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UNIQUENESS IN AN INVERSE PROBLEM FOR ONE-DIMENSIONAL FRACTIONAL DIFFUSION EQUATION

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Abstract. We consider a one-dimensional fractional diffusion equation: $\partial_t^\alpha u(x,t) = \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}(x,t)\right)$, $0 < x < \ell$, where $0 < \alpha < 1$ and ∂_t^α denotes the Caputo derivative in time of order α. We attach the homogeneous Neumann boundary condition at $x = 0, \ell$ and the initial value given by the Dirac delta function. We prove that α and $p(x)$, $0 < x < \ell$, are uniquely determined by data $u(0,t)$, $0 < t < T$. The uniqueness result is a theoretical background in experimentally determining the order α of many anomalous diffusion phenomena which are important for example in the environmental engineering. The proof is based on the eigenfunction expansion of the weak solution to the initial value/boundary value problem and the Gel'fand-Levitan theory.

§1. Introduction.

Recently there are many anomalous diffusion phenomena observed which show different aspects from the classical diffusion. For example, Adams and Gelhar [1] pointed that field data in the saturated zone of a highly heterogeneous aquifer are not well simulated by the classical advection-diffusion equation which is based on

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the random walk, and the data indicate "slower" diffusion than the classical one. The slow diffusion is characterized by the long-tailed profile in spatial distribution of densities as the time passes. Also see Zhou and Selim [42]. Such slow diffusion is called the anomalous diffusion. Since [1], there have been many studies for better models, because from the practical viewpoint, the anomalous diffusion is seriously concerned e.g., with the quantitative environmental problems such as evaluation of underground contaminants. In particular, Berkowitz, Scher and Silliman [4], Y. Hatano and N. Hatano [11] have applied the continuous-time random walk to the underground environmental problem.

For applying the continuous-time random walk, we have to determine some parameters in the continuous-time random walk, and there appears an important parameter characterizing in the large-time behaviour of a waiting-time distribution function. We can refer to Y. Hatano and N. Hatano [11] where the authors fit the parameter by data of columun experiments at laboratory. See also Xiong, G. Huang and Q. Huang [40], and Berkowitz, Cortis, Dentz and Scher [3] as a survey. Although there have been many works which are concerned more experimentally with the continuous-time random walk, there are very few mathematical analyses for the parameter identification. The continuous-time random walk is a microscopic model for the anomalous diffusion, while from it, we can derive a macroscopic model equation, e.g., Metzler and Klafter [26] (pp.14-18), Roman and Alemany [34], Sokolov, Klafter and Blumen [36]. The derivation corresponds to the way with which the classical diffusion equation is derived from the random walk, and as a macroscopic model from the continuous-time random walk, we have a fractional diffusion equation:

$$\partial_t^\alpha u(x,t) = \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}(x,t)\right), \quad 0 < x < \ell,\ t > 0, \tag{1.1}$$

where the diffusion coefficient $p(x)$ describes the heterogeneity of the medium, $\alpha > 0$, and $\partial_t^\alpha u(x,t)$ means the Caputo derivative :

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{\partial u}{\partial s}(x,s)ds. \tag{1.2}$$

See e.g., Kilbas, Srivastava and Trujillo [15], Podlubny [31] for the definition and properties of the Caputo derivative.

In the slow diffusion, we can take $0 < \alpha < 1$. The fractional order α is related with the parameter specifying the large-time behaviour of the waiting-time distribution function. As related papers, see Giona, Gerbelli and Roman [8], Giona and Roman [9], Mainardi [21] - [23], Metzler, Glöckle and Nonnenmacher [25], Metzler and Klafter [27], Nigmatullin [29], Roman [33] and see section 10.10 in Podlubny [31].

The main purpose of this paper is to establish the uniqueness in determining α and $p(x)$ by means of observation data $u(0,t)$, $0 < t < T$ at one end point. By our uniqueness result, we expect that by experiments, we can identify an important parameter α and $p(x)$ characterizing the anomalous diffusion.

There are many works on the forward problem for fractional diffusion equations such as an initial value/ boundary value problem and we refer to Bazhlekova [2], Eidelman and Kochubei [6], Metzler and Klafter [27], Gorenflo, Luchko and Zabrejko [10], Hanyga [12], Luchko [19], [20] and the references therein. Also see Prüss [32] (e.g., Section 2 of Chapter I) as a monograph. However, to the authors' best knowledge, there are very few works on inverse problems for fractional diffusion equations in spite of the physical and practical importance, and our uniqueness is the first mathematical result for the coefficient inverse problem for a fractional differential equation.

The paper is composed of 3 sections and an appendix. In section 2, we formulate our inverse problem and state the uniqueness in the inverse problem as main result. In section 3, we complete the proof of the main result. Appendix is devoted to the proof of the unique existence of the weak solution.

§2. Formulation and the main result.

We consider the following fractional partial differential equation.

$$\partial_t^\alpha u(x,t) = \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}(x,t)\right), \quad 0 < x < \ell,\ 0 < t < T, \tag{2.1}$$

$$u(x,0) = \delta(x), \tag{2.2}$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\ell,t) = 0,\ \ 0 < t \le T. \tag{2.3}$$

Here $T > 0, \ell > 0$ are fixed and $\delta(x)$ is the Dirac delta function,

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{\partial u}{\partial s}(x,s)ds$$

(e.g., [15], [31]). We assume that $p \in C^2[0,\ell]$ and $0 < \alpha < 1$. The initial condition (2.2) means that we start experiments by setting up a density profile concentrating at $x = 0$, and the boundary condition (2.3) requires no fluxes at the both end points.

We discuss

Inverse problem. Determine the order $\alpha \in (0,1)$ of the time derivative and the diffusion coefficient $p(x)$ from boundary data $u(0,t)$, $0 < t \le T$.

Due to the irregular initial value in (2.2), we have to consider a weak solution to (2.1) - (2.3) which is defined below. In terms of the weak solution, we can state our main result.

Theorem 2.1. *Let us assume $p, q \in C^2[0, \ell]$, $p, q > 0$ on $[0, \ell]$, $\alpha, \beta \in (0, 1)$. Let u be the weak solution to (2.1) - (2.3), and let v be the weak solution to (2.4) with the same initial and boundary conditions as (2.2) and (2.3):*

$$\partial_t^\beta v(x,t) = \frac{\partial}{\partial x}\left(q(x)\frac{\partial v}{\partial x}(x,t)\right), \quad 0 < x < \ell,\, 0 < t < T. \tag{2.4}$$

Then $u(0,t) = v(0,t), 0 < t \le T$ with some $T > 0$, implies $\alpha = \beta$ and $p(x) = q(x)$, $0 \le x \le \ell$.

In the case of $\alpha = \beta = 1$, our inverse problem is concerned with the one-dimensional diffusion equation and we can refer to Isakov and Kindermann [14], Murayama [28], Pierce [30], Suzuki [37], [38], Suzuki and Murayama [39]. As source books for inverse problems for partial differential equations without fractional order derivatives, see for example, Isakov [13], Klibanov and Timonov [16] and Lavrent'ev, Romanov and Shishat·skiĭ[17], Romanov [35].

In Luchko [20] and Podlubny [31] for example, solutions to initial value-boundary value problems for fractional diffusion equations are constructed by the eigenfunction expansions or the Laplace transform, etc., and in [20] such a formally constructed solution is proved to be a unique weak solution in a suitable sense, but in [20] initial values must be smoother and the Dirac delta function can not be discussed. Even in our case (2.2), one can easily construct formal solutions by the same method (see (2.11)). However, to the authors' best knowledge, there are no works on any relevant definitions and the unique existence of weak solution in the case of singular initial values such as in (2.2), although the initial condition (2.2) describes a pointwise density profile and so is quite physical. Thus it is necessary that first we have to introduce a relevant definition for the weak solution to (2.1) - (2.3) and second verify that the formally constructed solution is really the weak solution in that sense, which requires an independent and non-trivial work. For it, we need function spaces.

Now we define the weak solution to (2.1) - (2.3). First we define an operator A_p in $L^2(0,\ell)$ by

$$\begin{cases} (A_p\psi)(x) = -\dfrac{d}{dx}\left(p(x)\dfrac{d}{dx}\psi(x)\right), \; 0 < x < \ell, \\ \mathcal{D}(A_p) = \left\{\psi \in H^2(0,\ell); \dfrac{d\psi}{dx}(0) = \dfrac{d\psi}{dx}(\ell) = 0\right\}. \end{cases} \tag{2.5}$$

It is known that the operator A_p has only real and simple eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$, and with suitable numbering, we have

$$0 = \lambda_1 < \lambda_2 < \cdots, \quad \lim_{n\to\infty} \lambda_n = \infty.$$

Moreover by means of the Liouville transform (e.g., Yosida [41], Levitan and Sargsjan [18]), we see the following asymptotic:

$$\lambda_n = \left(\int_0^\ell \frac{1}{\sqrt{p(x)}}dx\right)^{-2} n^2\pi^2 + O(1), \; n \to \infty. \tag{2.6}$$

By φ_n we denote the eigenfunction corresponding to λ_n which satisfies $\varphi_n(0) = 1$. Henceforth $(\cdot,\cdot)$ denotes the scalar product in $L^2(0,\ell)$ and we set $\|\varphi\|_{L^2(0,\ell)} = \|\varphi\| = (\varphi,\varphi)^{\frac{1}{2}}$. We define

$$\rho_n = \|\varphi_n\|^{-2}.$$

Then, for each $v \in L^2(0,\ell)$, we have the eigenfunction expansion :

$$\psi = \sum_{n=1}^\infty \rho_n(\psi,\varphi_n)\varphi_n.$$

Moreover $\{\rho_n\}_{n\in\mathbb{N}}$ satisfies the asymptotic behaviour: there exists a constant $c_0 > 0$ such that

$$\rho_n = c_0 + o(1), \ n \to \infty, \tag{2.7}$$

which is derived by the Liouville transform (e.g., [41], [18]).

Now we arbitrarily choose a constant $M > 0$ and define the operator $A_{p,M}$ in $L^2(0,\ell)$ as follows :

$$\begin{cases} (A_{p,M}\psi)(x) = -\dfrac{d}{dx}\left(p(x)\dfrac{d}{dx}\psi(x)\right) + M\psi, \quad 0 < x < \ell, \\ \mathcal{D}(A_{p,M}) = \left\{\psi \in H^2(0,\ell); \dfrac{d\psi}{dx}(0) = \dfrac{d\psi}{dx}(\ell) = 0\right\}. \end{cases}$$

Then the set of all the eigenvalues of $A_{p,M}$ is $\{\lambda_n + M\}_{n\in\mathbb{N}}$, and we set $\lambda_n^{(M)} = \lambda_n + M$. Then we have $\lambda_n^{(M)} > 0$, $n \in \mathbb{N}$.

We define the function space $\mathcal{D}(A_{p,M}^\kappa)$ for $\kappa > 0$ by

$$\mathcal{D}(A_{p,M}^\kappa) = \left\{\psi \in L^2(0,\ell); \sum_{n=1}^\infty \rho_n |\lambda_n^{(M)}|^{2\kappa} |(\psi,\varphi_n)|^2 < \infty\right\}.$$

Then we see that $\mathcal{D}(A_{p,M}^\kappa)$ is a Banach space with the norm :

$$||\psi||_{\mathcal{D}(A_{p,M}^\kappa)} = \left\{\sum_{n=1}^\infty \rho_n |\lambda_n^{(M)}|^{2\kappa} |(\psi,\varphi_n)|^2\right\}^{\frac{1}{2}}.$$

We have $\mathcal{D}(A_{p,M}^\kappa) = H^{2\kappa}(0,\ell)$ if $0 \le \kappa < \frac{3}{4}$. Since $\mathcal{D}(A_{p,M}^\kappa) \subset L^2(0,\ell)$, identifying the dual $L^2(0,\ell)'$ with itself, we have $\mathcal{D}(A_{p,M}^\kappa) \subset L^2(0,\ell) \subset (\mathcal{D}(A_{p,M}^\kappa))'$. Here $(\mathcal{D}(A_{p,M}^\kappa))'$ denotes the dual space, which consists of bounded linear functionals on the Banach space $\mathcal{D}(A_{p,M}^\kappa)$. Henceforth we set $\mathcal{D}(A_{p,M}^{-\kappa}) = (\mathcal{D}(A_{p,M}^\kappa))'$. For $f \in \mathcal{D}(A_{p,M}^{-\kappa})$ and $\psi \in \mathcal{D}(A_{p,M}^\kappa)$, by ${}_{-\kappa}< f,\psi >_\kappa$ we denote the value which is obtained by operating f to ψ. We note that $\mathcal{D}(A_{p,M}^{-\kappa})$ is a Banach space with the norm :

$$||f||_{\mathcal{D}(A_{p,M}^{-\kappa})} = \left\{\sum_{n=1}^\infty \rho_n |\lambda_n^{(M)}|^{-2\kappa} |{}_{-\kappa}< f,\varphi_n >_\kappa|^2\right\}^{\frac{1}{2}}.$$

Now we fix $0 < \epsilon < \frac{1}{2}$. By the Sobolev embedding theorem, we have $\delta \in \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})$ and $\delta = \sum_{n=1}^{\infty} \rho_n \varphi_n$ in $\mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})$. We set $< \cdot, \cdot >= {}_{-\frac{1}{4}-\epsilon}< \cdot, \cdot >_{\frac{1}{4}+\epsilon}$. We note

$$< f, \psi >= (f, \psi) \text{ if } f \in L^2(0,\ell) \text{ and } \psi \in \mathcal{D}(A_{p,M}^{\frac{1}{4}+\epsilon})$$

(e.g., Chapter V in Brezis [5]).

Let us define the weak solution to system (2.1) - (2.3) as follows.

Definition. *We call that u is a weak solution to (2.1) - (2.3) if the following conditions hold :*

$$\begin{cases} u(\cdot,t) \in L^2(0,\ell) \text{ for } \ 0 < t \le T, \\ u \in C([0,T]; \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})), \\ \dfrac{\partial}{\partial t} u, \ \partial_t^{\alpha} u, \ A_{p,M} u \in C((0,T]; \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})), \end{cases} \tag{2.8}$$

$$\lim_{t\to 0} ||u(\cdot,t) - \delta||_{\mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})} = 0, \tag{2.9}$$

$$< \partial_t^{\alpha} u(\cdot,t), \psi > +(u(\cdot,t), A_p \psi) = 0 \text{ for } t \in (0,T], \ \psi \in \mathcal{D}(A_p). \tag{2.10}$$

Remark. Let u be a sufficiently smooth weak solution. Then, integrating (2.10) by parts, we have

$$\begin{aligned} &0 =< \partial_t^{\alpha} u(\cdot,t), \psi > +(u(\cdot,t), A_p \psi) \\ &= \left(\partial_t^{\alpha} u - \frac{\partial}{\partial x} \left(p(x) \frac{\partial u}{\partial x} \right), \psi \right) + \left[\psi(x) p(x) \frac{\partial u}{\partial x}(x,t) \right]_{x=0}^{x=\ell} \end{aligned}$$

for $\psi \in \mathcal{D}(A_p)$. Taking $\psi \in C_0^{\infty}(0,\ell)$, we see that $\partial_t^{\alpha} u(x,t) = \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right)$ for $x \in (0,\ell)$ and $t \in (0,T]$. Since we arbitrarily choose $\psi(0)$ and $\psi(\ell)$ within $\psi \in \mathcal{D}(A_p)$, we obtain $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\ell,t) = 0$ for $t \in (0,T]$. Therefore the smooth weak solution satisfies (2.1) and (2.3) in a usual sense.

Proposition 2.1. *There exists a unique weak solution to (2.1) - (2.3) and*

$$u(x,t) = \sum_{n=1}^{\infty} \rho_n E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n(x) \quad \textit{in } C([0,T]; \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\varepsilon})). \tag{2.11}$$

Here for $\alpha > 0$ and $\beta \in \mathbb{R}$, the Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{2.12}$$

(e.g., [15], [31]) and Γ is the gamma function. We note that $E_{\alpha,\beta}(z)$ is an entire function in $z \in \mathbb{C}$ (e.g., [15]). Moreover we notice that the regularity of our weak solution is sufficient in proving Theorem 2.1. The proof of Proposition 2.1 is done in a setting similar to the formulation of weak solutions for partial differential equations (e.g., Brezis [5]) and given in Appendix.

§3. Proof of Theorem 2.1.

We will use the following result on the Mittag-Leffler function.

Lemma 3.1. *If $\alpha < 2$, β is an arbitrary real number and μ satisfies $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$, then there exists a constant $C_1 > 0$ such that*

$$|E_{\alpha,\beta}(z)| \le \frac{C_1}{1+|z|}, \; z \in \mathbb{C}, \, \mu \le |\arg(z)| \le \pi.$$

For the proof, we refer to Theorem 1.6 (p.35) in Podlubny [31] for example.

By Proposition 2.1, the weak solutions u and v are given by

$$u(x,t) = \sum_{n=1}^{\infty} \rho_n E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n(x) \tag{3.1}$$

and

$$v(x,t) = \sum_{n=1}^{\infty} \sigma_n E_{\beta,1}(-\mu_n t^\beta)\psi_n(x). \tag{3.2}$$

Here $0 = \lambda_1 < \lambda_2 < \cdots$, $n \in \mathbb{N}$ are all the eigenvalues of the operator A_p defined by (2.5) and φ_n is the eigenfunction corresponding to λ_n with $\varphi_n(0) = 1$ and we set $\rho_n = ||\varphi_n||^{-2}_{L^2(0,\ell)}$, while $0 = \mu_1 < \mu_2 < \cdots$ are all the eigenvalues of A_q, ψ_n is the eigenfunction corresponding to μ_n with $\psi_n(0) = 1$, and we set $\sigma_n = ||\psi_n||^{-2}_{L^2(0,\ell)}$.

Let $t_0 > 0$ be arbitrarily fixed. By the Sobolev embedding theorem, we have

$$\|\varphi_n\|_{C[0,\ell]} \le C_0' \|\varphi_n\|_{H^{\frac{1}{2}+2\varepsilon}(0,\ell)}$$

with sufficiently small $\varepsilon > 0$. Moreover we see that

$$\|\varphi_n\|_{H^{\frac{1}{2}+2\varepsilon}(0,\ell)} \le C_0' \|A_{p,M}^{\frac{1}{4}+\varepsilon}\varphi_n\|_{L^2(0,\ell)} = C_0' |\lambda_n^{(M)}|^{\frac{1}{4}+\varepsilon} \frac{1}{\sqrt{\rho_n}}.$$

Hence by Lemma 3.1, (2.6) and (2.7), we have

$$\sum_{n=1}^{\infty} \max_{0\le x\le \ell} |\rho_n E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n(x)| \le C_0 \sum_{n=1}^{\infty} \sqrt{\rho_n} |\lambda_n^{(M)}|^{\frac{1}{4}+\varepsilon} \frac{1}{1+|\lambda_n t^\alpha|} < \infty$$

for $t_0 \le t \le T$. Therefore we see that the series on the right-hand sides of (3.1) and (3.2) are convergent uniformly in $x \in [0,\ell]$ and $t \in [t_0, T]$.

Consequently, assuming that $u(0,t) = v(0,t)$ for $0 < t \le T$, we have

$$\sum_{n=1}^{\infty} \rho_n E_{\alpha,1}(-\lambda_n t^\alpha) = \sum_{n=1}^{\infty} \sigma_n E_{\beta,1}(-\mu_n t^\beta), \quad 0 < t \le T. \tag{3.3}$$

Since we see that from Lemma 3.1, (2.6) and (2.7) that the both sides of this equation are analytic in Re $t > 0$, we have

$$\sum_{n=1}^{\infty} \rho_n E_{\alpha,1}(-\lambda_n t^\alpha) = \sum_{n=1}^{\infty} \sigma_n E_{\beta,1}(-\mu_n t^\beta), \quad t > 0.$$

For $E_{\alpha,1}(z)$, we have the following asymptotic behaviour

$$E_{\alpha,1}(-t) = \frac{1}{t\Gamma(1-\alpha)} + O(|t|^{-2}) \quad \text{as } t > 0, \to \infty. \tag{3.4}$$

(e.g., Theorem 1.4 (pp.33-34) in [31]).

First Step. First we will deduce $\alpha = \beta$ and

$$\int_0^\ell \frac{1}{\sqrt{p(x)}} dx = \int_0^\ell \frac{1}{\sqrt{q(x)}} dx.$$

Since $\lambda_1 = \mu_1 = 0$ and $\lambda_n > 0$, $\mu_n > 0$ for $n = 2, 3, 4, ...$, we have

$$\sum_{n=1}^\infty \rho_n E_{\alpha,1}(-\lambda_n t^\alpha) = \rho_1 + \sum_{n=2}^\infty \rho_n \left[\frac{1}{\Gamma(1-\alpha)} \cdot \frac{1}{\lambda_n t^\alpha} + \left\{ E_{\alpha,1}(-\lambda_n t^\alpha) - \frac{1}{\Gamma(1-\alpha)} \cdot \frac{1}{\lambda_n t^\alpha} \right\} \right].$$

By (3.4) and $\lambda_n > 0$ for $n \ge 2$, there exists a constant $C_1 > 0$ such that

$$\left| E_{\alpha,1}(-\lambda_n t^\alpha) - \frac{1}{\Gamma(1-\alpha)} \frac{1}{\lambda_n t^\alpha} \right| \le \frac{C_1}{\lambda_n^2 t^{2\alpha}}, \quad n \ge 2$$

for sufficiently large t. Taking the summation for $n = 1, 2, \cdots$, by (2.6) and (2.7) we have

$$\sum_{n=1}^\infty \rho_n \left| E_{\alpha,1}(-\lambda_n t^\alpha) - \frac{1}{\Gamma(1-\alpha)} \cdot \frac{1}{\lambda_n t^\alpha} \right| \le \frac{C_2}{t^{2\alpha}}$$

with some $C_2 > 0$. Then we have

$$\sum_{n=1}^\infty \rho_n E_{\alpha,1}(-\lambda_n t^\alpha) = \rho_1 + \sum_{n=2}^\infty \rho_n \frac{1}{\Gamma(1-\alpha)} \frac{1}{\lambda_n t^\alpha} + O\left(\frac{1}{t^{2\alpha}}\right). \tag{3.5}$$

Similarly arguing for $\sum_{n=1}^\infty \sigma_n E_{\beta,1}(-\mu_n t^\beta)$, we have

$$\rho_1 + \frac{1}{t^\alpha} \sum_{n=2}^\infty \rho_n \frac{1}{\Gamma(1-\alpha)} \frac{1}{\lambda_n} + O\left(\frac{1}{t^{2\alpha}}\right) = \sigma_1 + \frac{1}{t^\beta} \sum_{n=2}^\infty \sigma_n \frac{1}{\Gamma(1-\beta)} \frac{1}{\mu_n} + O\left(\frac{1}{t^{2\beta}}\right)$$

as $t \to \infty$. This means that $\alpha = \beta$ and $\rho_1 = \sigma_1$. In fact, letting $t \to \infty$, we see that $\rho_1 = \sigma_1$. Let $\alpha > \beta$. Then the multiplication by t^β yields

$$-\frac{t^\beta}{t^\alpha} \left(\sum_{n=2}^\infty \rho_n \frac{1}{\Gamma(1-\alpha)} \frac{1}{\lambda_n} \right) + O\left(\frac{t^\beta}{t^{2\alpha}}\right) + \sum_{n=2}^\infty \sigma_n \frac{1}{\Gamma(1-\beta)} \frac{1}{\mu_n} + O\left(\frac{1}{t^\beta}\right) = 0.$$

Then, letting $t \to \infty$, by $\alpha > \beta$, we have

$$\sum_{n=2}^\infty \sigma_n \frac{1}{\Gamma(1-\beta)} \frac{1}{\mu_n} = 0.$$

By $\sigma_n > 0$ and $\mu_n > 0$ for $n \ge 2$, this is impossible. Hence we see that $\alpha > \beta$ is impossible. Similarly $\beta > \alpha$ is impossible. Therefore $\alpha = \beta$ follows.

Hence we have

$$\sum_{n=2}^\infty \rho_n E_{\alpha,1}(-\lambda_n t^\alpha) = \sum_{n=2}^\infty \sigma_n E_{\alpha,1}(-\mu_n t^\alpha), \quad t > 0. \tag{3.6}$$

Second Step. We will prove $\lambda_n = \mu_n$, $n \in \mathbb{N}$. For it, we take the Laplace transform and we can

$$\int_0^\infty e^{-zt} E_{\alpha,1}(-\lambda_n t^\alpha) dt = \frac{z^{\alpha-1}}{z^\alpha + \lambda_n}, \quad \text{Re } z > 0. \tag{3.7}$$

In fact, we can take the Laplace transforms termwise in (2.12) to obtain

$$\int_0^\infty e^{-zt} E_{\alpha,1}(-\lambda_n t^\alpha) dt = \frac{z^{\alpha-1}}{z^\alpha + \lambda_n}, \quad \text{Re } z > \lambda_n^{\frac{1}{\alpha}}$$

(cf. formula (1.80) on p.21 in [31]). Since $\sup_{t\ge 0} |E_{\alpha,1}(-\lambda_n t^\alpha)| < \infty$ by Lemma 3.1, we see that $\int_0^\infty e^{-zt} E_{\alpha,1}(-\lambda_n t^\alpha) dt$ is analytic with respect to z in Re $z > 0$. Therefore the analytic continuation yields (3.7) for Re $z > 0$.

By Lemma 3.1, (2.6), (2.7) and the Lebesgue convergence theorem, noting that

$$\left| e^{-t\mathrm{Re}\, z} \sum_{n=2}^\infty \rho_n E_{\alpha,1}(-\lambda_n t^\alpha) \right| \le C_1' e^{-t\mathrm{Re}\, z} \left(\sum_{n=2}^\infty \frac{1}{|\lambda_n|} \right) \frac{1}{t^\alpha} \le \frac{C_1''}{t^\alpha} e^{-t\mathrm{Re}\, z}, \quad t > 0,$$

and $e^{-t\mathrm{Re}\, z} t^{-\alpha}$ is integrable in $t \in (0,\infty)$ for fixed z satisfying Re $z > 0$, we have

$$\int_0^\infty e^{-zt} \sum_{n=2}^\infty \rho_n E_{\alpha,1}(-\lambda_n t^\alpha) dt = \sum_{n=2}^\infty \rho_n \frac{z^{\alpha-1}}{z^\alpha + \lambda_n}, \quad \text{Re } z > 0.$$

Similarly

$$\int_0^\infty e^{-zt} \sum_{n=2}^\infty \sigma_n E_{\alpha,1}(-\mu_n t^\alpha) dt = \sum_{n=2}^\infty \sigma_n \frac{z^{\alpha-1}}{z^\alpha + \mu_n}, \quad \text{Re } z > 0.$$

Hence (3.6) yields

$$\sum_{n=2}^\infty \frac{\rho_n}{z^\alpha + \lambda_n} = \sum_{n=2}^\infty \frac{\sigma_n}{z^\alpha + \mu_n}, \quad \text{Re } z > 0.$$

That is,

$$\sum_{n=2}^\infty \frac{\rho_n}{\eta + \lambda_n} = \sum_{n=2}^\infty \frac{\sigma_n}{\eta + \mu_n}, \quad \text{Re } \eta > 0. \tag{3.8}$$

By (2.6) and (2.7), we can analytically continue the both sides of (3.8) in η, so that (3.8) holds for $\eta \in \mathbb{C} \setminus (\{-\lambda_n\}_{n\geq 2} \cup \{-\mu_n\}_{n\geq 2})$.

Now we deduce $\lambda_2 = \mu_2$ from (3.8). Let us assume $\lambda_2 \neq \mu_2$. Without loss of generality, we can assume that $\lambda_2 < \mu_2$. Then we can take a suitable disk which includes $-\lambda_2$ and does not include $\{-\lambda_n\}_{n\geq 3} \cup \{-\mu_n\}_{n\geq 2}$. Integrating (3.8) in a disk, we have

$$2\pi i \rho_2 = 0.$$

This is contradiction because of $\rho_2 \neq 0$. Then we obtain $\lambda_2 = \mu_2$. Repeating this argument, we can obtain

$$\lambda_n = \mu_n, \quad n = 2, 3, 4, ...$$

Moreover by (2.6) we see that

$$\int_0^\ell \frac{1}{\sqrt{p(x)}} dx = \int_0^\ell \frac{1}{\sqrt{q(x)}} dx. \tag{3.9}$$

Third Step. In order to prove that $p = q$ on $[0, \ell]$, we apply the Gel'fand-Levitan theory. For it, we have to transform (2.1) to the canonical form by means of the Liouville transform (e.g., Yosida [41]). The argument in this step is a modification of Murayama [28]. We note that a modification is necessary because the argument in [28] is based for the eigenfunction expansion in the case of $\alpha = 1$ which is different from the case $0 < \alpha < 1$.

By (3.9), we set

$$\ell_0 = \int_0^\ell \frac{1}{\sqrt{p(x)}} dx = \int_0^\ell \frac{1}{\sqrt{q(x)}} dx.$$

By the Liouville transform, we have

$$z = z(x) = \int_0^x \frac{1}{\sqrt{p(\xi)}} d\xi$$

and

$$\widetilde{u}(z,t) = u(x,t)p(x)^{1/4},$$

system (2.1) - (2.3) is transformed to

$$\begin{cases} \partial_t^\alpha \widetilde{u} + \left(a - \dfrac{\partial^2}{\partial z^2}\right)\widetilde{u} = 0, \quad 0 < z < \ell_0,\, 0 < t < T, \\ \dfrac{\partial \widetilde{u}}{\partial z}(0,t) - h\widetilde{u}(0,t) = 0, \quad 0 < t < T, \\ \dfrac{\partial \widetilde{u}}{\partial z}(\ell_0,t) + H\widetilde{u}(\ell_0,t) = 0, \quad 0 < t < T, \\ \widetilde{u}(z,0) = \delta(z)f(z), \quad 0 < z < \ell_0, \end{cases}$$

where

$$a(z) = \frac{1}{f(z)}\frac{d^2}{dz^2}f(z), \quad f(z) = p(x)^{1/4} \tag{3.10}$$

and

$$h = \frac{1}{f(0)}\frac{df}{dx}(0), \quad H = -\frac{1}{f(\ell_0)}\frac{df}{dx}(\ell_0). \tag{3.11}$$

Similarly, by

$$w = w(y) = \int_0^y \frac{1}{\sqrt{q(\xi)}}d\xi$$

and

$$\widetilde{v}(z,t) = v(y,t)q(y)^{1/4},$$

system (2.2) - (2.4) is transformed to

$$\begin{cases} \partial_t^\alpha \widetilde{v} + \left(b - \dfrac{\partial^2}{\partial w^2}\right)\widetilde{v} = 0, \quad 0 < w < \ell_0,\, 0 < t < T, \\ \dfrac{\partial \widetilde{v}}{\partial w}(0,t) - j\widetilde{v}(0,t) = 0, \quad 0 < t < T, \\ \dfrac{\partial \widetilde{v}}{\partial w}(\ell_0,t) + J\widetilde{v}(\ell_0,t) = 0, \quad 0 < t < T, \\ \widetilde{v}(w,0) = \delta(w)g(w), \quad 0 < w < \ell_0, \end{cases}$$

where

$$b(w) = \frac{1}{g(w)}\frac{d^2}{dw^2}g(w), \quad g(w) = q(y)^{1/4} \tag{3.12}$$

and

$$j = \frac{1}{g(0)}\frac{dg}{dw}(0), \quad J = -\frac{1}{g(\ell_0)}\frac{dg}{dw}(\ell_0). \tag{3.13}$$

Then $u(0,t) = v(0,t)$, $0 < t < T$ is equivalent to

$$p(0)^{-1/4}\widetilde{u}(0,t) = q(0)^{-1/4}\widetilde{v}(0,t), \quad 0 < t < T. \tag{3.14}$$

We will define an operator $A_{a,h,H}$ in $L^2(0,\ell_0)$ by

$$\begin{cases} (A_{a,h,H}\psi)(z) = -\dfrac{\partial^2}{\partial z^2}\psi + a(z)\psi(z), \quad 0 < z < \ell_0, \\ \mathcal{D}(A_{a,h,H}) = \left\{\psi \in H^2(0,\ell_0);\ \dfrac{d\psi}{dz}(0) - h\psi(0) = \dfrac{d\psi}{dz}(\ell_0) + H\psi(\ell_0) = 0\right\} \end{cases}$$

and we define an operator $A_{b,j,J}$ similarly. By $\sigma(A_{a,h,H})$, we denote the set of all the eigenvalues of $A_{a,h,H}$. Since the Liouville transform does not change the eigenvalues, by $\sigma(A_p) = \sigma(A_q)$ we obtain

$$\sigma(A_{a,h,H}) = \sigma(A_{b,j,J}) = \{\lambda_n\}_{n\in\mathbb{N}}. \tag{3.15}$$

Let $\widetilde{\varphi}_n$ and $\widetilde{\psi}_n$, $n \in \mathbb{N}$ be the corresponding eigenfunctions of $A_{a,h,H}$ and $A_{b,j,J}$ for λ_n respectively such that $\widetilde{\varphi}_n(0) = \widetilde{\psi}_n(0) = 1$. We set

$$\widetilde{\rho}_n = \frac{1}{\|\widetilde{\varphi}_n\|^2_{L^2(0,\ell_0)}}, \quad \widetilde{\sigma}_n = \frac{1}{\|\widetilde{\psi}_n\|^2_{L^2(0,\ell_0)}}.$$

Similarly to Proposition 2.1, noting that $\widetilde{u}(z,0) = \delta(z)p(x)^{\frac{1}{4}}$ and $\widetilde{v}(w,0) = \delta(w)q(y)^{\frac{1}{4}}$, we obtain

$$\begin{cases} \widetilde{u}(z,t) = p(0)^{1/4}\displaystyle\sum_{n=1}^{\infty}\widetilde{\rho}_n E_{\alpha,1}(-\lambda_n t^{\alpha})\widetilde{\varphi}_n(z), \\ \widetilde{v}(z,t) = q(0)^{1/4}\displaystyle\sum_{n=1}^{\infty}\widetilde{\sigma}_n E_{\alpha,1}(-\lambda_n t^{\alpha})\widetilde{\psi}_n(z), \end{cases} \tag{3.16}$$

where the convergences are understood in a corresponding space to (2.8). Moreover it is known (e.g., [18]) that $\sup_{n\in\mathbb{N}}\widetilde{\rho}_n$, $\sup_{n\in\mathbb{N}}\widetilde{\sigma}_n < \infty$. Therefore by (2.6), (2.7) and Lemma 3.1, we can prove that the series on the right-hand sides of (3.16) are convergent in $C((0,T];C[0,\ell_0])$.

Hence (3.14) yields

$$\sum_{n=1}^{\infty} \widetilde{\rho}_n E_{\alpha,1}(-\lambda_n t^\alpha) = \sum_{n=1}^{\infty} \widetilde{\sigma}_n E_{\alpha,1}(-\lambda_n t^\alpha), \quad 0 < t \le T.$$

Similarly to (3.8), we can argue to obtain

$$\sum_{n=1}^{\infty} \frac{\widetilde{\rho}_n}{\eta + \lambda_n} = \sum_{n=1}^{\infty} \frac{\widetilde{\sigma}_n}{\eta + \lambda_n}, \quad \eta \in \mathbb{C} \setminus \{-\lambda_n\}_{n\in\mathbb{N}}.$$

Integrating the both sides in a sufficiently small disk centred at $-\lambda_n$, we see that

$$\widetilde{\rho}_n = \widetilde{\sigma}_n, \qquad n \in \mathbb{N}. \tag{3.17}$$

By (3.15) and (3.17), we apply the Gel'fand-Levitan theory (e.g., Theorem 1.4.2 (p.21) in Freiling and Yurko [7], Marchenko [24]) to have

$$a(z) = b(z), \quad 0 \le z \le \ell_0, \quad h = j,\, H = J. \tag{3.18}$$

Finally we have to derive $p(x) = q(x)$, $0 \le x \le \ell$ from (3.18). The argument is same as in Murayama [28] and we repeat it for the completeness. We first have

$$\ell = \int_0^{\ell_0} \frac{dx}{dz} dz = \int_0^{\ell_0} \sqrt{p(x)} dz = \int_0^{\ell_0} f(z)^2 dz$$

and similarly

$$\ell = \int_0^{\ell_0} g(z)^2 dz.$$

On the other hand, we can prove that a positive solution $e = e(z)$ to

$$\begin{cases} \dfrac{d^2 e}{dz^2}(z) = a(z)e(z), \quad 0 < z < \ell_0, \\ \dfrac{1}{e(0)} \dfrac{de}{dz}(0) = h, \quad \displaystyle\int_0^{\ell_0} e(z)^2 dz = \ell, \end{cases}$$

is unique. Consequently we have

$$g(z) = f(z), \qquad 0 \le z \le \ell_0$$

by (3.10) - (3.13) and (3.18). Therefore, since

$$\frac{dz}{dx} = \frac{1}{f(z)^2}, \quad 0 \le x \le \ell,\ z(0) = 0$$

and

$$\frac{dw}{dx} = \frac{1}{g(w)^2}, \quad 0 \le x \le \ell,\ w(0) = 0,$$

we obtain $w(x) = z(x)$, $0 \le x \le \ell$. Therefore

$$q(x) = \left(\frac{dw}{dx}(x)\right)^{-2} = \left(\frac{dz}{dx}(x)\right)^{-2} = p(x), \quad 0 \le x \le \ell.$$

Thus the proof of Theorem 2.1 is completed.

Appendix. Proof of Proposition 2.1.

First Step. We will prove the uniqueness of the weak solutions to system (2.1) -(2.3).

Let u be a weak solution with $u(\cdot, 0) = 0$. We set

$$v_n(t) = < u(\cdot, t), \varphi_n >,\ 0 < t \le T.$$

By $u \in C([0, T]; \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon}))$, we see that $v_n \in C[0, T]$ and $v_n(0) = 0$. By $u(\cdot, t) \in L^2(0, \ell),\ t \in (0, T]$, we have $v_n(t) = (u(\cdot, t), \varphi_n)$ for $t \in (0, T]$. Therefore (2.10) implies

$$< \partial_t^\alpha u(\cdot, t), \varphi_n > + (u(\cdot, t), A_p\varphi_n) = 0,\ 0 < t \le T,$$

that is,

$$< \partial_t^\alpha u(\cdot, t), \varphi_n > + \lambda_n v_n(t) = 0,\ 0 < t \le T. \tag{1}$$

Now we prove the following.

Lemma 1. *We have*

$$< \partial_t^\alpha u(\cdot,t), \varphi_n >= \partial_t^\alpha (u(\cdot,t), \varphi_n),\ 0 < t \le T.$$

Proof. Since the third condition in (2.8) yields

$$\partial_t^\alpha u(\cdot,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial u}{\partial s}(\cdot,s)ds \in \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon}), \quad 0 < t \le T,$$

setting

$$J_{\varepsilon_1,\varepsilon_2}(\cdot,t) = \frac{1}{\Gamma(1-\alpha)} \int_{\varepsilon_1}^{t-\varepsilon_2} (t-s)^{-\alpha} \frac{\partial u}{\partial s}(\cdot,s)ds,$$

we have $\lim_{\varepsilon_1,\varepsilon_2\to 0, \varepsilon_1,\varepsilon_2>0} J_{\varepsilon_1,\varepsilon_2}(\cdot,t) = \partial_t^\alpha u(\cdot,t)$ in $\mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})$ for $0 < t \le T$. Approximating $J_{\varepsilon_1,\varepsilon_2}(\cdot,t)$ by the Riemann sum, in terms of $\frac{\partial u}{\partial s} \in C([\varepsilon_1, T-\varepsilon_2]; \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon}))$, we can see

$$< J_{\varepsilon_1,\varepsilon_2}(\cdot,t), \varphi_n >= \frac{1}{\Gamma(1-\alpha)} \int_{\varepsilon_1}^{t-\varepsilon_2} (t-s)^{-\alpha} \left\langle \frac{\partial u}{\partial s}(\cdot,s), \varphi_n \right\rangle ds,\ 0 < t \le T.$$

Hence letting $\varepsilon_1, \varepsilon_2 \to 0$, by (2.8) we have

$$< \partial_t^\alpha u(\cdot,t), \varphi_n >= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \left\langle \frac{\partial u}{\partial s}(\cdot,s), \varphi_n \right\rangle ds,\ 0 < t \le T.$$

Moreover (2.8) yields

$$\left\langle \frac{\partial u}{\partial s} u(\cdot,s), \varphi_n \right\rangle = \frac{\partial}{\partial s} < u(\cdot,s), \varphi_n >= \frac{\partial}{\partial s}(u(\cdot,s), \varphi_n),\ 0 < s \le T.$$

Then we have

$$< \partial_t^\alpha u(\cdot,t), \varphi_n >= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial}{\partial s}(u(\cdot,s), \varphi_n)ds = \partial_t^\alpha (u(\cdot,s), \varphi_n),\ 0 < t \le T.$$

Thus the proof of the lemma is completed.

Applying Lemma 1 in (1), we have

$$\partial_t^\alpha v_n(t) + \lambda_n v_n(t) = 0,\ 0 < t \le T,\ v_n(0) = 0.$$

The uniqueness of the initial value problem for the fractional ordinary differential equation (e.g., Kilbas, Srivastava and Trujillo [15], Chapter 3 in Podlubny [31]) implies $v_n(t) = 0$ for $0 \le t \le T$ and $n \in \mathbb{N}$. Since $\{\varphi_n\}_{n\in\mathbb{N}}$ is complete in $L^2(0,\ell)$, we see that $u(\cdot,t) = 0$ for $0 \le t \le T$. The proof of the uniqueness is completed.

Second Step. Next, we will verify that the representation (2.11) gives the weak solution to system (2.1) - (2.3). In the following, we set

$$\widetilde{u}(x,t) = \sum_{n=1}^{\infty} \rho_n E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n(x).$$

Next we show

Lemma 2. *Let* $\lambda > 0$.

(i)

$$\frac{d}{dt}E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha), \quad t > 0,\, \alpha > 0.$$

(ii)

$$\partial_t^\alpha E_{\alpha,1}(-\lambda t^\alpha) = -\lambda E_{\alpha,1}(-\lambda t^\alpha), \quad t > 0,\, 0 < \alpha < 1.$$

By noting that $E_{\alpha,1}(z)$ is an entire fucntion in $z \in \mathbb{C}$, the proof of the lemma follows directly by the termwise differentiation of (2.12) and

$$\partial_t^\alpha t^{\alpha k} = \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+1-\alpha)}t^{-\alpha+\alpha k}, \quad 0 < \alpha < 1,\, k \in \mathbb{N}.$$

Now we will prove that $\widetilde{u}$ satisfies (2.8).

(i) Verification of $\widetilde{u} \in C([0,T];\mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon}))$:

Let us fix $t \in [0,T]$. It follows from (2.6), (2.7) and Lemma 3.1 that

$$\sum_{n=1}^{\infty} \frac{1}{|\lambda_n^{(M)}|^{\frac{1}{2}+2\epsilon}}\rho_n|E_{\alpha,1}(-\lambda_n t^\alpha)|^2 < \infty.$$

Thus for fixed $t \in [0,T]$, we have $\widetilde{u}(\cdot,t) \in \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})$. For $t, t+h \in [0,T]$, we have

$$\begin{aligned} &||\widetilde{u}(\cdot,t+h) - \widetilde{u}(\cdot,t)||^2_{\mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})} \\ &= \sum_{n=1}^{\infty} \frac{1}{|\lambda_n^{(M)}|^{\frac{1}{2}+2\epsilon}}\rho_n|E_{\alpha,1}(-\lambda_n(t+h)^\alpha) - E_{\alpha,1}(-\lambda_n t^\alpha)|^2. \end{aligned} \tag{2}$$

Here it follows from Lemma 3.1 that $|E_{\alpha,1}(-\lambda_n(t+h)^\alpha)-E_{\alpha,1}(-\lambda_n t^\alpha)|^2$ is uniformly bounded for $n \in \mathbb{N}$. Thus using the Lebesgue convergence theorem, in terms of (2.6) and (2.7), we have

$$\lim_{h\to 0} ||\widetilde{u}(\cdot, t+h) - \widetilde{u}(\cdot, t)||_{\mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})} = 0.$$

Therefore $\widetilde{u} \in C([0,T]; \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon}))$.

(ii) Verification of $\widetilde{u}(\cdot, t) \in L^2(0,\ell)$ **for** $t \in (0,T]$:

For fixed $t \in (0,T]$, Lemma 3.1, (2.6) and (2.7) yield

$$||\widetilde{u}(\cdot,t)||^2_{L^2(0,\ell)} = \sum_{n=1}^{\infty} \rho_n |E_{\alpha,1}(-\lambda_n t^\alpha)|^2 \le \sum_{n=1}^{\infty} \rho_n \left(\frac{C_1}{1+|\lambda_n t^\alpha|}\right)^2 < \infty,$$

which means that $\widetilde{u}(\cdot,t) \in L^2(0,\ell)$ for $t \in (0,T]$.

(iii) Verification of $\frac{\partial \widetilde{u}}{\partial t} \in C((0,T]; \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon}))$:

First, we consider

$$U(x,t) = \sum_{n=1}^{\infty} \rho_n \frac{d}{dt}(E_{\alpha,1}(-\lambda_n t^\alpha))\varphi_n(x)$$

for $t \in (0,T]$. By Lemma 2 (i), we have

$$U(x,t) = \sum_{n=1}^{\infty} \rho_n (-\lambda_n) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)\varphi_n(x).$$

By Lemma 3.1, (2.6) and (2.7), we have

$$\sum_{n=1}^{\infty} \frac{1}{|\lambda_n^{(M)}|^{\frac{1}{2}+2\epsilon}} \rho_n |(-\lambda_n) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha))|^2 < \infty, \quad 0 < t \le T.$$

Thus $U(\cdot,t) \in \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})$ for $t \in (0,T]$.

Next we have

$$\left|\left|\frac{\widetilde{u}(\cdot,t+h)-\widetilde{u}(\cdot,t)}{h} - U(\cdot,t)\right|\right|^2_{\mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})}$$

(3)

$$= \sum_{n=1}^{\infty} \frac{1}{|\lambda_n^{(M)}|^{\frac{1}{2}+2\epsilon}} \rho_n \left|\frac{E_{\alpha,1}(-\lambda_n(t+h)^\alpha) - E_{\alpha,1}(-\lambda_n t^\alpha)}{h} - \frac{d}{dt}(E_{\alpha,1}(-\lambda_n t^\alpha))\right|^2.$$

Since the mean value theorem implies that

$$\left|\frac{E_{\alpha,1}(-\lambda_n(t+h)^\alpha)-E_{\alpha,1}(-\lambda_n t^\alpha)}{h}\right| = \left|\frac{dE_{\alpha,1}(-\lambda_n\eta^\alpha)}{d\eta}|_{\eta=t+\theta h}\right|$$

with some $\theta \in [0,1]$, we have

$$\left|\frac{E_{\alpha,1}(-\lambda_n(t+h)^\alpha)-E_{\alpha,1}(-\lambda_n t^\alpha)}{h} - \frac{d}{dt}(E_{\alpha,1}(-\lambda_n t^\alpha))\right|^2$$

is uniformly bounded for $n \in \mathbb{N}$ from Lemma 3.1 and Lemma 2 (i). Therefore, by the Lebesgue convergence theorem, the left-hand side of (3) tends to 0 for $h \to 0$. Hence $\frac{\partial \widetilde{u}}{\partial t}(\cdot,t)$ exists and is equal to $U(\cdot,t) \in \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})$ for $0 < t \le T$:

$$\frac{\partial \widetilde{u}}{\partial t}(\cdot,t) = \sum_{n=1}^{\infty} \rho_n(-\lambda_n)t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^\alpha)\varphi_n, \quad 0<t\le T. \tag{4}$$

The continuity of $\frac{\partial \widetilde{u}}{\partial t}(\cdot,t)$ in $t \in (0,T]$ is proved similarly to (2). Therefore $\frac{\partial \widetilde{u}}{\partial t} \in C((0,T];\mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon}))$ is proved.

(iv) Verification of $\partial_t^\alpha \widetilde{u} \in C((0,T];\mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon}))$:

Let us fix $t \in (0,T]$. For $0 < s < t$, by Lemmata 3.1 and 2 (i), the following estimation hold :

$$\left\|(t-s)^{-\alpha}\frac{\partial \widetilde{u}}{\partial s}(\cdot,s)\right\|^2_{\mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})} = (t-s)^{-2\alpha}\sum_{n=1}^{\infty}\frac{1}{|\lambda_n^{(M)}|^{\frac{1}{2}+2\epsilon}}\rho_n|\lambda_n|^2 s^{2\alpha-2}|E_{\alpha,\alpha}(-\lambda_n s^\alpha)|^2$$
$$\le C_2 s^{2\alpha-2}(t-s)^{-2\alpha}\sum_{n=1}^{\infty}\frac{1}{|\lambda_n^{(M)}|^{\frac{1}{2}+2\epsilon}}\frac{|\lambda_n|^2}{(1+|\lambda_n|s^\alpha)^2},$$

where $C_2 > 0$ is some constant. On the other hand,

$$\frac{1}{|\lambda_n^{(M)}|^{\frac{1}{2}+2\epsilon}}\frac{|\lambda_n|^2}{(1+|\lambda_n|s^\alpha)^2} = \frac{1}{|\lambda_n^{(M)}|^{\frac{1}{2}+\epsilon}}\frac{|\lambda_n|^\epsilon}{|\lambda_n^{(M)}|^\epsilon}\frac{|\lambda_n|^{2-\epsilon}}{(1+|\lambda_n|s^\alpha)^{2-\epsilon}}\frac{1}{(1+|\lambda_n|s^\alpha)^\epsilon}$$
$$\le \frac{1}{|\lambda_n^{(M)}|^{\frac{1}{2}+\epsilon}}\frac{|\lambda_n|^\epsilon}{|\lambda_n^{(M)}|^\epsilon}\frac{(|\lambda_n|s^\alpha)^{2-\epsilon}}{(1+|\lambda_n|s^\alpha)^{2-\epsilon}}\frac{1}{s^{(2-\varepsilon)\alpha}} \le \frac{1}{|\lambda_n^{(M)}|^{\frac{1}{2}+\epsilon}}\frac{|\lambda_n|^\epsilon}{|\lambda_n^{(M)}|^\epsilon}s^{-(2-\epsilon)\alpha},$$

so that

$$\left\|(t-s)^{-\alpha}\frac{\partial \widetilde{u}}{\partial s}(\cdot,s)\right\|_{\mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})} \le C_3 s^{-1+\frac{1}{2}\epsilon\alpha}(t-s)^{-\alpha}$$

with some constant $C_3 > 0$. Therefore $||(t-s)^{-\alpha}\frac{\partial \widetilde{u}}{\partial s}(\cdot, s)||_{\mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})}$ is integrable over the interval $s \in (0,t)$. Then $\partial_t^\alpha \widetilde{u}(\cdot,t) \in \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})$ exists. From (4) and Lemma 2 (i), in terms of the Lebesgue convergence theorem, we can prove

$$\partial_t^\alpha \widetilde{u}(\cdot,t) = \sum_{n=1}^\infty \rho_n(-\lambda_n)E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n, \quad 0 < t \le T. \tag{5}$$

The continuity of $\partial_t^\alpha \widetilde{u}(\cdot,t)$ in $t \in (0,T]$ is proved similarly to (2). Therefore, $\partial_t^\alpha \widetilde{u} \in C((0,T];\mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon}))$ is verified.

(v) Verification of (2.9):

Since

$$\delta = \sum_{n=1}^\infty \rho_n\varphi_n \quad \text{in } \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon}),$$

we have

$$||\widetilde{u}(\cdot,t)-\delta||^2_{\mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})} = \sum_{n=1}^\infty \frac{1}{|\lambda_n^{(M)}|^{\frac{1}{2}+2\epsilon}}\rho_n|E_{\alpha,1}(-\lambda_n t^\alpha)-1|^2.$$

Taking $t \to 0$, by Lemma 3.1 and the Lebesgue convergence theorem, we verify (2.9).

(vi) Verification of (2.10):

Let us take $\psi \in \mathcal{D}(A_p)$ arbitrarily. Then we have $\psi = \sum_{n=1}^\infty \rho_n(\psi,\varphi_n)\varphi_n$ in $\mathcal{D}(A_p)$. Then by (5), we have

$$\langle \partial_t^\alpha \widetilde{u}(\cdot,t),\psi\rangle = \left\langle \sum_{n=1}^\infty \rho_n(-\lambda_n)E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n, \sum_{m=1}^\infty \rho_m(\psi,\varphi_m)\varphi_m \right\rangle$$
$$= \sum_{n=1}^\infty \rho_n(-\lambda_n)E_{\alpha,1}(-\lambda_n t^\alpha)(\psi,\varphi_n), \quad 0 < t \le T.$$

On the other hand,

$$(\widetilde{u}(\cdot,t),A_p\psi) = \left(\sum_{n=1}^\infty \rho_n E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n, \sum_{m=1}^\infty \lambda_m\rho_m(\psi,\varphi_m)\varphi_m\right)$$
$$= \sum_{n=1}^\infty \rho_n\lambda_n E_{\alpha,1}(-\lambda_n t^\alpha)(\psi,\varphi_n),$$

which means (2.10).

From (i)-(vi) and the uniqueness of weak solution, the eigenfunction expansion (2.11) gives the weak solution.

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