the same method, we show the Second Main Theorem for hypersurfaces in m-subgeneral position ($m \geq 2$) in $\mathbb{P}^2(\mathbb{C})$ such that any two hypersurfaces intersect transversally. We say that hypersurfaces are in m-subgeneral position if the intersection of any $m+1$ hypersurfaces is empty. In the case where hypersurfaces are hyperplanes, E. I. Nochka proved the Second Main Theorem in [8]. The approach that we employ is different from Nochka's one (see Theorem 5).

Now, we state our main theorem precisely. Let s_0, $\ldots$, s_n be homogeneous polynomials of degree d in $\mathbb{C}[X_0, \ldots, X_n]$ such that

$$\det \left(\frac{\partial s_j}{\partial X_k} \right)_{0 \leq j,k \leq n} \not\equiv 0.$$

Then we construct the meromorphic connection $\widetilde{\nabla} = d + \widetilde{\Gamma}$ on $\mathbb{C}^{n+1}$ defined by

$$\sum_{0\le\lambda\le n} \frac{\partial s_\kappa}{\partial z_\lambda}\, \widetilde{\Gamma}^{\lambda}_{i,j} = \frac{\partial^2 s_\kappa}{\partial z_i \partial z_j}.$$

This meromorphic connection induces the meromorphic partial projective connection ∇ on $\mathbb{P}^n(\mathbb{C})$ (see §3).

Theorem 1 (Main Theorem)**.** *Let σ_k, $k = 1, \dots, q$ be elements of linear system $|\{s_0, \dots, s_n\}|$ such that σ_k is smooth, and $\sum_{1\le k\le q} \sigma_k$ is a simple normal crossing divisor.*

Assume that $X_0^{d-l_0}|s_0$, $\dots$, $X_n^{d-l_n}|s_n$ for non-negative integers $l_j \le d$, $j = 0, \dots, n$. Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a non-constant holomorphic map whose image is neither contained in a support of an element of linear system $|\{s_0, \dots, s_n\}|$ nor contained in the polar locus of ∇. Then we have

$$\begin{aligned}&\left(q - \frac{n+1}{d} - \frac{1}{2d}(n-1)n(n+1+l_0+\cdots+l_n)\right) T_f(r, dH)\\ &\quad\le \sum_{1\le i\le q} N_n(r, f^*\sigma_i) + S_f(r),\end{aligned}$$

where H is a hyperplane bundle on $\mathbb{P}^n(\mathbb{C})$, and $S_f(r) = O(\log^+ T_f(r) + \log^+ r)\|$. Here "$\|$" means that the inequality holds for all $r \in (0, +\infty)$ possibly except for subset with finite Lebesgue measure.

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2. Notation

We introduce some functions which play an important role in the Nevanlinna theory. Let E be an effective divisor on $\mathbb{C}$. We write $E = \sum m_j P_j$, where $\{P_j\}$ is a set of discrete points in $\mathbb{C}$ and m_j are positive integers. Put $n_k(r, E) = \sum_{|P_j|<r} \min\{k, m_j\}$. We define the counting function of E by

$$N_k(r, E) = \int_1^r \frac{n_k(t, E)}{t} dt.$$

Let X be a complex projective algebraic manifold, and let D be an effective divisor on X. Put $L = [D]$, where $[D]$ denotes the line bundle defined by D. Let σ be a holomorphic section of L such that the zero divisor (σ) of σ equals D. By $\operatorname{supp} D$ we denote the support of D. Let $f : \mathbb{C} \to X$ be a non-constant holomorphic map. We define the proximity function of D by

$$m_f(r, D) = \int_0^{2\pi} \log \frac{1}{\|\sigma(f(re^{i\theta}))\|} \frac{d\theta}{2\pi},$$

where $\| \cdot \|$ is a Hermitian metric in L. Let $R(L, \| \cdot \|)$ be the curvature form of the metrized line bundle $(L, \| \cdot \|)$ representing the first Chern class. Then we define the characteristic function of L by

$$T_f(r, L) = \int_1^r \frac{dt}{t} \int_{\Delta(t)} f^* R(L, \| \cdot \|),$$

where $\Delta(t) = \{z \in \mathbb{C} \,|\, |z| < t\}$. We set $T_f(r) = T_f(r, L)$ for an ample L on X. The equation

$$T_f(r, L) = N(r, f^* D) + m_f(r, D) + O(1)$$

is fundamental in the Nevanlinna theory; it is called the First Main Theorem (cf. Noguchi and Ochiai [11], Chapter V, §2). If $X = \mathbb{P}(\mathbb{C})$, f is a meromorphic function on $\mathbb{C}$. Then we have the lemma on logarithmic derivative (cf. Noguchi and Ochiai [11], Chapter VI, §1)

$$\int_0^{2\pi} \log^+ \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| d\theta \leq S_f(r),$$

where $\log^+ r = \max\{0, \log r\}$, and $S_f(r) = O(\log^+ T_f(r) + \log^+ r)\|$. Here "$\|$" means that the inequality holds for all $r \in (0, +\infty)$ possibly except for a subset with finite Lebesgue measure.

Let X be an n-dimensional complex manifold, and let x be a point of X. Let f be a holomorphic map from a neighborhood of $0 \in \mathbb{C}$ to X such that $f(0) = x$. Let $H(\mathbb{C}, X)_x$ denote the set of all those holomorphic mappings. Take a holomorphic local cordinate system $(z_1, \cdots, z_n)$ about x, and put $f_i = z_i \circ f$, $g_i = z_i \circ g$ for $f, g \in H(\mathbb{C}, X)_x$. Then we write $f \overset{k}{\sim} g$ if

$$\frac{d^j}{dz^j} f_i(0) = \frac{d^j}{dz^j} g_i(0), \quad 0 \leq j \leq k, \quad 1 \leq i \leq n.$$

This equivalence relation does not depend on the choice of a holomorphic local cordinate system. Let $j_{k,x}(f)$ denote the equivalence class of $f \in H(\mathbb{C}, X)_x$ and set

$$J_k(X)_x = H(\mathbb{C}, X)_x / \overset{k}{\sim} .$$

Define

$$J_k(X) = \bigcup_{x \in X} J_k(X)_x.$$

Let $p : J_k(X) \to X$ be the natural projection. Then $J_k(X)$ naturally carries a structure of a complex manifold, and the triple $(J_k(X), p, X)$ forms a holomorphic fiber bundle over X. This holomorphic fiber bundle is called the k-jet bundle over X. Let $f : U \to X$ be a holomorphic map from an open set U in $\mathbb{C}$ to X. Then there exists a holomorphic map $J_k(f) : U \to J_k(X)$ such that $J_k(f)(z) = j_{k,f(z)}(f)$, $z \in U$. We call $J_k(f)$ the lifting of order k of f. A holomorphic (resp. meromorphic) functional on $J_k(X)$ which is a polynomial on every fiber is called holomorphic (resp. meromorphic) k-jet differential on X.

3. Meromorphic partial projective connections and Totally geodesic hypersurfaces

In this section, we recall some definitions and properties of meromorphic partial projective connections and totally geodesic hypersurfaces. A reference for this section is J.-P. Demailly [3], §11.

Let X be an n-dimensional complex projective algebraic manifold. Let $\{U_j\}_{1 \le j \le N}$ be an affine open covering of X.

Definition 1. A meromorphic partial projective connection ∇ relative to an affine open covering $\{U_j\}_{1 \le j \le N}$ of X is a collection of meromorphic connections ∇_j on U_j, satisfying

$$\nabla_j - \nabla_k = \alpha_{jk} \otimes \mathrm{Id}_{T_X} + \mathrm{Id}_{T_X} \otimes \beta_{jk},$$

for all $1 \le j, k \le N$, where α_{jk}, β_{jk} are meromorphic one-forms on $U_j \cap U_k$. We write $\nabla = \{(\nabla_j, U_j)\}_{1 \le j \le N}$.

Let S_j be the smallest subvariety of X such that ∇_j is a holomorphic connection on $U_j \setminus S_j \cap U_j$. We set $\mathrm{supp}\,(\nabla)_\infty = \bigcup_{1 \le j \le N} S_j$ and call it the polar locus of ∇.

Example 1. Let $\{U_j\}_{0 \le j \le n}$ be an affine open covering of $\mathbb{P}^n(\mathbb{C})$ such that $U_j = \{[X_0 : \cdots : X_n] \in \mathbb{P}^n(\mathbb{C}) | X_j \neq 0\}$. There is a canonical isomorphism $U_j \simeq \mathbb{C}^n$ and we take flat connections d_j on U_j. Then $\{(d_j, U_j)\}_{0 \le j \le n}$ is a partial projective connection on $\mathbb{P}^n(\mathbb{C})$. □

Let f be a holomorphic map from $\mathbb{C}$ to X, and let ∇ be a meromorphic partial projective connection relative to an affine covering $\{U_j\}$ of X. Assume that $f(\mathbb{C})$ is not contained in the polar locus of ∇. We write

$$\nabla_j^{(m)} = \overbrace{\nabla_j \circ \cdots \circ \nabla_j}^{m-\text{times}}.$$

By the definition of meromorphic partial projective connections, we have

$$\begin{aligned} & f'(z) \wedge \nabla_{j\, f'(z)} f'(z) \wedge \cdots \wedge \nabla_{j\, f'(z)}^{(n-1)} f'(z) \\ = \quad & f'(z) \wedge \nabla_{k\, f'(z)} f'(z) \wedge \cdots \wedge \nabla_{k\, f'(z)}^{(n-1)} f'(z) \in \bigwedge^n TX_{f(z)}, \end{aligned}$$

for $f(z) \in U_j \cap U_k \setminus \operatorname{supp}(\nabla)_\infty$.

Definition 2. The *Wronskian* of f relative to a meromorphic partial projective connection ∇ is defined by

$$W_\nabla(f)(z) = f'(z) \wedge \nabla_{f'(z)} f'(z) \wedge \cdots \wedge \nabla_{f'(z)}^{(n-1)} f'(z) \in \bigwedge^n TX_{f(z)},$$

where ∇ is ∇_j for $f(z) \in U_j \setminus \operatorname{supp}(\nabla)_\infty$.

Let $\mathbb{P}^n(\mathbb{C})$ be n-dimensional complex projective space, and let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n(\mathbb{C})$ be the canonical projection. Put $U_j = \{[X_0 : \cdots : X_n] \in \mathbb{P}^n(\mathbb{C}) \mid X_j \neq 0\}$ where $[X_0 : \ldots : X_n]$ is a homogeneous cordinate system of $\mathbb{P}^n(\mathbb{C})$. In U_j, we take a local cordinate system $(X_0/X_j, \cdots, X_{j-1}/X_j, X_{j+1}/X_j, \ldots, X_n/X_j)$. Let $\eta_j : U_j \to \mathbb{C}^{n+1}$ be a holomorphic map such that

$$\eta_j\left(\frac{X_0}{X_j}, \cdots, \frac{X_{j-1}}{X_j}, \frac{X_{j+1}}{X_j}, \cdots, \frac{X_n}{X_j}\right) = \left(\frac{X_0}{X_j}, \cdots, \overset{\text{j−th}}{\overset{\downarrow}{1}}, \cdots, \frac{X_n}{X_j}\right).$$

Then $\pi \circ \eta_j = \mathrm{Id}_{U_j}$. A meromorphic connection $\widetilde{\nabla}$ on $\mathbb{C}^{n+1}$ induces a meromorphic connection ∇_j on U_j by

$$\nabla_j = \pi_*(\eta_j^* \widetilde{\nabla}).$$

The following lemma is Corollary 11.10. of J.-P. Demailly [3]:

Lemma 1. *Let* $\widetilde{\nabla} = d + \widetilde{\Gamma}$ *be a memromorphic connection on* $\mathbb{C}^{n+1}$*, and let* $\varepsilon = \sum z_j \partial/\partial z_j$ *be the Euler vector field on* $\mathbb{C}^{n+1}$*. Then* $\{(\pi_*(\eta_j^* \widetilde{\nabla}), U_j)\}_{0 \le j \le n}$ *is a meromorphic partial projective connection on* $\mathbb{P}^n(\mathbb{C})$ *provided that*

(i) *the Christoffel symbols* $\widetilde{\Gamma}_{j\mu}^\lambda$ *of* $\widetilde{\nabla}$ *are homogeneous rational functions of degree* -1,

(ii) *on every intersection* $(\pi^{-1}U_i) \cap (\pi^{-1}U_j)$ $(i \neq j)$ *there are meromorphic functions* α, β *and meromorphic* 1*-forms* γ, ξ *on* $\mathbb{C}^{n+1} \setminus \{0\}$ *such that*

$$\widetilde{\Gamma}(\varepsilon, \upsilon) = \alpha\upsilon + \gamma(\upsilon)\varepsilon, \quad \widetilde{\Gamma}(\omega, \varepsilon) = \beta\omega + \xi(\omega)\varepsilon$$

for all vector fields υ, ω.

Proof. See the proof of Lemma 11.8. of J.-P. Demailly [3]. □

Let D be a reduced effective divisor of an n-dimensional complex projective algebraic manifold X, and let ∇ be a meromorphic connection. Take the holomorphic function s on an open set $U \subset X$ such that $D|_U = (s)$, and take a local cordinate system $(z_1, \dots, z_n)$ on U.

We define that D is totally geodesic with respect to ∇ on U if there exist meromorphic one-forms $a = \sum_{1 \leq j \leq n} a_j dz_j$, $b = \sum_{1 \leq j \leq n} b_j dz_j$ and a meromorphic two-form $c = \sum_{1 \leq j, \mu \leq n} c_{j\mu} dz_j \otimes dz_\mu$ such that no polar locus of a_j, b_j, or $c_{j\mu}$ $(1 \leq j, \mu \leq n)$ contains $\operatorname{supp} D|_U$, and

$$\nabla^*(ds) = d^2 s - ds \circ \Gamma = a \otimes ds + ds \otimes b + sc$$

in U, where ∇^* is the induced connection on T_X^*.

The following Lemma 2 was obtained by J.-P. Demailly [3].

Lemma 2. *Assume that D is totally geodesic with respect to ∇ on U, i.e., there exist meromorphic one-forms a, b and meromorphic two-form c on U such that*

$$\nabla^*(ds) = a \otimes ds + ds \otimes b + sc.$$

We take a holomorphic function β on U such that βa, βb, βc are holomorphic forms. Let V be a domain in $\mathbb{C}$. Let $f : V \to U$ be a holomorphic map. Then we have

$$\begin{aligned} \frac{d^k(s \circ f)}{dz^k}(z) =& \gamma_k(z)(s \circ f)(z) + \sum_{0 \leq l \leq k-2} \gamma_{l,k}(z)(ds \cdot \nabla_{f'}^{(l)} f')(z) \\ &+ (ds \cdot \nabla_{f'}^{(k-1)} f')(z), \qquad z \in V \end{aligned}$$

for $k \in \mathbb{N}$. Here γ_k and $\gamma_{l,k}$ are meromorphic functions on V such that $\beta^{k-1}(f)\gamma_k$ and $\beta^{k-l-1}(f)\gamma_{l,k}$ are holomorphic functions on V.

Proof. The lemma holds for $k=1$, because

$$\frac{ds\circ f}{dz}=ds\cdot f'.$$

We shall prove the lemma by induction over the order k, and so we assume that it has already been proved for $k-1$. Then we have

$$\begin{aligned}
(1)\qquad & \frac{d}{dz}\left(\frac{d^{k-1}(s\circ f)}{dz^{k-1}}\right)(z)\\
&= \frac{d}{dz}\Bigg(\gamma_{k-1}(z)(s\circ f)(z)+\sum_{0\le l\le k-3}\gamma_{l,k-1}(z)(ds\cdot\nabla_{f'}^{(l)}f')(z)\\
&\quad + (ds\cdot\nabla_{f'}^{(k-2)}f')(z)\Bigg)\\
&= \frac{d\gamma_{k-1}}{dz}(z)(s\circ f)(z)+\gamma_{k-1}(z)\frac{d(s\circ f)}{dz}(z)\\
&\quad + \sum_{0\le l\le k-3}\left(\frac{d\gamma_{l,k-1}}{dz}(z)(ds\cdot\nabla_{f'}^{(l)}f')(z)+\gamma_{l,k-1}(z)\frac{d}{dz}(ds\cdot\nabla_{f'}^{(l)}f')(z)\right)\\
&\quad + \frac{d}{dz}(ds\cdot\nabla_{f'}^{(k-2)}f')(z).
\end{aligned}$$

It follows that

$$\begin{aligned}
(2)\qquad & \frac{d}{dz}(ds\cdot\nabla_{f'}^{(l)}f')(z)\\
= & \ (ds\cdot\nabla_{f'}^{(l+1)}f')(z)+\nabla^*(ds)(f',\nabla_{f'}^{(l)}f')(z)\\
= & \ (ds\cdot\nabla_{f'}^{(l+1)}f')(z)+a(f')(z)(ds\cdot\nabla_{f'}^{(l)}f')(z)\\
& +(ds\cdot f')(z)b(\nabla_{f'}^{(l)}f')(z)+(s\circ f)(z)c(f',\nabla_{f'}^{(l)}f')(z).
\end{aligned}$$

By (1) and (2), we have the lemma. □

Let $\nabla=\{(\nabla_j,U_j)\}_{1\le j\le N}$ be a meromorphic partial projective connection relative to an affine covering $\{U_j\}_{1\le j\le N}$ of X, and let s_j be a holomorphic function on U_j such that $D|_{U_j}=(s_j)$. By the definition of the meromorphic partial projective connection,

$$\nabla_j-\nabla_k=\alpha_{jk}\otimes\mathrm{Id}_{T_X}+\mathrm{Id}_{T_X}\otimes\beta_{jk},$$

on $U_j\cap U_k$ with meromorphic one-forms α_{jk} and β_{jk}. Then we have

$$(\nabla_j^*-\nabla_k^*)\,ds_j=-ds_j\otimes\alpha_{jk}-\beta_{jk}\otimes ds_j.$$

This means that D is totally geodesic with respect to ∇_j on $U_j \cap U_k$ if D is totally geodesic with respect to ∇_k and $\operatorname{supp} D|_{U_j \cap U_k}$ is not contained in the polar loci of α_{jk}, β_{jk}.

Definition 3. Let ∇ be a meromorphic partial projective connection relative to an affine open covering $\{U_j\}$ of X. Let D be an effective divisor on X such that $\operatorname{supp} D|_{U_j} \not\subset \operatorname{supp}(\nabla)_\infty$. Then D is said to be totally geodesic with respect to ∇ if $D|_{U_j}$ is totally geodesic with respect to ∇_j on U_j for all j.

Let $s_0, \dots, s_n$ be homogeneous polynomials of $\mathbb{C}[X_0, \dots, X_n]$ such that $\deg(s_0) = \dots = \deg(s_n) = d$ and $\det(\partial s_j / \partial X_k)_{0 \le j,k \le n} \not\equiv 0$. We define a meromorphic connection $\widetilde{\nabla} = d + \widetilde{\Gamma}$ on $\mathbb{C}^{n+1}$ by

$$\sum_{0 \le \lambda \le n} \frac{\partial s_\kappa}{\partial X_\lambda} \widetilde{\Gamma}^{\lambda}_{i\,j} = \frac{\partial^2 s_\kappa}{\partial X_i \partial X_j}$$

for $0 \le i, j \le n$. Then $\widetilde{\nabla}^* ds_j \equiv 0$ for all $0 \le j \le n$. Let $U_j = \{[X_0 : \dots : X_n] \in \mathbb{P}^n(\mathbb{C}) \mid X_j \neq 0\}$ be an affine open subset of $\mathbb{P}^n(\mathbb{C})$. Let $\eta_j : U_j \to \mathbb{C}^{n+1} \setminus \{0\}$ be the canonical section of the fiber bundle $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ such that

$$\eta_j([X_0 : \dots : X_n]) = \left(\frac{X_0}{X_j}, \dots, \overset{\text{j−th}}{\overset{\downarrow}{1}}, \dots, \frac{X_n}{X_j} \right).$$

Then meromorphic connection $\widetilde{\nabla}$ induces a meromorphic partial projective connection $\nabla = \{(\pi_*(\eta_j^* \widetilde{\nabla}), U_j)\}$ on $\mathbb{P}^n(\mathbb{C})$ by Lemma 1 (see §11 of J.-P. Demailly [3]). A Reduced divisor s of the linear system $|\{s_0, \dots, s_n\}|$ is totally geodesic with respect to ∇ if $\operatorname{supp}(s)$ is not contained in $\operatorname{supp}(\nabla)_\infty$.

Remark 1. Let $(z_0, \dots, z_{j-1}, z_{j+1}, \dots, z_n)$ be a local cordinate system on U_j such that $z_k = X_k / X_j$. Put $\nabla_j = \pi_*(\eta_j^* \tilde{\nabla}) = d + \left(\Gamma^{\lambda}_{i\,\mu}\right)$ where $\Gamma^{\lambda}_{i\,\mu}$ is a Christoffel symbols with respect to this cordinate system. Then one can check that

$$\Gamma^{\lambda}_{i\,\mu} = \eta_j^* \widetilde{\Gamma}^{\lambda}_{i\,\mu} - z_\lambda \eta_j^* \widetilde{\Gamma}^{j}_{i\,\mu} \,,$$

$$\nabla^* d(\eta_j^* s_\kappa) = \deg(s_\kappa)\ \eta_j^* s_\kappa \sum_{i,\mu} \eta_j^* \widetilde{\Gamma}^{j}_{i\,\mu} dz_i dz_\mu$$

on U_j.

The following lemma was obtained by J.-P. Demailly [3].

Lemma 3. *Let $\nabla = \{(\nabla_i, U_i)\}_{0\leq i\leq n}$ be the meromorphic partial projective connection on $\mathbb{P}^n$ constructed as above. Let $f : \mathbb{C} \to \mathbb{P}^n$ be a non-constant holomorphic map such that the image of f is not contained in* $\operatorname{supp}(\nabla)_\infty$. *Then*

$$W_\nabla(f) = f' \wedge \nabla_{f'} f' \wedge \cdots \wedge \nabla_{f'}^{(n-1)} f' \equiv 0$$

if and only if $f(\mathbb{C})$ is contained in a support of an element of linear system $|\{s_0, \dots, s_n\}|$.

Proof. Let z be a point of $\mathbb{C}$ such that $f(z)$ is not contained in $\operatorname{supp}(\nabla)_\infty$. There exists j such that $f(z) \in U_j$. We can take a non-trivial solution $(\alpha_0, \cdots, \alpha_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ which sutisfies

$$\alpha_0\, \eta_j^* s_0(f(z)) + \cdots + \alpha_n\, \eta_j^* s_n(f(z)) = 0,$$
$$\alpha_0\, \eta_j^*(ds_0 \cdot f')(z) + \cdots + \alpha_n\, \eta_j^*(ds_n \cdot f')(z) = 0,$$
$$\alpha_0\, \eta_j^*(ds_0 \cdot \nabla_{f'}^{(l)} f')(z) + \cdots + \alpha_n\, \eta_j^*(ds_n \cdot \nabla_{f'}^{(l)} f')(z) = 0$$

for $1 \leq l \leq k-1$ where ∇ is ∇_j. Put $s = \alpha_0 s_0 + \cdots + \alpha_n s_n$, then we have

$$s(f(z)) = 0, \quad (ds \cdot f')(z) = 0, \quad (ds \cdot \nabla_{f'}^{(l)} f')(z) = 0 \tag{3}$$

for all $1 \leq l \leq k-1$. Assume that $W_\nabla(f) \equiv 0$. Then there exist holomorphic functions $a_0, \cdots, a_{k-1}$ for $k \leq n-1$ on a neighborfood of z such that

$$\nabla_{f'}^{(k)} f' = a_0 f' + a_1 \nabla_{f'} f' + \cdots + a_{k-1} \nabla_{f'}^{(k-1)} f' \tag{4}$$

on a neighborfood of z. By (3) and (4), we have

$$(ds \cdot \nabla_{f'}^{(l)} f')(z) = 0$$

for all $l \in \mathbb{N}$. Because $\operatorname{supp}(s)$ is totally geodesic with respect to ∇, we have

$$\frac{d^l(s \circ f)}{dz^l}(z) = 0$$

for all $l \in \mathbb{N}$ by Lemma 2. Therefore $s \circ f \equiv 0$. Then the image of f is contained in a support of (s).

Conversely, assume that s is an element of linear system $|\{s_0, \dots, s_n\}|$ such that $s \circ f \equiv 0$. Then, by Lemma 2,

$$(ds \cdot f')(z) = 0, \quad (ds \cdot \nabla_{f'}^{(l)} f')(z) = 0$$

for all $l \in \mathbb{N}$. So f', $\nabla_{f'}^{(l)} f'$ are elements of a kernel of ds. Because the dimension of $\operatorname{Ker}(ds)$ is less than $n-1$, we have $W_\nabla(f) \equiv 0$. $\square$

4. Proof of the Main Theorem

To prove the Second Main Theorem, we need Borel's lemma.

Lemma 4. *Let $h(r) > 0$ be a monotone increasing function in $r \geq 1$. Then, for arbitrary $\delta > 0$, we have*

$$\frac{dh(r)}{dr} \leq (h(r))^{1+\delta} \|.$$

Proof. See Noguchi-Ochiai [11], Chapter V, §5. □

Let X be an n-dimensional complex projective algebraic manifold, and let σ_i $(1 \leq i \leq q)$ be a holomorphic section of the holomorphic line bundle L_i on X. Let $\nabla = \{(\nabla_j, U_j)\}_{1 \leq j \leq N}$ be a meromorphic partial projective connection relative to an affine covering $\{U_j\}_{1 \leq j \leq N}$ of X. Let β be a holomorphic section of the holomorphic line bundle L on X such that $\beta \nabla_j$ is holomorphic on U_j for all $1 \leq j \leq N$.

Lemma 5. *Assume that (σ_i) is smooth and $\sum_{1 \leq i \leq q} (\sigma_i)$ is a simple normal crossing divisor of X. Assume that* $\operatorname{supp}(\sigma_j)$ *is not contained in* $\operatorname{supp}(\nabla)_\infty$ *and (σ_j) is totally geodesic with respect to ∇ for all $1 \leq j \leq q$. Let $f : \mathbb{C} \to X$ be a holomorphic map such that $f(\mathbb{C})$ is not contained in* $\operatorname{supp}(\nabla)_\infty$ *and the Wronskian $W_\nabla(f) \not\equiv 0$. Then we have*

$$\int_{|z|=r} \log^+ \frac{\|W_\nabla(f)(z)\|_{\bigwedge^n TX} \|\beta(f(z))\|_L^{n(n-1)/2}}{\prod_{i=1}^q \|\sigma_i(f(z))\|_{L_j}} \frac{d\theta}{2\pi} \leq S_f(r),$$

where $S_f(r) = O(\log^+ r + \log^+ T_f(r))\|$.

Proof. Take an open covering $\{V_j\}_{1 \leq j \leq N}$ such that $V_j \Subset U_j$ (i.e., V_j is contained in U_j and topological closure of V_j is compact), and take a partition of unity $\{\phi_j\}_{1 \leq j \leq N}$ subordinate to the covering $\{V_j\}_{1 \leq j \leq N}$. Take holomorphic functions $z_1, \ldots, z_n$ on U_j such that $dz_1, \ldots, dz_n$ are linearly independent and

$$U_j \cap \bigcup_{i=1}^q \operatorname{supp}(\sigma_i) = \{w \in U_j \,|\, z_1(w) \cdots z_p(w) = 0\},$$

for some p, $0 \leq p \leq n$. We put $f_l = z_l \circ f$, $(\nabla_{f'}^{(k)} f')_l = dz_l \cdot \nabla_{f'}^{(k)} f'$. Then we have

$$\phi_j(f) \log^+ \frac{\|f' \wedge \nabla_{f'} f' \wedge \cdots \wedge \nabla_{f'}{}^{(n-1)} f'\|_{\bigwedge^n TX} \|\beta(f)\|_L^{n(n-1)/2}}{\prod_{i=1}^q \|\sigma_i(f)\|_{L_j}}$$

$$= \phi_j(f) \log^+ \Bigg(\varphi_j(f) \|\beta(f)\|_L^{n(n-1)/2}$$

$$\times \left| \begin{array}{cccccc} \frac{f_1'}{f_1} & \cdots & \frac{f_p'}{f_p} & f_{p+1}' & \cdots & f_n' \\ \frac{((\nabla_j)_{f'} f')_1}{f_1} & \cdots & \frac{((\nabla_j)_{f'} f')_p}{f_p} & ((\nabla_j)_{f'} f')_{p+1} & \cdots & ((\nabla_j)_{f'} f')_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{((\nabla_j)_{f'}^{(n-1)} f')_1}{f_1} & \cdots & \frac{((\nabla_j)_{f'}^{(n-1)} f')_p}{f_p} & ((\nabla_j)_{f'}^{(n-1)} f')_{p+1} & \cdots & ((\nabla_j)_{f'}^{(n-1)} f')_n \end{array} \right| \Bigg)$$

on $f^{-1}(U_j)$ where φ_j is a C^∞-function on U_j. By Lemma 2,

$$((\nabla_j)_{f'}^{(l)} f')_i(z) = \sum_{0 \leq k \leq l+1} a_{i,l,k}(z) \frac{d^k f_i}{dz^k}(z),$$

for $1 \leq i \leq p$. Here $a_{i,l,k}$ are meromorphic functions on $f^{-1}(U_j)$ such that $a_{i,l,k}(z)(\beta \circ f(z))^l$ is a holomorphic function. So we have

$$\int_{|z|=r} \phi_j(f) \log^+ \frac{\|f' \wedge \nabla_{f'} f' \wedge \cdots \wedge \nabla_{f'}{}^{(n-1)} f'\|_{\bigwedge^n TX} \|\beta(f)\|_L^{n(n-1)/2}}{\prod_{i=1}^q \|\sigma_i(f)\|_{L_j}} \frac{d\theta}{2\pi}$$

$$\leq \int_{|z|=r} \Phi(f(z)) \frac{d\theta}{2\pi} + \sum_{1 \leq k \leq p} \sum_{1 \leq l \leq n} \int_{|z|=r} \log^+ \frac{|f_k^{(l)}(z)|}{|f_k(z)|} \frac{d\theta}{2\pi}$$

$$+ \sum_{p+1 \leq k \leq n} \sum_{1 \leq l \leq n} \int_{|z|=r} \log^+ |f_k^{(l)}(z)| \frac{d\theta}{2\pi},$$

where Φ is a bounded C^∞–function on X. By using the lemma on logarithmic derivative, we have

$$\int_{|z|=r} \log^+ \frac{|f_k^{(l)}(z)|}{|f_k(z)|} \frac{d\theta}{2\pi} \leq S_f(r),$$

$$\begin{aligned} \int_{|z|=r} \log^+ |f_k^{(l)}(z)| \frac{d\theta}{2\pi} &\leq \int_{|z|=r} \log^+ \frac{|f_k^{(l)}(z)|}{|f_k'(z)|} \frac{d\theta}{2\pi} + \int_{|z|=r} \log^+ |f_k'(z)| \frac{d\theta}{2\pi} \\ &\leq \int_{|z|=r} \log^+ |f_k'(z)| \frac{d\theta}{2\pi} + S_f(r). \end{aligned}$$

It follows that

$$\begin{aligned} \int_{|z|=r} \log^+ |f_k'(z)| \frac{d\theta}{2\pi} &= \frac{1}{2} \int_{|z|=r} \log^+ |f_k'(z)|^2 \frac{d\theta}{2\pi} \\ &\leq \frac{1}{2} \int_{|z|=r} \log^+ \|f'(z)\|_{TX}^2 \frac{d\theta}{2\pi} + O(1), \end{aligned}$$

where $\|\cdot\|_{TX}$ is a hermitian metric of TX.

By Lemma 4 and the concavity of log, we have that for $\delta > 0$,

$$
\begin{aligned}
&\frac{1}{2}\int_{|z|=r} \log^+ \|f'(z)\|^2_{TX}\frac{d\theta}{2\pi} \\
&\leq \frac{1}{2}\int_{|z|=r} \log\{\|f'(z)\|^2_{TX}+1\}\frac{d\theta}{2\pi} \\
&\leq \frac{1}{2}\log\left(1+\int_{|z|=r}\|f'(z)\|^2_{TX}\frac{d\theta}{2\pi}\right)+O(1) \\
&\leq \frac{1}{2}\log\left(1+\frac{1}{2\pi r}\frac{d}{dr}\int_{|z|\leq r}\|f'(z)\|^2_{TX}\frac{\sqrt{-1}}{2}dz\wedge d\bar{z}\right)+O(1) \\
&\leq \frac{1}{2}\log\left(1+\frac{1}{2\pi r}\left(\int_{|z|\leq r}\|f'(z)\|^2_{TX}\frac{\sqrt{-1}}{2}dz\wedge d\bar{z}\right)^{1+\delta}\right)+O(1)\| \\
&= \frac{1}{2}\log\left(1+\frac{r^\delta}{2\pi}\left(\frac{d}{dr}\int_1^r\frac{dt}{t}\int_{|z|\leq r}\|f'(z)\|^2_{TX}\frac{\sqrt{-1}}{2}dz\wedge d\bar{z}\right)^{1+\delta}\right)+O(1)\| \\
&\leq \frac{1}{2}\log\left(1+\frac{r^\delta}{2\pi}\left(\int_1^r\frac{dt}{t}\int_{|z|\leq r}\|f'(z)\|^2_{TX}\frac{\sqrt{-1}}{2}dz\wedge d\bar{z}\right)^{(1+\delta)^2}\right)+O(1)\| \\
&\leq S_f(r).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\int_{|z|=r}\log^+\frac{\|W_\nabla(f)(z)\|_{\bigwedge^n TX}\|\beta(f(z))\|_L^{n(n-1)/2}}{\prod_{i=1}^q\|\sigma_i(f(z))\|_{L_i}}\frac{d\theta}{2\pi} \\
&=\sum_j\int_{|z|=r}\phi_j(f(z))\log^+\frac{\|W_\nabla(f)(z)\|_{\bigwedge^n TX}\|\beta(f(z))\|_L^{n(n-1)/2}}{\prod_{i=1}^q\|\sigma_i(f(z))\|_{L_i}}\frac{d\theta}{2\pi} \\
&\leq S_f(r).
\end{aligned}
$$

□

Theorem 2. *Let K_X be the canonical line bundle of X. Under the hypothesis of Lemma 5, we have*

$$\sum_{1\leq i\leq q}T_f(r,L_i)+T_f(r,K_X)-\frac{1}{2}n(n-1)T_f(r,L)\leq\sum_{1\leq i\leq q}N_n(r,f^*(\sigma_i))+S_f(r).$$

Proof. We denote by $\operatorname{ord}_z(\sigma_j\circ f)$ the order of zero of $\sigma_j\circ f$ at the point of $z\in\mathbb{C}$. We denote by $\operatorname{ord}_z\beta(f)^{n(n-1)/2}W_\nabla(f)$ the order of zero of $\beta(f)^{n(n-1)/2}W_\nabla(f)$ at the point $z\in\mathbb{C}$. If $\operatorname{ord}_z(\sigma_j\circ f)\geq n+1$ for $z\in\mathbb{C}$, then $\operatorname{ord}_z\beta(f)^{n(n-1)/2}W_\nabla(f)\geq$

$\mathrm{ord}_z(\sigma_j \circ f) - n$ by Lemma 2. So we have, by the Nevanlinna First Main Theorem,

$$\sum_j T_f(r, L_j) - \sum_j N_n(r, f^*(\sigma_j)) - T_f(r, \bigwedge^n T_X) - \frac{1}{2}n(n-1)T_f(r, L)$$
$$\leq \int_{|z|=r} \log \frac{\|W_\nabla(f)(z)\|_{\bigwedge^n TX} \|\beta(f(z))\|_L^{n(n-1)/2}}{\prod_{i=1}^q \|\sigma_i(f(z))\|_{L_j}} \frac{d\theta}{2\pi}.$$

From Lemma 5 the theorem follows. □

Proof of the Main Theorem. We construct the meromorphic partial projective connection $\nabla = \{(\nabla_j, U_j)\}_{0 \leq j \leq n}$ on $\mathbb{P}^n(\mathbb{C})$ as in §2. By Cramer's rule, the degree of the pole divisor of each ∇_j is less than or equal to $l_0 + \cdots + l_n + n + 1$. The Main Theorem follows from Thorem 2 and $K_{\mathbb{P}^n} = -(n+1)H$. □

Now we show two typical corollaries. Define the defect

$$\delta_f((\sigma_j)) = 1 - \limsup_{r \to \infty} \frac{N(r, f^*\sigma_j)}{T_f(r, [D_j])}.$$

Corollary 1. (Defect Relation) *Under the hypothesis of Theorem 1, we have*

$$\sum_{1 \leq j \leq q} \delta_f((\sigma_j)) \leq \frac{n+1}{d} + \frac{1}{2d}(n-1)n(l_0 + \cdots + l_n + n + 1)$$

Proof. This is deduced from Theorem 1 and the same arguments in Noguchi-Ochiai [11], Chapter V, §5. □

Remark 2. When $q = 1$ and

$$\frac{n+1}{d} + \frac{1}{2d}(n-1)n(l_0 + \cdots + l_n + n + 1) < 1,$$

the holomorphic map omitting the hypersurface is algebraically degenerate by Corollary 1.

Corollary 2. (Ramification Theorem) *Assume that*

$$f^*\sigma_j \geq \mu_j \operatorname{supp}(f^*\sigma_j)$$

for some positive integers μ_j, $1 \leq j \leq q$. Under the hypothesis of Theorem 1, we have

$$\sum_{1 \leq j \leq q} \left(1 - \frac{n}{\mu_j}\right) \leq \frac{n+1}{d} + \frac{1}{2d}(n-1)n(l_0 + \cdots + l_n + n + 1).$$

Proof. This is deduced from Theorem 1 and the same arguments in Noguchi-Ochiai [11], Chapter V, §5. □

Example 2. Put $s_0 = X_0^d, \ldots, s_n = X_n^d \in \mathbb{C}[X_0, \ldots, X_n]$. Let $\sigma_1, \ldots, \sigma_q \in |\{s_0, \ldots, s_n\}|$ be smooth Fermat hypersurfaces such that the divisor $\sigma_1 + \cdots + \sigma_q$ is of simple normal crossing. By Theorem 1, we have

$$\left(q - \frac{n+1}{d} - \frac{1}{2d}(n-1)n(n+1)\right) T_f(r, dH) \leq \sum_{1 \leq i \leq q} N_n(r, f^*(\sigma_i)) + S_f(r).$$

□

Example 3. Put $s_0 = X_0^d, \ldots, s_{n-1} = X_{n-1}^d, s_n = X_n^{d-1}(\varepsilon_0 X_0 + \cdots + \varepsilon_n X_n) \subset \mathbb{C}[X_0, \ldots, X_n]$. Let $\sigma = s_0 + \cdots + s_n$. Assume that the hypersurface defined by σ in $\mathbb{P}^n(\mathbb{C})$ is smooth. Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map such that the image of f is Zariski dense in $\mathbb{P}^n(\mathbb{C})$. If

$$\frac{(n+1)(n^2 - n + 1)}{d} < 1,$$

the image of f intersects the hypersurface defined by σ. □

5. Restriction of the meromorphic partial projective connection

Let $s_0, \ldots, s_{n+p}$ be homogenious polynomials in $\mathbb{C}[X_0, \cdots, X_{n+p}]$ such that

$$\det \left(\frac{\partial s_j}{\partial X_k}\right)_{0 \leq j,k \leq n+q} \not\equiv 0.$$

Let $X \subset \mathbb{P}^{n+p}(\mathbb{C})$ be a smooth n-dimensional complete intersection for some hypersurfaces associated to elements of linear system $|s_0, \ldots, s_{n+p}|$. Then we construct the meromorphic partial projective connection ∇ associated to $\{s_0, \ldots s_{n+p}\}$ on $\mathbb{P}^{n+p}(\mathbb{C})$ as in §2. We may assume that ∇ is a meromorphic partial projective connection on X because elements of $|s_0, \ldots, s_{n+p}|$ is totally geodesic with respect to ∇.

For $\alpha = (\alpha_0, \ldots, \alpha_{n+p}) \in \mathbb{C}^{n+p+1}$ we put

$$s_\alpha = \alpha_0 s_0 + \cdots + \alpha_{n+p} s_{n+p}.$$

We denote the hypersurface in $\mathbb{P}^{n+p}(\mathbb{C})$ corresponding to s_α by Y_α.

The next lemma is due to Theorem 11.19. of J.-P. Demailly [3].

Lemma 6. *Let*

$$Z = Y_{\alpha^1} \cap \cdots \cap Y_{\alpha^p} \subset \mathbb{P}^{n+p}(\mathbb{C})$$

be a smooth n-dimensional complete intersection, for linearly independent elements $\alpha^1, \ldots, \alpha^p \in \mathbb{C}^{n+p+1}$ such that $ds_{\alpha^1} \wedge \cdots \wedge ds_{\alpha^p}$ does not vanish along Z. Assume that Z is not contained in $\{\det(\partial s_j/\partial X_k)_{0 \leq j,k \leq n+q} = 0\}$. Let $f : \mathbb{C} \to Z$ be a non-constant holomorphic map. Assume that $f(\mathbb{C})$ is not contained in $\{\det(\partial s_j/\partial X_k)_{0 \leq j,k \leq n+q} = 0\}$ nor contained in a hypersurface Y_α which satisfies $Z \not\subset Y_\alpha$. Then we have

$$W_\nabla(f) = f' \wedge \nabla_{f'} f' \wedge \cdots \wedge \nabla_{f'}^{(n-1)} f' \not\equiv 0.$$

Proof. See the proof of the Theorem 11.19. of J.-P. Demailly [3] □

Let $i : Z \to \mathbb{P}^{n+p}(\mathbb{C})$ be the inclusion map from Z to $\mathbb{P}^{n+p}(\mathbb{C})$, and let $H_Z = i^*H$ be the pull back of the hyperplane bundle on $\mathbb{P}^{n+p}(\mathbb{C})$.

Theorem 3. *Let $f : \mathbb{C} \to Z$ be a holomorphic map such that $f(\mathbb{C})$ is not contained in $\{\det(\partial s_j/\partial X_k)_{0 \leq j,k \leq n+q} = 0\}$ nor contained in a hypersurface Y_α which satisfies $Z \not\subset Y_\alpha$. Assume that $X_0^{d-l_0}|s_0, \ldots, X_{n+p}^{d-l_{n+p}}|s_{n+p}$ for $0 \leq l_j \leq d$. Let $\sigma_1, \ldots, \sigma_q$ be elements of a linear system $|\{s_j\}|$ such that $(i^*\sigma_j)$ is smooth and $\sum_{1 \leq j \leq q} i^*(\sigma_j)$ is a simply normal crossing divisor on Z. Then we have*

$$\left(q + p - \frac{n+p+1}{d} - \frac{1}{2d} n(n-1)(n+1+l_0+\cdots+l_{n+p})\right) T_f(r, dH_Z)$$
$$\leq \sum_{1 \leq j \leq q} N_n(r, f^*\sigma_j) + S_f(r).$$

Proof. Because the canonical line bundle $K_Z = (pd - n - p - 1)H_Z$, Theorem 2 implies the statement. □

Remark 3. In paricular, if $q = 0$ and

$$p - \frac{n+p+1}{d} - \frac{1}{2d} n(n-1)(n+1+l_0+\cdots+l_{n+p}) > 0,$$

then f is algebraically degenerate (cf. Theorem 11.19. of J.-P. Demailly [3])

6. Pull back of the meromorphic partial projective connection

Let X and $\widetilde{X}$ be n-dimensional complex projective algebraic manifolds. Let $\pi : \widetilde{X} \to X$ be a surjective holomorphic map. Then there exists a proper subvariety S of X such that $\widetilde{X} \setminus \pi^{-1}(S)$ and $X \setminus S$ are locally biholomorphic. Let $\nabla = \{(\nabla_j, U_j)\}_{1 \leq j \leq N}$ be a meromorphic partial projective connection on X relative to an affine open covering $\{U_j\}_{1\leq j\leq N}$ on X. We shall now construct the meromorphic partial projective connection $\widetilde{\nabla} = \{(\widetilde{\nabla}_j, \pi^{-1}U_j)\}_{1\leq j\leq N}$ on $\widetilde{X}$. Let $p \in \pi^{-1}U_k \subset \widetilde{X}$. Let $u, v \in \Gamma(V, T_X)$ be local holomorphic vector fields on a small neighborhood V of p. Then $V \setminus \pi^{-1}(S)$ is locally biholomorphic with $\pi(V) \setminus S$. We define

$$(\widetilde{\nabla}_k)_u v|_{V\setminus\pi^{-1}(S)} = (\pi_*|_{V\setminus\pi^{-1}S})^{-1}\,(\nabla_k)_{\pi_* u}\pi_* v|_{V\setminus\pi^{-1}S},$$

on $V \setminus \pi^{-1}S$. Then, the meromorphic vector field $(\widetilde{\nabla}_k)_u v|_{V\setminus\pi^{-1}(S)}$ on $V \setminus \pi^{-1}S$ is uniquely extended to the meromorphic vector field on V. In this way, we define the meromorphic connection $\widetilde{\nabla}_k$ on $\pi^{-1}(U_k)$. Let α_{ij} and β_{ij} be meromorphic one-forms on $U_i \cap U_j$ such that

$$\nabla_i - \nabla_j = \alpha_{ij} \otimes \mathrm{Id}_{TX} + \mathrm{Id}_{TX} \otimes \beta_{ij}.$$

Then we have

$$\widetilde{\nabla}_i - \widetilde{\nabla}_j = \pi^*\alpha_{ij} \otimes \mathrm{Id}_{T\widetilde{X}} + \mathrm{Id}_{T\widetilde{X}} \otimes \pi^*\beta_{ij}.$$

So $\widetilde{\nabla} = \{(\widetilde{\nabla}_j, \pi^{-1}U_j)\}_{1\leq j\leq N}$ is a meromorphic partial projective connection on $\widetilde{X}$ relative to an affine open covering $\{\pi^{-1}U_j\}_{1\leq j\leq N}$ of $\widetilde{X}$.

Assume that π is the blowing-up of X at a point of X. Let D be a reduced effective divisor in X such that $\operatorname{supp} D$ is not contained in $\operatorname{supp}(\nabla)_\infty$. Take a holomorphic function s_j on U_j such that $D|_{U_j} = (s_j)$.

Lemma 7. *Assume that D is totally geodesic with respect to ∇, and the strict transform of D under π is smooth. Then the strict transform of D is totally geodesic with respect to the meromorphic partial projective connection $\widetilde{\nabla} = \{(\widetilde{\nabla}_j, \pi^{-1}(U_j))\}_{1\leq j\leq N}$ on $\widetilde{X}$.*

Proof. There exist meromorphic one-forms a_j, b_j and meromorphic two-form c_j on U_j such that no polar locus of a_j, b_j, c_j does not contain $\operatorname{supp} D|_{U_j}$, and a_j, b_j, c_j satisfy

$$\nabla_j{}^* ds_j = a_j \otimes ds_j + ds_j \otimes b_j + s_j c_j,$$

for all $1 \leq j \leq N$. So we have

$$\widetilde{\nabla}_j^* d\pi^* s_j = \pi^* a_j \otimes d\pi^* s_j + d\pi^* s_j \otimes \pi^* b_j + \pi^* s_j \pi^* c_j.$$

Let E be an exceptional divisor of π. Let $\widetilde{D}$ be a strict transform of D under the blowing-up π. Then, $\operatorname{supp} \widetilde{D}$ is not contained in $\operatorname{supp}(\widetilde{\nabla})_\infty$. We may assume that there exists a holomorphic function e on $\pi^{-1}(U_j)$ such that $(e)=E|_{\pi^{-1}(U_j)}$. Then we have

$$\widetilde{D}|_V=\left(\frac{\pi^*s_j}{e^k}\right),$$

for some non-negative integer k. In $\pi^{-1}(U_j)$, it follows that

$$\begin{aligned}
d\left(\frac{\pi^*s_j}{e^k}\right)&=\frac{d\pi^*s_j}{e^k}-k\,\frac{\pi^*s_j}{e^k}\frac{de}{e},\\
\widetilde{\nabla}_j^*\left(\frac{d\pi^*s_j}{e^k}\right)&=d\left(\frac{1}{e^k}\right)\otimes d\pi^*s_j+\frac{1}{e^k}\widetilde{\nabla}_j^*d\pi^*s_j\\
&=-k\frac{de}{e}\otimes\frac{d\pi^*s_j}{e^k}+\frac{1}{e^k}(\pi^*a_j\otimes d\pi^*s_j+d\pi^*s_j\otimes\pi^*b_j+\pi^*s_j\,\pi^*c_j)\\
&=\left(\pi^*a_j-k\frac{de}{e}\right)\otimes\frac{d\pi^*s_j}{e^k}+\frac{d\pi^*s_j}{e^k}\otimes\pi^*b_j+\frac{\pi^*s_j}{e^k}\pi^*c_j,\\
\widetilde{\nabla}_j^*\left(-k\frac{\pi^*s_j}{e^k}\frac{de}{e}\right)&=-k\,d\left(\frac{\pi^*s_j}{e^k}\right)\otimes\frac{de}{e}-k\frac{\pi^*s_j}{e^k}\widetilde{\nabla}_j^*\left(\frac{de}{e}\right).
\end{aligned}$$

So we have

$$\begin{aligned}
\widetilde{\nabla}_j^*d\left(\frac{\pi^*s_j}{e^k}\right)&=\widetilde{\nabla}_j^*\left(\frac{d\pi^*s_j}{e^k}\right)+\widetilde{\nabla}_j^*\left(-k\frac{\pi^*s_j}{e^k}\frac{de}{e}\right)\\
&=\left(\pi^*a_j-k\frac{de}{e}\right)\otimes\frac{d\pi^*s_j}{e^k}+\frac{d\pi^*s_j}{e^k}\otimes\pi^*b_j-k\,d\left(\frac{\pi^*s_j}{e^k}\right)\otimes\frac{de}{e}\\
&\qquad+\frac{\pi^*s_j}{e^k}\left(\pi^*c_j-k\widetilde{\nabla}_j^*\frac{de}{e}\right)\\
&=\left(\pi^*a_j-k\frac{de}{e}\right)\otimes\left(d\left(\frac{\pi^*s_j}{e^k}\right)+k\,\frac{\pi^*s_j}{e^k}\frac{de}{e}\right)\\
&\qquad+\left(d\left(\frac{\pi^*s_j}{e^k}\right)+k\,\frac{\pi^*s_j}{e^k}\frac{de}{e}\right)\otimes\pi^*b_j-k\,d\left(\frac{\pi^*s_j}{e^k}\right)\otimes\frac{de}{e}\\
&\qquad+\frac{\pi^*s_j}{e^k}\left(\pi^*c_j-k\widetilde{\nabla}_j^*\frac{de}{e}\right)\\
&=\widetilde{a}_j\otimes d\left(\frac{\pi^*s_j}{e^k}\right)+d\left(\frac{\pi^*s_j}{e^k}\right)\otimes\widetilde{b}_j+\frac{\pi^*s_j}{e^k}\widetilde{c}_j,
\end{aligned}$$

where

$$\widetilde{a}_j=\pi^*a_j-k\frac{de}{e},\quad \widetilde{b}_j=\pi^*b_j-k\frac{de}{e},$$

$$\widetilde{c}_j=k\pi^*a_j\otimes\frac{de}{e}+k\frac{de}{e}\otimes\pi^*b_j+\pi^*c_j-k^2\frac{de}{e}\otimes\frac{de}{e}-k\,\widetilde{\nabla}_j^*\frac{de}{e}.$$

$\square$

7. The Second Main Theorem for singular divisors

Let X be an n-dimensional complex algebraic projective manifold, and let π be the blowing-up of X at the point p of X. Let U be an affine open neighborhood of p, and let ∇ be a meromorphic connection on U. Let $z_1, \dots, z_n$ be holomorphic functions on U such that $dz_1, \dots, dz_n$ are linearly independent on U and $p = \{x \in U \mid z_1(x) = \cdots = z_n(x) = 0\}$. Then,

$$\pi^{-1}(U) \simeq \{(x, [y_1 : \cdots : y_n]) \in U \times \mathbb{P}^{n-1}(\mathbb{C}) \mid z_i(x)y_j = z_j(x)y_i \quad \text{for all } i, j\}.$$

Let $\widetilde{U}_k = \{(x, [y_1 : \cdots : y_n]) \in \pi^{-1}(U) \mid y_k \neq 0\}$ be an affine open set of $\widetilde{X}$. Define a holomorphic function $u_i = y_i/y_k$ on $\widetilde{U}_k$. Then $du_1, \dots, du_{k-1}, dz_k, du_{k+1}, \dots, du_n$ are linearly independent on $\widetilde{U}_k$, and $E|_{\widetilde{U}_k} = (z_k)$ where E is the exceptional divisor of π. We shall now show that $W_{\widetilde{\nabla}}$ has only logarithmic poles on the exceptional divisor. Here $\widetilde{\nabla}$ is the pull back of ∇.

We may assume "$k = 1$" without loss of generality.

We have

$$\pi_* \left(\frac{\partial}{\partial z_1} \, \frac{\partial}{\partial u_2} \cdots \frac{\partial}{\partial u_n} \right) = \left(\frac{\partial}{\partial z_1} \cdots \frac{\partial}{\partial z_n} \right) \begin{pmatrix} 1 & 0 & \cdots & 0 \\ u_2 & z_1 & & \\ \vdots & & \ddots & \text{\Large 0} \\ u_n & \text{\Large 0} & & z_1 \end{pmatrix}.$$

We denote the above Jacobian matrix by A. Then we have

$$A^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -u_2/z_1 & 1/z_1 & & \\ \vdots & & \ddots & \text{\Large 0} \\ -u_n/z_1 & \text{\Large 0} & & 1/z_1 \end{pmatrix}.$$

Put $\nabla = d + \Gamma$ where $\Gamma = ({\Gamma^\lambda}_\mu)_{1 \le \lambda, \mu \le n}$ is a connection form with respect to the frame $\partial/\partial z_1, \dots, \partial/\partial z_n$. Let $\widetilde{\Gamma} = ({\widetilde{\Gamma}^\lambda}_\mu)_{1 \le \lambda, \mu \le n}$ be the connection form of the meromorphic connection $\widetilde{\nabla}$ on $\widetilde{U}_1$ with respect to the frame $\partial/\partial z_1, \partial/\partial u_2, \dots, \partial/\partial u_n$. Then we have $\widetilde{\Gamma} = A^{-1} dA + A^{-1} \pi^* \Gamma A$. Since

$$\frac{d\pi^* z_1}{z_1} = \frac{dz_1}{z_1}, \qquad \frac{d\pi^* z_j}{z_1} = du_j + u_j \frac{dz_1}{z_1},$$

we have

$$\frac{\pi^*\Gamma^\lambda_\mu}{z_1} = \phi_{1,\lambda,\mu}\frac{dz_1}{z_1} + \sum_{j=2}^n \phi_{j,\lambda,\mu} du_j$$

where $\phi_{j,\lambda,\mu}$ is a meromorphic function on $\widetilde{U}_1$. Let β be a holomorphic function on U such that $\beta\,\nabla$ is a holomorphic connection on U. It follows that $\pi^*\beta\,\phi_{1,\lambda,\mu}$ is holomorphic function. So $\pi^*\beta\, A^{-1}\pi^*\Gamma A$ has only logarithmic poles on the exceptional divisor.

It follows that

$$\begin{aligned} A^{-1}dA &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -u_2/z_1 & 1/z_1 & & \\ \vdots & & \ddots & \text{\huge 0} \\ -u_n/z_1 & \text{\huge 0} & & 1/z_1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ du_2 & dz_1 & & \\ \vdots & & \ddots & \text{\huge 0} \\ du_n & \text{\huge 0} & & dz_1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ (du_2)/z_1 & (dz_1)/z_1 & & \\ \vdots & & \ddots & \text{\huge 0} \\ (du_n)/z_1 & \text{\huge 0} & & (dz_1)/z_1 \end{pmatrix}. \end{aligned}$$

We define meromorphic connection $\widetilde{\nabla}_1$ on $\widetilde{U}_1$ by

$$\widetilde{\nabla}_1 = \widetilde{\nabla} - \frac{dz_1}{z_1} \otimes \mathrm{Id}_{T\widetilde{X}} - \mathrm{Id}_{T\widetilde{X}} \otimes \frac{dz_1}{z_1}.$$

Then $\{(\widetilde{\nabla}, \widetilde{U}_1), (\widetilde{\nabla}_1, \widetilde{U}_1)\}$ is a meromorphic partial projective connection on $\widetilde{U}_1$, so $W_{\widetilde{\nabla}} = W_{\widetilde{\nabla}_1}$. One sees that

$$
\begin{aligned}
\widetilde{\nabla}_1 &= d + A^{-1}\pi^*\Gamma A + A^{-1}dA - \frac{dz_1}{z_1} \otimes \mathrm{Id}_{T\widetilde{X}} - \mathrm{Id}_{T\widetilde{X}} \otimes \frac{dz_1}{z_1} \\
&= d + A^{-1}\pi^*\Gamma A + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ (du_2)/z_1 & (dz_1)/z_1 & & \\ \vdots & & \ddots & \mathbf{0} \\ (du_n)/z_1 & \mathbf{0} & & (dz_1)/z_1 \end{pmatrix} \\
&\quad - \frac{dz_1}{z_1}\begin{pmatrix} 1 & & & \mathbf{0} \\ & \ddots & & \\ & & \ddots & \\ \mathbf{0} & & & 1 \end{pmatrix} - \begin{pmatrix} (dz_1)/z_1 & 0 & \cdots & 0 \\ (du_2)/z_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ (du_n)/z_1 & 0 & \cdots & 0 \end{pmatrix} \\
&= d + A^{-1}\pi^*\Gamma A - \begin{pmatrix} (2dz_1)/z_1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathbf{0} & \\ 0 & & & \end{pmatrix}.
\end{aligned}
$$

Therefore $\pi^*\beta\,\widetilde{\nabla}_1$ has only logarithmic poles on the exceptional divisor. In the same way as above, we can construct the meromorphic connection $\widetilde{\nabla}_k$ on $\widetilde{U}_k$ for every k, $1 \leq k \leq n$. Then we obtain the meromorphic partial projective connection $\{(\widetilde{\nabla}_i, \widetilde{U}_i)\}_{1\leq i\leq n}$ on $\pi^{-1}(U)$ such that each $\pi^*\beta\,\widetilde{\nabla}_i$ has only logarithmic poles on the exceptional divisor.

Let X be an n-dimensional complex algebraic projective manifold, and let S be a reduced effective divisor on X such that the singular locus of S is $\{x_1, \ldots, x_p\} \subset X$. Let $\pi : \widetilde{X} \to X$ be the blowing-up at $\{x_1, \ldots, x_p\}$. Let E_i, $1 \leq i \leq p$ be irreducible divisors of $\widetilde{X}$ such that $E = \bigcup_{1\leq i\leq p} E_i$ is an exceptional divisor of π and $\pi(E_i) = x_i$. Let $\nabla = \{(\nabla_j, U_j)\}_{1\leq j\leq N}$ be a meromorphic partial projective connection on X relative to an affine open covering $\{U_j\}_{1\leq j\leq N}$ of X. Assume that S is not contained in the polar locus of ∇, and S is totally geodesic with respect to ∇. Let $\beta \in \Gamma(X, L)$ be a holomorphic section of a line bundle L on X such that $\beta\nabla_j$ is holomorphic for every j.

Lemma 8. *Let $f : \mathbb{C} \to X$ be a non-constant holomorphic map such that $f(\mathbb{C})$ is not contained in the polar locus of ∇ and $W_\nabla(f) \not\equiv 0$. Let $\widetilde{f} : \mathbb{C} \to \widetilde{X}$ denote the lift of f. Assume that the proper transform $\widetilde{S}$ of S is non-singular, and $\widetilde{S}$ intersects E transversally. Then we have*

$$\int_{|z|=r} \log^+ \frac{\|W_{\widetilde{\nabla}}(\widetilde{f})(z)\|_{\bigwedge^n T\widetilde{X}} \|\beta(f)(z)\|_L^{n(n-1)/2}}{\|\widetilde{S}(\widetilde{f})(z)\|_{[\widetilde{S}]}} \frac{d\theta}{2\pi} \leq S_f(r).$$

Proof. By shrinking U_i for $1 \leq i \leq N$, we may assume that each U_i has at most one singular point of S. Let $I = \{1, 2, \cdots, N\}$, and let I' and I'' be subsets of I such that

$$I' = \{i \in I | U_i \cap \{x_1, \cdots, x_p\} \neq \emptyset\},$$

$$I'' = I \setminus I' = \{i \in I | U_i \cap \{x_1, \cdots, x_p\} = \emptyset\}.$$

Let $\widetilde{\nabla} = \{(\pi^* \nabla_i, \pi^{-1}(U_i))\}_{1 \leq i \leq N}$ be the pull back of ∇. If $j \in I'$, there exists one $x_\nu \in U_j$. By the above argument, we can construct a meromorphic partial projective connection $\{(\widetilde{\nabla}_{j,k}, \widetilde{U}_{j,k})\}_{1 \leq k \leq n}$ on $\pi^{-1}(U_j)$ such that each $\pi^* \beta \widetilde{\nabla}_{j,k}$ has only logarithmic poles on E_ν. Here $\{\widetilde{U}_{j,k}\}_{1 \leq k \leq n}$ is the affine open covering of $\pi^{-1}(U_j)$. We define meromorphic partial projective connection $\{(\widehat{\nabla}_j, \Omega_j)\}_{1 \leq j \leq N'}$ on $\widetilde{X}$ by

$$\{(\widetilde{\nabla}_{j,k}, \widetilde{U}_{j,k})\}_{j \in I', 1 \leq k \leq n} \cup \{(\widetilde{\nabla}_j, \pi^{-1} U_j)\}_{j \in I''},$$

where $(\widehat{\nabla}_j, \Omega_j)$ is equal to $(\widetilde{\nabla}_{j,k}, \widetilde{U}_{j,k})$, $j \in I'$, or equal to $(\widetilde{\nabla}_j, \pi^{-1} U_j)$, $j \in I''$. Then it follows that the Wronskian of $\widetilde{\nabla}$ is equal to the Wronskian of $\widehat{\nabla}$. Take an open covering $\{V_j\}_{1 \leq j \leq N'}$ of $\widetilde{X}$ such that $V_j \Subset \Omega_j$, and take a partition of unity $\{\phi_j\}_{1 \leq j \leq N'}$ subordinate to the open covering $\{V_j\}_{1 \leq j \leq N'}$. If $\Omega_i = \widetilde{U}_{j,k}$ for some $j \in I', 1 \leq k \leq n$, then $\widetilde{S}$ intersects exceptional divisor E transversally in Ω_i. So we can take holomorphic functions $X_1, \cdots, X_n$ in Ω_i such that $(X_1) = E|_{\Omega_i}$ and $(X_2) = \widetilde{S}|_{\Omega_i}$, and $dX_1, \cdots, dX_n$ are linearly independent. We trivialize the n-jet bundle of $\widetilde{X}$ on Ω_i by

$$X_1^{(1)}, \cdots, X_n^{(1)}, X_1^{(2)} \cdots, X_n^{(2)}, \cdots, X_n^{(n)},$$

where $dX_l = X_l^{(1)}$ and $dX_l^{(k)} = X_l^{(k+1)}$. There exists a meromorphic n-jet differential ω on Ω_i such that

$$W_{\widehat{\nabla}_i} = \omega \frac{\partial}{\partial X_1} \wedge \cdots \wedge \frac{\partial}{\partial X_n}.$$

Let $\widetilde{\beta}_i$ be a holomorphic function in Ω_i such that $(\widetilde{\beta}_i) = \pi^*\beta|_{\Omega_i}$. From Lemma 2 it follows that

$$\widetilde{\beta}_i^{\;n(n-1)/2}\frac{\omega}{X_2}$$

is logarithmic n-jet differential along the divisor $(X_1)+(X_2)$ (cf. Noguchi [10]). So it follows that

$$\int_{|z|=r}\phi_i(\widetilde{f})\log^+\frac{\|W_{\widetilde{\nabla}}(f)(z)\|_{\bigwedge^n T\widetilde{X}}\|\beta(f)(z)\|_L^{n(n-1)/2}}{\|\widetilde{S}(\widetilde{f})(z)\|_{[\widetilde{S}]}}\frac{d\theta}{2\pi}\leq S_{\widetilde{f}}(r),$$

by the same arguments as that in the proof of Lemma 5. If $\Omega_i = \pi^{-1}(U_j)$ for some $j \in I$", the above inequality also holds. Because $\pi : \widetilde{X} \to X$ is a bimeromorphic map, we have $S_f(r) = S_{\widetilde{f}}(r)$. So we completes the proof. □

Theorem 4. *Under the hypothesis of Lemma 8, we have*

$$\begin{aligned}
&T_{\widetilde{f}}(r,[\widetilde{S}]) + T_f(r,K_X) + (n-1)\sum_{i=1}^{p}T_{\widetilde{f}}(r,[E_i])\\
\leq\quad & N_n(r,\widetilde{f}^*\widetilde{S}) + \frac{1}{2}n(n-1)T_f(r,L) + \frac{1}{2}n(n-1)\sum_{i=1}^{p}N(r,\widetilde{f}^*E_i) + S_f(r).
\end{aligned}$$

Proof. Let $e_j \in \Gamma(X,[E_j])$ be a holomorphic section of E_j such that $(e_j) = E_j$. By Lemma 8, it follows that

$$\begin{aligned}
&\int_{|z|=r}\log^+\frac{\|W_{\widetilde{\nabla}}(\widetilde{f})\|_{\bigwedge^n T\widetilde{X}}\|\beta(f)\|_L^{n(n-1)/2}\prod_{i=1}^p\|e_i(\widetilde{f})\|_{E_i}^{n(n-1)/2}}{\|\widetilde{S}(\widetilde{f})\|_{[\widetilde{S}]}}\frac{d\theta}{2\pi}\\
&\leq -\frac{1}{2}n(n-1)\sum_{i=1}^{p}m_{\widetilde{f}}(r,E_i) + S_f(r).
\end{aligned}$$

Then we have

$$\begin{aligned}
&T_{\widetilde{f}}(r,[\widetilde{S}]) + T_{\widetilde{f}}(r,K_{\widetilde{X}}) - \frac{1}{2}n(n-1)\sum_{i=1}^{p}T_{\widetilde{f}}(r,[E_i]) - \frac{1}{2}n(n-1)T_f(r,L)\\
&\quad\leq N_n(r,\widetilde{f}^*\widetilde{S}) - \frac{1}{2}n(n-1)\sum_{i=1}^{p}m_{\widetilde{f}}(r,E_i) + S_f(r).
\end{aligned}$$

Since $N(r,\widetilde{f}^*E_i) = T_{\widetilde{f}}(r,[E_i]) - m_{\widetilde{f}}(r,E_i)$ and $K_{\widetilde{X}} = K_X + (n-1)\sum_{i=1}^p E_i$, We complete the proof. □

Example 4. Let

$$S = \{X_0^{d-2}(\varepsilon_1 X_1^2 + \varepsilon_2 X_2^2) + X_1^d + X_2^d = 0\},$$

$$\varepsilon_1 \neq 0,\ \varepsilon_2 \neq 0,\ |\varepsilon_1| \neq |\varepsilon_2|.$$

Then S is smooth except the point $[1:0:0]$. Let

$$\pi : (\mathbb{P}^2(\mathbb{C}))^{\sim} \to \mathbb{P}^2(\mathbb{C})$$

be the blowing-up at $[1:0:0] \in \mathbb{P}^2(\mathbb{C})$ and $\widetilde{S}$ be the proper transformation of S. One can check that $\widetilde{S}$ is non-singular, and $\widetilde{S}$ intersects transversally the exceptional divisor E of π. Let $f : \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ be a holomorphic map and $\widetilde{f} : \mathbb{C} \to (\mathbb{P}^2(\mathbb{C}))^{\sim}$ be the lift of f. Put $s_0 = X_0^{d-2}(\varepsilon_1 X_1^2 + \varepsilon_2 X_2^2)$, $s_1 = X_1^d$, $s_2 = X_2^d$. Then we construct a meromorphic partial projective connection ∇ on $\mathbb{P}^2(\mathbb{C})$ as in §2. Then the degree of the pole of ∇ is five. By Theorem 5, we have

$$T_{\widetilde{f}}(r, [\widetilde{S}]) + T_{\widetilde{f}}(r, [E]) - 8T_f(r, H) \leq N_2(r, \widetilde{f}^*\widetilde{S}) + N(r, \widetilde{f}^*E) + S_f(r),$$

where H is a hyperplane bundle of $\mathbb{P}^2(\mathbb{C})$. Because $\pi^*S = \widetilde{S} + 2E$, and $T_f(r, H) = T_{\widetilde{f}}(r, \pi^*H) \geq T_{\widetilde{f}}(r, E)$, we have

$$\left(1 - \frac{9}{d}\right) T_f(r, H) \leq N_2(r, \tilde{f}^*\widetilde{S}) + N(r, \tilde{f}^*E) + S_f(r).$$

$\square$

Now we prove the Second Main Theorem for smooth hypersurfaces in $\mathbb{P}^2(\mathbb{C})$ which are not normal crossing.

Let $s_0, s_1, s_2 \in \mathbb{C}[X_0, X_1, X_2]$ be homogeneous polynomials of degree d such that $\det(\partial s_j/\partial X_k)_{0 \leq j,k \leq 2} \not\equiv 0$, and $X_0^{d-l_0} | s_0, X_1^{d-l_1} | s_1, X_2^{d-l_2} | s_2$ for $0 \leq l_0, l_1, l_2 \leq d$. Let $\sigma_0, \ldots, \sigma_q$ be elements of linear system $|\{s_0, s_1, s_2\}|$ such that σ_j is a non-singular divisor in $\mathbb{P}^2(\mathbb{C})$. Assume that σ_j intersects σ_k transversally for all $1 \leq j \neq k \leq q$. Take finitely many points $x_1, \ldots, x_p \in \mathbb{P}^2(\mathbb{C})$ such that $\sum_{i=1}^q (\sigma_i)$ is simple normal crossing in $\mathbb{P}^2(\mathbb{C}) \setminus \{x_1, \ldots, x_p\}$. Let $\pi : (\mathbb{P}^2(\mathbb{C}))^{\sim} \to \mathbb{P}^2(\mathbb{C})$ be the blowing-up at $\{x_1, \ldots, x_p\}$, and let $E = \sum_{i=1}^p E_i$ be the exceptional divisor of π, where E_i is irreducible and $\pi(E_i) = x_i$. Let $\widetilde{\sigma}_i$ be the proper transform of σ_i under the blowing-up π.

Theorem 5. (a) *Let H be the hyperplane bundle on $\mathbb{P}^2(\mathbb{C})$. Let $f : \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ be a holomorphic map such that $f(\mathbb{C})$ is neither contained in the support of elements of $|\{s_0, s_1, s_2\}|$ nor in $\{\det(\partial s_j/\partial X_k) = 0\}$. Let $\widetilde{f} : \mathbb{C} \to (\mathbb{P}^2(\mathbb{C}))^{\sim}$ be the lift of f.*

(i) *When $d=1$, we have*

$$\sum_{i=1}^{p} T_{\widetilde{f}}(r,[\widetilde{\sigma}_i]) + \sum_{i=1}^{p} T_{\widetilde{f}}(r,[E_i]) - 3T_f(r,H) \tag{5}$$
$$\leq \sum_{i=1}^{q} N_2(r,\widetilde{f}^*\widetilde{\sigma}) + \sum_{i=1}^{p} N(r,\widetilde{f}^*E_i) + S_f(r).$$

(ii) *When $d \geq 2$, we have*

$$\sum_{i=1}^{p} T_{\widetilde{f}}(r,[\widetilde{\sigma}_i]) + \sum_{i=1}^{p} T_{\widetilde{f}}(r,[E_i]) - (6+l_0+l_1+l_2)T_f(r,H) \tag{6}$$
$$\leq \sum_{i=1}^{q} N_2(r,\widetilde{f}^*\widetilde{\sigma}) + \sum_{i=1}^{p} N(r,\widetilde{f}^*E_i) + S_f(r).$$

(b) *Furhtermore, we assume that $\sigma_1,\ldots,\sigma_q$ are in m-subgeneral position. Let $H_1,H_2,H_3 \subset \mathbb{P}^2(\mathbb{C})$ be hyperplanes in general position which do not pass through $\{x_1,\ldots,x_p\}$.*

(i) *When $d=1$, we have*

$$(q-3)T_f(r,H) \leq \sum_{i=1}^{q} N_2(r,\widetilde{f}^*\widetilde{\sigma}_i) + m\sum_{i=1}^{p} N(r,\widetilde{f}^*E_i) \tag{7}$$
$$+\frac{m-1}{2}\sum_{i=1}^{3} N_2(r,f^*H_i) + S_f(r).$$

(ii) *When $d \geq 2$, we have*

$$\left(q-\frac{6+l_0+l_1+l_2}{d}\right)T_f(r,H) \leq \sum_{i=1}^{p} N_2(r,\widetilde{f}^*\widetilde{\sigma}_i) + m\sum_{i=1}^{p} N(r,\widetilde{f}^*E_i) \tag{8}$$
$$+\frac{m-1}{2}\sum_{i=1}^{3} N_2(r,f^*H_i) + S_f(r).$$

Proof. We construct the meromorphic partial projective connection ∇ on $\mathbb{P}^2(\mathbb{C})$ as in §2. Then $\sigma_1,\ldots,\sigma_q$ are totally geodesic with respect to ∇. When $d=1$, the pole degree of ∇ is 0. When $d \geq 2$, the pole degree of ∇ is $3+l_0+l_1+l_2$. Because σ_i intersects σ_j transversally for all $1 \leq i \neq j \leq q$, the divisor $\widetilde{\sigma}_1+\cdots+\widetilde{\sigma}_q+E$ is of simple normal crossing, and the divisor $\widetilde{\sigma}_1+\cdots+\widetilde{\sigma}_q$ is smooth. We have (5) and (6) by the same arguments as that in the proof of Theorem 5.

If $\sigma_1,\ldots,\sigma_q$ are in m-subgeneral position. We have

$$\sum_{i=1}^{q} \pi^*\sigma_i \leq \sum_{i=1}^{q} \widetilde{\sigma}_i + m\sum_{i=1}^{p} E_i,$$

on $(\mathbb{P}^2(\mathbb{C}))\tilde{}$. It follows that

$$\sum_{i=1}^{q} T_f(r, [\sigma_i]) \le \sum_{i=1}^{q} T_{\widetilde{f}}(r, [\widetilde{\sigma}_i]) + m \sum_{i=1}^{p} T_{\widetilde{f}}(r, [E_i]). \tag{9}$$

Take hyperplanes $\{L_i', L_i''\}_{1 \le i \le p}$ in $\mathbb{P}^2(\mathbb{C})$ such that L_i' and L_i'' pass through x_i, and the divisor $H_1 + H_2 + H_3 + \sum_{i=1}^{p}(L_i' + L_i'')$ is in general position. Then, by Cartan's Second Main Theorem (cf. [2]) we have

$$\begin{aligned} & \sum_{i=1}^{3} T_f(r, [H_i]) + \sum_{i=1}^{p} T_f(r, [L_i'] + [L_i'']) - 3T_f(r, H) \\ \le \quad & \sum_{i=1}^{3} N_2(r, H_i) + \sum_{i=1}^{p} N_2(r, f^*(L_i' + L_i'')) + S_f(r). \end{aligned}$$

Let $\widetilde{L}_i'$ and $\widetilde{L}_i''$ be proper transforms of L_i', L_i'' under the blowing-up π. Since

$$T_f(r, L_i' + L_i'') = T_{\widetilde{f}}(r, \widetilde{L}_i' + \widetilde{L}_i'') + 2T_{\widetilde{f}}(r, E_i)$$

and

$$N(r, f^*(L_i' + L_i'')) = N(r, \widetilde{f}^*(\widetilde{L}_i' + \widetilde{L}_i'') + 2N(r, \widetilde{f}^* E_i),$$

we have

$$\begin{aligned} & 2\sum_{i=1}^{p} T_{\widetilde{f}}(r, E_i) + \sum_{i=1}^{p} T_{\widetilde{f}}(r, \widetilde{L}_i' + \widetilde{L}_i'') \\ \le \quad & 2\sum_{i=1}^{p} N(r, \widetilde{f}^* E_i) + \sum_{i=1}^{p} N(r, \widetilde{f}^*(\widetilde{L}_i' + \widetilde{L}_i'')) + \sum_{i=1}^{3} N_2(r, f^* H_i) + S_f(r). \end{aligned}$$

By the First Main Theorem,

$$T_{\widetilde{f}}(r, [\widetilde{L}_i'] + [\widetilde{L}_i'']) \ge N(r, \widetilde{f}^*(\widetilde{L}_i' + \widetilde{L}_i'')).$$

Therefore we have

$$\sum_{i=1}^{p} T_{\widetilde{f}}(r, E_i) \le \sum_{i=1}^{p} N(r, \widetilde{f}^* E_i) + \frac{1}{2} \sum_{i=1}^{3} N_2(r, f^* H_i). \tag{10}$$

Then we deduce (7) and (8) from (5), (6), (9), and (10). $\square$

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