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Abstract

We study whether a (Dedekind) cut has a model or not for the logical space of the logical system MPCL and for relations satisfying the MPC.1 law. The results depend on whether the quantity system is well-ordered and has the largest element or not. We apply the results to show a condition for a consistent subset to have a model. Another application is an alternative proof for the fact that the MPC.1 law is a characteristic law of the logical space of MPCL.

Contents

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1 Introduction

As proved in [7] and illustrated in [8], each logical system yields a $\{0,1\}$-valued functional logical space $(A,\mathcal{F})$ in the sense of [6] under certain reasonable conditions, where A is the set of the sentences and $\mathcal{F}$ is a set of mappings of A into $\{0,1\}$ induced by the semantics of the logical system. Meanwhile, let $\preccurlyeq$ be a relation on the set A^* of all finite sequences of elements of A. Then a (Dedekind) cut of A by $\preccurlyeq$ is a pair (X,Y) of subsets X,Y of A which satisfies $\alpha \not\preccurlyeq \beta$ for each pair (α,β) of elements α,β of A^* such that $\alpha \subseteq X$ and $\beta \subseteq Y$, where α,β are regarded as subsets of A. Also, an $\mathcal{F}$-model of the cut (X,Y) is an element $f \in \mathcal{F}$ which satisfies $X \subseteq f^{-1}\{1\}$ and $Y \subseteq f^{-1}\{0\}$.

The main purpose of this paper is to study whether the cut (X,Y) has an $\mathcal{F}$-model or not for the logical space $(A,\mathcal{F})$ of the logical system MPCL defined in [8] and for relations $\preccurlyeq$ which satisfy the law introduced and called the MPC.1 law in [9] and are contained in the validity relation $\preccurlyeq_{\mathcal{F}}$ of $(A,\mathcal{F})$. The results depend on a parameter $\mathbb{P}$ of MPCL which is called the quantity system and defined as a totally ordered commutative monoid; namely they depend on whether $\mathbb{P}$ is well-ordered and has the largest element or not.

The validity relation $\preccurlyeq_{\mathcal{F}}$ satisfies the MPC.1 law, and therefore the results apply to $\preccurlyeq_{\mathcal{F}}$. Furthermore, a subset X of A is consistent if and only if $(X,\emptyset)$ is a cut of A by $\preccurlyeq_{\mathcal{F}}$. Also, $(X,\emptyset)$ has an $\mathcal{F}$-model if and only if X has a model in a usual sense. Thus we have a condition for a consistent subset to have a model. We can also apply our results to obtain a condition for a deduction system (R,D) to be $\mathcal{F}$-complete. Suppose (R,D) is $\mathcal{F}$-sound. Then by a general results in [6], (R,D) is $\mathcal{F}$-complete if and only if the deduction relation $\preccurlyeq_{R,D}$ satisfies a characteristic law of $(A,\mathcal{F})$. As shown in [9], the MPC.1 law is a characteristic law of $(A,\mathcal{F})$. Our results may be used to obtain an alternative proof of the fact.

Our method of constructing an $\mathcal{F}$-model of a cut (X,Y) is inspired by Henkin's proof [10] of Gödel's completeness theorem [3]. We first extend (X,Y) to a cut $(X\cup Z,Y)$ by a certain subset Z of A (cf. Lemma 5.1). Next we extend $(X\cup Z,Y)$ to a cut (P,Q) which is maximal with respect to a certain order between cuts. Then (P,Q) satisfies conditions such as the ones described by Lemma 5.5. These conditions enable us to construct an $\mathcal{F}$-model of (P,Q) (cf. Lemma 5.10). The semantics of MPCL is parameterized by a $\mathbb{P}$-valued measure $|U|$ for the sets U of the entities which satisfies the pigeonhole principle. In constructing an $\mathcal{F}$-model, therefore, we have to construct such a measure. This is accomplished by using Lemma 5.6 which is an expression of the pigeonhole principle in terms of (P, Q). This method is essentially due to [11]. In the course of the construction of an $\mathcal{F}$-model, we need to deal with occurrences of variables, for the sake of which Lemma 4.19 supplies a concept of alternatives.

In [9], following the method of [11], resolution trees are used in proving that the MPC.1 law is a characteristic law of MPCL. The advantages of using (Dedekind) cuts instead of resolution trees are as follows. First, the cut method yields results not only on characteristic laws (cf. §8) but also on models (cf. §5) and the classification of the logical space (cf. §7). Secondly, the $\mathcal{F}$-model which we will construct is 'larger' than that constructed by resolution trees, and it is hoped that this will be used to prove an incompleteness theorem for MPCL like Gödel's original [4]. Lastly, Lemma 5.4 can have no counterpart in the resolution tree method, and it simplifies an argument used in resolution trees. On the other hand, the method using cuts requires different conditions (cf. Assumption 3.1) in comparison with those in [9].

This paper is organized as follows. Section 2 collects notation, terminology and basic facts about logical spaces and logical systems. In Section 3 we define the logical system MPCL. Section 4 introduces the MPC.1 law. Section 5 is devoted to the proof of the main result of this paper, and deals with the case where the quantity system is well-ordered and has the largest element. Section 6 deals with the remaining case. Sections 7 and 8 contain applications of the main result to the classification and characteristic laws of the logical space.

2 Preliminaries

The notation and terminology in §2.1–2.4 are due to [6] and [7]. In §2.5 we argue on the extension of formal languages and its relation to the logical systems. In §2.6 we define the parallelism relation on a formal language satisfying the variable operation condition.

2.1 Logical spaces

Let A be a set. A **logic** on A is a relation R between A^* and A, where A^* is the set of all finite sequences of elements of A. A **deduction system** on A is a pair (R, D) of a logic R on A and a subset D of A. Here we denote elements of A^* by $\alpha, \beta, \ldots$. When $\alpha = a_1 \cdots a_n$, we will denote the subset $\{a_1, \ldots, a_n\}$ of A also by α. A subset B of A is said to be **closed** under R, if the following holds:

$$\alpha \subseteq B,\ y \in A,\ \alpha\, R\, y \implies y \in B.$$

For each $X \in \mathcal{P}A$ there exists the smallest of the subsets of A which contain X and are closed under R. We denote it by $[X]_R$ and call it the **R-closure** of X.[1] We define the logic R^D by

$$\alpha\, R^D\, y \iff [\alpha \cup D]_R \ni y$$

for each $\alpha \in A^*$, $y \in A$. We call R^D the **D-closure** of R. Furthermore, the **deduction relation** $\preccurlyeq_{R,D}$ on A^* is defined by

$$\alpha \preccurlyeq_{R,D} \beta \iff [\alpha \cup D]_R \supseteq \bigcap_{y \in \beta} [\{y\} \cup D]_R$$

for each $(\alpha, \beta) \in A^* \times A^*$.

A **logical space** is a pair $(A, \mathcal{B})$ of a non-empty set A and a subset $\mathcal{B}$ of $\mathcal{P}A$. We call $\bigcap_{B \in \mathcal{B}} B$ the **$\mathcal{B}$-core**. A logic R on A is called a **$\mathcal{B}$-logic**, if each $B \in \mathcal{B}$ is closed under R. There exists the largest $\mathcal{B}$-logic on A by [6, Theorem 6.1] . Let C be the $\mathcal{B}$-core, Q be the largest logic on A and (R, D) be a deduction system on A. Then

- (R, D) is said to be **$\mathcal{B}$-sound** if $R^D \subseteq Q$.
- (R, D) is said to be **$\mathcal{B}$-sufficient** if $Q \subseteq R^D$.
- (R, D) is said to be **$\mathcal{B}$-complete** if $R^D = Q$.
- (R, D) is said to be **$\mathcal{B}$-core-complete** if $C = [D]_R$.

A subset X of A is said to be **$\mathcal{B}$-consistent** if $[X]_Q \neq A$. A **$\mathcal{B}$-model** of a subset X of A is a set $B \in \mathcal{B} - \{A\}$ containing X. A **$\mathcal{B}$-model** of a pair $(X, Y) \in \mathcal{P}A \times \mathcal{P}A$ is an element $B \in \mathcal{B} - \{A\}$ satisfying $X \subseteq B$ and $Y \subseteq A - B$.

Let A be a set and $\preccurlyeq$ be a relation on A^*. A pair $(X, Y) \in \mathcal{P}A \times \mathcal{P}A$ is called a **cut** of A by $\preccurlyeq$, if $\alpha \not\preccurlyeq \beta$ for each $\alpha \subseteq X$ and $\beta \subseteq Y$. We say that (X, Y) is **finite** if both X and Y are finite sets.

Let $(A, \mathcal{B})$ be a logical space and X be a subset of A. We denote the set of finite subsets of X by $\mathcal{P}'X$. Then X is said to be **super-covered** by $\mathcal{B}$, if for each $Y \in \mathcal{P}'X$ there exists an element $B \in \mathcal{B}$ such that $Y \subseteq B \subseteq X$. Furthermore, $\mathcal{B}$ is said to be **quasi-finitary**, if every subset of A which is super-covered by $\mathcal{B}$ belongs to $\mathcal{B}$. We denote by $\mathcal{B}^\cap$ the smallest of the $\cap$-closed subsets of $\mathcal{P}A$ which contain $\mathcal{B}$, and call it the **$\cap$-closure** of $\mathcal{B}$.[2] Also, we denote by $\overline{\mathcal{B}^\cap}$ the smallest of the subsets of $\mathcal{P}A$ which contains $\mathcal{B}$ and

[1] Consult [6, §4].

[2] The $\cap$-closure of $\mathcal{B}$ exists by [6, Theorem 2.5].

are $\cap$-closed and quasi-finitary, and call it the **quasi-finitary $\cap$-closure** of $\mathcal{B}$.[3]

Logical spaces $(A, \mathcal{B})$ are put into the following three **classes**.

Class 1. $\overline{\mathcal{B}^\cap} = \mathcal{B}$, that is, $\mathcal{B}$ is $\cap$-closed in $\mathcal{P}A$ and quasi-finitary.

Class 2. $\overline{\mathcal{B}^\cap} = \mathcal{B}^\cap \neq \mathcal{B}$, that is, $\mathcal{B}$ is not $\cap$-closed in $\mathcal{P}A$ and the $\cap$-closure $\mathcal{B}^\cap$ of $\mathcal{B}$ is quasi-finitary.

Class 3. $\overline{\mathcal{B}^\cap} \neq \mathcal{B}^\cap$, that is, the $\cap$-closure $\mathcal{B}$ of $\mathcal{B}$ in $\mathcal{P}A$ is not quasi-finitary.

A **$\mathbb{B}$-valued functional logical space** is a pair $(A, \mathcal{F})$ of a non-empty set A and a subset $\mathcal{F}$ of $A \to \mathbb{B}$, where $\mathbb{B}$ is a lattice which has the least element and the largest element, and is non-trivial in the sense that $\#\mathbb{B} \geq 2$. For each $f \in \mathcal{F}$ and each $a \in A$, we define $A_{f,a}$ by

$$A_{f,a} = \{x \in A \mid fx \geq a\},$$

and we define $\mathcal{B}_\mathcal{F} \subseteq \mathcal{P}A$ by

$$\mathcal{B}_\mathcal{F} = \begin{cases} \{A_{f,a} \mid f \in \mathcal{F},\ a \in \mathbb{B}\} & \text{if } \mathcal{F} \neq \emptyset, \\ \{A\} & \text{if } \mathcal{F} = \emptyset. \end{cases}$$

Then $(A, \mathcal{B}_\mathcal{F})$ is a logical space. We say the $\mathcal{F}$-core to mean the $\mathcal{B}_\mathcal{F}$-core, the $\mathcal{F}$-logics for the $\mathcal{B}_\mathcal{F}$-logics, and so on.

Remark 2.1 Let $(A, \mathcal{F})$ be a $\mathbb{T}$-valued functional logical space, where $\mathbb{T} = \{0, 1\}$. Then a subset X of A has an $\mathcal{F}$-model if and only if there exists an element $f \in \mathcal{F}$ satisfying $fx = 1$ for each $x \in X$. Also, a pair $(X, Y) \in \mathcal{P}A \times \mathcal{P}A$ has an $\mathcal{F}$-model if and only if there exists an element $f \in \mathcal{F}$ satisfying $fx = 1$ for each $x \in X$ and $fy = 0$ for each $y \in Y$.

Let $(A, \mathcal{F})$ be a $\mathbb{B}$-valued functional logical space. We define $\vec{A} = A^* \times A^*$, denote each element (α, β) of $\vec{A}$ by $\alpha \to \beta$ and call it a **sequent**. We define for each $f \in \mathcal{F}$ the **f-validity relation** $\preccurlyeq_f$ on A^* by

$$\alpha \preccurlyeq_f \beta \iff \inf f\alpha \leq \sup f\beta,$$

and define the **$\mathcal{F}$-validity relation** $\preccurlyeq_\mathcal{F}$ on A^* by

$$\alpha \preccurlyeq_\mathcal{F} \beta \iff \alpha \preccurlyeq_f \beta \text{ for every } f \in \mathcal{F}.$$

Then we define a subset $\vec{A}_f$ of $\vec{A}$ by $\vec{A}_f = \{\alpha \to \beta \in \vec{A} \mid \alpha \preccurlyeq_f \beta\}$ for each $f \in \mathcal{F}$, and we define $\vec{\mathcal{F}} \subseteq \mathcal{P}\vec{A}$ by $\vec{\mathcal{F}} = \{\vec{A}_f \mid f \in \mathcal{F}\}$. Thus $(\vec{A}, \vec{\mathcal{F}})$ is a logical space, which we call the **sequent logical space** accompanying $(A, \mathcal{F})$. A deduction system $(\vec{R}, \vec{D})$ on $\vec{A}$ is called a **characteristic law** of $(A, \mathcal{F})$ if $(\vec{R}, \vec{D})$ is $\vec{\mathcal{F}}$-core-complete. We say that a relation R on A^* **satisfies** a deduction system $(\vec{R}, \vec{D})$ on $\vec{A}$, if R, as a subset of $\vec{A} = A^* \times A^*$, is closed under $\vec{R}$ and contains $\vec{D}$.

[3] The quasi-finitary $\cap$-closure of $\mathcal{B}$ exists by [6, Theorem 2.7].

2.2 Sorted algebras

For each set A and each natural number n, an n-ary **operation** on A is a mapping α of a subset D of A^n into A. The set D is called the **domain** of α and denoted by $\mathrm{Dom}\,\alpha$, while the image αD is denoted by $\mathrm{Im}\,\alpha$. The number n is called an **arity** of α, and so if $D = \emptyset$, every natural number is an arity of α. We say that α is **total** if $D = A^n$. A subset B of A is said to be **closed** under the operation α if $\alpha(a_1, \ldots, a_n) \in B$ for each $(a_1, \ldots, a_n) \in B^n \cap D$. If B is closed under α, the **restriction** $\alpha|_{B^n \cap D}$ of α to B becomes an operation on B.

An **algebra** is a set A equipped with a family $(\alpha_\lambda)_{\lambda \in L}$ of operations on A. We often identify the operation α_λ with its index λ. We sometimes call A an L-algebra. The algebra $(A, (\alpha_\lambda)_{\lambda \in L})$ is said to be **total** if α_λ is total for every $\lambda \in L$.

Let $(A, (\alpha_\lambda)_{\lambda \in L})$ be an algebra. If a subset B of A is closed under α_λ for each $\lambda \in L$, then B becomes an algebra equipped with the family $(\beta_\lambda)_{\lambda \in L}$ consisting of restrictions β_λ of α_λ to B. Such an algebra $(B, (\beta_\lambda)_{\lambda \in L})$ is called a **subalgebra** of A. Also, an algebra $(A, (\alpha_\mu)_{\mu \in M})$ is obtained by reducing $(\alpha_\lambda)_{\lambda \in L}$ to $(\alpha_\mu)_{\mu \in M}$ for a subset M of L. Such an algebra will be called an **M-reduct** of A.

Let $(A, (\alpha_\lambda)_{\lambda \in L})$ be an algebra. For each subset S of A, the intersection of all subalgebras of A which contain S is the smallest of the subalgebras of A which contain S. We denote it by $[S]$ and call it the **closure** of S or the subalgebra **generated** by S. Define the subsets $S_n\,(n = 0, 1, \ldots)$ of A inductively as follows. First $S_0 = S$. Next for each $n \geq 1$, S_n is the set of all elements $\alpha_\lambda(a_1, \ldots, a_m)$ with $\lambda \in L, (a_1, \ldots, a_m) \in \mathrm{Dom}\,\alpha_\lambda$, and $a_i \in S_{l_i}\,(i = 1, \ldots, m)$ for some non-negative integers $l_1, \ldots, l_m$ such that $n = 1 + \sum_{i=1}^{m} l_i$. Then it is easy to show $[S] = \bigcup_{n \geq 0} S_n$. We call $S_n\,(n = 0, 1, \ldots)$ the **descendants** of S.

Two algebras A and B are said to be **similar**, if $(\alpha_\lambda)_{\lambda \in L}$ and $(\beta_\lambda)_{\lambda \in L}$ are indexed by the same set L, and α_λ and β_λ have a common arity for each $\lambda \in L$.

Let $(A, (\alpha_\lambda)_{\lambda \in L})$ and $(B, (\beta_\lambda)_{\lambda \in L})$ be similar algebras. Then a mapping f of A into B is called a **holomorphism** if it satisfies the following two conditions for all $\lambda \in L$, where n_λ denotes an arity common to α_λ and β_λ:

- If $(a_1, \ldots, a_{n_\lambda}) \in \mathrm{Dom}\,\alpha_\lambda$, then $(fa_1, \ldots, fa_{n_\lambda}) \in \mathrm{Dom}\,\beta_\lambda$ and $f(\alpha(a_1, \ldots, a_{n_\lambda})) = \beta(fa_1, \ldots, fa_{n_\lambda})$.
- If $(a_1, \ldots, a_{n_\lambda}) \in A^{n_\lambda}$ and $(fa_1, \ldots, fa_{n_\lambda}) \in \mathrm{Dom}\,\beta_\lambda$, then $(a_1, \ldots, a_{n_\lambda}) \in \mathrm{Dom}\,\alpha_\lambda$.

A bijective holomorphism is called an **isomorphism**.

$$f(\alpha_\lambda(a_1, \ldots, a_{n_\lambda})) = \beta_\lambda(fa_1, \ldots, fa_{n_\lambda}).$$

A **sorted algebra** is an algebra A equipped with an algebra T similar to A and a holomorphism σ of A into T. We call T and σ the **sorter** and the **sorting** of the sorted algebra A. For each subset S of A and each $t \in T$, we define the **t-part** S_t of S to be the inverse image $\{a \in S \mid \sigma a = t\}$ of t in S by σ.

Let (A, T, σ) and (B, T, τ) be sorted algebras with the same sorter T. Then a mapping f of A into B is said to be **sort-consistent**, if it satisfies $\tau f = \sigma$, or equivalently $f(A_t) \subseteq B_t$ for all $t \in T$.

A sorted algebra (A, T, σ) is said to be **universal** or called a **USA** (**universal sorted algebra**) if A has a subset S which satisfies the following two conditions, the latter being called the **universality**.

- $A = [S]$.
- If (A', T, σ') is a sorted algebra and φ is a mapping of S into A' satisfying $\sigma'\varphi = \sigma|_S$, then there exists a sort-consistent holomorphism f of A into A' which extends φ.

We call S as above the set of the **primes** of A. It is known that every sorted algebra has at most one prime set and that f in the above condition is uniquely determined by φ.

Theorem 2.1 Let S be a set, T be an algebra, and τ be a mapping of S into T. Then there exists a USA (A, T, σ, S) with $\sigma|_S = \tau$. If (A', T, σ', S) is also a USA with $\sigma'|_S = \tau$, then there exists a sort-consistent isomorphism of A onto A' extending id_S.

Proof Consult [7, Theorem 2.1].

Theorem 2.2 Let (A, T, σ, S) be a USA on an algebra $(A, (\alpha_\lambda)_{\lambda \in L})$. Then the algebra is **free** over S, or S is its **basis**, in the sense that the following holds:

1. $A = [S]$.
2. $S \cap \bigcup_{\lambda \in L} \operatorname{Im} \alpha_\lambda = \emptyset$, that is, no element $a \in S$ has an expression $a = \alpha_\lambda(a_1, \dots, a_k)$ with $\lambda \in L$ and $(a_1, \dots, a_k) \in \operatorname{Dom} \alpha_\lambda$.
3. Each element $a \in A - S$ has a unique expression $a = \alpha_\lambda(a_1, \dots, a_k)$ with $\lambda \in L$ and $(a_1, \dots, a_k) \in \operatorname{Dom} \alpha_\lambda$, which we call the **word form** of a.

If an algebra $(A, (\alpha_\lambda)_{\lambda \in L})$ has a basis S, then A is the direct union $\coprod_{n=0}^{\infty} S_n$ of the descendants S_n $(n = 0, 1, \dots)$ of S, and so for each element $a \in A$, there exists a unique non-negative integer n satisfying $a \in S_n$, which we call the **rank** of a and denote by $\operatorname{Rank} a$, and if $\operatorname{Rank} a \geq 1$, then the unique word form $\alpha_\lambda(a_1, \dots, a_k)$ of a satisfies $\operatorname{Rank} a = 1 + \sum_{j=1}^{k} \operatorname{Rank} a_j$.

Proof Consult [7, Theorem 2.2].

Let (A, T, σ) be a sorted algebra and V be a non-empty set. Define $A^V = \bigcup_{t \in T}(V \to A_t)$. Then we can construct a sorted algebra (A^V, T, ρ) as follows. First define the sorting ρ of A^V into T by $\rho b = t$ for each $b \in V \to A_t$ and each $t \in T$. Then

$$\rho b = \sigma(bv) \tag{2.1}$$

for each $b \in A^V$ and each $v \in V$. Let $(\alpha_\lambda)_{\lambda \in L}$ and $(\tau_\lambda)_{\lambda \in L}$ be the operations of A and T respectively, and let n_λ be an arity of α_λ and τ_λ. For each $\lambda \in L$, define the operation β_λ on A^V as follows. First define the domain of β_λ to be

$$D_\lambda = \big\{(b_1, \dots, b_{n_\lambda}) \in (A^V)^{n_\lambda} \,\big|\, (\rho b_1, \dots, \rho b_{n_\lambda}) \in \operatorname{Dom} \tau_\lambda\big\}.$$

If $(b_1, \dots, b_{n_\lambda}) \in D_\lambda$, then $\big(\sigma(b_1 v), \dots, \sigma(b_{n_\lambda} v)\big) = (\rho b_1, \dots, \rho b_{n_\lambda}) \in \operatorname{Dom} \tau_\lambda$ by (2.1), so $(b_1 v, \dots, b_{n_\lambda} v) \in \operatorname{Dom} \alpha_\lambda$ for each $v \in V$, and we can define the mapping $\beta_\lambda(b_1, \dots, b_{n_\lambda})$ of V into A by

$$\big(\beta_\lambda(b_1, \dots, b_{n_\lambda})\big)v = \alpha_\lambda(b_1 v, \dots, b_{n_\lambda} v) \tag{2.2}$$

for each $v \in V$. Furthermore by (2.1)

$$\sigma\big(\alpha_\lambda(b_1 v, \dots, b_{n_\lambda} v)\big) = \tau_\lambda\big(\sigma(b_1 v), \dots, \sigma(b_{n_\lambda} v)\big) = \tau_\lambda(\rho b_1, \dots, \rho b_{n_\lambda}), \tag{2.3}$$

and $t = \tau_\lambda(\rho b_1, \dots, \rho b_{n_\lambda})$ is not varied by $v \in V$, hence $\beta_\lambda(b_1, \dots, b_{n_\lambda}) \in V \to A_t \subseteq A^V$. Thus β_λ is an operation on A^V for each $\lambda \in L$, and so $(A^V, (\beta_\lambda)_{\lambda \in L})$ becomes an algebra. Furthermore, by (2.1), (2.2) and (2.3), we have

$$\begin{aligned}\rho\big(\beta_\lambda(b_1, \dots, b_{n_\lambda})\big) &= \sigma\big(\big(\beta_\lambda(b_1, \dots, b_{n_\lambda})\big)v\big) \\ &= \sigma\big(\alpha_\lambda(b_1 v, \dots, b_{n_\lambda} v)\big) = \tau_\lambda(\rho b_1, \dots, \rho b_{n_\lambda})\end{aligned}$$

with any element $v \in V$, and so ρ is a holomorphism of A^V into T. Thus we have constructed the sorted algebra (A^V, T, ρ), which we call the **V-power** of A. Furthermore, it follows from (2.1) and (2.2) that for each $v \in V$ the mapping $b \mapsto bv$ of A^V into A is a sort-consistent holomorphism, which we call the **projection** by v.

2.3 Logical systems

A **formal language** is a universal sorted algebra (A, T, σ, S) equipped with subsets C and $X \neq \emptyset$ of S and a set Γ which satisfy the following three conditions.

- The prime set S is the direct union $C \amalg X$ of C and X.

- Let $(\tau_\lambda)_{\lambda\in L}$ be the operations of the sorter T. Then its index set L is contained in the subset $\Gamma \cup \Gamma X$ of the free semigroup over $\Gamma \amalg S$.

- The arity of each operation τ_λ with $\lambda \in L \cap \Gamma X$ is equal to 1.

We call C and X the sets of the **constants** and **variables** respectively. Henceforth, we identify each index $\lambda \in L \cap \Gamma X$ with the operation τ_λ, call it a **variable operation**, and denote its domain by T_λ.

Let $(A, T, \sigma, S, C, X, \Gamma)$ be a formal language and $(\tau_\lambda)_{\lambda\in L}$ be the operations of T. Define $M = L \cap \Gamma$ and let T_M be the M-reduct of T. Then, a sorted algebra W is called a **denotable world** for A, if it satisfies the following two conditions.

- The sorter of W is equal to T_M.

- $W_t \neq \emptyset$ for each $t \in \sigma S$.

A **C-denotation** into the denotable world W for A is a mapping Φ of C into W which satisfies $\Phi(C_t) \subseteq W_t$ for each $t \in T$. There is at least one C-denotation. If $C = \emptyset$, then since $\emptyset \to W = \{\emptyset\}$ by the set-theoretical definition of $Y \to Z$, $\emptyset$ is the unique C-denotation. Similarly, an **X-denotation** into W is a mapping ν of X into W which satisfies $\nu(X_t) \subseteq W_t$ for each $t \in T$. We denote the set of all X-denotations into W by $V_{X,W}$. Then $V_{X,W} \neq \emptyset$, and so we can construct the $V_{X,W}$-power $(W^{V_{X,W}}, T_M, \rho)$ of W as described in §2.2. Let $(\beta_\lambda)_{\lambda\in M}$ be the operations of $W^{V_{X,W}}$.

An **interpretation** of the set $L \cap \Gamma X$ of the variable operations on the denotable world W for A is a mapping I_W which assigns each $\lambda \in L \cap \Gamma x$ with $x \in X$ a mapping

$$\lambda_W \in \left(\bigcup_{t\in T_\lambda} (W_{\sigma x} \to W_t) \right) \to W$$

which satisfies

$$\lambda_W(W_{\sigma x} \to W_t) \subseteq W_{\lambda t}$$

for each $t \in T_\lambda$. We call $\lambda_W = I_W(\lambda)$ the **meaning** of λ on W under the interpretation I_W. Then we can define the unary operation β_λ on $W^{V_{X,W}}$ for each $\lambda \in L \cap \Gamma X$ as follows, and extending the operations of $W^{V_{X,W}}$ from $(\beta_\lambda)_{\lambda\in M}$ to $(\beta_\lambda)_{\lambda\in L}$, we can construct the sorted algebra $(W^{V_{X,W}}, T, \rho)$. First we define, for each pair (x, w) of $x \in X$ and $w \in W_{\sigma x}$, the transformation $\nu \mapsto (x/w)\nu$ on $V_{X,W}$ by

$$\big((x/w)\nu\big)y = \begin{cases} \nu y & \text{if } y \in X - \{x\}, \\ w & \text{if } y = x. \end{cases}$$

We call the transformation (x/w) the **redenotation** for x by w. Next we define, for each quadruple (t, φ, x, ν) consisting of $t \in T, \varphi \in V_{X,W} \to W_t, x \in X$ and $\nu \in V_{X,W}$, the mapping $\varphi\big((x/\Box)\nu\big)$ of $W_{\sigma x}$ into W_t by

$$\big(\varphi\big((x/\Box)\nu\big)\big)w = \varphi\big((x/w)\nu\big) \tag{2.4}$$

for each $w \in W_{\sigma x}$. We finally define for each $\lambda \in L \cap \Gamma X$ the unary operation β_λ on $W^{V_{X,W}}$ as follows. Suppose $\lambda \in \Gamma x$ with $x \in X$. First we define

$$\operatorname{Dom} \beta_\lambda = \bigcup_{t \in T_\lambda} (V_{X,W} \to W_t).$$

Next for each $t \in T_\lambda$ and each $\varphi \in V_{X,W} \to W_t$ we define $\beta_\lambda\varphi$ to be the element of $V_{X,W} \to W_{\lambda t}$ such that

$$(\beta_\lambda\varphi)\nu = \lambda_W\big(\varphi\big((x/\Box)\nu\big)\big)$$

for each $\nu \in V_{X,W}$. Since $\varphi\big((x/\Box)\nu\big) \in W_{\sigma x} \to W_t$ and $\lambda_W(W_{\sigma x} \to W_t) \subseteq W_{\lambda t}$, certainly $(\beta_\lambda\varphi)\nu \in W_{\lambda t}$. Since $V_{X,W} \to W_t$ is the t-part of $W^{V_{X,W}}$ for each $t \in T$, we have thus constructed the sorted algebra $(W^{V_{X,W}}, T', \rho)$.

Now let Φ be a C-denotation into W. Then we can construct a sort-consistent holomorphism Φ^* of A into $W^{V_{X,W}}$ as follows. First we define the mapping φ of $S = C \amalg X$ into $V_{X,W} \to W$ so that

$$(\varphi a)\nu = \begin{cases} \Phi a & \text{when } a \in C, \\ \nu a & \text{when } a \in X \end{cases}$$

for each $\nu \in V_{X,W}$. Then $\varphi S_t \subseteq V_{X,W} \to W_t$ for each $t \in T$ because $\Phi(C_t) \subseteq W_t$ and $\nu(W_t) \subseteq W_t$, and so φ maps S into $W^{V_{X,W}}$ and satisfies $\rho\varphi = \sigma|_S$. Therefore by the universality of A, there exists a unique sort-consistent holomorphism of A into $W^{V_{X,W}}$ which extends φ. We call it the **metadenotation** determined by Φ and denote it by Φ^*. Since Φ^* is an extension of φ,

$$(\Phi^* a)\nu = \begin{cases} \Phi a & \text{when } a \in C, \\ \nu a & \text{when } a \in X \end{cases}$$

for each $\nu \in V_{X,W}$.

A **logical system**[4] is a triple $(A, \mathcal{W}, (I_W)_{W \in \mathcal{W}})$ of a formal language $(A, T, \sigma, S, C, X, \Gamma)$, a non-empty collection $\mathcal{W}$ of denotable worlds for A, and a family $(I_W)_{W \in \mathcal{W}}$ of interpretations I_W on $W \in \mathcal{W}$.

Suppose the logical system $(A, \mathcal{W}, (I_W)_{W \in \mathcal{W}})$ satisfies the following condition:

- For an element $\phi \in T$, the ϕ-part of A is non-empty, and the ϕ-part W_ϕ of each $W \in \mathcal{W}$ is equal to $\mathbb{T} = \{0, 1\}$.

[4] For some other kinds of formalization of a logical system, the reader may consult [2] for example.

Then we call ϕ a **truth** and call the elements of A_ϕ the **ϕ-sentences**.

Suppose $(A, \mathcal{W}, (I_W)_{W\in\mathcal{W}})$ is a logical system with a truth ϕ. Then we can construct a non-empty subset $\mathcal{F}$ of $A_\phi \to \mathbb{T}$ as follows. Let $W \in \mathcal{W}$ be a denotable world and Φ be a C-denotation into W. Then since the metadenotation Φ^* is sort-consistent and the ϕ-part $V_{X,W} \to W_\phi$ of $W^{V_{X,W}}$ is equal to $V_{X,W} \to \mathbb{T}$ because $W_\phi = \mathbb{T}$, we have $\Phi^*(A_\phi) \subseteq V_{X,W} \to \mathbb{T}$, and so for each $v \in V_{X,W}$, we obtain the mapping $a \mapsto (\Phi^* a)v$ of A_ϕ into $\mathbb{T}$. We define $\mathcal{F}$ to be the set of all those mappings obtained from all possible triples (W, Φ, v) of denotable worlds $W \in \mathcal{W}$ and C-denotations Φ into W and $v \in V_{X,W}$.

Thus we have seen above that each logical system $(A, \mathcal{W}, (I_W)_{W\in\mathcal{W}})$ with a truth ϕ yields the pair $(A_\phi, \mathcal{F})$ of A_ϕ and the subset $\mathcal{F} \neq \emptyset$ of $A_\phi \to \mathbb{T}$. We call $(A_\phi, \mathcal{F})$ the **ϕ-sentential functional logical space** associated with $(A, \mathcal{W}, (I_W)_{W\in\mathcal{W}})$.

2.4 Occurrences and substitutions

Let $(A, (\alpha_\lambda)_{\lambda\in L})$ be an algebra. If, for two elements a and b of A, there exists an element $\lambda \in L$ such that $a = \alpha_\lambda(\dots, b, \dots)$, then we write $b \prec a$. If $b \prec a$ or $b = a$, we write $b \preceq a$. If there exists a sequence $(b_i)_{i=0,\dots,n}$ $(n \geq 0)$ of elements of A such that $b_0 = a, b_n = b$ and $b_i \preceq b_{i-1}$ for $i = 1, \dots, n$, then we say that b **occurs** in a and call the sequence an **occurrence** of b in a.

In the rest of this subsection, let $(A, T, \sigma, S, C, X, \Gamma)$ be a formal language, and $(\alpha_\lambda)_{\lambda\in L}$ and $(\tau_\lambda)_{\lambda\in L}$ be the operations of A and T respectively. Then L is contained in the set of the formal products of the elements of $\Gamma \amalg S$. For each element λ of L, let S^λ denote the set of the elements of S which occur in λ as defined above.

Let $a \in A$ and $s \in S$. Then an occurrence $(s_i)_{i=0,\dots,n}$ of s in a is said to be **free**, if $\{s_0, \dots, s_n\} \cap \operatorname{Im} \alpha_\lambda = \emptyset$ for each $\lambda \in L$ such that $s \in S^\lambda$. If there exists a free occurrence of s in a, we say that s **occurs free** in a or write $s \ll a$. For each subset X of S, we define $X^a_{\text{free}} = \{x \in X \,|\, x \ll a\}$. Let $b \in A$. Then the occurrence $(s_i)_{i=0,\dots,n}$ of s in a is said to be **free from** b, if $\{s_0, \dots, s_n\} \cap \operatorname{Im} \alpha_\lambda = \emptyset$ for each $\lambda \in L$ such that $(S^\lambda)^b_{\text{free}} \neq \emptyset$. We say that s is **free from b in** a, if every free occurrence of s in a is free from b.

Let $s \in S$ and $c \in A$ with $\sigma s = \sigma c$. Then, for each element a of A, we can define the element $a(s/c)$ of A with $\sigma(a(s/c)) = \sigma a$ by induction on the rank r of a as follows. If $r = 0$, then $a \in S$, and so we define

$$a(s/c) = \begin{cases} c & \text{if } a = s, \\ a & \text{if } a \neq s, \end{cases}$$

hence $\sigma(a(s/c)) = \sigma a$ as desired. Suppose $r \geq 1$. Then a has a unique word form $\alpha_\lambda(a_1, \dots, a_k)$ and r is larger than the ranks of $a_1, \dots, a_k$, so $a_i(s/c)$ has already been defined and satisfies $\sigma(a_i(s/c)) = \sigma a_i$ for $i =$

$1, \dots, k$. Since $(\sigma a_1, \dots, \sigma a_k)$ belongs to $\operatorname{Dom} \tau_\lambda$, so does $\bigl(\sigma(a_1(s/c)), \dots, \sigma(a_k(s/c))\bigr)$ hence $(a_1(s/c), \dots, a_k(s/c)) \in \operatorname{Dom} \alpha_\lambda$, and so we define

$$a(s/c) = \begin{cases} \alpha_\lambda(a_1(s/c), \dots, a_k(s/c)) & \text{if } s \notin S^\lambda, \\ a & \text{if } s \in S^\lambda. \end{cases}$$

Then even when $a(s/c) \neq a$, we have

$$\begin{aligned} \sigma(a(s/c)) &= \sigma\bigl(\alpha_\lambda(a_1(s/c), \dots, a_k(s/c))\bigr) \\ &= \tau_\lambda\bigl(\sigma(a_1(s/c)), \dots, \sigma(a_k(s/c))\bigr) = \tau_\lambda(\sigma a_1, \dots, \sigma a_k) = \sigma a \end{aligned}$$

as desired. The definition of $a(s/c)$ by induction is complete. We call the transformation $a \mapsto a(s/c)$ on A the **substitution of c for s**. Since $\sigma(a(s/c)) = \sigma a$, the substitution is sort-consistent.

For each subset B of A and element $a \in A$, let B^a denote the set of the elements of B which occur in a. Furthermore define $L^a = \{\lambda \in L \mid (\operatorname{Im} \alpha_\lambda)^a \neq \emptyset\}$. If $\lambda \in L^a$, then we say that λ **occurs** in a.

Lemma 2.1 For each element $a \in A$, S^a is a finite set.

Proof Consult [7, Lemma 4.1].

Lemma 2.2 If $a = \alpha_\lambda(a_1, \dots, a_k) \in A$, then $L^a = \{\lambda\} \cup \bigcup_{j=1}^k L^{a_j}$. If $a \in S$, then $L^a = \emptyset$. For each element $a \in A$, L^a is a finite set.

Proof Consult [9, Proposition 1].

Lemma 2.3 If $a = \alpha_\lambda(a_1, \dots, a_k) \in A$, then $S^a_{\text{free}} = \bigcup_{j=1}^k S^{a_j}_{\text{free}} - S^\lambda$. If $a \in S$, then $S^a_{\text{free}} = \{a\}$.

Proof Consult [9, Proposition 1].

Lemma 2.4 If $a, b \in A$ and $(S^\lambda)^b_{\text{free}} = \emptyset$ for each $\lambda \in L^a$, then every element of S is free from b in a.

Proof Consult [9, Proposition 1].

Lemma 2.5 Let $a, b, c \in A$, $s \in S$ and assume that $\sigma s = \sigma c$ and $b = a(s/c)$, where (s/c) denotes the substitution of c for s. Then $S^b_{\text{free}} \subseteq S^c_{\text{free}} \cup (S^a_{\text{free}} - \{s\})$ and $L^b \subseteq L^a \cup L^c$.

Proof Consult [9, Proposition 1].

Lemma 2.6 Let $a \in A$ and $s \in S$. If $s \not\ll a$ then s is free from any element $b \in A$ in a.

Proof There is no free occurrence of s in a, so the conclusion is immediate by the definition. ■

Lemma 2.7 Let $a \in A$ and $s \in S$. Then s is free from s in a, and $a(s/s) = a$ holds.

Proof Let $(s_i)_{i=0,\dots,n}$be a free occurrence of s in a, if any. If $(S^\lambda)^s_{\text{free}} \neq \emptyset$, then $s \in S^\lambda$, hence $\{s_0, \dots, s_n\} \cap \operatorname{Im} \alpha_\lambda = \emptyset$. We can prove $a(s/s) = a$ easily by induction on Rank a. ■

Lemma 2.8 Let $a \in A$, $s, r \in S$, and $\sigma s = \sigma r$. Then $L^{a(s/r)} \subseteq L^a$.

Proof By Lemma 2.5, $L^{a(s/r)} \subseteq L^a \cup L^r$. By Lemma 2.2, $L^r = \emptyset$. ■

Lemma 2.9 Let $a \in A$, $s, r \in S$, and $\sigma s = \sigma r$. Then Rank $a(s/r) =$ Rank a.

Proof We use induction on Rank a. If Rank $a = 0$, then $a(s/r)$ is equal to either a or r. Hence $a(s/r) \in S$, that is, Rank $a(s/r) = 0$.

We assume that Rank $a \geq 1$. By Theorem 2.2, a has a unique word form $\alpha_\lambda(a_1, \dots, a_k)$, and Rank $a_j <$ Rank a for $j = 1, \dots, k$. If $s \in S^\lambda$, then $a(s/r) = a$, hence the conclusion follows. Suppose $s \notin S^\lambda$. Then $a(s/r) = \alpha_\lambda(a_1(s/r), \dots, a_k(s/r))$. Therefore, by the inductive hypothesis, Rank $a(s/r) = 1 + \sum_{j=1}^k$ Rank $a_j(s/r) = 1 + \sum_{j=1}^k$ Rank $a_j(s/r) =$ Rank a. ■

Lemma 2.10 Let $a = \alpha_\lambda(a_1, \dots, a_k) \in A$, $s \in S$, and $b \in A$. Then s is free from b in a if and only if $s \not\ll a$ or the following two conditions hold:

1. s is free from b in a_j for $j = 1, \dots, k$.
2. $(S^\lambda)^b_{\text{free}} = \emptyset$.

Proof This proof is based essentially on [5, Theorem 3.16.6]. If $s \not\ll a$, then s is free from b in a by Lemma 2.6. If $s \ll a$ and s is free from b in a, then the conditions 1 and 2 hold by [7, Lemma 4.3].

We assume that $s \ll a$ and that the conditions 1 and 2 hold. In order to prove that s is free from b in a, we show that $\{s_0, \dots, s_n\} \cap \operatorname{Im} \alpha_\mu = \emptyset$ for each free occurrence $(s_i)_{i=0,\dots,n}$ of s in a and each $\mu \in L$ satisfying $(S^\mu)^b_{\text{free}} \neq \emptyset$. Since $a \neq s$ by the uniqueness of the word form of a, we can assume that $s_0 \neq s_1$. Then $s_1 \in \{a_1, \dots, a_k\}$, hence $(s_i)_{i=1,\dots,n}$ is a free occurrence of s in a_j for some $j \in \{1, \dots, k\}$. By the condition 1, $(s_i)_{i=1,\dots,n}$ is free from b in a_j. Therefore $\{s_1, \dots, s_n\} \cap \operatorname{Im} \alpha_\mu = \emptyset$. Since $\lambda \neq \mu$ by the condition 2, $s_0 = a \notin \operatorname{Im} \alpha_\mu$. Therefore $\{s_0, \dots, s_n\} \cap \operatorname{Im} \alpha_\mu = \emptyset$ ■

Lemma 2.11 Let $a, b, c \in A$, $s, x \in S$ and assume that $\sigma s = \sigma c$. If $x \not\ll c$ and x is free from b in a, then x is free from b in $a(s/c)$.

Proof We use induction on Rank a. First we assume that Rank $a = 0$, that is, $a \in S$. If $a = s$ then $a(s/c) = c$, hence the conclusion follows from Lemma 2.6. If $a \neq s$ then $a(s/c) = a$, and the conclusion is immediate.

Next we assume that Rank $a \geq 1$. By Theorem 2.2, a has a unique word form $\alpha_\lambda(a_1, \ldots, a_k)$, and Rank $a_j <$ Rank a for $j = 1, \ldots, k$. If $s \in S^\lambda$ then $a(s/c) = a$, so the conclusion is immediate. Therefore we may assume that $s \notin S^\lambda$. Then we have $a(s/c) = \alpha_\lambda(a_1(s/c), \ldots, a_k(s/c))$.

First we consider the case where $x \ll a$. By Theorem 2.10, x is free from b in a_j for $j = 1, \ldots, k$ and $(S^\lambda)^b_{\text{free}} = \emptyset$. Then x is free from b in $a_j(s/c)$ for $j = 1, \ldots, k$ by the inductive hypothesis. Hence, again by Theorem 2.10, x is free from b in $a(s/c)$.

Next we consider the case where $x \not\ll a$. If $x \not\ll a(s/c)$ then x is free from b in $a(s/c)$ by Lemma 2.6, hence we may assume that $x \ll a(s/c)$. Furthermore, if $x \in S^\lambda$ then $x \not\ll a(s/c)$ by Lemma 2.3, so we may assume in addition that $x \notin S^\lambda$. Since $x \not\ll a$, it follows that $x \not\ll a_j$ for $j = 1, \ldots, k$ by Lemma 2.3. Since $x \not\ll c$, it follows that $x \not\ll a_j(s/c)$ by Lemma 2.5. Therefore, again by Lemma 2.3, $x \not\ll a(s/c)$. A contradiction. ■

Lemma 2.12 Let $a, c \in A$, $s \in S$ and $\sigma s = \sigma c$. If $s \not\ll a$ then $a(s/c) = a$.

Proof We use induction on Rank a. If Rank $a = 0$, then $a \in S$. Since $s \not\ll a$, it follows that $a \neq s$. Therefore $a(s/c) = a$. Next suppose Rank $a \geq 1$. By Theorem 2.2, a has a unique word form $\alpha_\lambda(a_1, \ldots, a_k)$, and Rank $a_j <$ Rank a for $j = 1, \ldots, k$. If $s \in S^\lambda$, then $a(s/c) = a$. If $s \notin S^\lambda$, then $s \not\ll a_j$ for $j = 1, \ldots, k$ by Lemma 2.3. Therefore $a(s/c) = \alpha_\lambda(a_1(s/c), \ldots, a_k(s/c)) = \alpha_\lambda(a_1, \ldots, a_k) = a$ by the inductive hypothesis. ■

Lemma 2.13 Let $a, b \in A$, $r, s \in S$ and $\sigma r = \sigma s$. If $b = a(s/r)$, $r \not\ll a$ and s is free from r in a, then $a = b(r/s)$, $s \not\ll b$ and r is free from s in b.

Proof This proof is based on [5, Theorem 3.17.6]. First we prove that $s \not\ll b$. If $r \neq s$, then $s \not\ll b$ by Lemma 2.5. If $r = s$, then $b = a$ by Lemma 2.7, hence $s \not\ll b$ by the assumption.

Next we prove that $a = b(r/s)$ and that r is free from s in b by induction on Rank a. First we assume that Rank $a = 0$, that is, $a \in S$. If $a = s$, then $b = a(s/r) = r$, hence $b(r/s) = s = a$. r is free from s in r by the definition. Suppose $a \neq s$. Then $b = a(s/r) = a$. Since $r \not\ll a$ it follows that $a \neq r$, hence $a(r/s) = a$. r is free from s in a by Lemma 2.6.

Henceforth we assume that Rank $a \geq 1$. By Theorem 2.2, a has a unique word form $\alpha_\lambda(a_1, \ldots, a_k)$, and Rank $a_j <$ Rank a for $j = 1, \ldots, k$. First suppose $s \notin S^\lambda$ and $r \notin S^\lambda$. Then $b = a(s/r) = \alpha_\lambda(a_1(s/r), \ldots, a_k(s/r))$. By Lemma 2.3 and Theorem 2.10, $r \not\ll a_j$ and s is free from r in a_j for $j = 1, \ldots, k$. By the inductive hypothesis, $a_j(s/r)(r/s) = a_j$ for $j = 1, \ldots, k$. Therefore $b(r/s) = a$. r is free from s in $a_j(s/r)$ for $j = 1, \ldots, k$ by the inductive hypothesis. $(S^\lambda)^s_{\text{free}} = \emptyset$ because $S^s_{\text{free}} = \{s\}$. Therefore r is free from s in b by Theorem 2.10. Next suppose $s \in S^\lambda$. Then $b = a(s/r) = a$. Since $r \not\ll a$, it follows that $a(r/s) = a$ by Lemma 2.12, and it follows that r is free from s in b by Lemma 2.6. Finally suppose $r \in S^\lambda$. Then, since s is free from r in a, it follows that $s \not\ll a$, hence $b = a(s/r) = a$ by Lemma 2.12. Therefore, by the same argument as above, $a(r/s) = a$ and r is free from s in b. ■

2.5 Extension of logical systems

In this subsection we argue on the extension of formal languages and its relation to the logical systems. The following theorem is based on [5, Theorem 4.7.1].

Theorem 2.3 Let $(A, T, \sigma, S, C, X, \Gamma)$ and $(A', T', \sigma', S', C', X', \Gamma')$ be formal languages, $(A, \mathcal{W}, (I_W)_{W \in \mathcal{W}})$ and $(A', \mathcal{W}', (I'_{W'})_{W' \in \mathcal{W}'})$ be logical systems, and L and L' be the indices of T and T' respectively. Assume the following:

1. $L \subseteq L'$.
2. A is a subalgebra of the L-reduct of A'.
3. T is the L-reduct of T'.
4. $\sigma = \sigma'|_A$.
5. $C \subseteq C'$, $X \subseteq X'$.
6. $\Gamma \subseteq \Gamma'$.
7. $W \in \mathcal{W}$, $W' \in \mathcal{W}'$.
8. W is the $L \cap \Gamma$-reduct of W'.
9. $I_W(\lambda)$ is the restriction of $I'_{W'}(\lambda)$ for each $\lambda \in L \cap \Gamma X$.

Let Φ be a C-denotation into W, Φ' be a C'-denotation into W', ν be an X-denotation into W, and ν' be an X'-denotation into W', and assume that Φ, ν are the restriction of Φ', ν' to C, X, respectively. Then $(\Phi^* a)\nu = (\Phi'^* a)\nu'$ for each $a \in A$.

Proof We use induction on $\operatorname{Rank} a$. First we assume that $\operatorname{Rank} a = 0$, that is, $a \in S = C \cup X$. If $a \in C$ then $(\Phi^* a)\nu = \Phi a = \Phi' a = (\Phi'^* a)\nu'$. If $a \in X$ then $(\Phi^* a)\nu = \nu a = \nu' a = (\Phi'^* a)\nu'$.

Henceforth we assume that $\operatorname{Rank} a \geq 1$. Let $(\alpha_\lambda)_{\lambda \in L}, (\alpha'_{\lambda'})_{\lambda' \in L'}, (\omega_\lambda)_{\lambda \in L}$ and $(\omega'_{\lambda'})_{\lambda' \in L'}$ be the operations of A, A', W and W', respectively. By Theorem 2.2, a has a unique word form $\alpha_\lambda(a_1, \dots, a_k)$, and $\operatorname{Rank} a_j <$

$\operatorname{Rank} a$ for $j = 1, \ldots, k$. Assume that $\lambda \in L \cap \Gamma$. Then $(\Phi^* a_j)v = (\Phi'^* a_j)v'$ for $j = 1, \ldots, k$ by the inductive hypothesis. Hence

$$\begin{aligned}(\Phi^* a)v &= \omega_\lambda((\Phi^* a_1)v, \ldots, (\Phi^* a_k)v) \\ &= \omega'_\lambda((\Phi'^* a_1)v', \ldots, (\Phi'^* a_k)v') \\ &= (\Phi'^* a)v'.\end{aligned}$$

Assume that $\lambda \in L \cap \Gamma X$, that is, $\lambda = \gamma x$ for some $\gamma \in \Gamma$ and $x \in X$. Then $a = \alpha_\lambda a_1$ because α_λ is unary. The mappings $(\Phi^* a_1)\big((x/\Box)v\big) \in W_{\sigma x} \to W_{\sigma a_1}$ and $(\Phi'^* a_1)\big((x/\Box)v'\big) \in W'_{\sigma' x} \to W'_{\sigma' a_1}$ are defined by (2.4). Since $W_{\sigma x} = W'_{\sigma' x}$ and $W_{\sigma a_1} = W'_{\sigma' a_1}$ by the assumption, it follows that $W_{\sigma x} \to W_{\sigma a_1} = W'_{\sigma' x} \to W'_{\sigma' a_1}$. For each $w \in W_{\sigma x}$, $(x/w)v$ is the restriction of $(x/w)v'$, hence $(\Phi^* a_1)\big((x/w)v\big) = (\Phi'^* a_1)\big((x/w)v'\big)$ by the inductive hypothesis. Therefore we have $(\Phi^* a_1)\big((x/\Box)v\big) = (\Phi'^* a_1)\big((x/\Box)v'\big)$. Let β_λ and β'_λ be the operations indexed by λ on the metaworlds $W^{V_{X,W}}$ and $W'^{V_{X',W'}}$, respectively. Then it follows that

$$\begin{aligned}(\Phi^* a)v &= \big(\beta_\lambda(\Phi^* a_1)\big)v \\ &= I_W(\lambda)\big((\Phi^* a_1)\big((x/\Box)v\big)\big) \\ &= I'_{W'}(\lambda)\big((\Phi'^* a_1)\big((x/\Box)v'\big)\big) \\ &= \big(\beta'_\lambda(\Phi'^* a_1)\big)v' \\ &= (\Phi'^* a)v'.\end{aligned}$$

■

2.6 Parallelism

Let $(A, T, \sigma, S, C, X, \Gamma)$ be a formal language, $(\alpha_\lambda)_{\lambda \in L}$ and $(\tau_\lambda)_{\lambda \in L}$ be the operations of A and T respectively. Assume the following **variable operation condition**:

- For each $\gamma \in \Gamma$ and $x, y \in X$ satisfying $\gamma x, \gamma y \in L$ and $\sigma x = \sigma y$, the operations $\tau_{\gamma x}$ and $\tau_{\gamma y}$ on T are equal as mappings.

Remark 2.2 This condition is satisfied by various formal languages including those of first-order predicate logic and typed lambda calculus[5] as well as MPCL.

A relation R on A is said to be **sort-consistent** if $a \mathrel{R} a' \implies \sigma a = \sigma a'$ for each $a, a' \in A$. A **congruence relation** on A is an equivalence relation R on A satisfying the following for each $\lambda \in L$.

$$\left.\begin{array}{l}(a_1, \ldots, a_k) \in \operatorname{Dom} \alpha_\lambda, \\ a_j \mathrel{R} a'_j \ (j = 1, \ldots, k)\end{array}\right\} \implies \left\{\begin{array}{l}(a'_1, \ldots, a'_k) \in \operatorname{Dom} \alpha_\lambda, \\ \alpha_\lambda(a_1, \ldots, a_k) \mathrel{R} \alpha_\lambda(a'_1, \ldots, a'_k).\end{array}\right.$$

[5] These are treated as examples of logical systems in [7, §5].

Recall that A is the direct union $\coprod_{n=0}^{\infty} S_n$ of the descendants S_n of S. We define the sort-consistent equivalence relation $\|_n$ on each S_n inductively, then we define the relation $\|$ to be the union of $\|_n$ and call it the **parallelism relation**. First we define $\|_0$ to be the equality relation. For $n \geq 1$, we define $\|_n$ to be the transitive closure of the union of the following two relations $P_{n,1}$ and $P_{n,2}$.

- $a \, P_{n,1} a'$ if and only if $a = \alpha_\lambda(a_1, \ldots, a_k)$, $a' = \alpha_\lambda(a'_1, \ldots, a'_k)$, where $\lambda \in L$, $(a_1, \ldots, a_k)$, $(a'_1, \ldots, a'_k) \in \mathrm{Dom}\, \alpha_\lambda$, $a_j, a'_j \in S_{l_j}$ and $a_j \|_{l_j} a'_j$ for $j = 1, \ldots, k$.

- $a \, P_{n,2} a'$ if and only if $a = \alpha_{\gamma x} a_1$, $a' = \alpha_{\gamma y} a'_1$, where $\gamma \in \Gamma$, $x, y \in X$, $\gamma x, \gamma y \in L$, $\sigma x = \sigma y$, $a_1 \in \mathrm{Dom}\, \alpha_{\gamma x}$, $a'_1 \in \mathrm{Dom}\, \alpha_{\gamma y}$, $a'_1 = a_1(x/y)$, $y \not\ll a_1$ and x is free from y in a_1.[6]

If $a \, P_{n,1} a'$, then $\sigma a_j = \sigma a'_j$ for $j = 1, \ldots, k$ because $\|_{l_j}$ is sort-consistent. Hence $\sigma a = \tau_\lambda(\sigma a_1, \ldots, \sigma a_k) = \tau_\lambda(\sigma a'_1, \ldots, \sigma a'_k) = \sigma a'$. If $a \, P_{n,2} a'$, then $\sigma a_1 = \sigma(a_1(x/y)) = \sigma a'_1$, hence $\sigma a = \tau_{\gamma x}(\sigma a_1) = \tau_{\gamma y}(\sigma a'_1) = \sigma a'$ by the variable operation condition. Therefore the relation $\|_n$ is sort-consistent. The relation $P_{n,1}$ is an equivalence relation. The relation $P_{n,2}$ is symmetric by Theorem 2.13. Therefore the relation $\|_n$ is an equivalence relation.

We say that a is **parallel** to a' if $a \| a'$.

Theorem 2.4 The parallelism relation $\|$ on A is the smallest of the sort-consistent congruence relations R on A satisfying

$$\alpha_{\gamma x} a \, R \, \alpha_{\gamma y} a(x/y) \tag{2.5}$$

for each $a \in A$, $x, y \in X$ and $\gamma \in \Gamma$ such that $\gamma x, \gamma y \in L$, $\sigma x = \sigma y$, $a \in \mathrm{Dom}\, \alpha_{\gamma x}$, $y \not\ll a$ and x is free from y in a.

Proof Let $P_{n,1}$ and $P_{n,2}$ be as in the definition of the parallelism relation. As shown in the definition, the parallelism relation is a sort-consistent equivalence relation.

First we prove that the parallelism relation is a congruence relation. Let $\lambda \in L$, $(a_1, \ldots, a_k) \in \mathrm{Dom}\, \alpha_\lambda$, and $a_j \| a'_j$ for $j = 1, \ldots, k$. Then $\sigma a_j = \sigma a'_j$ for $j = 1, \ldots, k$, hence $(a'_1, \ldots, a'_k) \in \mathrm{Dom}\, \alpha_\lambda$. Therefore $\alpha_\lambda(a_1, \ldots, a_k) \, P_{n,1} \, \alpha_\lambda(a'_1, \ldots, a'_k)$, where $n = \mathrm{Rank}\, \alpha_\lambda(a_1, \ldots, a_k)$. Hence $\alpha_\lambda(a_1, \ldots, a_k) \| \alpha_\lambda(a'_1, \ldots, a'_k)$.

Next we prove that the parallelism relation satisfies (2.5). Let $x, y \in X$, $\gamma \in \Gamma$, $\gamma x, \gamma y \in L$, $\sigma x = \sigma y$, $a \in \mathrm{Dom}\, \alpha_{\gamma x}$, $y \not\ll a$ and x is free from y in a. Since $\sigma a \in \mathrm{Dom}\, \tau_{\gamma x}$ and $\sigma(a(x/y)) = \sigma a$, by the variable operation condition it follows that $\sigma(a(x/y)) \in \mathrm{Dom}\, \tau_{\gamma y}$, hence $a(x/y) \in$

[6] Recall that the variable operations are unary. The definition of the substitution shows that $\sigma a_1 = \sigma(a_1(x/y))$. Also recall that $\mathrm{Rank}\, a_1 = \mathrm{Rank}\, a_1(x/y)$ by Lemma 2.9.

$\mathrm{Dom}\,\alpha_{\gamma y}$. Therefore $\alpha_{\gamma x}a\, P_{n,2}\alpha_{\gamma y}a(x/y)$, where $n = \mathrm{Rank}\,\alpha_{\gamma x}a$. Hence $\alpha_{\gamma x}a \parallel \alpha_{\gamma y}a(x/y)$.

Finally we prove that the parallelism relation is the smallest in the sense of Theorem 2.4. Let R be a sort-consistent congruence relation satisfying (2.5). We assume $a \parallel a'$ and prove $a\,R\,a'$ by induction on $\mathrm{Rank}\,a$. Recall that $\mathrm{Rank}\,a = \mathrm{Rank}\,a'$ by the definition of the parallelism relation. First we assume that $\mathrm{Rank}\,a = 0$. Then $a = a'$, hence $a\,R\,a'$ because R is reflexive. Next we assume that $\mathrm{Rank}\,a = n \geq 1$. Then there exist elements $b_0, \ldots, b_m \in A$ satisfying $b_0 = a$, $b_m = a'$, and $b_{i-1}\,P_{n,1}b_i$ or $b_{i-1}\,P_{n,2}b_i$ for $i = 1, \ldots, m$. If $b_{i-1}\,P_{n,1}b_i$, then $b_{i-1} = \alpha_\lambda(c_1, \ldots, c_k)$, $b_i = \alpha_\lambda(c'_1, \ldots, c'_k)$ for some $\lambda \in L$, $c_1, \ldots, c_k, c'_1, \ldots, c'_k \in A$, and $c_j \parallel c'_j$ for $j = 1, \ldots, k$. By the inductive hypothesis, $c_j\,R\,c'_j$ for $j = 1, \ldots, k$. Therefore $b_{i-1}\,R\,b_i$ because R is a congruence relation. If $b_{i-1}\,P_{n,2}b_i$, then $b_{i-1} = \alpha_{\gamma x}c$, $b_i = \alpha_{\gamma y}c(x/y)$ for some $\gamma \in \Gamma$, $x, y \in X$ and $c \in A$ such that $\gamma x, \gamma y \in L$, $\sigma x = \sigma y$, $y \not\ll c$ and x is free from y in c. Therefore $b_{i-1}\,R\,b_i$ because R satisfies (2.5). Since R is transitive, $a\,R\,a'$ as required.

3 Logical system MPCL

We define MPC languages, and MPC worlds denotable for an MPC language. The interpretation on each MPC world is naturally defined. Thus the logical system MPCL is defined. The formulation of MPCL is due to [8].

3.1 Quantities and measures

A **quantity system** is a set $\mathbb{P}$ equipped with a total binary associative and commutative operation $(x, y) \mapsto x + y$ with the identity element 0 and an order $\leq$ which satisfy the following two conditions.

- If elements $p, p', q, q' \in \mathbb{P}$ satisfy $p \leq p'$ and $q \leq q'$, then $p + q \leq p' + q'$.
- $0 \leq p$ for every element p of $\mathbb{P}$, that is to say, $0 = \min \mathbb{P}$.

The quantity system $\mathbb{P}$ is said to be **linear** if the order $\leq$ is linear.

Let $(\mathbb{P}, +, 0, \leq)$ be a quantity system and $Q \subseteq \mathbb{P}$. Since $\mathbb{P}$ is a $+$-algebra, the subalgebra $[Q \cup \{0\}]$ generated by $Q \cup \{0\}$ is defined. Then $[Q \cup \{0\}]$ equipped with the restriction of $\leq$ to it is a quantity system.

Lemma 3.1 Let $(\mathbb{P}, +, 0, \leq)$ be a linear quantity system and Q be a finite subset of $\mathbb{P}$. Then $[Q]$ is well-ordered with respect to $\leq$.

Proof Consult [8, Theorem 2.1] or [1, Corollary 1.2].

Let S be a set and $(\mathbb{P}, +, 0, \leq)$ be a quantity system. Then a $\mathbb{P}$**-measure** on S is a mapping $\mathsf{X} \mapsto |\mathsf{X}|$ of $\mathcal{P}\mathsf{S}$ into $\mathbb{P}$ which satisfies the following three conditions for all $\mathsf{X}, \mathsf{Y} \in \mathcal{P}\mathsf{S}$.

- $\mathsf{X} \neq \emptyset \iff |\mathsf{X}| > 0$.
- $\mathsf{X} \subseteq \mathsf{Y} \implies |\mathsf{X}| \leq |\mathsf{Y}|$.
- $|\mathsf{X} \cup \mathsf{Y}| \leq |\mathsf{X}| + |\mathsf{Y}|$.

Lemma 3.2 Let S be a set, $(\mathbb{P}, +, 0, \leq)$ be a quantity system, R be a relation between $\mathcal{P}\mathsf{S}$ and $\mathbb{P}$, and $0 \neq \acute{o} \in \mathbb{P}$. Assume the following conditions:

- $\mathsf{X} = \emptyset$ if and only if $\mathsf{X}\,\mathsf{R}\,0$.
- If $\mathsf{X} \subseteq \mathsf{Y}$ and $\mathsf{Y}\,\mathsf{R}\,\mathfrak{a}$, then $\mathsf{X}\,\mathsf{R}\,\mathfrak{a}$.
- If $\mathsf{X}\,\mathsf{R}\,\mathfrak{a}$ and $\mathsf{Y}\,\mathsf{R}\,\mathfrak{b}$, then $(\mathsf{X} \cup \mathsf{Y})\,\mathsf{R}\,(\mathfrak{a} + \mathfrak{b})$.
- For each $\mathsf{X} \in \mathcal{P}\mathsf{S}$, $\min\big(\{\mathfrak{a} \in \mathbb{P} \,|\, \mathsf{X}\,\mathsf{R}\,\mathfrak{a}\} \cup \{\acute{o}\}\big)$ exists.

Then the mapping $\mathsf{X} \mapsto \min\big(\{\mathfrak{a} \in \mathbb{P} \,|\, \mathsf{X}\,\mathsf{R}\,\mathfrak{a}\} \cup \{\acute{o}\}\big)$ is a $\mathbb{P}$-measure on S.

Proof Consult [8, Theorem 2.2].

3.2 MPC language

Here we define the formal language of MPCL. First we take arbitrary three sets $\mathbb{S}, \mathbb{C}, \mathbb{X}$ such that $\mathbb{S} = \mathbb{C} \amalg \mathbb{X}$ and $\mathbb{X} \neq 0$. Next we take an arbitrary set K equipped with a specific element π. We call K the set of **cases** and in particular call π the **nominative case**. Next we take two arbitrary distinct symbols δ and ε not contained in K, and define $\mathsf{T} = \{\delta, \varepsilon\} \cup \mathcal{P}\mathsf{K}$. Next we take a mapping τ of $\mathbb{S}$ into T such that the inverse image $\mathbb{X}_\varepsilon = \{x \in \mathbb{X} \,|\, \tau x = \varepsilon\}$ of ε in $\mathbb{X}$ is not empty. Next we take an arbitrary quantity system $(\mathbb{P}, +, 0, \leq)$ with $\#\mathbb{P} > 1$, then let $\mathfrak{P}$ be a subset of $\mathcal{P}\mathbb{P}$. Next we take a copy $\neg\mathfrak{P} = \{\neg\mathfrak{p} \,|\, \mathfrak{p} \in \mathfrak{P}\}$ of the set $\mathfrak{P}$ such that $\neg\mathfrak{P} \cap \mathfrak{P} = \emptyset$, and define $\mathfrak{Q} = \neg\mathfrak{P} \amalg \mathfrak{P}$, which we call the set of the **quantifiers**. Also we take an arbitrary symbol $\breve{o} \notin \mathfrak{Q}$. Next we let $(n_\mathfrak{f})_{\mathfrak{f} \in \mathfrak{F}}$ be a family of non-negative integers indexed by a set $\mathfrak{F}$. Finally we define the nine kinds of operations on T as follows.

1. **The family of binary operations** $\breve{o}\mathsf{k}$ $(\mathsf{k} \in \mathsf{K})$**.**

$$\operatorname{Dom} \breve{o}\mathsf{k} = \{\varepsilon\} \times \{\mathsf{P} \in \mathcal{P}\mathsf{K} \,|\, \mathsf{k} \in \mathsf{P}\}, \qquad \varepsilon\, \breve{o}\mathsf{k}\, \mathsf{P} = \mathsf{P} - \{\mathsf{k}\}.$$

2. **The family of binary operations** $\mathfrak{x}\mathsf{k}$ $((\mathfrak{x}, \mathsf{k}) \in \mathfrak{Q} \times \mathsf{K})$**.**

$$\operatorname{Dom} \mathfrak{x}\mathsf{k} = \{\delta, \varepsilon\} \times \{\mathsf{P} \in \mathcal{P}\mathsf{K} \,|\, \mathsf{k} \in \mathsf{P}\}, \qquad \delta\, \mathfrak{x}\mathsf{k}\, \mathsf{P} = \varepsilon\, \mathfrak{x}\mathsf{k}\, \mathsf{P} = \mathsf{P} - \{\mathsf{k}\}.$$

3. The three binary operations $\wedge, \vee$ and $\Rightarrow$.

$$\mathrm{Dom} \wedge = \mathrm{Dom} \vee = \mathrm{Dom} \Rightarrow = (\mathcal{P}K)^2, \quad P \wedge Q = P \vee Q = P \Rightarrow Q = P \cup Q.$$

4. The unary operation $\Diamond$.

$$\mathrm{Dom} \Diamond = \mathcal{P}K, \qquad\qquad P^{\Diamond} = P.$$

5. The unary operation $\triangle$.

$$\mathrm{Dom} \triangle = \{\delta, \varepsilon\}, \qquad\qquad \delta\triangle = \varepsilon\triangle = \{\pi\}.$$

6. The two binary operations $\sqcap$ and $\sqcup$.

$$\mathrm{Dom} \sqcap = \mathrm{Dom} \sqcup = \{\delta, \varepsilon\}^2, \quad \xi \sqcap \eta = \xi \sqcup \eta = \delta \text{ for each } (\xi, \eta) \in \{\delta, \varepsilon\}^2.$$

7. The unary operation $\Box$.

$$\mathrm{Dom} \Box = \{\delta, \varepsilon\}, \qquad\qquad \delta^{\Box} = \varepsilon^{\Box} = \delta.$$

8. The family of operations $\mathfrak{f} \in \mathfrak{F}$.

$$\mathrm{Dom} \mathfrak{f} = \{\varepsilon\}^{n_{\mathfrak{f}}}, \qquad\qquad \mathfrak{f}(\varepsilon, \ldots, \varepsilon) = \varepsilon.$$

9. The family of unary operations Ωx $(x \in \mathbb{X}_\varepsilon)$.

$$\mathrm{Dom} \Omega x = \{\emptyset\}, \qquad\qquad \emptyset\, \Omega x = \delta.$$

We let T be the algebra equipped with the above nine kinds of operations. Thus we have chosen a set $\mathbb{S}$, an algebra T, and a mapping τ of $\mathbb{S}$ into T. Therefore by Theorem 2.1, there exists the USA (A, T, σ, S) with $\sigma|_{\mathbb{S}} = \tau$, which is unique up to sort-consistent isomorphism. The operations of T and A are both indexed by the set

$$L = \{\breve{o}k, \mathfrak{r}k, \wedge, \vee, \Rightarrow, \Diamond, \triangle, \sqcap, \sqcup, \Box, \mathfrak{f}, \Omega x \,|\, k \in K, \mathfrak{r} \in \mathfrak{Q}, \mathfrak{f} \in \mathfrak{F}, x \in \mathbb{X}_\varepsilon\},$$

and so if we define

$$\Gamma = \{\breve{o}k, \mathfrak{r}k, \wedge, \vee, \Rightarrow, \Diamond, \triangle, \sqcap, \sqcup, \Box, \mathfrak{f}, \Omega \,|\, k \in K, \mathfrak{r} \in \mathfrak{Q}, \mathfrak{f} \in \mathfrak{F}\},$$

then we may consider that L is contained in the subset $\Gamma \cup \Gamma\mathbb{X}$ of the free semigroup over $\Gamma \amalg \mathbb{S}$ with $L \cap \Gamma\mathbb{X} = \{\Omega x \,|\, x \in \mathbb{X}_\varepsilon\}$. Therefore $(A, T, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ is a formal language, which we call the **MPC language**. Its variable operations Ωx $(x \in \mathbb{X}_\varepsilon)$ are called the **nominalizers**.

Since (A, T, σ) is a sorted algebra, A is divided into its t-parts A_t $(t \in T)$, and since $T = \{\delta, \varepsilon\} \cup \mathcal{P}K$, we have

$$A = A_\delta \cup A_\varepsilon \cup \bigcup_{P \in \mathcal{P}K} A_P,$$

so we define

$$G = A_\delta \cup A_\varepsilon, \qquad\qquad H = \bigcup_{P \in \mathcal{P}K} A_P.$$

We call G the set of the **nominals** and call H the set of the **predicates**. For each $f \in H$, we denote by K_f the element $P \in \mathcal{P}K$ satisfying $f \in A_P$ and call it the **range** of f.

Since (A, T, σ) is a sorted algebra, the following also holds on the domains and images of the operations in the operation system L of A.

1. $\mathrm{Dom}\, \breve{o}k = A_\varepsilon \times \bigcup_{k \in P \in \mathcal{P}K} A_P$ for each $k \in K$. If $a \in A_\varepsilon$ and $f \in A_P$ with $k \in P \in \mathcal{P}K$, then $a\, \breve{o}k\, f \in A_{P-\{k\}}$.
2. $\mathrm{Dom}\, \mathfrak{x}k = G \times \bigcup_{k \in P \in \mathcal{P}K} A_P$ for each $(\mathfrak{x}, k) \in \mathfrak{L} \times K$. If $a \in G$ and $f \in A_P$ with $k \in P \in \mathcal{P}K$, then $a\, \lambda k\, f \in A_{P-\{k\}}$.
3. $\mathrm{Dom} \wedge = \mathrm{Dom} \vee = \mathrm{Dom} \Rightarrow = H^2$. If $f \in A_P$ and $g \in A_Q$ with $P, Q \in \mathcal{P}K$, then $f \wedge g, f \vee g, f \Rightarrow g \in A_{P \cup Q}$.
4. $\mathrm{Dom} \lozenge = H$. If $f \in A_P$ with $P \in \mathcal{P}K$, then $f^\lozenge \in A_P$.
5. $\mathrm{Dom} \triangle = G$, $\mathrm{Im} \triangle \subseteq A_{\{\pi\}}$.
6. $\mathrm{Dom} \sqcap = \mathrm{Dom} \sqcup = G^2$, $\mathrm{Im} \sqcap \subseteq A_\delta$, $\mathrm{Im} \sqcup \subseteq A_\delta$.
7. $\mathrm{Dom} \square = G$, $\mathrm{Im} \square \subseteq A_\delta$.
8. $\mathrm{Dom}\, \mathfrak{f} = (A_\varepsilon)^{n_\mathfrak{f}}$, $\mathrm{Im}\, \mathfrak{f} \subseteq A_\varepsilon$ for each $\mathfrak{f} \in \mathfrak{F}$.
9. $\mathrm{Dom}\, \Omega x = A_\emptyset$, $\mathrm{Im}\, \Omega x \subseteq A_\delta$ for each $x \in \mathbb{X}_\varepsilon$.

Assumption 3.1 In this paper we assume the following conditions.

1. The quantity system $\mathbb{P}$ is linear.
2. The range K_f of each predicates $f \in H$ is a finite set.
3. The set $\mathbb{X}_\varepsilon$ has the same cardinality as A.
4. The set $\mathfrak{P}$ is the set of the unions of a finite number of intervals of $\mathbb{P}$ on the following list:

$$\begin{aligned} (p \rightarrow) &= \{x \in \mathbb{P} \mid p < x\}, \\ (p, q] &= \{x \in \mathbb{P} \mid p < x \leq q\}, \qquad \text{where } p, q \in \mathbb{P}. \\ (\leftarrow q] &= \{x \in \mathbb{P} \mid x \leq q\}, \end{aligned}$$

For each $X \in \mathcal{P}\mathbb{P}$, we denote by X° the complement $\mathbb{P} - X$. Then $\mathfrak{P}$ is closed under the three set-theoretical operations $\cap, \cup, \circ$ on $\mathcal{P}\mathbb{P}$.

Remark 3.1 The condition 3 in Assumption 3.1 implies that $\mathbb{X}_\varepsilon$ is an infinite set. The condition 3 can be satisfied, for example, if $\mathbb{X}_\varepsilon$ is an infinite set and $\#\mathbb{X}_\varepsilon = \#\mathbb{S} = \#\mathsf{L}$.[7] Instead of the condition 3, in [9, p. 2], both A_ε and $\mathbb{X}_\varepsilon$ are assumed to be enumerable.

Remark 3.2 Let ∞ denote the largest element of $\mathbb{P}$, provided it exists. By the conditions 1 and 4 in Assumption 3.1, if an element $\mathfrak{p}$ of $\mathfrak{P}$ is connected, then the **endpoints** of $\mathfrak{p}$ are uniquely determined as follows.

- If $\mathfrak{p} \in \{\emptyset, \mathbb{P}\}$, then $\mathfrak{p}$ has no endpoint.
- If $\mathfrak{p} = (\mathsf{p} \rightarrow)$ and $\mathsf{p} \neq \infty$, then p is the endpoint of $\mathfrak{p}$.
- If $\mathfrak{p} = (\mathsf{p}, \mathsf{q}]$ and $\mathsf{p} < \mathsf{q} \neq \infty$, then p and q are the endpoints of $\mathfrak{p}$.
- If $\mathfrak{p} = (\leftarrow \mathsf{q}]$ and $\mathsf{q} \neq \infty$, then q is the endpoint of $\mathfrak{p}$.

This is well-defined because one and only one of the above cases holds. Again by Assumption 3.1, each $\mathfrak{p} \in \mathfrak{P}$ is uniquely expressed as the union of a finite numbers of the distinct **connected components**. In view of this, we say that $\mathsf{p} \in \mathbb{P}$ is an **endpoint** of $\mathfrak{p} \in \mathfrak{P}$ if p is an endpoint of some connected component of $\mathfrak{p}$. For each $\mathsf{a} \in \mathsf{A}$ and $\mathsf{p} \in \mathbb{P}$, we say that p **occurs** in a if there exist elements $\mathfrak{p} \in \mathfrak{P}$ and $\mathsf{k} \in \mathsf{K}$ such that p is an endpoint of $\mathfrak{p}$, and $\mathfrak{p}\mathsf{k}$ or $\neg\mathfrak{p}\mathsf{k}$ occurs in a. We denote by $\mathbb{P}^{\mathsf{a}}$ the set of elements of $\mathbb{P}$ which occur in a. For each subset B of A, we define $\mathbb{P}^{\mathsf{B}} = \bigcup_{\mathsf{a} \in \mathsf{B}} \mathbb{P}^{\mathsf{a}}$.

We will use the following abbreviation for quantifiers:

$$\begin{array}{lll} \underline{\mathsf{p}} = \neg(\leftarrow \mathsf{p}], & \overline{\mathsf{p}} = (\mathsf{p} \rightarrow), & \text{for each } \mathsf{p} \in \mathbb{P}, \\ \forall = \underline{0}, & \exists = \overline{0}, & \text{where } 0 = \min \mathbb{P}. \end{array}$$

We use one as an abbreviation for $(\mathsf{x}\,\breve{\mathsf{o}}\pi\,\mathsf{x}\triangle)\,\Omega\mathsf{x}$, where x is an arbitrary fixed element of $\mathbb{X}_\varepsilon$.

3.3 MPC worlds

Let $(\mathsf{A}, \mathsf{T}, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ be an MPC language defined in §3.2. Here we define the domain $\mathcal{W}$ of the denotable worlds for A. Define

$$\mathsf{M} = \mathsf{L} \cap \Gamma = \{\breve{\mathsf{o}}\mathsf{k}, \mathfrak{x}\mathsf{k}, \wedge, \vee, \Rightarrow, \Diamond, \triangle, \sqcap, \sqcup, \square, \mathfrak{f} \,|\, \mathsf{k} \in \mathsf{K}, \mathfrak{x} \in \mathfrak{Q}, \mathfrak{f} \in \mathfrak{F}\},$$

[7] By [5, Theorem 3.2.1], if κ is an infinite cardinal satisfying $\#\mathbb{S} \leq \kappa$ and $\#\mathsf{L} \leq \kappa$, then $\#(\mathbb{S}_n) \leq \kappa$ for each descendants $\mathbb{S}_n$ of $\mathbb{S}$, hence $\#\mathsf{A} \leq \kappa$. Related results may be found in [12] or [13].

and let T_M be the M-reduct of T.

First we take an arbitrary non-empty set S, and define

$$W = (S \to \mathbb{T}) \cup S \cup \bigcup_{P \in \mathcal{P}K} \big((P \to S) \to \mathbb{T}\big).$$

We call S the **base** of W.

Next we define the sorting ρ of W into $T = \{\delta, \varepsilon\} \cup \mathcal{P}K$ so that the t-parts W_t ($t \in T_M$) satisfy $W_\delta = S \to \mathbb{T}$, $W_\varepsilon = S$, and $W_P = (P \to S) \to \mathbb{T}$ for each $P \in \mathcal{P}K$. In particular $W_\emptyset = \mathbb{T}$. We call $W_\delta \cup W_\varepsilon$ the set of the **entities**, while we call $\bigcup_{P \in \mathcal{P}K} W_P$ the set of the **affairs**.

Next we define a family of operations on W indexed by M. The definition depends on two parameters. The one is an arbitrary $\mathbb{P}$-measure $X \mapsto |X|$ on S. The other is an arbitrary reflexive relation $\exists$ on S, which we call the **basic relation** of W. In order to define the operations, we first extend $\exists$ to the relation between $(S \to \mathbb{T}) \cup S$ and S by

$$a \exists b \iff ab = 1$$

for each $a \in S \to \mathbb{T}$ and each $b \in S$. Next, when $s \in S$ and $k \in P \in \mathcal{P}K$, we define for each $\theta \in (P - \{k\}) \to S$ the element $(k/s)\theta \in P \to S$ by

$$\big((k/s)\theta\big)l = \begin{cases} \theta l & \text{if } l \in P - \{k\}, \\ s & \text{if } l = k. \end{cases}$$

If $P = \{k\}$, then $(P - \{k\}) \to S = \{\emptyset\}$, so we denote $(k/s)\theta$ by (k/s). Next we define $\neg(\neg\mathfrak{p}) = \mathfrak{p}$ for each $\mathfrak{p} \in \mathfrak{P}$. Thus, if $\mathfrak{x} \in \mathfrak{P}$ then $\neg\mathfrak{x} \in \neg\mathfrak{P}$, while if $\mathfrak{x} \in \neg\mathfrak{P}$ then $\neg\mathfrak{x} \in \mathfrak{P}$. Finally we define the eight kinds of operations on W as follows.

1. **The family of binary operations** $\breve{o}k$ $(k \in K)$**.**

$$\mathrm{Dom}\, \breve{o}k = S \times \bigcup_{k \in P \in \mathcal{P}K} \big((P \to S) \to \mathbb{T}\big).$$

For each $s \in S$ and each $f \in (P \to S) \to \mathbb{T}$ with $k \in P \in \mathcal{P}K$, we define $s\, \breve{o}k\, f$ to be the element of $\big((P - \{k\}) \to S\big) \to \mathbb{T}$ such that

$$(s\, \breve{o}k\, f)\theta = f\big((k/s)\theta\big)$$

for each $\theta \in (P - \{k\}) \to S$.

2. **The family of binary operations** $\mathfrak{x}k$ $((\mathfrak{x}, k) \in \mathfrak{Q} \times K)$**.**

$$\mathrm{Dom}\, \mathfrak{x}k = \big((S \to \mathbb{T}) \cup S\big) \times \bigcup_{k \in P \in \mathcal{P}K} \big((P \to S) \to \mathbb{T}\big).$$

For each $s \in (S \to \mathbb{T}) \cup S$ and each $f \in (P \to S) \to \mathbb{T}$ with $k \in P \in \mathcal{P}K$, we define $s\,\mathfrak{x}k\,f$ to be the element of $\big((P - \{k\}) \to S\big) \to \mathbb{T}$ such that

$$(a\,\mathfrak{x}k\,f)\theta = 1 \iff \begin{cases} \big|\big\{s \in S \,\big|\, a \,\exists\, s,\ f\big((k/s)\theta\big) = 0\big\}\big| \in \neg\mathfrak{x} & \text{if } \mathfrak{x} \in \neg\mathfrak{P}, \\ \big|\big\{s \in S \,\big|\, a \,\exists\, s,\ f((k/s)\theta) = 1\big\}\big| \in \mathfrak{x} & \text{if } \mathfrak{x} \in \mathfrak{P} \end{cases}$$

for each $\theta \in (P - \{k\}) \to S$. Notice that $f\big((k/s)\theta\big) = (s\,\breve{o}k\,f)\theta$.

3. The three binary operations $\wedge, \vee$ and $\Rightarrow$.

$$\mathrm{Dom} \wedge = \mathrm{Dom} \vee = \mathrm{Dom} \Rightarrow\ = \left(\bigcup_{P \in \mathcal{P}K} \big((P \to S) \to \mathbb{T}\big) \right)^2.$$

For each $f \in (P \to S) \to \mathbb{T}$ and each $g \in (Q \to S) \to \mathbb{T}$ with $P, Q \in \mathcal{P}K$, we define $f \wedge g, f \vee g, f \Rightarrow g$ to be the elements of $\big((P \cup Q) \to S\big) \to \mathbb{T}$ such that

$$\begin{aligned} (f \wedge g)\theta &= f(\theta|_P) \wedge (\theta|_Q), \\ (f \vee g)\theta &= f(\theta|_P) \vee (\theta|_Q), \\ (f \Rightarrow g)\theta &= f(\theta|_P) \Rightarrow (\theta|_Q) \end{aligned}$$

for each $\theta \in (P \cup Q) \to S$, where $\wedge, \vee$ and $\Rightarrow$ on the right-hand sides of the equations are the meet, join, and implication on the Boolean lattice $\mathbb{T}$ defined by $a \wedge b = \inf\{a, b\}$, $a \vee b = \sup\{a, b\}$ and $a \Rightarrow b = \sup\{1 - a, b\}$ for all $a, b \in \mathbb{T}$.

4. The unary operation $\Diamond$.

$$\mathrm{Dom}\,\Diamond = \bigcup_{P \in \mathcal{P}K} \big((P \to S) \to \mathbb{T}\big).$$

For each $f \in (P \to S) \to \mathbb{T}$ with $P \in \mathcal{P}K$, we define $f^\Diamond$ to be the element of $(P \to S) \to \mathbb{T}$ such that

$$(f^\Diamond)\theta = (f\theta)^\Diamond$$

for each $\theta \in P \to S$, where $\Diamond$ on the right-hand side of the equation is the complement on the Boolean lattice $\mathbb{T}$ defined by $a^\Diamond = 1 - a$ for all $a \in \mathbb{T}$.

5. The unary operation $\triangle$.

$$\mathrm{Dom}\,\triangle = (S \to \mathbb{T}) \cup S.$$

For each $a \in (S \to \mathbb{T}) \cup S$, we define $a\triangle$ to be the element of $(\{\pi\} \to S) \to \mathbb{T}$ such that

$$(a\triangle)\theta = 1 \iff a \,\exists\, \theta\pi$$

for each $\theta \in \{\pi\} \to S$.

6. The two binary operations $\sqcap, \sqcup$.

$$\operatorname{Dom} \sqcap = \operatorname{Dom} \sqcup = \big((S \to \mathbb{T}) \cup S\big)^2.$$

For each $(a, b) \in \big((S \to \mathbb{T}) \cup S\big)^2$, we define $a \sqcap b$ and $a \sqcup b$ to be the elements of $S \to \mathbb{T}$ such that

$$\begin{aligned} a \sqcap b \ni s &\iff a \ni s \text{ and } b \ni s, \\ a \sqcup b \ni s &\iff a \ni s \text{ or } b \ni s \end{aligned}$$

for each $s \in S$.

7. The family of operations $\mathfrak{f} \in \mathfrak{F}$.

$$\operatorname{Dom} \mathfrak{f} = S^{n_\mathfrak{f}}.$$

For each $(s_1, \ldots, s_n) \in S^{n_\mathfrak{f}}$, we define $\mathfrak{f}(s_1, \ldots, s_{n_\mathfrak{f}})$ to be an arbitrary element of S.

8. The unary operation $\square$.

$$\operatorname{Dom} \square = (S \to \mathbb{T}) \cup S.$$

For each $a \in (S \to \mathbb{T}) \cup S$, we define $a^\square$ to be the element of $S \to \mathbb{T}$ such that

$$a^\square \ni s \iff a \not\ni s$$

for each $s \in S$.

We let W be the algebra equipped with the above eight kinds of operations. Then (W, T_M, ρ) becomes a sorted algebra and satisfies $W_t \neq \emptyset$ for all $t \in T_M$. Therefore W is a denotable world for A.

We call the sorted algebras constructed as above the **MPC worlds** cognizable by the MPC language $(A, T, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ and denote by $\mathcal{W}$ the collection of all such worlds.

3.4 Interpretations of the nominalizers

Let $(A, T, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ be an MPC language defined in §3.2, and let $\mathcal{W}$ be the collection of the denotable worlds for A defined in §3.3. Following §2.3, here we define the interpretation I_W of the set $L \cap \Gamma\mathbb{X}$ of the variable operations on each $W \in \mathcal{W}$, and thereby complete the definition of MPCL.

Let $\lambda \in L \cap \Gamma\mathbb{X}$. Since $L \cap \Gamma\mathbb{X}$ consists of the nominalizers, $\lambda = \Omega x$ for some $x \in \mathbb{X}_\varepsilon$, and so the domain T_λ of λ on T is equal to $\{\emptyset\}$ and $\lambda\emptyset = \delta$. Moreover $W_\delta = S \to \mathbb{T} = W_{\sigma x} \to W_\emptyset$. Thus, $\mathrm{I}_W(\lambda) = \lambda_W$ is a mapping of $W_{\sigma x} \to W_\emptyset$ into itself, and so we define λ_W to be the identity mapping of $W_{\sigma x} \to W_\emptyset$. Then the domain of the operation β_λ on $W^{V_{\mathbb{X}}, W}$ corresponding to the index

λ is equal to $V_{\mathbb{X},W} \to W_\emptyset = V_{\mathbb{X},W} \to \mathbb{T}$, and for each $\varphi \in V_{\mathbb{X},W} \to \mathbb{T}$ we have $\beta_\lambda\varphi \in V_{\mathbb{X},W} \to W_\delta = V_{\mathbb{X},W} \to (S \to \mathbb{T})$ with $(\beta_\lambda\varphi)\nu = \varphi\big((x/\square)\nu\big)$ for each $\nu \in V_{\mathbb{X},W}$, hence $\big((\beta_\lambda\varphi)\nu\big)s = \varphi\big((x/s)\nu\big)$ for each $s \in S$.

Since $\lambda = \Omega x$ ($x \in \mathbb{X}_\varepsilon$) and we will denote $\beta_\lambda\varphi$ by $\varphi\,\Omega x$, we conclude that the domain of the nominalizer Ωx on $W^{V_{\mathbb{X},W}}$ is equal to $V_{\mathbb{X},W} \to \mathbb{T}$, the image $\varphi\,\Omega x$ of $\varphi \in V_{\mathbb{X},W} \to \mathbb{T}$ belongs to $V_{\mathbb{X},W} \to (S \to \mathbb{T})$, so $(\varphi\,\Omega x)\nu \in S \to \mathbb{T}$ for each $\nu \in V_{\mathbb{X},W}$, and the following holds for each $s \in S$:

$$\big((\varphi\,\Omega x)\nu\big)s = \varphi\big((x/s)\nu\big). \tag{3.1}$$

This completes the definition of the logical system MPCL.

3.5 Predicate logical spaces

Let $(A, T, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ be an MPC language and $(A, \mathcal{W}, (I_W)_{W\in\mathcal{W}})$ be the logical system MPCL on it. The $\emptyset$-part of each $W \in \mathcal{W}$ is equal to $(\emptyset \to W_\varepsilon) \to \mathbb{T}$, and is identified with $\mathbb{T}$ because $\emptyset \to W_\varepsilon$ is a singleton. Therefore $(A, \mathcal{W}, (I_W)_{W\in\mathcal{W}})$ together with the truth $\emptyset$ yields the $\emptyset$-sentential functional logical space $(A_\emptyset, \mathcal{F})$ associated with the logical system, as we have seen in §2.3. Notice that $\varphi \in \mathcal{F}$ if and only if φ is a mapping $a \mapsto (\Phi^*a)\nu$ for some MPC world $W \in \mathcal{W}$, $\mathbb{C}$-denotation Φ into W, and $\mathbb{X}$-denotation ν into W.

In this subsection we define another functional logical space. Recall that $H = \bigcup_{P\in\mathcal{P}K} A_P$ is the set of the predicates of A. Let $W \in \mathcal{W}$, Φ be a $\mathbb{C}$-denotation into W, and ν be a $\mathbb{X}$-denotation into W. Then, for each $f \in H$, $(\Phi^*f)\nu \in W_{K_f} = (K_f \to W_\varepsilon) \to \mathbb{T}$. Hence $\big((\Phi^*f)\nu\big)(\theta|_{K_f}) \in \mathbb{T}$ for each $\theta \in K \to W_\varepsilon$. We define $\mathcal{G}$ to be the set of mappings $f \mapsto \big((\Phi^*f)\nu\big)(\theta|_{K_f})$ obtained from all possible quadruples (W, Φ, ν, θ) of $W \in \mathcal{W}$, $\mathbb{C}$-denotations Φ into W, $\mathbb{X}$-denotations ν into W, and $\theta \in K \to W_\varepsilon$. Thus $(H, \mathcal{G})$ is a $\mathbb{T}$-valued functional logical space, which we call the **predicate logical space** associated with $(A, \mathcal{W}, (I_W)_{W\in\mathcal{W}})$. The **$\mathcal{G}$-validity relation** $\preccurlyeq_\mathcal{G}$ on H^* is defined by $\alpha \preccurlyeq_\mathcal{G} \beta \iff \inf_{f\in\alpha}\big((\Phi^*f)\nu\big)(\theta|_{K_f}) \le \sup_{g\in\beta}\big((\Phi^*g)\nu\big)(\theta|_{K_g})$ for every $W \in \mathcal{W}$, $\mathbb{C}$-denotation Φ into W, $\mathbb{X}$-denotation f into W, and $\theta \in K \to W_\varepsilon$.

If $h \in A_\emptyset$, then $\theta|_{K_h} \in \emptyset \to W_\varepsilon$, and since $\emptyset \to W_\varepsilon$ is a singleton we identify $\big((\Phi^*h)\nu\big)(\theta|_{K_h}) \in \mathbb{T}$ with $(\Phi^*h)\nu$. Thus $(H, \mathcal{G})$ is an extension of $(A_\emptyset, \mathcal{F})$ in the sense that $A_\emptyset \subseteq H$ and $\mathcal{F} = \{\varphi|_{A_\emptyset} \mid \varphi \in \mathcal{G}\}$.

Remark 3.3 By Remark 2.1, a pair $(X, Y) \in \mathcal{P}H \times \mathcal{P}H$ has a $\mathcal{G}$-model if and only if there exists a quadruple (W, Φ, ν, θ) of an MPC world $W \in \mathcal{W}$ denotable for A, a $\mathbb{C}$-denotation Φ into W, an $\mathbb{X}$-denotation ν into W and an element $\theta \in K \to W_\varepsilon$ satisfying $\big((\Phi^*f)\nu\big)(\theta|_{K_f}) = 1$ for each $f \in X$ and $\big((\Phi^*g)\nu\big)(\theta|_{K_g}) = 0$ for each $g \in Y$. Similarly, a pair $(X, Y) \in \mathcal{P}(A_\emptyset) \times \mathcal{P}(A_\emptyset)$ has an $\mathcal{F}$-model if and only if there exists a triple (W, Φ, ν) which satisfies $(\Phi^*f)\nu = 1$ for each $f \in X$ and $(\Phi^*g)\nu = 0$ for each $g \in Y$. Moreover, $(X, Y) \in \mathcal{P}(A_\emptyset) \times \mathcal{P}(A_\emptyset)$ has a $\mathcal{G}$-model if and only if it has an $\mathcal{F}$-model.

3.6 Structure of MPC worlds

Let $(A, T, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ be an MPC language and $(A, \mathcal{W}, (I_W)_{W \in \mathcal{W}})$ be the logical system MPCL on it.

Lemma 3.3 Let $W \in \mathcal{W}$, $a \in W_\varepsilon$, $b \in W_\delta \cup W_\varepsilon$ and $\ni$ be the basic relation of W. Then the following holds.

$$a \breve{o}\pi\, b\triangle = 1 \iff b \ni a.$$

Proof We have $a \breve{o}\pi\, b\triangle = (b\triangle)(\pi/a)$ by the definition of the operation $\breve{o}\pi$. It follows that $(b\triangle)(\pi/a) = 1 \iff b \ni a$ by the definition of the operation $\triangle$. ■

Lemma 3.4 Let $(H, \mathcal{G})$ be the predicate logical space associated with $(A, \mathcal{W}, (I_W)_{W \in \mathcal{W}})$. Then $\#\mathcal{G} > 1$.

Proof Let x, y be distinct elements of $\mathbb{X}_\varepsilon$. We construct an MPC world $W \in \mathcal{W}$ as follows. Define the base S of W by $S = \{s_1, s_2\}$. We can define the basic relation $\ni$ so that $s_1 \not\ni s_2$. Define a $\mathbb{P}$-measure arbitrarily. Define the operations $\mathfrak{f} \in \mathfrak{F}$ arbitrarily. Next we define a $\mathbb{C}$-denotation Φ into W arbitrarily. Finally we define $\mathbb{X}$-denotations ν, ν' so that $\nu x = \nu y = \nu' x = s_1$ and $\nu' y = s_2$ hold. By Lemma 3.3, $s_1 \breve{o}\pi\, s_1\triangle = 1$ if and only if $s_1 \ni s_1$. On the other hand, $s_2 \breve{o}\pi\, s_1\triangle = 1$ if and only if $s_1 \ni s_2$. Since the basic relation $\ni$ is reflexive, $\Phi^*(y \breve{o}\pi\, x\triangle)\nu = s_1 \breve{o}\pi\, s_1\triangle = 1$. Since $s_1 \not\ni s_2$, $\Phi^*(y \breve{o}\pi\, x\triangle)\nu' = s_2 \breve{o}\pi\, s_1\triangle = 0$. Therefore, two quadruples (W, Φ, ν, θ) and (W, Φ, ν', θ) for an arbitrary $\theta \in K \to S$ induce two distinct elements of $\mathcal{G}$. ■

Theorem 3.1 Let $W \in \mathcal{W}$, Φ be a $\mathbb{C}$-denotation into W and ν be an $\mathbb{X}$-denotation into W. Then $(\Phi^* \mathrm{one})\nu$ is equal to the largest element 1 of W_δ, while $\big(\Phi^*(\mathrm{one}^\square)\big)\nu$ is equal to the least element 0 of W_δ.

Proof Consult [8, Theorem 3.19].

Lemma 3.5 Let $a \in G$, $p \in \mathbb{P}$, $W \in \mathcal{W}$, Φ be a $\mathbb{C}$-denotation into W and ν be an $\mathbb{X}$-denotation into W. Then

$$\big(\Phi^*(a\, \overline{p}\pi\, \mathrm{one}\triangle)\big)\nu = 1 \iff |\{s \in S \mid (\Phi^* a)\nu \ni s\}| > p,$$

where S, $\ni$ and $|\cdot|$ are the base, the basic relation and the $\mathbb{P}$-measure of W, respectively.

Proof We have $\big(\Phi^*(a\, \overline{p}\pi\, \mathrm{one}\triangle)\big)\nu = (\Phi^* a)\nu\, \overline{p}\pi\, (\Phi^* \mathrm{one})\nu\triangle$, and

$$\begin{aligned}
&(\Phi^* a)\nu\, \overline{p}\pi\, (\Phi^* \mathrm{one})\nu\triangle = 1 \\
&\iff |\{s \in S \mid (\Phi^* a)\nu \ni s,\ \big((\Phi^* \mathrm{one})\nu\triangle\big)(\pi/s) = 1\}| > p \\
&\iff |\{s \in S \mid (\Phi^* a)\nu \ni s,\ (\Phi^* \mathrm{one})\nu \ni s\}| > p \\
&\iff |\{s \in S \mid (\Phi^* a)\nu \ni s\}| > p,
\end{aligned}$$

because $(\Phi^*\mathsf{one})\nu \ni \mathsf{s}$ by Theorem 3.1. ■

Lemma 3.6 Let $\mathsf{f}, \mathsf{g} \in \mathsf{H}$, $W \in \mathcal{W}$, Φ be a $\mathbb{C}$-denotation into W and ν be an $\mathbb{X}$-denotation into W. Then the following holds.

$$\begin{aligned}
\big(\Phi^*(\mathsf{f} \wedge \mathsf{g})\big)\nu &= (\Phi^*\mathsf{f})\nu \wedge (\Phi^*\mathsf{g})\nu, \\
\big(\Phi^*(\mathsf{f} \vee \mathsf{g})\big)\nu &= (\Phi^*\mathsf{f})\nu \vee (\Phi^*\mathsf{g})\nu, \\
\big(\Phi^*(\mathsf{f} \Rightarrow \mathsf{g})\big)\nu &= (\Phi^*\mathsf{f})\nu \Rightarrow (\Phi^*\mathsf{g})\nu, \\
(\Phi^*\mathsf{f}^\Diamond)\nu &= \big((\Phi^*\mathsf{f})\nu\big)^\Diamond.
\end{aligned}$$

Proof The conclusion follows from the fact that the mapping $\mathsf{f} \mapsto (\Phi^*\mathsf{f})\nu$ is a holomorphism with respect to $\wedge, \vee, \Rightarrow$ and $\Diamond$. ■

Lemma 3.7 $A_\varepsilon = [\mathbb{S}_\varepsilon]_{\mathfrak{F}}$, where $[\mathbb{S}_\varepsilon]_{\mathfrak{F}}$ is the closure of $\mathbb{S}_\varepsilon$ in the $\mathfrak{F}$-reduct $A_{\mathfrak{F}}$ of A.

Proof Consult [8, §2.2].

Lemma 3.8 Let $W \in \mathcal{W}$, Φ be a $\mathbb{C}$-denotation into W and ν be an $\mathbb{X}$-denotation into W. Assume that the base of W is equal to A_ε, each operation $\mathfrak{f} \in \mathfrak{F}$ on W is equal to $\mathfrak{f}$ on A, and that Φ and ν are the identity mappings when restricted to $\mathbb{C}_\varepsilon$ and $\mathbb{X}_\varepsilon$, respectively. Then $(\Phi^*a)\nu = a$ for all $a \in A_\varepsilon$.

Proof Recall that $A_\varepsilon = [\mathbb{S}_\varepsilon]_{\mathfrak{F}}$ by Lemma 3.7. In order to prove $(\Phi^*a)\nu = a$, we use induction on $\operatorname{Rank} a$. First we assume that $\operatorname{Rank} a = 0$, that is, $a \in \mathbb{S}_\varepsilon$. If $a \in \mathbb{C}_\varepsilon$, then $(\Phi^*a)\nu = \Phi a = a$ by the assumption for Φ. If $a \in \mathbb{X}_\varepsilon$, then $(\Phi^*a)\nu = \nu a = a$ by the assumption for ν. Next we assume that $\operatorname{Rank} a \geq 1$. By the uniqueness of the word form of a, we have $a = \mathfrak{f}(a_1, \ldots, a_k)$, where $\mathfrak{f} \in \mathfrak{F}$, $(a_1, \ldots, a_k) \in \operatorname{Dom} \mathfrak{f}$ and $\operatorname{Rank} a_j < \operatorname{Rank} a$ for $j = 1, \ldots, k$. Then $(\Phi^*a_j)\nu = a_j$ by the inductive hypothesis. Hence,

$$\begin{aligned}
(\Phi^*a)\nu &= \mathfrak{f}((\Phi^*a_1)\nu, \ldots, (\Phi^*a_k)\nu) \\
&= \mathfrak{f}(a_1, \ldots, a_k) = a.
\end{aligned}$$

■

Lemma 3.9 Let $k_1, \ldots, k_n$ be distinct elements of K, $f \in K_{\{k_1,\ldots,k_n\}}$ and $\theta \in \{k_1, \ldots, k_n\} \to W_\varepsilon$. Then the following holds:

$$f\theta = (\theta k_i \,\check{o}\, k_i)_{i=1,\ldots,n} f.$$

Proof Consult [8, Corollary 3.5.2].

3.7 Occurrences in MPC languages

Let $(A, T, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ be an MPC language satisfying Assumption 3.1 and $(A, \mathcal{W}, (I_W)_{W \in \mathcal{W}})$ be the logical system MPCL on it.

Lemma 3.10 For each $\mu \in L$, the following holds.

$$\mathbb{S}^{\mu} = \begin{cases} \emptyset & \text{if } \mu \in \Gamma, \\ x & \text{if } \mu = \Omega x. \end{cases}$$

Proof Recall that L is a subset of the free semigroup over $\Gamma \amalg \mathbb{S}$. If $\mu \in \Gamma$, then only μ occurs in μ but $\mu \notin \mathbb{S}$. If $\mu = \Omega x$, then the only element of $\mathbb{S}$ which occurs in μ is x. ■

Lemma 3.11 If $a, b \in A_{\varepsilon}$ and $x \in \mathbb{X}_{\varepsilon}$, then x is free from b in a.

Proof From Lemma 3.7 it follows that $L^a \subseteq \mathfrak{F}$. By Lemma 3.10, $S^{\mathfrak{f}} = \emptyset$ for each $\mathfrak{f} \in \mathfrak{F}$. Hence x is free from b in a by Lemma 2.4. ■

Lemma 3.12 Let $\mu \in L \cap \Gamma$, $(a_1, \ldots, a_n) \in \operatorname{Dom} \mu$, $b \in A_{\varepsilon}$, $x \in \mathbb{X}_{\varepsilon}$ and x is free from b in a_i $(i = 1, \ldots, n)$. Then x is free from b in $\mu(a_1, \ldots, a_n)$.

Proof We have $S^{\mu} = \emptyset$ by Lemma 3.10. Therefore x is free from b in $\mu(a_1, \ldots, a_n)$ by Theorem 2.10. ■

Lemma 3.13 For each $a \in A$, $\mathbb{P}^a$ is a finite set.

Proof By Lemma 2.2, L^a is a finite set. Since each $\mathfrak{p} \in \mathfrak{P}$ has at most finite endpoints, $\mathbb{P}^a$ is a finite set. ■

Lemma 3.14 Let $a, b \in A$ and $\mathfrak{Q}K = \{\mathfrak{r}k \,|\, \mathfrak{r} \in \mathfrak{Q}, k \in K\}$. If $L^a \cap \mathfrak{Q}K \subseteq L^b \cap \mathfrak{Q}K$, then $\mathbb{P}^a \subseteq \mathbb{P}^b$.

Proof Let $p \in \mathbb{P}$ and suppose $p \in \mathbb{P}^a$. Then there exist elements $\mathfrak{p} \in \mathfrak{P}$ and $k \in K$ such that p is an endpoint of $\mathfrak{p}$, and $\mathfrak{p}k$ or $\neg\mathfrak{p}k$ occurs in a. Since $L^a \cap \mathfrak{Q}K \subseteq L^b \cap \mathfrak{Q}K$, $\mathfrak{p}k$ or $\neg\mathfrak{p}k$ occurs in b. Hence $p \in \mathbb{P}^b$. ■

Lemma 3.15 Let $a, a' \in A$. If a is parallel to a', then $\mathbb{P}^a = \mathbb{P}^{a'}$.

Proof Define a relation R on A by $c \mathrel{R} c'$ if and only if $L^c \cap \mathfrak{Q}K = L^{c'} \cap \mathfrak{Q}K$ and $\sigma c = \sigma c'$, where $\mathfrak{Q}K = \{\mathfrak{r}k \,|\, \mathfrak{r} \in \mathfrak{Q}, k \in K\}$. Then R is sort-consistent. We prove that R is a congruence relation satisfying (2.5) in Theorem 2.4.

Suppose $(a_1, \ldots, a_k) \in \operatorname{Dom} \mu$ and $a_j \mathrel{R} a'_j$ for $j = 1, \ldots, k$. Then $(a_1, \ldots, a_k) \in \operatorname{Dom} \mu$ because $\sigma a_j = \sigma a'_j$ for $j = 1, \ldots, k$. Let $c = \mu(a_1, \ldots, a_k)$ and $c' = \mu(a'_1, \ldots, a'_k)$. Then we have by Lemma 2.2 $L^c \cap \mathfrak{Q}K = (\{\mu\} \cap \mathfrak{Q}K) \cup$

$\left(\bigcup_{j=1}^{k} L^{a_j} \cap \mathfrak{Q}K\right) = (\{\mu\} \cap \mathfrak{Q}K) \cup \left(\bigcup_{j=1}^{k} L^{a'_j} \cap \mathfrak{Q}K\right) = L^{c'} \cap \mathfrak{Q}K$. Since (A, T, σ) is a sorted algebra, $\sigma c = \sigma c'$.

Next suppose $f, g \in A_\emptyset$, $x, y \in \mathbb{X}_\varepsilon$, $g = f(x/y)$, $y \not\ll f$ and x is free from y in f. Then $g(y/x) = f$ by Lemma 2.13, hence $L^f = L^g$ by Lemma 2.8. Let $c = f\,\Omega x$ and $c' = g\,\Omega y$. Since the nominalizers do not belong to $\mathfrak{Q}K$, $L^c \cap \mathfrak{Q}K = L^{c'} \cap \mathfrak{Q}K$. By the definition of the nominalizers, $\sigma c = \sigma c'$.

Therefore, by Theorem 2.4, if a is parallel to a' then $a\,R\,a'$, in particular $L^a \cap \mathfrak{Q}K = L^{a'} \cap \mathfrak{Q}K$, hence $\mathbb{P}^a = \mathbb{P}^{a'}$ by Lemma 3.14. ■

Lemma 3.16 Let $a \in A$, $x \in \mathbb{X}_\varepsilon$, $c \in A_\varepsilon$, and $b = a(x/c)$. Then $\mathbb{P}^b \subseteq \mathbb{P}^a$.

Proof By Lemma 2.5, $L^b \subseteq L^a \cup L^c$. For each $\mathfrak{x} \in \mathfrak{Q}$ and each $k \in K$, by Lemma 3.7, $\mathfrak{x}k \notin L^c$. Hence it follows that if $\mathfrak{x}k$ occurs in b, it occurs in a. Therefore $\mathbb{P}^b \subseteq \mathbb{P}^a$. ■

4 MPC.1 relations

Let $(A, T, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ be an MPC language satisfying Assumption 3.1. In this section, we introduce the MPC.1 law, and show the properties of the relations which satisfy the MPC.1 law. The definition of the MPC.1 law is due to [9].

4.1 Definition

Recall that $G = A_\delta \cup A_\varepsilon$ is the set of nominals and $H = \bigcup_{P \in \mathcal{P}K} A_P$ the set of predicates. We denote by H^* the set of all sequences $f_1 \cdots f_n$ of elements $f_1, \ldots, f_n$ of H of arbitrary finite length $n \geq 0$. We denote elements of H^* by $\alpha, \beta, \ldots$. When $\alpha = f_1 \cdots f_n$, we will denote the subset $\{f_1, \ldots, f_n\}$ of H also by α.

Let $\preccurlyeq$ be a relation on H^*. We denote by $\asymp$ the intersection of the restriction of $\preccurlyeq$ to $H \times H$ and its dual. That is to say, $f \asymp g$ if and only if $f \preccurlyeq g$ and $f \succcurlyeq g$ for each $f, g \in H$. We call $\preccurlyeq$ an **MPC.1 relation** if it satisfies the following **MPC.1 law**. The collection of the former nine laws is called the **Boolean law**:

$$f \preccurlyeq f, \qquad \text{(repetition law)}$$

$$\left.\begin{array}{l} \alpha \preccurlyeq \beta \implies f\alpha \preccurlyeq \beta, \\ \alpha \succcurlyeq \beta \implies f\alpha \succcurlyeq \beta, \end{array}\right\} \qquad \text{(weakening law)}$$

$$\left.\begin{array}{l} ff\alpha \preccurlyeq \beta \implies f\alpha \preccurlyeq \beta, \\ ff\alpha \succcurlyeq \beta \implies f\alpha \succcurlyeq \beta, \end{array}\right\} \qquad \text{(contraction law)}$$

$$\left.\begin{array}{l} \alpha fg\beta \preccurlyeq \gamma \implies \alpha gf\beta \preccurlyeq \gamma, \\ \alpha fg\beta \succcurlyeq \gamma \implies \alpha gf\beta \succcurlyeq \gamma, \end{array}\right\} \qquad \text{(exchange law)}$$

$$\left.\begin{array}{r}\alpha \preccurlyeq f\gamma,\\ f\beta \preccurlyeq \delta\end{array}\right\} \implies \alpha\beta \preccurlyeq \gamma\delta, \qquad \text{(strong cut law)}$$

$$f \wedge g \preccurlyeq f, \quad f \wedge g \preccurlyeq g, \quad fg \preccurlyeq f \wedge g, \qquad \text{(conjunction law)}$$

$$f \vee g \succcurlyeq f, \quad f \vee g \succcurlyeq g, \quad fg \succcurlyeq f \vee g, \qquad \text{(disjunction law)}$$

$$f^{\lozenge} \preccurlyeq f \Rightarrow g, \quad g \preccurlyeq f \Rightarrow g, \quad f \Rightarrow g \preccurlyeq f^{\lozenge} g, \qquad \text{(implication law)}$$

$$ff^{\lozenge} \preccurlyeq\ , \quad ff^{\lozenge} \succcurlyeq\ . \qquad \text{(negation law)}$$

The remaining twenty six laws are proper to MPCL.

$$\preccurlyeq f \implies \ \preccurlyeq a\,\breve{o}k\,f, \qquad \text{(case+ law)}$$

where $a \in A_\varepsilon$ and $k \in K_f$.

$$\preccurlyeq x\,\breve{o}k\,f \implies \ \preccurlyeq f, \qquad \text{(case− law)}$$

where $x \in \mathbb{X}_\varepsilon, k \in K_f$, and $x \not\ll f$.

$$\preccurlyeq f \implies \ \preccurlyeq \mathit{one}\,\forall\pi\,(f\,\Omega x)\triangle, \qquad \text{(}\forall\text{+ law)}$$

where $f \in A_\emptyset$ and $x \in \mathbb{X}_\varepsilon$.

$$\preccurlyeq a\,\breve{o}\pi\,a\triangle, \qquad \text{(= law)}$$

where $a \in A_\varepsilon$.

$$a\,\lambda k\,(b\,\breve{o}l\,f) \asymp b\,\breve{o}l\,(a\,\lambda k\,f), \qquad (\mathfrak{Q}, \breve{o}\text{ law})$$

where $a \in G, b \in A_\varepsilon, f \in H, k, l \in K_f, k \neq l$, and $\lambda \in \{\breve{o}\} \cup \mathfrak{Q}$. Also $a \in A_\varepsilon$ in case $\lambda = \breve{o}$.

$$(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(f \wedge g) \asymp (a_i\,\breve{o}k_i)_{i=1,\ldots,m} f \wedge (a_i\,\breve{o}k_i)_{i=n+1,\ldots,l}\, g, \qquad (\wedge\text{ law})$$

$$(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(f \vee g) \asymp (a_i\,\breve{o}k_i)_{i=1,\ldots,m} f \vee (a_i\,\breve{o}k_i)_{i=n+1,\ldots,l}\, g, \qquad (\vee\text{ law})$$

$$(a_i\,\breve{o}k_i)_{i=1,\ldots,l}(f \Rightarrow g) \asymp (a_i\,\breve{o}k_i)_{i=1,\ldots,m} f \Rightarrow (a_i\,\breve{o}k_i)_{i=n+1,\ldots,l}\, g, \qquad (\Rightarrow\text{ law})$$

where $a_1, \ldots, a_l \in A_\varepsilon$, $f, g \in H$, and $k_1, \ldots, k_l$ are distinct cases such that $k_1, \ldots, k_n \in K_f - K_g, k_{n+1}, \ldots, k_m \in K_f \cap K_g$, and $k_{m+1}, \ldots, k_l \in K_g - K_f$ $(0 \leq n \leq m \leq l)$. Also, $(a_i\,\breve{o}k_i)_{i=1,\ldots,l}h$ is an abbreviation for $a_1\,\breve{o}k_1\,(a_2\,\breve{o}k_2\,(\ldots(a_l\,\breve{o}k_l h)\ldots))$.

$$(a_i\,\breve{o}k_i)_{i=1,\ldots,n}(f^{\lozenge}) \asymp \big((a_i\,\breve{o}k_i)_{i=1,\ldots,n} f\big)^{\lozenge}, \qquad (\lozenge\text{ law})$$

where $a_1, \ldots, a_n \in A_\varepsilon, f \in H$, and $k_1, \ldots, k_n$ are distinct cases in K_f.

$$a\,\neg\mathfrak{p}k\,f \asymp a\,\mathfrak{p}k\,f^{\lozenge}, \qquad (\neg\text{ law})$$

$$a\,\mathfrak{p}^{\circ}k\,f \asymp (a\,\mathfrak{p}k\,f)^{\lozenge}, \qquad (\circ\text{ law})$$

where $a \in G, f \in H, k \in K_f$, and $\mathfrak{p} \in \mathfrak{P}$.

$$a\,(\mathfrak{p} \cap \mathfrak{q})k\,f \asymp a\,\mathfrak{p}k\,f \wedge a\,\mathfrak{q}k\,f, \tag{$\cap$ law}$$

$$a\,(\mathfrak{p} \cup \mathfrak{q})k\,f \asymp a\,\mathfrak{p}k\,f \vee a\,\mathfrak{q}k\,f, \tag{$\cup$ law}$$

where $a \in G, f \in H, k \in K_f$ and $\mathfrak{p}, \mathfrak{q} \in \mathfrak{P}$.

$$a\,\overline{p}k\,f \asymp a\,\overline{p}\pi\,\big((x\,\breve{o}k\,f)\,\Omega x\big)\triangle, \tag{$\mathfrak{P}$ law}$$

where $a \in G, f \in H, x \in \mathbb{X}_\varepsilon, p \in \mathbb{P}, K_f = \{k\}$ and $x \not\ll f$.

$$a\,\overline{p}\pi\,b\triangle \asymp (a \sqcap b)\,\overline{p}\pi\,\mathit{one}\triangle, \tag{$\triangle$ law}$$

where $a, b \in G$, and $p \in \mathbb{P}$.

$$f, \mathit{one}\,\forall\pi\,\big((f \Rightarrow g)\,\Omega x\big)\triangle \preccurlyeq \mathit{one}\,\forall\pi\,(g\,\Omega x)\triangle, \tag{$\forall, \Rightarrow$ law}$$

where $f, g \in A_\emptyset, x \in \mathbb{X}_\varepsilon$, and $x \not\ll f$.

$$\mathit{one}\,\forall\pi\,\Big(\big((x\,\breve{o}\pi\,a\triangle) \Rightarrow (x\,\breve{o}k\,f)\big)\,\Omega x\Big)\triangle \preccurlyeq a\,\forall k\,f, \tag{$\forall$ law}$$

where $x \in \mathbb{X}_\varepsilon, a \in G, f \in H, K_f = \{k\}$, and $x \not\ll a, f$.

$$a\,\forall\pi\,b\triangle, a\,\overline{p}k\,f \preccurlyeq b\,\overline{p}k\,f, \tag{$\forall, \mathfrak{P}$ law}$$

where $a, b \in G, f \in H, k \in K_f$, and $p \in \mathbb{P}$.

$$(a \sqcup b)\,\overline{p + q}k\,f \preccurlyeq a, \overline{p}k\,f, b\,\overline{q}k\,f, \tag{$\sqcup, +$ law}$$

where $a, b \in G, f \in H, k \in K_f$, and $p, q \in \mathbb{P}$.

$$\mathit{one}^{\square}\,\overline{p}k\,f \preccurlyeq\ , \tag{$\mathit{one}^{\square}$ law}$$

where $f \in H, k \in K_f$, and $p \in \mathbb{P}$.

$$b\,\breve{o}\pi\,a\triangle \preccurlyeq a\,\exists\pi\,\mathit{one}\triangle, \tag{$\exists$ law}$$

where $a \in G$ and $b \in A_\varepsilon$.

$$(a \sqcap b)\triangle \asymp a\triangle \wedge b\triangle, \tag{$\sqcap$ law}$$

$$(a \sqcup b)\triangle \asymp a\triangle \vee b\triangle, \tag{$\sqcup$ law}$$

$$(a^{\square})\triangle \asymp (a\triangle)^{\Diamond}, \tag{$\square$ law}$$

where $a, b \in G$.

$$a\,\breve{o}\pi\,(f\,\Omega x)\triangle \asymp f(x/a), \tag{Ω law}$$

where $a \in A_\varepsilon, f \in A_\emptyset, x \in \mathbb{X}$, and x is free from a in f.

$$\mathit{one}\,\forall\pi\,(f\,\Omega x)\triangle \preccurlyeq f, \tag{$\forall-$ law}$$

where $f \in A_\emptyset$ and $x \in \mathbb{X}_\varepsilon$.

This completes the list of the MPC.1 law.

Remark 4.1 Notice that the MPC.1 law is regarded as a deduction system on $H^* \times H^*$.

Theorem 4.1 The $\mathcal{G}$-validity relation $\preccurlyeq_{\mathcal{G}}$ of the predicate logical space $(H, \mathcal{G})$ is an MPC.1 relation.

Proof Consult [9, Theorem 2].

4.2 Properties of MPC.1 relations

In this subsection, let $\preccurlyeq$ be an MPC.1 relation.

Lemma 4.1 The following holds:

- $\alpha f g \beta \preccurlyeq \gamma \iff \alpha, f \wedge g, \beta \preccurlyeq \gamma.$
- $\alpha f g \beta \succcurlyeq \gamma \iff \alpha, f \vee g, \beta \succcurlyeq \gamma.$
- $\begin{cases} \alpha \preccurlyeq f\beta \iff f^{\Diamond}\alpha \preccurlyeq \beta, \\ \alpha \succcurlyeq f\beta \iff f^{\Diamond}\alpha \succcurlyeq \beta. \end{cases}$
- $f\alpha \preccurlyeq g\beta \iff \alpha \preccurlyeq f \Rightarrow g, \beta.$
- $\left.\begin{array}{l} f_1 \wedge \cdots \wedge f_n \asymp (\ldots(f_1 \wedge f_2)\ldots) \wedge f_n, \\ f_1 \vee \cdots \vee f_n \asymp (\ldots(f_1 \vee f_2)\ldots) \vee f_n, \end{array}\right\}$ irrespective of the order of applying the operations $\wedge$ and $\vee$ on the left-hand side of $\asymp$.
- $\left.\begin{array}{l} f_1 \preccurlyeq g_1, \\ f_2 \preccurlyeq g_2 \end{array}\right\} \implies \begin{cases} f_1 \wedge f_2 \preccurlyeq g_1 \wedge g_2, \\ f_1 \vee f_2 \preccurlyeq g_1 \vee g_2. \end{cases}$
- $\alpha \preccurlyeq \beta \iff \alpha \preccurlyeq f \wedge f^{\Diamond}, \beta \iff f \vee f^{\Diamond}, \alpha \preccurlyeq \beta.$

Proof Consult [9, Lemma 2.1].

Lemma 4.2 Let $a_1, \ldots, a_n \in A_\varepsilon$, $f_1, \ldots, f_m \in H$, and $k_1, \ldots, k_n$ be distinct cases in $K_{f_1} \cap \cdots \cap K_{f_m}$. Then the following holds irrespective of the order of applying the operations $\wedge$ and $\vee$:

$$(a\,\breve{o}k_i)_{i=1,\ldots,n}(f_1 \wedge \cdots \wedge f_m) \asymp (a\,\breve{o}k_i)_{i=1,\ldots,n}f_1 \wedge \cdots \wedge (a\,\breve{o}k_i)_{i=1,\ldots,n}f_m, \quad \text{(generalized } \wedge \text{ law)}$$
$$(a\,\breve{o}k_i)_{i=1,\ldots,n}(f_1 \vee \cdots \vee f_m) \asymp (a\,\breve{o}k_i)_{i=1,\ldots,n}f_1 \vee \cdots \vee (a\,\breve{o}k_i)_{i=1,\ldots,n}f_m. \quad \text{(generalized } \vee \text{ law)}$$

Proof Consult [9, Lemma 2.2].

Lemma 4.3 Let $a \in G$, $f \in H$, $k \in K_f$ and $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \in \mathfrak{P}$. Then the following holds irrespective of the order of applying the operations $\wedge, \vee$:

$$a\,(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n)k\,f \asymp a\,\mathfrak{p}_1 k\,f \wedge \cdots \wedge a\,\mathfrak{p}_n k\,f, \quad \text{(generalized } \cap \text{ law)}$$
$$a\,(\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n)k\,f \asymp a\,\mathfrak{p}_1 k\,f \vee \cdots \vee a\,\mathfrak{p}_n k\,f, \quad \text{(generalized } \cup \text{ law)}$$

Proof Consult [9, Lemma 2.3].

Lemma 4.4 Let $f_1, \dots, f_m, g_1, \dots, g_n \in H$, $\alpha, \beta \in H^*$, $a \in A_\varepsilon$ and $k \in K$. Assume that k belongs to the ranges of $f_1, \dots, f_m, g_1, \dots, g_n$ but does not belong to those of the predicates in $\alpha \cup \beta$. Then the following holds:

$$\begin{aligned} & f_1 \cdots f_m \alpha \preccurlyeq g_1 \cdots g_n \beta \\ & \implies a\,\breve{o}k\, f_1, \dots, a\,\breve{o}k\, f_m, \alpha \preccurlyeq a\,\breve{o}k\, g_1, \dots, a\,\breve{o}k\, g_n, \beta. \end{aligned}$$

(generalized case+ law)

Proof Consult [9, Lemma 2.6].

Lemma 4.5 Let $f_1, \dots, f_m, g_1, \dots, g_n \in H$, $\alpha, \beta \in H^*$, $x \in \mathbb{X}_\varepsilon$ and $k \in K$. Assume that k belongs to the ranges of $f_1, \dots, f_m, g_1, \dots, g_n$ but does not belong to those of the predicates in $\alpha \cup \beta$ and x does not occur free in the predicates in $\{f_1, \dots, f_m, g_1, \dots, g_n\} \cup \alpha \cup \beta$. Then the following holds:

$$\begin{aligned} & x\,\breve{o}k\, f_1, \dots, x\,\breve{o}k\, f_m, \alpha \preccurlyeq x\,\breve{o}k\, g_1, \dots, x\,\breve{o}k\, g_n, \beta \\ & \implies f_1 \cdots f_m \alpha \preccurlyeq g_1 \cdots g_n \beta. \end{aligned}$$

(generalized case− law)

Proof Consult [9, Lemma 2.7].

Lemma 4.6 Let $x \in \mathbb{X}_\varepsilon$, $a, b_1, \dots, b_n \in G$, $\alpha, \beta \in (A_\emptyset)^*$, $f \in H$, $k \in K_f$, $p, q_1, \dots, q_n \in \mathbb{P}$, and assume that x does not occur free in the elements of $\{a, b_1, \dots, b_n\} \cup \alpha \cup \beta$ and that $p \geq \sum_{i=1}^n q_i$ holds, where if $n = 0$ then $\sum_{i=1}^n q_i = 0$ by definition. Then the following holds:

$$\begin{aligned} & x\,\breve{o}\pi\, a\triangle, \alpha \preccurlyeq x\,\breve{o}\pi\, b_1\triangle, \dots, x\,\breve{o}\pi\, b_n\triangle, \beta \\ & \implies a\,\overline{p}k\, f, \alpha \preccurlyeq b_1\,\overline{q_1}k\, f, \dots, b_n\,\overline{q_n}k\, f, \beta. \end{aligned}$$

(pigeonhole principle)

Remark 4.2 When $n = 1$ and $q_1 = p$, the following holds:

$$x\,\breve{o}\pi\, a\triangle \preccurlyeq x\,\breve{o}\pi\, b_1\triangle \implies a\,\overline{p}k\, f \preccurlyeq b_1\,\overline{p}k\, f. \tag{4.1}$$

Proof Consult [9, Lemma 2.8].

Lemma 4.7 Let $a_1, \dots, a_n \in A_\varepsilon$, $f \in H$, and $k_1, \dots, k_n$ be distinct cases in K_f. Then the following holds for every $\rho \in \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group on the letters $1, \dots, n$:

$$(a_i\,\breve{o}k_i)_{i=1,\dots,n} f \asymp (a_{\rho i}\,\breve{o}k_{\rho i})_{i=1,\dots,n} f.$$

(permutation law)

Proof Consult [9, Lemma 2.9].

Lemma 4.8 Let $a_1, \dots, a_n \in A_\varepsilon$, $f, g \in H$ and $k_1, \dots, k_n$ be distinct cases in $K_f \cap K_g$. If $f \preccurlyeq g$, then $(a_i\,\breve{o}k_i)_{i=1,\dots,n} f \preccurlyeq (a_i\,\breve{o}k_i)_{i=1,\dots,n} g$.

Proof Apply the generalized case+ law to $f \preccurlyeq g$, n times. ■

Lemma 4.9 Let $f_1, \ldots, f_n \in H$. Then the following holds:

$$f_1 \ldots f_n \preccurlyeq f_1 \wedge \cdots \wedge f_n,$$
$$f_1 \ldots f_n \succcurlyeq f_1 \vee \cdots \vee f_n.$$

Proof We prove the first equation by induction on n. If $n = 1$, the conclusion is the repetition law itself. Assume $n \geq 2$. We have

$$f_1 \ldots f_{n-1} \preccurlyeq f_1 \wedge \cdots \wedge f_{n-1}$$

by the inductive hypothesis, and

$$f_1 \wedge \cdots \wedge f_{n-1}, f_n \preccurlyeq f_1 \wedge \cdots \wedge f_n$$

by the conjunction law. Applying the strong cut law to the above two equations, we have the conclusion.

A similar argument holds for the second equation. ■

Lemma 4.10 Let $a \in G$, $b_1, \ldots, b_n \in A_\varepsilon$, $f \in H$, $k, k_1, \ldots, k_n$ be distinct cases in K_f, and $\lambda \in \{\breve{o}\} \cup \mathfrak{Q}$. Also assume $a \in A_\varepsilon$ in case $\lambda = \breve{o}$. Then the following holds:

$$a\,\lambda k\left((b_i\,\breve{o}k_i)_{i=1,\ldots,n} f\right) \asymp (b_i\,\breve{o}k_i)_{i=1,\ldots,n}(a\,\lambda k\, f). \quad \text{(generalized } \mathfrak{Q}, \breve{o} \text{ law)}$$

Proof We use induction on n. If $n = 0$, then the conclusion follows from the repetition law. Suppose $n \geq 1$. We have

$$a\,\lambda k\left((b_i\,\breve{o}k_i)_{i=2,\ldots,n} f\right) \asymp (b_i\,\breve{o}k_i)_{i=2,\ldots,n}(a\,\lambda k\, f)$$

by the inductive hypothesis, hence

$$b_1\,\breve{o}k_1\left(a\,\lambda k\left((b_i\,\breve{o}k_i)_{i=2,\ldots,n} f\right)\right) \asymp (b_i\,\breve{o}k_i)_{i=1,\ldots,n}(a\,\lambda k\, f)$$

by Lemma 4.8. We have

$$a\,\lambda k\left((b_i\,\breve{o}k_i)_{i=1,\ldots,n} f\right) \asymp b_1\,\breve{o}k_1\left(a\,\lambda k\left((b_i\,\breve{o}k_i)_{i=2,\ldots,n} f\right)\right)$$

by the $\mathfrak{Q}, \breve{o}$ law. Applying the strong cut law to the above two equations, we have the conclusion. ■

Lemma 4.11 Let $f, g \in H$ and $x \in \mathbb{X}_\varepsilon$. If $f \preccurlyeq g$, then $(f\,\Omega x)\triangle \preccurlyeq (g\,\Omega x)\triangle$.

Proof Notice that x is free from x in both f and g by Lemma 2.7, and also that $f(x/x) = f$ and $g(x/x) = g$. We have

$$x\,\breve{o}\pi\,(f\,\Omega x)\triangle \asymp f,$$
$$x\,\breve{o}\pi\,(g\,\Omega x)\triangle \asymp g$$

by the Ω law. Applying the strong cut law to $f \preccurlyeq g$ with the above two equations, we have

$$x\,\breve{o}\pi\,(f\,\Omega x)\triangle \preccurlyeq x\,\breve{o}\pi\,(g\,\Omega x)\triangle.$$

Since x does not occur free in $(f\,\Omega x)\triangle$ nor in $(g\,\Omega x)\triangle$, we have

$$(f\,\Omega x)\triangle \preccurlyeq (g\,\Omega x)\triangle$$

by the generalized case$-$ law. ■

Lemma 4.12 Let $a_1, \ldots, a_n \in A_\varepsilon$, $f, f_1, \ldots, f_m \in H$, and $k_1, \ldots, k_n$ be distinct cases of $K_f \cap K_{f_1} \cap \cdots \cap K_{f_m}$. If $f \asymp f_1 \wedge \cdots \wedge f_m$ then

$$(a_i\,\breve{o}k_i)_{i=1,\ldots,n}f \asymp (a_i\,\breve{o}k_i)_{i=1,\ldots,n}f_1 \wedge \cdots \wedge (a_i\,\breve{o}k_i)_{i=1,\ldots,n}f_m,$$

while if $f \asymp f_1 \vee \cdots \vee f_m$ then

$$(a_i\,\breve{o}k_i)_{i=1,\ldots,n}f \asymp (a_i\,\breve{o}k_i)_{i=1,\ldots,n}f_1 \vee \cdots \vee (a_i\,\breve{o}k_i)_{i=1,\ldots,n}f_m.$$

Proof Suppose $f \asymp f_1 \wedge \cdots \wedge f_m$. Then we have

$$(a_i\,\breve{o}k_i)_{i=1,\ldots,n}f \asymp (a_i\,\breve{o}k_i)_{i=1,\ldots,n}(f_1 \wedge \cdots \wedge f_m)$$

by Lemma 4.8. We have

$$(a_i\,\breve{o}k_i)_{i=1,\ldots,n}(f_1 \wedge \cdots \wedge f_m) \asymp (a_i\,\breve{o}k_i)_{i=1,\ldots,n}f_1 \wedge \cdots \wedge (a_i\,\breve{o}k_i)_{i=1,\ldots,n}f_m$$

by the generalized $\wedge$ law. Applying the strong cut law to the above two equations, we have the first conclusion.

A similar argument holds for the second equation. ■

Lemma 4.13 Let $a_1, \ldots, a_n \in A_\varepsilon$, $f, g \in H$, and $k_1, \ldots, k_n$ be distinct cases of $K_f \cap K_g$. If $f \asymp g^\Diamond$ then $(a_i\,\breve{o}k_i)_{i=1,\ldots,n}f \asymp \big((a_i\,\breve{o}k_i)_{i=1,\ldots,n}g\big)^\Diamond$.

Proof Since $f \asymp g^\Diamond$, we have

$$(a_i\,\breve{o}k_i)_{i=1,\ldots,n}f \asymp (a_i\,\breve{o}k_i)_{i=1,\ldots,n}g^\Diamond$$

by Lemma 4.8. We have

$$(a_i\,\breve{o}k_i)_{i=1,\ldots,n}g^\Diamond \asymp \big((a_i\,\breve{o}k_i)_{i=1,\ldots,n}g\big)^\Diamond$$

by the $\Diamond$ law. Applying the strong cut law to the above two equations, we have the conclusion. ■

Lemma 4.14 Let $a_1, \ldots, a_n \in A_\varepsilon$, $f \in H$, $k, k_1, \ldots, k_n$ be distinct cases of K_f, and $\mathfrak{p}_1, \ldots, \mathfrak{p}_m \in \mathfrak{P}$. Then the following holds:

$$\begin{aligned}
&(a_i\,\breve{o}k_i)_{i=1,\ldots,n}(a\,(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m)k\,f) \\
&\quad \asymp (a_i\,\breve{o}k_i)_{i=1,\ldots,n}(a\,\mathfrak{p}_1 k\,f) \wedge \cdots \wedge (a_i\,\breve{o}k_i)_{i=1,\ldots,n}(a\,\mathfrak{p}_m k\,f), \\
&(a_i\,\breve{o}k_i)_{i=1,\ldots,n}(a\,(\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_m)k\,f) \\
&\quad \asymp (a_i\,\breve{o}k_i)_{i=1,\ldots,n}(a\,\mathfrak{p}_1 k\,f) \vee \cdots \vee (a_i\,\breve{o}k_i)_{i=1,\ldots,n}(a\,\mathfrak{p}_m k\,f).
\end{aligned}$$

Proof Let $h = a\,(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m)k\,f$, and $h_j = a\,\mathfrak{p}_j k\,f$ for $j = 1, \ldots, m$. We have $h \asymp h_1 \wedge \cdots \wedge h_m$ by the generalized $\cap$ law. It follows that $K_h = K_{h_1} = \cdots = K_{h_m} = K_f - \{k\}$, and that $k_1, \ldots, k_n$ are distinct cases of K_h. Therefore we have the first conclusion by Lemma 4.12.

A similar argument holds for the second conclusion. ■

Lemma 4.15 Let $a_1, \ldots, a_n \in A_\varepsilon$, $f \in H$, $k, k_1, \ldots, k_n$ be distinct cases of K_f, and $\mathfrak{p} \in \mathfrak{P}$. Then the following holds:

$$\begin{aligned}
(a_i\,\breve{o}k_i)_{i=1,\ldots,n}(a\,\neg\mathfrak{p}k\,f) &\asymp (a_i\,\breve{o}k_i)_{i=1,\ldots,n}(a\,\mathfrak{p}k\,f^{\Diamond}), \\
(a_i\,\breve{o}k_i)_{i=1,\ldots,n}(a\,\mathfrak{p}^{\circ}k\,f) &\asymp \big((a_i\,\breve{o}k_i)_{i=1,\ldots,n}(a\,\mathfrak{p}k\,f)\big)^{\Diamond}.
\end{aligned}$$

Proof We have $a\,\neg\mathfrak{p}k\,f \asymp a\,\mathfrak{p}k\,f^{\Diamond}$ by the $\neg$ law, hence we have the first conclusion by Lemma 4.8.

We have $a\,\mathfrak{p}^{\circ}k\,f \asymp (a\,\mathfrak{p}k\,f)^{\Diamond}$ by the $\circ$ law. It follows that $k_1, \ldots, k_n$ are distinct cases of the ranges of both $a\,\mathfrak{p}^{\circ}k\,f$ and $(a\,\mathfrak{p}k\,f)^{\Diamond}$. Therefore we have the second conclusion by Lemma 4.13. ■

Lemma 4.16 Let $a \in A_\varepsilon$, $b, c \in G$. Then the following holds:

$$\begin{aligned}
a\,\breve{o}\pi\,(b \sqcap c)\triangle &\asymp a\,\breve{o}\pi\,b\triangle \wedge a\,\breve{o}\pi\,c\triangle, \\
a\,\breve{o}\pi\,(b \sqcup c)\triangle &\asymp a\,\breve{o}\pi\,b\triangle \vee a\,\breve{o}\pi\,c\triangle, \\
a\,\breve{o}\pi\,b^{\Box}\triangle &\asymp (a\,\breve{o}\pi\,b\triangle)^{\Diamond}.
\end{aligned}$$

Proof We have $(b \sqcap c)\triangle \asymp b\triangle \wedge c\triangle$ by the $\sqcap$ law, hence we have the first conclusion.

The second conclusion is proved similarly.

We have $b^{\Box}\triangle \asymp (b\triangle)^{\Diamond}$ by the $\Diamond$ law, hence we have the third conclusion by Lemma 4.13. ■

Lemma 4.17 Let $a \in G$, $f \in H$, $x \in \mathbb{X}_\varepsilon$, $k \in K_f$ and $p \in \mathbb{P}$. Also, let $a_1, \ldots, a_n \in A_\varepsilon$ and $k_1, \ldots, k_n$ be the set of distinct cases in $K_f - \{k\}$. Assume $x \not\lessdot (a_i\,\breve{o}k_i)_{i=1,\ldots,n}f$. Then the following holds:

$$(a_i\,\breve{o}k_i)_{i=1,\ldots,n}(a\,\overline{p}k\,f) \asymp \big(a \sqcap \big(x\,\breve{o}k\,(a_i\,\breve{o}k_i)_{i=1,\ldots,n}f\big)\Omega x\big)\,\overline{p}\pi\,one\triangle.$$

Proof Consult [9, Lemma 2.10].

4.3 Alternative lemma

Let $(A, T, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ be an MPC language satisfying Assumption 3.1, $(A, \mathcal{W}, (I_W)_{W \in \mathcal{W}})$ be the logical system MPCL on it, and $\preccurlyeq$ be an MPC.1 relation contained in the validity relation $\preccurlyeq_{\mathcal{G}}$ of the predicate logical space $(H, \mathcal{G})$.

Lemma 4.18 Let $a, b \in G$, $f, g \in H$, $\lambda \in \{\breve{o}\} \cup \mathfrak{Q}$ and $k \in K_f \cap K_g$. Assume also $a, b \in A_\varepsilon$ in case $\lambda = \breve{o}$. If $a\triangle \asymp b\triangle$ and $f \asymp g$, then $a\,\lambda k\, f \asymp b\,\lambda k\, g$.

Proof (i) First we consider the case where $\lambda = \breve{o}$. In this case a and b must belong to A_ε. If $a = b$, then we have the conclusion $a\,\breve{o}k\, f \asymp b\,\breve{o}k\, g$ by Lemma 4.8. So it suffices to prove that $a = b$. Assume $a \neq b$ to deduce a contradiction. Since $a\triangle \asymp b\triangle$, we have $a\,\breve{o}\pi\, a\triangle \asymp a\,\breve{o}\pi\, b\triangle$ by Lemma 4.8. We construct an MPC world W denotable for A as follows. Define the base S of W by $S = A_\varepsilon$, let $\exists$ be the equality relation on S, and define a $\mathbb{P}$-measure arbitrarily. Next we define a $\mathbb{C}$-denotation Φ into W such that $\Phi c = c$ for each $c \in \mathbb{C}_\varepsilon$, and an $\mathbb{X}$-denotation ν into W such that $\nu x = x$ for each $x \in \mathbb{X}_\varepsilon$. Then $(\Phi^* a)\nu = a$ and $(\Phi^* b)\nu = b$ by Lemma 3.8. Since $a \exists a$ and $b \not\exists a$, we have $\big(\Phi^*(a\,\breve{o}\pi\, a\triangle)\big)\nu = 1$ and $\big(\Phi^*(a\,\breve{o}\pi\, b\triangle)\big)\nu = 0$ by Lemma 3.3. This contradicts that $\preccurlyeq$ is contained in $\preccurlyeq_{\mathcal{G}}$.

(ii) Next we consider the case where $\lambda = \overline{p}$ with $p \in \mathbb{P}$. Take $x \in \mathbb{X}_\varepsilon$ which does not occur free in f nor in g. This can be done because $\mathbb{X}_\varepsilon$ is an infinite set. By the $\mathfrak{P}$ law and the $\triangle$ law, we have

$$a\,\overline{p}kf \asymp \big(a \sqcap (x\,\breve{o}k\, f)\,\Omega x\big)\,\overline{p}\pi\, one\triangle, \tag{4.2}$$

$$b\,\overline{p}kg \asymp \big(b \sqcap (x\,\breve{o}k\, g)\,\Omega x\big)\,\overline{p}\pi\, one\triangle. \tag{4.3}$$

Since $f \asymp g$, we have $x\,\breve{o}k\, f \asymp x\,\breve{o}k\, g$ by Lemma 4.8, hence

$$\big((x\,\breve{o}k\, f)\,\Omega x\big)\triangle \asymp \big((x\,\breve{o}k\, g)\,\Omega x\big)\triangle$$

by Lemma 4.11. This together with $a\triangle \asymp b\triangle$ implies

$$a\triangle \wedge \big((x\,\breve{o}k\, f)\,\Omega x\big)\triangle \asymp b\triangle \wedge \big((x\,\breve{o}k\, g)\,\Omega x\big)\triangle$$

by Lemma 4.1. We have

$$\big(a \sqcap (x\,\breve{o}k\, f)\,\Omega x\big)\triangle \asymp a\triangle \wedge \big((x\,\breve{o}k\, f)\,\Omega x\big)\triangle,$$
$$\big(b \sqcap (x\,\breve{o}k\, g)\,\Omega x\big)\triangle \asymp b\triangle \wedge \big((x\,\breve{o}k\, g)\,\Omega x\big)\triangle$$

by the $\sqcap$ law. Applying the strong cut law to the above three equations, we have

$$\big(a \sqcap (x\,\breve{o}k\, f)\,\Omega x\big)\triangle \asymp \big(b \sqcap (x\,\breve{o}k\, g)\,\Omega x\big)\triangle.$$

Take $y \in \mathbb{X}_\varepsilon$ which does not occur free in $\big(a \sqcap (x\,\breve{o}k\, f)\,\Omega x\big)\triangle$ nor in $\big(b \sqcap (x\,\breve{o}k\, g)\,\Omega x\big)\triangle$. Then

$$y\,\breve{o}\pi\,\big(a \sqcap (x\,\breve{o}k\, f)\,\Omega x\big)\triangle \asymp y\,\breve{o}\pi\,\big(b \sqcap (x\,\breve{o}k\, g)\,\Omega x\big)\triangle$$

by Lemma 4.8, hence we have

$$\big(\mathfrak{a} \sqcap (x\,\breve{o}k\,f)\,\Omega x\big)\,\overline{p}\pi\,one\triangle \asymp \big(\mathfrak{b} \sqcap (x\,\breve{o}k\,g)\,\Omega x\big)\,\overline{p}\pi\,one\triangle$$

by the pigeonhole principle or (4.1). Applying the strong cut law to (4.2), (4.3) and the above equation, we have

$$\mathfrak{a}\,\overline{p}kf \asymp \mathfrak{b}\,\overline{p}kg,$$

which is the conclusion.

(iii) The case where $\lambda = (\leftarrow p]$ with $p \in \mathbb{P}$. We have $\mathfrak{a}\,\overline{p}k\,f \asymp \mathfrak{b}\,\overline{p}k\,g$ by the case (ii), hence

$$(\mathfrak{a}\,\overline{p}k\,f)^{\Diamond} \asymp (\mathfrak{b}\,\overline{p}k\,g)^{\Diamond}$$

by Lemma 4.1. We have

$$\mathfrak{a}\,(\leftarrow p]k\,f \asymp (\mathfrak{a}\,\overline{p}k\,f)^{\Diamond},$$
$$\mathfrak{b}\,(\leftarrow p]k\,g \asymp (\mathfrak{b}\,\overline{p}k\,g)^{\Diamond},$$

by the $\circ$ law. Applying the strong cut law to the above three equations, we have

$$\mathfrak{a}\,(\leftarrow p]k\,f \asymp \mathfrak{b}\,(\leftarrow p]k\,g,$$

which is the conclusion.

(iv) The case where $\lambda = (p, q]$ with $p, q \in \mathbb{P}$. We have

$$\mathfrak{a}\,\overline{p}k\,f \asymp \mathfrak{b}\,\overline{p}k\,g$$

by the case (ii), and

$$\mathfrak{a}\,(\leftarrow q]k\,f \asymp \mathfrak{b}\,(\leftarrow q]k\,g$$

by the case (iii), hence we have

$$\mathfrak{a}\,\overline{p}k\,f \wedge \mathfrak{a}\,(\leftarrow q]k\,f \asymp \mathfrak{b}\,\overline{p}k\,g \wedge \mathfrak{a}\,(\leftarrow q]k\,g$$

by Lemma 4.1. We have

$$\mathfrak{a}\,(p, q]k\,f \asymp \mathfrak{a}\,\overline{p}k\,f \wedge \mathfrak{a}\,(\leftarrow q]k\,f,$$
$$\mathfrak{b}\,(p, q]k\,g \asymp \mathfrak{b}\,\overline{p}k\,g \wedge \mathfrak{a}\,(\leftarrow q]k\,g$$

by the $\cap$ law. Applying the strong cut law to the above three equations, we have

$$\mathfrak{a}\,(p, q]k\,f \asymp \mathfrak{b}\,(p, q]k\,g,$$

which is the conclusion.

(v) The case where $\lambda = \mathfrak{p} \in \mathfrak{P}$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be the connected components of $\mathfrak{p}$. Then, for each $i \in \{1, \ldots, m\}$, $\mathfrak{p}_i$ is of the form $\overline{p}$, $(\leftarrow p]$, or $(p, q]$, where $p, q \in \mathbb{P}$. So we have

$$\mathfrak{a}\,\mathfrak{p}_i k\,f \asymp \mathfrak{b}\,\mathfrak{p}_i k\,g$$

by the cases (ii)-(iv), hence

$$\mathfrak{a}\,\mathfrak{p}_1 k\, f \vee \cdots \vee \mathfrak{a}\,\mathfrak{p}_m k\, f \asymp \mathfrak{b}\,\mathfrak{p}_1 k\, \mathfrak{g} \vee \cdots \vee \mathfrak{b}\,\mathfrak{p}_m k\, \mathfrak{g}$$

by Lemma 4.1. We have

$$\mathfrak{a}\,\mathfrak{p}k\, f \asymp \mathfrak{a}\,\mathfrak{p}_1 k\, f \vee \cdots \vee \mathfrak{a}\,\mathfrak{p}_m k\, f,$$
$$\mathfrak{b}\,\mathfrak{p}k\, \mathfrak{g} \asymp \mathfrak{b}\,\mathfrak{p}_1 k\, \mathfrak{g} \vee \cdots \vee \mathfrak{b}\,\mathfrak{p}_m k\, \mathfrak{g}$$

by the generalized $\cup$ law. Applying the strong cut law to the above three equations, we have

$$\mathfrak{a}\,\mathfrak{p}k\, f \asymp \mathfrak{b}\,\mathfrak{p}k\, \mathfrak{g},$$

which is the conclusion.

(vi) Finally we consider the case where $\lambda = \neg\mathfrak{p}$ with $\mathfrak{p} \in \mathfrak{P}$. We have $f^{\Diamond} \asymp \mathfrak{g}^{\Diamond}$ by Lemma 4.1, hence

$$\mathfrak{a}\,\mathfrak{p}k\, f^{\Diamond} \asymp \mathfrak{b}\,\mathfrak{p}k\, \mathfrak{g}^{\Diamond}$$

by the case (v). We have

$$\mathfrak{a}\,\neg\mathfrak{p}k\, f \asymp \mathfrak{a}\,\mathfrak{p}k\, f^{\Diamond},$$
$$\mathfrak{b}\,\neg\mathfrak{p}k\, \mathfrak{g} \asymp \mathfrak{b}\,\mathfrak{p}k\, \mathfrak{g}^{\Diamond}$$

by the $\neg$ law. Applying the strong cut law to the above three equations, we have

$$\mathfrak{a}\,\neg\mathfrak{p}k\, f \asymp \mathfrak{b}\,\neg\mathfrak{p}k\, \mathfrak{g},$$

which is the conclusion. ■

Lemma 4.19 If $\mathfrak{a} \in A$, $\mathfrak{b} \in A_{\varepsilon}$ and $x \in \mathbb{X}_{\varepsilon}$, then there exists an element $\hat{\mathfrak{a}} \in A$ parallel to $\mathfrak{a}$ satisfying the following conditions:

- x is free from $\mathfrak{b}$ in $\hat{\mathfrak{a}}$.
- If $\mathfrak{a} \in A_{\varepsilon}$, then $\mathfrak{a} = \hat{\mathfrak{a}}$.
- If $\mathfrak{a} \in H$, then $\mathfrak{a} \asymp \hat{\mathfrak{a}}$.
- If $\mathfrak{a} \in G$, then $\mathfrak{a}\triangle \asymp \hat{\mathfrak{a}}\triangle$.

We call such $\hat{\mathfrak{a}}$ an $(x, \mathfrak{b})$-**alternative** of $\mathfrak{a}$.

Remark 4.3 By the definition of the parallelism relation, $\mathfrak{a} \parallel \hat{\mathfrak{a}}$ implies $\sigma\mathfrak{a} = \sigma\hat{\mathfrak{a}}$. In particular, if $\mathfrak{a} \in H$ then $\hat{\mathfrak{a}} \in H$, while if $\mathfrak{a} \in G$ then $\hat{\mathfrak{a}} \in G$.

Proof We use induction on $\mathrm{Rank}\, a$. First we assume that $\mathrm{Rank}\, a = 0$, that is, $a \in \mathbb{S}$. Let $\hat{a} = a$. Then $a \parallel \hat{a}$. x is free from b in $\hat{a}$ by Lemma 3.11. By the repetition law, if $a \in H$ then $a \asymp \hat{a}$, while if $a \in G$ then $a\triangle \asymp \hat{a}\triangle$.

Henceforth we assume that $\mathrm{Rank}\, a \geq 1$. Then, by Theorem 2.2, a has a unique word form $a = \mu(a_1, \ldots, a_n)$, and $\mathrm{Rank}\, a_i < \mathrm{Rank}\, a$ for $i = 1, \ldots, n$. For each a_i there exists an element $\widehat{a_i} \in A$ parallel to a_i satisfying the conditions of Lemma 4.19 by the inductive hypothesis.

(i) **The case where $\mu \in L \cap \Gamma$.** Let $\hat{a} = \mu(\widehat{a_1}, \ldots, \widehat{a_n})$. Then $\hat{a}$ is parallel to a by Remark 2.4, and x is free from b in $\hat{a}$ by Lemma 3.12.

If $a \in A_\varepsilon$, then $\mu = \mathfrak{f} \in \mathfrak{F}$ by Lemma 3.7, so that $a = \mathfrak{f}(a_1, \ldots, a_n)$ and $\hat{a} = \mathfrak{f}(\widehat{a_1}, \ldots, \widehat{a_n})$, hence $a_1, \ldots, a_n \in A_\varepsilon$. We have $a_i = \widehat{a_i}$ by the inductive hypothesis, hence $\mathfrak{f}(a_1, \ldots, a_n) = \mathfrak{f}(\widehat{a_1}, \ldots, \widehat{a_n})$.

Next we will prove that if $a \in H$ then $a \asymp \hat{a}$. Here μ must be one of λk $(\lambda \in \{\breve{o}\} \cup \mathfrak{Q}, k \in K), \wedge, \vee, \Rightarrow, \lozenge$ and $\triangle$.

Assume $\mu = \lambda k$ where $\lambda \in \{\breve{o}\} \cup \mathfrak{Q}$ and $k \in K$. Then we have $a = a_1\, \lambda k\, a_2$ and $\hat{a} = \widehat{a_1}\, \lambda k\, \widehat{a_2}$, hence $a_1 \in G$, $a_2 \in H$. We have $a_1\triangle \asymp \widehat{a_1}\triangle$ and $a_2 \asymp \widehat{a_2}$ by the inductive hypothesis. Hence $a_1\, \lambda k\, a_2 \asymp \widehat{a_1}\, \lambda k\, \widehat{a_2}$ by Lemma 4.18.

Assume $\mu = \wedge$. Then $a = a_1 \wedge a_2$ and $\hat{a} = \widehat{a_1} \wedge \widehat{a_2}$, hence $a_1, a_2 \in H$. We have $a_1 \asymp \widehat{a_1}$ and $a_2 \asymp \widehat{a_2}$ by the inductive hypothesis. Hence $a_1 \wedge a_2 \asymp \widehat{a_1} \wedge \widehat{a_2}$ by Lemma 4.1. Similar arguments hold when $\mu = \vee$ or $\Rightarrow$.

Assume $\mu = \lozenge$. Then we have $a = a_1^{\lozenge}$ and $\hat{a} = \widehat{a_1}^{\lozenge}$, hence $a_1 \in H$. We have $a_1 \asymp \widehat{a_1}$ by the inductive hypothesis. Hence $a_1^{\lozenge} \asymp \widehat{a_1}^{\lozenge}$ by Lemma 4.1.

Assume $\mu = \triangle$. Then we have $a = a_1\triangle$ and $\hat{a} = \widehat{a_1}\triangle$, hence $a_1 \in G$. By the inductive hypothesis, it is clear that $a_1\triangle \asymp \widehat{a_1}\triangle$.

In what follows we will prove that if $a \in G$ then $a\triangle \asymp \hat{a}\triangle$. Here μ must be one of $\sqcap, \sqcup, \square$ and $\mathfrak{f}$ $(\mathfrak{f} \in \mathfrak{F})$.

Assume $\mu = \sqcap$. Then we have $a = a_1 \sqcap a_2$ and $\hat{a} = \widehat{a_1} \sqcap \widehat{a_2}$, hence $a_1, a_2 \in G$. We have $a_1\triangle \asymp \widehat{a_1}\triangle$ and $a_2\triangle \asymp \widehat{a_2}\triangle$ by the inductive hypothesis. Hence

$$a_1\triangle \wedge a_2\triangle \asymp \widehat{a_1}\triangle \wedge \widehat{a_2}\triangle$$

by Lemma 4.1. We have

$$(a_1 \sqcap a_2)\triangle \asymp a_1\triangle \wedge a_2\triangle,$$
$$(\widehat{a_1} \sqcap \widehat{a_2})\triangle \asymp \widehat{a_1}\triangle \wedge \widehat{a_2}\triangle$$

by the $\sqcap$ law. Applying the strong cut law to the above three equations, we have $(a_1 \sqcap a_2)\triangle \asymp (\widehat{a_1} \sqcap \widehat{a_2})\triangle$. A similar argument holds when $\mu = \sqcup$.

Next assume $\mu = \square$. Then $a = a_1^{\square}$ and $\hat{a} = \widehat{a_1}^{\square}$, hence $a_1 \in G$. We have $a_1\triangle \asymp \widehat{a_1}\triangle$ by the inductive hypothesis. Hence

$$(a_1\triangle)^{\lozenge} \asymp (\widehat{a_1}\triangle)^{\lozenge}$$

by Lemma 4.1. We have

$$(\mathfrak{a}_1^{\Box})\triangle \asymp (\mathfrak{a}_1\triangle)^{\Diamond},$$
$$(\hat{\mathfrak{a}_1}^{\Box})\triangle \asymp (\hat{\mathfrak{a}_1}\triangle)^{\Diamond}$$

by the $\Box$ law. Applying the strong cut law to the above three equations, we have $(\mathfrak{a}_1^{\Box})\triangle \asymp (\hat{\mathfrak{a}_1}^{\Box})\triangle$.

Next assume $\mu = \mathfrak{f} \in \mathfrak{F}$. Then $\mathfrak{a} = \mathfrak{f}(\mathfrak{a}_1, \dots, \mathfrak{a}_n)$ and $\hat{\mathfrak{a}} = \mathfrak{f}(\hat{\mathfrak{a}_1}, \dots, \hat{\mathfrak{a}_n})$, hence $\mathfrak{a}_1, \dots, \mathfrak{a}_n \in \mathrm{A}_\varepsilon$. For each $i \in \{1, \dots, n\}$, we have $\mathfrak{a}_i = \hat{\mathfrak{a}_i}$ by the inductive hypothesis. It is clear that $\mathfrak{f}(\mathfrak{a}_1, \dots, \mathfrak{a}_n) = \mathfrak{f}(\hat{\mathfrak{a}_1}, \dots, \hat{\mathfrak{a}_n})$, hence $\mathfrak{f}(\mathfrak{a}_1, \dots, \mathfrak{a}_n)\triangle \asymp \mathfrak{f}(\hat{\mathfrak{a}_1}, \dots, \hat{\mathfrak{a}_n})\triangle$ by the repetition law.

(ii) **The case where $\mu \in \mathrm{L} \cap \Gamma\mathbb{X}$.** In this case we have $\mathfrak{a} = \mathfrak{a}_1\,\Omega y$ for some $y \in \mathbb{X}_\varepsilon$, hence $\mathfrak{a}_1 \in \mathrm{H}$. By Lemmas 2.1 and 2.2, we can take $z \in \mathbb{X}_\varepsilon$ such that $z \neq x, z \not\lll \mathfrak{b}, z \not\lll \hat{\mathfrak{a}_1}$, and $z \notin \mathbb{S}^\nu$ for each $\nu \in \mathrm{L}^{\hat{\mathfrak{a}_1}}$. Then, by Lemma 2.4, y is free from z in $\hat{\mathfrak{a}_1}$. Let $\hat{\mathfrak{a}} = \hat{\mathfrak{a}_1}(y/z)\,\Omega z$, where (y/z) is the substitution of z for y. From the inductive hypothesis it follows that $\mathfrak{a}_1 \parallel \hat{\mathfrak{a}_1}$, hence $\mathfrak{a}_1\,\Omega y \parallel \hat{\mathfrak{a}_1}\,\Omega y$ by Remark 2.4. Since $z \not\lll \hat{\mathfrak{a}_1}$ and y is free from z in $\hat{\mathfrak{a}_1}$, we have $\hat{\mathfrak{a}_1}\,\Omega y \parallel \hat{\mathfrak{a}_1}(y/z)\,\Omega z$ by Remark 2.4. Therefore $\mathfrak{a} \parallel \hat{\mathfrak{a}}$. By the inductive hypothesis, x is free from $\mathfrak{b}$ in $\hat{\mathfrak{a}_1}$. Since $z \neq x$, it follows that $x \not\lll z$, hence x is free from $\mathfrak{b}$ in $\hat{\mathfrak{a}_1}(y/z)$ by Lemma 2.11. Since $z \not\lll \mathfrak{b}$, we have $(\mathrm{S}^{\Omega z})^{\mathfrak{b}}_{\text{free}} = \emptyset$, hence x is free from $\mathfrak{b}$ in $\hat{\mathfrak{a}_1}(y/z)\,\Omega z$ by Theorem 2.10. Finally we will prove that $\mathfrak{a}\triangle \asymp \hat{\mathfrak{a}}\triangle$. We have $\mathfrak{a}_1 \asymp \hat{\mathfrak{a}_1}$ by the inductive hypothesis, hence

$$(\mathfrak{a}_1\,\Omega y)\triangle \asymp (\hat{\mathfrak{a}_1}\,\Omega y)\triangle \tag{4.4}$$

by Lemma 4.11. Since y is free from z in $\hat{\mathfrak{a}_1}$, we have

$$z\,\breve{o}\pi\,(\hat{\mathfrak{a}_1}\,\Omega y)\triangle \asymp \hat{\mathfrak{a}_1}(y/z),$$
$$z\,\breve{o}\pi\,\big(\hat{\mathfrak{a}_1}(y/z)\,\Omega z\big)\triangle \asymp \hat{\mathfrak{a}_1}(y/z)$$

by the Ω law. Applying the strong cut law to these two equations, we have

$$z\,\breve{o}\pi\,(\hat{\mathfrak{a}_1}\,\Omega y)\triangle \asymp z\,\breve{o}\pi\,\big(\hat{\mathfrak{a}_1}(y/z)\,\Omega z\big)\triangle.$$

Since z does not occur free in $(\hat{\mathfrak{a}_1}\,\Omega y)\triangle$ nor in $\big(\hat{\mathfrak{a}_1}(y/z)\,\Omega z\big)\triangle$, we have

$$(\hat{\mathfrak{a}_1}\,\Omega y)\triangle \asymp \big(\hat{\mathfrak{a}_1}(y/z)\,\Omega z\big)\triangle \tag{4.5}$$

by the generalized case$-$ law. Applying the strong cut law to (4.4) and (4.5), we have $(\mathfrak{a}_1\,\Omega y)\triangle \asymp \big(\hat{\mathfrak{a}_1}(y/z)\,\Omega z\big)\triangle$, that is, $\mathfrak{a}\triangle \asymp \hat{\mathfrak{a}}\triangle$. ■

Lemma 4.20 Let $\mathrm{f} \in \mathrm{A}_\emptyset$, $x \in \mathbb{X}_\varepsilon$, $\mathfrak{a} \in \mathrm{A}_\varepsilon$, and $\mathfrak{g}$ be an $(x, \mathfrak{a})$-alternative of f. Then $\mathfrak{a}\,\breve{o}\pi\,(\mathrm{f}\,\Omega x)\triangle \asymp \mathfrak{g}(x/\mathfrak{a})$.

Proof By Lemma 4.19, $f \asymp g$. Hence we have $(f\,\Omega x)\triangle \asymp (g\,\Omega x)\triangle$ by Lemma 4.11, and

$$a\,\breve{o}\pi\,(f\,\Omega x)\triangle \asymp a\,\breve{o}\pi\,(g\,\Omega x)\triangle$$

by Lemma 4.8. Since x is free from a in g by Lemma 4.19, we have

$$a\,\breve{o}\pi\,(g\,\Omega x)\triangle \asymp g(x/a)$$

by the Ω law. Applying the strong cut law to the above two equations, we have the conclusion. ■

5 The existence theorem

Let $(A, T, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ be an MPC language satisfying Assumption 3.1, $(A, \mathcal{W}, (I_W)_{W\in\mathcal{W}})$ be the logical system MPCL on it, and $\preccurlyeq$ be an MPC.1 relation contained in the validity relation $\preccurlyeq_{\mathcal{G}}$ of the predicate logical space $(H, \mathcal{G})$.

Theorem 5.1 Let $X, Y \subseteq A_\emptyset$, and (X, Y) be a cut of H by $\preccurlyeq$. Assume that $[\mathbb{P}^{X\cup Y} \cup \{0\}]$ is well-ordered and that $\mathbb{P}^{X\cup Y}$ has an upper bound in $\mathbb{P}$. Furthermore, assume that there exist κ many elements of $\mathbb{X}_\varepsilon$ which do not occur free in the predicates in $X \cup Y$, where $\kappa = \#A$. Then there exists an $\mathcal{F}$-model of (X, Y).

Remark 5.1 The assumption on $\mathbb{X}_\varepsilon$ in Theorem 5.1 is satisfied, for example, if the cut (X, Y) is finite.

Before proving Theorem 5.1, we derive the following corollary.

Corollary 5.1 Assume the quantity system $\mathbb{P}$ of A is well-ordered and has the largest element ∞. Let (X, Y) be a cut of H by $\preccurlyeq$. Furthermore, assume that there exist κ many elements of $\mathbb{X}_\varepsilon$ which do not occur free in the predicates in $X \cup Y$, where $\kappa = \#A$. Then there exists a $\mathcal{G}$-model of (X, Y).

Proof Let $K' = \bigcup_{h\in H} K_h$, and let $\lhd$ be a total order on K'. We can take distinct elements $x_k \in \mathbb{X}_\varepsilon$ $(k \in K')$ which do not occur free in the predicates in $X \cup Y$. This can be done because $\#K' \leq \#A$.[8] For each $h \in H$, we will define an element $\bar{h} \in A_\emptyset$ as follows. The range K_h is a finite set, so let $k_1, \ldots, k_l$ be the set of distinct cases in K_h satisfying $k_1 \rhd k_2 \rhd \cdots \rhd k_l$, and define $\bar{h} = (x_{k_i}\,\breve{o}k_i)_{i=1,\ldots,l}h$. Notice that if $K_h = \emptyset$ then $\bar{h} = h$. Let $\bar{X} = \{\bar{f} \,|\, f \in X\}$ and $\bar{Y} = \{\bar{g} \,|\, g \in Y\}$. We will prove that $(\bar{X}, \bar{Y})$ is a cut of H by $\preccurlyeq$. Assume that there exist elements $f_1, \ldots, f_m \in X$ and $g_1, \ldots, g_n \in Y$ satisfying $\bar{f_1} \ldots \bar{f_m} \preccurlyeq \bar{g_1} \ldots \bar{g_n}$, to deduce a contradiction. The set $\bigcup_{i=1}^m K_{f_i} \cup$

[8] If $f, f' \in H$, $k \in K_f$, $k' \in K_{f'}$, $x, x' \in \mathbb{X}_\varepsilon$ and $k \neq k'$, then $x\,\breve{o}k\,f$ and $x'\,\breve{o}k'\,f'$ are distinct elements of H. Therefore $\#(\bigcup_{f\in H} K_f) \leq \#A$.

$\bigcup_{j=1}^{n} K_{g_j}$ is a finite set, so let N be its cardinality. Applying the generalized case– law N times to $\bar{f_1} \dots \bar{f_m} \preccurlyeq \bar{g_1} \dots \bar{g_n}$, we have $f_1 \dots f_m \preccurlyeq g_1 \dots g_n$. This contradicts that (X, Y) is a cut.

Since $\mathbb{P}$ is well-ordered and has the largest element, it is obvious that $[\mathbb{P}^{\bar{X} \cup \bar{Y}} \cup \{0\}]$ is well-ordered and $\mathbb{P}^{\bar{X} \cup \bar{Y}}$ has an upper bound in $\mathbb{P}$. We may assume that there exist κ many elements of $\mathbb{X}_\varepsilon$ which do not occur free in the predicates in $\bar{X} \cup \bar{Y}$, where $\kappa = \#A$. Therefore, by Theorem 5.1, there exists an MPC world $W \in \mathcal{W}$ with a $\mathbb{C}$-denotation Φ into W and an $\mathbb{X}$-denotation ν into W satisfying $(\Phi^* \bar{f})\nu = 1$ for each $\bar{f} \in \bar{X}$ and $(\Phi^* \bar{g})\nu = 0$ for each $\bar{g} \in \bar{Y}$. Define $\theta \in K \to W_\varepsilon$ so that $\theta k = (\Phi^* x_k)\nu$ holds for each $k \in K'$. For each $h \in H$, we have $\bar{h} = (x_{k_i}\, \breve{o} k_i)_{i=1,\dots,l} h \in A_\emptyset$, and by Lemma 3.9 we have

$$\begin{aligned}
\big((\Phi^* h)\nu\big)(\theta|_{K_h}) &= (\theta k_i\, \breve{o} k_i)_{i=1,\dots,l} (\Phi^* h)\nu \\
&= \big((\Phi^* x_{k_i})\nu\, \breve{o} k_i\big)_{i=1,\dots,l} (\Phi^* h)\nu \\
&= \big(\Phi^* \big((x_{k_i}\, \breve{o} k_i)_{i=1,\dots,l} h\big)\big)\nu \\
&= (\Phi^* \bar{h})\nu.
\end{aligned}$$

Therefore, if $f \in X$ then $\big((\Phi^* f)\nu\big)(\theta|_{K_f}) = (\Phi^* \bar{f})\nu = 1$, while if $g \in Y$ then $\big((\Phi^* g)\nu\big)(\theta|_{K_g}) = (\Phi^* \bar{g})\nu = 0$. ■

The rest of this section is devoted to the proof of Theorem 5.1.

We start the proof by constructing a set $Z \subseteq A_\emptyset$ as follows. We can well-order all the sequences $(a, b_1, \dots, b_m; p, q_1, \dots, q_m)$ such that $m \geq 0$, $a, b_1, \dots, b_m \in G$, $p, q_1, \dots, q_m \in \mathbb{P}^{X \cup Y} \cup \{0\}$, $\mathbb{P}^a \cup \bigcup_{i=1}^{m} \mathbb{P}^{b_i} \subseteq \mathbb{P}^{X \cup Y} \cup \{0\}$ and $p \geq \sum_{i=1}^{m} q_i$. Let $\kappa = \#A$. Notice that there exist κ many such sequences because $\#G = \#A$ and $\#\mathbb{P} \leq \#A$.[9] We denote the j-th sequence by D_j. We will define an element $h_j \in A_\emptyset$ for each ordinal $j < \kappa$ inductively as follows. Suppose h_l is defined for each $l < j$. We can take $x_j \in \mathbb{X}_\varepsilon$ which does not occur free in the elements in $X \cup Y \cup \{h_l \,|\, l < j\} \cup \{a, b_1, \dots, b_m\}$, where $D_j = (a, b_1, \dots, b_m; p, q_1, \dots, q_m)$. Then we define $h_j = f \Rightarrow g$, where

$$\begin{aligned}
&f = a\, \overline{p} \pi\, \mathrm{one} \triangle \wedge (b_1\, \overline{q_1} \pi\, \mathrm{one} \triangle)^\Diamond \wedge \dots \wedge (b_m\, \overline{q_m} \pi\, \mathrm{one} \triangle)^\Diamond, \\
&g = x_j\, \breve{o} \pi\, a \triangle \wedge (x_j\, \breve{o} \pi\, b_1 \triangle)^\Diamond \wedge \dots \wedge (x_j\, \breve{o} \pi\, b_m \triangle)^\Diamond.
\end{aligned}$$

We define $Z = \{h_j \,|\, j < \kappa\}$.

Remark 5.2 By the way of the construction of Z, the following condition holds:

- If $a, b_1, \dots, b_m \in G$, $p, q_1, \dots, q_m \in \mathbb{P}^{X \cup Y} \cup \{0\}$, $\mathbb{P}^a \cup \bigcup_{i=1}^{m} \mathbb{P}^{b_i} \subseteq \mathbb{P}^{X \cup Y} \cup \{0\}$ and $p \geq \sum_{i=1}^{m} q_i$, then there exist elements $h \in Z$ and

[9] $\#\mathbb{X}_\varepsilon \leq \#G \leq \#A = \#\mathbb{X}_\varepsilon$ by the condition 3 in Assumption 3.1. If $p, q \in \mathbb{P}$ and $x \in \mathbb{X}_\varepsilon$, then $x\, \overline{p} \pi\, x \triangle$ and $x\, \overline{q} \pi\, x \triangle$ are distinct elements of H.

$x \in \mathbb{X}_\varepsilon$ satisfying $h = f \Rightarrow g$, where

$$f = a\,\overline{p}\pi\,\mathsf{one}\,\triangle \wedge (b_1\,\overline{q_1}\pi\,\mathsf{one}\,\triangle)^\Diamond \wedge \cdots \wedge (b_m\,\overline{q_m}\pi\,\mathsf{one}\,\triangle)^\Diamond,$$
$$g = x\,\breve{o}\pi\,a\,\triangle \wedge (x\,\breve{o}\pi\,b_1\,\triangle)^\Diamond \wedge \cdots \wedge (x\,\breve{o}\pi\,b_m\,\triangle)^\Diamond.$$

Remark 5.3 From the way of the construction of Z, it follows that $\mathbb{P}^Z \subseteq \mathbb{P}^{X \cup Y} \cup \{0\}$.

Lemma 5.1 $(X \cup Z, Y)$ is a cut of H by $\preccurlyeq$.

Proof Assume that $(X \cup Z, Y)$ is not a cut to deduce a contradiction. Let n be the smallest integer such that

$$\alpha, h_{j_1}, \ldots, h_{j_n} \preccurlyeq \beta \tag{5.1}$$

holds for some $\alpha \subseteq X$, $\beta \subseteq Y$ and some ordinals $j_1 < \cdots < j_n$. Then $n \geq 1$ because (X, Y) is a cut of H by $\preccurlyeq$.

By the way of the construction of h_{j_n}, there exist a sequence $D_{j_n} = (a, b_1, \ldots, b_m;\ p, q_1, \ldots, q_m)$ and an element $x_{j_n} \in \mathbb{X}_\varepsilon$ such that $h_{j_n} = f_{j_n} \Rightarrow g_{j_n}$, where

$$f_{j_n} = a\,\overline{p}\pi\,\mathsf{one}\,\triangle \wedge (b_1\,\overline{q_1}\pi\,\mathsf{one}\,\triangle)^\Diamond \wedge \cdots \wedge (b_m\,\overline{q_m}\pi\,\mathsf{one}\,\triangle)^\Diamond,$$
$$g_{j_n} = x_{j_n}\breve{o}\pi\,a\,\triangle \wedge (x_{j_n}\breve{o}\pi\,b_1\,\triangle)^\Diamond \wedge \cdots \wedge (x_{j_n}\breve{o}\pi\,b_m\,\triangle)^\Diamond.$$

We have

$$\preccurlyeq f_{j_n}, h_{j_n}, \tag{5.2}$$
$$g_{j_n} \preccurlyeq h_{j_n}. \tag{5.3}$$

by the implication law and Lemma 4.1. We have

$$\alpha, h_{j_1}, \ldots, h_{j_{n-1}} \preccurlyeq \beta, f_{j_n} \tag{5.4}$$

by applying the strong cut law to (5.1) and (5.2), and

$$\alpha, h_{j_1}, \ldots, h_{j_{n-1}}, g_{j_n} \preccurlyeq \beta \tag{5.5}$$

by applying the strong cut law to (5.1) and (5.3). We have

$$\alpha, h_{j_1}, \ldots, h_{j_{n-1}}, x_{j_n}\breve{o}\pi\,a\,\triangle \preccurlyeq \beta, x_{j_n}\breve{o}\pi\,b_1\,\triangle, \ldots, x_{j_n}\breve{o}\pi\,b_m\,\triangle$$

by (5.5) and Lemma 4.1. Since x_{j_n} does not occur free in the elements in $\alpha \cup \beta \cup \{h_{j_1}, \ldots, h_{j_{n-1}}, a, b_1, \ldots, b_m\}$ and $p \geq \sum_{i=1}^m q_i$ holds, we have

$$\alpha, h_{j_1}, \ldots, h_{j_{n-1}}, a\,\overline{p}\pi\,\mathsf{one}\,\triangle \preccurlyeq \beta, b_1\,\overline{q_1}\pi\,\mathsf{one}\,\triangle, \ldots, b_m\,\overline{q_m}\pi\,\mathsf{one}\,\triangle$$

by the pigeonhole principle, hence

$$\alpha, h_{j_1}, \ldots, h_{j_{n-1}}, f_{j_n} \preccurlyeq \beta \tag{5.6}$$

by Lemma 4.1. Applying the strong cut law to (5.4) and (5.6), we have

$$\alpha, h_{j_1}, \ldots, h_{j_{n-1}} \preccurlyeq \beta.$$

This contradicts that n is the smallest. ■

Lemma 5.2 A partial order $\leq$ on the set of cuts of H by $\preccurlyeq$ is defined by

$$(P_1, Q_1) \leq (P_2, Q_2) \Longleftrightarrow \begin{cases} P_1 \subseteq P_2,\ Q_1 \subseteq Q_2, \text{ and} \\ \mathbb{P}^{P_1 \cup Q_1} \cup \{0\} = \mathbb{P}^{P_2 \cup Q_2} \cup \{0\}. \end{cases}$$

Then the order $\leq$ is inductive.

Proof Let I be a non-empty set and $\big((P_i, Q_i)\big)_{i \in I}$ be totally ordered. Define $P = \bigcup_{i \in I} P_i$, $Q = \bigcup_{i \in I} Q_i$. Then

$$\begin{aligned} & p \in \mathbb{P}^{P \cup Q} \cup \{0\} \\ & \Longleftrightarrow p \in \mathbb{P}^{P_i \cup Q_i} \cup \{0\} \text{ for some } i \in I \\ & \Longleftrightarrow p \in \mathbb{P}^{P_i \cup Q_i} \cup \{0\} \text{ for all } i \in I. \end{aligned}$$

Assume that $\alpha \preccurlyeq \beta$ for some $\alpha \subseteq P$, $\beta \subseteq Q$. Then there exists an element $i \in I$ such that $\alpha \subseteq P_i$ and $\beta \subseteq Q_i$. This contradicts that (P_i, Q_i) is a cut. Therefore (P, Q) is a cut, and it is the least upper bound with respect to $\leq$. ■

By Lemma 5.2, there exists a $\leq$-maximal cut (P, Q) of H by $\preccurlyeq$ such that $(X \cup Z, Y) \leq (P, Q)$.

Remark 5.4 By the definition of $\leq$ and Remark 5.3, $X \cup Z \subseteq P$, $Y \subseteq Q$ and $\mathbb{P}^{P \cup Q} \cup \{0\} = \mathbb{P}^{X \cup Y} \cup \{0\}$.

Lemma 5.3 $P \cap Q = \emptyset$.

Proof Assume that $h \in P \cap Q$. Since $h \preccurlyeq h$ by the repetition law, this contradicts that (P, Q) is a cut of H by $\preccurlyeq$. ■

Lemma 5.4 Let $f \in H$, $\alpha \in H^*$ and assume that $\mathbb{P}^f \subseteq \mathbb{P}^{P \cup Q} \cup \{0\}$. If $\alpha \subseteq P$ and $\alpha \preccurlyeq f$ then $f \in P$, while if $\alpha \subseteq Q$ and $f \preccurlyeq \alpha$ then $f \in Q$.

Proof Assume that $\alpha \subseteq P$, $\alpha \preccurlyeq f$ and that $f \notin P$ to deduce a contradiction. Since (P, Q) is a maximal cut with respect to $\leq$ and $\mathbb{P}^{P \cup \{f\} \cup Q} \cup \{0\} = \mathbb{P}^{P \cup Q} \cup \{0\}$, it follows that $(P \cup \{f\}, Q)$ is not a cut. Hence $f\beta \preccurlyeq \gamma$ for some $\beta \subseteq P$, $\gamma \subseteq Q$. Applying the strong cut law to this with $\alpha \preccurlyeq f$, we have $\alpha\beta \preccurlyeq \gamma$, which contradicts that (P, Q) is a cut because $\alpha \subseteq P$.

A similar argument holds for the latter assertion. ■

Lemma 5.5 Let $f, f_1, \ldots, f_n \in H$. Then the following holds:

1. If $f_1 \wedge \cdots \wedge f_n \in P$, then $f_i \in P$ for all $i \in \{1, \ldots, n\}$.

2. If $f_1 \wedge \cdots \wedge f_n \in Q$, then $f_i \in Q$ for some $i \in \{1, \ldots, n\}$.

3. If $f_1 \vee \cdots \vee f_n \in P$, then $f_i \in P$ for some $i \in \{1, \ldots, n\}$.

4. If $f_1 \wedge \cdots \wedge f_n \in Q$, then $f_i \in Q$ for all $i \in \{1, \ldots, n\}$.

5. If $f^{\diamondsuit} \in P$, then $f \in Q$.

6. If $f^{\diamondsuit} \in Q$, then $f \in P$.

Proof 1. Assume that $f_i \notin P$ for some i to deduce a contradiction. Since (P, Q) is a maximal cut with respect to $\leq$ and $\mathbb{P}^{P \cup \{f_i\} \cup Q} = \mathbb{P}^{P \cup Q}$, it follows that $(P \cup \{f_i\}, Q)$ is not a cut. Hence $f_i \alpha \preccurlyeq \beta$ for some $\alpha \subseteq P$, $\beta \subseteq Q$. We have $f_1 \ldots f_n \alpha \preccurlyeq \beta$ by the weakening law, and $f_1 \wedge \cdots \wedge f_n\, \alpha \preccurlyeq \beta$ by Lemma 4.1. This contradicts that (P, Q) is a cut because $f_1 \wedge \cdots \wedge f_n \in P$.

2. Assume that $f_i \notin Q$ for all i to deduce a contradiction. For each i, it follows that $(P, Q \cup \{f_i\})$ is not a cut. Hence $\alpha_i \preccurlyeq f_i \beta_i$ for some $\alpha_i \subseteq P$, $\beta_i \subseteq Q$. We have $f_1 \ldots f_n \preccurlyeq f_1 \wedge \cdots \wedge f_n$ by Lemma 4.9. By applying the strong cut law repeatedly, we have $\alpha_1 \ldots \alpha_n \preccurlyeq f_1 \wedge \cdots \wedge f_n\, \beta_1 \ldots \beta_n$, which contradicts that (P, Q) is a cut because $f_1 \wedge \cdots \wedge f_n \in Q$.

3. Assume that $f_i \notin P$ for all i to deduce a contradiction. For each i, it follows that $(P \cup \{f_i\}, Q)$ is not a cut. Hence $f_i \alpha_i \preccurlyeq \beta_i$ for some $\alpha_i \subseteq P$, $\beta_i \subseteq Q$. We have $f_1 \vee \cdots \vee f_n \preccurlyeq f_1 \ldots f_n$ by Lemma 4.9. By applying the strong cut law repeatedly, we have $f_1 \vee \cdots \vee f_n\, \alpha_1 \ldots \alpha_n \preccurlyeq \beta_1 \ldots \beta_n$, which contradicts that (P, Q) is a cut because $f_1 \vee \cdots \vee f_n \in P$.

4. Assume that $f_i \notin Q$ for some i to deduce a contradiction. It follows that $(P, Q \cup \{f_i\})$ is not a cut. Hence $\alpha \preccurlyeq f_i \beta$ for some $\alpha \subseteq P$, $\beta \subseteq Q$. We have $\alpha \preccurlyeq f_1 \ldots f_n \beta$ by the weakening law, and $\alpha \preccurlyeq f_1 \vee \cdots \vee f_n\, \beta$ by Lemma 4.1. This contradicts that (P, Q) is a cut because $f_1 \wedge \cdots \wedge f_n \in Q$.

5. Assume that $f \notin Q$ to deduce a contradiction. It follows that $(P, Q \cup \{f\})$ is not a cut. Hence $\alpha \preccurlyeq f \beta$ for some $\alpha \subseteq P$, $\beta \subseteq Q$. By Lemma 4.1 we have $f^{\diamondsuit} \alpha \preccurlyeq \beta$, which contradicts that (P, Q) is a cut because $f^{\diamondsuit} \in P$.

6. Assume that $f \notin P$ to deduce a contradiction. It follows that $(P \cup \{f\}, Q)$ is not a cut. Hence $f \alpha \preccurlyeq \beta$ for some $\alpha \subseteq P$, $\beta \subseteq Q$. By Lemma 4.1 we have $\alpha \preccurlyeq f^{\diamondsuit} \beta$, which contradicts that (P, Q) is a cut because $f^{\diamondsuit} \in Q$. ■

Lemma 5.6 Let $a, b_1, \ldots, b_m \in G$ and $p, q_1, \ldots, q_m \in \mathbb{P}$. If $a\, \overline{p} \pi\, \mathrm{one} \triangle \in P$, $b_1\, \overline{q_1} \pi\, \mathrm{one} \triangle, \ldots, b_m\, \overline{q_m} \pi\, \mathrm{one} \triangle \in Q$ and $p \geq \sum_{i=1}^{m} q_i$, then $x\, \breve{o} \pi\, a \triangle \in P$, $x\, \breve{o} \pi\, b_1 \triangle, \ldots, x\, \breve{o} \pi\, b_m \triangle \in Q$ for some $x \in \mathbb{X}_{\varepsilon}$.

Proof Since $a\, \overline{p} \pi\, \mathrm{one} \triangle \in P$, it follows that $\mathbb{P}^{a} \subseteq \mathbb{P}^{P \cup Q} \subseteq \mathbb{P}^{X \cup Y} \cup \{0\}$ by Remark 5.4. Similarly, $\mathbb{P}^{b_i} \subseteq \mathbb{P}^{X \cup Y} \cup \{0\}$ for $i = 1, \ldots, m$. And also

$p, q_1, \ldots, q_m \in \mathbb{P}^{X \cup Y} \cup \{0\}$. Since $p \geq \sum_{i=1}^{m} q_i$, there exist elements $x \in \mathbb{X}_\varepsilon$ and $h \in Z$ satisfying $h = f \Rightarrow g$, where

$$\begin{aligned} f &= a\,\overline{p}\pi\,\mathrm{one}\,\triangle \wedge (b_1\,\overline{q_1}\pi\,\mathrm{one}\,\triangle)^\diamondsuit \wedge \cdots \wedge (b_m\,\overline{q_m}\pi\,\mathrm{one}\,\triangle)^\diamondsuit, \\ g &= x\,\breve{o}\pi\,a\,\triangle \wedge (x\,\breve{o}\pi\,b_1\,\triangle)^\diamondsuit \wedge \cdots \wedge (x\,\breve{o}\pi\,b_m\,\triangle)^\diamondsuit \end{aligned}$$

by Remark 5.2. Since $h \in Z \subseteq P$, it follows that $h \notin Q$ by Lemma 5.3. Therefore we have

$$\alpha \preccurlyeq \beta, h \tag{5.7}$$

for some $\alpha \subseteq P$, $\beta \subseteq Q$.

We assume $x\,\breve{o}\pi\,a\triangle \notin P$ to deduce a contradiction. Since (P, Q) is a maximal cut, we have

$$\alpha', x\,\breve{o}\pi\,a\triangle \preccurlyeq \beta'$$

for some $\alpha' \subseteq P$, $\beta' \subseteq Q$. Hence we have

$$\alpha', x\,\breve{o}\pi\,a\triangle \preccurlyeq \beta', x\,\breve{o}\pi\,b_1\triangle, \ldots, x\,\breve{o}\pi\,b_m\triangle$$

by the weakening law, and

$$\alpha', (x\,\breve{o}\pi\,a\triangle \wedge (x\,\breve{o}\pi\,b_1\triangle)^\diamondsuit \wedge (x\,\breve{o}\pi\,b_m\triangle)^\diamondsuit) \preccurlyeq \beta',$$

that is,

$$\alpha', g \preccurlyeq \beta' \tag{5.8}$$

by Lemma 4.1. We have

$$f, h \preccurlyeq g \tag{5.9}$$

by the implication law. Applying the strong cut law to (5.7), (5.8) and (5.9), we have

$$\alpha, \alpha', f \preccurlyeq \beta, \beta'.$$

Therefore we have

$$\alpha, \alpha', a\,\overline{p}\pi\,\mathrm{one}\,\triangle \preccurlyeq \beta, \beta', b_1\,\overline{q_1}\pi\,\mathrm{one}\,\triangle, \ldots, b_m\,\overline{q_m}\pi\,\mathrm{one}\,\triangle$$

by Lemma 4.1. The predicates in the left-hand side are contained in P, while those in the right-hand side are contained in Q. This contradicts that (P, Q) is a cut. A similar argument holds when $x\,\breve{o}\pi\,b_i\triangle \notin Q$ for some $i \in \{1, \ldots, m\}$. ∎

Lemma 5.7 Let $a \in G$. If $a\,\exists\pi\,\mathrm{one}\triangle \in Q$, then $b\,\breve{o}\pi\,a\triangle \in Q$ for all $b \in A_\varepsilon$.

Proof We have $\mathsf{b}\,\breve{o}\pi\,\mathsf{a}\triangle \preccurlyeq \mathsf{a}\,\exists\pi\,\mathsf{one}\triangle$ by the $\exists$ law. Hence, by Lemma 5.4, we have the conclusion. ■

Here we will construct an MPC world W denotable for A. In order to construct W, it suffices to define the base S, the basic relation $\exists$ on S, the $\mathbb{P}$-measure $|\cdot|$ on S and the family of operations $\mathfrak{f} \in \mathfrak{F}$. Let $S = A_\varepsilon$. For each $\mathfrak{f} \in \mathfrak{F}$, we define the operation $\mathfrak{f}$ on W to be the same as on A. We define the basic relation by

$$\mathsf{b} \exists \mathsf{a} \iff \mathsf{a}\,\breve{o}\pi\,\mathsf{b}\triangle \notin Q.$$

We have $\preccurlyeq \mathsf{a}\,\breve{o}\pi\,\mathsf{a}\triangle$ by the $=$ law. Hence, by Lemmas 5.4 and 5.3, $\mathsf{a}\,\breve{o}\pi\,\mathsf{a}\triangle \notin Q$. Therefore $\exists$ is reflexive.

We define the $\mathbb{P}$-measure as follows. First, for each $\mathsf{a} \in G$, we define $S^{\mathsf{a}} \in \mathcal{P}S$ by

$$S^{\mathsf{a}} = \{\mathsf{s} \in S \mid \mathsf{s}\,\breve{o}\pi\,\mathsf{a}\triangle \notin Q\}. \tag{5.10}$$

Next we define a relation R between $\mathcal{P}S$ and $\mathbb{P}$ by

$$U\,R\,p \iff \begin{cases} \text{There exist elements } b_1,\dots,b_m \in G \text{ and} \\ q_1,\dots,q_m \in \mathbb{P} \text{ satisfying} \\ U \subseteq \bigcup_{i=1}^m S^{b_i}, \\ p = \sum_{i=1}^m q_i, \text{ and} \\ b_i\,\overline{q_i}\pi\,\mathsf{one}\triangle \in Q \text{ for } i = 1,\dots,m. \end{cases}$$

If $m = 0$, then $\bigcup_{i=1}^m S^{b_i} = \emptyset$ and $\sum_{i=1}^m q_i = 0$. Therefore $\emptyset\,R\,0$. If $U \subseteq V$ and $V\,R\,p$, then $U\,R\,p$. If $U\,R\,p$ and $V\,R\,q$, then $(U \cup V)\,R\,(p+q)$. Next we define an element $\acute{o}$ to be an arbitrary element of $\mathbb{P}$ larger than any element of $\mathbb{P}^{X\cup Y} \cup \{0\}$. Such an element exists because $\mathbb{P}^{X\cup Y}$ is bounded and $\infty \notin \mathbb{P}^{X\cup Y}$. For each $U \in \mathcal{P}S$, $\min\big(\{p \in \mathbb{P} \mid U\,R\,p\} \cup \{\acute{o}\}\big)$ exists because $\{p \in \mathbb{P} \mid U\,R\,p\} \subseteq [\mathbb{P}^{X\cup Y} \cup \{0\}]$ and $[\mathbb{P}^{X\cup Y} \cup \{0\}]$ is well-ordered. In order to apply Lemma 3.2 to the relation R, we will prove that $U\,R\,0$ implies $U = \emptyset$. Let $U\,R\,0$. Then there exist elements $b_1,\dots,b_m \in G$, $q_1,\dots,q_m \in \mathbb{P}$ satisfying the above conditions. Since $\sum_{i=1}^m q_i = 0$, it follows that $q_1 = \cdots = q_m = 0$, hence $\mathsf{c}\,\breve{o}\pi\,b_i\triangle \in Q$ for all $\mathsf{c} \in A_\varepsilon$ by Lemma 5.7. Therefore $U \subseteq \bigcup_{i=1}^m S^{b_i} = \emptyset$ by (5.10). Thus, by Lemma 3.2, we define the $\mathbb{P}$-measure by

$$|U| = \min\big(\{p \in \mathbb{P} \mid U\,R\,p\} \cup \{\acute{o}\}\big). \tag{5.11}$$

This completes the construction of W.

Next we define a $\mathbb{C}$-denotation Φ into W as follows. For each $\mathsf{a} \in \mathbb{C}_\varepsilon$, we define $\Phi\mathsf{a} = \mathsf{a}$. For each $\mathsf{a} \in \mathbb{C}_\delta$, we define $\Phi\mathsf{a} \in S \to \mathbb{T}$ by

$$(\Phi\mathsf{a})\mathsf{s} = 1 \iff \mathsf{s}\,\breve{o}\pi\,\mathsf{a}\triangle \notin Q$$

for each $s \in S$. For each $f \in \mathbb{C} \cap H$, we define $\Phi f \in (K_f \to S) \to \mathbb{T}$ by

$$(\Phi f)\theta = 1 \iff (\theta k_i \,\breve{o} k_i)_{i=1,\ldots,l} f \notin Q$$

for each $\theta \in K_f \to S$, where $K_f = \{k_1, \ldots, k_l\}$ and $k_1, \ldots, k_l$ are distinct. The definition of Φf is irrelevant to the ordering of $k_1, \ldots, k_l$ by virtue of the repetition law and Lemma 5.5. We define an $\mathbb{X}$-denotation ν into W similarly as follows. For each $a \in \mathbb{X}_\varepsilon$, we define $\nu a = a$. For each $a \in \mathbb{X}_\delta$, we define $\nu a \in S \to \mathbb{T}$ by

$$(\nu a)s = 1 \iff s \,\breve{o} \pi \, a \triangle \notin Q$$

for each $s \in S$. For each $f \in \mathbb{X} \cap H$, we define $\nu f \in (K_f \to S) \to \mathbb{T}$ by

$$(\nu f)\theta = 1 \iff (\theta k_i \,\breve{o} k_i)_{i=1,\ldots,l} f \notin Q$$

for each $\theta \in K_f \to S$, where $K_f = \{k_1, \ldots, k_l\}$ and $k_1, \ldots, k_l$ are distinct.

Remark 5.5 Let $f, g \in H$. If $f \asymp g$, then $(\Phi^* f)\nu = (\Phi^* g)\nu$ because $\preccurlyeq$ is contained in $\preccurlyeq_g$.

Lemma 5.8 Let ∞ be the largest element of $\mathbb{P}$, if it exists. There exists a mapping I of $L \amalg A$ into $\mathbb{Z}_{\geq 0}$ which satisfies the following conditions:

1. If $\mu \in L$ and $(a_1, \ldots, a_n) \in \operatorname{Dom} \mu$, then $I(\mu(a_1, \ldots, a_n)) = I\mu + Ia_1 + \cdots + Ia_n$.
2. If $a \in \{\breve{o}k, \triangle, \mathfrak{f} \,|\, k \in K, \mathfrak{f} \in \mathfrak{F}\} \amalg \mathbb{S}$, then $Ia = 0$.
3. If $a \in \{\wedge, \vee, \Rightarrow, \Diamond, \sqcap, \sqcup, \Box, \Omega x \,|\, x \in \mathbb{X}_\varepsilon\}$, then $Ia = 1$.
4. If $p \in \mathbb{P} - \{\infty\}$, then $I(\overline{p}k) = 4$ for each $k \in K$.
5. If $p \in \mathbb{P} - \{\infty\}$, then $I((\leftarrow p]k) = 5$ for each $k \in K$.
6. If $\mathfrak{p}$ is a connected quantifier in $\mathfrak{P}$ other than those dealt with in (4) and (5), then $I(\mathfrak{p}k) = 6$ for each $k \in K$.
7. If $\mathfrak{p}$ is a disconnected quantifier in $\mathfrak{P}$, then $I(\mathfrak{p}k) = 7$ for each $k \in K$.
8. If $\mathfrak{x}$ is a quantifier in $\neg\mathfrak{P}$, then $I(\mathfrak{x}k) = 9$ for each $k \in K$.

Proof Consult [9].

Lemma 5.9 If $a \in A, x \in \mathbb{X}_\varepsilon$, and $b \in A_\varepsilon$, then $I(a(x/b)) = Ia$.

Proof Consult [9].

Remark 5.6 Let $\mathsf{a}, \mathsf{b} \in \mathsf{A}$. By Remark 2.4, the condition 1 of Lemma 5.8 and Lemma 5.9, it follows that if a is parallel to b, then $\mathrm{I}\mathsf{a} = \mathrm{I}\mathsf{b}$.

Lemma 5.10 Suppose W, Φ and ν are defined as above. For each $\mathsf{h} \in \mathsf{A}_\emptyset$, if $\mathsf{h} \in \mathsf{P}$ then $(\Phi^*\mathsf{h})\nu = 1$, while if $\mathsf{h} \in \mathsf{Q}$ then $(\Phi^*\mathsf{h})\nu = 0$.

Proof We use induction on $\mathrm{I}\mathsf{h}$ defined by Lemma 5.8. By Theorem 2.2, we can determine $\mathsf{l} \geq 0$, $\mathsf{a}_1, \ldots, \mathsf{a}_\mathsf{l} \in \mathsf{A}_\varepsilon$, $\mathsf{k}_1, \ldots, \mathsf{k}_\mathsf{l} \in \mathsf{K}$ and $\mathsf{h}' \in \mathsf{H} - \bigcup_{\mathsf{k}\in\mathsf{K}} \mathrm{Im}\,\breve{o}\mathsf{k}$ satisfying $\mathsf{h} = (\mathsf{a}_\mathsf{i}\,\breve{o}\mathsf{k}_\mathsf{i})_{\mathsf{i}=1,\ldots,\mathsf{l}}\mathsf{h}'$. Then, by Lemma 5.8, it follows that $\mathrm{I}\mathsf{h} = \mathrm{I}\mathsf{h}'$. Let $\mathsf{a}_1, \ldots, \mathsf{a}_\mathsf{l}$, $\mathsf{k}_1, \ldots, \mathsf{k}_\mathsf{l}$ and h' be determined as above throughout this proof. Also, we write $(\mathsf{a}_\mathsf{i}\,\breve{o}\mathsf{k}_\mathsf{i})_\mathsf{i}\mathsf{h}'$ if there is no ambiguity. Since $\mathsf{h}' \in \mathsf{H} - \bigcup_{\mathsf{k}\in\mathsf{K}} \mathrm{Im}\,\breve{o}\mathsf{k}$, either $\mathsf{h}' \in \mathbb{S} \cap \mathsf{H}$ or h' is one of the word forms $\mathsf{a}\,\mathfrak{r}\mathsf{k}\,\mathsf{f}$ ($\mathfrak{r} \in \mathfrak{Q}$), $\mathsf{f} \wedge \mathsf{g}$, $\mathsf{f} \vee \mathsf{g}$, $\mathsf{f} \Rightarrow \mathsf{g}$, $\mathsf{f}^\Diamond$ and $\mathsf{c}\triangle$. If $\mathsf{h}' = \mathsf{c}\triangle$, then either $\mathsf{c} \in \mathbb{S}_\delta \cup \mathsf{A}_\varepsilon$ or c is in one of the word forms $\mathsf{a} \sqcap \mathsf{b}$, $\mathsf{a} \sqcup \mathsf{b}$, $\mathsf{a}^\Box$ and $\mathsf{f}\,\Omega\mathsf{x}$ ($\mathsf{x} \in \mathbb{X}_\varepsilon$).

First we will assume that $\mathrm{I}\mathsf{h} = 0$. Then either $\mathsf{h}' \in \mathbb{S} \cap \mathsf{H}$ or $\mathsf{h}' = \mathsf{c}\triangle$ for some $\mathsf{c} \in \mathbb{S}_\delta \cup \mathsf{A}_\varepsilon$. Suppose $\mathsf{h}' \in \mathbb{S} \cap \mathsf{H}$. Notice that $\mathsf{K}_{\mathsf{h}'} = \{\mathsf{k}_1, \ldots, \mathsf{k}_\mathsf{l}\}$ because $\mathsf{h} \in \mathsf{A}_\emptyset$. We define $\theta \in \mathsf{K}_{\mathsf{h}'} \to \mathsf{S}$ by $\theta\mathsf{k}_\mathsf{i} = \mathsf{a}_\mathsf{i}$ for $\mathsf{i} = 1, \ldots, \mathsf{l}$. Then we have

$$\begin{aligned}
(\Phi^*\mathsf{h})\nu &= \big(\Phi^*\big((\mathsf{a}_\mathsf{i}\,\breve{o}\mathsf{k}_\mathsf{i})_\mathsf{i}\,\mathsf{h}'\big)\big)\nu \\
&= \big((\Phi^*\mathsf{a}_\mathsf{i})\nu\,\breve{o}\mathsf{k}_\mathsf{i}\big)_\mathsf{i}\,(\Phi^*\mathsf{h}')\nu \\
&= (\mathsf{a}_\mathsf{i}\,\breve{o}\mathsf{k}_\mathsf{i})_\mathsf{i}\,(\Phi^*\mathsf{h}')\nu && \text{(by Lemma 3.8)} \\
&= (\theta\mathsf{k}_\mathsf{i}\,\breve{o}\mathsf{k}_\mathsf{i})_\mathsf{i}\,(\Phi^*\mathsf{h}')\nu.
\end{aligned}$$

If $\mathsf{h} \in \mathsf{P}$, then $\mathsf{h} \notin \mathsf{Q}$ by Lemma 5.3, hence $(\Phi^*\mathsf{h})\nu = 1$. If $\mathsf{h} \in \mathsf{Q}$, then $(\Phi^*\mathsf{h})\nu = 0$. Next suppose $\mathsf{h}' = \mathsf{c}\triangle$ for some $\mathsf{c} \in \mathbb{S}_\delta \cup \mathsf{A}_\varepsilon$. Then we have $\mathsf{h} = \mathsf{a}_1\,\breve{o}\pi\,\mathsf{c}\triangle$. We have

$$\begin{aligned}
(\Phi^*\mathsf{h})\nu &= \big(\Phi^*(\mathsf{a}_1\,\breve{o}\pi\,\mathsf{c}\triangle)\big)\nu \\
&= (\Phi^*\mathsf{a}_1)\nu\,\breve{o}\pi\,\big((\Phi^*\mathsf{c})\nu\big)\triangle \\
&= \mathsf{a}_1\,\breve{o}\pi\,\big((\Phi^*\mathsf{c})\nu\big)\triangle && \text{(by Lemma 3.8)}
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{a}_1\,\breve{o}\pi\,\big((\Phi^*\mathsf{c})\nu\big)\triangle = 1 &\iff \big((\Phi^*\mathsf{c})\nu\big)\,\exists\,\mathsf{a}_1 \\
&\iff \mathsf{a}_1\,\breve{o}\pi\,\mathsf{c}\triangle \notin \mathsf{Q}
\end{aligned}$$

by the definition of Φ and ν. If $\mathsf{h} \in \mathsf{P}$, then $\mathsf{h} \notin \mathsf{Q}$ by Lemma 5.3, hence $(\Phi^*\mathsf{h})\nu = 1$. If $\mathsf{h} \in \mathsf{Q}$, then $(\Phi^*\mathsf{h})\nu = 0$.

Henceforth we will assume that $\mathrm{I}\mathsf{h} \geq 1$. The case where $\mathsf{h}' = \mathsf{a}\,\mathfrak{r}\mathsf{k}\,\mathsf{f}$ with $\mathfrak{r} \in \mathfrak{Q}$ is divided into the cases (i)–(vi) below. The case (i) deals with the case where $\mathfrak{r} \in \neg\mathfrak{P}$, where the case (ii) deals with the case where $\mathfrak{r} \in \mathfrak{P}$ but $\mathfrak{r}$ is disconnected. If $\mathfrak{r} \in \mathfrak{P}$ is connected, then by Assumption 3.1 $\mathfrak{r}$ is in one of the four shapes $\mathbb{P}$, $(p, q]$ with $p < q \neq \infty$ or $p = q = 0$,[10] $(\leftarrow p]$ with $p \neq \infty$, and $\overline{p}$ with $p \neq \infty$. These cases are dealt with by the cases (iii)–(vi) respectively.

(i) The case where $h' = a \neg \mathfrak{p} k f$ with $\mathfrak{p} \in \mathfrak{P}$. In this case have $h = (a_i \breve{o} k_i)_i (a \neg \mathfrak{p} k f)$. Let $\grave{h} = (a_i \breve{o} k_i)_i (a \mathfrak{p} k f^{\Diamond})$. By Lemma 4.15 it follows that $h \asymp \grave{h}$. By Remark 5.5, $(\Phi^* h)\nu = (\Phi^* \grave{h})\nu$. By Lemma 5.4, if $h \in P$ then $\grave{h} \in P$, while if $h \in Q$ then $\grave{h} \in Q$. We have

$$\begin{aligned} Ih &= I(a \neg \mathfrak{p} k f) = Ia + I(\neg \mathfrak{p} k) + If = Ia + 9 + If \\ &> Ia + 7 + If + 1 = Ia + 7 + I(f^{\Diamond}) \\ &\geq Ia + I(\mathfrak{p} k) + I(f^{\Diamond}) = I(a \mathfrak{p} k f^{\Diamond}) = I\grave{h}, \end{aligned}$$

hence if $\grave{h} \in P$ then $(\Phi^* \grave{h})\nu = 1$ while if $\grave{h} \in Q$ then $(\Phi^* \grave{h})\nu = 0$ by the inductive hypothesis. Therefore the conclusion follows.

(ii) The case where $h' = a \mathfrak{p} k f$ with disconnected $\mathfrak{p} \in \mathfrak{P}$. In this case we have $h = (a_i \breve{o} k_i)_i (a \mathfrak{p} k f)$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be the connected components of $\mathfrak{p}$, and $h_j = (a_i \breve{o} k_i)_i (a \mathfrak{p}_j k f)$ for $j = 1, \ldots, m$. Then $\mathbb{P}^{h_j} \subseteq \mathbb{P}^h$. By Lemma 4.14 it follows that $h \asymp h_1 \vee \cdots \vee h_m$. By Remark 5.5 and Lemma 3.6, $(\Phi^* h)\nu = \big(\Phi^*(h_1 \vee \cdots \vee h_m)\big)\nu = (\Phi^* h_1)\nu \vee \cdots \vee (\Phi^* h_m)\nu$. By Lemmas 5.4 and 5.5, if $h \in P$ then $h_j \in P$ for some $j \in \{1, \ldots, m\}$, while if $h \in Q$ then $h_j \in Q$ for all $j \in \{1, \ldots, m\}$. For each $j \in \{1, \ldots, m\}$, we have

$$\begin{aligned} Ih &= I(a \mathfrak{p} k f) = Ia + I(\mathfrak{p} k) + If = Ia + 7 + If \\ &> Ia + 6 + If \geq Ia + I(\mathfrak{p}_j k) + If = I(a \mathfrak{p}_j k f) = Ih_j, \end{aligned}$$

hence if $h_j \in P$ then $(\Phi^* h_j)\nu = 1$ while if $h_j \in Q$ then $(\Phi^* h_j)\nu = 0$ by the inductive hypothesis. Therefore the conclusion follows.

(iii) The case where $h' = a \mathbb{P} k f$. Here we have $h = (a_i \breve{o} k_i)_i (a \mathbb{P} k f)$. Let $h_1 = (a_i \breve{o} k_i)_i (a \overline{0} k f)$, $h_2 = (a_i \breve{o} k_i)_i (a (\leftarrow 0] k f)$. Then $\mathbb{P}^{h_j} \subseteq \mathbb{P}^h \cup \{0\}$ for $j = 1, 2$. By Lemma 4.14 it follows that $h \asymp h_1 \vee h_2$. By Remark 5.5 and Lemma 3.6, $(\Phi^* h)\nu = \big(\Phi^*(h_1 \vee h_2)\big)\nu = (\Phi^* h_1)\nu \vee (\Phi^* h_2)\nu$. By Lemmas 5.4 and 5.5, if $h \in P$ then $h_1 \in P$ or $h_2 \in P$, while if $h \in Q$ then $h_1 \in Q$ and $h_2 \in Q$. We have

$$\begin{aligned} Ih &= I(a \mathbb{P} k f) = Ia + I(\mathbb{P} k) + If = Ia + 6 + If \\ &> \begin{cases} Ia + 4 + If \geq Ia + I(\overline{0} k) + If = I(a \overline{0} k f) = Ih_1 \\ Ia + 5 + If \geq Ia + I((\leftarrow 0] k) + If = I(a (\leftarrow 0] k f) = Ih_2, \end{cases} \end{aligned}$$

hence if $h_j \in P$ then $(\Phi^* h_j)\nu = 1$ while if $h_j \in Q$ then $(\Phi^* h_j)\nu = 0$ by the inductive hypothesis $(j = 1, 2)$. Therefore the conclusion follows.

(iv) The case where $h' = a (p, q] k f$ with $p, q \in \mathbb{P}$ such that either $p < q \neq \infty$ or $p = q = 0$. Here we have $h = (a_i \breve{o} k_i)_i (a (p, q] k f)$. Let $h_1 =$

[10] In this argument $\emptyset \in \mathfrak{P}$ is treated as $(0, 0]$.

$(a_i \,\breve{o}k_i)_i\,(a\,\overline{p}k\,f)$, $h_2 = (a_i\,\breve{o}k_i)_i\,(a\,(\leftarrow q]k\,f)$. By Lemma 4.14 it follows that $h \asymp h_1 \wedge h_2$. By Remark 5.5 and Lemma 3.6, $(\Phi^* h)\nu = \big(\Phi^*(h_1 \wedge h_2)\big)\nu = (\Phi^* h_1)\nu \wedge (\Phi^* h_2)\nu$. By Lemmas 5.4 and 5.5, if $h \in P$ then $h_1 \in P$ and $h_2 \in P$, while if $h \in Q$ then $h_1 \in Q$ or $h_2 \in Q$. We have

$$\begin{aligned} Ih &= I(a\,(p,q]k\,f) = Ia + I((p,q]k) + If = Ia + 6 + If \\ &> \begin{cases} Ia + 4 + If \geq Ia + I(\overline{p}k) + If = I(a\,\overline{p}k\,f) = Ih_1 \\ Ia + 5 + If \geq Ia + I((\leftarrow q]k) + If = I(a\,(\leftarrow q]k\,f) = Ih_2, \end{cases} \end{aligned}$$

hence if $h_j \in P$ then $(\Phi^* h_j)\nu = 1$ while if $h_j \in Q$ then $(\Phi^* h_j)\nu = 0$ by the inductive hypothesis $(j = 1, 2)$. Therefore the conclusion follows.

(v) The case where $h' = a\,(\leftarrow p]k\,f$ with $p \in \mathbb{P} - \{\infty\}$. In this case we have $h = (a_i\,\breve{o}k_i)_i\,(a\,(\leftarrow p]k\,f)$. Let $\grave{h} = (a_i\,\breve{o}k_i)_i\,(a\,\overline{p}k\,f)$. By Lemma 4.15 it follows that $h \asymp \grave{h}^{\Diamond}$. By Remark 5.5 and Lemma 3.6, $(\Phi^* h)\nu = (\Phi^* \grave{h}^{\Diamond})\nu = \big((\Phi^* \grave{h})\nu\big)^{\Diamond}$. By Lemmas 5.4 and 5.5, if $h \in P$ then $\grave{h} \in Q$, while if $h \in Q$ then $\grave{h} \in P$. We have

$$\begin{aligned} Ih &= I(a\,(\leftarrow p]k\,f) = Ia + I((\leftarrow p]k) + If = Ia + 5 + If \\ &> Ia + 4 + If = Ia + I(\overline{p}k) + If = I(a\,\overline{p}k\,f) = I\grave{h}, \end{aligned}$$

hence if $\grave{h} \in P$ then $(\Phi^* \grave{h})\nu = 1$ while if $\grave{h} \in Q$ then $(\Phi^* \grave{h})\nu = 0$ by the inductive hypothesis. Therefore the conclusion follows.

(vi) The case where $h' = a\,\overline{p}k\,f$ with $p \in \mathbb{P} - \{\infty\}$. In this case we have $h = (a_i\,\breve{o}k_i)_i\,(a\,\overline{p}k\,f)$. Let $g = (a_i\,\breve{o}k_i)_i\,f$. Take $x \in \mathbb{X}_\varepsilon$ which does not occur free in g, and then let $c = a \sqcap \big((x\,\breve{o}k\,g)\,\Omega x\big)$ and $U_c = \{s \in S \mid (\Phi^* c)\nu \ni s\}$. By Lemma 4.17 it follows that $h \asymp c\,\overline{p}\pi\,\mathsf{one}\triangle$. By Remark 5.5, $(\Phi^* h)\nu = \big(\Phi^*(c\,\overline{p}\pi\,\mathsf{one}\triangle)\big)\nu$. By Lemma 3.5, $\big(\Phi^*(c\,\overline{p}\pi\,\mathsf{one}\triangle)\big)\nu = 1$ if and only if $|U_c| > p$. Therefore, $(\Phi^* h)\nu = 1$ if and only if $|U_c| > p$.

Suppose $h \in P$. We assume that $|U_c| \leq p$ to deduce a contradiction. Since $p \in \mathbb{P}^{X \cup Y}$, $p \neq \acute{o}$. By (5.11), there exist elements $b_1, \ldots, b_m \in G$ and $q_1, \ldots, q_m \in \mathbb{P}$ such that $U_c \subseteq \bigcup_{j=1}^m S^{b_j}$, $|U_c| = \sum_{j=1}^m q_j$ and $b_j\,\overline{q_j}\pi\,\mathsf{one}\triangle \in Q$ for $j = 1, \ldots, m$. Since $h \asymp c\,\overline{p}\pi\,\mathsf{one}\triangle$, by Lemma 5.4, $c\,\overline{p}\pi\,\mathsf{one}\triangle \in P$. Since $p \geq |U_c| = \sum_{j=1}^m q_j$, there exists by Lemma 5.6 an element $y \in \mathbb{X}_\varepsilon$ such that $y\,\breve{o}\pi\,a\triangle \in P$ and $y\,\breve{o}\pi\,b_1\triangle, \ldots, y\,\breve{o}\pi\,b_m\triangle \in Q$. We have

$$\begin{aligned} Ih &= I(a\,\overline{p}k\,f) = Ia + I(\overline{p}k) + If = Ia + 4 + If = Ia + 4 + Ig \\ &> Ia + 2 + Ig = Ia + I(\sqcap) + Ig + I(\Omega x) \\ &= I\big(a \sqcap \big((x\,\breve{o}k\,g)\,\Omega x\big)\big) = Ic = I(y\,\breve{o}\pi\,c\triangle), \end{aligned}$$

hence $(\Phi^*(y\,\breve{o}\pi\,c\triangle))\nu = 1$ by the inductive hypothesis. By Lemma 3.3 and Lemma 3.8 it follows that $(\Phi^* c)\nu \ni y$, that is, $y \in U_c$. Besides, $y\,\breve{o}\pi\,b_j\triangle \in Q$ for $j = 1, \ldots, m$, hence $y \notin S^{b_j}$ by (5.10). This contradicts that $U_c \subseteq \bigcup_{j=1}^m S^{b_j}$.

Suppose $h \in Q$. We will prove that $|U_c| \leq p$. Since $h \asymp c\,\overline{p}\pi\,one\triangle$, by Lemma 5.4, $c\,\overline{p}\pi\,one\triangle \in Q$. Let $s \in S$ and suppose $s \notin S^c$. Then $s\,\breve{o}\pi\,c\triangle \in Q$ by (5.10). We have

$$Ih > Ic = I(s\,\breve{o}\pi\,c\triangle),$$

hence $(\Phi^*(s\,\breve{o}\pi\,c\triangle))\nu = 0$ by the inductive hypothesis. By Lemma 3.3 and Lemma 3.8 it follows that $(\Phi^* c)\nu \not\ni s$, that is, $s \notin U_c$. Thus we have $U_c \subseteq S^c$. This together with $c\,\overline{p}\pi\,one\triangle \in Q$ shows that $|U_c| \leq p$ by (5.11).

(vii) The case where $h' = f \wedge g$, $f \vee g$ or $f \Rightarrow g$ with $f, g \in H$. Assume that $h = (a_i\,\breve{o}k_i)_{i=1,\ldots,l}(f \wedge g)$. There exists an element $\rho \in \mathfrak{S}_l$ such that $K_f - K_g = \{k_{\rho 1}, \ldots, k_{\rho n}\}$, $K_f \cap K_g = \{k_{\rho(n+1)}, \ldots, k_{\rho m}\}$ and that $K_g - K_f = \{k_{\rho(m+1)}, \ldots, k_{\rho l}\}$. Let $h_f = (a_{\rho i}\breve{o}k_{\rho i})_{i=1,\ldots,m} f$, and $h_g = (a_{\rho i}\breve{o}k_{\rho i})_{i=n+1,\ldots,l}\, g$. By the permutation law and the generalized $\wedge$ law, it follows that $h \asymp h_f \wedge h_g$. By Remark 5.5 and Lemma 3.6, $(\Phi^* h)\nu = \big(\Phi^*(h_f \wedge h_g)\big)\nu = (\Phi^* h_f)\nu \wedge (\Phi^* h_g)\nu$. By Lemmas 5.4 and 5.5, if $h \in P$ then $h_f \in P$ and $h_g \in P$, while if $h \in Q$ then $h_f \in Q$ or $h_g \in Q$. We have

$$Ih = I(f \wedge g) = If + I\wedge + Ig = If + 1 + Ig > \begin{cases} If = Ih_f \\ Ig = Ih_g, \end{cases}$$

hence if $h_j \in P$ then $(\Phi^* h_j)\nu = 1$ while if $h_j \in Q$ then $(\Phi^* h_j)\nu = 0$ by the inductive hypothesis ($j \in \{f, g\}$). Therefore the conclusion follows.

Similar arguments hold for the case where $h' = f \vee g$ or $f \Rightarrow g$.

(viii) The case where $h' = f^{\Diamond}$ with $f \in H$. In this case we have $h = (a_i\,\breve{o}k_i)_i\,(f^{\Diamond})$. Let $\grave{h} = (a_i\,\breve{o}k_i)_i\, f$. By the $\Diamond$ law, it follows that $h \asymp \grave{h}^{\Diamond}$. By Remark 5.5 and Lemma 3.6, $(\Phi^* h)\nu = \big(\Phi^*(\grave{h}^{\Diamond})\big)\nu = \big((\Phi^*\grave{h})\nu\big)^{\Diamond}$. By Lemmas 5.4 and 5.5, if $h \in P$ then $\grave{h} \in Q$ while if $h \in Q$ then $\grave{h} \in P$. We have

$$Ih = (f^{\Diamond}) = If + I\Diamond = If + 1 > If = I\grave{h},$$

hence if $\grave{h} \in P$ then $(\Phi^*\grave{h})\nu = 1$ while if $\grave{h} \in Q$ then $(\Phi^*\grave{h})\nu = 0$ by the inductive hypothesis. Therefore the conclusion follows.

(ix) The case where $h' = (a \sqcap b)\triangle$ or $(a \sqcup b)\triangle$ with $a, b \in G$. Assume that $h' = (a \sqcap b)\triangle$. Then we have $h = a_1\,\breve{o}\pi\,(a \sqcap b)\triangle$. Let $h_a = a_1\,\breve{o}\pi\,a\triangle$, $h_b = a_1\,\breve{o}\pi\,b\triangle$. By Lemma 4.16 it follows that $h \asymp h_a \wedge h_b$. By Remark 5.5 and Lemma 3.6, $(\Phi^* h)\nu = \big(\Phi^*(h_a \wedge h_b)\big)\nu = (\Phi^* h_f)\nu \wedge (\Phi^* h_g)\nu$. By Lemmas 5.4 and 5.5, if $h \in P$ then $h_a \in P$ and $h_b \in P$, while if $h \in Q$ then $h_a \in Q$ or $h_b \in Q$. We have

$$Ih = I((a \sqcap b)\triangle) = Ia + I\sqcap + Ib = Ia + 1 + Ib > \begin{cases} Ia = Ih_a \\ Ib = Ih_b, \end{cases}$$

hence if $h_j \in P$ then $(\Phi^* h_j)\nu = 1$ while if $h_j \in Q$ then $(\Phi^* h_j)\nu = 0$ by the inductive hypothesis ($j \in \{a, b\}$). Therefore the conclusion follows. A similar argument holds when $h' = (a \sqcup b)\triangle$.

(x) The case where $h' = a^{\Box}\triangle$ with $a \in G$. In this case we have $h = a_1\, \breve{o}\pi\, a^{\Box}\triangle$. Let $\grave{h} = a_1\, \breve{o}\pi\, a\triangle$. By Lemma 4.16 it follows that $h \asymp \grave{h}^{\Diamond}$. By Remark 5.5 and Lemma 3.6, $(\Phi^* h)v = \big(\Phi^*(\grave{h}^{\Diamond})\big)v = \big((\Phi^*\grave{h})v\big)^{\Diamond}$. By Lemmas 5.4 and 5.5, if $h \in P$ then $\grave{h} \in Q$ while if $h \in Q$ then $\grave{h} \in P$. We have

$$\mathrm{I}h = \mathrm{I}(a^{\Box}\triangle) = \mathrm{I}a + \mathrm{I}\Box = \mathrm{I}a + 1 > \mathrm{I}a = \mathrm{I}\grave{h},$$

hence if $\grave{h} \in P$ then $(\Phi^*\grave{h})v = 1$ while if $\grave{h} \in Q$ then $(\Phi^*\grave{h})v = 0$. Therefore the conclusion follows.

(xi) The case where $h' = (f\,\Omega x)\triangle$ with $f \in A_{\emptyset}$, $x \in \mathbb{X}_{\varepsilon}$. In this case we have $h = a_1\, \breve{o}\pi\,(f\,\Omega x)\triangle$. By Lemma 4.19, there exists an (x, a_1)-alternative $g \in A_{\emptyset}$ of f. Let $\grave{h} = g(x/a_1)$. Since f is parallel to g, $\mathbb{P}^{\grave{h}} \subseteq \mathbb{P}^{h}$ by Lemmas 2.2, 3.15 and 3.16. By Lemma 4.20 it follows that $h \asymp \grave{h}$. By Remark 5.5, $(\Phi^* h)v = (\Phi^*\grave{h})v$. By Lemma 5.4, if $h \in P$ then $\grave{h} \in P$, while if $h \in Q$ then $\grave{h} \in Q$. By Remark 5.6, $\mathrm{I}f = \mathrm{I}g$. We have $\mathrm{I}g = \mathrm{I}(g(x/a_1))$ by Lemma 5.9. Therefore we have

$$\begin{aligned} \mathrm{I}h &= \mathrm{I}((f\,\Omega x)\triangle) = \mathrm{I}f + \mathrm{I}(\Omega x) = \mathrm{I}f + 1 \\ &> \mathrm{I}f = \mathrm{I}g = \mathrm{I}(g(x/a_1)) = \mathrm{I}\grave{h}, \end{aligned}$$

hence if $\grave{h} \in P$ then $(\Phi^*\grave{h})v = 1$ while if $\grave{h} \in Q$ then $(\Phi^*\grave{h})v = 0$ by the inductive hypothesis. Therefore the conclusion follows. ■

Thus we have completed the proof of Theorem 5.1.

6 The non-existence theorem

Let $(A, T, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ be an MPC language satisfying Assumption 3.1, $(A, \mathcal{W}, (\mathrm{I}_W)_{W\in\mathcal{W}})$ be the logical system MPCL on it, and $\preccurlyeq$ be an MPC.1 relation contained in the validity relation $\preccurlyeq_{\mathcal{G}}$ of the predicate logical space $(H, \mathcal{G})$.

In Corollary 5.1 we dealt with the case where $\mathbb{P}$ is well-ordered and has the largest element. The following theorem deals with the remaining case.

Theorem 6.1 Assume that the quantity system $\mathbb{P}$ of A is not well-ordered or does not have the largest element. Then there exists a cut $(X, Y) \in \mathcal{P}(A_{\emptyset}) \times \mathcal{P}(A_{\emptyset})$ of H by $\preccurlyeq$ which has no $\mathcal{F}$-model.

Remark 6.1 By Remark 3.3, the above cut (X, Y) has no $\mathcal{G}$-model either.

Proof We define subsets P, Q of $\mathbb{P}$ as follows. If $\mathbb{P}$ is not well-ordered, then let Q be an arbitrary non-empty subset of $\mathbb{P}$ which does not have the smallest element, and let $P = \{p \in \mathbb{P} \,|\, p < q$ for every $q \in Q\}$. Otherwise (in this case $\mathbb{P}$ does not have the largest element), let $P = \mathbb{P}$ and $Q = \emptyset$. Take an element $x \in \mathbb{X}_{\varepsilon}$ arbitrarily, and let $X = \{x\,\overline{p}\pi\,\mathsf{one}\triangle \,|\, p \in P\}$, $Y = \{x\,\overline{q}\pi\,\mathsf{one}\triangle \,|\, q \in Q\}$.

First we will prove that (X, Y) is a cut of H by $\preccurlyeq$. Assume $\alpha \preccurlyeq \beta$ for some $\alpha \subseteq X$, $\beta \subseteq Y$ to deduce a contradiction. Then there exist elements $p_1, \ldots, p_m \in P$ and $q_1, \ldots, q_n \in Q$ satisfying $\alpha = f_1 \ldots f_m$ and $\beta = g_1 \ldots g_n$, where $f_i = x\,\overline{p_i}\pi\,\mathrm{one}\triangle$ for $i = 1, \ldots, m$ and $g_j = x\,\overline{q_j}\pi\,\mathrm{one}\triangle$ for $j = 1, \ldots, n$. There exists an element $r \in \mathbb{P}$ such that $p_i < r$ for every $i \in \{1, \ldots, m\}$ and $r < q_j$ for every $j \in \{1, \ldots, n\}$ by the definition of P and Q. We can assume that $r \neq 0$ because $0 \notin Q$.

We will construct an MPC world W_r as follows. Define $S = \{s\}$ where s is arbitrary. Let $\ni$ be the identity relation on S. Define the $\mathbb{P}$-measure by $|\{s\}| = r$, $|\emptyset| = 0$. Let Φ be an arbitrary $\mathbb{C}$-denotation into W_r, and ν be an arbitrary $\mathbb{X}$-denotation into W_r. We have $(\Phi^* f_i)\nu = 1$ and $(\Phi^* g_j)\nu = 0$ by Lemma 3.5. This contradicts that $\preccurlyeq$ is contained in $\preccurlyeq_{\mathcal{G}}$. Therefore (X, Y) is a cut of H by $\preccurlyeq$.

Next we will prove that (X, Y) has no $\mathcal{F}$-model. Assume that there exists a triple (W, Φ, ν) which satisfies $(\Phi^* f)\nu = 1$ for each $f \in X$ and $(\Phi^* g)\nu = 0$ for each $g \in Y$, to deduce a contradiction. Let $U_x = \{s \in S \,|\, (\Phi^* x)\nu \ni s\}$, where S is the base, $\ni$ is the basic relation, and $|\cdot|$ is the $\mathbb{P}$-measure of W. For each $p \in P$, $x\,\overline{p}\pi\,\mathrm{one}\triangle \in X$, hence $\big(\Phi^*(x\,\overline{p}\pi\,\mathrm{one}\triangle)\big)\nu = 1$. Therefore $|U_x| > p$ by Lemma 3.5. For each $q \in Q$, $x\,\overline{q}\pi\,\mathrm{one}\triangle \in X$, hence $\big(\Phi^*(x\,\overline{q}\pi\,\mathrm{one}\triangle)\big)\nu = 0$. Therefore $|U_x| \leq q$ by Lemma 3.5. If $\mathbb{P}$ is not well-ordered, then this contradicts that Q does not have the smallest element. Otherwise, this contradicts that $\mathbb{P}$ does not have the largest element. ■

7 Classification

Let $(A, T, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ be an MPC language satisfying Assumption 3.1, and $(A, \mathcal{W}, (I_W)_{W \in \mathcal{W}})$ be the logical system MPCL on it. Recall that the $\mathcal{G}$-validity relation $\preccurlyeq_{\mathcal{G}}$ of the predicate logical space $(H, \mathcal{G})$ is an MPC.1 relation by Theorem 4.1.

In this section we apply the results in §5 and §6 to determine which class $(H, \mathcal{G})$ belongs to.

Lemma 7.1 For each $(f, g) \in H \times H$, there exists an element $h \in H$ such that $h \succcurlyeq_{\mathcal{G}} f$, $h \succcurlyeq_{\mathcal{G}} f$, and $fg \succcurlyeq_{\mathcal{G}} h$. Let $\alpha \in H^*$. Then $\alpha \preccurlyeq_{\mathcal{G}} f$ for every element $f \in H$ if and only if $\alpha \preccurlyeq_{\mathcal{G}}$.

Proof The former assertion holds for $h = f \vee g$ by the disjunction law.

If $\alpha \preccurlyeq_{\mathcal{G}}$, then $\alpha \preccurlyeq_{\mathcal{G}} h$ for all $h \in H$ by the weakening law.

Next we assume that $\alpha \preccurlyeq_{\mathcal{G}} h$ for all $h \in H$. Then $\alpha \preccurlyeq_{\mathcal{G}} f$ and $\alpha \preccurlyeq_{\mathcal{G}} f^{\diamondsuit}$ for some $f \in H$. We have $ff^{\diamondsuit} \preccurlyeq_{\mathcal{G}}$ by the negation law. Applying the strong cut law to the above three equations, we have $\alpha \preccurlyeq_{\mathcal{G}}$. ■

Lemma 7.2 Let $X \subseteq H$. Then X is $\mathcal{G}$-inconsistent if and only if there exists an element $\alpha \in H^*$ such that $\alpha \subseteq H$ and $\alpha \preccurlyeq_{\mathcal{G}}$.

Proof By [6, Theorems 6.5, 6.7], the largest $\mathcal{G}$-logic is the restriction of $\preccurlyeq_{\mathcal{G}}$.[11] Moreover, by [6, Theorem 8.2], X is $\mathcal{G}$-inconsistent if and only if there exists an element $\alpha \in H^*$ such that $\alpha \subseteq H$ and $\alpha \preccurlyeq_{\mathcal{G}} h$ for all $h \in H$. Lemma 7.1 shows that $\alpha \preccurlyeq_{\mathcal{G}} h$ for all $h \in H$ if and only if $\alpha \preccurlyeq_{\mathcal{G}}$. ■

Lemma 7.3 Let $X, Y \subseteq H$. Then (X, Y) is a cut of H by $\preccurlyeq_{\mathcal{G}}$ if and only if $X \cup Y^{\lozenge}$ is $\mathcal{G}$-consistent, where $Y^{\lozenge} = \{g^{\lozenge} \mid g \in Y\}$.

Proof Suppose (X, Y) is not a cut. Then there exist a sequence $\alpha \subseteq X$ and elements $g_1, \ldots, g_n \in Y$ satisfying $\alpha \preccurlyeq_{\mathcal{F}} g_1 \ldots g_n$. Hence $\alpha g_1^{\lozenge} \ldots g_n^{\lozenge} \preccurlyeq_{\mathcal{G}}$ by Lemma 4.1. Therefore $X \cup Y^{\lozenge}$ is $\mathcal{G}$-inconsistent by Lemma 7.2. The opposite direction is proved similarly. ■

Lemma 7.4 Let $X, Y \subseteq H$. Then (X, Y) has a $\mathcal{G}$-model if and only if $X \cup Y^{\lozenge}$ has a $\mathcal{G}$-model, where $Y^{\lozenge} = \{g^{\lozenge} \mid g \in Y\}$.

Proof For each $f \in H$, $W \in \mathcal{W}$, $\mathbb{C}$-denotation Φ into W, $\mathbb{X}$-denotation ν into W and $\theta \in K \to W_{\varepsilon}$, we have

$$\begin{aligned}\big((\Phi^* f^{\lozenge})\nu\big)(\theta|_{K_f}) &= \big((\Phi^* f)\nu\big)^{\lozenge}(\theta|_{K_f}) \\ &= \Big(\big((\Phi^* f)\nu\big)(\theta|_{K_f})\Big)^{\lozenge}\end{aligned}$$

by Lemma 3.6 and the definition of $\lozenge$ on W. Hence

$$\big((\Phi^* f)\nu\big)(\theta|_{K_f}) = 0 \iff \big((\Phi^* f^{\lozenge})\nu\big)(\theta|_{K_f}) = 1.$$

Therefore, a $\mathcal{G}$-model of (X, Y) is a $\mathcal{G}$-model of $X \cup Y^{\lozenge}$, and vice versa. ■

Lemma 7.5 We can obtain an MPC language $(A', T', \sigma', \mathbb{S}', \mathbb{C}, \mathbb{X}', \Gamma)$ by extending the set $\mathbb{X}_{\varepsilon}$ to $\mathbb{X}'_{\varepsilon}$. Let $(A', \mathcal{W}', (I'_{W'})_{W' \in \mathcal{W}'})$ be the logical system MPCL on A' and $(H', \mathcal{G}')$ be the predicate logical space associated with the logical system. Then the $\mathcal{G}$-validity relation $\preccurlyeq_{\mathcal{G}}$ is the restriction of the $\mathcal{G}'$-validity relation $\preccurlyeq_{\mathcal{G}'}$ to $H^* \times H^*$. Moreover, let X be a subset of H. Then X is $\mathcal{G}$-consistent if and only if X is $\mathcal{G}'$-consistent. Also X has a $\mathcal{G}$-model if and only if X has a $\mathcal{G}'$-model.

Proof Let L' be the index set of T. Then $L' = L \cup \{\Omega x \mid x \in \mathbb{X}_{\varepsilon}\}$, hence $L \cap \Gamma = L' \cap \Gamma$. A and T are the L-reduct of A' and T' respectively. By the definition of the MPC language A', it follows that $\sigma = \sigma'|_A$ and that $\mathbb{X} \subseteq \mathbb{X}'$. An MPC world denotable for A is regarded as an MPC world denotable for A', and vice versa. Therefore $\mathcal{W} = \mathcal{W}'$. For each $W \in \mathcal{W}$ and each $x \in \mathbb{X}_{\varepsilon}$, $I_W(\Omega x)$ and $I'_W(\Omega x)$ are the same. Each $\mathbb{X}$-denotation into

[11]Lemma 7.1 shows that $\preccurlyeq_{\mathcal{G}}$ satisfies the quasi-disjunction law and the lower quasi-end law defined in [6].

W is extended to an $\mathbb{X}'$-denotation into W. For each $\mathbb{X}'$-denotation ν' into W, the restriction $\nu'|_{\mathbb{X}}$ is an $\mathbb{X}$-denotation into W. Therefore, by Theorem 2.3, it follows that $\mathcal{G} = \{\varphi|_{\mathsf{H}} \mid \varphi \in \mathcal{G}'\}$. This shows that, for each $\alpha, \beta \in \mathsf{H}^*$, $\alpha \preccurlyeq_{\mathcal{G}} \beta$ if and only if $\alpha \preccurlyeq_{\mathcal{G}'} \beta$.

For each $\mathsf{X} \subseteq \mathsf{H}$,

$$\begin{aligned} & \mathsf{X} \text{ is } \mathcal{G}\text{-consistent} \\ & \iff \alpha \not\preccurlyeq_{\mathcal{G}} \text{ for every } \alpha \subseteq \mathsf{X} \\ & \iff \alpha \not\preccurlyeq_{\mathcal{G}'} \text{ for every } \alpha \subseteq \mathsf{X} \\ & \iff \mathsf{X} \text{ is } \mathcal{G}'\text{-consistent} \end{aligned}$$

and

$$\begin{aligned} & \mathsf{X} \text{ has a } \mathcal{G}\text{-model} \\ & \iff \mathsf{X} \subseteq \varphi^{-1}1 \text{ for some } \varphi \in \mathcal{G} \\ & \iff \mathsf{X} \subseteq \varphi'^{-1}1 \text{ for some } \varphi' \in \mathcal{G}' \\ & \iff \mathsf{X} \text{ has a } \mathcal{G}'\text{-model.} \end{aligned}$$

■

Theorem 7.1 The predicate logical space $(\mathsf{H}, \mathcal{G})$ belongs to Class 2 or 3. It belongs to Class 2 if and only if the quantity system $\mathbb{P}$ of A is well-ordered and has the largest element.

Proof By [6, Theorem 8.9], $(\mathsf{H}, \mathcal{G})$ belongs to Class 1 or 2 if and only if every $\mathcal{G}$-consistent subset X of H has a $\mathcal{G}$-model. By Lemma 3.4, we have $\#\mathcal{G} > 1$. Hence $(\mathsf{H}, \mathcal{G})$ does not belong to Class 1 by [6, Remark 6.3].

Suppose $\mathbb{P}$ is well-ordered and has the largest element. Let X be a $\mathcal{G}$-consistent subset of H, and $\kappa = \#\mathsf{A}$. By virtue of Lemma 7.5, we may assume that there exists κ many elements of $\mathbb{X}_\varepsilon$ which do not occur in the predicates of X. $(\mathsf{X}, \emptyset)$ is a cut of H by $\preccurlyeq_{\mathcal{G}}$ by Lemma 7.3. Hence $(\mathsf{X}, \emptyset)$ has a $\mathcal{G}$-model by Corollary 5.1. Therefore X has a $\mathcal{G}$-model by Lemma 7.4.

Next suppose $\mathbb{P}$ is not well-ordered or does not have the largest element. Then there exists a cut (X, Y) of H by $\preccurlyeq_{\mathcal{G}}$ which has no $\mathcal{G}$-model by Theorem 6.1 and Remark 6.1. Hence $\mathsf{X} \cup \mathsf{Y}^{\diamond}$ is a $\mathcal{G}$-consistent set which has no $\mathcal{G}$-model by Lemma 7.3 and Lemma 7.4. ■

Remark 7.1 An argument similar to the proof of Theorem 7.1 holds for the $\emptyset$-sentential functional logical space $(\mathsf{A}_\emptyset, \mathcal{F})$, and it belongs to Class 2 if and only if the quantity system $\mathbb{P}$ of A is well-ordered and has the largest element.

8 A characteristic law

Let $(\mathsf{A}, \mathsf{T}, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ be an MPC language satisfying Assumption 3.1, $(\mathsf{A}, \mathcal{W}, (\mathrm{I}_W)_{W \in \mathcal{W}})$ be the logical system MPCL on it, and $(\mathsf{H}, \mathcal{G})$ be the predicate logical space associated with the logical system.

In this section we apply Theorem 5.1 to show that the MPC.1 law is a characteristic law of $(\mathsf{H}, \mathcal{G})$.

Theorem 8.1 Let $(\mathsf{A}, \mathcal{F})$ be a $\mathbb{T}$-valued functional logical space and $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ be a deduction system on $\vec{\mathsf{A}}$, where $\vec{\mathsf{A}} = \mathsf{A}^* \times \mathsf{A}^*$. Assume the following:

1. The $\mathcal{F}$-validity relation $\preccurlyeq_{\mathcal{F}}$ satisfies $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$.
2. Every finite cut of A by every relation which satisfies $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ and is contained in $\preccurlyeq_{\mathcal{F}}$ has an $\mathcal{F}$-model.

Then $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ together with the weakening law, contraction law, and exchange law forms a characteristic law of $(\mathsf{A}, \mathcal{F})$.

Proof Consult [6, Theorem 7.13].

Theorem 8.2 The MPC.1 law is a characteristic law of $(\mathsf{H}, \mathcal{G})$.

Proof The $\mathcal{G}$-validity relation $\preccurlyeq_{\mathcal{G}}$ satisfies the MPC.1 law by Theorem 4.1.

In view of Theorem 8.1, it suffices to show that every finite cut of H by every MPC.1 relation contained in $\preccurlyeq_{\mathcal{G}}$ has a $\mathcal{G}$-model.

Let $\preccurlyeq$ be an MPC.1 relation contained in $\preccurlyeq_{\mathcal{G}}$ and (X, Y) be a finite cut of H by $\preccurlyeq$. Since (X, Y) is finite, there exist κ many elements of $\mathbb{X}_\varepsilon$ which do not occur in the predicates in $\mathsf{X} \cup \mathsf{Y}$, where $\kappa = \#\mathsf{A}$. By Lemma 3.13 it follows that $\mathbb{P}^{\mathsf{X} \cup \mathsf{Y}}$ is finite, hence $[\mathbb{P}^{\mathsf{X} \cup \mathsf{Y}} \cup \{0\}]$ is well-ordered by Lemma 3.1. Therefore, by Theorem 5.1, (X, Y) has a $\mathcal{G}$-model. ∎

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