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Partial Cauchy data for general second order elliptic operators in two dimensions

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PARTIAL CAUCHY DATA FOR GENERAL SECOND ORDER ELLIPTIC OPERATORS IN TWO DIMENSIONS

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Abstract. We consider the problem of determining the coefficients of a first-order perturbation of the Laplacian in two dimensions by measuring the corresponding Cauchy data on an arbitrary open subset of the boundary. From this information we obtained a coupled system of $\partial_{\overline{z}}$ and ∂_z which the coefficients satisfy. As a corollary we show for the magnetic Schrödinger equation that the magnetic field and the electric potential are uniquely determined by measuring the partial Cauchy data on an arbitrary part of the boundary. We also show that the coefficients of any real vector field perturbation of the Laplacian, the convection terms, are uniquely determined by their partial Cauchy data.

1. Introduction

We consider the problem of determining a complex-valued potential q and complex-valued coefficients A and B in a bounded two dimensional domain from the Cauchy data measured on an arbitrary open subset of the boundary for the associated second order elliptic operator $\Delta + 2A\frac{\partial}{\partial z} + 2B\frac{\partial}{\partial \overline{z}} + q$. Specific cases of interest are the magnetic Schrödinger operator and the Laplacian with convection terms. We remark that general second order elliptic operators can be reduced to this form by using isothermal coordinates (e.g., [20]). The case of the conductivity equation has been considered in [12]. For global uniqueness results in the two dimensional case for the conductivity equation with full data measurements under different regularity assumptions see [1], [4], [16]. Such a problem originates from [7].

Below we more precisely formulate our inverse problem under consideration. Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with smooth boundary $\partial\Omega = \cup_{k=1}^{\mathcal{N}}\gamma_k$, where γ_k, $1 \le k \le \mathcal{N}$, are connected components and smooth closed contours and let ν be the unit outward normal vector to $\partial\Omega$. We denote $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$. Henceforth we set $i = \sqrt{-1}$, $x_1, x_2 \in \mathbf{R}$, $z = x_1 + ix_2$, $\overline{z}$ denotes the complex conjugate of $z \in \mathbf{C}$, and we identify $x = (x_1, x_2) \in \mathbf{R}^2$ with $z = x_1 + ix_2 \in \mathbf{C}$. We also denote $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2})$, $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2})$.

Henceforth let $(A, B, q), (A_j, B_j, q_j) \in C^{5+\alpha}(\overline{\Omega}) \times C^{5+\alpha}(\overline{\Omega}) \times C^{4+\alpha}(\overline{\Omega})$, $j = 1, 2$ for some $\alpha > 0$ and let A, B, q, A_j, B_j, q_j be complex-valued.

Let a function $u \in H^1(\Omega)$ be a solution of the Dirichlet problem

$$L(x, D)u = \Delta u + 2A\frac{\partial u}{\partial z} + 2B\frac{\partial u}{\partial \overline{z}} + qu = 0 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = f, \tag{1.1}$$

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where $f \in H^{\frac{1}{2}}(\partial\Omega)$ is a given boundary input. The Dirichlet-to-Neumann (DN) map, assuming that 0 is not a Dirichlet eigenvalue, is defined by

$$\Lambda_{A,B,q}(f) = \frac{\partial u}{\partial \nu}\Big|_{\widetilde{\Gamma}}. \tag{1.2}$$

More generally we define the set of Cauchy data for a bounded potential q by:

$$\widehat{C_{A,B,q}} = \left\{ \left(u|_{\partial\Omega}, \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} \right) \mid (\Delta + 2A\frac{\partial}{\partial z} + 2B\frac{\partial u}{\partial \overline{z}} + q)u = 0 \text{ in } \Omega, \quad u \in H^1(\Omega) \right\}. \tag{1.3}$$

We have $\widehat{C_{A,B,q}} \subset H^{\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$.

Let $\widetilde{\Gamma} \subset \partial\Omega$ be a fixed non-empty open subset of the boundary and $\Gamma_0 = \partial\Omega \setminus \overline{\widetilde{\Gamma}}$.

Our main result proves that the coefficients must satisfy a system of $\overline{\partial}$ and ∂ if the set of Cauchy data for the pairs coincide. Consider the following sets of Cauchy data on $\widetilde{\Gamma}$ for $j = 1, 2$:

$$\mathcal{C}_{(A_j,B_j,q_j)} = \left\{ \left(u|_{\widetilde{\Gamma}}, \frac{\partial u}{\partial \nu}\Big|_{\widetilde{\Gamma}} \right) \mid (\Delta + 2A_j\frac{\partial}{\partial z} + 2B_j\frac{\partial}{\partial \overline{z}} + q_j)u = 0 \,\text{in}\, \Omega, \; u|_{\Gamma_0} = 0, \; u \in H^1(\Omega) \right\}. \tag{1.4}$$

Our main result is:

Theorem 1.1. *Assume $\mathcal{C}_{(A_1,B_1,q_1)} = \mathcal{C}_{(A_2,B_2,q_2)}$. Then*

$$A_1 = A_2, \quad B_1 = B_2 \quad \text{on} \quad \widetilde{\Gamma}, \tag{1.5}$$

$$-2\tfrac{\partial}{\partial z}(A_1 - A_2) - (B_1 - B_2)A_1 - (A_1 - A_2)B_2 + (q_1 - q_2) = 0 \quad \text{in } \Omega, \tag{1.6}$$

$$-2\tfrac{\partial}{\partial \overline{z}}(B_1 - B_2) - (A_1 - A_2)B_1 - (B_1 - B_2)A_2 + (q_1 - q_2) = 0 \quad \text{in } \Omega. \tag{1.7}$$

Remark. *In the case that $A_1 = A_2$ and $B_1 = B_2$ in Ω, then Theorem 1.1 yields $q_1 = q_2$, which is a main result in [12]. We can not uniquely determine (A, B, q). Really, in order to see this, consider the function $\eta \in C^\infty(\overline{\Omega})$ such that $\eta|_{\tilde{\Gamma}} = \frac{\partial \eta}{\partial \nu}|_{\tilde{\Gamma}} = 0$. Then it is easy to check the operators $L(x, D)$ and $e^\eta L(x, D)e^{-\eta}$ have the same Dirichlet-to-Neumann map.*

In the case of a simply connected domain Ω, from Theorem 1.1 we can derive the following corollary which gives a necessary and sufficient condition for the coincidence of the Dirichlet-to-Neumann maps.

Corollary 1.1. *Let Ω be a simply connected domain. Then $\mathcal{C}_{(A_1,B_1,q_1)} = \mathcal{C}_{(A_2,B_2,q_2)}$ if and only if there exists a function $\eta \in C^{5+\alpha}(\bar{\Omega})$, $\eta|_{\tilde{\Gamma}} = \frac{\partial \eta}{\partial \nu}|_{\tilde{\Gamma}} = 0$ such that*

$$L_1(x, D) = e^{-\eta}L_2(x, D)e^{\eta}. \tag{1.8}$$

Proof. It is sufficient to prove (1.8) from $\mathcal{C}_{(A_1,B_1,q_1)} = \mathcal{C}_{(A_2,B_2,q_2)}$, because the converse is directly proved. By (1.6), (1.7), $\frac{\partial}{\partial z}(A_1 - A_2) = \frac{\partial}{\partial \bar{z}}(B_1 - B_2)$, since the domain Ω is assumed to be simply connected, there exists a function $\eta \in C^{5+\alpha}(\bar{\Omega})$ such that

$$A_1 - A_2 = 2\frac{\partial \eta}{\partial \bar{z}}, \quad B_1 - B_2 = 2\frac{\partial \eta}{\partial z}.$$

The existence of such an η is proved as follows. Let us set $f = A_1 - A_2$ and $g = B_1 - B_2$. Then $\frac{\partial f}{\partial z} = \frac{\partial g}{\partial \overline{z}}$. Therefore we can define a one-form a by $a = fd\overline{z} + gdz$. Then we have

$$da = \frac{\partial f}{\partial z} dz \wedge d\overline{z} + \frac{\partial g}{\partial \overline{z}} d\overline{z} \wedge dz = \frac{\partial f}{\partial z} dz \wedge d\overline{z} - \frac{\partial g}{\partial \overline{z}} dz \wedge d\overline{z} = 0$$

by $\frac{\partial f}{\partial z} = \frac{\partial g}{\partial \overline{z}}$. Therefore in a simply connected domain Ω, we can choose η such that $a = d\eta = \frac{\partial \eta}{\partial z} dz + \frac{\partial \eta}{\partial \overline{z}} d\overline{z}$. By $a = fd\overline{z} + gdz$, we have $f = \frac{\partial \eta}{\partial \overline{z}}$ and $g = \frac{\partial \eta}{\partial z}$. Thus the existence of η is proved.

Therefore by (1.6)

$$q_1 = q_2 + \Delta\eta + 4\frac{\partial \eta}{\partial z}\frac{\partial \eta}{\partial \overline{z}} + 2\frac{\partial \eta}{\partial z} A_2 + 2\frac{\partial \eta}{\partial \overline{z}} B_2. \tag{1.9}$$

The operator $L_1(x, D)$ given by (1.8) has the Laplace operator as the principal part, the coefficients of $\frac{\partial}{\partial z}$ equals to $2A_2 + 4\frac{\partial \eta}{\partial \overline{z}}$, the coefficient of $\frac{\partial}{\partial \overline{z}}$ equals to $2B_2 + 4\frac{\partial \eta}{\partial z}$ and the coefficient of the zero order term is given by the right-hand side of (1.9). By (1.5) we have that $\frac{\partial \eta}{\partial \nu}|_{\tilde{\Gamma}} = 0$ and $\eta|_{\tilde{\Gamma}} = 0$. The proof of the corollary is completed. □

We now apply our result to the case of the magnetic Schrödinger operator. Denote $\widetilde{A} = (\widetilde{A}_1, \widetilde{A}_2)$, $\widetilde{\mathcal{A}} = \widetilde{A}_1 - i\widetilde{A}_2$, rot $\widetilde{A} = \frac{\partial \widetilde{A}_2}{\partial x_1} - \frac{\partial \widetilde{A}_1}{\partial x_2}$, $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$. Consider the magnetic Schrödinger operator

$$\mathcal{L}_{\widetilde{A},\widetilde{q}}(x, D) = \sum_{k=1}^{2}(D_k + \widetilde{A}_k)^2 + \widetilde{q}. \tag{1.10}$$

Let us define the following set of partial Cauchy data

$$\widetilde{C}_{\widetilde{A}^{(j)},\widetilde{q}^{(j)}} = \left\{ (u|_{\tilde{\Gamma}}, \frac{\partial u}{\partial \nu}|_{\tilde{\Gamma}}) | \mathcal{L}_{\widetilde{A}^{(j)},\widetilde{q}^{(j)}}(x, D)u = 0 \text{ in } \Omega,\ u|_{\Gamma_0} = 0, u \in H^1(\Omega) \right\}.$$

Corollary 1.2. *Let $\alpha > 0$, real-valued vector fields $\widetilde{A}^{(1)}, \widetilde{A}^{(2)} \in C^{5+\alpha}(\overline{\Omega})$ and complex-valued potentials $\widetilde{q}^{(1)}, \widetilde{q}^{(2)} \in C^{4+\alpha}(\overline{\Omega})$ be such that $\widetilde{C}_{\widetilde{A}^{(1)},\widetilde{q}^{(1)}} = \widetilde{C}_{\widetilde{A}^{(2)},\widetilde{q}^{(2)}}$. Then $\widetilde{q}^{(1)} = \widetilde{q}^{(2)}$ and rot $\widetilde{A}^{(1)}$ = rot $\widetilde{A}^{(2)}$.*

Proof. A straightforward calculation gives

$$\begin{aligned}\mathcal{L}_{\widetilde{A},\widetilde{q}}(x, D) &= -\Delta + \frac{2}{i}\widetilde{A}_1\frac{\partial}{\partial x_1} + \frac{2}{i}\widetilde{A}_2\frac{\partial}{\partial x_2} + |\widetilde{A}|^2 + \frac{1}{i}\frac{\partial \widetilde{A}_1}{\partial x_1} + \frac{1}{i}\frac{\partial \widetilde{A}_2}{\partial x_2} + \widetilde{q} \\ &= -\Delta + \frac{2}{i}\overline{\widetilde{\mathcal{A}}}\frac{\partial}{\partial z} + \frac{2}{i}\widetilde{\mathcal{A}}\frac{\partial}{\partial \overline{z}} + \frac{2}{i}\frac{\partial \overline{\widetilde{\mathcal{A}}}}{\partial z} - \text{rot}\,\widetilde{A} + |\widetilde{A}|^2 + \widetilde{q}.\end{aligned} \tag{1.11}$$

Then the operator $\mathcal{L}_{\widetilde{A},\widetilde{q}}(x, D)$ is a particular case of (1.1). Suppose that the vector fields $\widetilde{A}^{(1)}$, $\widetilde{A}^{(2)}$ and the potentials $\widetilde{q}^{(1)}, \widetilde{q}^{(2)}$ have the same Dirichlet-to-Neumann map. Taking into account that $A_j = -\frac{1}{i}\overline{\widetilde{\mathcal{A}}^{(j)}}$, $B_j = -\frac{1}{i}\widetilde{\mathcal{A}}^{(j)}$, $q_j = -(\frac{2}{i}\frac{\partial \overline{\widetilde{\mathcal{A}}^{(j)}}}{\partial z} - \text{rot}\,\widetilde{A}^{(j)} + |\widetilde{A}^{(j)}|^2 + \widetilde{q}^{(j)})$, we see that (1.6) gives

$$\text{rot}\,\widetilde{A}^{(1)} - \text{rot}\,\widetilde{A}^{(2)} + \widetilde{q}^{(2)} - \widetilde{q}^{(1)} \equiv 0$$

and (1.7) gives

$$\frac{2}{i}\frac{\partial\widetilde{\mathcal{A}}^{(1)}}{\partial\overline{z}}-\frac{2}{i}\frac{\partial\widetilde{\mathcal{A}}^{(2)}}{\partial\overline{z}}-\frac{2}{i}\frac{\partial\overline{\widetilde{\mathcal{A}}^{(1)}}}{\partial z}+\frac{2}{i}\frac{\partial\overline{\widetilde{\mathcal{A}}^{(2)}}}{\partial z}+\mathrm{rot}\,\widetilde{A}^{(1)}-\mathrm{rot}\,\widetilde{A}^{(2)}+\widetilde{q}^{(2)}-\widetilde{q}^{(1)}\equiv 0. \tag{1.12}$$

Using the identity $\frac{2}{i}\frac{\partial\mathcal{A}}{\partial\overline{z}}-\frac{2}{i}\frac{\partial\overline{\mathcal{A}}}{\partial z}=-2\mathrm{rot}\,\widetilde{A}$ we transform (1.12) to the form

$$-(\mathrm{rot}\,\widetilde{A}^{(1)}-\mathrm{rot}\,\widetilde{A}^{(2)})+\widetilde{q}^{(2)}-\widetilde{q}^{(1)}\equiv 0.$$

The proof of the corollary is completed. □

We remark that this corollary generalizes the result which we obtained in [12] where it was assumed that the magnetic potential $\widetilde{A_k}$ is zero. A global uniqueness result for full data in this case was given in [5]. As for the uniqueness by partial Cauchy data, see [6] and [14].

Corollary 1.2 is new even in the case when the data is measured on the whole boundary. In two dimensions, Sun proved in [19] that for measurements on the whole boundary uniqueness holds assuming that both the magnetic potential and the electric potential are small. Kang and Uhlmann proved global uniqueness for the case of measurements on the whole boundary for a special case of the magnetic Schrödinger equation, namely the Pauli Hamiltonian [13]. In dimension $n\ge 3$ global uniqueness was shown in [17] for the case of full data. The regularity assumptions in the result were improved by Salo in [18]. The case of partial data was considered in [10] with an improvement on the regularity of the coefficients in [15].

Our main theorem implies that the Dirichlet-to-Neumann map can uniquely determine any two of (A,B,q). First we can prove that A and B are uniquely determined if q is known. We discuss the uniqueness for Laplace operators with convection terms:

$$L(x,D)u=\Delta u+a(x)\frac{\partial u}{\partial x_1}+b(x)\frac{\partial u}{\partial x_2}+q(x)u. \tag{1.13}$$

Here a, b, q are complex-valued functions. Let us define the following set of partial Cauchy data

$$\widetilde{C}_{a^{(j)},b^{(j)}}=\left\{(u|_{\widetilde{\Gamma}},\frac{\partial u}{\partial\nu}|_{\widetilde{\Gamma}})|\Delta u+a^{(j)}(x)\frac{\partial u}{\partial x_1}+b^{(j)}(x)\frac{\partial u}{\partial x_2}+q(x)u=0 \text{ in } \Omega,\ u|_{\Gamma_0}=0, u\in H^1(\Omega)\right\}.$$

We have

Corollary 1.3. *Let* $\alpha>0$ *and two pairs of complex-valued coefficients* $(a^{(1)},b^{(1)})\in C^{5+\alpha}(\overline{\Omega})\times C^{5+\alpha}(\overline{\Omega})$ *and* $(a^{(2)},b^{(2)})\in C^{5+\alpha}(\overline{\Omega})\times C^{5+\alpha}(\overline{\Omega})$ *be such that* $\widetilde{C}_{a^{(1)},b^{(1)}}=\widetilde{C}_{a^{(2)},b^{(2)}}$. *Then* $(a^{(1)},b^{(1)})\equiv(a^{(2)},b^{(2)})$.

Proof. Taking into account that $\frac{\partial}{\partial x_1}=(\frac{\partial}{\partial z}+\frac{\partial}{\partial\overline{z}})$ and $\frac{\partial}{\partial x_2}=i(\frac{\partial}{\partial z}-\frac{\partial}{\partial\overline{z}})$, we can rewrite the operator (1.13) in the form

$$L(x,D)u=\Delta u+(a(x)+ib(x))\frac{\partial u}{\partial z}+(a(x)-ib(x))\frac{\partial u}{\partial\overline{z}}+q(x)u.$$

The pairs $(a^{(1)},b^{(1)})$ and $(a^{(2)},b^{(2)})$ be such that corresponding operators defined by (1.13) have the same Dirichlet-to-Neumann map. Denote $2A_k(x)=a^{(k)}(x)+ib^{(k)}(x)$ and $2B_k(x)=$

$a^{(k)}(x) - ib^{(k)}(x)$. By (1.6), we have

$$-2\frac{\partial}{\partial z}(A_1 - A_2) - (B_1 - B_2)A_1 - (A_1 - A_2)B_2 = 0 \quad \text{in } \Omega, \tag{1.14}$$

$$-2\frac{\partial}{\partial \overline{z}}(B_1 - B_2) - (A_1 - A_2)B_1 - (B_1 - B_2)A_2 = 0 \quad \text{in } \Omega. \tag{1.15}$$

Applying to equation (1.14) the operator $2\frac{\partial}{\partial \overline{z}}$ and to equation (1.15) the operator $2\frac{\partial}{\partial z}$ we have

$$-\Delta(A_1 - A_2) - 2\frac{\partial}{\partial \overline{z}}((B_1 - B_2)A_1 + (A_1 - A_2)B_2) = 0 \quad \text{in } \Omega, \tag{1.16}$$

$$-\Delta(B_1 - B_2) - 2\frac{\partial}{\partial z}((A_1 - A_2)B_1 + (B_1 - B_2)A_2) = 0 \quad \text{in } \Omega. \tag{1.17}$$

By (1.5)

$$(A_1 - A_2)|_{\widetilde{\Gamma}} = (B_1 - B_2)|_{\widetilde{\Gamma}} = 0.$$

Using these identities and equations (1.14), (1.15) we obtain

$$\frac{\partial(A_1 - A_2)}{\partial\nu}|_{\widetilde{\Gamma}} = \frac{\partial(B_1 - B_2)}{\partial\nu}|_{\widetilde{\Gamma}} = 0.$$

The uniqueness of the Cauchy problem for the system (1.16)-(1.17) can be proved in the standard way by a Carleman estimate (e.g., [11]). Therefore we have $A_1 = A_2$ and $B_1 = B_2$ in Ω. □

We remark that this generalizes the result of [9] who proved this result assuming that the measurements are made on the whole boundary. In dimension $n \geq 3$ global uniqueness was shown in [8] for the case of full data.

Similarly to Corollary 1.2, we can prove that the Dirichlet-to-Neumann map can uniquely determine a potential q and one of A and B in (1.1).

Corollary 1.4. *For $j = 1, 2$, let $(A_j, B_j, q_j) \in C^{5+\alpha}(\overline{\Omega}) \times C^{5+\alpha}(\overline{\Omega}) \times C^{4+\alpha}(\overline{\Omega})$ for some $\alpha > 0$ and be complex-valued. We assume either $A_1 = A_2$ or $B_1 = B_2$ in Ω. Then $\mathcal{C}_{(A_1,B_1,q_1)} = \mathcal{C}_{(A_2,B_2,q_2)}$ implies $(A_1, B_1, q_1) = (A_2, B_2, q_2)$.*

The proof of Theorem 1.1 follows the general method of [12]. In this case we need to prove a new Carleman estimate with degenerate harmonic weights to construct appropriate complex geometrical optics solutions. These solutions have a different form to take into account the first order terms. The new form of these solutions complicates considerably the arguments, especially the asymptotic expansions needed to analyze the behavior of the solutions. In Section 2 we prove the Carleman estimate which we need. In Section 3 we state the estimates and asymptotics which we will use in the construction of the complex geometrical optics solutions. This construction is done in Section 4. The proof of Theorem 1.1 is finished in Section 5.

2. Carleman estimate

Notations

$i = \sqrt{-1}$, $x_1, x_2, \xi_1, \xi_2 \in \mathbf{R}$, $z = x_1 + ix_2$, $\zeta = \xi_1 + i\xi_2$, $\overline{z}$ denotes the complex conjugate of $z \in \mathbf{C}$, $D_k = \frac{1}{i}\frac{\partial}{\partial x_k}$. We identify $x = (x_1, x_2) \in \mathbf{R}^2$ with $z = x_1 + ix_2 \in \mathbf{C}$. We set $\partial_z = \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2})$, $\partial_{\overline{z}} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2})$, $\mathcal{O}_\epsilon = \{x \in \Omega | \text{dist}(x, \partial\Omega) \leq \epsilon\}$. Throughout the paper we use both notations ∂_z and $\frac{\partial}{\partial z}$, etc. and for example we denote $\partial_{\bar z}^2 = \frac{\partial^2}{\partial \bar z^2}$. We say that a function $a(x)$ is antiholomorphic if $\partial_z a(x) = 0$. The tangential derivative on the boundary is given by $\frac{\partial}{\partial \vec\tau} = \nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2}$, with $\nu = (\nu_1, \nu_2)$ the unit outer normal to $\partial\Omega$, $B(\widehat{x}, \delta) = \{x \in \mathbf{R}^2 | |x - \widehat{x}| < \delta\}$, $S(\widehat{x}, \delta) = \{x \in \mathbf{R}^2 | |x - \widehat{x}| = \delta\}$, $f(x) : \mathbf{R}^2 \to \mathbf{R}^1$, f'' is the Hessian matrix with entries $\frac{\partial^2 f}{\partial x_i \partial x_j}$, $\|\cdot\|^2_{H^{k,\tau}(\Omega)} = \|\cdot\|^2_{H^k(\Omega)} + |\tau|^{2k}\|\cdot\|^2_{L^2(\Omega)}$, $(\cdot,\cdot)_{H^{k,\tau}(\Omega)} = (\cdot,\cdot)_{H^k(\Omega)} + |\tau|^{2k}(\cdot,\cdot)_{L^2(\Omega)}$, $\mathcal{L}(X,Y)$ denotes the Banach space of all bounded linear operators from a Banach space X to another Banach space Y. By $o_X(\frac{1}{\tau^\kappa})$ we denote a function $f(\tau,\cdot)$ such that

$$\|f(\tau,\cdot)\|_X = o(\frac{1}{\tau^\kappa}) \quad \text{as } |\tau| \to +\infty.$$

Let $\Phi(z) = \varphi(x_1,x_2) + i\psi(x_1,x_2) \in C^5(\overline{\Omega})$ with real-valued φ and ψ satisfy

$$\frac{\partial \Phi}{\partial \bar z}(z) = 0 \quad \text{in } \Omega, \quad \operatorname{Im}\Phi|_{\Gamma_0} = 0. \tag{2.1}$$

Denote by $\mathcal{H}$ the set of all the critical points of the function Φ

$$\mathcal{H} = \{z \in \overline{\Omega} | \frac{\partial \Phi}{\partial z}(z) = 0\}.$$

Assume that Φ has no critical points on $\overline{\tilde\Gamma}$, and that all critical points are nondegenerate:

$$\mathcal{H} \cap \partial\Omega \subset \Gamma_0, \quad \frac{\partial^2 \Phi}{\partial z^2}(z) \neq 0, \quad \forall z \in \mathcal{H}. \tag{2.2}$$

Then Φ has only a finite number of critical points and we can set:

$$\mathcal{H} \setminus \Gamma_0 = \{\widetilde{x}_1, ..., \widetilde{x}_\ell\}, \quad \mathcal{H} \cap \Gamma_0 = \{\widetilde{x}_{\ell+1}, ..., \widetilde{x}_{\ell+\ell'}\}. \tag{2.3}$$

The following proposition was proved in [12]:

Proposition 2.1. *Let $\widetilde{x}$ be an arbitrary point in Ω. There exists a sequence of functions $\{\Phi_\epsilon\}_{\epsilon\in(0,1)}$ satisfying (2.1) such that all the critical points of Φ_ϵ are nondegenerate and there exists a sequence $\{\widetilde{x}_\epsilon\}, \epsilon \in (0,1)$ such that*

$$\widetilde{x}_\epsilon \in \mathcal{H}_\epsilon = \{z \in \overline{\Omega} | \frac{\partial \Phi_\epsilon}{\partial z}(z) = 0\}, \quad \widetilde{x}_\epsilon \to \widetilde{x} \quad \textit{as } \epsilon \to +0.$$

Moreover for any j from $\{1, \dots, \mathcal{N}\}$ we have

$$\mathcal{H}_\epsilon \cap \gamma_j = \emptyset \quad \textit{if } \gamma_j \cap \tilde\Gamma \neq \emptyset,$$

$$\mathcal{H}_\epsilon \cap \gamma_j \subset \Gamma_0 \quad \textit{if } \gamma_j \cap \tilde\Gamma = \emptyset,$$

$$Im\,\Phi_\epsilon(\widetilde{x}_\epsilon) \notin \{Im\,\Phi_\epsilon(x) | x \in \mathcal{H}_\epsilon \setminus \{\widetilde{x}_\epsilon\}\} \textit{ and } Im\,\Phi_\epsilon(\widetilde{x}_\epsilon) \neq 0.$$

In order to prove (1.5) we need the following proposition:

Proposition 2.2. *Let $\Gamma_* \subset\subset \widetilde{\Gamma}$ be an arc with the left endpoint x_- and the right endpoint x_+ orientating clockwise. For any $\widehat{x} \in Int\,\Gamma_*$ there exists a function $\Phi(z)$ which satisfies (2.1), (2.2), $Im\,\Phi|_{\partial\Omega\setminus\Gamma_*} = 0$ and*

$$\widehat{x} \in \mathcal{G} = \{x \in \Gamma_* | \quad \frac{\partial Im\,\Phi}{\partial \vec\tau}(x) = 0\}, \quad card\mathcal{G} < \infty, \tag{2.4}$$

(2.5) $$(\frac{\partial}{\partial\vec\tau})^2 Im\,\Phi(x)\neq 0 \quad \forall x\in\mathcal{G}\setminus\{x_-,x_+\},$$

Moreover

(2.6) $$Im\,\Phi(\widehat x)\neq Im\,\Phi(x) \quad \forall x\in\mathcal{G}\setminus\{\widehat x\}\ \ and \quad Im\,\Phi(\widehat x)\neq 0.$$

(2.7) $$(\frac{\partial}{\partial\vec\tau+0})^6 Im\,\Phi(x_-)\neq 0, \quad (\frac{\partial}{\partial\vec\tau-0})^6 Im\,\Phi(x_+)\neq 0.$$

Proof. Denote $\Gamma_0^*=\partial\Omega\setminus\Gamma_*$. Let $\widehat x_-,\widehat x_+\in\partial\Omega$ be points such that the arc $[\widehat x_-,\widehat x_+]\subset(x_-,x_+)$ and $\widehat x\in(\widehat x_-,\widehat x_+)$ be an arbitrary point and x_0 be another fixed point from the interval $(\widehat x,\widehat x_+)$. We claim that there exists a pair $(\varphi,\psi)\in C^5(\overline\Omega)\times C^5(\overline\Omega)$ which solves the system of Cauchy-Riemann equations in Ω such that

A) $\psi|_{\Gamma_0^*}=0, |\frac{\partial\varphi}{\partial\vec\tau}|_{\gamma_j\setminus\Gamma_*}>0$ if$\gamma_j\cap\Gamma_*\neq\emptyset$, $\frac{\partial\psi}{\partial\vec\tau}(\widehat x)=0, (\frac{\partial}{\partial\vec\tau})^2\psi(\widehat x)\neq 0,$

A′) $(\frac{\partial}{\partial\vec\tau+0})^6\psi(x_-)\neq 0, \quad (\frac{\partial}{\partial\vec\tau-0})^6\psi(x_+)\neq 0,$

B) The restriction of the function ψ on the arc $[\widehat x_-,\widehat x_+]$ is a Morse function,

C) $\frac{\partial\psi}{\partial\vec\tau}>0$ on $(x_-,\widehat x_-]$, $\quad \frac{\partial\psi}{\partial\vec\tau}<0$ on $[\widehat x_+,x_+)$,

D) $\psi(\widehat x)\notin\{\psi(x)|x\in\partial\Omega\setminus\{\widehat x\},\frac{\partial\psi}{\partial\vec\tau}(x)=0\}$,

E) if $\gamma_j\cap\Gamma_*=\emptyset$, then the restriction of the function φ on γ_j has only two nondegenerate critical points.

Such a pair of functions may be constructed in the following way. Let $\gamma_1\cap\Gamma_*\neq\emptyset$ and $\gamma_j\cap\Gamma_*=\emptyset$ for all $j\in\{2,\dots\mathcal{N}\}$. First by Corollary 6.1 for some $\alpha\in(0,1)$, there exists a solution $(\widetilde\varphi,\widetilde\psi)\in C^{5+\alpha}(\overline\Omega)\times C^{5+\alpha}(\overline\Omega)$ to the Cauchy-Riemann equations with the following boundary data

$$\widetilde\psi|_{\partial\Omega\setminus[x_0,\widehat x_+]}=\psi_*, \quad \frac{\partial\widetilde\varphi}{\partial\vec\tau}|_{\gamma_0\setminus[x_0,\widehat x_+]}<\beta<0$$

and such that if $\gamma_j\cap\Gamma_*=\emptyset$ the function $\tilde\varphi$ has only two nondegenerate critical points located on the contour γ_j. The function ψ_* has the following properties: $\psi_*|_{\Gamma_0^*}=0$, $\frac{\partial\psi_*}{\partial\vec\tau}>0$ on $(x_-,\widehat x_-]$, $\frac{\partial\psi_*}{\partial\vec\tau}<0$ on $[\widehat x_+,x_+)$. The function ψ_* on the set $[\widehat x_-,x_0]$ has only one critical point $\widehat x$ and $\psi_*(\hat x)\neq 0$. On the set $(x_0,\hat x_+)$ the Cauchy data is not fixed. The restriction of the function $\widetilde\psi$ on $[x_0,\widehat x_+]$ can be approximated in the space $C^{5+\alpha}(\overline{[x_0,\widehat x_+]})$ by a sequence of Morse functions $\{g_\epsilon\}_{\epsilon\in(0,1)}$ such that

$$(\frac{\partial}{\partial\vec\tau})^k\widetilde\psi(x)=(\frac{\partial}{\partial\vec\tau})^k g_\epsilon(x) \quad x\in\{\widehat x_+,x_0\}, \quad k\in\{0,1,\dots,5\},$$

and

$$\psi_*(\hat x)\notin\{g_\epsilon(x)|\frac{\partial g_\epsilon(x)}{\partial\vec\tau}=0\}.$$

Let us consider some arc $\mathcal{J} \subset\subset (x_-, \widehat{x}_-)$. On this arc we have $\frac{\partial\widetilde{\psi}}{\partial\vec{\tau}} > 0$, say,

$$\frac{\partial\widetilde{\psi}}{\partial\vec{\tau}} > \beta' > 0 \quad \text{on } \mathcal{J} \quad \text{for some positive} \quad \beta'. \tag{2.8}$$

Let $(\varphi_\epsilon, \psi_\epsilon) \in C^{5+\alpha}(\overline{\Omega}) \times C^{5+\alpha}(\overline{\Omega})$ be a solution to the Cauchy-Riemann equations with boundary data $\psi_\epsilon = 0$ on $\partial\Omega \setminus (\mathcal{J} \cup [x_0, \widehat{x}_+])$ and $\psi_\epsilon = g_\epsilon - \widetilde{\psi}$ on $[x_0, \widehat{x}_+]$ and on $\mathcal{J}$ the Cauchy data is chosen in such a way that

$$\|\psi_\epsilon\|_{C^{5+\alpha}(\overline{\partial\Omega})} + \|\varphi_\epsilon\|_{C^{5+\alpha}(\overline{\partial\Omega})} \to 0 \text{ as } \quad \|g_\epsilon - \widetilde{\psi}\|_{C^{5+\alpha}([x_0, \widehat{x}_+])} \to 0. \tag{2.9}$$

By (2.8), (2.9) for all small positive ϵ, the restriction of the function $\widetilde{\psi} + \psi_\epsilon$ on $\partial\Omega$ satisfies

$$(\widetilde{\psi} + \psi_\epsilon)|_{\Gamma_0^*} = 0, \quad \frac{\partial(\widetilde{\psi} + \psi_\epsilon)}{\partial\nu}|_{\gamma_0\setminus[x_0,\hat{x}_+]} < 0, \quad \frac{\partial(\widetilde{\psi} + \psi_\epsilon)}{\partial\vec{\tau}} > 0 \text{ on } [x_-, \hat{x}_-],$$

$$\frac{\partial(\widetilde{\psi} + \psi_\epsilon)}{\partial\vec{\tau}} < 0 \text{ on } [\hat{x}_+, x_+], \quad (\widetilde{\psi} + \psi_\epsilon)|_{[x_0,\widehat{x}_+]} = g_\epsilon, \quad (\widetilde{\psi} + \psi_\epsilon)|_{[\hat{x}_-,x_0]} = \psi_*.$$

If $j \geq 2$ then the restriction of the function $\varphi_\epsilon + \widetilde{\varphi}$ on γ_j has only two critical points located on the contour $\gamma_j \subset \Gamma_0^*$. These critical points are nondegenerate if ϵ is sufficiently small.

Therefore the restriction of the function $(\widetilde{\psi} + \psi_\epsilon)$ on Γ_* has a finite number of critical points. Some of these points may be the critical points of $(\widetilde{\psi} + \psi_\epsilon)$ considered as the function on $\overline{\Omega}$. We change slightly the function $(\widetilde{\psi} + \psi_\epsilon)$ such that all of its critical points are in Ω. Suppose that function $\widetilde{\psi} + \psi_\epsilon$ has critical points on Γ_*. Then these critical points should be among the set of critical points of the function g_ϵ, otherwise it would be the point $\widehat{x}$. We denote these points as $\widehat{x}_1, \ldots, \widehat{x}_m$. Let $(\widehat{\varphi}, \widehat{\psi}) \in C^{5+\alpha}(\overline{\Omega}) \times C^{5+\alpha}(\overline{\Omega})$ be a solution to the Cauchy-Riemann problem (6.1) with the following boundary data

$$\widehat{\psi}|_{\Gamma_0^*} = 0, \ \widehat{\psi}(\widehat{x}) = 1, \ \widehat{\psi}|_{\mathcal{G}\setminus\{\widehat{x}\}} = 0, \quad |\frac{\partial\widehat{\psi}}{\partial\nu}||_{\overline{\gamma_0\setminus\mathcal{J}}} > 0.$$

For all small positive ϵ_1 the function $\widetilde{\psi} + \psi_\epsilon + \epsilon_1\widehat{\psi}$ does not have a critical point on $\partial\Omega$ and the restriction of this function on $\widetilde{\Gamma}$ has a finite number of nondegenerate critical points. Therefore we take $(\widetilde{\varphi} + \varphi_\epsilon + \epsilon_1\widehat{\varphi}, \widetilde{\psi} + \psi_\epsilon + \epsilon_1\widehat{\psi})$ as the pairs of functions satisfying A) - E).

The function $\varphi + i\psi$ with pair (φ, ψ) satisfying conditions A)-E) satisfies all the hypotheses of Proposition 2.2 except that some of its critical points might possibly be degenerate. In order to fix this problem we consider the perturbation of the function $\varphi + i\psi$ which is constructed in the following way. By Proposition 6.2, there exists a function w holomorphic in Ω, such that

$$\operatorname{Im} w|_{\Gamma_0^*} = 0, \ w|_{\mathcal{H}_0} = \frac{\partial w}{\partial z}|_{\mathcal{H}_0} = 0, \quad \frac{\partial^2 w}{\partial z^2}|_{\mathcal{H}_0} \neq 0. \tag{2.10}$$

Denote $\Phi_\delta = \varphi + i\psi + \delta w$. For all sufficiently small positive δ, we have

$$\mathcal{H}_0 \subset \mathcal{H}_\delta \equiv \{x \in \Omega | \frac{\partial}{\partial z}\Phi_\delta(x) = 0\}.$$

We now show that for all sufficiently small positive δ, all critical points of the function Φ_δ are nondegenerate. Let $\widetilde{x}$ be a critical point of the function $\varphi + i\psi$. If $\widetilde{x}$ is a nondegenerate critical point, by the implicit function theorem, there exists a ball $B(\widetilde{x}, \delta_1)$ such that the function Φ_δ in this ball has only one nondegenerate critical point for all small δ. Let $\widetilde{x}$ be a degenerate critical point of $\varphi + i\psi$. Without loss of generality we may assume that $\widetilde{x} = 0$. In some neighborhood of 0, we have $\frac{\partial \Phi_\delta}{\partial z} = \sum_{k=1}^\infty c_k z^{k+\widehat{k}} - \delta \sum_{k=1}^\infty b_k z^k$ for some natural number $\widehat{k}$ and some $c_1 \neq 0$. Moreover (2.10) implies $b_1 \neq 0$. Let $(x_{1,\delta}, x_{2,\delta}) \in \mathcal{H}_\delta$ and $z_\delta = x_{1,\delta} + ix_{2,\delta} \to 0$. Then either

$$z_\delta = 0 \quad \text{or} \quad z_\delta^{\widehat{k}} = \delta b_1/c_1 + o(\delta) \quad \text{as } \delta \to 0. \tag{2.11}$$

Therefore $\frac{\partial^2 \Phi_\delta}{\partial z^2}(z_\delta) \neq 0$ for all sufficiently small δ. □

The following proposition was proven in [12]:

Proposition 2.3. *Let Φ satisfy (2.1) and (2.2). Let $\widetilde{f} \in L^2(\Omega)$, and $\widetilde{v} \in H^1(\Omega)$ be a solution to*

$$2\frac{\partial}{\partial z}\widetilde{v} - \tau\frac{\partial \Phi}{\partial z}\widetilde{v} = \widetilde{f} \quad \text{in } \Omega \tag{2.12}$$

or $\widetilde{v}$ be a solution to

$$2\frac{\partial}{\partial \overline{z}}\widetilde{v} - \tau\frac{\partial \overline{\Phi}}{\partial \overline{z}}\widetilde{v} = \widetilde{f} \quad \text{in } \Omega. \tag{2.13}$$

In the case (2.12) we have

$$\begin{aligned} &\|\frac{\partial}{\partial x_1}(e^{-i\tau\psi}\widetilde{v})\|^2_{L^2(\Omega)} - \tau\int_{\partial\Omega}(\nabla\varphi, \nu)|\widetilde{v}|^2 d\sigma \\ &+Re\int_{\partial\Omega} i\left(\left(\nu_2\frac{\partial}{\partial x_1} - \nu_1\frac{\partial}{\partial x_2}\right)\widetilde{v}\right)\overline{\widetilde{v}}d\sigma + \|\frac{\partial}{\partial x_2}(e^{-i\tau\psi}\widetilde{v})\|^2_{L^2(\Omega)} = \|\widetilde{f}\|^2_{L^2(\Omega)}. \end{aligned} \tag{2.14}$$

In the case (2.13) we have

$$\begin{aligned} \|\frac{\partial}{\partial x_1}(e^{i\tau\psi}\widetilde{v})\|_{L^2(\Omega)} - \tau\int_{\partial\Omega}(\nabla\varphi, \nu)|\widetilde{v}|^2 d\sigma + Re\int_{\partial\Omega} i\left(\left(-\nu_2\frac{\partial}{\partial x_1} + \nu_1\frac{\partial}{\partial x_2}\right)\widetilde{v}\right)\overline{\widetilde{v}}d\sigma \\ +\|\frac{\partial}{\partial x_2}(e^{i\tau\psi}\widetilde{v})\|^2_{L^2(\Omega)} = \|\widetilde{f}\|^2_{L^2(\Omega)}. \end{aligned} \tag{2.15}$$

Let $\alpha \in (0,1)$ and $\mathcal{A}, \mathcal{B} \in C^{6+\alpha}(\overline{\Omega})$ be two complex-valued solutions to the boundary value problem:

$$2\frac{\partial \mathcal{A}}{\partial \overline{z}} = -A \quad \text{in } \Omega,\ \text{Im}\,\mathcal{A}|_{\Gamma_0} = 0, \quad 2\frac{\partial \mathcal{B}}{\partial z} = -B \quad \text{in } \Omega,\ \text{Im}\,\mathcal{B}|_{\Gamma_0} = 0. \tag{2.16}$$

Consider the boundary value problem

$$\begin{cases} \mathcal{K}(x,D)u = (4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} + 2A\frac{\partial}{\partial z} + 2B\frac{\partial}{\partial \overline{z}})u = f \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

For this problem we have the following Carleman estimate with boundary terms.

Proposition 2.4. *Suppose that Φ satisfies (2.1), (2.2), $u \in H_0^1(\Omega)$ and the coefficients $A, B \in \{D \in C^1(\overline{\Omega}) | \|D\|_{C^1(\overline{\Omega})} \leq K\}$. Then there exist $\tau_0 = \tau_0(K, \Phi)$ and $C = C(K, \Phi)$ independent of u and τ such that for all $|\tau| > \tau_0$:*

$$
\begin{aligned}
&|\tau|\|ue^{\tau\varphi}\|^2_{L^2(\Omega)}+\|ue^{\tau\varphi}\|^2_{H^1(\Omega)}+\|\frac{\partial u}{\partial\nu}e^{\tau\varphi}\|^2_{L^2(\Gamma_0)}+\tau^2\||\frac{\partial\Phi}{\partial z}|ue^{\tau\varphi}\|^2_{L^2(\Omega)}\\
&\le C_1(\|(\mathcal{K}(x,D)u)e^{\tau\varphi}\|^2_{L^2(\Omega)}+|\tau|\int_{\widetilde\Gamma}|\frac{\partial u}{\partial\nu}|^2e^{2\tau\varphi}d\sigma). \qquad (2.17)
\end{aligned}
$$

Proof. Denote $\widetilde v=ue^{\tau\varphi},\mathcal{K}(x,D)u=f$. Observe that $\varphi(x_1,x_2)=\frac12(\Phi(z)+\overline{\Phi(z)})$. Therefore

$$
\begin{aligned}
&e^{\tau\varphi}\mathcal{K}(x,D)(e^{-\tau\varphi}\widetilde v)=(2\frac{\partial}{\partial z}-(\tau\frac{\partial\Phi}{\partial z}-B))(2\frac{\partial}{\partial\overline z}-(\tau\frac{\partial\overline\Phi}{\partial\overline z}-A))\widetilde v+(-2\frac{\partial A}{\partial z}-AB)\widetilde v=\\
&(2\frac{\partial}{\partial\overline z}-(\tau\frac{\partial\overline\Phi}{\partial\overline z}-A))(2\frac{\partial}{\partial z}-(\tau\frac{\partial\Phi}{\partial z}-B))\widetilde v+(-2\frac{\partial B}{\partial\overline z}-AB)\widetilde v=fe^{\tau\varphi}. \qquad (2.18)
\end{aligned}
$$

Denote $\widetilde w_1=\overline{\widetilde Q(z)}(2\frac{\partial}{\partial\overline z}-\tau\frac{\partial\overline\Phi}{\partial\overline z}+A)\widetilde v,\widetilde w_2=Q(z)(2\frac{\partial}{\partial z}-\tau\frac{\partial\Phi}{\partial z}+B)\widetilde v$, where $Q(z),\widetilde Q(z)\in C^2(\overline\Omega)$ are some holomorphic functions in Ω and will be specified below. Thanks to the zero Dirichlet boundary condition for u we have

$$
\widetilde w_1|_{\partial\Omega}=2\overline{\widetilde Q(z)}\frac{\partial\widetilde v}{\partial\overline z}|_{\partial\Omega}=(\nu_1+i\nu_2)\overline{\widetilde Q(z)}\frac{\partial\widetilde v}{\partial\nu}|_{\partial\Omega},\ \widetilde w_2|_{\partial\Omega}=2Q(z)\frac{\partial\widetilde v}{\partial z}|_{\partial\Omega}=(\nu_1-i\nu_2)Q(z)\frac{\partial\widetilde v}{\partial\nu}|_{\partial\Omega}.
$$

By Proposition 2.3 we have the following integral equalities:

$$
\begin{aligned}
&\|(\frac{\partial}{\partial x_1}-i\tau\frac{\partial\psi}{\partial x_1})(\widetilde w_1e^{\mathcal{B}})\|^2_{L^2(\Omega)}-\tau\int_{\partial\Omega}(\nabla\varphi,\nu)|\widetilde Q|^2|\frac{\partial\widetilde v}{\partial\nu}e^{\mathcal{B}}|^2d\sigma \qquad (2.19)\\
&+\mathrm{Re}\int_{\partial\Omega}i((\nu_2\frac{\partial}{\partial x_1}-\nu_1\frac{\partial}{\partial x_2})(\widetilde w_1e^{\mathcal{B}}))\overline{\widetilde w_1e^{\mathcal{B}}}d\sigma+\\
&+\|(\frac{\partial}{\partial x_2}-i\tau\frac{\partial\psi}{\partial x_2})(\widetilde w_1e^{\mathcal{B}})\|^2_{L^2(\Omega)}=\|\widetilde Q(fe^{\tau\varphi}+(2\frac{\partial A}{\partial z}+AB)\widetilde v)e^{\mathcal{B}}\|^2_{L^2(\Omega)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\|(\frac{\partial}{\partial x_1}+i\tau\frac{\partial\psi}{\partial x_1})(\widetilde w_2e^{\mathcal{A}})\|^2_{L^2(\Omega)}-\tau\int_{\partial\Omega}(\nabla\varphi,\nu)|Q|^2|\frac{\partial\widetilde v}{\partial\nu}e^{\mathcal{A}}|^2d\sigma\\
&+\mathrm{Re}\int_{\partial\Omega}i((-\nu_2\frac{\partial}{\partial x_1}+\nu_1\frac{\partial}{\partial x_2})(\widetilde w_2e^{\mathcal{A}}))\overline{\widetilde w_2e^{\mathcal{A}}}d\sigma+ \qquad (2.20)\\
&+\|(\frac{\partial}{\partial x_2}+i\tau\frac{\partial\psi}{\partial x_2})(\widetilde w_2e^{\mathcal{A}})\|^2_{L^2(\Omega)}=\|Q(fe^{\tau\varphi}+(2\frac{\partial B}{\partial\overline z}+AB)\widetilde v)e^{\mathcal{A}}\|^2_{L^2(\Omega)}.
\end{aligned}
$$

We now simplify the integral $\mathrm{Re}\, i\int_{\partial\Omega}((\nu_2\frac{\partial}{\partial x_1}-\nu_1\frac{\partial}{\partial x_2})(\widetilde w_1e^{\mathcal{B}}))\overline{\widetilde w_1e^{\mathcal{B}}}d\sigma$. We recall that $\widetilde v=ue^{\tau\varphi}$ and $\widetilde w_1=\overline{\widetilde Q(z)}(\nu_1+i\nu_2)\frac{\partial\widetilde v}{\partial\nu}=\overline{\widetilde Q(z)}(\nu_1+i\nu_2)\frac{\partial u}{\partial\nu}e^{\tau\varphi}$. Denote $R+iP=\overline{\widetilde Q(z)}(\nu_1+i\nu_2)e^{\mathcal{B}}$.

Therefore

$$\begin{aligned} \text{Re}\int_{\partial\Omega} i((\nu_2\frac{\partial}{\partial x_1}-\nu_1\frac{\partial}{\partial x_2})(\widetilde{w}_1 e^{\mathcal{B}}))\overline{\widetilde{w}_1 e^{\mathcal{B}}}d\sigma = \\ \text{Re}\int_{\partial\Omega} i((\nu_2\frac{\partial}{\partial x_1}-\nu_1\frac{\partial}{\partial x_2})[(R+iP)\frac{\partial u}{\partial \nu}e^{\tau\varphi}])(R-iP)\frac{\partial u}{\partial\nu}e^{\tau\varphi}d\sigma = \\ \text{Re}\int_{\partial\Omega} i[(\nu_2\frac{\partial}{\partial x_1}-\nu_1\frac{\partial}{\partial x_2})(R+iP)]|\frac{\partial\widetilde{v}}{\partial\nu}|^2(R-iP)d\sigma + \\ \text{Re}\int_{\partial\Omega}\frac{i}{2}(R^2+P^2)(\nu_2\frac{\partial}{\partial x_1}-\nu_1\frac{\partial}{\partial x_2})|\frac{\partial\widetilde{v}}{\partial\nu}|^2d\sigma = \\ \int_{\partial\Omega}(\frac{\partial R}{\partial\vec{\tau}}P-\frac{\partial P}{\partial\vec{\tau}}R)|\frac{\partial\widetilde{v}}{\partial\nu}|^2d\sigma. \end{aligned} \tag{2.21}$$

Let us simplify the integral $\text{Re}\int_{\partial\Omega} i((-\nu_2\frac{\partial}{\partial x_1}+\nu_1\frac{\partial}{\partial x_2})(\widetilde{w}_2e^{\mathcal{A}}))\overline{\widetilde{w}_2e^{\mathcal{A}}}d\sigma$. We recall that $\widetilde{v} = ue^{\tau\varphi}$ and $\widetilde{w}_2 = (\nu_1-i\nu_2)Q(z)\frac{\partial\widetilde{v}}{\partial\nu} = (\nu_1-i\nu_2)Q(z)\frac{\partial u}{\partial\nu}e^{\tau\varphi}$. Denote $\tilde{R}+i\tilde{P} = Q(z)(\nu_1-i\nu_2)e^{\mathcal{A}}$. Thus

$$\begin{aligned} \text{Re}\int_{\partial\Omega} i((-\nu_2\frac{\partial}{\partial x_1}+\nu_1\frac{\partial}{\partial x_2})(\widetilde{w}_2 e^{\mathcal{A}}))\overline{\widetilde{w}_2 e^{\mathcal{A}}}d\sigma = \\ \text{Re}\int_{\partial\Omega} i((-\nu_2\frac{\partial}{\partial x_1}+\nu_1\frac{\partial}{\partial x_2})[(\widetilde{R}+i\widetilde{P})\frac{\partial u}{\partial \nu}e^{\tau\varphi}])(\widetilde{R}-i\widetilde{P})\frac{\partial u}{\partial\nu}e^{\tau\varphi}d\sigma = \\ \text{Re}\int_{\partial\Omega} i[(-\nu_2\frac{\partial}{\partial x_1}+\nu_1\frac{\partial}{\partial x_2})(\widetilde{R}+i\widetilde{P})]|\frac{\partial\widetilde{v}}{\partial\nu}|^2(\widetilde{R}-i\widetilde{P})d\sigma - \\ \text{Re}\int_{\partial\Omega}\frac{i}{2}(\widetilde{R}^2+\widetilde{P}^2)(\nu_2\frac{\partial}{\partial x_1}-\nu_1\frac{\partial}{\partial x_2})|\frac{\partial\widetilde{v}}{\partial\nu}|^2d\sigma = \\ \int_{\partial\Omega}(\frac{\partial \widetilde{R}}{\partial\vec{\tau}}\widetilde{P}-\frac{\partial \widetilde{P}}{\partial\vec{\tau}}\widetilde{R})|\frac{\partial\widetilde{v}}{\partial\nu}|^2d\sigma. \end{aligned} \tag{2.22}$$

Using the above formula we obtain

$$\begin{aligned} \|(\frac{\partial}{\partial x_1}+i\tau\frac{\partial\psi}{\partial x_1})(\widetilde{w}_2e^{\mathcal{A}})\|^2_{L^2(\Omega)}+\|(\frac{\partial}{\partial x_2}+i\tau\frac{\partial\psi}{\partial x_2})(\widetilde{w}_2e^{\mathcal{A}})\|^2_{L^2(\Omega)} \\ -\tau\int_{\partial\Omega}(\nu,\nabla\varphi)|\widetilde{Q}|^2|\frac{\partial\widetilde{v}}{\partial\nu}e^{\mathcal{B}}|^2d\sigma-\tau\int_{\partial\Omega}(\nu,\nabla\varphi)|Q|^2|\frac{\partial\widetilde{v}}{\partial\nu}e^{\mathcal{A}}|^2d\sigma \\ +\|(\frac{\partial}{\partial x_1}-i\tau\frac{\partial\psi}{\partial x_2})(\widetilde{w}_1e^{\mathcal{B}})\|^2_{L^2(\Omega)}+\|(\frac{\partial}{\partial x_2}-i\tau\frac{\partial\psi}{\partial x_2})(\widetilde{w}_1e^{\mathcal{B}})\|^2_{L^2(\Omega)} \\ +\int_{\partial\Omega}(\frac{\partial R}{\partial\vec{\tau}}P-\frac{\partial P}{\partial\vec{\tau}}R)|\frac{\partial\widetilde{v}}{\partial\nu}e^{\mathcal{B}}|^2d\sigma+\int_{\partial\Omega}(\frac{\partial \widetilde{R}}{\partial\vec{\tau}}\widetilde{P}-\frac{\partial \widetilde{P}}{\partial\vec{\tau}}\widetilde{R})|\frac{\partial\widetilde{v}}{\partial\nu}e^{\mathcal{A}}|^2d\sigma = \\ \|Q(fe^{\tau\varphi}+(2\frac{\partial A}{\partial z}+AB)\widetilde{v})e^{\mathcal{B}}\|^2_{L^2(\Omega)}+\|Q(fe^{\tau\varphi}+(2\frac{\partial B}{\partial\overline{z}}+AB)\widetilde{v})e^{\mathcal{A}}\|^2_{L^2(\Omega)}. \end{aligned} \tag{2.23}$$

We can rewrite (2.23) in the form

$$\|\frac{\partial}{\partial x_1}(e^{i\psi\tau}\widetilde{w}_2e^{\mathcal{A}})\|^2_{L^2(\Omega)}+\|\frac{\partial}{\partial x_2}(e^{i\psi\tau}\widetilde{w}_2e^{\mathcal{A}})\|^2_{L^2(\Omega)}$$
$$-\tau\int_{\partial\Omega}(\nu,\nabla\varphi)|\widetilde{Q}|^2|\frac{\partial\widetilde{v}}{\partial\nu}e^{\mathcal{B}}|^2d\sigma-\tau\int_{\partial\Omega}(\nu,\nabla\varphi)|Q|^2|\frac{\partial\widetilde{v}}{\partial\nu}e^{\mathcal{A}}|^2d\sigma$$
$$+\|\frac{\partial}{\partial x_1}(e^{-i\psi\tau}\widetilde{w}_1e^{\mathcal{B}})\|^2_{L^2(\Omega)}+\|\frac{\partial}{\partial x_2}(e^{-i\psi\tau}\widetilde{w}_1e^{\mathcal{B}})\|^2_{L^2(\Omega)}$$
$$+\int_{\partial\Omega}(\frac{\partial R}{\partial\vec{\tau}}P-\frac{\partial P}{\partial\vec{\tau}}R)|\frac{\partial\widetilde{v}}{\partial\nu}e^{\mathcal{B}}|^2d\sigma+\int_{\partial\Omega}(\frac{\partial\widetilde{R}}{\partial\vec{\tau}}\widetilde{P}-\frac{\partial\widetilde{P}}{\partial\vec{\tau}}\widetilde{R})|\frac{\partial\widetilde{v}}{\partial\nu}e^{\mathcal{A}}|^2d\sigma=$$
$$\|\widetilde{Q}(fe^{\tau\varphi}+(2\frac{\partial A}{\partial z}+AB)\widetilde{v})e^{\mathcal{B}}\|^2_{L^2(\Omega)}+\|Q(fe^{\tau\varphi}+(2\frac{\partial B}{\partial\overline{z}}+AB)\widetilde{v})e^{\mathcal{A}}\|^2_{L^2(\Omega)}. \tag{2.24}$$

Next we show that it is possible to make a choice of functions $\tilde{Q}$ and Q such that

$$(\frac{\partial R}{\partial\vec{\tau}}P-\frac{\partial P}{\partial\vec{\tau}}R)>0 \text{ on } \overline{\Gamma_0},\quad (\frac{\partial\widetilde{R}}{\partial\vec{\tau}}\widetilde{P}-\frac{\partial\widetilde{P}}{\partial\vec{\tau}}\widetilde{R})>0 \text{ on } \overline{\Gamma_0}. \tag{2.25}$$

Let γ_j be a contour from $\partial\Omega$. We parametrize the curve γ_j by a length parameter s starting from one fixed point on γ_j: $x(s):[0,\ell_j]\to\gamma_j$. Here by ℓ_j we denote the total length of γ_j: We note that $\frac{\partial R}{\partial\vec{\tau}}=\frac{d}{ds}R(x(s))$. If there exists a holomorphic function $\tilde{Q}$ such that $R(x(s))=\ell_j sin(s/\ell_j)$, $P(x(s))=\ell_j cos(s/\ell_j)$. Then

$$\frac{\partial R}{\partial\vec{\tau}}P-\frac{\partial P}{\partial\vec{\tau}}R=\ell_j \text{ on } \gamma_j\quad\forall j\in\{1,\dots,\mathcal{N}\}. \tag{2.26}$$

Taking into account that $R+iP=\overline{\widetilde{Q}(z)}(\nu_1+i\nu_2)e^{\mathcal{B}}$ we set

$$b_1=\mathrm{Re}\left\{\frac{\ell_j sin(s/\ell_j)-i\ell_j cos(s/\ell_j)}{(\nu_1-i\nu_2)}e^{-\mathcal{B}}\right\},\quad b_2=\mathrm{Im}\left\{\frac{\ell_j sin(s/\ell_j)-i\ell_j cos(s/\ell_j)}{(\nu_1-i\nu_2)}e^{-\mathcal{B}}\right\}. \tag{2.27}$$

Using Proposition 5.1 in [12] we choose $\widetilde{Q}(z)$ such that on $\overline{\Gamma_0}$ the function $\tilde{Q}(z)$ is close to b_1+ib_2 in the norm of the space $C^1(\overline{\Gamma_0})$. Then by (2.27) we have the first inequality in (2.25). The choice of the function Q may be done in the similar way.

By (2.25) we obtain from (2.24) that

$$\|\frac{\partial\tilde{v}}{\partial\nu}\|_{L^2(\Gamma_0)}\le C_2(\|fe^{\tau\varphi}\|_{L^2(\Omega)}+\|\tilde{v}\|_{L^2(\Omega)}). \tag{2.28}$$

From now on we assume that $\tilde{Q}=Q=1$. Observe that there exists a positive constant C_3, independent of τ, such that

$$\frac{1}{C_3}(\|\widetilde{w}_1\|^2_{L^2(\Omega)}+\|\widetilde{w}_2\|^2_{L^2(\Omega)})\le\frac{1}{2}\|\frac{\partial}{\partial x_1}(e^{i\tau\psi}\widetilde{w}_2e^{\mathcal{A}})\|^2_{L^2(\Omega)}+\frac{1}{2}\|\frac{\partial}{\partial x_2}(e^{i\tau\psi}\widetilde{w}_2e^{\mathcal{A}})\|^2_{L^2(\Omega)}$$
$$-\tau\int_{\partial\Omega_-}(\nu,\nabla\varphi)|\widetilde{Q}|^2|\frac{\partial\widetilde{v}}{\partial\nu}|^2|e^{\mathcal{B}}|^2d\sigma-\tau\int_{\partial\Omega_-}(\nu,\nabla\varphi)|Q|^2|\frac{\partial\widetilde{v}}{\partial\nu}|^2|e^{\mathcal{A}}|^2d\sigma$$
$$+\frac{1}{2}\|\frac{\partial}{\partial x_1}(e^{-i\tau\psi}\widetilde{w}_1e^{\mathcal{B}})\|^2_{L^2(\Omega)}+\frac{1}{2}\|\frac{\partial}{\partial x_2}(e^{-i\tau\psi}\widetilde{w}_1e^{\mathcal{B}})\|^2_{L^2(\Omega)}. \tag{2.29}$$

Here $\partial\Omega_- = \{x \in \partial\Omega | (\nu, \nabla\varphi) < 0\}$. (Since $\int_{\partial\Omega} \frac{\partial\varphi}{\partial\nu} d\sigma = 0 = \int_{\tilde\Gamma} \frac{\partial\varphi}{\partial\nu} d\sigma$ the domain $\partial\Omega_-$ is not empty.) Observe that function $\bar u$ satisfy the equation $\overline{\mathcal{K}(x,D)\overline{u}} = \overline{f}$. Denoting $w_1^* = \overline{\tilde Q(z)}(2\frac{\partial}{\partial\bar z} - \tau\frac{\partial\bar\Phi}{\partial\bar z} + \overline{B})\overline{\tilde v}$, $w_2^* = \tilde Q(z)(2\frac{\partial}{\partial z} - \tau\frac{\partial\Phi}{\partial z} + \overline{A})\overline{\tilde v}$ and repeating the above arguments we have the analog of (2.29):

$$\frac{1}{C_4}(\|w_1^*\|^2_{L^2(\Omega)} + \|w_2^*\|^2_{L^2(\Omega)}) \le \frac{1}{2}\|\frac{\partial}{\partial x_1}(e^{i\tau\psi}w_2^* e^{\overline{\mathcal{B}}})\|^2_{L^2(\Omega)} + \frac{1}{2}\|\frac{\partial}{\partial x_2}(e^{i\tau\psi}w_2^* e^{\overline{\mathcal{B}}})\|^2_{L^2(\Omega)}$$
$$-\tau\int_{\partial\Omega_-}(\nu,\nabla\varphi)|\widetilde{Q}|^2|\frac{\partial\overline{\widetilde{v}}}{\partial\nu}|^2|e^{\overline{\mathcal{A}}}|^2 d\sigma - \tau\int_{\partial\Omega_-}(\nu,\nabla\varphi)|Q|^2|\frac{\partial\overline{\widetilde{v}}}{\partial\nu}|^2|e^{\overline{\mathcal{B}}}|^2 d\sigma$$
$$+\frac{1}{2}\|\frac{\partial}{\partial x_1}(e^{-i\tau\psi}w_1^* e^{\overline{\mathcal{A}}})\|^2_{L^2(\Omega)} + \frac{1}{2}\|\frac{\partial}{\partial x_2}(e^{-i\tau\psi}w_1^* e^{\mathcal{A}})\|^2_{L^2(\Omega)}. \tag{2.30}$$

By (2.29), (2.30) and the definitions of $\widetilde{w}_1, w_1^*$ and $\widetilde{w}_2, w_2^*$ we have

$$\|\frac{\partial \operatorname{Re}\widetilde{v}}{\partial z} - \tau\frac{\partial\Phi}{\partial z}\operatorname{Re}\widetilde{v}\|^2_{L^2(\Omega)} + \|2\frac{\partial \operatorname{Im}\widetilde{v}}{\partial z} - \tau\frac{\partial\Phi}{\partial z}\operatorname{Im}\widetilde{v}\|^2_{L^2(\Omega)}$$
$$\le C_5(\|\widetilde{w}_1\|^2_{L^2(\Omega)} + \|\widetilde{w}_2\|^2_{L^2(\Omega)} + \|\widetilde{v}\|^2_{L^2(\Omega)}).$$

Therefore

$$\|\nabla\widetilde{v}\|^2_{L^2(\Omega)} + \tau^2\|\frac{\partial\Phi}{\partial z}\widetilde{v}\|^2_{L^2(\Omega)} \tag{2.31}$$
$$\le C_6(\|\widetilde{w}_1\|^2_{L^2(\Omega)} + \|\widetilde{w}_2\|^2_{L^2(\Omega)} + \|\widetilde{v}\|^2_{L^2(\Omega)}).$$

Now since by assumption (2.2) the function Φ has zeros of at most second order, there exists a constant $C_7 > 0$ independent of τ such that

$$\tau\|\widetilde{v}\|^2_{L^2(\Omega)} \le C_7(\|\widetilde{v}\|^2_{H^1(\Omega)} + \tau^2\||\frac{\partial\Phi}{\partial z}|\widetilde{v}\|^2_{L^2(\Omega)}). \tag{2.32}$$

By (2.31) and (2.32)

$$\tau\|\widetilde{v}\|^2_{L^2(\Omega)} + \|\widetilde{v}\|^2_{H^1(\Omega)} + \tau^2\||\frac{\partial\Phi}{\partial z}|\widetilde{v}\|^2_{L^2(\Omega)} \le C_8(\|\widetilde{w}_1\|^2_{L^2(\Omega)} + \|\widetilde{w}_2\|^2_{L^2(\Omega)} + \|\widetilde{v}\|^2_{L^2(\Omega)}). \tag{2.33}$$

By (2.33) and (2.28) we obtain from (2.24), (2.29)

$$\frac{1}{C_9}(\tau\|\widetilde{v}\|^2_{L^2(\Omega)} + \|\widetilde{v}\|^2_{H^1(\Omega)} + \tau^2\||\frac{\partial\Phi}{\partial z}|\widetilde{v}\|^2_{L^2(\Omega)})$$
$$-\tau\int_{\partial\Omega}(\nu,\nabla\varphi)|\frac{\partial\widetilde{v}}{\partial\nu}|^2|e^{\mathcal{B}}|^2 d\sigma - \tau\int_{\partial\Omega}(\nu,\nabla\varphi)|\frac{\partial\widetilde{v}}{\partial\nu}|^2|e^{\mathcal{A}}|^2 d\sigma$$
$$\le \|fe^{s\varphi}\|^2_{L^2(\Omega)} + \tau\int_{\widetilde\Gamma}|(\nu,\nabla\varphi)||\frac{\partial\widetilde{v}}{\partial\nu}|^2(|e^{\mathcal{B}}|^2 + |e^{\mathcal{A}}|^2)d\sigma. \tag{2.34}$$

This concludes the proof of the proposition. □

As a corollary we derive a Carleman inequality the function u which satisfies the integral equality

$$(u, \mathcal{K}(x,D)^* w)_{L^2(\Omega)} + (f, w)_{H^{1,\tau}(\Omega)} + (ge^{\tau\varphi}, e^{-\tau\varphi}w)_{H^{\frac{1}{2},\tau}(\widetilde\Gamma)} = 0 \tag{2.35}$$

which holds true for all $w \in \mathcal{X} = \{w \in H^1(\Omega) | w|_{\Gamma_0} = 0, \mathcal{K}(x,D)^* w \in L^2(\Omega)\}$. We have

Corollary 2.1. *Suppose that Φ satisfies (2.1), (2.2), $f \in H^1(\Omega), g \in H^{\frac12}(\widetilde\Gamma)$, $u \in L^2(\Omega)$ and the coefficients $A, B \in \{C \in C^1(\overline{\Omega}) \vert \Vert C\Vert_{C^1(\overline{\Omega})} \le K\}$. Then there exist $\tau_0 = \tau_0(K,\Phi)$ and $C = C(K,\Phi)$, independent of u and τ, such that for solutions of (2.35):*

$$\Vert ue^{\tau\varphi}\Vert^2_{L^2(\Omega)} \le C_{10}\vert\tau\vert(\Vert fe^{\tau\varphi}\Vert^2_{H^{1,\tau}(\Omega)} + \Vert ge^{\tau\varphi}\Vert^2_{H^{\frac12,\tau}(\widetilde\Gamma)}) \quad \forall \vert\tau\vert \ge \tau_0. \tag{2.36}$$

Proof. Let ϵ be some positive number. Consider the extremal problem

$$J_\epsilon(w) = \frac12\Vert we^{-\tau\varphi}\Vert^2_{L^2(\Omega)} + \frac{1}{2\epsilon}\Vert \mathcal{K}(x,D)^*w - ue^{2\tau\varphi}\Vert^2_{L^2(\Omega)} + \frac{1}{2\vert\tau\vert}\Vert we^{-\tau\varphi}\Vert^2_{L^2(\widetilde\Gamma)} \to \inf, \tag{2.37}$$

$$w \in \hat{\mathcal{X}} = \{w \in H^{\frac12}(\Omega) \vert \mathcal{K}(x,D)^*w \in L^2(\Omega), w\vert_{\Gamma_0} = 0\}. \tag{2.38}$$

There exists a unique solution to (2.37), (2.38) which we denote by $\widehat{w}_\epsilon$. By Fermat's theorem

$$J'_\epsilon(\widehat{w}_\epsilon)[\delta] = 0 \quad \forall \delta \in \hat{\mathcal{X}}.$$

Using the notation $p_\epsilon = \frac{1}{\epsilon}(\mathcal{K}(x,D)^*\widehat{w}_\epsilon - ue^{2\tau\varphi})$ this implies

$$\mathcal{K}(x,D)p_\epsilon + \widehat{w}_\epsilon e^{-2\tau\varphi} = 0 \quad \text{in } \Omega, \quad p_\epsilon\vert_{\partial\Omega} = 0, \quad \frac{\partial p_\epsilon}{\partial\nu}\vert_{\widetilde\Gamma} = \frac{\widehat{w}_\epsilon}{\vert\tau\vert}e^{-2\tau\varphi}. \tag{2.39}$$

By Proposition 2.4 we have

$$\begin{aligned}\vert\tau\vert\Vert p_\epsilon e^{\tau\varphi}\Vert^2_{L^2(\Omega)} + \Vert p_\epsilon e^{\tau\varphi}\Vert^2_{H^1(\Omega)} + \Vert\frac{\partial p_\epsilon}{\partial\nu}e^{\tau\varphi}\Vert^2_{L^2(\Gamma_0)} + \tau^2\Vert\vert\frac{\partial\Phi}{\partial z}\vert p_\epsilon e^{\tau\varphi}\Vert^2_{L^2(\Omega)}\\ \le C_{11}(\Vert\widehat{w}_\epsilon e^{-\tau\varphi}\Vert^2_{L^2(\Omega)} + \frac{1}{\vert\tau\vert}\int_{\widetilde\Gamma}\vert\widehat{w}_\epsilon\vert^2 e^{-2\tau\varphi}d\sigma) \le 2C_{11}J_\epsilon(\widehat{w}_\epsilon).\end{aligned} \tag{2.40}$$

Taking the scalar product of equation (2.39) with $\widehat{w}_\epsilon$ we obtain

$$2J_\epsilon(\widehat{w}_\epsilon) + (ue^{2\tau\varphi}, p_\epsilon)_{L^2(\Omega)} = 0.$$

Applying to the second term of the above equality estimate (2.40) we have

$$\vert\tau\vert J_\epsilon(\widehat{w}_\epsilon) \le C_{12}\Vert ue^{\tau\varphi}\Vert^2_{L^2(\Omega)}.$$

Using this estimate we pass to the limit in (2.39) as ϵ goes to zero:

$$\mathcal{K}(x,D)p + \widehat{w}e^{-2\tau\varphi} = 0 \quad \text{in } \Omega, \quad p\vert_{\partial\Omega} = 0, \quad \frac{\partial p}{\partial\nu}\vert_{\widetilde\Gamma} = \frac{\widehat{w}}{\vert\tau\vert}e^{-2\tau\varphi}, \tag{2.41}$$

$$\mathcal{K}(x,D)^*\widehat{w} - ue^{2\tau\varphi} = 0 \quad \text{in } \Omega, \quad \widehat{w}\vert_{\Gamma_0} = 0, \tag{2.42}$$

and

$$\vert\tau\vert\Vert\widehat{w}e^{-\tau\varphi}\Vert^2_{L^2(\Omega)} + \Vert\widehat{w}e^{-\tau\varphi}\Vert^2_{L^2(\widetilde\Gamma)} \le C_{13}\Vert ue^{\tau\varphi}\Vert^2_{L^2(\Omega)}. \tag{2.43}$$

Since $\widehat{w} \in L^2(\Omega)$ we have $p \in H^2(\Omega)$, $\frac{\partial p}{\partial\nu} \in H^{\frac32}(\partial\Omega)$ and therefore $\widehat{w} \in H^{\frac12}(\partial\Omega)$. By (2.40)-(2.43) we obtain:

$$\Vert\widehat{w}e^{-\tau\varphi}\Vert_{H^{\frac12,\tau}(\partial\Omega)} \le C_{14}\vert\tau\vert^{\frac12}\Vert ue^{\tau\varphi}\Vert_{L^2(\Omega)}. \tag{2.44}$$

Taking the scalar product of (2.42) with function $\hat{w}e^{-2\tau\varphi}$ and using the estimates (2.44), (2.43) we obtain

$$\frac{1}{|\tau|}\|\nabla\widehat{w}e^{-\tau\varphi}\|^2_{L^2(\Omega)}+|\tau|\|\widehat{w}e^{-\tau\varphi}\|^2_{L^2(\Omega)}+\frac{1}{|\tau|}\|\widehat{w}e^{-\tau\varphi}\|^2_{H^{\frac12,\tau}(\tilde\Gamma)}\le C_{15}\|ue^{\tau\varphi}\|^2_{L^2(\Omega)}. \tag{2.45}$$

From this estimate and a standard duality argument, the statement of Corollary 2.1 follows immediately. □

Consider the following boundary value problem

$$\frac{\partial a}{\partial\bar z}=0\quad\text{in}\,\Omega,\quad \frac{\partial d}{\partial z}=0\quad\text{in}\,\Omega,\quad (a(z)e^{\mathcal{A}}+d(\bar z)e^{\mathcal{B}})|_{\Gamma_0}=\beta. \tag{2.46}$$

The existence of such functions $a(z)$ and $d(\bar z)$ is given by the following proposition:

Proposition 2.5. *Let $\alpha\in(0,1)$, functions $\mathcal{A}$ and $\mathcal{B}$ be as in (2.16). If $\beta\in C^{5+\alpha}(\overline{\Gamma}_0)$ the problem (2.46) has at least one solution $(a,d)\in C^{5+\alpha}(\overline{\Omega})\times C^{5+\alpha}(\overline{\Omega})$ such that*

$$\|(a,d)\|_{C^{5+\alpha}(\overline{\Omega})\times C^{5+\alpha}(\overline{\Omega})}\le C_{16}\|\beta\|_{C^{5+\alpha}(\overline{\Gamma}_0)}. \tag{2.47}$$

If $\beta\in H^{\frac12}(\Gamma_0)$, then the problem (2.46) has at least one solution $(a,d)\in H^1(\Omega)\times H^1(\Omega)$ such that

$$\|(a,d)\|_{H^1(\Omega)\times H^1(\Omega)}\le C_{18}\|\beta\|_{H^{\frac12}(\Gamma_0)}. \tag{2.48}$$

Proof. Let $\tilde\Omega$ be a domain in $\mathbb{R}^2$ with smooth boundary such that $\Omega\subset\tilde\Omega$ and there exits an open subdomain $\tilde\Gamma_0\subset\partial\tilde\Omega$ satisfying $\overline{\Gamma}_0\subset\tilde\Gamma_0$. Denote $\Gamma^*=\partial\tilde\Omega\setminus\tilde\Gamma_0$. We extend functions $\mathcal{A},\mathcal{B}$ on $\tilde\Gamma_0$ keeping the regularity and we extend function β on $\tilde\Gamma_0$ in such a way that $\|\beta\|_{H^{\frac12}(\tilde\Gamma_0)}\le C_{19}\|\beta\|_{H^{\frac12}(\Gamma_0)}$ or $\|\beta\|_{C^{5+\alpha}(\overline{\tilde\Gamma_0})}\le C_{19}\|\beta\|_{C^{5+\alpha}(\overline{\Gamma_0})}$ where the constant C_{19} is independent of β. By the trace theorem there exist a constant C_{20} independent of β, and a pair $(r,\tilde r)$ such that $(re^{\mathcal{A}}+\tilde re^{\mathcal{B}})|_{\tilde\Gamma_0}=\beta$ and if $\beta\in H^{\frac12}(\tilde\Gamma_0)$ then $(r,\tilde r)\in H^1(\Omega)\times H^1(\Omega)$ and

$$\|(r,\tilde r)\|_{H^1(\Omega)\times H^1(\Omega)}\le C_{20}\|\beta\|_{H^{\frac12}(\Gamma_0)}$$

and if $\beta\in C^{5+\alpha}(\tilde\Gamma_0)$ then $(r,\tilde r)\in C^{5+\alpha}(\overline{\Omega})\times C^{5+\alpha}(\overline{\Omega})$ and

$$\|(r,\tilde r)\|_{C^{5+\alpha}(\overline{\Omega})\times C^{5+\alpha}(\overline{\Omega})}\le C_{21}\|\beta\|_{C^{5+\alpha}(\Gamma_0)}.$$

Let $f=\frac{\partial r}{\partial z}$ and $\tilde f=\frac{\partial\tilde r}{\partial\bar z}$. Consider the extremal problem

$$J_\epsilon(p,\tilde p)=\|(p,\tilde p)\|^2_{L^2(\tilde\Omega)}+\frac{1}{\epsilon}\|\frac{\partial p}{\partial z}-f\|^2_{L^2(\tilde\Omega)}+\frac{1}{\epsilon}\|\frac{\partial\tilde p}{\partial\bar z}-\tilde f\|^2_{L^2(\tilde\Omega)}\to\inf,\quad (p,\tilde p)\in\mathcal{K},$$

where $\mathcal{K}=\{(h_1,h_2)\in L^2(\tilde\Omega)\times L^2(\tilde\Omega)|(h_1e^{\mathcal{A}}+h_2e^{\mathcal{B}})|_{\tilde\Gamma_0}=0\}$. Denote the solution to this extremal problem as $(p_\epsilon,\tilde p_\epsilon)$. Then

$$J'_\epsilon(p_\epsilon,\tilde p_\epsilon)(\delta,\tilde\delta)=0\quad\forall(\delta,\tilde\delta)\in\mathcal{K}.$$

Hence

$$((p_\epsilon,\tilde p_\epsilon),(\delta,\tilde\delta))_{L^2(\tilde\Omega)}+\frac{1}{\epsilon}(\frac{\partial p_\epsilon}{\partial z}-f,\frac{\partial\delta}{\partial z})_{L^2(\tilde\Omega)}+\frac{1}{\epsilon}(\frac{\partial\tilde p_\epsilon}{\partial\bar z}-\tilde f,\frac{\partial\tilde\delta}{\partial\bar z})_{L^2(\tilde\Omega)}=0\quad\forall(\delta,\tilde\delta)\in\mathcal{K}. \tag{2.49}$$

Denote $P_\epsilon = -\frac{1}{\epsilon}(\frac{\partial p_\epsilon}{\partial z} - f), \tilde P_\epsilon = -\frac{1}{\epsilon}(\frac{\partial \tilde p_\epsilon}{\partial \bar z} - \tilde f)$. Form (2.49) we obtain

$$(2.50)\quad \frac{\partial P_\epsilon}{\partial \bar z} = p_\epsilon, \quad \frac{\partial \tilde P_\epsilon}{\partial z} = \tilde p_\epsilon, \quad P_\epsilon|_{\Gamma^*} = \tilde P_\epsilon|_{\Gamma^*} = 0, ((\nu_1 + i\nu_2)P_\epsilon e^{\mathcal{B}} - (\nu_1 - i\nu_2)\tilde P_\epsilon e^{\mathcal{A}})|_{\tilde\Gamma_0} = 0.$$

We claim that there exists a constant C_{22} independent of ϵ such that

$$(2.51)\qquad \|(P_\epsilon, \tilde P_\epsilon)\|_{H^1(\tilde\Omega)} \le C_{22}(\|(p_\epsilon, \tilde p_\epsilon)\|_{L^2(\tilde\Omega)} + \|(P_\epsilon, \tilde P_\epsilon)\|_{L^2(\tilde\Omega)}).$$

It clearly suffices to prove the estimate (2.51) locally assuming that $\mathrm{supp}\,(p_\epsilon, \tilde p_\epsilon)$ is in the small neighborhood of zero and the vector $(0,1)$ is orthogonal to $\partial\Omega$ on the intersection of this neighborhood with boundary. Using the conformal transformation we may assume that $\partial\Omega \cap \mathrm{supp}\, P_\epsilon, \partial\Omega \cap \mathrm{supp}\, \tilde P_\epsilon \subset \{x_1 = 0\}$. In order to prove this fact we consider the system of equations

$$(2.52)\qquad \frac{\partial \mathbf{u}}{\partial x_2} + \hat B \frac{\partial \mathbf{u}}{\partial x_1} = \mathbf{F}, \quad \mathrm{supp}\,\mathbf{u} \subset B(0,\delta) \cap \{x | x_2 \ge 0\}.$$

Here $\mathbf{u} = (u_1, u_2, u_3, u_4) = (\mathrm{Re}\, P_\epsilon, \mathrm{Im}\, P_\epsilon, \mathrm{Re}\, \tilde P_\epsilon, \mathrm{Im}\, \tilde P_\epsilon)$, $\mathbf{F} = 2(\mathrm{Re}\, p_\epsilon, \mathrm{Im}\, p_\epsilon, \mathrm{Re}\, \tilde p_\epsilon, \mathrm{Im}\, \tilde p_\epsilon)$, $\hat B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. The matrix $\hat B$ has two eigenvalues $\pm i$ and four linearly independent eigenvectors:

$$\mathbf{q}_3 = (0,0,1,i), \mathbf{q}_4 = (1,-i,0,0) \quad \text{corresponding to eigenvalue } -i,$$

$$\mathbf{q}_1 = (1,i,0,0), \mathbf{q}_2 = (0,0,1,-i) \quad \text{corresponding to eigenvalue } i.$$

We set $\mathbf{r}_1 = (\nu_1 e^{\mathcal{B}}, -\nu_2 e^{\mathcal{B}}, -\nu_1 e^{\mathcal{A}}, -\nu_2 e^{\mathcal{A}})$, $\mathbf{r}_2 = (\nu_2 e^{\mathcal{B}}, \nu_1 e^{\mathcal{B}}, \nu_2 e^{\mathcal{A}}, -\nu_1 e^{\mathcal{A}})$. Consider the matrix $D = \{d_{j\ell}\}$ where $d_{j\ell} = r_j \cdot q_\ell$. We have

$$D = \begin{pmatrix} (\nu_1 - i\nu_2)e^{\mathcal{B}} & -(\nu_1 - i\nu_2)e^{\mathcal{A}} \\ (\nu_2 + i\nu_1)e^{\mathcal{B}} & (\nu_2 + i\nu_1)e^{\mathcal{A}} \end{pmatrix}.$$

Since the Lopatinski determinant $det\, D \ne 0$ we obtain (2.51) (see e.g. [21]).

Suppose that for any $\tilde C$ one can find ϵ such that the estimate

$$\|(P_\epsilon, \tilde P_\epsilon)\|_{H^1(\tilde\Omega)} \le \tilde C \|(p_\epsilon, \tilde p_\epsilon)\|_{L^2(\tilde\Omega)}$$

fails. That is, for all $\epsilon \in (0,1)$, there exist p_ϵ, $\tilde p_\epsilon$, P_ϵ, $\tilde P_\epsilon$, $C_\epsilon > 0$ such that $\lim_{\epsilon\to 0} C_\epsilon = \infty$ and

$$\left\| \frac{(p_\epsilon, \tilde p_\epsilon)}{\|(P_\epsilon, \tilde P_\epsilon)\|_{H^1(\tilde\Omega)}} \right\|_{L^2(\tilde\Omega)} < \frac{1}{C_\epsilon}.$$

We set $(Q_\epsilon, \tilde Q_\epsilon) = (P_\epsilon, \tilde P_\epsilon)/\|(P_\epsilon, \tilde P_\epsilon)\|_{H^1(\tilde\Omega)}$ and $(q_\epsilon, \tilde q_\epsilon) = (p_\epsilon, \tilde p_\epsilon)/\|(P_\epsilon, \tilde P_\epsilon)\|_{H^1(\tilde\Omega)}$. Then $\|(q_\epsilon, \tilde q_\epsilon)\|_{L^2(\tilde\Omega)} \to 0$ as $\epsilon \to 0$. Passing to the limit in (2.50),

$$\frac{\partial Q}{\partial \bar z} = 0 \quad \text{in}\, \tilde\Omega, \quad \frac{\partial \tilde Q}{\partial z} = 0 \quad \text{in}\, \tilde\Omega, \quad Q|_{\Gamma^*} = \tilde Q|_{\Gamma^*} = 0.$$

By the uniqueness of the Cauchy problem for the operator ∂_z we have $Q = \tilde Q = 0$. On the other hand, since $\|(Q_\epsilon, \tilde Q_\epsilon)\|_{H^1(\tilde\Omega)} = 1$, we can extract a subsequence, denoted by the same letters, which is convergent in $L^2(\tilde\Omega)$. Therefore the sequence $(Q_\epsilon,\tilde Q_\epsilon)$ converges to zero in $L^2(\tilde\Omega)$. By (2.51), we have $1/C_{23}\le \|(q_\epsilon,\tilde q_\epsilon)\|_{L^2(\tilde\Omega)}+\|(Q_\epsilon,\tilde Q_\epsilon)\|_{L^2(\tilde\Omega)}$. Therefore $\liminf_{\epsilon\to 0}\|(Q_\epsilon,\tilde Q_\epsilon)\|_{L^2(\tilde\Omega)}\ne 0$, and this is a contradiction. Hence

$$\|(P_\epsilon,\tilde P_\epsilon)\|_{H^1(\tilde\Omega)}\le C_{24}\|(p_\epsilon,\tilde p_\epsilon)\|_{L^2(\tilde\Omega)},\quad \forall\epsilon>0.$$

Let us plug in (2.49) the function $(p_\epsilon,\tilde p_\epsilon)$ instead of $(\delta,\tilde\delta)$. Then, by the above inequality, in view of the definitions of P_ϵ and $\tilde P_\epsilon$, we have

$$\|(p_\epsilon,\tilde p_\epsilon)\|^2_{L^2(\tilde\Omega)}\le C_{25}((f,\tilde f),(P_\epsilon,\tilde P_\epsilon))_{L^2(\tilde\Omega)}\le C_{26}\|(f,\tilde f)\|_{L^2(\tilde\Omega)}\|(P_\epsilon,\tilde P_\epsilon)\|_{L^2(\tilde\Omega)}$$
$$\le C_{27}\|(f,\tilde f)\|_{L^2(\tilde\Omega)}\|(p_\epsilon,\tilde p_\epsilon)\|_{L^2(\tilde\Omega)}.$$

This inequality implies that the sequence $(p_\epsilon,\tilde p_\epsilon)$ is bounded in $L^2(\tilde\Omega)$ and

$$(\frac{\partial p_\epsilon}{\partial z},\frac{\partial\tilde p_\epsilon}{\partial\bar z})\to(f,\tilde f)\quad\text{in } L^2(\tilde\Omega)\times L^2(\tilde\Omega).$$

Then we construct a solution to (2.46) such that

$$\|(p,\tilde p)\|_{L^2(\tilde\Omega)}\le C_{28}\|(f,\tilde f)\|_{L^2(\tilde\Omega)}. \tag{2.53}$$

Observe that we can write the boundary value problem

$$\frac{\partial p}{\partial z}=f\quad\text{in }\Omega,\quad \frac{\partial\tilde p}{\partial\bar z}=\tilde f\quad\text{in }\Omega,\quad (pe^{\mathcal A}+\tilde pe^{\mathcal B})|_{\tilde\Gamma_0}=0$$

in the form of (2.52) with $\mathbf u=(\mathrm{Re}\,p,\mathrm{Im}\,p,\mathrm{Re}\,\tilde p,\mathrm{Im}\,\tilde p),\mathbf F=2(\mathrm{Re}\,f,\mathrm{Im}\,f,\mathrm{Re}\,\tilde f,\mathrm{Im}\,\tilde f)$. We set $\mathbf r_1=(e^{\mathcal A},-e^{\mathcal A},-e^{\mathcal B},-e^{\mathcal B})$, $\mathbf r_2=(e^{\mathcal A},e^{\mathcal A},e^{\mathcal B},-e^{\mathcal B})$. Consider the matrix $D=\{d_{j\ell}\}$ where $d_{j\ell}=r_j\cdot q_\ell$. We have

$$D=\begin{pmatrix} e^{\mathcal B} & -e^{\mathcal A}\\ e^{\mathcal B} & e^{\mathcal A}\end{pmatrix}.$$

Since the Lopatinski determinant $det\,D\ne 0$ the estimate (2.53) imply (2.47) and (2.48) (see e.g., [21] Theorem 4.1.2.). This completes the proof of the proposition. □

Consider the following problem

$$L(x,D)u=fe^{\tau\varphi}\quad\text{in }\Omega,\quad u|_{\Gamma_0}=ge^{\tau\varphi}. \tag{2.54}$$

We have

Proposition 2.6. *Let $A,B\in C^{5+\alpha}(\overline\Omega),q\in L^\infty(\Omega)$ and ϵ,α be a small positive numbers. There exists $\tau_0>0$ such that for all $|\tau|>\tau_0$ there exists a solution to the boundary value problem (2.54) such that*

$$\frac{1}{\sqrt{|\tau|}}\|\nabla ue^{-\tau\varphi}\|_{L^2(\Omega)}+\sqrt{|\tau|}\|ue^{-\tau\varphi}\|_{L^2(\Omega)}\le C_{29}(\|f\|_{L^2(\Omega)}+\|g\|_{H^{\frac12,\tau}(\Gamma_0)}). \tag{2.55}$$

Let ϵ be a sufficiently small positive number. If $supp f\subset G_\epsilon=\{x\in\Omega|dist(x,\mathcal H\setminus\Gamma_0)>\epsilon\}$ and $g=0$ then there exists $\tau_0>0$ such that for all $|\tau|>\tau_0$ there exists a solution to the boundary value problem (2.54) such that

$$\|\nabla ue^{-\tau\varphi}\|_{L^2(\Omega)}+|\tau|\|ue^{-\tau\varphi}\|_{L^2(\Omega)}\le C_{30}(\epsilon)\|f\|_{L^2(\Omega)}. \tag{2.56}$$

Proof. First we reduce problem (2.54) to the case $g = 0$. Let $r(z)$ be a holomorphic function and $\widetilde{r}(\overline{z})$ be an antiholomorphic function such that $(e^{\mathcal{A}}r + e^{\mathcal{B}}\widetilde{r})|_{\Gamma_0} = g$ where $\mathcal{A}, \mathcal{B} \in C^{6+\alpha}(\overline{\Omega})$ be as in (2.16). The existence of such functions $r, \widetilde{r}$ follows from Proposition 2.5, and these functions can be chosen in such a way that

$$\|r\|_{H^1(\Omega)} + \|\widetilde{r}\|_{H^1(\Omega)} \le C_{31}\|g\|_{H^{\frac12}(\Gamma_0)}.$$

We look for solution u in the form

$$u = (e^{\mathcal{A}+\tau\Phi}r + e^{\mathcal{B}+\tau\overline{\Phi}}\widetilde{r}) + \widetilde{u},$$

where

$$L(x,D)\widetilde{u} = \widetilde{f}e^{\tau\varphi} \quad \text{in } \Omega, \quad \widetilde{u}|_{\Gamma_0} = 0 \tag{2.57}$$

and $\widetilde{f} = f - (q - 2\frac{\partial A}{\partial z} - AB)e^{\mathcal{A}}re^{i\tau\psi} - (q - 2\frac{\partial B}{\partial \overline{z}} - AB)e^{\mathcal{B}}\widetilde{r}e^{-i\tau\psi}$.

In order to prove (2.55) we consider the following extremal problem:

$$\widetilde{I}_\epsilon(u) = \frac12\|ue^{-\tau\varphi}\|^2_{H^{1,\tau}(\Omega)} + \frac{1}{2\epsilon}\|L(x,D)u - \tilde{f}e^{\tau\varphi}\|^2_{L^2(\Omega)} + \frac12\|ue^{-\tau\varphi}\|^2_{H^{\frac12,\tau}(\widetilde{\Gamma})} \to \inf, \tag{2.58}$$

$$u \in \mathcal{Y} = \{w \in H^1(\Omega)| w|_{\Gamma_0} = 0, L(x,D)w \in L^2(\Omega)\}. \tag{2.59}$$

There exists a unique solution to problem (2.58), (2.59) which we denote as $\widehat{u}_\epsilon$. By Fermat's theorem

$$\widetilde{I}'_\epsilon(\widehat{u}_\epsilon)[\delta] = 0 \quad \forall \delta \in \mathcal{Y}. \tag{2.60}$$

Let $p_\epsilon = \frac{1}{\epsilon}(L(x,D)\hat{u}_\epsilon - \tilde{f}e^{\tau\varphi})$. Applying Corollary 2.1 we obtain from (2.60)

$$\frac{1}{|\tau|}\|p_\epsilon e^{\tau\varphi}\|^2_{L^2(\Omega)} \le C_{32}(\|\widehat{u}_\epsilon e^{-\tau\varphi}\|^2_{H^{1,\tau}(\Omega)} + \|\widehat{u}_\epsilon e^{-\tau\varphi}\|^2_{H^{\frac12,\tau}(\widetilde{\Gamma})}) \le 2C_{32}\widetilde{I}_\epsilon(\widehat{u}_\epsilon). \tag{2.61}$$

Taking in (2.60) with $\delta = \widehat{u}_\epsilon$ we obtain

$$2\widetilde{I}_\epsilon(\widehat{u}_\epsilon) + (\tilde{f}e^{\tau\varphi}, p_\epsilon)_{L^2(\Omega)} = 0.$$

Applying to this equality estimate (2.61) we have

$$\widetilde{I}_\epsilon(\widehat{u}_\epsilon) \le C_\epsilon|\tau|\|\tilde{f}\|^2_{L^2(\Omega)}.$$

Using this estimate we pass to the limit as $\epsilon \to +0$. We obtain

$$L(x,D)u - \tilde{f}e^{\tau\varphi} = 0 \quad \text{in } \Omega, \quad u|_{\Gamma_0} = 0, \tag{2.62}$$

and

$$\|ue^{-\tau\varphi}\|^2_{H^{1,\tau}(\Omega)} + \|ue^{-\tau\varphi}\|^2_{L^2(\widetilde{\Gamma})} \le C_{33}|\tau|\|\tilde{f}\|^2_{L^2(\Omega)}. \tag{2.63}$$

Since $\|\tilde{f}\|_{L^2(\Omega)} \le C_{34}(\|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac12}(\Gamma_0)})$, inequality (2.63) implies (2.55). In order to prove (2.56) we consider the following extremal problem

$$\widetilde{J}_\epsilon(u) = \frac12\|ue^{-\tau\varphi}\|^2_{L^2(\Omega)} + \frac{1}{2\epsilon}\|L(x,D)u - fe^{\tau\varphi}\|^2_{L^2(\Omega)} + \frac{1}{2|\tau|}\|ue^{-\tau\varphi}\|^2_{L^2(\widetilde{\Gamma})} \to \inf, \tag{2.64}$$

$$u \in \widetilde{\mathcal{X}} = \{w \in H^{\frac12}(\Omega)| w|_{\Gamma_0} = 0, L(x,D)w \in L^2(\Omega)\}. \tag{2.65}$$

There exists a unique solution to problem (2.64), (2.65) which we denote as $\widehat{u}_\epsilon$. By Fermat's theorem

$$\widetilde{J}'_\epsilon(\widehat{u}_\epsilon)[\delta]=0 \quad \forall \delta\in\widetilde{\mathcal{X}}.$$

This equality implies

$$L(x,D)^*p_\epsilon+\widehat{u}_\epsilon e^{-2\tau\varphi}=0 \quad \text{in } \Omega, \quad \widehat{p}_\epsilon\vert_{\partial\Omega}=0, \quad \frac{\partial p_\epsilon}{\partial\nu}\vert_{\widetilde{\Gamma}}=\frac{\widehat{u}_\epsilon}{\vert\tau\vert}e^{-2\tau\varphi}. \tag{2.66}$$

By Proposition 2.4

$$\frac{1}{\vert\tau\vert}\Vert p_\epsilon e^{\tau\varphi}\Vert^2_{H^{1,\tau}(\Omega)}+\Vert\frac{\partial p_\epsilon}{\partial\nu}e^{\tau\varphi}\Vert^2_{L^2(\Gamma_0)}+\tau^2\Vert\vert\frac{\partial\Phi}{\partial z}\vert p_\epsilon e^{\tau\varphi}\Vert^2_{L^2(\Omega)}$$

$$\le C_{35}(\Vert\widehat{u}_\epsilon e^{-\tau\varphi}\Vert^2_{L^2(\Omega)}+\frac{1}{\vert\tau\vert}\int_{\widetilde{\Gamma}}\vert\widehat{u}_\epsilon\vert^2e^{-2\tau\varphi}d\sigma)\le C_{36}\widetilde{J}_\epsilon(\widehat{u}_\epsilon). \tag{2.67}$$

Taking the scalar product of equation (2.66) with $\widehat{u}_\epsilon$ we obtain

$$2\widetilde{J}_\epsilon(\widehat{u}_\epsilon)+(fe^{\tau\varphi},p_\epsilon)_{L^2(\Omega)}=0.$$

Applying to this equality estimate (2.67) we have

$$\vert\tau\vert^2\widetilde{J}_\epsilon(\widehat{u}_\epsilon)\le C_\epsilon\Vert f\Vert^2_{L^2(\Omega)}. \tag{2.68}$$

Using this estimate we pass to the limit in (2.66):

$$L(x,D)^*p+ue^{-2\tau\varphi}=0 \quad \text{in } \Omega, \quad p\vert_{\partial\Omega}=0, \quad \frac{\partial p}{\partial\nu}\vert_{\widetilde{\Gamma}}=\frac{u}{\vert\tau\vert}e^{-2\tau\varphi}, \tag{2.69}$$

$$L(x,D)u-fe^{-\tau\varphi}=0 \quad \text{in } \Omega, \quad u\vert_{\Gamma_0}=0. \tag{2.70}$$

Moreover (2.68) implies

$$\vert\tau\vert^2\Vert ue^{-\tau\varphi}\Vert^2_{L^2(\Omega)}+\Vert ue^{-\tau\varphi}\Vert^2_{L^2(\widetilde{\Gamma})}\le C_{37}\Vert f\Vert^2_{L^2(\Omega)}. \tag{2.71}$$

This finishes the proof of the proposition. □

3. Estimates and Asymptotics

In this section we prove some estimates and obtain asymptotic expansions needed in the construction of the complex geometrical optics solutions in Section 4.

Consider the operator

$$L_1(x,D)=4\frac{\partial}{\partial z}\frac{\partial}{\partial\overline{z}}+2A_1\frac{\partial}{\partial z}+2B_1\frac{\partial}{\partial\overline{z}}+q_1=$$

$$(2\frac{\partial}{\partial z}+B_1)(2\frac{\partial}{\partial\overline{z}}+A_1)+q_1-2\frac{\partial A_1}{\partial z}-A_1B_1=$$

$$(2\frac{\partial}{\partial\overline{z}}+A_1)(2\frac{\partial}{\partial z}+B_1)+q_1-2\frac{\partial B_1}{\partial\overline{z}}-A_1B_1. \tag{3.1}$$

Let $\mathcal{A}_1,\mathcal{B}_1,\mathcal{A}_2,\mathcal{B}_2\in C^{6+\alpha}(\overline{\Omega})$ with some $\alpha\in(0,1)$ satisfy

$$2\frac{\partial\mathcal{A}_1}{\partial\overline{z}}=-A_1 \quad \text{in } \Omega, \ \operatorname{Im}\mathcal{A}_1\vert_{\Gamma_0}=0, \quad 2\frac{\partial\mathcal{B}_1}{\partial z}=-B_1 \quad \text{in } \Omega, \ \operatorname{Im}\mathcal{B}_1\vert_{\Gamma_0}=0 \tag{3.2}$$

$$2\frac{\partial \mathcal{A}_2}{\partial z} = \overline{A_2} \quad \text{in } \Omega, \ \operatorname{Im} \mathcal{A}_2|_{\Gamma_0} = 0, \quad 2\frac{\partial \mathcal{B}_2}{\partial \overline{z}} = \overline{B}_2 \quad \text{in } \Omega, \ \operatorname{Im} \mathcal{B}_2|_{\Gamma_0} = 0.$$

Observe that

$$(2\frac{\partial}{\partial \overline{z}} + A_1)e^{\mathcal{A}_1} = 0 \quad \text{in } \Omega, \quad (2\frac{\partial}{\partial z} + B_1)e^{\mathcal{B}_1} = 0 \quad \text{in } \Omega.$$

Therefore if $a(z), \Phi(z)$ are holomorphic functions and $b(\overline{z})$ is an antiholomorphic function, we have

$$L_1(x,D)(e^{\mathcal{A}_1}a(z)e^{\tau\Phi}) = (q_1 - 2\frac{\partial A_1}{\partial z} - A_1B_1)e^{\mathcal{A}_1}ae^{\tau\Phi},$$

$$L_1(x,D)(e^{\mathcal{B}_1}b(\overline{z})e^{\tau\overline{\Phi}}) = (q_1 - 2\frac{\partial B_1}{\partial \overline{z}} - A_1B_1)e^{\mathcal{B}_1}be^{\tau\overline{\Phi}}.$$

Let us introduce the operators:

$$\partial_{\overline{z}}^{-1}g = \frac{1}{2\pi i}\int_\Omega \frac{g(\xi_1,\xi_2)}{\zeta - z}d\zeta\wedge d\overline{\zeta} = -\frac{1}{\pi}\int_\Omega \frac{g(\xi_1,\xi_2)}{\zeta - z}d\xi_1 d\xi_2,$$

$$\partial_{z}^{-1}g = -\frac{1}{2\pi i}\overline{\int_\Omega \frac{\overline{g}(\xi_1,\xi_2)}{\zeta - z}d\zeta\wedge d\overline{\zeta}} = -\frac{1}{\pi}\int_\Omega \frac{g(\xi_1,\xi_2)}{\overline{\zeta} - \overline{z}}d\xi_1 d\xi_2.$$

We have (e.g., p.47, 56, 72 in [20]):

Proposition 3.1. A) *Let $m \ge 0$ be an integer number and $\alpha \in (0,1)$. The operators $\partial_{\overline{z}}^{-1}, \partial_z^{-1} \in \mathcal{L}(C^{m+\alpha}(\overline{\Omega}), C^{m+\alpha+1}(\overline{\Omega}))$.*
B) *Let $1 \le p \le 2$ and $1 < \gamma < \frac{2p}{2-p}$. Then $\partial_{\overline{z}}^{-1}, \partial_z^{-1} \in \mathcal{L}(L^p(\Omega), L^\gamma(\Omega))$.*
C)*Let $1 < p < \infty$. Then $\partial_{\overline{z}}^{-1}, \partial_z^{-1} \in \mathcal{L}(L^p(\Omega), W^1_p(\Omega))$.*

In fact, **C**) is seen as follows. Set $f = \partial_{\overline{z}}^{-1}g$ with $g \in L^p(\Omega)$. Then $g = \frac{\partial f}{\partial \overline{z}} \in L^p(\Omega)$. Then Theorem 1.16 (p.72 in [20]) yields $\frac{\partial f}{\partial z} \in L^p(\Omega)$. From $\frac{\partial f}{\partial \overline{z}}, \frac{\partial f}{\partial z} \in L^p(\Omega)$, it follows that $\nabla f \in L^p(\Omega)$, that is, $f \in W^1_p(\Omega)$.

Assume that $\mathcal{A}, \mathcal{B}$ satisfy (2.16). Setting $T_B g = e^{\mathcal{B}}\partial_z^{-1}(e^{-\mathcal{B}}g)$ and $P_A g = e^{\mathcal{A}}\partial_{\overline{z}}^{-1}(e^{-\mathcal{A}}g)$, we have

$$(2\frac{\partial}{\partial z} + B)T_B g = g \quad \text{in } \Omega, \quad (2\frac{\partial}{\partial \overline{z}} + A)P_A g = g \quad \text{in } \Omega.$$

We define two other operators:

$$\mathcal{R}_{\tau,A}g = \frac{1}{2}e^{\mathcal{A}}e^{\tau(\overline{\Phi}-\Phi)}\partial_{\overline{z}}^{-1}(ge^{-\mathcal{A}}e^{\tau(\Phi-\overline{\Phi})}), \ \widetilde{\mathcal{R}}_{\tau,B}g = \frac{1}{2}e^{\mathcal{B}}e^{\tau(\overline{\Phi}-\Phi)}\partial_z^{-1}(ge^{-\mathcal{B}}e^{\tau(\Phi-\overline{\Phi})}). \tag{3.3}$$

The following proposition follows from straightforward calculations.

Proposition 3.2. *Let $g \in C^\alpha(\overline{\Omega})$ for some positive α. The function $\mathcal{R}_{\tau,A}g$ is a solution to*

$$2\frac{\partial}{\partial \overline{z}}\mathcal{R}_{\tau,A}g - 2\tau\frac{\partial\overline{\Phi}}{\partial \overline{z}}\mathcal{R}_{\tau,A}g + A\mathcal{R}_{\tau,A}g = g \quad in \quad \Omega. \tag{3.4}$$

The function $\widetilde{\mathcal{R}}_{\tau,B}g$ solves

$$2\frac{\partial}{\partial z}\widetilde{\mathcal{R}}_{\tau,B}g + 2\tau\frac{\partial\Phi}{\partial z}\widetilde{\mathcal{R}}_{\tau,B}g + B\widetilde{\mathcal{R}}_{\tau,B}g = g \quad in\ \Omega. \tag{3.5}$$

We have:

Proposition 3.3. *Let $g \in C^2(\Omega), g|_{\mathcal{O}_\epsilon} = 0$ and $g|_{\mathcal{H}} = 0$. Then for any $1 \le p < \infty$*

$$\left\|\mathcal{R}_{\tau,A}g + \frac{g}{2\tau\overline{\partial_z\Phi}}\right\|_{L^p(\Omega)} + \left\|\widetilde{\mathcal{R}}_{\tau,B}g - \frac{g}{2\tau\partial_z\Phi}\right\|_{L^p(\Omega)} = o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \to +\infty. \tag{3.6}$$

Proof. We give a proof of the asymptotic formula for $\widetilde{\mathcal{R}}_{\tau,B}g$. The proof for the $\mathcal{R}_{\tau,A}g$ is similar. Let $\tilde{g}(\zeta,\overline{\zeta}) = ge^{-\mathcal{B}}$. Then

$$\begin{aligned} e^{-\mathcal{B}}\mathcal{R}_{\tau,B}g &= -\frac{e^{\tau(\overline{\Phi}-\Phi)}}{\pi}\int_\Omega \frac{g(\zeta,\overline{\zeta})}{\overline{\zeta}-\overline{z}}e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2 \\ &= -\frac{e^{\tau(\overline{\Phi}-\Phi)}}{\pi}\lim_{\delta\to+0}\int_{\Omega\setminus B(z,\delta)} \frac{g(\zeta,\overline{\zeta})}{\overline{\zeta}-\overline{z}}e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2. \end{aligned}$$

Let $z = x_1 + ix_2$ and (x_1, x_2) be not a critical point of the function Φ. Then

$$\begin{aligned} e^{-\mathcal{B}}\mathcal{R}_{\tau,B}g &= -\frac{e^{\tau(\overline{\Phi}-\Phi)}}{\pi\tau}\lim_{\delta\to+0}\int_{\Omega\setminus B(z,\delta)} \frac{\tilde{g}(\zeta,\overline{\zeta})}{\overline{\zeta}-\overline{z}}\frac{\partial_\zeta e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}}{\partial_\zeta\Phi(\zeta)}d\xi_1 d\xi_2 \\ &= \frac{e^{\tau(\overline{\Phi}-\Phi)}}{\pi\tau}\lim_{\delta\to+0}\int_{\Omega\setminus B(z,\delta)} \frac{1}{\overline{\zeta}-\overline{z}}\frac{\partial}{\partial\zeta}\left(\frac{\tilde{g}(\zeta,\overline{\zeta})}{\partial_\zeta\Phi(\zeta)}\right)e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2 \\ &- \frac{e^{\tau(\overline{\Phi}-\Phi)}}{\pi\tau}\lim_{\delta\to+0}\int_{S(z,\delta)} \frac{\tilde{g}(\zeta,\overline{\zeta})}{\overline{\zeta}-\overline{z}}\frac{(\widetilde{\nu}_1 - i\widetilde{\nu}_2)}{2\partial_\zeta\Phi(\zeta)}e^{\tau(\Phi(\zeta)-\Phi(\overline{\zeta}))}d\xi_1 d\xi_2. \end{aligned}$$

Since $\tilde{g}|_{\mathcal{H}} = 0$, we have

$$\left|\frac{\partial}{\partial\zeta}\left(\frac{\tilde{g}(\zeta,\overline{\zeta})}{\partial_\zeta\Phi(\zeta)}\right)\right| \le C\sum_{k=1}^{\ell}\frac{\|\tilde{g}\|_{C^1(\overline{\Omega})}}{|\zeta-\widetilde{x}_k|} \in L^p(\Omega) \quad \forall p \in (1,2). \tag{3.7}$$

Hence

$$e^{-\mathcal{B}}\mathcal{R}_{\tau,B}g = \frac{e^{\tau(\overline{\Phi}-\Phi)}}{\pi\tau}\int_\Omega \frac{1}{\overline{\zeta}-\overline{z}}\frac{\partial}{\partial\zeta}\left(\frac{\tilde{g}(\zeta,\overline{\zeta})}{\partial_\zeta\Phi(\zeta)}\right)e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2 - \frac{\tilde{g}(z,\bar{z})}{\tau\partial_z\Phi(z)}.$$

Denote $G_\tau(x) = \int_\Omega \frac{1}{\overline{\zeta}-\overline{z}}\frac{\partial}{\partial\zeta}\left(\frac{\tilde{g}(\zeta,\overline{\zeta})}{\partial_\zeta\Phi(\zeta)}\right)e^{\tau(\Phi(\zeta)-\Phi(\overline{\zeta}))}d\xi_1 d\xi_2$. By the stationary phase argument, we see that

$$G_\tau(x) \longrightarrow 0 \ \text{ as } |\tau| \to +\infty \quad \forall x \in \overline{\Omega}. \tag{3.8}$$

Denote

$$F(\xi_1,\xi_2) = \left|\frac{\partial_{\overline{\zeta}}\tilde{g}(\zeta,\bar{\zeta})}{\partial_\zeta\Phi(\zeta)}\right|.$$

Clearly

$$|G_\tau(x)| \le \int_\Omega \frac{|F(\xi_1,\xi_2)|}{|z-\zeta|}d\xi_1 d\xi_2 \quad a.e. \text{ in } \Omega \quad \forall\tau. \tag{3.9}$$

By (3.7) F belongs to $L^p(\Omega)$ for any $p \in (1,2)$. For $f \in L^p(\mathbf{R}^2)$, we set

$$I_r f(z) = \int_{\mathbf{R}^2}|z-\zeta|^{-\frac{2}{r}}f(\zeta,\bar{\zeta})d\xi_1 d\xi_2.$$

Then, by the Hardy-Littlewood-Sobolev inequality, if $r > 1$ and $\frac{1}{r} = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)$ for $1 < p < q < \infty$, then

$$\|I_r f\|_{L^q(\mathbf{R}^2)} \le C_{p,q}\|f\|_{L^p(\mathbf{R}^2)}.$$

Set $r = 2$. Then we have to choose $\frac{1}{p} - \frac{1}{q} = \frac{1}{2}$, that is, we can arbitrarily choose $p > 2$ close to 2, so that q is arbitrarily large. Hence $\int_\Omega \frac{F}{|z-\zeta|} d\xi_1 d\xi_2$ belongs to $L^q(\Omega)$ with positive q. By (3.8), (3.9) and the dominated convergence theorem

$$G_\tau \to 0 \quad \text{in } L^q(\Omega) \qquad \forall q \in (1,\infty).$$

The proof of the proposition is finished. □

Using the stationary phase argument (e.g., Bleistein and Handelsman [2]), we will show

Proposition 3.4. *Let $g \in L^1(\Omega)$ and a function Φ satisfy (2.1),(2.2). Then*

$$lim_{|\tau|\to+\infty} \int_\Omega g e^{\tau(\Phi(z)-\overline{\Phi(z)})} dx = 0.$$

Proof. Let $\{g_k\}_{k=1}^\infty \in C_0^\infty(\Omega)$ be a sequence of functions such that $g_k \to g$ in $L^1(\Omega)$. Let $\epsilon > 0$ be an arbitrary number. Suppose that $\widehat{j}$ is large enough such that $\|g - g_{\widehat{j}}\|_{L^1(\Omega)} \le \frac{\epsilon}{2}$. Then

$$|\int_\Omega g e^{\tau(\Phi(z)-\overline{\Phi(z)})} dx| \le |\int_\Omega (g - g_{\widehat{j}}) e^{\tau(\Phi(z)-\overline{\Phi(z)})} dx| + |\int_\Omega g_{\widehat{j}} e^{\tau(\Phi(z)-\overline{\Phi(z)})} dx|.$$

The first term on the right-hand side of this inequality is less then $\epsilon/2$ and the second goes to zero as $|\tau|$ approaches to infinity by the stationary phase argument (see e.g. [2]). □

We now consider the contribution from the critical points.

Proposition 3.5. *Let Φ satisfy (2.1) and (2.2). Let $g \in C^{4+\alpha}(\overline{\Omega})$ for some $\alpha > 0$, $g|_{\mathcal{O}_\epsilon} = 0$ and $g|_{\mathcal{H}} = 0$. Then there exist constants p_k such that*

$$\int_\Omega g e^{\tau(\Phi(z)-\overline{\Phi(z)})} dx = \frac{1}{\tau^2} \sum_{k=1}^{\ell} p_k e^{2\tau i\psi(\widetilde{x}_k)} + o(\frac{1}{\tau^2}) \quad as\ |\tau| \to +\infty. \tag{3.10}$$

Proof. Let $\delta > 0$ be a sufficiently small number and $\tilde{e}_k \in C_0^\infty(B(\widetilde{x}_k, \delta)), \tilde{e}_k|_{B(\widetilde{x}_k,\delta/2)} \equiv 1$. By the stationary phase argument

$$I(\tau) = \int_\Omega g e^{\tau(\Phi-\overline{\Phi})} dx = \sum_{k=1}^{\ell} \int_{B(\widetilde{x}_k,\delta)} \tilde{e}_k g e^{\tau(\Phi-\overline{\Phi})} dx + o(\frac{1}{\tau^2}) =$$

$$\sum_{k=1}^{\ell} e^{2i\tau\psi(\widetilde{x}_k)} \int_{B(\widetilde{x}_k,\delta)} \tilde{e}_k g e^{\tau(\Phi-\overline{\Phi})-2i\tau\psi(\widetilde{x}_k)} dx + o(\frac{1}{\tau^2}) \quad \text{as } |\tau| \to +\infty.$$

Since all the critical points of Φ are nondegenerate, in some neighborhood of $\widetilde{x}_k$ one can take local coordinates such that $\Phi - \overline{\Phi} - 2i\tau\psi(\widetilde{x}_k) = z^2 - \overline{z}^2$. Therefore

$$I(\tau) = \sum_{k=1}^{\ell} e^{2i\tau\psi(\widetilde{x}_k)} \int_{B(0,\delta')} q_k e^{\tau(z^2-\overline{z}^2)} dx + o(\frac{1}{\tau^2}) \quad \text{as } |\tau| \to +\infty,$$

where $q_k \in C^4_0(B(0,\delta'))$ and $q_k(0)=0$. Hence there exist functions $r_{1,k}, r_{2,k} \in C^3_0(B(0,\delta'))$ such that $q_k = 2zr_{1,k} + 2\overline{z}r_{2,k}$. Integrating by parts, one can decompose $I(\tau)$ as

$$I(\tau) = -\frac{1}{\tau}\sum_{k=1}^{\ell} e^{2i\tau\psi(\widetilde{x}_k)}\int_{B(0,\delta')}(\frac{\partial r_{1,k}}{\partial z} - \frac{\partial r_{2,k}}{\partial \overline{z}})e^{\tau(z^2-\overline{z}^2)}dx + o(\frac{1}{\tau^2}) =$$

$$-\frac{1}{\tau}\sum_{k=1}^{\ell} e^{2i\tau\psi(\widetilde{x}_k)}\int_{B(0,\delta')}(\frac{\partial r_{1,k}}{\partial z} - \frac{\partial r_{2,k}}{\partial \overline{z}})(0)\chi(x)e^{\tau(z^2-\overline{z}^2)}dx + o(\frac{1}{\tau^2})$$

$$-\frac{1}{\tau}\sum_{k=1}^{\ell} e^{2i\tau\psi(\widetilde{x}_k)}\int_{B(0,\delta')}\widetilde{q}_k e^{\tau(z^2-\overline{z}^2)}dx + o(\frac{1}{\tau^2}) \quad \text{as } |\tau|\rightarrow+\infty,$$

where $\chi, \widetilde{q}_k \in C^2_0(B(0,\delta'))$, $\chi|_{B(0,\delta'/2)} \equiv 1$ and $\widetilde{q}_k(0)=0$. Hence there exist functions $\widetilde{r}_{1,k}, \widetilde{r}_{2,k} \in C^1_0(B(0,\delta'))$ such that $\widetilde{q}_k = 2z\widetilde{r}_{1,k} + 2\overline{z}\widetilde{r}_{2,k}$. Integrating by parts and applying Proposition 3.3 we obtain

$$\lim_{|\tau|\rightarrow+\infty}\tau\int_{B(0,\delta')}\widetilde{q}_k e^{\tau(z^2-\overline{z}^2)}dx = -\sum_{k=1}^{\ell} e^{2i\tau\psi(\widetilde{x}_k)}\lim_{|\tau|\rightarrow+\infty}\int_{B(0,\delta')}(\frac{\partial \widetilde{r}_{1,k}}{\partial z} - \frac{\partial \widetilde{r}_{2,k}}{\partial \overline{z}})e^{\tau(z^2-\overline{z}^2)}dx = 0.$$

Therefore (3.10) follows from a standard application of stationary phase. The proof of the proposition is completed. □

Proposition 3.6. *Let $0<\epsilon'<\epsilon$, a function Φ satisfy (2.1), (2.2) and $\overline{\mathcal{O}}_\epsilon \cap (\mathcal{H}\setminus\Gamma_0)=\emptyset$. Suppose that $g \in C^\alpha(\overline{\Omega})$ for some $\alpha>0$, $g|_{\mathcal{O}_\epsilon}=0$ and $g|_{\mathcal{H}}=0$. Then*

$$|\tau|\|\widetilde{\mathcal{R}}_{\tau,B}g\|_{L^\infty(\mathcal{O}_{\epsilon'})} + \|\nabla\widetilde{\mathcal{R}}_{\tau,B}g\|_{L^\infty(\mathcal{O}_{\epsilon'})} \le C_1(\epsilon',\alpha)\|g\|_{C^\alpha(\overline{\Omega})\cap H^1(\Omega)}. \tag{3.11}$$

Moreover

$$\|\nabla\widetilde{\mathcal{R}}_{\tau,B}g\|_{L^2(\Omega)} + |\tau|^{\frac{1}{2}}\|\widetilde{\mathcal{R}}_{\tau,B}g\|_{L^2(\Omega)} + |\tau|\|\frac{\partial\Phi}{\partial z}\widetilde{\mathcal{R}}_{\tau,B}g\|_{L^2(\Omega)} \le C_2(\epsilon',\alpha)\|g\|_{C^\alpha(\overline{\Omega})\cap H^1(\Omega)}. \tag{3.12}$$

Proof. Denote $\widetilde{g} = ge^{-\mathcal{B}}$. Let $x=(x_1,x_2)$ be an arbitrary point from $\mathcal{O}_{\epsilon'}$ and $z = x_1+ix_2$. Then

$$-\pi\partial_z^{-1}(e^{\tau(\Phi-\overline{\Phi})}\widetilde{g}) = \int_\Omega \frac{\widetilde{g}e^{\tau(\Phi-\overline{\Phi})}}{\overline{\zeta}-\overline{z}}d\xi_1 d\xi_2 = \lim_{\delta\rightarrow+0}\sum_{k=1}^{\ell}\int_{\Omega\setminus B(\widetilde{x}_k,\delta)}\frac{\widetilde{g}e^{\tau(\Phi-\overline{\Phi})}}{\overline{\zeta}-\overline{z}}d\xi_1 d\xi_2.$$

Integrating by parts and taking δ sufficiently small we have

$$\begin{aligned}-\pi\partial_z^{-1}(e^{\tau(\Phi-\overline{\Phi})}\widetilde{g}) = &-\frac{1}{\tau}\lim_{\delta\rightarrow+0}\int_{\Omega\setminus\cup_{k=1}^{\ell}B(\widetilde{x}_k,\delta)}\frac{\frac{\partial\widetilde{g}}{\partial\zeta}}{(\overline{\zeta}-\overline{z})\frac{\partial\Phi}{\partial\zeta}}e^{\tau(\Phi-\overline{\Phi})}d\xi_1 d\xi_2 \\ &+\frac{1}{\tau}\lim_{\delta\rightarrow+0}\int_{\Omega\setminus\cup_{k=1}^{\ell}B(\widetilde{x}_k,\delta)}\frac{\widetilde{g}\frac{\partial^2\Phi}{\partial\zeta^2}}{(\overline{\zeta}-\overline{z})(\frac{\partial\Phi}{\partial\zeta})^2}e^{\tau(\Phi-\overline{\Phi})}d\xi_1 d\xi_2 \\ &+\frac{1}{2\tau}\lim_{\delta\rightarrow+0}\int_{\cup_{k=1}^{\ell}S(\widetilde{x}_k,\delta)}(\widetilde{\nu}_1-i\widetilde{\nu}_2)\frac{\widetilde{g}}{(\overline{\zeta}-\overline{z})\frac{\partial\Phi}{\partial\zeta}}e^{\tau(\Phi-\overline{\Phi})}d\sigma.\end{aligned} \tag{3.13}$$

Since $g|_{\mathcal{H}} = 0$ and all the critical points of Φ are nondegenerate we have that $\|g\|_{C^0(S(\widetilde{x}_k,\delta))} \le \delta^\alpha \|g\|_{C^\alpha(\overline{\Omega})}$. Therefore

$$\frac{1}{2\tau}\lim_{\delta\to+0}\int_{\cup_{k=1}^{\ell}S(\widetilde{x}_k,\delta)}(\widetilde{\nu}_1 - i\widetilde{\nu}_2)\frac{\widetilde{g}}{(\overline{\zeta}-\overline{z})\frac{\partial\Phi}{\partial\zeta}}e^{\tau(\Phi-\overline{\Phi})}d\sigma = 0.$$

Since $|\frac{\widetilde{g}\frac{\partial^2\Phi}{\partial\zeta^2}}{(\frac{\partial\Phi}{\partial\zeta})^2}(\zeta,\bar{\zeta})| \le C_3\|\widetilde{g}\|_{C^\alpha(\overline{\Omega})}\sum_{k=1}^{\ell}\frac{1}{|\xi-\widetilde{x}_k|^{2-\alpha}}$ we see that $\frac{\widetilde{g}\frac{\partial^2\Phi}{\partial\zeta^2}}{(\frac{\partial\Phi}{\partial\zeta})^2}(\zeta,\bar{\zeta}) \in L^1(\Omega)$ and

$$-\pi\partial_z^{-1}(e^{\tau(\Phi-\overline{\Phi})}\widetilde{g}) = -\frac{1}{\tau}\int_\Omega\frac{\frac{\partial\widetilde{g}}{\partial\zeta}}{(\overline{\zeta}-\overline{z})\frac{\partial\Phi}{\partial\zeta}}e^{\tau(\Phi-\overline{\Phi})}d\xi_1d\xi_2$$

$$+\frac{1}{\tau}\int_\Omega\frac{\widetilde{g}\frac{\partial^2\Phi}{\partial\zeta^2}}{(\overline{\zeta}-\overline{z})(\frac{\partial\Phi}{\partial\zeta})^2}e^{\tau(\Phi-\overline{\Phi})}d\xi_1d\xi_2. \tag{3.14}$$

From this equality and definition (3.3) of the operator $\widetilde{\mathcal{R}}_{\tau,B}$, the estimate (3.11) follows immediately. In order to prove (3.12) we observe

$$\frac{\partial\widetilde{\mathcal{R}}_{\tau,B}g}{\partial\overline{z}} = \frac{\partial\mathcal{B}}{\partial\overline{z}}\widetilde{\mathcal{R}}_{\tau,B}g + \widetilde{\mathcal{R}}_{\tau,B}\{\frac{\partial g}{\partial\overline{z}} - \frac{\partial\mathcal{B}}{\partial\overline{z}}g\} + \frac{\tau}{2\pi}e^{\tau(\overline{\Phi}-\Phi)+\mathcal{B}}\int_\Omega\frac{\frac{\partial\overline{\Phi}(\zeta)}{\partial\overline{\zeta}} - \frac{\partial\overline{\Phi}(z)}{\partial\overline{z}}}{\overline{\zeta}-\overline{z}}\tilde{g}e^{\tau(\Phi-\overline{\Phi})}d\xi_1d\xi_2.$$

By Proposition 3.1

$$\|\frac{\partial\mathcal{B}}{\partial\overline{z}}\widetilde{\mathcal{R}}_{\tau,B}g + \widetilde{\mathcal{R}}_{\tau,B}\{\frac{\partial g}{\partial\overline{z}} - \frac{\partial\mathcal{B}}{\partial\overline{z}}g\}\|_{L^2(\Omega)} \le C_4\|g\|_{H^1(\Omega)}.$$

Using arguments similar to (3.13), (3.14) we obtain

$$\|\frac{\tau}{2\pi}\int_\Omega\frac{\frac{\partial\overline{\Phi}(\zeta)}{\partial\overline{\zeta}} - \frac{\partial\overline{\Phi}(z)}{\partial\overline{z}}}{\overline{\zeta}-\overline{z}}\tilde{g}e^{\tau(\Phi-\overline{\Phi})}d\xi_1d\xi_2\|_{L^2(\Omega)} \le C_5\|g\|_{C^\alpha(\overline{\Omega})\cap H^1(\Omega)}.$$

Hence

$$\|\frac{\partial\widetilde{\mathcal{R}}_{\tau,B}g}{\partial\overline{z}}\|_{L^2(\Omega)} \le C_6\|g\|_{C^\alpha(\overline{\Omega})\cap H^1(\Omega)}.$$

Combining this estimate with (3.11) we conclude

$$\|\nabla\widetilde{\mathcal{R}}_{\tau,B}g\|_{L^2(\Omega)} \le C_7\|g\|_{C^\alpha(\overline{\Omega})\cap H^1(\Omega)}.$$

Using this estimate and equation (3.4) we have

$$|\tau|\|\frac{\partial\Phi}{\partial z}\widetilde{\mathcal{R}}_{\tau,B}g\|_{L^2(\Omega)} \le C_8\|g\|_{C^\alpha(\overline{\Omega})\cap H^1(\Omega)},$$

finishing the proof of the proposition. □

Let $e_1, e_2 \in C^\infty(\overline{\Omega})$ be functions such that

$$e_1 + e_2 = 1 \quad \text{in } \Omega, \tag{3.15}$$

e_2 vanishes in some neighborhood of $\mathcal{H}\setminus\Gamma_0$ and e_1 vanishes in a neighborhood of $\partial\Omega$.

Proposition 3.7. *Let for some $\alpha \in (0,1)$ $A, B \in C^{5+\alpha}(\overline{\Omega})$, functions $\mathcal{A}, \mathcal{B} \in C^{6+\alpha}(\overline{\Omega})$ satisfy (2.16), functions e_1, e_2 defined in (3.15), a function g belong to $L^p(\Omega)$ for some $p > 2$, supp $g \subset\subset$ supp e_1. We define the function u by*

$$u = \widetilde{\mathcal{R}}_{\tau,B}(e_1(P_A g - \widetilde{M}e^{\mathcal{A}})) + \frac{e_2(P_A g - \widetilde{M}e^{\mathcal{A}})}{2\tau\partial_z\Phi},$$

where $\widetilde{M} = \widetilde{M}(z)$ is a polynomial such that $(P_A g - \widetilde{M}e^{\mathcal{A}})|_{\mathcal{H}} = 0$ and $\frac{\partial^k}{\partial z^k}(P_A g - \widetilde{M}e^{\mathcal{A}})|_{\mathcal{H}} = 0$ for any k from $\{1, \dots, 4\}$. Then we have

$$\mathcal{P}(x, D)(ue^{\tau\Phi}) \triangleq (2\frac{\partial}{\partial \overline{z}} + A)(2\frac{\partial}{\partial z} + B)(ue^{\tau\Phi}) = ge^{\tau\Phi} + \frac{e^{\tau\varphi}}{|\tau|}h_\tau \quad \text{as } |\tau| \to +\infty, \tag{3.16}$$

where

$$\|h_\tau\|_{L^\infty(\Omega)} \le C_9(p)\|g\|_{L^p(\Omega)}$$

and

$$\frac{1}{|\tau|^{\frac{1}{2}}}\|\nabla u\|_{L^2(\Omega)} + |\tau|^{\frac{1}{2}}\|u\|_{L^2(\Omega)} + \|u\|_{H^{1,\tau}(\mathcal{O}_{\epsilon'})} \le C_{10}\|g\|_{L^p(\Omega)}. \tag{3.17}$$

Proof. By Proposition 3.1 $P_A g$ belongs to $W^1_p(\Omega)$. Since $p > 2$, by the Sobolev embedding theorem there exists $\alpha > 0$ such that $P_A g \in C^\alpha(\overline{\Omega})$. By properties of elliptic operators and the fact that supp $e_2 \cap$ supp $g = \{\emptyset\}$ we have that $P_A g \in C^5(\text{supp}\, e_2)$. The estimate (3.17) follows from Proposition 3.6. Short calculations give

$$\mathcal{P}(x, D)(ue^{\tau\Phi}) = ge^{\tau\Phi} + \frac{e^{\tau\Phi}}{\tau}\mathcal{P}(x, D)\left(\frac{e_2(P_A g - \widetilde{M}e^{\mathcal{A}})}{2\partial_z\Phi}\right). \tag{3.18}$$

This formula implies (3.16) with $h_\tau = e^{i\tau\psi}\mathcal{P}(x, D)\left(\frac{e_2(P_A g - \widetilde{M}e^{\mathcal{A}})}{2\partial_z\Phi}\right)/\text{sign}\,\tau$. □

The following proposition will play a critically important role in construction of complex geometric optic solutions.

Proposition 3.8. *Let $f \in L^p(\Omega)$ for some $p > 2$, ϵ' be a small positive number such that $O_{\epsilon'} \cap (\mathcal{H} \setminus \Gamma_0) = \emptyset$. Then there exists τ_0 such that for all $|\tau| > \tau_0$ there exists a solution to the boundary value problem*

$$L(x, D)w = fe^{\tau\Phi} \quad \text{in } \Omega, \quad w|_{\Gamma_0} = qe^{\tau\varphi}/\tau \tag{3.19}$$

such that

$$\sqrt{|\tau|}\|we^{-\tau\varphi}\|_{L^2(\Omega)} + \frac{1}{\sqrt{|\tau|}}\|\nabla we^{-\tau\varphi}\|_{L^2(\Omega)} + \|we^{-\tau\varphi}\|_{H^{1,\tau}(\mathcal{O}_{\epsilon'})} \le C_{11}(\|f\|_{L^p(\Omega)} + \|q\|_{H^{\frac{1}{2}}(\Gamma_0)}).$$

Proof. Let $\chi \in C_0^\infty(\Omega)$ be equal to one in some neighborhood of the set $\mathcal{H}\setminus\Gamma_0$. By Proposition 2.6 there exists a solution to the problem (3.19) with inhomogeneous term $(1 - \chi)f$ and boundary data q/τ such that

$$\|w_1 e^{-\tau\varphi}\|_{H^{1,\tau}(\Omega)} \le C_{12}(\|f\|_{L^2(\Omega)} + \|q\|_{H^{\frac{1}{2}}(\Gamma_0)}). \tag{3.20}$$

Denote $w_2 = \widetilde{\mathcal{R}}_{\tau,B}(e_1(P_A(\chi f) - \widetilde{M}e^{\mathcal{A}})) + \frac{e_2(P_A(\chi f) - \widetilde{M}e^{\mathcal{A}})}{2\tau\partial_z\Phi}$ where $\widetilde{M} = \widetilde{M}(z)$ is a polynomial such that $(P_A g - \widetilde{M}e^{\mathcal{A}})|_{\mathcal{H}} = 0$ and $\frac{\partial^k}{\partial z^k}(P_A g - \widetilde{M}e^{\mathcal{A}})|_{\mathcal{H}} = 0$ for any k from $\{1, \dots, 4\}$. Let q_τ

be the restriction of w_2 on Γ_0. By (3.12) there exists a constant C_{13} independent of τ such that

$$|\tau|\|q_\tau\|_{C^1(\overline{\Gamma_0})} \le C_{13}\|f\|_{L^p(\Omega)}. \tag{3.21}$$

By Proposition 3.7 there exists a constant C_{14} independent of τ such that

$$\sqrt{|\tau|}\|w_2 e^{-\tau\varphi}\|_{L^2(\Omega)} + \frac{1}{\sqrt{|\tau|}}\|\nabla w_2 e^{-\tau\varphi}\|_{L^2(\Omega)} + \|w_2 e^{-\tau\varphi}\|_{H^{1,\tau}(\mathcal{O}_{\epsilon'})} \le C_{14}\|f\|_{L^p(\Omega)}. \tag{3.22}$$

Let $\widetilde{a}_\tau, \widetilde{b}_\tau \in H^1(\Omega)$ be holomorphic and an antiholomorphic functions respectively such that $(\widetilde{a}_\tau e^{\mathcal{A}} + \widetilde{b}_\tau e^{\mathcal{B}})|_{\Gamma_0} = -q_\tau$. By (3.21) and Proposition 2.5 there exist constants C_{15}, C_{16} independent of τ such that

$$\|\widetilde{a}_\tau\|_{H^1(\Omega)} + \|\widetilde{b}_\tau\|_{H^1(\Omega)} \le C_{15}\|q_\tau\|_{C^1(\overline{\Gamma_0})} \le C_{16}\frac{\|f\|_{L^p(\Omega)}}{|\tau|}. \tag{3.23}$$

The function $W = (w_2 + \widetilde{a}_\tau e^{\mathcal{A}})e^{\tau\Phi} + \widetilde{b}_\tau e^{\mathcal{B}+\tau\overline{\Phi}}$ satisfies

$$L(x,D)W = \chi f e^{\tau\Phi} + e^{\tau\varphi}\frac{\widetilde{h}_\tau}{\sqrt{|\tau|}} \quad \text{in } \Omega, \quad W|_{\Gamma_0} = 0,$$

where

$$\|\widetilde{h}_\tau\|_{L^2(\Omega)} \le C_{17}\|f\|_{L^2(\Omega)} \tag{3.24}$$

with some constant C_{17} independent of τ. By (3.22), (3.23)

$$\sqrt{|\tau|}\|W e^{-\tau\varphi}\|_{L^2(\Omega)} + \frac{1}{\sqrt{|\tau|}}\|\nabla W e^{-\tau\varphi}\|_{L^2(\Omega)} + \|W e^{-\tau\varphi}\|_{H^{1,\tau}(\mathcal{O}_{\epsilon'})} \le C_{18}\|f\|_{L^p(\Omega)}. \tag{3.25}$$

Let $\widetilde{W}$ be a solution to problem (2.54) with inhomogeneous term and boundary data $f = -\frac{\widetilde{h}_\tau}{\sqrt{|\tau|}}, g \equiv 0$ respectively given by Proposition 2.6. The estimate (2.55) has the form

$$\|\widetilde{W} e^{-\tau\varphi}\|_{H^{1,\tau}(\Omega)} \le C_{19}\|\widetilde{h}_\tau\|_{L^2(\Omega)} \le C_{20}\|f\|_{L^2(\Omega)}. \tag{3.26}$$

Then the function $w_1 + W + \widetilde{W}$ solves (3.19). The estimate (3.18) follows form (3.20), (3.25) and (3.26). The proof of the proposition is completed. □

4. Complex Geometrical Optics Solutions

For a vector field (A_1, B_1) and potential q_1 we will construct solutions to the boundary value problem

$$L_1(x,D)u_1 = 0 \quad \text{in } \Omega, \quad u_1|_{\Gamma_0} = 0 \tag{4.1}$$

of the form

$$u_1(x) = a_\tau(z)e^{\mathcal{A}_1+\tau\Phi} + d_\tau(\overline{z})e^{\mathcal{B}_1+\tau\overline{\Phi}} + u_{11}e^{\tau\varphi} + u_{12}e^{\tau\varphi}. \tag{4.2}$$

Here $\mathcal{A}_1$ and $\mathcal{B}_1$ are defined by (2.16) respectively for A_1 and B_1, $a_\tau(z) = a(z) + \frac{a_1(z)}{\tau} + \frac{a_{2,\tau}(z)}{\tau^2}$, $d_\tau(\overline{z}) = d(\overline{z}) + \frac{d_1(\overline{z})}{\tau} + \frac{d_{2,\tau}(\overline{z})}{\tau^2}$,

$$a, d \in C^{5+\alpha}(\overline{\Omega}), \quad \frac{\partial a}{\partial \overline{z}} = 0 \text{ in } \Omega, \quad \frac{\partial d}{\partial z} = 0 \text{ in } \Omega, \tag{4.3}$$

$$(a(z)e^{\mathcal{A}_1}+d(\bar z)e^{\mathcal{B}_1})|_{\Gamma_0}=0. \tag{4.4}$$

Suppose in addition that

$$\frac{\partial^k a}{\partial z^k}|_{\mathcal{H}\cap\partial\Omega}=0,\quad \frac{\partial^k d}{\partial \bar z^k}|_{\mathcal{H}\cap\partial\Omega}=0\quad \forall k\in\{0,\dots,5\}. \tag{4.5}$$

The existence of such functions $a(z)$ and $d(\bar z)$ is given by Proposition 2.5.

Denote

$$g_1=T_{B_1}((q_1-2\frac{\partial B_1}{\partial \overline{z}}-A_1B_1)de^{\mathcal{B}_1})-M_2(\overline{z})e^{\mathcal{B}_1},\quad g_2=P_{A_1}((q_1-2\frac{\partial A_1}{\partial z}-A_1B_1)ae^{\mathcal{A}_1})-M_1(z)e^{\mathcal{A}_1},$$

where $M_1(z)$ and $M_2(\overline{z})$ are polynomials such that

$$\frac{\partial^k g_1}{\partial \bar z^k}|_{\mathcal{H}}=\frac{\partial^k g_2}{\partial z^k}|_{\mathcal{H}}=0\quad \forall k\in\{0,\dots,5\}. \tag{4.6}$$

Thanks to our assumptions on regularity of A_1,B_1 and q, the functions g_1,g_2 belong to $C^{6+\alpha}(\overline{\Omega})$.

Note that by (4.6), (4.5)

$$\frac{\partial^{k+j}}{\partial z^k\partial \bar z^j}g_1|_{\mathcal{H}\cap\partial\Omega}=\frac{\partial^{k+j}}{\partial z^k\partial \bar z^j}g_2|_{\mathcal{H}\cap\partial\Omega}=0\quad \text{if } k+j\le 5. \tag{4.7}$$

The function $a_1(z)$ is holomorphic in Ω and $d_1(\overline{z})$ is antiholomorphic in Ω and

$$a_1(z)e^{\mathcal{A}_1}+d_1(\overline{z})e^{\mathcal{B}_1}=\frac{g_1}{2\overline{\partial_z\Phi}}+\frac{g_2}{2\partial_z\Phi}\quad \text{on}\,\Gamma_0.$$

The existence of such functions is given by Proposition 2.5. Observe that by (4.7) the functions $\frac{e_2g_1}{\overline{\partial_z\Phi}},\frac{e_2g_2}{\partial_z\Phi}$ belong to the space $C^4(\overline{\Omega})$. Let

$$\hat g_1=T_{B_1}((q_1-2\frac{\partial B_1}{\partial \overline{z}}-A_1B_1)d_1e^{\mathcal{B}_1})-\hat M_2(\overline{z})e^{\mathcal{B}_1},\quad \hat g_2=P_{A_1}((q_1-2\frac{\partial A_1}{\partial z}-A_1B_1)a_1e^{\mathcal{A}_1})-\hat M_1(z)e^{\mathcal{A}_1},$$

where $\hat M_1(z)$ and $\hat M_2(\overline{z})$ are polynomials such that

$$\frac{\partial^k \hat g_1}{\partial \bar z^k}|_{\mathcal{H}}=\frac{\partial^k \hat g_2}{\partial z^k}|_{\mathcal{H}}=0\quad \forall k\in\{0,\dots,3\}. \tag{4.8}$$

The function u_{11} is given by

$$\begin{aligned}u_{11}=&-e^{-i\tau\psi}\mathcal{R}_{-\tau,A_1}\left\{e_1(g_1+\hat g_1/\tau)\right\}-e^{-i\tau\psi}\frac{e_2(g_1+\frac{\hat g_1}{\tau})}{2\tau\overline{\partial_z\Phi}}+\frac{e^{-i\tau\psi}}{4\tau^2\overline{\partial_z\Phi}}L_1(x,D)\left(\frac{e_2g_1}{\overline{\partial_z\Phi}}\right)\\ &-e^{i\tau\psi}\widetilde{\mathcal{R}}_{\tau,B_1}\left\{e_1(g_2+\hat g_2/\tau)\right\}-e^{i\tau\psi}\frac{e_2(g_2+\frac{\hat g_2}{\tau})}{2\tau\partial_z\Phi}+\frac{e^{i\tau\psi}}{4\tau^2\partial_z\Phi}L_1(x,D)\left(\frac{e_2g_2}{\partial_z\Phi}\right),\end{aligned} \tag{4.9}$$

Now let us determine the functions u_{12}, $a_2(z)$ and $d_2(\overline{z})$.

First we can obtain the following asymptotic formulae for any x from the boundary of Ω:

(4.10)

$$\mathcal{R}_{-\tau,A_1}\left\{e_1g_1\right\}=\frac{1}{2\tau^2}\frac{e^{\mathcal{A}_1+2i\tau\psi}}{|\det\operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}}\sum_{k=1}^{\ell}\left(\frac{e^{-2i\tau\psi(\tilde x_k)}q_{1,k}(\tilde x_k)}{(z-\tilde z_k)^2}+\frac{e^{-2i\tau\psi(\tilde x_k)}m_{1,k}(\tilde x_k)}{(\tilde z_k-z)}\right)+\mathcal{W}_{\tau,1},$$

(4.11)

$$\tilde{\mathcal{R}}_{\tau,B_1}\{e_1g_2\} = \frac{1}{2\tau^2}\frac{e^{\mathcal{B}_1-2i\tau\psi}}{|\det \operatorname{Im}\Phi''(\tilde{x}_k)|^{\frac12}}\sum_{k=1}^{\ell}\left(\frac{e^{2i\tau\psi(\tilde{x}_k)}\tilde{q}_{1,k}(\tilde{x}_k)}{(\bar{z}-\bar{\tilde{z}}_k)^2}+\frac{e^{2i\tau\psi(\tilde{x}_k)}\tilde{m}_{1,k}(\tilde{x}_k)}{(\bar{\tilde{z}}_k-\bar{z})}\right)+\mathcal{W}_{\tau,2},$$

where

$$q_{1,k}=\frac{\partial_z\tilde{g}_1(\tilde{x}_k)}{4\partial_z^2\Phi(\tilde{x}_k)},\quad m_{1,k}=\frac18\left(\frac{\partial_z\tilde{g}_1(\tilde{x}_k)}{\partial_z^2\Phi(\tilde{x}_k)}\frac{\partial_z^3\Phi(\tilde{x}_k)}{\partial_z^2\Phi(\tilde{x}_k)}+\frac{\partial_{\bar{z}}^2\tilde{g}_1(\tilde{x}_k)}{\overline{\partial_z^2\Phi(\tilde{x}_k)}}-\frac{\partial_z^2\tilde{g}_1(\tilde{x}_k)}{\partial_z^2\Phi(\tilde{x}_k)}\right),$$

$$\tilde{q}_{1,k}=\frac{\partial_{\bar{z}}\tilde{g}_2(\tilde{x}_k)}{4\overline{\partial_z^2\Phi(\tilde{x}_k)}},\quad \tilde{m}_{1,k}=\frac18\left(\frac{\partial_{\bar{z}}\tilde{g}_2(\tilde{x}_k)}{\overline{\partial_z^2\Phi(\tilde{x}_k)}}\frac{\partial_{\bar{z}}^3\overline{\Phi}(\tilde{x}_k)}{\partial_{\bar{z}}^2\overline{\Phi}(\tilde{x}_k)}-\frac{\partial_{\bar{z}}^2\tilde{g}_2(\tilde{x}_k)}{\overline{\partial_z^2\Phi(\tilde{x}_k)}}+\frac{\partial_z^2\tilde{g}_2(\tilde{x}_k)}{\partial_z^2\Phi(\tilde{x}_k)}\right),$$

$\tilde{g}_1 = e^{-\mathcal{A}_1}g_1$, $\tilde{g}_2 = e^{-\mathcal{B}_1}g_2$ and $\mathcal{W}_{\tau,1},\mathcal{W}_{\tau,2}\in H^{\frac12}(\Gamma_0)$ satisfy

$$\|\mathcal{W}_{\tau,1}\|_{H^{\frac12}(\Gamma_0)}+\|\mathcal{W}_{\tau,2}\|_{H^{\frac12}(\Gamma_0)}=o(\frac{1}{\tau^2})\quad \text{as}\,|\tau|\to+\infty. \tag{4.12}$$

The proof of (4.10) and (4.11) is given in Section 7.

Denote

$$p_k = e^{\mathcal{A}_1}\left(\frac{q_{1,-k}(\tilde{x}_{-k})}{(z-\tilde{z}_{-k})^2}+\frac{m_{1,-k}(\tilde{x}_{-k})}{(\tilde{z}_{-k}-z)}\right)\quad k\in\{-1,\dots,-\ell\},$$

$$p_k = e^{\mathcal{B}_1}\left(\frac{\tilde{q}_{1,k}(\tilde{x}_k)}{(\bar{z}-\bar{\tilde{z}}_k)^2}+\frac{\tilde{m}_{1,k}(\tilde{x}_k)}{(\bar{\tilde{z}}_k-\bar{z})}\right)\quad k\in\{1,\dots,\ell\}.$$

Thanks to Proposition 2.5 we can define functions $a_{2,k}(z)\in C^2(\overline{\Omega})$ and $d_{2,k}(\overline{z})\in C^2(\overline{\Omega})$ satisfying

$$a_{2,k}(z)e^{\mathcal{A}_1}+d_{2,k}(\overline{z})e^{\mathcal{B}_1}=p_k\quad \text{on}\,\Gamma_0,\ \forall k\in\{\pm1,\cdots\pm\ell\}. \tag{4.13}$$

Straightforward computations give

$$\begin{aligned}&L_1(x,D)((a(z)+\tfrac{a_1(z)}{\tau})e^{\mathcal{A}_1+\tau\Phi}+(d(\bar{z})+\tfrac{d_1(\bar{z})}{\tau})e^{\mathcal{B}_1+\tau\overline{\Phi}}+e^{\tau\varphi}u_{11})\\&=(q_1-2\tfrac{\partial A_1}{\partial z}-A_1B_1)e^{\tau\Phi}\left(-\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1(g_2+\hat{g}_2/\tau)\}-\tfrac{e_2(g_2+\hat{g}_2/\tau)}{2\tau\partial_z\Phi}\right)\\&+(q_1-2\tfrac{\partial B_1}{\partial\bar{z}}-A_1B_1)e^{\tau\overline{\Phi}}\left(-\mathcal{R}_{-\tau,A_1}\{e_1(g_1+\hat{g}_1/\tau)\}-\tfrac{e_2(g_1+\hat{g}_1/\tau)}{2\tau\overline{\partial_z\Phi}}\right)\\&+\tfrac{e^{\tau\Phi}}{\tau^2}L_1(x,D)\left(\tfrac{1}{4\partial_z\Phi}L_1(x,D)\left(\tfrac{e_2g_2}{\partial_z\Phi}\right)\right)+\tfrac{e^{\tau\overline{\Phi}}}{\tau^2}L_1(x,D)\left(\tfrac{1}{4\overline{\partial_z\Phi}}L_1(x,D)\left(\tfrac{e_2g_1}{\overline{\partial_z\Phi}}\right)\right).\end{aligned} \tag{4.14}$$

Using Proposition 3.3 we transform the right-hand side of (4.14) as follows:

$$\begin{aligned}&L_1(x,D)((a(z)+\tfrac{a_1(z)}{\tau})e^{\mathcal{A}_1+\tau\Phi}+(d(\bar{z})+\tfrac{d_1(\bar{z})}{\tau})e^{\mathcal{B}_1+\tau\overline{\Phi}}+u_{11}e^{\tau\varphi})\\&=-(q_1-2\tfrac{\partial A_1}{\partial z}-A_1B_1)e^{\tau\Phi}\tfrac{g_1}{2\tau\partial_z\Phi}\\&-(q_1-2\tfrac{\partial B_1}{\partial\bar{z}}-A_1B_1)e^{\tau\overline{\Phi}}\tfrac{g_2}{2\tau\overline{\partial_z\Phi}}+o_{L^4(\Omega)}\left(\tfrac{1}{\tau}\right)\quad\text{as}\,|\tau|\to+\infty.\end{aligned} \tag{4.15}$$

We are looking for u_{12} in the form $u_{12}=u_0+u_{-1}$. The function u_{-1} is given by

$$u_{-1} = \tfrac{e^{i\tau\psi}}{\tau}\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1g_5\}+\tfrac{e^{-i\tau\psi}}{\tau}\mathcal{R}_{-\tau,A_1}\{e_1g_6\}+\tfrac{e_2g_5e^{i\tau\psi}}{2\tau^2\partial_z\Phi}+\tfrac{e_2g_6e^{-i\tau\psi}}{2\tau^2\overline{\partial_z\Phi}}, \tag{4.16}$$

where

(4.17)
$$g_5=\frac{P_{A_1}((q_1-2\frac{\partial A_1}{\partial z}-A_1B_1)g_1)-M_5(z)e^{\mathcal{A}_1}}{2\partial_z\Phi},\, g_6=\frac{T_{B_1}((q_1-2\frac{\partial B_1}{\partial \overline{z}}-A_1B_1)g_2)-M_6(\overline{z})e^{\mathcal{B}_1}}{2\overline{\partial_z\Phi}},$$

where $M_5(z), M_6(\overline{z})$ are polynomials such that

$$g_5|_{\mathcal{H}}=g_6|_{\mathcal{H}}=\nabla g_5|_{\mathcal{H}}=\nabla g_6|_{\mathcal{H}}=0.$$

Using Proposition 2.5 we introduce functions $a_{2,0}, d_{2,0}\in C^2(\bar\Omega)$ (holomorphic and antiholomorphic respectively) such that

$$a_{2,0}(z)e^{\mathcal{A}_1}+d_{2,0}(\overline{z})e^{\mathcal{B}_1}=\frac{g_5}{2\partial_z\Phi}+\frac{g_6}{2\overline{\partial_z\Phi}}\quad\text{on}\,\Gamma_0. \tag{4.18}$$

Next we claim that

$$\mathcal{R}_{-\tau,A_1}\{e_1g_6\}|_{\Gamma_0}=o(\frac{1}{\tau})\text{ as }|\tau|\to+\infty,\ \widetilde{\mathcal{R}}_{\tau,B_1}\{e_1g_5\}|_{\Gamma_0}=o(\frac{1}{\tau})\quad\text{as }|\tau|\to+\infty. \tag{4.19}$$

In order to see this, let us introduce the function $\mathcal{F}$ with domain Γ_0:

$$\begin{aligned}\mathcal{F}&=2e^{-\mathcal{A}_1}e^{\tau(\Phi-\overline{\Phi})}\mathcal{R}_{-\tau,A_1}\{e_1g_6\}\\&=\partial_{\overline{z}}^{-1}(e_1e^{-\mathcal{A}_1+\tau(\Phi-\overline{\Phi})}\frac{T_{B_1}((q_1-2\frac{\partial B_1}{\partial\overline{z}}-A_1B_1)g_2)-M_6e^{\mathcal{B}_1}}{2\overline{\partial_z\Phi}}).\end{aligned}$$

Denoting $r(x)=e^{\mathcal{A}_1}\frac{T_{B_1}((q_1-2\frac{\partial B_1}{\partial\overline{z}}-A_1B_1)g_2)-M_6(\overline{z})e^{\mathcal{B}_1}}{2\overline{\partial_z\Phi}}$ we have

$$\mathcal{F}(x)=-\frac{1}{\pi}\int_\Omega\frac{e_1(x)r(x)e^{2i\tau\psi}}{\overline{\zeta}-\overline{z}}d\xi_1d\xi_2=\frac{1}{2i\pi\tau}\int_\Omega\sum_{k=1}^2\frac{\partial}{\partial x_k}\left(\frac{\partial\psi}{\partial x_k}\frac{e_1(x)}{|\nabla\psi|^2}\frac{r(x)}{\overline{\zeta}-\overline{z}}\right)e^{2i\tau\psi}d\xi_1d\xi_2.$$

Since $\sum_{k=1}^2\frac{\partial}{\partial x_k}(\frac{\frac{\partial\psi}{\partial x_k}}{|\nabla\psi|^2}\frac{e_1(x)r(x)}{\overline{\zeta}-\overline{z}})\in L^1(\Omega)$, we have $\mathcal{F}=o(\frac{1}{\tau})$. This proves (4.19).

Now we finish the construction of the functions $a_{2,\tau}(z)$ and $d_{2,\tau}(\overline{z})$ by setting

$$d_{2,\tau}(\overline{z})=d_{2,0}(\overline{z})+\frac{1}{2}\sum_{k=1}^{\ell}\left(\frac{d_{2,k}(\overline{z})e^{2i\tau\psi(\widetilde{x}_k)}}{|\det\operatorname{Im}\Phi''(\tilde{x}_k)|^{\frac12}}+\frac{d_{2,-k}(\overline{z})e^{-2i\tau\psi(\widetilde{x}_k)}}{|\det\operatorname{Im}\Phi''(\tilde{x}_k)|^{\frac12}}\right),$$

$$a_{2,\tau}(z)=a_{2,0}(z)+\frac{1}{2}\sum_{k=1}^{\ell}\left(\frac{a_{2,k}(z)e^{2i\tau\psi(\widetilde{x}_k)}}{|\det\operatorname{Im}\Phi''(\tilde{x}_k)|^{\frac12}}+\frac{a_{2,-k}(z)e^{-2i\tau\psi(\widetilde{x}_k)}}{|\det\operatorname{Im}\Phi''(\tilde{x}_k)|^{\frac12}}\right),$$

where $a_{2,k}, d_{2,k}$ satisfy (4.13). In order to complete the construction of a solution to (4.1) we define u_0 as the solution to the inhomogeneous problem

$$L_1(x,D)(u_0e^{\tau\varphi})=h_1e^{\tau\varphi}\quad\text{in }\Omega, \tag{4.20}$$

$$u_0e^{\tau\varphi}=e^{\tau\varphi}\mathbf{m}_1\quad\text{on }\Gamma_0, \tag{4.21}$$

where

$$\begin{aligned}h_1&=-e^{-\tau\varphi}L_1(x,D)(a_\tau(z)e^{\mathcal{A}_1+\tau\Phi}+d_\tau(\overline{z})e^{\mathcal{B}_1+\tau\overline{\Phi}}+u_{11}e^{\tau\varphi}+u_{-1}e^{\tau\varphi}),\\ \mathbf{m}_1&=-e^{-\tau\varphi}(a_\tau(z)e^{\mathcal{A}_1+\tau\Phi}+d_\tau(\overline{z})e^{\mathcal{B}_1+\tau\overline{\Phi}}+u_{11}e^{\tau\varphi}+u_{-1}e^{\tau\varphi})|_{\Gamma_0}.\end{aligned}$$

Observe that by (4.15) - (4.17)

$$\|h_1\|_{L^4(\Omega)} = o(\frac{1}{\tau}) \quad \text{as} \, |\tau| \to +\infty$$

and by (4.8), (4.12), (4.13), (4.18)

$$\|u_0\|_{H^{\frac{1}{2}}(\Gamma_0)} = o(\frac{1}{\tau^2}) \quad \text{as} \, |\tau| \to +\infty. \tag{4.22}$$

By Proposition 2.6 and Proposition 3.8 there exists a solution to (4.20), (4.21) such that

$$\frac{1}{\sqrt{|\tau|}}\|u_0\|_{H^1(\Omega)} + \sqrt{|\tau|}\|u_0\|_{L^2(\Omega)} + \|u_0\|_{H^{1,\tau}(\mathcal{O}_\epsilon)} = o(\frac{1}{\tau}) \quad \text{as} \, |\tau| \to +\infty. \tag{4.23}$$

4.1. **Complex geometrical optics solutions for the adjoint operator.** Consider the operator $L_2(x,D) = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z} + 2A_2\frac{\partial}{\partial z} + 2B_2\frac{\partial}{\partial \bar z} + q_2$. Its adjoint has the form

$$\begin{aligned} L_2(x,D)^* &= 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z} - 2\overline{A_2}\frac{\partial}{\partial \bar z} - 2\overline{B_2}\frac{\partial}{\partial z} + \overline{q_2} - 2\frac{\partial \overline{A}_2}{\partial \bar z} - \frac{\partial \overline{B}_2}{\partial z} \\ &= (2\frac{\partial}{\partial z} - \overline{A_2})(2\frac{\partial}{\partial \bar z} - \overline{B_2}) + q_2 - 2\frac{\partial \overline{A_2}}{\partial \bar z} - \overline{A_2B_2} \\ &= (2\frac{\partial}{\partial \bar z} - \overline{B_2})(2\frac{\partial}{\partial z} - \overline{A_2}) + q_2 - 2\frac{\partial \overline{B_2}}{\partial z} - \overline{A_2B_2}. \end{aligned}$$

Next we construct solution to the following boundary value problem:

$$L_2(x,D)^* v = 0 \quad \text{in} \, \Omega, \quad v|_{\Gamma_0} = 0. \tag{4.24}$$

We construct solutions to (4.24) of the form

$$v(x) = b_\tau(z)e^{\mathcal{B}_2 - \tau\Phi} + c_\tau(\bar z)e^{\mathcal{A}_2 - \tau\overline{\Phi}} + v_{11}e^{-\tau\varphi} + v_{12}e^{-\tau\varphi}, \quad v|_{\Gamma_0} = 0. \tag{4.25}$$

Here $\mathcal{A}_2, \mathcal{B}_2 \in C^{6+\alpha}(\overline{\Omega})$ satisfy

$$2\frac{\partial \mathcal{A}_2}{\partial z} = \overline{A_2} \quad \text{in} \, \Omega, \quad \text{Im}\, \mathcal{A}_2|_{\Gamma_0} = 0, \quad 2\frac{\partial \mathcal{B}_2}{\partial \bar z} = \overline{B_2} \quad \text{in} \, \Omega, \quad \text{Im}\, \mathcal{B}_2|_{\Gamma_0} = 0,$$

and $b_\tau(z) = b(z) + \frac{b_1(z)}{\tau} + \frac{b_{2,\tau}(z)}{\tau^2}$, $c_\tau(\bar z) = c(\bar z) + \frac{c_1(\bar z)}{\tau} + \frac{c_{2,\tau}(\bar z)}{\tau^2}$ and

$$b, c \in C^{5+\alpha}(\overline{\Omega}), \ \frac{\partial b}{\partial \bar z} = 0 \, \text{in} \, \Omega, \ \frac{\partial c}{\partial z} = 0 \, \text{in} \, \Omega, \tag{4.26}$$

$$(b(z)e^{\mathcal{B}_2} + c(\bar z)e^{\mathcal{A}_2})|_{\Gamma_0} = 0, \tag{4.27}$$

$$\frac{\partial^k b}{\partial z^k}|_{\mathcal{H}\cap\partial\Omega} = 0, \quad \frac{\partial^k c}{\partial \bar z^k}|_{\mathcal{H}\cap\partial\Omega} = 0 \quad k \in \{0,\dots,5\}. \tag{4.28}$$

The existence of the functions b and c is given by Proposition 2.5. Denote

$$g_3 = P_{-\overline{B_2}}((\overline{q}_2 - 2\frac{\partial \overline{A_2}}{\partial \bar z} - \overline{A_2B_2})be^{\mathcal{B}_2}) - M_3(z)e^{\mathcal{B}_2}, \quad g_4 = T_{-\overline{A_2}}((\overline{q} - 2\frac{\partial \overline{B_2}}{\partial z} - \overline{A_2B_2})ce^{\mathcal{A}_2}) - M_4(\bar z)e^{\mathcal{A}_2},$$

where the polynomials $M_3(z), M_4(\bar z)$ are chosen such that

$$\frac{\partial^k g_3}{\partial z^k}|_{\mathcal{H}} = \frac{\partial^k g_4}{\partial \bar z^k}|_{\mathcal{H}} = 0 \quad \forall k \in \{0,\dots,5\}. \tag{4.29}$$

By (4.29), (4.28)

$$\frac{\partial^{k+j}}{\partial z^k\partial\bar z^j}g_3|_{\mathcal H\cap\partial\Omega}=\frac{\partial^{k+j}}{\partial z^k\partial\bar z^j}g_4|_{\mathcal H\cap\partial\Omega}=0\quad\forall k+j\le 5. \tag{4.30}$$

Observe that by (4.30) $\frac{g_3}{\partial_z\Phi},\frac{g_4}{\partial_{\bar z}\Phi}\in C^{4+\alpha}(\overline\Omega)$. Using Proposition 2.5 we introduce a holomorphic function $b_1(z)\in C^2(\overline\Omega)$ and an antiholomorphic function $c_1(\bar z)\in C^2(\overline\Omega)$ such that

$$b_1e^{\mathcal B_2}+c_1e^{\mathcal A_2}=\frac{e_2g_3}{2\partial_z\Phi}+\frac{e_2g_4}{2\overline{\partial_z\Phi}}\quad\text{on }\Gamma_0. \tag{4.31}$$

Let

$$\hat g_3=P_{-\overline{B_2}}((\overline q_2-2\frac{\partial\overline{A_2}}{\partial\bar z}-\overline{A_2B_2})b_1e^{\mathcal B_2})-\hat M_3(z)e^{\mathcal B_2},\quad \hat g_4=T_{-\overline{A_2}}((\overline q-2\frac{\partial\overline{B_2}}{\partial z}-\overline{A_2B_2})c_1e^{\mathcal A_2})-\hat M_4(\bar z)e^{\mathcal A_2},$$

where the polynomials $\hat M_3(z),\hat M_4(\bar z)$ are chosen such that

$$\frac{\partial^k\hat g_3}{\partial z^k}|_{\mathcal H}=\frac{\partial^k\hat g_4}{\partial\bar z^k}|_{\mathcal H}=0\quad\forall k\in\{0,\dots,3\}. \tag{4.32}$$

The function v_{11} is defined by

$$\begin{aligned}v_{11}\ &=-e^{-i\tau\psi}\widetilde{\mathcal R}_{-\tau,-\overline{A_2}}\{e_1(g_3+\hat g_3/\tau)\}+\tfrac{e^{-i\tau\psi}e_2(g_3+\hat g_3/\tau)}{2\tau\partial_z\Phi}-e^{i\tau\psi}\mathcal R_{\tau,-\overline{B_2}}\{e_1(g_4+\hat g_4/\tau)\}+\tfrac{e^{i\tau\psi}e_2(g_4+\hat g_4/\tau)}{2\tau\overline{\partial_z\Phi}}\\ &-\tfrac{e^{-i\tau\psi}}{4\tau^2\partial_z\Phi}L_2(x,D)^*\left(\tfrac{e_2g_3}{\partial_z\Phi}\right)-\tfrac{e^{i\tau\psi}}{4\tau^2\overline{\partial_z\Phi}}L_2(x,D)^*\left(\tfrac{e_2g_4}{\overline{\partial_z\Phi}}\right).\end{aligned} \tag{4.33}$$

Here we set

$$\mathcal R_{\tau,-\overline{B_2}}\{g\}=\frac12e^{\mathcal B_2}e^{\tau(\overline\Phi-\Phi)}\partial_{\bar z}^{-1}(ge^{-\mathcal B_2}e^{\tau(\Phi-\overline\Phi)})$$
$$\widetilde{\mathcal R}_{-\tau,-\overline{A_2}}\{g\}=\frac12e^{\mathcal A_2}e^{\tau(\Phi-\overline\Phi)}\partial_z^{-1}(ge^{-\mathcal A_2}e^{\tau(\overline\Phi-\Phi)})$$

provided that $A_2,B_2,\mathcal A_2,\mathcal B_2$ satisfy (3.2). By Proposition 7.1 the following asymptotic formulae hold:

$$\mathcal R_{\tau,-\overline B_2}\{e_1g_3\}|_{\Gamma_0}=\frac{1}{2\tau^2}\frac{e^{-\overline{\mathcal B_2}-2\tau i\psi}}{|\det\operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}}\sum_{k=1}^{\ell}\left(\frac{e^{2i\tau\psi(\tilde x_k)}\tilde r_{1,k}(\tilde x_k)}{(z-\tilde z_k)^2}+\frac{e^{2i\tau\psi(\tilde x_k)}\tilde t_{1,k}(\tilde x_k)}{(\tilde z_k-z)}\right)+\widetilde{\mathcal W}_{1,\tau}, \tag{4.34}$$

$$\widetilde{\mathcal R}_{-\tau,-\overline A_2}\{e_1g_4\}|_{\Gamma_0}=\frac{1}{2\tau^2}\frac{e^{-\overline{\mathcal A_2}+2\tau i\psi}}{|\det\operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}}\sum_{k=1}^{\ell}\left(\frac{e^{-2i\tau\psi(\tilde x_k)}r_{1,k}(\tilde x_k)}{(\bar z-\bar{\tilde z}_k)^2}+\frac{e^{-2i\tau\psi(\tilde x_k)}t_{1,k}(\tilde x_k)}{(\bar{\tilde z}_k-\bar z)}\right)+\widetilde{\mathcal W}_{2,\tau}, \tag{4.35}$$

where

$$r_{1,k}=-\frac{\partial_z\tilde g_4(\tilde x_k)}{4\partial_z^2\Phi(\tilde x_k)},\quad t_{1,k}=\frac18\left(-\frac{\partial_z\tilde g_4(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}\frac{\partial_z^3\Phi(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}-\frac{\partial_{\bar z}^2\tilde g_4(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}+\frac{\partial_z^2\tilde g_4(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}\right),$$

$$\tilde r_{1,k}=-\frac{\partial_{\bar z}\tilde g_3(\tilde x_k)}{4\overline{\partial_z^2\Phi(\tilde x_k)}},\quad \tilde t_{1,k}=\frac18\left(-\frac{\partial_{\bar z}\tilde g_3(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}\frac{\partial_{\bar z}^3\overline\Phi(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}+\frac{\partial_{\bar z}^2\tilde g_3(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}-\frac{\partial_z^2\tilde g_3(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}\right),$$

$\tilde g_3=e^{\overline{\mathcal A_2}}g_3,\tilde g_4=e^{\overline{\mathcal B_2}}g_4$. Here the functions $\widetilde{\mathcal W}_{\tau,1},\widetilde{\mathcal W}_{\tau,2}\in H^{\frac12}(\Gamma_0)$ satisfy

$$\|\widetilde{\mathcal W}_{\tau,1}\|_{H^{\frac12}(\Gamma_0)}+\|\widetilde{\mathcal W}_{\tau,2}\|_{H^{\frac12}(\Gamma_0)}=o(\frac{1}{\tau^2})\quad\text{as }|\tau|\to+\infty. \tag{4.36}$$

Using Proposition 2.5 we define the holomorphic functions $b_{2,k}(z) \in C^2(\bar\Omega)$ and antiholomorphic $c_{2,k}(\overline{z}) \in C^2(\bar\Omega)$ such that

$$b_{2,k}(z)e^{\mathcal{B}_2} + c_{2,k}(\overline{z})e^{\mathcal{A}_2} = \widetilde{p}_k \quad \text{on}\,\Gamma_0,\ \forall k \in \{\pm 1, \cdots \pm \ell\}, \tag{4.37}$$

where $\widetilde{p}_k$ is defined as

$$\widetilde{p}_k(z) = e^{-\overline{\mathcal{B}_2}}\left(\frac{\tilde r_{1,k}(\tilde x_k)}{(z-\tilde z_k)^2} + \frac{\tilde t_{1,k}(\tilde x_k)}{(\tilde z_k - z)}\right) \quad \forall k \in \{1,\dots,\ell\},$$

$$\widetilde{p}_k(\overline{z}) = e^{-\overline{\mathcal{A}_2}}\left(\frac{r_{1,-k}(\tilde x_{-k})}{(\overline{z}-\overline{\tilde z_{-k}})^2} + \frac{t_{1,-k}(\tilde x_{-k})}{(\overline{\tilde z_{-k}} - \overline{z})}\right) \quad k \in \{-1,\dots,-\ell\}.$$

Similarly to (4.15) we obtain

$$L_2(x,D)^*((b(z)+\tfrac{b_1(z)}{\tau})e^{\mathcal{B}_2-\tau\Phi(z)} + (\overline{b(z)} + \tfrac{c_1(\overline{z})}{\tau})e^{\mathcal{A}_2-\tau\overline{\Phi(z)}} + v_{11}e^{-\tau\varphi})$$
$$= \tfrac{g_3 e^{-\tau\Phi}}{2\tau\partial_z\Phi}(\overline{q}_2 - 2\tfrac{\partial\overline{A_2}}{\partial\overline{z}} - \overline{A_2B_2}) - \tfrac{g_4 e^{-\tau\overline{\Phi}}}{2\tau\overline{\partial_z\Phi}}(\overline{q}_2 - 2\tfrac{\partial\overline{B_2}}{\partial z} - \overline{A_2B_2}) + o_{L^4(\Omega)}\left(\tfrac{1}{\tau}\right). \tag{4.38}$$

We are looking for v_{12} in the form $v_{12} = v_0 + v_{-1}$. The function v_{-1} is given by

$$v_{-1} = -\frac{e^{\tau i\psi}}{\tau}\mathcal{R}_{\tau,-\overline{B}_2}\{e_1g_7\} - \frac{e^{-\tau i\psi}}{\tau}\widetilde{\mathcal{R}}_{-\tau,-\overline{A}_2}\{e_1g_8\} + \frac{e_2g_7}{2\tau^2\overline{\partial_z\Phi}} + \frac{e_2g_8}{2\tau^2\partial_z\Phi}, \tag{4.39}$$

where

$$g_7 = \frac{P_{-\overline{B_2}}((q_2 - 2\frac{\partial\overline{B_2}}{\partial z} - \overline{A_2B_2})g_3) - M_7(z)e^{\mathcal{B}_2}}{2\partial_z\Phi},\ g_8 = \frac{T_{-\overline{A_2}}((q_2 - 2\frac{\partial\overline{A_2}}{\partial\overline{z}} - \overline{A_2B_2})g_4) - M_8(\overline{z})e^{\mathcal{A}_2}}{2\overline{\partial_z\Phi}}, \tag{4.40}$$

and $M_7(z), M_8(\overline{z})$ are polynomials such that

$$g_7|_{\mathcal{H}} = g_8|_{\mathcal{H}} = \nabla g_7|_{\mathcal{H}} = \nabla g_8|_{\mathcal{H}} = 0. \tag{4.41}$$

Using Proposition 2.5 we introduce functions $b_{2,0}, c_{2,0} \in C^2(\bar\Omega)$ such that

$$b_{2,0}(z)e^{\mathcal{B}_2} + c_{2,0}(\overline{z})e^{\mathcal{A}_2} = \frac{g_7}{2\overline{\partial_z\Phi}} + \frac{g_8}{2\partial_z\Phi} \quad \text{on}\,\Gamma_0. \tag{4.42}$$

Similarly to (4.19) we have

$$(\frac{1}{\tau}\mathcal{R}_{\tau,-\overline{B}_2}\{e_1g_7\} + \frac{1}{\tau}\widetilde{\mathcal{R}}_{-\tau,-\overline{A}_2}\{e_1g_8\})|_{\Gamma_0} = o(\frac{1}{\tau^2}) \quad \text{as}\,|\tau| \to +\infty.$$

Now we finish the construction of the functions $b_{2,\tau}(z)$ and $c_{2,\tau}(\overline{z})$ by setting

$$b_{2,\tau}(\overline{z}) = b_{2,0}(\overline{z}) + \frac{1}{2}\sum_{k=1}^{\ell}\left(\frac{b_{2,k}(\overline{z})e^{2i\tau\psi(\widetilde{x}_k)}}{|\det\operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}} + \frac{b_{2,-k}(\overline{z})e^{-2i\tau\psi(\widetilde{x}_k)}}{|\det\operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}}\right) \tag{4.43}$$

and

$$c_{2,\tau}(z) = c_{2,0}(z) + \frac{1}{2}\sum_{k=1}^{\ell}\left(\frac{c_{2,k}(z)e^{2i\tau\psi(\widetilde{x}_k)}}{|\det\operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}} + \frac{c_{2,-k}(z)e^{-2i\tau\psi(\widetilde{x}_k)}}{|\det\operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}}\right), \tag{4.44}$$

where $b_{2,k}, c_{2,k}$ are defined in (4.37).

Consider the following boundary value problem

$$L_2(x,D)^*(e^{-\tau\varphi}v_0) = h_2 e^{-\tau\varphi} \quad \text{in } \Omega, \tag{4.45}$$

$$e^{-\tau\varphi}v_0|_{\Gamma_0} = \mathbf{m}_2 e^{-\tau\varphi}, \tag{4.46}$$

where

$$h_2 = -e^{\tau\varphi}L_2(x,D)^*(b_\tau(z)e^{\mathcal{B}_2-\tau\Phi} + c_\tau(\overline{z})e^{\mathcal{A}_2-\tau\overline{\Phi}} + v_{11}e^{-\tau\varphi} + v_{-1}e^{-\tau\varphi})$$

and

$$\mathbf{m}_2 = -e^{\tau\varphi}(b_\tau(z)e^{\mathcal{A}_2-\tau\Phi} + c_\tau(\overline{z})e^{\mathcal{B}_2-\tau\overline{\Phi}} + v_{11}e^{-\tau\varphi} + v_{-1}e^{-\tau\varphi}).$$

By (4.38)-(4.40) we can estimate the norm of the function h_2 as

$$\|h_2\|_{L^4(\Omega)} = o(\frac{1}{\tau}) \quad \text{as } |\tau| \to +\infty. \tag{4.47}$$

By (4.43), (4.44), (4.37), (4.36), (4.26) we have

$$\|v_0\|_{H^{\frac{1}{2}}(\Gamma_0)} = o(\frac{1}{\tau^2}) \quad \text{as } |\tau| \to +\infty. \tag{4.48}$$

Thanks to (4.47), (4.48), by Proposition 2.6 and Proposition 3.8 for sufficiently small positive ϵ there exists a solution to problem (4.45), (4.46) such that

$$\frac{1}{\sqrt{|\tau|}}\|v_0\|_{H^1(\Omega)} + \sqrt{|\tau|}\|v_0\|_{L^2(\Omega)} + \|v_0\|_{H^{1,\tau}(\mathcal{O}_\epsilon)} = o(\frac{1}{\tau}) \quad \text{as } |\tau| \to +\infty. \tag{4.49}$$

5. End of the proof of the main theorem

Let u_1 be a complex geometrical optics solution as in (4.2). Let u_2 be a solution to the following boundary value problem

$$L_2(x,D)u_2 = 0 \quad \text{in } \Omega, \quad u_2|_{\partial\Omega} = u_1|_{\partial\Omega}, \quad \frac{\partial u_2}{\partial\nu}|_{\tilde\Gamma} = \frac{\partial u_1}{\partial\nu}|_{\tilde\Gamma}. \tag{5.1}$$

Setting $u = u_1 - u_2, q = q_1 - q_2$ we have

$$L_2(x,D)u + 2(A_1 - A_2)\frac{\partial u_1}{\partial z} + 2(B_1 - B_2)\frac{\partial u_1}{\partial\overline{z}} + qu_1 = 0 \quad \text{in } \Omega, \tag{5.2}$$

$$u|_{\partial\Omega} = 0, \quad \frac{\partial u}{\partial\nu}|_{\tilde\Gamma} = 0. \tag{5.3}$$

Let v be a solution to (4.24) in the form (4.25). Taking the scalar product of (5.2) with $\overline{v}$ in $L^2(\Omega)$ we obtain

$$0 = \int_\Omega (2(A_1 - A_2)\frac{\partial u_1}{\partial z} + 2(B_1 - B_2)\frac{\partial u_1}{\partial\overline{z}} + qu_1)\overline{v}dx. \tag{5.4}$$

Our goal is to get the asymptotic formula for the right hand side of (5.4). We have

Proposition 5.1. *The following asymptotic formula is valid as $|\tau|\to+\infty$:*

$$(qu,v)_{L^2(\Omega)}=\int_\Omega(qa\bar ce^{(\mathcal{A}_1+\overline{\mathcal{A}_2})}+qd\bar be^{(\mathcal{B}_1+\overline{\mathcal{B}_2})})dx \tag{5.5}$$
$$+\int_\Omega(\frac{q}{\tau}(a_1b+a\overline{c_1})e^{(\mathcal{A}_1-\overline{\mathcal{B}_2})}+\frac{q}{\tau}(\overline{a}\overline{b_1}+\bar bd_1)e^{(\mathcal{A}_2-\overline{\mathcal{B}_1})})dx$$
$$+\frac{1}{\tau}\int_\Omega\left(\frac{a\overline{g_4}e^{\mathcal{A}_1}}{2\partial_z\Phi}-\frac{\bar cg_2e^{\overline{\mathcal{A}_2}}}{2\partial_z\Phi}-\frac{\bar bg_1e^{\overline{\mathcal{B}_2}}}{2\partial_{\bar z}\overline{\Phi}}+\frac{d\overline{g_3}e^{\mathcal{B}_1}}{2\overline{\partial_z\Phi}}\right)dx$$
$$+2\pi\sum_{k=1}^{\ell}\frac{(qa\bar b)(\widetilde{x}_k)e^{(\mathcal{A}_1+\overline{\mathcal{B}_2}+2\tau iIm\Phi)(\widetilde{x}_k)}+(qd\bar c)(\widetilde{x}_k)e^{(\mathcal{B}_1+\overline{\mathcal{A}_2}-2i\tau Im\Phi)(\widetilde{x}_k)}}{\tau|det\,Im\,\Phi''(\widetilde{x}_k)|^{\frac12}}+$$
$$\frac{1}{2\tau i}\int_{\partial\Omega}qa\bar be^{\mathcal{A}_1+\overline{\mathcal{B}_2}+2\tau i\psi}\frac{(\nu,\nabla\psi)}{|\nabla\psi|^2}d\sigma-\frac{1}{2\tau i}\int_{\partial\Omega}qd\bar ce^{\mathcal{B}_1+\overline{\mathcal{A}_2}-2\tau i\psi}\frac{(\nu,\nabla\psi)}{|\nabla\psi|^2}d\sigma+o(\frac{1}{\tau}).$$

Proof. By (4.2), (4.9) and Proposition 3.3 we have

$$u_1(x)=(a(z)+\frac{a_1(z)}{\tau})e^{\mathcal{A}_1+\tau\Phi}+(d(\bar z)+\frac{d_1(\overline{z})}{\tau})e^{\mathcal{B}_1+\tau\overline{\Phi}}-\frac{g_1e^{\tau\overline{\Phi}}}{2\tau\overline{\partial_z\Phi}}-\frac{g_2e^{\tau\Phi}}{2\tau\partial_z\Phi}+o_{L^2(\Omega)}(\frac{1}{\tau}). \tag{5.6}$$

Using (4.25), (4.33) and Proposition 3.3 we obtain

$$v(x)=(b(z)+\frac{b_1(z)}{\tau})e^{\mathcal{B}_2-\tau\Phi}+(c(\bar z)+\frac{c_1(\overline{z})}{\tau})e^{\mathcal{A}_2-\tau\overline{\Phi}}+\frac{g_4e^{-\tau\overline{\Phi}}}{2\tau\overline{\partial_z\Phi}}+\frac{g_3e^{-\tau\Phi}}{2\tau\partial_z\Phi}+o_{L^2(\Omega)}(\frac{1}{\tau}). \tag{5.7}$$

By (5.6), (5.7) we obtain

$$(qu,v)_{L^2(\Omega)}=(q((a+\frac{a_1}{\tau})e^{\mathcal{A}_1+\tau\Phi}+(d+\frac{d_1}{\tau})e^{\mathcal{B}_1+\tau\overline{\Phi}}-\frac{g_1e^{\tau\overline{\Phi}}}{2\tau\overline{\partial_z\Phi}}-\frac{g_2e^{\tau\Phi}}{2\tau\partial_z\Phi}+o_{L^2(\Omega)}(\frac{1}{\tau})),$$
$$(b+\frac{b_1}{\tau})e^{\mathcal{B}_2-\tau\Phi}+(c+\frac{c_1}{\tau})e^{\mathcal{A}_2-\tau\overline{\Phi}}+\frac{g_4e^{-\tau\overline{\Phi}}}{2\tau\overline{\partial_z\Phi}}+\frac{g_3e^{-\tau\Phi}}{2\tau\partial_z\Phi}+o_{L^2(\Omega)}(\frac{1}{\tau}))_{L^2(\Omega)}=$$
$$\int_\Omega\left(q(d\bar b+\frac{1}{\tau}(d_1\bar b+d\bar b_1))e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}+q(a\bar c+\frac{1}{\tau}(a\overline{c_1}+a_1\bar c))e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}\right)dx$$
$$+\frac{1}{\tau}\int_\Omega\left(\frac{a\overline{g_4}e^{\mathcal{A}_1}}{2\partial_z\Phi}-\frac{\bar cg_2e^{\overline{\mathcal{A}_2}}}{2\partial_z\Phi}-\frac{\bar bg_1e^{\overline{\mathcal{B}_2}}}{2\partial_{\bar z}\overline{\Phi}}+\frac{d\overline{g_3}e^{\mathcal{B}_1}}{2\overline{\partial_z\Phi}}\right)dx$$
$$+\int_\Omega q\left(a\bar be^{\mathcal{A}_1+\overline{\mathcal{B}_2}+\tau(\Phi-\overline{\Phi})}+d\bar ce^{\mathcal{B}_1+\overline{\mathcal{A}_2}+\tau(\overline{\Phi}-\Phi)}\right)dx+o(\frac{1}{\tau}).$$

Applying the stationary phase argument to the last integral in the right hand side of this formula we finish the proof of Proposition 5.1. □

We set

$$\mathcal{U}(x)=a_\tau(z)e^{\mathcal{A}_1(x)+\tau\Phi(z)}+d_\tau(\overline{z})e^{\mathcal{B}_1(x)+\tau\overline{\Phi(z)}},\ \mathcal{V}(x)=b_\tau(z)e^{\mathcal{B}_2(x)-\tau\Phi(z)}+c_\tau(\overline{z})e^{\mathcal{A}_2(x)-\tau\overline{\Phi(z)}}. \tag{5.8}$$

Short calculations give:

$$\begin{aligned}
I_1 &\equiv 2((A_1-A_2)\frac{\partial \mathcal{U}}{\partial z},\mathcal{V})_{L^2(\Omega)}\\
&= (2(A_1-A_2)(\left(\frac{\partial \mathcal{A}_1}{\partial z}+\tau\frac{\partial \Phi}{\partial z}\right)a_\tau+\frac{\partial a_\tau}{\partial z})e^{\mathcal{A}_1+\tau\Phi}+d_\tau\frac{\partial \mathcal{B}_1}{\partial z}e^{\mathcal{B}_1+\tau\overline{\Phi}},\\
&\qquad b_\tau e^{\mathcal{B}_2-\tau\Phi}+c_\tau e^{\mathcal{A}_2-\tau\overline{\Phi}})_{L^2(\Omega)}+\frac{1}{\tau}\mathcal{I}_1(\partial\Omega)\\
&= \sum_{k=1}^{3}\tau^{2-k}\kappa_k-\int_\Omega (A_1-A_2)B_1 d_\tau(\overline{z})\overline{c_\tau(\overline{z})}e^{\mathcal{B}_1+\overline{\mathcal{A}_2}-2i\tau\psi}dx\\
&\quad -\left(2\frac{\partial}{\partial z}(A_1-A_2)a_\tau e^{\mathcal{A}_1+\tau\Phi}, b_\tau e^{\mathcal{B}_2-\tau\Phi}\right)_{L^2(\Omega)}\\
&\quad -(2(A_1-A_2)a_\tau e^{\mathcal{A}_1+\tau\Phi},\frac{\partial \mathcal{B}_2}{\partial \overline{z}}b_\tau e^{\mathcal{B}_2-\tau\Phi})_{L^2(\Omega)}\\
&\quad +\int_{\partial\Omega}(A_1-A_2)(\nu_1-i\nu_2)a_\tau\overline{b_\tau}e^{\mathcal{A}_1+\overline{\mathcal{B}_2}+2i\tau\psi}d\sigma+o\left(\frac{1}{\tau}\right)\\
&= \sum_{k=1}^{3}\tau^{2-k}\kappa_k+\int_\Omega\Big\{-(A_1-A_2)B_1 d_\tau(\overline{z})\overline{c_\tau(\overline{z})}e^{\mathcal{B}_1+\overline{\mathcal{A}_2}-2i\tau\psi}\\
&\quad -(A_1-A_2)B_2 a_\tau\overline{b_\tau(z)}e^{\mathcal{A}_1+\overline{\mathcal{B}_2}+2i\tau\psi}-2\frac{\partial}{\partial z}(A_1-A_2)a_\tau\overline{b_\tau(z)}e^{\mathcal{A}_1+\overline{\mathcal{B}_2}+2i\tau\psi}\Big\}dx\\
&\quad +\int_{\partial\Omega}(A_1-A_2)(\nu_1-i\nu_2)a_\tau\overline{b_\tau}e^{\mathcal{A}_1+\overline{\mathcal{B}_2}+2i\tau\psi}d\sigma+\frac{1}{\tau}\mathcal{I}_1(\partial\Omega)+o\left(\frac{1}{\tau}\right) \qquad (5.9)
\end{aligned}$$

and

$$I_2 \equiv ((B_1 - B_2)\frac{\partial \mathcal{U}}{\partial \overline{z}}, \mathcal{V})_{L^2(\Omega)}$$

$$= (2(B_1 - B_2)(a_\tau e^{\mathcal{A}_1+\tau\Phi}\frac{\partial \mathcal{A}_1}{\partial \overline{z}} + \frac{\partial}{\partial \overline{z}}\left(d_\tau e^{\mathcal{B}_1+\tau\overline{\Phi}}\right)), b_\tau e^{\mathcal{B}_2-\tau\Phi} + c_\tau e^{\mathcal{A}_2-\tau\overline{\Phi}})_{L^2(\Omega)}$$

$$= \sum_{k=1}^{3}\tau^{2-k}\widetilde{\kappa}_k + \int_\Omega 2(B_1 - B_2)\frac{\partial \mathcal{A}_1}{\partial \overline{z}} a_\tau \overline{b_\tau} e^{\mathcal{A}_1+\overline{\mathcal{B}_2}-2\tau i\psi}dx - \Big(2\frac{\partial}{\partial \overline{z}}(B_1 - B_2)d_\tau e^{\mathcal{B}_1+\tau\overline{\Phi}},$$

$$b_\tau e^{\mathcal{B}_2-\tau\Phi} + c_\tau e^{\mathcal{A}_2-\tau\overline{\Phi}}\Big)_{L^2(\Omega)} - \left(2(B_1 - B_2)d_\tau e^{\mathcal{B}_1+\tau\overline{\Phi}}, \frac{\partial \mathcal{A}_2}{\partial z}c_\tau e^{\mathcal{A}_2-\tau\overline{\Phi}}\right)_{L^2(\Omega)}$$

$$+ \int_{\partial\Omega}(B_1 - B_2)(\nu_1 + i\nu_2)d_\tau(\overline{z})\overline{c_\tau(\overline{z})}e^{\mathcal{B}_1+\overline{\mathcal{A}_2}-2i\tau\psi}d\sigma + \frac{1}{\tau}\mathcal{I}_2(\partial\Omega) + o\left(\frac{1}{\tau}\right)$$

$$= \sum_{k=1}^{3}\tau^{2-k}\widetilde{\kappa}_k + \int_\Omega \Big\{-(B_1 - B_2)A_1 a_\tau \overline{b_\tau} e^{\mathcal{A}_1+\overline{\mathcal{B}_2}+2\tau i\psi}$$

$$- 2\frac{\partial}{\partial \overline{z}}(B_1 - B_2)d_\tau(\overline{z})\overline{c_\tau(\overline{z})}e^{\mathcal{B}_1+\overline{\mathcal{A}_2}-2i\tau\psi} - (B_1 - B_2)d_\tau(\overline{z})\overline{c_\tau(\overline{z})}A_2 e^{\mathcal{B}_1+\overline{\mathcal{A}_2}-2i\tau\psi}\Big\}dx$$

$$+ \int_{\partial\Omega}(B_1 - B_2)(\nu_1 + i\nu_2)d_\tau(\overline{z})\overline{c_\tau(\overline{z})}e^{\mathcal{B}_1+\overline{\mathcal{A}_2}-2i\tau\psi}d\sigma + \frac{1}{\tau}\mathcal{I}_2(\partial\Omega) + o\left(\frac{1}{\tau}\right). \tag{5.10}$$

Here $\kappa_k, \widetilde{\kappa}_k$ are some constants independent of τ but may be dependent on A_j, B_j, Φ. The terms $\mathcal{I}_1(\partial\Omega), \mathcal{I}_2(\partial\Omega)$ are given by formulae

$$\mathcal{I}_1(\partial\Omega) = \int_\Omega (A_1 - A_2)e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}\frac{\partial \Phi}{\partial z}\overline{c(\overline{z})}\sum_{k=1}^{\ell}\left(\frac{a_{2,k}e^{-2\tau i\psi(\tilde{x}_k)}}{|\det \operatorname{Im}\Phi''(\tilde{x}_k)|^{\frac{1}{2}}} + \frac{a_{2,-k}e^{2\tau i\psi(\tilde{x}_k)}}{|\det \operatorname{Im}\Phi''(\tilde{x}_k)|^{\frac{1}{2}}}\right)dx$$

$$+ \int_\Omega (A_1 - A_2)e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}\frac{\partial \Phi}{\partial z}a(z)\sum_{k=1}^{\ell}\left(\overline{\frac{c_{2,k}e^{2\tau i\psi(\tilde{x}_k)}}{|\det \operatorname{Im}\Phi''(\tilde{x}_k)|^{\frac{1}{2}}} + \frac{c_{2,-k}e^{-2\tau i\psi(\tilde{x}_k)}}{|\det \operatorname{Im}\Phi''(\tilde{x}_k)|^{\frac{1}{2}}}}\right)dx =$$

$$-2\int_\Omega \frac{\partial}{\partial \overline{z}}e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}\frac{\partial \Phi}{\partial z}\overline{c(\overline{z})}\sum_{k=1}^{\ell}\left(\frac{a_{2,k}e^{-2\tau i\psi(\tilde{x}_k)}}{|\det \operatorname{Im}\Phi''(\tilde{x}_k)|^{\frac{1}{2}}} + \frac{a_{2,-k}e^{2\tau i\psi(\tilde{x}_k)}}{|\det \operatorname{Im}\Phi''(\tilde{x}_k)|^{\frac{1}{2}}}\right)dx$$

$$-2\int_\Omega \frac{\partial}{\partial \overline{z}}e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}\frac{\partial \Phi}{\partial z}a(z)\sum_{k=1}^{\ell}\left(\overline{\frac{c_{2,k}e^{2\tau i\psi(\tilde{x}_k)}}{|\det \operatorname{Im}\Phi''(\tilde{x}_k)|^{\frac{1}{2}}} + \frac{c_{2,-k}e^{-2\tau i\psi(\tilde{x}_k)}}{|\det \operatorname{Im}\Phi''(\tilde{x}_k)|^{\frac{1}{2}}}}\right)dx =$$

$$-2\int_{\partial\Omega}(\nu_1 + i\nu_2)e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}\frac{\partial \Phi}{\partial z}\overline{c(\overline{z})}\sum_{k=1}^{\ell}\left(\frac{a_{2,k}e^{-2\tau i\psi(\tilde{x}_k)}}{|\det \operatorname{Im}\Phi''(\tilde{x}_k)|^{\frac{1}{2}}} + \frac{a_{2,-k}e^{2\tau i\psi(\tilde{x}_k)}}{|\det \operatorname{Im}\Phi''(\tilde{x}_k)|^{\frac{1}{2}}}\right)d\sigma$$

$$-2\int_{\partial\Omega}(\nu_1 + i\nu_2)e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}\frac{\partial \Phi}{\partial z}a(z)\sum_{k=1}^{\ell}\left(\overline{\frac{c_{2,k}e^{2\tau i\psi(\tilde{x}_k)}}{|\det \operatorname{Im}\Phi''(\tilde{x}_k)|^{\frac{1}{2}}} + \frac{c_{2,-k}e^{-2\tau i\psi(\tilde{x}_k)}}{|\det \operatorname{Im}\Phi''(\tilde{x}_k)|^{\frac{1}{2}}}}\right)d\sigma \tag{5.11}$$

and

$$\mathcal{I}_2(\partial\Omega) = \int_\Omega (B_1 - B_2) e^{\mathcal{B}_1+\overline{\mathcal{B}_2}} \frac{\partial\overline{\Phi}}{\partial\bar z}\overline{b(z)} \sum_{k=1}^{\ell} \left(\frac{d_{2,k}e^{-2\tau i\psi(\tilde x_k)}}{|\det \mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}} + \frac{d_{2,-k}e^{2\tau i\psi(\tilde x_k)}}{|\det \mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}} \right) dx$$

$$+ \int_\Omega (B_1 - B_2) e^{\mathcal{B}_1+\overline{\mathcal{B}_2}} \frac{\partial\overline{\Phi}}{\partial\bar z} d(\bar z) \sum_{k=1}^{\ell} \overline{\left(\frac{b_{2,k}e^{2\tau i\psi(\tilde x_k)}}{|\det \mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}} + \frac{b_{2,-k}e^{-2\tau i\psi(\tilde x_k)}}{|\det \mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}} \right)} dx =$$

$$-2\int_\Omega \frac{\partial}{\partial z} e^{\mathcal{B}_1+\overline{\mathcal{B}_2}} \frac{\partial\overline{\Phi}}{\partial\bar z}\overline{b(z)} \sum_{k=1}^{\ell} \left(\frac{d_{2,k}e^{-2\tau i\psi(\tilde x_k)}}{|\det \mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}} + \frac{d_{2,-k}e^{2\tau i\psi(\tilde x_k)}}{|\det \mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}} \right) dx$$

$$-2\int_\Omega \frac{\partial}{\partial z} e^{\mathcal{B}_1+\overline{\mathcal{B}_2}} \frac{\partial\Phi}{\partial z} d(\bar z) \sum_{k=1}^{\ell} \overline{\left(\frac{b_{2,k}e^{2\tau i\psi(\tilde x_k)}}{|\det \mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}} + \frac{b_{2,-k}e^{-2\tau i\psi(\tilde x_k)}}{|\det \mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}} \right)} dx =$$

$$-2\int_{\partial\Omega} (\nu_1 - i\nu_2) e^{\mathcal{B}_1+\overline{\mathcal{B}_2}} \frac{\partial\overline{\Phi}}{\partial\bar z}\overline{b(z)} \sum_{k=1}^{\ell} \left(\frac{d_{2,k}e^{-2\tau i\psi(\tilde x_k)}}{|\det \mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}} + \frac{d_{2,-k}e^{2\tau i\psi(\tilde x_k)}}{|\det \mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}} \right) d\sigma$$

$$-2\int_{\partial\Omega} (\nu_1 - i\nu_2) e^{\mathcal{B}_1+\overline{\mathcal{B}_2}} \frac{\partial\overline{\Phi}}{\partial\bar z} d(\bar z) \sum_{k=1}^{\ell} \overline{\left(\frac{b_{2,k}e^{2\tau i\psi(\tilde x_k)}}{|\det \mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}} + \frac{b_{2,-k}e^{-2\tau i\psi(\tilde x_k)}}{|\det \mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}} \right)} d\sigma. \tag{5.12}$$

Denote

$$U_1 = -e^{\tau\overline{\Phi}}\mathcal{R}_{-\tau,A_1}\{e_1g_1\}, \quad U_2 = -e^{\tau\Phi}\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1g_2\}.$$

A short calculation gives

$$2\frac{\partial U_1}{\partial \bar z} = (-e_1g_1 + A_1\mathcal{R}_{-\tau,A_1}\{e_1g_1\})e^{\tau\overline{\Phi}} \tag{5.13}$$

and

$$2\frac{\partial U_2}{\partial z} = (-e_1g_2 + B_1\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1g_2\})e^{\tau\Phi}. \tag{5.14}$$

We calculate

$$\frac{\partial}{\partial z}\mathcal{R}_{-\tau,A_1}\{e_1g_1\} = \frac{\partial\mathcal{A}_1}{\partial z}\mathcal{R}_{-\tau,A_1}\{e_1g_1\} + \tau\frac{\partial\Phi}{\partial z}\mathcal{R}_{-\tau,A_1}\{e_1g_1\} + \mathcal{R}_{-\tau,A_1}\{\frac{\partial(e_1g_1)}{\partial z}\}$$

$$-\mathcal{R}_{-\tau,A_1}\{e_1g_1\frac{\partial\mathcal{A}_1}{\partial z}\} - \tau\mathcal{R}_{-\tau,A_1}\{\frac{\partial\Phi}{\partial z}e_1g_1\} = \mathcal{R}_{-\tau,A_1}\{\frac{\partial(e_1g_1)}{\partial z}\}$$

$$+\tau\frac{e^{\mathcal{A}_1}}{2\pi}e^{-\tau(\overline{\Phi}-\Phi)}\int_\Omega \frac{\frac{\partial\Phi}{\partial\zeta}(\zeta) - \frac{\partial\Phi}{\partial z}(z)}{\zeta - z}(e_1g_1e^{-\mathcal{A}_1})(\xi_1,\xi_2)e^{\tau(\overline{\Phi(\zeta)}-\Phi(\zeta))}d\xi_1d\xi_2$$

$$+\frac{e^{\mathcal{A}_1}}{2\pi}e^{-\tau(\overline{\Phi}-\Phi)}\int_\Omega \frac{\frac{\partial\mathcal{A}_1}{\partial\zeta}(\zeta,\bar\zeta) - \frac{\partial\mathcal{A}_1}{\partial z}(z,\bar z)}{\zeta - z}(e_1g_1e^{-\mathcal{A}_1})(\xi_1,\xi_2)e^{\tau(\overline{\Phi(\zeta)}-\Phi(\zeta))}d\xi_1d\xi_2. \tag{5.15}$$

Denote $\mathcal{P}(x,D) = 2(A_1 - A_2)\frac{\partial}{\partial z} + 2(B_1 - B_2)\frac{\partial}{\partial \bar z}$. Let

$$\mathfrak{G}(x,g,\mathcal{A},\tau) = -\frac{1}{2\pi}\int_\Omega \frac{(\tau\frac{\partial\Phi(\zeta)}{\partial\zeta} + \frac{\partial\mathcal{A}(\zeta,\bar\zeta)}{\partial\zeta}) - (\tau\frac{\partial\Phi(z)}{\partial z} + \frac{\partial\mathcal{A}(z,\bar z)}{\partial z})}{\zeta - z}e_1ge^{-\mathcal{A}}e^{\tau(\overline{\Phi}-\Phi)}d\xi_1d\xi_2,$$

We set $\mathfrak{G}_1(x,\tau) = \mathfrak{G}(x,g_1,\mathcal{A}_1,\tau)$, $\mathfrak{G}_2(x,\tau) = \overline{\mathfrak{G}(x,\bar g_2,\overline{\mathcal{B}_1},\tau)}$, $\mathfrak{G}_3(x,\tau) = \overline{\mathfrak{G}(x,\overline{g_4},\overline{\mathcal{A}_2},-\tau)}$, $\mathfrak{G}_4 = \mathfrak{G}(x,g_3,\mathcal{B}_2,-\tau)$.

By (3.15), (4.6), (5.15) and Proposition 3.3 we have

$$\frac{\partial}{\partial z}\mathcal{R}_{-\tau,A_1}\{e_1g_1\} = \frac{\frac{\partial(e_1g_1)}{\partial z}}{2\tau\overline{\partial_z\Phi}} - e^{\mathcal{A}_1}e^{-\tau(\overline{\Phi}-\Phi)}\mathfrak{G}_1(\cdot,\tau) + o_{L^2(\Omega)}(\frac{1}{\tau}). \tag{5.16}$$

Simple computations provide the formula

$$\frac{\partial}{\partial \overline{z}}\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1g_2\} = \frac{\partial\mathcal{B}_1}{\partial\overline{z}}\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1g_2\} + \tau\frac{\partial\overline{\Phi}}{\partial\overline{z}}\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1g_2\} + \widetilde{\mathcal{R}}_{\tau,B_1}\{\frac{\partial(e_1g_2)}{\partial\overline{z}}\}$$
$$-\tau\widetilde{\mathcal{R}}_{\tau,B_1}\{\frac{\partial\overline{\Phi}}{\partial\overline{z}}e_1g_2\} - \widetilde{\mathcal{R}}_{\tau,B_1}\{\frac{\partial\mathcal{B}_1}{\partial\overline{z}}e_1g_2\} = \widetilde{\mathcal{R}}_{\tau,B_1}\{\frac{\partial(e_1g_2)}{\partial\overline{z}}\}$$
$$+\tau\frac{e^{\mathcal{B}_1}}{2\pi}e^{\tau(\overline{\Phi}-\Phi)}\int_\Omega\frac{\frac{\partial\overline{\Phi}}{\partial\overline{\zeta}}(\overline{\zeta}) - \frac{\partial\overline{\Phi}}{\partial\overline{z}}(\overline{z})}{\overline{\zeta}-\overline{z}}(e_1g_2e^{-\mathcal{B}_1})(\xi_1,\xi_2)e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1d\xi_2$$
$$+\frac{e^{\mathcal{B}_1}}{2\pi}e^{\tau(\overline{\Phi}-\Phi)}\int_\Omega\frac{\frac{\partial\mathcal{B}_1}{\partial\overline{\zeta}}(\zeta,\overline{\zeta}) - \frac{\partial\mathcal{B}_1}{\partial\overline{z}}(z,\overline{z})}{\overline{\zeta}-\overline{z}}(e_1g_2e^{-\mathcal{B}_1})(\xi_1,\xi_2)e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1d\xi_2. \tag{5.17}$$

By (3.15), (4.6), (5.17), Proposition 3.3 we have

$$\frac{\partial}{\partial \overline{z}}\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1g_2\} = \frac{\frac{\partial(e_1g_2)}{\partial\overline{z}}}{2\tau\partial_z\Phi} - e^{\mathcal{B}_1}e^{\tau(\overline{\Phi}-\Phi)}\mathfrak{G}_2(\cdot,\tau) + o_{L^2(\Omega)}(\frac{1}{\tau}). \tag{5.18}$$

Denote

$$V_2 = -e^{-\tau\overline{\Phi}}\mathcal{R}_{\tau,-\overline{B}_2}\{e_1g_3\}, \quad V_1 = -e^{-\tau\Phi}\widetilde{\mathcal{R}}_{-\tau,\overline{A}_2}\{e_1g_4\}.$$

We have

Proposition 5.2. *There exist two numbers κ,κ_0 independent of τ such that the following asymptotic formula holds true:*

$$(\mathcal{P}(x,D)(U_1+U_2), b_\tau e^{\mathcal{B}_2-\tau\Phi} + c_\tau e^{\mathcal{A}_2-\tau\overline{\Phi}})_{L^2(\Omega)} + (\mathcal{P}(x,D)(a_\tau e^{\mathcal{A}_1+\tau\Phi} + d_\tau e^{\mathcal{B}_1+\tau\overline{\Phi}}), V_1+V_2)_{L^2(\Omega)} =$$
$$\kappa + \frac{\kappa_0}{\tau} - 2\int_{\partial\Omega}(\nu_1+i\nu_2)e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}\overline{c_\tau(\overline{z})}\mathfrak{G}_1(x,\tau)d\sigma - 2\int_{\partial\Omega}(\nu_1-i\nu_2)e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}d_\tau(\overline{z})\overline{\mathfrak{G}_4(x,\tau)}d\sigma$$
$$-2\int_{\partial\Omega}a_\tau(z)\overline{\mathfrak{G}_3(x,\tau)}(\nu_1+i\nu_2)e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}d\sigma - 2\int_{\partial\Omega}(\nu_1-i\nu_2)e^{\mathcal{B}_1+\overline{\mathcal{B}}_2}\overline{b_\tau(z)}\mathfrak{G}_2(x,\tau)d\sigma. \tag{5.19}$$

The proof of this proposition is given at the end of Section 7.

By (5.13), (5.14), (5.16), (5.18) and Proposition 3.4 there exists a constant $\mathcal{C}_0$ independent of τ such that

$$(\quad \mathcal{P}(x,D)(U_1+U_2), V_1+V_2)_{L^2(\Omega)} = ((A_1-A_2)(-2\left(\frac{\frac{\partial(e_1g_1)}{\partial z}}{2\tau\overline{\partial_z\Phi}} - e^{\mathcal{A}_1}e^{-\tau(\bar{\Phi}-\Phi)}\mathfrak{G}_1 + o_{L^2(\Omega)}(\frac{1}{\tau})\right)e^{\tau\overline{\Phi}}$$
$$+ \quad (-e_1g_2 + \frac{B_1e_1g_2}{2\tau\partial_z\Phi} + o_{L^2(\Omega)}(\frac{1}{\tau}))e^{\tau\Phi}), V_1+V_2)_{L^2(\Omega)}$$
$$+ \quad ((B_1-B_2)(-e_1g_1 + A_1\frac{e_1g_1}{2\tau\overline{\partial_z\Phi}} + o_{L^2(\Omega)}(\frac{1}{\tau}))e^{\tau\overline{\Phi}}, V_1+V_2)_{L^2(\Omega)}$$
$$- \quad (2(B_1-B_2)(\frac{\frac{\partial(e_1g_2)}{\partial\overline{z}}}{2\tau\partial_z\Phi} - e^{\mathcal{B}_1}e^{\tau(\bar{\Phi}-\Phi)}\mathfrak{G}_2 + o_{L^2(\Omega)}(\frac{1}{\tau}))e^{\tau\Phi}, V_1+V_2)_{L^2(\Omega)} = \frac{\mathcal{C}_0}{\tau} + o(\frac{1}{\tau}) \quad \text{as } |\tau|\to+\infty.$$

Next we claim that

$$(\mathcal{P}(x,D)(u_0e^{\tau\varphi}),v)_{L^2(\Omega)} = o(\frac{1}{\tau}) \quad \text{as } |\tau|\to+\infty, \tag{5.20}$$

and

$$(\mathcal{P}(x,D)u,v_0e^{-\tau\varphi})_{L^2(\Omega)} = o(\frac{1}{\tau}) \quad \text{as } |\tau|\to+\infty. \tag{5.21}$$

Let us first prove (5.21). By (4.23) and (4.49), we have

$$(\mathcal{P}(x,D)u,v_0e^{-\tau\varphi})_{L^2(\Omega)} = (\mathcal{P}(x,D)\mathcal{U},v_0e^{-\tau\varphi})_{L^2(\Omega)} + o(\frac{1}{\tau}) \quad \text{as } |\tau|\to+\infty. \tag{5.22}$$

We remind that the function $\mathcal{U}$ and $\mathcal{V}$ are defined by (5.8). By (4.49) we obtain from (5.22)
(5.23)

$$(\mathcal{P}(x,D)u,v_0e^{-\tau\varphi})_{L^2(\Omega)} = \tau\int_\Omega 2\chi(\frac{\partial\Phi}{\partial z}(A_1-A_2)ae^{\mathcal{A}_1+i\tau\psi}+\frac{\partial\overline{\Phi}}{\partial\overline{z}}(B_1-B_2)be^{\mathcal{B}_1-i\tau\psi})\overline{v_0}dx+o(\frac{1}{\tau})$$

as $|\tau|\to+\infty$. Here $\chi\in C_0^\infty(\overline{\Omega})$ is a function such that $\chi\equiv 1$ in some neighborhood of $\overline{\operatorname{supp} e_2, \mathcal{H}\setminus\partial\Omega\subset supp\, e_2}$. The functions $v_{0,+}=e^{-i\tau\psi}\overline{v_0}$ and $v_{0,-}=e^{i\tau\psi}\overline{v_0}$ satisfy $e^{\tau\Phi}\overline{L_2(x,D)^*(e^{-\tau\overline{\Phi}}v_{0,+})}=\overline{h}_2e^{i\tau\psi}$ and $e^{\tau\overline{\Phi}}\overline{L_2(x,D)^*(e^{-\tau\Phi}v_{0,-})}=\overline{h}_2e^{-i\tau\psi}$. More explicitly there exist two first-order operators $\mathcal{P}_k(x,D)$ such that

$$e^{\tau\overline{\Phi}}\overline{L_2(x,D)^*(e^{-\tau\overline{\Phi}}v_{0,+})}=\Delta\overline{v_{0,+}}-2\tau\frac{\partial\Phi}{\partial z}(2\frac{\partial\overline{v_{0,+}}}{\partial\overline{z}}-A_2\overline{v}_{0,+})+\mathcal{P}_1(x,D)\overline{v_{0,+}}=o_{L^2(\Omega)}(\frac{1}{\tau}) \quad \text{as } |\tau|\to+\infty$$

and

$$e^{\tau\Phi}\overline{L_2(x,D)^*(e^{-\tau\Phi}v_{0,-})}=\Delta\overline{v_{0,-}}-2\tau\frac{\partial\overline{\Phi}}{\partial\overline{z}}(2\frac{\partial\overline{v_{0,-}}}{\partial z}-B_2\overline{v}_{0,-})+\mathcal{P}_2(x,D)\overline{v_{0,-}}=o_{L^2(\Omega)}(\frac{1}{\tau}) \quad \text{as } |\tau|\to+\infty.$$

Let $\chi_1\in C_0^\infty(\overline{\Omega})$ be a function such that $\chi_1\equiv 1$ on supp χ. Taking the scalar product of the first equation with $\chi_1 g$ where $g\in C^2(\overline{\Omega})$ we obtain

$$\int_\Omega\tau\frac{\partial\Phi}{\partial z}\overline{v_{0,+}}\chi_1(2\frac{\partial}{\partial\overline{z}}-A_2)gdx=o(\frac{1}{\tau})-\int_\Omega(\overline{v_{0,+}}(\Delta+\mathcal{P}_1(x,D)^*)(\chi_1 g)-\tau\overline{v_{0,+}}\frac{\partial\Phi}{\partial z}g(\frac{\partial}{\partial\overline{z}}-A_2)\chi_1)dx.$$

By (4.49) we have

$$\int_\Omega\tau\frac{\partial\Phi}{\partial z}\overline{v_{0,+}}\chi_1(2\frac{\partial}{\partial\overline{z}}-A_2)gdx=o(\frac{1}{\tau}) \quad \text{as } |\tau|\to+\infty. \tag{5.24}$$

Taking the scalar product of the second equation with $\chi_1 g$ where $g\in C^2(\overline{\Omega})$ we have

$$\int_\Omega\tau\frac{\partial\Phi}{\partial\overline{z}}\overline{v_{0,-}}\chi_1(2\frac{\partial}{\partial z}-B_2)gdx=o(\frac{1}{\tau})-\int_\Omega(\overline{v_{0,-}}(\Delta+\mathcal{P}_2(x,D)^*)(\chi_1 g)-\tau\overline{v_{0,-}}g\frac{\partial\Phi}{\partial\overline{z}}(2\frac{\partial}{\partial z}-B_2)\chi_1)dx.$$

By (4.49) we obtain

$$\int_\Omega\tau\frac{\partial\Phi}{\partial\overline{z}}\overline{v_{0,-}}\chi_1(2\frac{\partial}{\partial z}-B_2)gdx=o(\frac{1}{\tau}) \quad \text{as } |\tau|\to+\infty. \tag{5.25}$$

Taking g such that $(2\frac{\partial}{\partial\overline{z}}-A_2)g=(A_1-A_2)e^{\mathcal{A}_1}a(z)$ in (5.24) and g such that $(2\frac{\partial}{\partial z}-B_2)g=(B_1-B_2)b(\overline{z})e^{\mathcal{B}_1}$ in (5.25) from (5.22) we obtain (5.21).

In order to prove (5.20) we observe:

$$(\mathcal{P}(x,D)(u_0e^{\tau\varphi}),v)=(\mathcal{P}(x,D)(u_0e^{\tau\varphi}),\mathcal{V})_{L^2(\Omega)}+o(\frac{1}{\tau})=(\mathcal{P}(x,D)(u_0e^{\tau\varphi}),\chi\mathcal{V})_{L^2(\Omega)}+o(\frac{1}{\tau})$$

$$=(u_0e^{\tau\varphi},\mathcal{P}(x,D)^*(\chi\mathcal{V}))_{L^2(\Omega)}+o(\frac{1}{\tau})\quad \text{as } |\tau|\to+\infty. \tag{5.26}$$

Then we can finish the proof of (5.20) using arguments similar to (5.23)-(5.24).

Denote $\mathcal{M}_1=\frac{1}{4\partial_{\bar z}\overline{\Phi}}L_1(x,D)(\frac{e_2g_1}{\partial_{\bar z}\overline{\Phi}})$, $\mathcal{M}_2=\frac{1}{4\partial_z\Phi}L_1(x,D)(\frac{e_2g_2}{\partial_z\Phi})$, $\mathcal{M}_3=-\frac{1}{4\partial_z\Phi}L_2(x,D)^*\left(\frac{e_2g_3}{\partial_z\Phi}\right)$, $\mathcal{M}_4=-\frac{1}{4\partial_{\bar z}\overline{\Phi}}L_2(x,D)^*\left(\frac{e_2g_4}{\partial_{\bar z}\overline{\Phi}}\right)$. Then there exists a constant $\mathcal{C}$ independent of τ such that

$$(\mathcal{P}(x,D)(e^{\tau\Phi}\frac{\mathcal{M}_1}{\tau^2}+e^{\tau\overline{\Phi}}\frac{\mathcal{M}_2}{\tau^2}),v)_{L^2(\Omega)}+(\mathcal{P}(x,D)u,e^{-\tau\Phi}\frac{\mathcal{M}_3}{\tau^2}+e^{-\tau\overline{\Phi}}\frac{\mathcal{M}_4}{\tau^2})_{L^2(\Omega)}=\frac{\mathcal{C}}{\tau}+o(\frac{1}{\tau})\quad \text{as } |\tau|\to+\infty. \tag{5.27}$$

Denote $\mathcal{X}_1=-\frac{e_2g_1}{2\partial_{\bar z}\overline{\Phi}}$, $\mathcal{X}_2=-\frac{e_2g_2}{2\partial_z\Phi}$, $\mathcal{X}_3=\frac{e_2g_3}{2\partial_z\Phi}$, $\mathcal{X}_4=\frac{e_2g_4}{2\partial_{\bar z}\overline{\Phi}}$. Then, using the stationary phase argument we conclude

$$(\mathcal{P}(x,D)(e^{\tau\Phi}\frac{\mathcal{X}_2}{\tau}+e^{\tau\overline{\Phi}}\frac{\mathcal{X}_1}{\tau}),v)_{L^2(\Omega)}+(\mathcal{P}(x,D)u,e^{-\tau\Phi}\frac{\mathcal{X}_3}{\tau}+e^{-\tau\overline{\Phi}}\frac{\mathcal{X}_4}{\tau})_{L^2(\Omega)}= \tag{5.28}$$

$$\mathcal{C}_0+\frac{\mathcal{C}_{-1}}{\tau}+\frac{1}{\tau}\int_{\widetilde\Gamma}((A_1-A_2)\frac{\partial\Phi}{\partial z}\mathcal{X}_2\bar b e^{\overline{\mathcal{B}_2}}e^{2\tau i\psi}-(B_1-B_2)\frac{\partial\overline{\Phi}}{\partial\bar z}\mathcal{X}_1 b e^{\overline{\mathcal{A}_2}}e^{-2\tau i\psi})\frac{(\nabla\psi,\nu)}{2i|\nabla\psi|^2}d\sigma$$

$$+\frac{1}{\tau}\int_{\widetilde\Gamma}((A_1-A_2)\frac{\partial\Phi}{\partial z}\overline{\mathcal{X}_3}ae^{\mathcal{A}_1}e^{2\tau i\psi}-(B_1-B_2)\frac{\partial\overline{\Phi}}{\partial\bar z}\overline{\mathcal{X}_4}\bar a e^{\mathcal{B}_1}e^{-2\tau i\psi})\frac{(\nabla\psi,\nu)}{2i|\nabla\psi|^2}d\sigma+o(\frac{1}{\tau})\quad \text{as } |\tau|\to+\infty.$$

Next we show that

Proposition 5.3. *Under the conditions of Theorem 1.1*

$$A_1=A_2,\quad B_1=B_2\quad \text{on } \widetilde\Gamma \tag{5.29}$$

and

$$\mathfrak{I}(\Phi,a,b,c,d)=\int_{\widetilde\Gamma}\left\{(\nu_1+i\nu_2)\frac{\partial\Phi}{\partial z}a(z)\overline{c(\bar z)}e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}+(\nu_1-i\nu_2)\frac{\partial\overline{\Phi}}{\partial\bar z}d(\bar z)\overline{b(z)}e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}\right\}d\sigma=0. \tag{5.30}$$

Proof. Let $\widehat x$ be an arbitrary point from $Int\,\widetilde\Gamma$ and Γ_* be an arc containing $\widehat x$ such that $\Gamma_*\subset\subset\widetilde\Gamma$. By Proposition 2.2 there exists a weight function Φ satisfying (2.4) and (2.6). Then the boundary integrals in (5.9), (5.10) have the following asymptotic:

$$\int_{\widetilde\Gamma}(B_1-B_2)d_\tau(\bar z)\overline{c_\tau(\bar z)}e^{\mathcal{B}_1+\overline{\mathcal{A}_2}-2i\tau\psi}d\sigma+\int_{\widetilde\Gamma}(A_1-A_2)(\nu_1-i\nu_2)a_\tau(z)\overline{b_\tau(z)}e^{\mathcal{A}_1+\overline{\mathcal{B}_2}+2i\tau\psi}d\sigma=$$

$$\sum_{x\in\mathcal{G}\setminus\{x_-,x_+\}}\left\{\left(\frac{2\pi}{i\frac{\partial^2\psi}{\partial\vec\tau^2}(x)}\right)^{\frac12}(\bar a b(B_1-B_2))(x)\frac{e^{-2\tau i\psi(x)}}{\sqrt\tau}+\left(\frac{2\pi}{-i\frac{\partial^2\psi}{\partial\vec\tau^2}(x)}\right)^{\frac12}(a\bar b(A_1-A_2))(x)\frac{e^{2\tau i\psi(x)}}{\sqrt\tau}\right\}$$

$$+O(\frac{1}{\tau})\quad \text{as } |\tau|\to+\infty. \tag{5.31}$$

We remind that the set $\mathcal{G}$ is introduced in (2.4). Moreover, in order to avoid the contribution from the points $x_\pm$ we chose functions a, b in such a way that
(5.32)
$$\frac{\partial^{|\beta|}}{\partial x_1^{\beta_1}\partial x_2^{\beta_2}}a(x_\pm) = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1}\partial x_2^{\beta_2}}b(x_\pm) = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1}\partial x_2^{\beta_2}}c(x_\pm) = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1}\partial x_2^{\beta_2}}d(x_\pm) = 0 \quad \forall k \in \{0, \dots, 5\}.$$

Let $\tilde\chi_1 \in C^\infty(\partial\Omega)$ be a function such that it is equal 1 near points $x_\pm$ and has support located in a small neighborhood of these points. Then

$$\int_{\Gamma_*} \tilde\chi_1 (B_1 - B_2) d_\tau \overline{c_\tau} e^{\mathcal{B}_1 + \overline{\mathcal{A}_2} - 2i\tau\psi} d\sigma + \int_{\Gamma_*} \tilde\chi_1 (A_1 - A_2)(\nu_1 - i\nu_2) a_\tau \overline{b_\tau} e^{\mathcal{A}_1 + \overline{\mathcal{B}_2} + 2i\tau\psi} d\sigma =$$

$$\int_{\Gamma_*} \frac{\tilde\chi_1 (B_1 - B_2) d_\tau \overline{c_\tau} e^{\mathcal{B}_1 + \overline{\mathcal{A}_2}}}{-2i\tau \frac{\partial\psi}{\partial\vec\tau}} \frac{\partial e^{-2i\tau\psi}}{\partial\vec\tau} d\sigma + \int_{\Gamma_*} \frac{\tilde\chi_1 (A_1 - A_2)(\nu_1 - i\nu_2) a_\tau \overline{b_\tau} e^{\mathcal{A}_1 + \overline{\mathcal{B}_2}}}{2i\tau \frac{\partial\psi}{\partial\vec\tau}} \frac{\partial e^{2i\tau\psi}}{\partial\vec\tau} d\sigma =$$

$$\int_{\Gamma_*} \frac{\partial}{\partial\vec\tau}\left(\frac{\tilde\chi_1 (B_1 - B_2) d_\tau \overline{c_\tau} e^{\mathcal{B}_1 + \overline{\mathcal{A}_2}}}{2i\tau \frac{\partial\psi}{\partial\vec\tau}} \right) e^{-2i\tau\psi} d\sigma$$

$$- \int_{\Gamma_*} \frac{\partial}{\partial\vec\tau}\left(\frac{\tilde\chi_1 (A_1 - A_2)(\nu_1 - i\nu_2) a_\tau \overline{b_\tau} e^{\mathcal{A}_1 + \overline{\mathcal{B}_2}}}{2i\tau \frac{\partial\psi}{\partial\vec\tau}} \right) e^{2i\tau\psi} d\sigma = O(\frac{1}{\tau}).$$

In order to obtain the last equality used that by (5.32) and (2.7) the functions

$$\frac{\partial}{\partial\vec\tau}\left(\frac{\tilde\chi_1 (B_1 - B_2) d_\tau \overline{c_\tau} e^{\mathcal{B}_1 + \overline{\mathcal{A}_2}}}{2i \frac{\partial\psi}{\partial\vec\tau}} \right), \quad \frac{\partial}{\partial\vec\tau}\left(\frac{\tilde\chi_1 (A_1 - A_2)(\nu_1 - i\nu_2) a_\tau \overline{b_\tau} e^{\mathcal{A}_1 + \overline{\mathcal{B}_2}}}{2i \frac{\partial\psi}{\partial\vec\tau}} \right)$$

are bounded. By (5.5), (5.9)-(5.12), (5.19)-(5.21), (5.27) - (5.28) and (5.31), we can represent the right-hand side of (5.4) as

$$O(\frac{1}{\tau}) = \tau F_1 + F_0 + \sum_{x \in \mathcal{G} \setminus \{x_-, x_+\}} \left(\left(\frac{2\pi}{i \frac{\partial^2\psi}{\partial\vec\tau^2}(x)} \right)^{\frac{1}{2}} (\overline{a} b (B_1 - B_2))(x) \frac{e^{-2\tau i\psi(x)}}{\sqrt{\tau}} \right.$$
$$\left. + \left(\frac{2\pi}{-i \frac{\partial^2\psi}{\partial\vec\tau^2}(x)} \right)^{\frac{1}{2}} (a\overline{b}(A_1 - A_2))(x) \frac{e^{2\tau i\psi(x)}}{\sqrt{\tau}} \right).$$

Taking into account that F_1 is equal to the left-hand side of (5.30) we obtain the equality (5.30). Using (2.6) and applying Bohr's theorem (e.g., [3], p.393), we obtain (5.29). □

Denote

$$\mathcal{Q}_- = -(B_1 - B_2)A_1 - (A_1 - A_2)B_2 - 2\frac{\partial}{\partial z}(A_1 - A_2),$$
$$\mathcal{Q}_+ = -(A_1 - A_2)B_1 - (B_1 - B_2)A_2 - 2\frac{\partial}{\partial \overline{z}}(B_1 - B_2).$$

Thanks to (5.5), (5.9)-(5.12), (5.19)-(5.21), (5.27)-(5.29), (5.31), we can write down the right-hand side of (5.4) as

$$
\begin{aligned}
I_1 \quad + \quad & I_2 = \sum_{k=1}^{3} \tau^{2-k}(\kappa_k + \widetilde{\kappa}_k) + \kappa \\
+ \quad & \int_{\Gamma_0} (A_1 - A_2)(\nu_1 - i\nu_2) a_\tau \overline{b_\tau} e^{\mathcal{A}_1 + \overline{\mathcal{B}_2}} d\sigma + \int_{\Gamma_0} (B_1 - B_2)(\nu_1 + i\nu_2) d_\tau(\overline{z}) \overline{c_\tau(\overline{z})} e^{\mathcal{B}_1 + \overline{\mathcal{A}_2}} d\sigma \\
- \quad & \frac{1}{\tau} \int_{\Gamma_0} \mathcal{Q}_+ \overline{a} b e^{\mathcal{B}_1 + \overline{\mathcal{A}_2}} \frac{(\nabla\psi, \nu)}{|\nabla\psi|^2} d\sigma - \frac{1}{\tau} \int_{\Gamma_0} \mathcal{Q}_- a \overline{b} e^{\mathcal{A}_1 + \overline{\mathcal{B}_2}} \frac{(\nabla\psi, \nu)}{|\nabla\psi|^2} d\sigma \\
+ \quad & 2\pi \sum_{k=1}^{\ell} \frac{(\mathcal{Q}_+ a\overline{b})(\widetilde{x}_k) e^{(\mathcal{A}_1 + \overline{\mathcal{B}_2} + 2\tau i \mathrm{Im}\Phi)(\widetilde{x}_k)} + (\mathcal{Q}_- d\overline{c})(\widetilde{x}_k) e^{(\mathcal{B}_1 + \overline{\mathcal{A}_2} - 2i\tau \mathrm{Im}\Phi)(\widetilde{x}_k)}}{\tau |\det \mathrm{Im}\Phi''(\widetilde{x}_k)|^{\frac{1}{2}}} \\
+ \quad & \frac{1}{\tau}(\mathcal{I}_1(\partial\Omega) + \mathcal{I}_2(\partial\Omega)) \\
- \quad & 2 \int_{\partial\Omega} (\nu_1 + i\nu_2) e^{\mathcal{A}_1 + \overline{\mathcal{A}_2}} \overline{c_\tau(\overline{z})} \mathfrak{G}_1(x, \tau) d\sigma \\
- \quad & 2 \int_{\partial\Omega} (\nu_1 - i\nu_2) e^{\mathcal{B}_1 + \overline{\mathcal{B}_2}} d_\tau(\overline{z}) \overline{\mathfrak{G}_4(x, \tau)} d\sigma \\
- \quad & 2 \int_{\partial\Omega} (\nu_1 + i\nu_2) e^{\mathcal{A}_1 + \overline{\mathcal{A}_2}} a_\tau(z) \overline{\mathfrak{G}_3(x, \tau)} d\sigma
\end{aligned}
$$

$$
-2 \int_{\partial\Omega} (\nu_1 - i\nu_2) e^{\mathcal{B}_1 + \overline{\mathcal{B}}_2} \overline{b_\tau(z)} \mathfrak{G}_2(x, \tau) d\sigma + o(\frac{1}{\tau}) \quad \text{as } |\tau| \to +\infty. \tag{5.33}
$$

We note that κ_k and $\widetilde{\kappa}_k$ denote generic constants which are independent of τ. In order to transform some terms in the above equality, we need the following proposition:

Proposition 5.4. *There exist a holomorphic function $\Psi \in H^{\frac{1}{2}}(\Omega)$ and an antiholomorphic function $\tilde{\Psi} \in H^{\frac{1}{2}}(\Omega)$ such that*

$$
\Psi|_{\tilde{\Gamma}} = e^{\mathcal{A}_1 + \overline{\mathcal{A}_2}}, \quad \tilde{\Psi}|_{\tilde{\Gamma}} = e^{\mathcal{B}_1 + \overline{\mathcal{B}_2}} \tag{5.34}
$$

and

$$
e^{\mathcal{B}_1 + \overline{\mathcal{B}_2}} \Psi = e^{\mathcal{A}_1 + \overline{\mathcal{A}_2}} \tilde{\Psi} \quad \text{on } \Gamma_0. \tag{5.35}
$$

Proof. Consider the extremal problem:

$$
J(\Psi, \tilde{\Psi}) = \| e^{\mathcal{A}_1 + \overline{\mathcal{A}_2}} \frac{\partial \Phi}{\partial z} a\overline{c} - \Psi \|^2_{L^2(\tilde{\Gamma})} + \| e^{\mathcal{B}_1 + \overline{\mathcal{B}_2}} \frac{\partial \overline{\Phi}}{\partial \overline{z}} \overline{b} d - \tilde{\Psi} \|^2_{L^2(\tilde{\Gamma})} \to \inf,
$$

$$
\frac{\partial \Psi}{\partial \overline{z}} = 0 \quad \text{in } \Omega, \quad \frac{\partial \tilde{\Psi}}{\partial z} = 0 \quad \text{in } \Omega, \quad ((\nu_1 - i\nu_2)\Psi(z) + (\nu_1 + i\nu_2)\tilde{\Psi}(\overline{z}))|_{\Gamma_0} = 0. \tag{5.36}
$$

Here the functions a, b, c, d satisfy (4.3), (4.4), (4.26) and (4.27). Denote the unique solution to this extremal problem as $(\hat{\Psi}, \hat{\tilde{\Psi}})$. Applying the Lagrange principle to this extremal problem we obtain

$$
\mathrm{Re}(e^{\mathcal{A}_1 + \overline{\mathcal{A}_2}} \frac{\partial \Phi}{\partial z} a\overline{c} - \hat{\Psi}, \delta)_{L^2(\tilde{\Gamma})} + \mathrm{Re}(e^{\mathcal{B}_1 + \overline{\mathcal{B}_2}} \frac{\partial \overline{\Phi}}{\partial \overline{z}} \overline{b} d - \hat{\tilde{\Psi}}, \tilde{\delta})_{L^2(\tilde{\Gamma})} = 0 \tag{5.37}
$$

for any δ from $H^{\frac{1}{2}}(\Omega)$ such that

$$\frac{\partial \delta}{\partial \bar z}=0 \quad \text{in}\,\Omega, \quad \frac{\partial \tilde\delta}{\partial z}=0 \quad \text{in}\,\Omega, \quad (\nu_1-i\nu_2)\delta|_{\Gamma_0}=-(\nu_1+i\nu_2)\tilde\delta|_{\Gamma_0}$$

and there exist two functions $P,\tilde P\in H^{\frac{1}{2}}(\Omega)$ such that

$$\frac{\partial P}{\partial \bar z}=0 \quad \text{in } \Omega, \quad \frac{\partial \tilde P}{\partial z}=0 \quad \text{in } \Omega, \tag{5.38}$$

$$(\nu_1+i\nu_2)P=\overline{e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}\frac{\partial \Phi}{\partial z}a\bar c-\hat\Psi} \quad \text{on}\,\tilde\Gamma, \quad (\nu_1-i\nu_2)\tilde P=\overline{e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}\frac{\partial \overline{\Phi}}{\partial \bar z}\bar bd-\hat{\tilde\Psi}} \quad \text{on}\,\tilde\Gamma, \tag{5.39}$$

$$(P-\tilde P)|_{\Gamma_0}=0. \tag{5.40}$$

Denote $\Psi_0(z)=\frac{1}{2i}(P(z)-\overline{\tilde P(\bar z)}), \Phi_0(z)=\frac{1}{2}(P(z)+\overline{\tilde P(\bar z)})$. By (5.40)

$$\operatorname{Im}\Psi|_{\Gamma_0}=\operatorname{Im}\Phi|_{\Gamma_0}=0.$$

Hence

$$P=(\Phi_0+i\Psi_0), \quad \overline{\tilde P}=(\Phi_0-i\Psi_0). \tag{5.41}$$

From (5.37)

$$\operatorname{Re}(e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}\frac{\partial \Phi}{\partial z}a\bar c-\hat\Psi,\hat\Psi)_{L^2(\tilde\Gamma)}+\operatorname{Re}(e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}\frac{\partial \overline{\Phi}}{\partial \bar z}\bar bd-\hat{\tilde\Psi},\hat{\tilde\Psi})_{L^2(\tilde\Gamma)}=0. \tag{5.42}$$

By (5.38), (5.39) and (5.41), we have

$$H_1=\operatorname{Re}(e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}\frac{\partial \Phi}{\partial z}a\bar c-\hat\Psi,e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}\frac{\partial \Phi}{\partial z}a\bar c)_{L^2(\tilde\Gamma)}+\operatorname{Re}(e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}\frac{\partial \overline{\Phi}}{\partial \bar z}\bar bd-\hat{\tilde\Psi},e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}\frac{\partial \overline{\Phi}}{\partial \bar z}\bar bd)_{L^2(\tilde\Gamma)}$$

$$\operatorname{Re}((\nu_1-i\nu_2)\bar P,e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}\frac{\partial \Phi}{\partial z}a\bar c)_{L^2(\tilde\Gamma)}+\operatorname{Re}((\nu_1+i\nu_2)\overline{\tilde P},e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}\frac{\partial \overline{\Phi}}{\partial \bar z}\bar bd)_{L^2(\tilde\Gamma)}=$$

$$2\operatorname{Re}((\nu_1-i\nu_2)\overline{(\Phi_0+i\Psi_0)},e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}\frac{\partial \Phi}{\partial z}a\bar c)_{L^2(\tilde\Gamma)}+2\operatorname{Re}((\nu_1+i\nu_2)(\Phi_0-i\Psi_0),e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}\frac{\partial \overline{\Phi}}{\partial \bar z}\bar bd)_{L^2(\tilde\Gamma)}.$$

We can rewrite

$$2\operatorname{Re}((\nu_1-i\nu_2)\overline{\Phi_0},e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}\frac{\partial \Phi}{\partial z}a\bar c)_{L^2(\tilde\Gamma)}+2\operatorname{Re}((\nu_1+i\nu_2)\Phi_0,e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}\frac{\partial \overline{\Phi}}{\partial \bar z}\bar bd)_{L^2(\tilde\Gamma)}=$$

$$\mathfrak{I}(\Phi,\Phi_0a,b,c,\Phi_0d)+\overline{\mathfrak{I}(\Phi,\Phi_0a,b,c,\Phi_0d)} \tag{5.43}$$

and

$$2\mathrm{Re}((\nu_1-i\nu_2)\overline{(i\Psi_0)},e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}\frac{\partial\Phi}{\partial z}a\bar c)_{L^2(\tilde\Gamma)}+2\mathrm{Re}((\nu_1+i\nu_2)(-i\Psi_0),e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}\frac{\partial\overline{\Phi}}{\partial\bar z}\bar bd)_{L^2(\tilde\Gamma)}=$$
$$-2\mathrm{Im}((\nu_1-i\nu_2)\bar ac\overline{\Psi_0},e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}\frac{\partial\Phi}{\partial z})_{L^2(\tilde\Gamma)}-2\mathrm{Im}((\nu_1+i\nu_2)b\bar d\Psi_0,e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}\frac{\partial\overline{\Phi}}{\partial\bar z})_{L^2(\tilde\Gamma)}=$$
$$-\frac{1}{i}\int_\Omega((\nu_1-i\nu_2)\bar ac\overline{\Psi_0}\frac{\partial\overline{\Phi}}{\partial\bar z}e^{\overline{\mathcal{A}_1}-\mathcal{A}_2}-(\nu_1+i\nu_2)a\bar c\Psi_0\frac{\partial\Phi}{\partial z}e^{\mathcal{A}_1+\overline{\mathcal{A}_2}})dx$$
$$-\frac{1}{i}\int_\Omega((\nu_1+i\nu_2)b\bar d\Psi_0\frac{\partial\Phi}{\partial z}e^{\overline{\mathcal{B}_1}+\mathcal{B}_2}-(\nu_1-i\nu_2)\bar bd\overline{\Psi_0}\frac{\partial\overline{\Phi}}{\partial\bar z}e^{\mathcal{B}_1+\overline{\mathcal{B}_2}})dx=$$
$$\frac{1}{i}(-\mathfrak{I}(\Psi,a\Psi_0,b,c,d\Psi_0)+\overline{\mathfrak{I}(\Psi,a\Psi_0,b,c,d\Psi_0)}).\tag{5.44}$$

Then by (5.43), (5.44) and Proposition 5.3, $H_1=0$. Taking into account (5.42) we obtain that $J(\hat\Psi,\hat\Psi)=0$. Consequently (5.34) is proved. From (4.3), (4.4), (4.26), (4.27) and (5.36) we obtain (5.35). The proof of the proposition is completed. □

Thanks to Proposition 5.4, we can rewrite equality (5.33) as

$$o(\frac{1}{\tau})=\sum_{k=1}^{3}\tau^{2-k}\widetilde{F}_k\tag{5.45}$$
$$+\ 2\pi\sum_{k=1}^{\ell}\frac{1}{\tau|\det\mathrm{Im}\Phi''(\tilde x_k)|^{\frac12}}\Big\{((\mathcal{Q}_++(q_1-q_2))a\bar b)(\tilde x_k)e^{(\mathcal{A}_1+\overline{\mathcal{B}_2}+2\tau i\mathrm{Im}\Phi)(\tilde x_k)}$$
$$+\ ((\mathcal{Q}_-+(q_1-q_2))d\bar c)(\tilde x_k)e^{(\mathcal{B}_1+\overline{\mathcal{A}_2}-2i\tau\mathrm{Im}\Phi)(\tilde x_k)}\Big\}$$
$$-\ \frac{2}{\tau}\int_{\Gamma_0}(\nu_1+i\nu_2)(e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}-\Psi)\frac{\partial\Phi}{\partial z}a(z)\overline{\sum_{k=1}^{\ell}\left(\frac{c_{2,k}e^{2\tau i\psi(\tilde x_k)}}{|\det\mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}}+\frac{c_{2,-k}e^{-2\tau i\psi(\tilde x_k)}}{|\det\mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}}\right)}d\sigma$$
$$-\ \frac{2}{\tau}\int_{\Gamma_0}(\nu_1-i\nu_2)(e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}-\tilde\Psi)\frac{\partial\overline{\Phi}}{\partial\bar z}\overline{b(z)}\sum_{k=1}^{\ell}\left(\frac{d_{2,k}e^{-2\tau i\psi(\tilde x_k)}}{|\det\mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}}+\frac{d_{2,-k}e^{2\tau i\psi(\tilde x_k)}}{|\det\mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}}\right)d\sigma$$
$$-\ \frac{2}{\tau}\int_{\Gamma_0}(\nu_1+i\nu_2)(e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}-\Psi)\frac{\partial\Phi}{\partial z}c(\bar z)\sum_{k=1}^{\ell}\left(\frac{a_{2,k}e^{-2\tau i\psi(\tilde x_k)}}{|\det\mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}}+\frac{a_{2,-k}e^{2\tau i\psi(\tilde x_k)}}{|\det\mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}}\right)d\sigma$$
$$-\ \frac{2}{\tau}\int_{\Gamma_0}(\nu_1-i\nu_2)(e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}-\tilde\Psi)\frac{\partial\overline{\Phi}}{\partial\bar z}d(\bar z)\overline{\sum_{k=1}^{\ell}\left(\frac{b_{2,k}e^{2\tau i\psi(\tilde x_k)}}{|\det\mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}}+\frac{b_{2,-k}e^{-2\tau i\psi(\tilde x_k)}}{|\det\mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}}\right)}d\sigma$$
$$-\ 2\int_{\Gamma_0}(\nu_1+i\nu_2)(e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}-\Psi)\overline{c_\tau(\bar z)}\mathfrak{G}_1(x,\tau)d\sigma-2\int_{\partial\Omega}(\nu_1-i\nu_2)(e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}-\tilde\Psi)d_\tau(\bar z)\overline{\mathfrak{G}_4(x,\tau)}d\sigma$$
$$-\ 2\int_{\Gamma_0}a_\tau(z)(\nu_1+i\nu_2)(e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}-\Psi)\overline{\mathfrak{G}_3(x,\tau)}d\sigma-2\int_{\partial\Omega}(\nu_1-i\nu_2)(e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}-\tilde\Psi)\overline{b_\tau(z)}\mathfrak{G}_2(x,\tau)d\sigma.$$

Here $\widetilde{F}_k$ denote constants which are independent of τ.

Observe that

$$(\nu_1+i\nu_2)\frac{\partial\Phi}{\partial z}=-(\nu_1-i\nu_2)\frac{\partial\bar\Phi}{\partial\bar z}\quad \text{on } \Gamma_0. \tag{5.46}$$

Really $\frac{\partial}{\partial\bar z}=\frac12(\frac{\partial}{\partial x_1}+i\frac{\partial}{\partial x_2})(\varphi+i\psi)=\frac12(\frac{\partial\varphi}{\partial x_1}-\frac{\partial\psi}{\partial x_2})+\frac{i}{2}(\frac{\partial\varphi}{\partial x_2}+\frac{\partial\psi}{\partial x_1})$. Hence $\frac{\partial\varphi}{\partial x_1}=\frac{\partial\psi}{\partial x_2}$, $\frac{\partial\varphi}{\partial x_2}=-\frac{\partial\psi}{\partial x_1}$, $\frac{\partial\varphi}{\partial\nu}=-\frac{\partial\psi}{\partial\vec\tau}$ and $\frac{\partial\psi}{\partial\nu}=\frac{\partial\varphi}{\partial\vec\tau}$. Observe that $(\nu_1+i\nu_2)\frac{\partial}{\partial z}=\frac12(\nu_1\frac{\partial}{\partial x_1}+\nu_2\frac{\partial}{\partial x_2})+\frac{i}{2}(\nu_2\frac{\partial}{\partial x_1}-\nu_1\frac{\partial}{\partial x_2})=\frac12(\frac{\partial}{\partial\nu}+i\frac{\partial}{\partial\vec\tau})$ and $(\nu_1-i\nu_2)\frac{\partial}{\partial\bar z}=\frac12(\frac{\partial}{\partial\nu}-i\frac{\partial}{\partial\vec\tau})$. Hence

$$(\nu_1+i\nu_2)\frac{\partial\Phi}{\partial z}=\frac12(\frac{\partial}{\partial\nu}+i\frac{\partial}{\partial\vec\tau})(\varphi+i\psi)=\frac12(\frac{\partial\varphi}{\partial\nu}-\frac{\partial\psi}{\partial\vec\tau})+\frac{i}{2}(\frac{\partial\varphi}{\partial\vec\tau}+\frac{\partial\psi}{\partial\nu})=-\frac{\partial\psi}{\partial\vec\tau}+i\frac{\partial\varphi}{\partial\vec\tau}.$$

Therefore

$$(\nu_1-i\nu_2)\frac{\partial\overline\Phi}{\partial\bar z}=\overline{\frac{\partial\psi}{\partial\vec\tau}+i\frac{\partial\varphi}{\partial\vec\tau}}=-\frac{\partial\psi}{\partial\vec\tau}-i\frac{\partial\varphi}{\partial\vec\tau}.$$

Taking into account that $\psi|_{\Gamma_0}=0$ we obtain (5.46).

Then, using (5.46), on Γ_0 we have

$$\begin{aligned}
&-(\nu_1+i\nu_2)\frac{\partial\Phi}{\partial z}(e^{\mathcal A_1+\overline{\mathcal A_2}}-\Psi)c(\bar z)\sum_{k=1}^{\ell}\left(\frac{a_{2,k}e^{-2\tau i\psi(\tilde x_k)}}{|\det \operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}}+\frac{a_{2,-k}e^{2\tau i\psi(\tilde x_k)}}{|\det \operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}}\right)\\
&-(\nu_1-i\nu_2)\frac{\partial\overline\Phi}{\partial\bar z}(e^{\mathcal B_1+\overline{\mathcal B_2}}-\tilde\Psi)\overline{b(z)}\left(\sum_{k=1}^{\ell}(\frac{d_{2,k}e^{-2\tau i\psi(\tilde x_k)}}{|\det \operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}}+\frac{d_{2,-k}e^{2\tau i\psi(\tilde x_k)}}{|\det \operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}}\right)=\\
&-\sum_{k=1}^{\ell}(\nu_1+i\nu_2)\frac{\partial\Phi}{\partial z}\left((e^{\mathcal A_1+\overline{\mathcal A_2}}-\Psi)c(\bar z)\frac{a_{2,-k}e^{2\tau i\psi(\tilde x_k)}}{|\det \operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}}+(e^{\mathcal B_1+\overline{\mathcal A_2}}-\tilde\Psi e^{\overline{\mathcal A_2}-\overline{\mathcal B_2}})c(\bar z)\frac{d_{2,-k}e^{2\tau i\psi(\tilde x_k)}}{|\det \operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}}\right)\\
&-\sum_{k=1}^{\ell}(\nu_1-i\nu_2)\frac{\partial\overline\Phi}{\partial\bar z}\overline{b(z)}\left((e^{\mathcal A_1+\overline{\mathcal B_2}}-\Psi e^{\overline{\mathcal B_2}-\overline{\mathcal A_2}})\frac{a_{2,k}e^{-2\tau i\psi(\tilde x_k)}}{|\det \operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}}+(e^{\mathcal B_1+\overline{\mathcal B_2}}-\tilde\Psi)\frac{d_{2,k}e^{-2\tau i\psi(\tilde x_k)}}{|\det \operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}}\right)=\\
&-\sum_{k=1}^{\ell}(\nu_1+i\nu_2)\frac{\partial\Phi}{\partial z}\left((e^{\mathcal A_1+\overline{\mathcal A_2}}-\Psi)c(\bar z)\frac{a_{2,-k}e^{2\tau i\psi(\tilde x_k)}}{|\det \operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}}+(e^{\mathcal B_1+\overline{\mathcal A_2}}-\Psi e^{-\mathcal A_1+\mathcal B_1})c(\bar z)\frac{d_{2,=k}e^{2\tau i\psi(\tilde x_k)}}{|\det \operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}}\right)\\
&-\sum_{k=1}^{\ell}(\nu_1-i\nu_2)\frac{\partial\overline\Phi}{\partial\bar z}\overline{b(z)}\left((e^{\mathcal A_1+\overline{\mathcal B_2}}-\tilde\Psi e^{\mathcal A_1-\mathcal B_1})\frac{a_{2,k}e^{-2\tau i\psi(\tilde x_k)}}{|\det \operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}}+(e^{\mathcal B_1+\overline{\mathcal B_2}}-\tilde\Psi)\frac{d_{2,k}e^{-2\tau i\psi(\tilde x_k)}}{|\det \operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}}\right)=\\
&-\sum_{k=1}^{\ell}(\nu_1+i\nu_2)\frac{\partial\Phi}{\partial z}(e^{\overline{\mathcal A_2}}-\Psi e^{-\mathcal A_1})\frac{c(\bar z)p_{-k}}{|\det \operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}}e^{2\tau i\psi(\tilde x_k)}\\
&-\sum_{k=1}^{\ell}(\nu_1-i\nu_2)\frac{\partial\overline\Phi}{\partial\bar z}(e^{\overline{\mathcal B_2}}-\tilde\Psi e^{-\mathcal B_1})\frac{\overline{b(z)}p_ke^{-2\tau i\psi(\tilde x_k)}}{|\det \operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}}
\end{aligned}\tag{5.47}$$

and

$$-(\nu_1+i\nu_2)(e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}-\Psi)\frac{\partial\Phi}{\partial z}a(z)\sum_{k=1}^{\ell}\left(\overline{\frac{c_{2,k}e^{2\tau i\psi(\tilde x_k)}}{|\det \mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}}+\frac{c_{2,-k}e^{-2\tau i\psi(\tilde x_k)}}{|\det \mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}}}\right)$$
$$-(\nu_1-i\nu_2)(e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}-\tilde\Psi)\frac{\partial\overline{\Phi}}{\partial \bar z}d(\bar z)\sum_{k=1}^{\ell}\left(\overline{\frac{b_{2,k}e^{2\tau i\psi(\tilde x_k)}}{|\det \mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}}+\frac{b_{2,-k}e^{-2\tau i\psi(\tilde x_k))}}{|\det \mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}}}\right)=$$
$$-\sum_{k=1}^{\ell}(\nu_1+i\nu_2)\frac{\partial\Phi}{\partial z}a(z)\left((e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}-\Psi)\frac{\overline{c_{2,-k}e^{2\tau i\psi(\tilde x_k)}}}{|\det \mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}}+(e^{\mathcal{A}_1+\overline{\mathcal{B}_2}}-e^{-\mathcal{B}_1+\mathcal{A}_1}\tilde\Psi)\frac{\overline{b_{2,-k}e^{2\tau i\psi(\tilde x_k)}}}{|\det \mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}}\right)$$
$$-\sum_{k=1}^{\ell}(\nu_1-i\nu_2)\frac{\partial\overline{\Phi}}{\partial \bar z}d(\bar z)\left((e^{\mathcal{B}_1+\overline{\mathcal{A}_2}}-\Psi e^{-\mathcal{A}_1+\mathcal{B}_1})\frac{\overline{c_{2,k}e^{-2\tau i\psi(\tilde x_k)}}}{|\det \mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}}+(e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}-\tilde\Psi)\frac{\overline{b_{2,k}e^{-2\tau i\psi(\tilde x_k)}}}{|\det \mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}}\right)=$$

$$-\sum_{k=1}^{\ell}(\nu_1+i\nu_2)\frac{\partial\Phi}{\partial z}a(z)\left((e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}-\Psi)\frac{\overline{c_{2,-k}e^{2\tau i\psi(\tilde x_k)}}}{|\det \mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}}+(e^{\mathcal{A}_1+\overline{\mathcal{B}_2}}-e^{\overline{\mathcal{B}_2}-\overline{\mathcal{A}_2}}\Psi)\frac{\overline{b_{2,-k}e^{2\tau i\psi(\tilde x_k)}}}{|\det \mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}}\right)$$
$$-\sum_{k=1}^{\ell}(\nu_1-i\nu_2)\frac{\partial\overline{\Phi}}{\partial \bar z}d(\bar z)\left((e^{\mathcal{B}_1+\overline{\mathcal{A}_2}}-\Psi e^{\overline{\mathcal{A}_2}-\overline{\mathcal{B}_2}})\frac{\overline{c_{2,k}e^{-2\tau i\psi(\tilde x_k)}}}{|\det \mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}}+(e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}-\tilde\Psi)\frac{\overline{b_{2,k}e^{-2\tau i\psi(\tilde x_k)}}}{|\det \mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}}\right)=$$
$$-\sum_{k=1}^{\ell}(\nu_1+i\nu_2)\frac{\partial\Phi}{\partial z}(e^{\mathcal{A}_1}-\Psi e^{-\overline{\mathcal{A}_2}})\frac{a(z)\overline{\tilde p_{-k}e^{2\tau i\psi(\tilde x_k)}}}{|\det \mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}}$$
$$-\sum_{k=1}^{\ell}(\nu_1-i\nu_2)\frac{\partial\overline{\Phi}}{\partial \bar z}(e^{\mathcal{B}_1}-\tilde\Psi e^{-\overline{\mathcal{B}_2}})\frac{d(\bar z)\overline{e^{-2\tau i\psi(\tilde x_k)}\tilde p_k}}{|\det \mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}}. \qquad (5.48)$$

Using (5.47), (5.48) and Proposition 7.2 in Section 7, we rewrite (5.45) as

$$
\begin{aligned}
o(\frac{1}{\tau}) &= \sum_{k=1}^{3}\tau^{2-k}\widetilde{F}_k + \\
2 & \pi\sum_{k=1}^{\ell}\frac{1}{\tau|\det \mathrm{Im}\Phi''(\widetilde{x}_k)|^{\frac12}}\Big\{((\mathcal{Q}_+ + (q_1-q_2))a\overline{b})(\widetilde{x}_k)e^{(\mathcal{A}_1+\overline{\mathcal{B}_2}+2\tau i\mathrm{Im}\Phi)(\widetilde{x}_k)} \\
&+ ((\mathcal{Q}_- + (q_1-q_2))d\overline{c})(\widetilde{x}_k)e^{(\mathcal{B}_1+\overline{\mathcal{A}_2}-2i\tau\mathrm{Im}\Phi)(\widetilde{x}_k)}\Big\} \\
&- \frac{1}{4\tau}\int_{\Gamma_0}(\nu_1-i\nu_2)(e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}-\tilde{\Psi})d(\overline{z})\overline{\sum_{k=1}^{\ell}\frac{e^{2i\tau\psi(\tilde{x}_k)}}{|\det \mathrm{Im}\,\Phi''(\tilde{x}_k)|^{\frac12}}\frac{\partial_z(g_3e^{-\mathcal{A}_2})(\tilde{x}_k)}{\tilde{z}_k-z}}d\sigma \\
&- \frac{1}{4\tau}\int_{\Gamma_0}(\nu_1+i\nu_2)(e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}-\Psi)\overline{c(\overline{z})}\sum_{k=1}^{\ell}\frac{e^{-2i\tau\psi(\tilde{x}_k)}}{|\det \mathrm{Im}\,\Phi''(\tilde{x}_k)|^{\frac12}}\frac{\partial_z(g_1e^{-\mathcal{A}_1})(\tilde{x}_k)}{\tilde{z}_k-z}d\sigma \\
&- \frac{1}{4\tau}\int_{\Gamma_0}(\nu_1+i\nu_2)(e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}-\Psi)a(z)\overline{\sum_{k=1}^{\ell}\frac{e^{-2i\tau\psi(\tilde{x}_k)}}{|\det \mathrm{Im}\,\Phi''(\tilde{x}_k)|^{\frac12}}\frac{\partial_{\overline{z}}(g_4e^{-\mathcal{B}_2})(\tilde{x}_k)}{\overline{\tilde{z}_k}-\overline{z}}}d\sigma \\
&- \frac{1}{4\tau}\int_{\Gamma_0}(\nu_1-i\nu_2)(e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}-\tilde{\Psi})\overline{b(z)}\sum_{k=1}^{\ell}\frac{e^{2i\tau\psi(\tilde{x}_k)}}{|\det \mathrm{Im}\,\Phi''(\tilde{x}_k)|^{\frac12}}\frac{\partial_{\overline{z}}(g_2e^{-\mathcal{B}_1})(\tilde{x}_k)}{\overline{\tilde{z}_k}-\overline{z}}d\sigma \\
&+ o(\frac{1}{\tau})\quad \text{as } |\tau|\to+\infty. \qquad (5.49)
\end{aligned}
$$

Using Proposition 5.3 we take $a=a_*, d=d_*$ and $b=b_*, c=c_*$ such that

$$\int_{\Gamma_0}(\nu_1-i\nu_2)(e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}-\tilde{\Psi})\overline{b_*(z)}\frac{1}{\overline{\tilde{z}_k}-\overline{z}}d\sigma=0 \quad\text{and}\quad \int_{\Gamma_0}(\nu_1+i\nu_2)(e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}-\Psi)a_*(z)\frac{1}{\tilde{z}_k-z}d\sigma=0.$$

Then we obtain from (5.49)

$$\sum_{k=1}^{\ell}\frac{((\mathcal{Q}_+ + (q_1-q_2))a_*\overline{b}_*)(\widetilde{x}_k)e^{(\mathcal{A}_1+\overline{\mathcal{B}_2}+2\tau i\mathrm{Im}\Phi)(\widetilde{x}_k)}}{|\det \mathrm{Im}\Phi''(\widetilde{x}_k)|^{\frac12}}=0. \qquad (5.50)$$

On the other hand using the Proposition 5.3 again we can take $a=a^*, d=d^*$ and $b=b^*, c=c^*$ such that

$$\int_{\Gamma_0}(\nu_1-i\nu_2)(e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}-\tilde{\Psi})d^*(\overline{z})\frac{1}{\overline{\tilde{z}_k}-\overline{z}}d\sigma=0 \quad\text{and}\quad \int_{\Gamma_0}(\nu_1+i\nu_2)(e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}-\Psi)\overline{c^*(\overline{z})}\frac{1}{\tilde{z}_k-z}d\sigma=0.$$

These equalities and (5.49) imply

$$\sum_{k=1}^{\ell}\frac{((\mathcal{Q}_- + (q_1-q_2))d^*\overline{c^*})(\widetilde{x}_k)e^{(\mathcal{B}_1+\overline{\mathcal{A}_2}-2i\tau\mathrm{Im}\Phi)(\widetilde{x}_k)}}{|\det \mathrm{Im}\Phi''(\widetilde{x}_k)|^{\frac12}}=0. \qquad (5.51)$$

Let x be an arbitrary point from Ω. Consider the sequence of the functions Φ_ϵ given by Proposition 2.1. Applying Bohr's theorem (e.g., [3], p.393), we obtain from (5.50) and (5.51) we have

$$((\mathcal{Q}_- + (q_1-q_2))d^*\overline{c^*})(\widetilde{x}_\epsilon)=0,\quad ((\mathcal{Q}_+ + (q_1-q_2))a_*\overline{b}_*)(\widetilde{x}_\epsilon)=0.$$

The proof of the theorem is completed. □

6. Appendix I

Consider the Cauchy problem for the Cauchy-Riemann equations

$$L(\phi,\psi)=(\frac{\partial\phi}{\partial x_1}-\frac{\partial\psi}{\partial x_2},\frac{\partial\phi}{\partial x_2}+\frac{\partial\psi}{\partial x_1})=0\quad\text{in }\Omega,\quad(\phi,\psi)\,|_{\Gamma_0}=(b_1(x),b_2(x)),\tag{6.1}$$

$$\frac{\partial^l}{\partial z^l}(\phi+i\psi)(\hat x_j)=c_{0,j},\quad\forall j\in\{1,\dots N\}\quad\text{and }\forall l\in\{0,\dots,5\}.$$

Here $\hat x_1,\dots\hat x_N$ be an arbitrary fixed points in Ω. We consider the pair b_1,b_2 and complex numbers $\vec C=(c_{0,1},c_{1,1},c_{2,1},c_{3,1},c_{4,1},c_{5,1}\dots c_{0,N},c_{1,N},c_{2,N},c_{3,N},c_{4,N},c_{5,N})$ as initial data for (6.1). The following proposition establishes the solvability of (6.1) for a dense set of Cauchy data.

Proposition 6.1. *There exists a set $\mathcal O\subset(C^5(\overline{\Gamma_0}))^2\times\mathbb C^{6N}$ such that for each $(b_1,b_2,\vec C)\in\mathcal O$, (6.1) has at least one solution $(\phi,\psi)\in(C^5(\overline\Omega))^2$ and $\overline{\mathcal O}=(C^5(\overline\Gamma))^2\times\mathbb C^{6N}$.*

Proof. Denote $B=(b_1,b_2)$ an arbitrary element of the space $C^7(\overline{\Gamma_0})\times C^7(\overline{\Gamma_0})$. Consider the following extremal problem

$$J_\epsilon(\phi,\psi)=\|(\phi,\psi)-B\|^4_{B_4^{\frac{27}{4}}(\Gamma_0)}+\epsilon\sum_{k=0}^3\|\frac{\partial^k(\phi,\psi)}{\partial\nu^k}\|^4_{B_4^{\frac{27}{4}-k}(\partial\Omega)}\tag{6.2}$$

$$+\frac1\epsilon\left\|\Delta^3L(\phi,\psi)\right\|^4_{L^4(\Omega)}+\sum_{j=1}^N\sum_{k=0}^5|\frac{\partial^k}{\partial z^k}(\phi+i\psi)(\hat x_j)-c_{k,j}|^2\to\inf,$$

$$(\phi,\psi)\in W_4^7(\Omega)\times W_4^7(\Omega).\tag{6.3}$$

Here B_k^l denotes the Besov space of the corresponding orders.

For each $\epsilon>0$ there exists a unique solution to (6.2), (6.3) which we denote as $(\widehat\phi_\epsilon,\widehat\psi_\epsilon)$. This fact can be proved by standard arguments. We fix $\epsilon>0$. Denote by $\mathcal U_{ad}$ the set of admissible elements of the problem (6.2), (6.3), namely

$$\mathcal U_{ad}=\{(\phi,\psi)\in W_4^7(\Omega)\times W_4^7(\Omega)|J_\epsilon(\phi,\psi)<\infty\}.$$

Denote $\hat J_\epsilon=\inf_{(\phi,\psi)\in W_4^7(\Omega)\times W_4^7(\Omega)}J_\epsilon(\phi,\psi)$. Clearly the pair $(0,0)\in\mathcal U_{ad}$. Therefore there exists a minimizing sequence $\{(\phi_k,\psi_k)\}_{k=1}^\infty\subset W_4^7(\Omega)\times W_4^7(\Omega)$ such that

$$\hat J_\epsilon=\lim_{k\to+\infty}J_\epsilon(\phi_k,\psi_k).$$

Observe that the minimizing sequence is bounded in $W_4^7(\Omega)\times W_4^7(\Omega)$. Indeed, since the sequence $\{\Delta^3L(\phi_k,\psi_k),L(\phi_k,\psi_k)|_{\partial\Omega},\dots,\frac{\partial^3}{\partial\nu^3}L(\phi_k,\psi_k)|_{\partial\Omega}\}$ is bounded in $L^4(\Omega)\times\Pi_{k=0}^3B_4^{\frac{27}{4}-k}(\partial\Omega)$ the standard elliptic L^p-estimate implies that the sequence $\{L(\phi_k,\psi_k)\}$ is bounded in the space $W_4^6(\Omega)\times W_4^6(\Omega)$. Taking into account that the sequence traces of the functions (ϕ_k,ψ_k) is bounded in the Besov space $B_4^{\frac{27}{4}}(\partial\Omega)\times B_4^{\frac{27}{4}}(\partial\Omega)$ and applying the estimates for elliptic operators one more time we obtain that $\{(\phi_k,\psi_k)\}$ bounded in $W_4^7(\Omega)\times W_4^7(\Omega)$. By the Sobolev imbedding theorem the sequence $\{(\phi_k,\psi_k)\}$ is bounded in $C^6(\overline\Omega)\times C^6(\overline\Omega)$. Then taking if necessary a subsequence, (which we denote again as $\{(\phi_k,\psi_k)\}$) we obtain

$$(\phi_k,\psi_k)\to(\widehat\phi_\epsilon,\widehat\psi_\epsilon)\quad\text{weakly in }W_4^7(\Omega)\times W_4^7(\Omega),$$

$$(\frac{\partial^j \phi_k}{\partial \nu^j}, \frac{\partial^j \psi_k}{\partial \nu^j}) \rightarrow (\frac{\partial^j \widehat{\phi}_\epsilon}{\partial \nu^j}, \frac{\partial^j \widehat{\psi}_\epsilon}{\partial \nu^j}) \text{ weakly in } B_4^{\frac{27}{4}-j}(\partial\Omega) \quad \forall j \in \{0,1,2,3\},$$
$$\frac{\partial^k}{\partial z^k}(\phi + i\psi)(\hat{x}_j) - c_{k,j} \rightarrow C_{k,j,\epsilon}, \quad k \in \{0,\dots,5\},$$
$$\Delta^3 L(\phi_k, \psi_k) \rightarrow r_\epsilon \text{ weakly in } L^4(\Omega), \quad L(\phi_k,\psi_k) \rightarrow \tilde{r}_\epsilon \text{ weakly in } W_4^6(\Omega).$$

Obviously, $r_\epsilon = \Delta^3 L(\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon), \tilde{r}_\epsilon = L(\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon)$. Then, since the norms in the spaces $L^4(\Omega)$ and $B_4^{\frac{27}{4}-k}(\partial\Omega)$ are lower semicontinuous with respect to weak convergence we obtain that

$$J_\epsilon(\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon) \le \lim_{k\rightarrow+\infty} J_\epsilon(\phi_k,\psi_k) = \hat{J}_\epsilon.$$

Thus the pair $(\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon)$ is a solution to the extremal problem (6.2), (6.3). Since the set of an admissible elements is convex and the functional J_ϵ is strictly convex, this solution is unique.

By Fermat's theorem we have

$$J'_\epsilon(\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon)[\tilde{\delta}] = 0, \quad \forall \tilde{\delta} \in W_4^7(\Omega) \times W_4^7(\Omega).$$

This equality can be written in the form

$$I'_{\Gamma_0,\frac{27}{4}}((\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon) - B)[\tilde{\delta}] + \epsilon \sum_{k=0}^{3} I'_{\partial\Omega,\frac{27}{4}-k}(\frac{\partial^k}{\partial\nu^k}(\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon))[\frac{\partial^k}{\partial \nu^k}\tilde{\delta}] + (p_\epsilon, \Delta^3 L\tilde{\delta})_{L^2(\Omega)} \tag{6.4}$$
$$+ \sum_{j=1}^{N}\sum_{k=0}^{5} (\frac{\partial^k}{\partial z^k}(\widehat{\phi}_\epsilon + i\widehat{\psi}_\epsilon)(\hat{x}_j) - c_{k,j})\overline{\frac{\partial^k}{\partial z^k}(\tilde{\delta}_1 + i\tilde{\delta}_2)(\hat{x}_j)} + \overline{(\frac{\partial^k}{\partial z^k}(\widehat{\phi}_\epsilon + i\widehat{\psi}_\epsilon)(\hat{x}_j) - c_{k,j})}\frac{\partial^k}{\partial z^k}(\tilde{\delta}_1 + i\tilde{\delta}_2)(\hat{x}_j) = 0,$$

where $p_\epsilon = \frac{4}{\epsilon}((\Delta^3(\frac{\partial\hat{\phi}_\epsilon}{\partial x_1} - \frac{\partial\hat{\psi}_\epsilon}{\partial x_2}))^3, (\Delta^3(\frac{\partial\hat{\phi}_\epsilon}{\partial x_2} + \frac{\partial\hat{\psi}_\epsilon}{\partial x_1}))^3)$ and $I'_{\Gamma^*,\kappa}(\hat{w})$ denotes the derivative of the functional $w \rightarrow \|w\|^4_{B_4^\kappa(\Gamma^*)}$ at $\hat{w}$.

Observe that the pair $J_\epsilon(\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon) \le J_\epsilon(0,0) = \|B\|^4_{B_4^{\frac{27}{4}}(\Gamma_0)} + \sum_{j=1}^N \sum_{k=0}^5 |c_{k,j}|^2$. This implies that the sequence $\{(\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon)\}$ is bounded in $B_4^{\frac{27}{4}}(\Gamma_0)$, the sequences $\{\frac{\partial^k}{\partial z^k}(\widehat{\phi}_\epsilon + i\widehat{\psi}_\epsilon)(\hat{x}_j) - c_{k,j}\}$ are bounded in $\mathbb{C}$, the sequence $\epsilon\sum_{k=0}^3 I'_{\partial\Omega,\frac{27}{4}-k}(\frac{\partial^k}{\partial\nu^k}(\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon))[\frac{\partial^k}{\partial\nu^k}\tilde{\delta}]$ converges to zero for any $\tilde{\delta}$ from $B_4^{\frac{27}{4}}(\partial\Omega)$. Then (6.4) implies that the sequence $\{p_\epsilon\}$ is bounded in $L^{\frac{4}{3}}(\Omega)$.

Therefore there exist $\mathcal{B} \in B_4^{\frac{27}{4}}(\Gamma_0)$, $C_{0,j}, C_{1,j}, \dots, C_{5,j} \in \mathbb{C}$ and $p = (p_1,p_2) \in L^{\frac{4}{3}}(\Omega)$ such that

$$(\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon) - B \rightharpoonup \mathcal{B} \quad \text{weakly in } B_4^{\frac{27}{4}}(\Gamma_0), \quad p_\epsilon \rightharpoonup p \quad \text{weakly in } L^{\frac{4}{3}}(\Omega), \tag{6.5}$$

$$\frac{\partial^k}{\partial z^k}(\widehat{\phi}_\epsilon + i\widehat{\psi}_\epsilon)(\hat{x}_j) - c_{k,j} \rightharpoonup C_{k,j} \quad k \in \{0,1,\dots,5\}, \, j \in \{1,\dots,N\}. \tag{6.6}$$

Passing to the limit in (6.4) we obtain

$$I'_{\Gamma_0,\frac{27}{4}}(\mathcal{B})[\tilde{\delta}] + (p, \Delta^3 L\tilde{\delta})_{L^2(\Omega)} + 2\mathrm{Re}\sum_{j=1}^{N}\sum_{k=0}^{5} C_{k,j}\overline{\frac{\partial^k}{\partial z^k}(\tilde{\delta}_1 + i\tilde{\delta}_2)(\hat{x}_j)} = 0 \quad \forall \tilde{\delta} \in W_4^7(\Omega) \times W_4^7(\Omega). \tag{6.7}$$

Next we claim that

$$\Delta^3 p = 0 \quad \text{in } \Omega \setminus \cup_{j=1}^N \{\hat{x}\} \tag{6.8}$$

in the sense of distributions. Suppose that (6.8) is already proved. This implies

$$(p,\Delta^3 L\tilde\delta)_{L^2(\Omega)}+2\mathrm{Re}\sum_{j=1}^N\sum_{k=0}^5 C_{k,j}\overline{\frac{\partial^k}{\partial z^k}(\tilde\delta_1+i\tilde\delta_2)(\hat x_j)}=0\quad\forall\tilde\delta_1,\tilde\delta_2\in C_0^\infty(\Omega).$$

If $p=(p_1,p_2)$, denoting $P=p_1-ip_2$, we have

$$\mathrm{Re}\,(\Delta^3 P,\partial_{\bar z}(\tilde\delta_1+i\tilde\delta_2))_{L^2(\Omega)}+\mathrm{Re}\sum_{j=1}^N\sum_{k=0}^5\overline{C_{k,j}}\frac{\partial^k}{\partial z^k}(\tilde\delta_1+i\tilde\delta_2)(\hat x_j)=0\quad\forall\tilde\delta_1,\tilde\delta_2\in C_0^\infty(\Omega).$$

Since by (6.8) supp $\Delta^3 P\subset\cup_{j=1}^N\{\hat x_j\}$ there exist some constants $m_{\beta,j}$ and $\hat\ell_j$ such that $\Delta^3 P=\sum_{j=1}^N\sum_{|\beta|=1}^{\hat\ell}m_{\beta,j}D^\beta\delta(x-\hat x_j)$. The above equality can be written in the form

$$-\sum_{|\beta|=1}^{\hat\ell_j}m_{\beta,j}\frac{\partial}{\partial\bar z}D^\beta\delta(x-\hat x_j)=\sum_{k=0}^5(-1)^k\overline{C_{k,j}}\frac{\partial^k}{\partial z^k}\delta(x-\hat x_j).$$

From this we obtain

$$C_{0,j}=C_{1,j}=\cdots=C_{5,j}=0\quad j\in\{1,\dots,N\}.\tag{6.9}$$

Therefore

$$\Delta^3 p=0\quad\text{in }\Omega.\tag{6.10}$$

This implies

$$(p,\Delta^3 L\tilde\delta)_{L^2(\Omega)}=0\quad\forall\tilde\delta\in W_4^7(\Omega),\quad L\tilde\delta|_{\partial\Omega}=\frac{\partial L\tilde\delta}{\partial\nu}|_{\partial\Omega}=\dots\frac{\partial^5 L\tilde\delta}{\partial\nu^5}|_{\partial\Omega}=0.$$

This equality and (6.7) yield

$$I'_{\Gamma_0,\frac{27}{4}}(\mathcal B)[\tilde\delta]=0\quad\forall\tilde\delta\in W_4^7(\Omega)\times W_4^7(\Omega),\quad L\tilde\delta|_{\partial\Omega}=\frac{\partial L\tilde\delta}{\partial\nu}|_{\partial\Omega}=\dots\frac{\partial^5 L\tilde\delta}{\partial\nu^5}|_{\partial\Omega}=0.\tag{6.11}$$

Then using the trace theorem we conclude that $\mathcal B=0$. Using this and (6.5) we obtain

$$(\widehat{\phi}_{\epsilon_k},\widehat{\psi}_{\epsilon_k})-B\rightharpoonup 0\quad\text{weakly in }B_4^{\frac{27}{4}}(\Gamma_0).\tag{6.12}$$

From (6.6) and (6.9) we obtain

$$\frac{\partial^k}{\partial z^k}(\widehat{\phi}_\epsilon+i\widehat{\psi}_\epsilon)(\hat x)\rightharpoonup c_{k,j}\quad k\in\{0,1,\dots,5\},\ j\in\{1,\dots,N\}.$$

By the Sobolev embedding theorem $B_4^{\frac{27}{4}}(\Gamma_0)\subset\subset C^5(\overline{\Gamma_0})$. Therefore (6.12) implies

$$(\widehat{\phi}_{\epsilon_k},\widehat{\psi}_{\epsilon_k})-B\to 0\quad\text{in }C^5(\overline{\Gamma_0}).\tag{6.13}$$

Let the pair $(\tilde\phi_{\epsilon_k},\tilde\psi_{\epsilon_k})$ be a solution to the boundary value problem

$$L(\tilde\phi_{\epsilon_k},\tilde\psi_{\epsilon_k})=L(\hat\phi_{\epsilon_k},\hat\psi_{\epsilon_k})\quad\text{in }\Omega,\quad\tilde\psi_{\epsilon_k}|_{\partial\Omega}=\psi^*_{\epsilon_k}.\tag{6.14}$$

Here $\psi^*_{\epsilon_k}$ is a smooth function such that $\psi^*_{\epsilon_k}|_{\Gamma_0}=0$ and the pair $(L(\hat\phi_{\epsilon_k},\hat\psi_{\epsilon_k}),\psi^*_{\epsilon_k})$ is orthogonal to all solutions of the adjoint problem (see [20]). Moreover since $L(\tilde\phi_{\epsilon_k},\tilde\psi_{\epsilon_k})\to 0$ in $W_4^6(\Omega)$ we may assume $\psi^*_{\epsilon_k}\to 0$ in $C^6(\partial\Omega)$. Among all possible solutions to problem (6.14) (clearly there is no unique solution to this problem) we choose one such that $\int_\Omega \tilde\phi_{\epsilon_k} dx = 0$. Thus we obtain

$$(\tilde\phi_{\epsilon_k}, \tilde\psi_{\epsilon_k}) \to 0 \quad \text{in } W^7_4(\Omega) \times W^7_4(\Omega). \tag{6.15}$$

Therefore the sequence $\{(\widehat{\phi}_{\epsilon_k} - \tilde\phi_{\epsilon_k}, \widehat{\psi}_{\epsilon_k} - \tilde\psi_{\epsilon_k})\}$ represents the desired approximation for the solution of the Cauchy problem (6.1).

Now we prove (6.8). Let $\widetilde{x}$ be an arbitrary point in $\Omega \setminus \cup_{j=1}^N \{\hat x_j\}$ and let $\widetilde\chi$ be a smooth function such that it is zero in some neighborhood of $\Gamma_0 \cup \cup_{j=1}^N \{\hat x_j\}$ and the set $\mathcal{A} = \{x \in \Omega | \widetilde\chi(x) = 1\}$ contains an open connected subset $\mathcal{F}$ such that $\widetilde x \in \mathcal{F}$ and $\tilde\Gamma \cap \overline{\mathcal{F}}$ is an open set in $\partial\Omega$. In addition we assume that $\mathrm{Int}(\mathrm{supp}\,\chi)$ is a simply connected domain. By (6.7) we have

$$0 = (p, \Delta^3 L(\widetilde\chi\tilde\delta))_{L^2(\Omega)} = (\widetilde\chi p, \Delta^3 L\tilde\delta)_{L^2(\Omega)} + (p, [\Delta^3 L, \widetilde\chi]\tilde\delta)_{L^2(\Omega)} \quad \forall \tilde\delta \in W^7_4(\Omega) \times W^7_4(\Omega). \tag{6.16}$$

Denote $L\tilde\delta = \hat\delta$. Consider the functional mapping $\hat\delta \in W^2_4(\mathrm{supp}\,\widetilde\chi)$ to $(p, [\Delta^3 L, \widetilde\chi]\tilde\delta)_{L^2(\Omega)}$, where

$$L\tilde\delta = \hat\delta \quad \text{in}\,\Omega, \quad \mathrm{Im}\,\tilde\delta|_{\mathcal{S}} = 0, \int_{\mathrm{supp}\,\widetilde\chi} \mathrm{Re}\,\tilde\delta dx = 0,$$

where $\mathcal{S}$ denotes the boundary of $\mathrm{supp}\,\widetilde\chi$. For each $\hat\delta \in W^2_4(\mathrm{supp}\,\widetilde\chi)$, there exists a unique solution $\tilde\delta \in W^3_4(\mathrm{supp}\,\widetilde\chi)$. Hence the functional is well-defined and continuous on $W^2_4(\mathrm{supp}\,\widetilde\chi)$. Therefore there exists $\mathbf{q}, \mathbf{r}, q_0 \in L^{\frac43}(\mathrm{supp}\,\widetilde\chi)$ such that $\int_{\mathrm{supp}\,\widetilde\chi}(\sum_{j,k=1}^2 r_{jk}\frac{\partial^2}{\partial x_j \partial x_k}\hat\delta + (\mathbf{q}, \hat\delta) + q_0\hat\delta)dx = (p, [\Delta^3 L, \widetilde\chi]\tilde\delta)_{L^2(\mathrm{supp}\,\widetilde\chi)}$.

Consider the boundary value problem

$$\Delta^3 \tilde P = \tilde f \quad \text{in } \mathrm{supp}\,\chi, \quad \tilde P|_{\mathcal{S}} = \frac{\partial \tilde P}{\partial\nu}|_{\mathcal{S}} = \frac{\partial^2 \tilde P}{\partial \nu^2}|_{\mathcal{S}} = 0.$$

Here $\tilde f = 2\mathrm{div}\,(\nabla\tilde{\mathbf{q}}) - q_0 - \sum_{j,k=1}^2 \frac{\partial^2}{\partial x_j\partial x_k} r_{jk}$. A solution to this problem exists and is unique, since $\tilde f \in (\mathring{W}^2_4(\mathrm{supp}\,\widetilde\chi))'$. Then $P \in \mathring{W}^1_{\frac43}(\mathrm{supp}\,\widetilde\chi)$. On the other hand, thanks to (6.16), $P = \tilde\chi p \in \mathring{W}^1_{\frac43}(\mathrm{supp}\,\widetilde\chi)$.

Next we take another smooth cut off function $\widetilde\chi_1$ such that $\mathrm{supp}\,\widetilde\chi_1 \subset \mathcal{A}$ and Int $(\mathrm{supp}\ \chi_1)$ is a simply connected domain. A neighborhood of $\widetilde x$ belongs to $\mathcal{A}_1 = \{x | \widetilde\chi_1 = 1\}$, the interior of $\mathcal{A}_1$ is connected, and $\overline{\mathrm{Int}\ \mathcal{A}_1} \cap \tilde\Gamma$ contains an open subset $\mathcal{O}$ in $\partial\Omega$. Similarly to (6.16) we have

$$(\widetilde\chi_1 p, \Delta^3 L\tilde\delta)_{L^2(\Omega)} - (p, [\Delta^3 L, \widetilde\chi_1]\tilde\delta)_{L^2(\Omega)} = 0 \quad \forall\tilde\delta \in W^7_4(\Omega).$$

This equality implies that $\widetilde\chi_1 p \in W^2_{\frac43}(\Omega)$, using a similar argument to the one above.

Next we take another smooth cut off function $\widetilde\chi_2$ such that $\mathrm{supp}\,\widetilde\chi_2 \subset \mathcal{A}_2$ and Int $(\mathrm{supp}\ \chi_2)$ is a simply connected domain. A neighborhood of $\widetilde x$ belongs to $\mathcal{A}_3 = \{x | \widetilde\chi_2 = 1\}$, the interior of $\mathcal{A}_1$ is connected, and $\overline{\mathrm{Int}\ \mathcal{A}_3} \cap \tilde\Gamma$ contains an open subset $\mathcal{O}$ in $\partial\Omega$. Similarly to (6.16) we have

$$(\widetilde\chi_2 p, \Delta^3 L\tilde\delta)_{L^2(\Omega)} - (p, [\Delta^3 L, \widetilde\chi_2]\tilde\delta)_{L^2(\Omega)} = 0 \quad \forall\tilde\delta \in W^7_4(\Omega).$$

This equality implies that $\widetilde\chi_2 p \in W^3_{\frac43}(\Omega)$, using a similar argument to the one above. Let ω be a domain such that $\omega \cap \Omega = \emptyset$, $\partial\omega \cap \partial\Omega \subset \mathcal{O}$ contains an open set in $\partial\Omega$.

We extend p on ω by zero. Then

$$(\Delta^3(\widetilde{\chi}_2 p), L\tilde{\delta})_{L^2(\Omega\cup\omega)} + (p, [\Delta^3 L, \widetilde{\chi}_2]\tilde{\delta})_{L^2(\Omega\cup\omega)} = 0.$$

Hence, since $[\Delta^3 L, \widetilde{\chi}_2]|_{\mathcal{A}_1} = 0$ we have

$$L^*\Delta^3(\widetilde{\chi}_2 p) = 0 \quad \text{in} \;\; \text{Int } \mathcal{A}_2 \cup \omega, \quad p|_\omega = 0.$$

By Holmgren's theorem $\Delta^3(\widetilde{\chi}_2 p)|_{\text{Int } \mathcal{A}_1} = 0$, that is, $(\Delta^3 p)(\widetilde{x}) = 0$. Thus (6.8) is proved. □

Consider the Cauchy problem for the Cauchy-Riemann equations

$$L(\phi,\psi) = (\frac{\partial \phi}{\partial x_1} - \frac{\partial \psi}{\partial x_2}, \frac{\partial \phi}{\partial x_2} + \frac{\partial \psi}{\partial x_1}) = 0 \quad \text{in } \Omega, \quad (\phi,\psi)\,|_{\Gamma_0} = (b(x), 0), \tag{6.17}$$

$$\frac{\partial^l}{\partial z^l}(\phi + i\psi)(\hat{x}_j) = c_{0,j}, \quad \forall j \in \{1, \dots N\} \quad \text{and } \forall l \in \{0, \dots, 5\}.$$

Here $\hat{x}_1, \dots \hat{x}_N$ be an arbitrary fixed points in Ω. We consider the function b and complex numbers $\vec{C} = (c_{0,1}, c_{1,1}, c_{2,1}, c_{3,1}, c_{4,1}, c_{5,1} \dots c_{0,N}, c_{1,N}, c_{2,N}, c_{3,N}, c_{4,N}, c_{5,N})$ as initial data for (6.1). The following proposition establishes the solvability of (6.1) for a dense set of Cauchy data.

Corollary 6.1. *There exists a set $\mathcal{O} \subset C^5(\overline{\Gamma_0}) \times \mathbb{C}^{6N}$ such that for each $(b, \vec{C}) \in \mathcal{O}$, (6.17) has at least one solution $(\phi,\psi) \in (C^5(\overline{\Omega}))^2$ and $\overline{\mathcal{O}} = C^5(\overline{\Gamma}) \times \mathbb{C}^{6N}$.*

The proof of Corollary 6.1 is similar to the proof of Proposition 6.1. The only difference is that instead of the extremal problem we have to use the following extremal problem

$$J_\epsilon(\phi,\psi) = \|\phi - b\|^4_{B_4^{\frac{27}{4}}(\Gamma_0)} + \epsilon \sum_{k=0}^{3} \|\frac{\partial^k(\phi,\psi)}{\partial \nu^k}\|^4_{B_4^{\frac{27}{4}-k}(\partial\Omega)}$$

$$+\frac{1}{\epsilon}\left\|\Delta^3 L(\phi,\psi)\right\|^4_{L^4(\Omega)} + \sum_{j=1}^{N}\sum_{k=0}^{5} |\frac{\partial^k}{\partial z^k}(\phi + i\psi)(\hat{x}_j) - c_{k,j}|^2 \to \inf,$$

$$(\phi,\psi) \in W_4^7(\Omega) \times W_4^7(\Omega), \quad \psi|_{\Gamma_0} = 0.$$

We have

Proposition 6.2. *Let $\alpha \in (0,1)$, $g_1, \dots, g_p$ be linearly independent functions from $L^2(\Gamma_0)$, $y_1, \dots, y_{\hat{k}}$ be some arbitrary points from Γ_0 and $\hat{x}$ be an arbitrary point from Ω. Then there exist a holomorphic function $a \in C^{5+\alpha}(\bar{\Omega})$ and an antiholomorphic function $d(\bar{z}) \in C^{5+\alpha}(\bar{\Omega})$ such that $(ae^{\mathcal{A}} + de^{\mathcal{B}})|_{\Gamma_0} = 0$ and*

$$\int_{\Gamma_0} a g_p d\sigma = 0 \quad p \in \{1, \dots, m\}; \quad \frac{\partial^k}{\partial x_1^k}\frac{\partial^j}{\partial x_2^j} a(y_{\hat{k}}) = 0 \quad k + j \leq 5$$

and

$$a(\hat{x}) \neq 0 \quad \textit{and} \quad d(\bar{\hat{x}}) \neq 0.$$

Proof. Consider the operator

$$R(\gamma) = (\int_{\Gamma_0} ag_1 d\sigma, \dots, \int_{\Gamma_0} ag_p d\sigma, a(y_1), \dots \frac{\partial^5 a}{\partial z^5}(y_1), \dots$$

$$a(y_{\hat{k}}), \dots, \frac{\partial^5 a}{\partial z^5}(y_{\hat{k}}), d(y_1), \dots \frac{\partial^5 d}{\partial \bar{z}^5}(y_1), \dots, d(y_{\hat{k}}), \dots \frac{\partial^5 d}{\partial \bar{z}^5}(y_{\hat{k}}), a(\hat{x}), d(\hat{x})).$$

Here $\gamma \in C_0^\infty(\tilde{\Gamma})$ and the functions a and d are solutions to the following problem

$$\frac{\partial a}{\partial \bar{z}} = 0, \quad \text{in } \Omega, \quad \frac{\partial d}{\partial z} = 0 \quad \text{in } \Omega, \quad (ae^{\mathcal{A}} + de^{\mathcal{B}})|_{\partial\Omega} = \gamma.$$

Consider the image of the operator R. Clearly it is closed. Let us show that the point $(0, \dots, 0, 1, 1)$ belongs to the image of the operator R. Let a holomorphic function a satisfy $\int_{\Gamma_0} ag_1 d\sigma = \dots = \int_{\Gamma_0} ag_p d\sigma = 0$ and

$$\frac{\partial^\beta}{\partial {x_1}^{\beta_1} \partial {x_2}^{\beta_2}} a(y_j) = 0 \quad \forall |\beta| \in \{0, \dots, 5\}, \quad j \in \{1, \dots, \hat{k}\}.$$

Consider the function $-e^{\mathcal{A}-\mathcal{B}}a(z)$ and the pair $(b_1, b_2) = (\mathrm{Re}\{-e^{\mathcal{A}-\mathcal{B}}a\}, \mathrm{Im}\{e^{\mathcal{A}-\mathcal{B}}a\})$. Using Proposition 6.1 we solve problem (6.1) with $l = 0$ approximately. Let $(\phi_\epsilon, \psi_\epsilon)$ be a sequence of functions such that

$$\frac{\partial}{\partial z}(\phi_\epsilon + i\psi_\epsilon) = 0 \quad \text{in}\, \Omega, \quad (\phi_\epsilon, \psi_\epsilon)|_{\Gamma_0} \to (b_1, b_2) \quad \text{in } C^{5+\alpha}(\overline{\Gamma}_0), \quad (\phi_\epsilon + i\psi_\epsilon)(\hat{x}) \to 1.$$

Denote $d_\epsilon = \phi_\epsilon - i\psi_\epsilon, \beta_\epsilon = ae^{\mathcal{A}} + d_\epsilon e^{\mathcal{B}}$. Then the sequence $\{\beta_\epsilon\}$ converges to zero in the space $C^{5+\alpha}(\Gamma_0)$.

By Proposition 2.5 there exists a solution to problem (2.46) with the initial data β_ϵ, which we denote as $\{\tilde{a}_\epsilon, \tilde{d}_\epsilon\}$ such that the sequence $\{\tilde{a}_\epsilon, \tilde{d}_\epsilon\}$ converges to zero in $(C^5(\overline{\Omega}))^2$. Denote by $\gamma_\epsilon = (a + \tilde{a}_\epsilon, d_\epsilon + \tilde{d}_\epsilon)|_{\Gamma_0}$. Clearly $R(\gamma_\epsilon)$ converges to $(0, \dots, 0, 1, 1)$. The proof of the proposition is completed. □

7. Appendix II. Asymptotic formulas

Proposition 7.1. *Under the conditions of theorem 1.1 for any point x from the boundary of Ω we have*

$$-\frac{1}{\pi}\int_\Omega \frac{e_1\tilde{g}_1 e^{-\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}}{\zeta - z} d\xi_1 d\xi_2 = \frac{1}{2\tau^2}\sum_{k=1}^{\ell} \frac{\frac{\partial_z \tilde{g}_1(\tilde{x}_k)}{\partial_z^2\Phi(\tilde{x}_k)}}{(\tilde{z}_k - z)^2} \frac{e^{-2i\tau\psi(\tilde{x}_k)}}{|det\, Im\, \Phi''(\tilde{x}_k)|^{\frac{1}{2}}} + o(\frac{1}{\tau^2}) +$$

$$\frac{1}{4\tau^2}\sum_{k=1}^{\ell} \frac{\frac{\partial_z \tilde{g}_1(\tilde{x}_k)}{\partial_z^2\Phi(\tilde{x}_k)}\frac{\partial_z^3\Phi(\tilde{x}_k)}{\partial_z^2\Phi(\tilde{x}_k)} + \frac{\partial_{\bar{z}}^2 \tilde{g}_1(\tilde{x}_k)}{\overline{\partial_z^2\Phi(\tilde{x}_k)}} - \frac{\partial_z^2 \tilde{g}_1(\tilde{x}_k)}{\partial_z^2\Phi(\tilde{x}_k)}}{(\tilde{z}_k - z)} \frac{e^{-2i\tau\psi(\tilde{x}_k)}}{|detIm\, \Phi''(\tilde{x}_k)|^{\frac{1}{2}}} \quad as\ |\tau| \to +\infty. \tag{7.1}$$

$$(7.2)\qquad -\frac{1}{\pi}\int_\Omega \frac{e_1\tilde g_2 e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}}{\bar\zeta-\bar z}d\xi_1 d\xi_2 = \frac{1}{2\tau^2}\sum_{k=1}^{\ell}\frac{\frac{\partial_{\bar z}\tilde g_2(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}}{(\overline{\tilde z_k}-\bar z)^2}\frac{e^{2i\tau\psi(\tilde x_k)}}{|det\, Im\,\Phi''(\tilde x_k)|^{\frac12}}$$
$$-\frac{1}{4\tau^2}\sum_{k=1}^{\ell}\frac{-\frac{\partial_{\bar z}\tilde g_2(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}\frac{\partial_{\bar z}^3\overline{\Phi}(\tilde x_k)}{\partial_{\bar z}^2\overline{\Phi}(\tilde x_k)}+\frac{\partial_{\bar z}^2\tilde g_2(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}-\frac{\partial_z^2\tilde g_2(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}}{\overline{\tilde z_k}-\bar z}\frac{e^{2i\tau\psi(\tilde x_k)}}{|det\, Im\,\Phi''(\tilde x_k)|^{\frac12}}+o(\frac{1}{\tau^2})$$
$$as\ |\tau|\to+\infty.$$

$$(7.3)\qquad -\frac{1}{\pi}\int_\Omega \frac{e_1\tilde g_3 e^{-\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}}{\bar\zeta-\bar z}d\xi_1 d\xi_2 = -\frac{1}{2\tau^2}\sum_{k=1}^{\ell}\frac{\frac{\partial_{\bar z}\tilde g_3(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}}{(\overline{\tilde z_k}-\bar z)^2}\frac{e^{-2i\tau\psi(\tilde x_k)}}{|det\, Im\,\Phi''(\tilde x_k)|^{\frac12}}$$
$$-\frac{1}{4\tau^2}\sum_{k=1}^{\ell}\frac{\frac{\partial_{\bar z}\tilde g_3(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}\frac{\partial_{\bar z}^3\overline{\Phi}(\tilde x_k)}{\partial_{\bar z}^2\overline{\Phi}(\tilde x_k)}-\frac{\partial_{\bar z}^2\tilde g_3(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}+\frac{\partial_z^2\tilde g_3(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}}{\overline{\tilde z_k}-\bar z}\frac{e^{-2i\tau\psi(\tilde x_k)}}{|det\, Im\,\Phi''(\tilde x_k)|^{\frac12}}+o(\frac{1}{\tau^2})$$
$$as\ |\tau|\to+\infty.$$

$$-\frac{1}{\pi}\int_\Omega \frac{e_1\tilde g_4 e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}}{\zeta-z}d\xi_1 d\xi_2 = -\frac{1}{2\tau^2}\sum_{k=1}^{\ell}\frac{\frac{\partial_z\tilde g_4(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}}{(\tilde z_k-z)^2}\frac{e^{2i\tau\psi(\tilde x_k)}}{|det\, Im\,\Phi''(\tilde x_k)|^{\frac12}}$$
$$-\frac{1}{4\tau^2}\sum_{k=1}^{\ell}\frac{\frac{\partial_z\tilde g_4(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}\frac{\partial_z^3\Phi(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}+\frac{\partial_{\bar z}^2\tilde g_4(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}-\frac{\partial_z^2\tilde g_4(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}}{\tilde z_k-z}\frac{e^{2i\tau\psi(\tilde x_k)}}{|det\, Im\,\Phi''(\tilde x_k)|^{\frac12}}+o(\frac{1}{\tau^2})$$
$$(7.4)\qquad as\ |\tau|\to+\infty.$$

Proof. Let $\delta>0$ be a sufficiently small number and $\tilde e_k\in C_0^\infty(B(\widetilde x_k,\delta)), \tilde e_k|_{B(\widetilde x_k,\delta/2)}\equiv 1$. We compute the asymptotic formulae of the following integral as $|\tau|$ goes to infinity:

$$-\ \frac{1}{\pi}\int_\Omega \frac{e_1\tilde g_1 e^{-\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}}{\zeta-z}d\xi_1 d\xi_2 =$$
$$-\ \frac{1}{\pi}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)}\frac{\tilde e_k\tilde g_1 e^{-\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}}{\zeta-z}d\xi_1 d\xi_2+o(\frac{1}{\tau^2}) =$$
$$-\ \frac{1}{\pi}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)}\tilde e_k\left\{\frac{\partial_z\tilde g_1(\tilde x_k)(\zeta-\tilde z_k)+\frac12\partial_{\bar z}^2\tilde g_1(\tilde x_k)(\bar\zeta-\overline{\tilde z_k})^2}{\zeta-z}\right.$$
$$+\ \left.\frac{\partial_{\bar z}\partial_z\tilde g_1(\tilde x_k)(\zeta-\tilde z_k)(\bar\zeta-\overline{\tilde z_k})+\frac12\partial_z^2\tilde g_1(\tilde x_k)(\zeta-\tilde z_k)^2}{\zeta-z}\right\}e^{-\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2+o(\frac{1}{\tau^2}) =$$

$$
\begin{aligned}
&- \frac{1}{\pi}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)} \tilde e_k \left\{ \frac{\frac{\partial_z \tilde g_1(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}(\partial_\zeta\Phi - \frac12\partial_z^3\Phi(\tilde x_k)(\zeta-\tilde z_k)^2) + \frac12 \frac{\partial_{\bar z}^2 \tilde g_1(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}\partial_{\bar\zeta}\overline{\Phi}(\bar\zeta - \overline{\tilde z_k})}{\zeta - z}\right.\\
&+ \left.\frac{\frac{\partial_{\bar z}\partial_z\tilde g_1(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}(\zeta-\tilde z_k)\partial_{\bar\zeta}\overline\Phi + \frac12\frac{\partial_z^2\tilde g_1(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}\partial_\zeta\Phi(\zeta-\tilde z_k)}{\zeta-z}\right\} e^{-\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2 + o(\frac{1}{\tau^2}) =\\
&- \frac{1}{\pi}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)} \tilde e_k \left\{ \frac{\frac{\partial_z \tilde g_1(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}(\partial_\zeta\Phi - \frac12\frac{\partial_z^3\Phi(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}\partial_\zeta\Phi(\zeta-\tilde z_k)) + \frac12 \frac{\partial_{\bar z}^2 \tilde g_1(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}\partial_{\bar\zeta}\overline{\Phi}(\bar\zeta - \overline{\tilde z_k})}{\zeta - z}\right.\\
&+ \left.\frac{\frac{\partial_{\bar z}\partial_z\tilde g_1(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}(\zeta-\tilde z_k)\partial_{\bar\zeta}\overline\Phi + \frac12\frac{\partial_z^2\tilde g_1(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}\partial_\zeta\Phi(\zeta-\tilde z_k)}{\zeta-z}\right\} e^{-\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2 + o(\frac{1}{\tau^2}) =\\
&- \frac{1}{\pi\tau}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)} \tilde e_k \left\{ \frac12\frac{-\frac{\partial_z \tilde g_1(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}\frac{\partial_z^3\Phi(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)} - \frac{\partial_{\bar z}^2 \tilde g_1(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}}{\zeta - z} - \frac{\frac{\partial_z \tilde g_1(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}(1 - \frac12\frac{\partial_z^3\Phi(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}(\zeta-\tilde z_k))}{(\zeta-z)^2}\right.\\
&+ \left.\frac12\frac{\frac{\partial_z^2\tilde g_1(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}}{\zeta-z} - \frac12\frac{\frac{\partial_z^2\tilde g_1(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}(\zeta-\tilde z_k)}{(\zeta-z)^2}\right\} e^{-\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2 + o(\frac{1}{\tau^2}).
\end{aligned}
\tag{7.5}
$$

By the stationary phase argument

$$
\int_{B(\tilde x_k,\delta)} \tilde e_k \frac{(\zeta-\tilde z_k)}{(\zeta-z)^2} e^{-\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2 = o(\frac{1}{\tau}) \quad \text{as } |\tau|\rightarrow+\infty.
$$

Finally we have

$$
\begin{aligned}
-\frac{1}{\pi}\int_\Omega \frac{e_1\tilde g_1 e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}}{\zeta-z}d\xi_1 d\xi_2 &= \frac{1}{\pi\tau}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)} \tilde e_k \frac{\frac{\partial_z \tilde g_1(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}}{(\zeta-z)^2} e^{-\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}dx + o(\frac{1}{\tau^2}) +\\
&\frac{1}{2\pi\tau}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)} \tilde e_k \frac{\frac{\partial_z \tilde g_1(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}\frac{\partial_z^3\Phi(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)} + \frac{\partial_{\bar z}^2 \tilde g_1(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}} - \frac{\partial_z^2 \tilde g_1(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}}{(\zeta-z)} e^{-\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}dx =\\
&+\frac{1}{2\tau^2}\sum_{k=1}^{\ell}\frac{\frac{\partial_z \tilde g_1(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}}{(\tilde z_k - z)^2}\frac{e^{-2i\tau\psi(\tilde x_k)}}{|\det \operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}} + o(\frac{1}{\tau^2}) +\\
&\frac{1}{4\tau^2}\sum_{k=1}^{\ell}\frac{\frac{\partial_z \tilde g_1(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}\frac{\partial_z^3\Phi(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)} + \frac{\partial_{\bar z}^2 \tilde g_1(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}} - \frac{\partial_z^2 \tilde g_1(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}}{(\tilde z_k - z)}\frac{e^{-2i\tau\psi(\tilde x_k)}}{|\det\operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}} \quad \text{as } |\tau|\rightarrow+\infty.
\end{aligned}
\tag{7.6}
$$

We compute the asymptotic of the following integral as $|\tau|$ goes to infinity:

$$
\begin{aligned}
&- \frac{1}{\pi}\int_\Omega \frac{e_1\tilde g_2 e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}}{\bar\zeta-\bar z}d\xi_1 d\xi_2 = \\
&- \frac{1}{\pi}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)} \frac{\tilde e_k\tilde g_2 e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}}{\bar\zeta-\bar z}d\xi_1 d\xi_2 + o(\frac{1}{\tau^2}) = \\
&- \frac{1}{\pi}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)} \tilde e_k\left\{\frac{\partial_{\bar z}\tilde g_2(\tilde x_k)(\overline{\zeta}-\overline{\tilde z_k})+\frac12\partial^2_{\bar z}\tilde g_2(\tilde x_k)(\overline{\zeta}-\overline{\tilde z_k})^2}{\bar\zeta-\bar z}\right.\\
&+ \left.\frac{\partial_{\bar z}\partial_z\tilde g_2(\tilde x_k)(\zeta-\tilde z_k)(\overline{\zeta}-\overline{\tilde z_k})+\frac12\partial^2_{z}\tilde g_2(\tilde x_k)(\zeta-\tilde z_k)^2}{\bar\zeta-\bar z}\right\}e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2 + o(\frac{1}{\tau^2}) = \\
&- \frac{1}{\pi}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)} \tilde e_k\left\{\frac{\frac{\partial_{\bar z}\tilde g_2(\tilde x_k)}{\partial^2_{\bar z}\Phi(\tilde x_k)}(\partial_{\bar\zeta}\overline{\Phi}-\frac12\partial^3_{\bar z}\overline{\Phi}(\tilde x_k)(\overline{\zeta}-\overline{\tilde z_k})^2)+\frac12\frac{\partial^2_{\bar z}\tilde g_2(\tilde x_k)}{\partial^2_{\bar z}\Phi(\tilde x_k)}\partial_{\bar\zeta}\overline{\Phi}(\overline{\zeta}-\overline{\tilde z_k})}{\overline{\zeta}-\bar z}\right.\\
&+ \left.\frac{\frac{\partial_{\bar z}\partial_z\tilde g_2(\tilde x_k)}{\partial^2_{z}\Phi(\tilde x_k)}(\overline{\zeta}-\overline{\tilde z_k})\partial_\zeta\Phi+\frac12\frac{\partial^2_{z}\tilde g_2(\tilde x_k)}{\partial^2_{z}\Phi(\tilde x_k)}\partial_\zeta\Phi(\tilde x_k)(\zeta-\tilde z_k)}{\bar\zeta-\bar z}\right\}e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2 + o(\frac{1}{\tau^2}) = \\
&- \frac{1}{\pi}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)} \tilde e_k\left\{\frac{\frac{\partial_{\bar z}\tilde g_2(\tilde x_k)}{\partial^2_{\bar z}\Phi(\tilde x_k)}(\partial_{\bar\zeta}\overline{\Phi}-\frac12\frac{\partial^3_{\bar z}\overline{\Phi}(\tilde x_k)}{\partial^2_{\bar z}\overline{\Phi}(\tilde x_k)}\partial_{\bar\zeta}\overline{\Phi}(\overline{\zeta}-\overline{\tilde z_k}))+\frac12\frac{\partial^2_{\bar z}\tilde g_2(\tilde x_k)}{\partial^2_{\bar z}\Phi(\tilde x_k)}\partial_{\bar\zeta}\overline{\Phi}(\overline{\zeta}-\overline{\tilde z_k})}{\overline{\zeta}-\bar z}\right.\\
&+ \left.\frac{\frac{\partial_{\bar z}\partial_z\tilde g_2(\tilde x_k)}{\partial^2_{z}\Phi(\tilde x_k)}(\overline{\zeta}-\overline{\tilde z_k})\partial_\zeta\Phi+\frac12\frac{\partial^2_{z}\tilde g_2(\tilde x_k)}{\partial^2_{z}\Phi(\tilde x_k)}\partial_\zeta\Phi(\zeta-\tilde z_k)}{\overline{\zeta}-\bar z}\right\}e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2 + o(\frac{1}{\tau^2}) = \\
&- \frac{1}{\pi\tau}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)} \tilde e_k\left\{\frac{-\frac12\frac{\partial_{\bar z}\tilde g_2(\tilde x_k)}{\partial^2_{\bar z}\Phi(\tilde x_k)}\frac{\partial^3_{\bar z}\overline{\Phi}(\tilde x_k)}{\partial^2_{\bar z}\overline{\Phi}(\tilde x_k)}+\frac12\frac{\partial^2_{\bar z}\tilde g_2(\tilde x_k)}{\partial^2_{\bar z}\Phi(\tilde x_k)}}{\overline{\zeta}-\bar z}-\frac{\frac{\partial_{\bar z}\tilde g_2(\tilde x_k)}{\partial^2_{\bar z}\Phi(\tilde x_k)}(1-\frac12\frac{\partial^3_{\bar z}\overline{\Phi}(\tilde x_k)}{\partial^2_{\bar z}\overline{\Phi}(\tilde x_k)}(\overline{\zeta}-\overline{\tilde z_k}))}{(\overline{\zeta}-\bar z)^2}\right.\\
&- \left.\frac12\frac{\frac{\partial^2_{\bar z}g_2(\tilde x_k)}{\partial^2_{\bar z}\Phi(\tilde x_k)}(\overline{\zeta}-\overline{\tilde z_k})}{(\overline{\zeta}-\bar z)^2}-\frac12\frac{\frac{\partial^2_{z}\tilde g_2(\tilde x_k)}{\partial^2_{z}\Phi(\tilde x_k)}}{\overline{\zeta}-\bar z}\right\}e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2 + o(\frac{1}{\tau^2}).
\end{aligned}
\tag{7.7}
$$

Observe that by the stationary phase argument

$$\int_{B(\tilde x_k,\delta)}\tilde e_k\frac{(\overline{\zeta}-\overline{\tilde z_k})}{(\overline{\zeta}-\overline{z})^2}e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2 = o(\frac{1}{\tau}) \quad \text{as } |\tau|\to+\infty.$$

Finally we have

$$
\begin{aligned}
- & \frac{1}{\pi}\int_\Omega \frac{e_1\tilde g_2 e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}}{\bar\zeta-\bar z}d\xi_1 d\xi_2 = \frac{1}{\pi\tau}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)}\tilde e_k\frac{\frac{\partial_{\bar z}\tilde g_2(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}}{(\bar\zeta-\bar z)^2}e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2\\
- & \frac{1}{\pi\tau}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)}\tilde e_k\frac{-\frac12\frac{\partial_{\bar z}\tilde g_2(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}\frac{\partial_{\bar z}^3\overline{\Phi}(\tilde x_k)}{\partial_{\bar z}^2\overline{\Phi}(\tilde x_k)}+\frac12\frac{\partial_{\bar z}^2\tilde g_2(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}-\frac12\frac{\partial_z^2\tilde g_2(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}}{\bar\zeta-\bar z}e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2+o(\frac{1}{\tau^2}) =\\
+ & \frac{1}{2\tau^2}\sum_{k=1}^{\ell}\frac{\frac{\partial_{\bar z}\tilde g_2(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}}{(\overline{\tilde z_k}-\bar z)^2}\frac{e^{2i\tau\psi(\tilde x_k)}}{|\det \operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}}\\
- & \frac{1}{4\tau^2}\sum_{k=1}^{\ell}\frac{-\frac{\partial_{\bar z}\tilde g_2(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}\frac{\partial_{\bar z}^3\overline{\Phi}(\tilde x_k)}{\partial_{\bar z}^2\overline{\Phi}(\tilde x_k)}+\frac{\partial_{\bar z}^2\tilde g_2(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}-\frac{\partial_z^2\tilde g_2(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}}{\overline{\tilde z_k}-\bar z}\frac{e^{2i\tau\psi(\tilde x_k)}}{|\det \operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}}+o(\frac{1}{\tau^2})\quad \text{as } |\tau|\to+\infty.
\end{aligned}
$$

We compute the asymptotic of the following integral as $|\tau|$ goes to infinity:

$$
\begin{aligned}
- & \frac{1}{\pi}\int_\Omega \frac{e_1\tilde g_4 e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}}{\zeta-z}d\xi_1 d\xi_2 =\\
- & \frac{1}{\pi}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)}\frac{\tilde e_k\tilde g_4 e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}}{\zeta-z}d\xi_1 d\xi_2+o(\frac{1}{\tau^2}) =\\
- & \frac{1}{\pi}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)}\tilde e_k\Bigg\{\frac{\partial_z\tilde g_4(\tilde x_k)(\zeta-\tilde z_k)+\frac12\partial_{\bar z}^2\tilde g_4(\tilde x_k)(\bar\zeta-\overline{\tilde z_k})^2}{\zeta-z}\\
+ & \frac{\partial_{\bar z}\partial_z\tilde g_4(\tilde x_k)(\zeta-\tilde z_k)(\bar\zeta-\overline{\tilde z_k})+\frac12\partial_z^2\tilde g_4(\tilde x_k)(\zeta-\tilde z_k)^2}{\zeta-z}\Bigg\}e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2+o(\frac{1}{\tau^2}) =\\
- & \frac{1}{\pi}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)}\tilde e_k\Bigg\{\frac{\frac{\partial_z\tilde g_4(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}(\partial_\zeta\Phi-\frac12\partial_z^3\Phi(\tilde x_k)(\zeta-\tilde z_k)^2)+\frac12\frac{\partial_{\bar z}^2\tilde g_4(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}\partial_{\bar\zeta}\overline{\Phi}(\bar\zeta-\overline{\tilde z_k})}{\zeta-z}\\
+ & \frac{\frac{\partial_{\bar z}\partial_z\tilde g_4(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}(\zeta-\tilde z_k)\partial_{\bar\zeta}\overline{\Phi}+\frac12\frac{\partial_z^2\tilde g_4(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}\partial_\zeta\Phi(\zeta-\tilde z_k)}{\zeta-z}\Bigg\}e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2+o(\frac{1}{\tau^2}) =\\
- & \frac{1}{\pi}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)}\tilde e_k\Bigg\{\frac{\frac{\partial_z\tilde g_4(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}(\partial_\zeta\Phi-\frac12\frac{\partial_z^3\Phi(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}\partial_\zeta\Phi(\zeta-\tilde z_k))+\frac12\frac{\partial_{\bar z}^2\tilde g_4(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}\partial_{\bar\zeta}\overline{\Phi}(\bar\zeta-\overline{\tilde z_k})}{\zeta-z}\\
+ & \frac{\frac{\partial_{\bar z}\partial_z\tilde g_4(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}(\zeta-\tilde z_k)\partial_{\bar\zeta}\overline{\Phi}+\frac12\frac{\partial_z^2\tilde g_4(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}\partial_\zeta\Phi(\zeta-\tilde z_k)}{\zeta-z}\Bigg\}e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2+o(\frac{1}{\tau^2}) =
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{\pi\tau}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)} \tilde e_k \left\{ \frac{\frac12 \frac{\partial_{\bar z}\tilde g_4(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}\frac{\partial_z^3\Phi(\tilde x_k)}{\partial_z\Phi(\tilde x_k)} + \frac12 \frac{\partial_{\bar z}^2\tilde g_4(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}}}{\zeta - z} + \frac{\frac{\partial_z\tilde g_4(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}(1-\frac12\frac{\partial_z^3\Phi(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}(\zeta-\tilde z_k))}{(\zeta-z)^2}\right.\\
&- \left.\frac{\frac12\frac{\partial_z^2\tilde g_4(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}}{\zeta - z} + \frac{\frac12\frac{\partial_z^2\tilde g_4(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}(\zeta-\tilde z_k)}{(\zeta-z)^2}\right\} e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2 + o(\frac{1}{\tau^2}).
\end{aligned}
\tag{7.8}
$$

Observe that

$$
\int_{B(\tilde x_k,\delta)} \tilde e_k \frac{(\zeta-\tilde z_k)}{(\zeta-z)^2} e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2 = o(\frac{1}{\tau}) \quad \text{as } |\tau|\to+\infty.
$$

Finally we have

$$
\begin{aligned}
&- \frac{1}{\pi}\int_\Omega \frac{e_1\tilde g_4 e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}}{\zeta - z}d\xi_1 d\xi_2 =\\
&- \frac{1}{\pi\tau}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)} \tilde e_k \left\{ \frac{\frac12 \frac{\partial_{\bar z}\tilde g_4(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}\frac{\partial_z^3\Phi(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)} + \frac12 \frac{\partial_{\bar z}^2\tilde g_4(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}} - \frac12\frac{\partial_z^2\tilde g_4(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}}{\zeta - z}\right.\\
&+ \left.\frac{\frac{\partial_z\tilde g_4(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}}{(\zeta-z)^2}\right\} e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2 + o(\frac{1}{\tau^2}) =\\
&- \frac{1}{4\tau^2}\sum_{k=1}^{\ell} \frac{\frac{\partial_{\bar z}\tilde g_4(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}\frac{\partial_z^3\Phi(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)} + \frac{\partial_{\bar z}^2\tilde g_4(\tilde x_k)}{\overline{\partial_z^2\Phi(\tilde x_k)}} - \frac{\partial_z^2\tilde g_4(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}}{\tilde z_k - z}\frac{e^{2i\tau\psi(\tilde x_k)}}{|\det \operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}}\\
&- \frac{1}{2\tau^2}\sum_{k=1}^{\ell} \frac{\frac{\partial_z\tilde g_4(\tilde x_k)}{\partial_z^2\Phi(\tilde x_k)}}{(\tilde z_k-z)^2}\frac{e^{2i\tau\psi(\tilde x_k)}}{|\det \operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}} + o(\frac{1}{\tau^2}) \quad \text{as } |\tau|\to+\infty.
\end{aligned}
\tag{7.9}
$$

We compute the asymptotic of the following integral as $|\tau|$ goes to infinity:

$$
\begin{aligned}
&- \frac{1}{\pi}\int_\Omega \frac{e_1\tilde g_3 e^{-\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}}{\bar\zeta - \bar z}d\xi_1 d\xi_2 =\\
&- \frac{1}{\pi}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)} \frac{\tilde e_k\tilde g_3 e^{-\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}}{\bar\zeta - \bar z}d\xi_1 d\xi_2 + o(\frac{1}{\tau^2}) =\\
&- \frac{1}{\pi}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)} \tilde e_k \left\{ \frac{\partial_{\bar z}\tilde g_3(\tilde x_k)(\overline{\zeta}-\overline{\tilde z_k}) + \frac12\partial_{\bar z}^2\tilde g_3(\tilde x_k)(\bar\zeta-\overline{\tilde z_k})^2}{\bar\zeta-\bar z}\right.\\
&+ \left.\frac{\partial_{\bar z}\partial_z\tilde g_3(\tilde x_k)(\zeta-\tilde z_k)(\bar\zeta-\overline{\tilde z_k}) + \frac12\partial_z^2\tilde g_3(\tilde x_k)(\zeta-\tilde z_k)^2}{\bar\zeta-\bar z}\right\} e^{-\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2 + o(\frac{1}{\tau^2}) =
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{\pi}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)} \tilde e_k \Bigg\{ \frac{\frac{\partial_{\bar z}\tilde g_3(\tilde x_k)}{\overline{\partial^2_z\Phi(\tilde x_k)}}(\partial_{\bar\zeta}\overline{\Phi} - \frac12 \partial^3_{\bar z}\overline{\Phi}(\tilde x_k)(\overline{\zeta}-\overline{\tilde z_k})^2) + \frac12 \frac{\partial^2_{\bar z}\tilde g_3(\tilde x_k)}{\overline{\partial^2_z\Phi(\tilde x_k)}}\partial_{\bar\zeta}\overline{\Phi}(\overline{\zeta}-\overline{\tilde z_k})}{\overline{\zeta}-\bar z}\\
&+ \frac{\frac{\partial_{\bar z}\partial_z\tilde g_3(\tilde x_k)}{\partial^2_z\Phi(\tilde x_k)}(\bar\zeta-\overline{\tilde z_k})\partial_\zeta\Phi + \frac12\frac{\partial^2_z\tilde g_3(\tilde x_k)}{\partial^2_z\Phi(\tilde x_k)}\partial_\zeta\Phi(\zeta-\tilde z_k)}{\bar\zeta-\bar z}\Bigg\} e^{-\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2 + o(\frac{1}{\tau^2}) =\\
&- \frac{1}{\pi}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)} \tilde e_k \Bigg\{ \frac{\frac{\partial_{\bar z}\tilde g_3(\tilde x_k)}{\overline{\partial^2_z\Phi(\tilde x_k)}}(\partial_{\bar\zeta}\overline{\Phi} - \frac12 \frac{\partial^3_{\bar z}\overline{\Phi}(\tilde x_k)}{\partial^2_{\bar z}\overline{\Phi}(\tilde x_k)}\partial_{\bar\zeta}\overline{\Phi}(\bar\zeta-\overline{\tilde z_k})) + \frac12 \frac{\partial^2_{\bar z} g_3(\tilde x_k)}{\overline{\partial^2_z\Phi(\tilde x_k)}}\partial_{\bar\zeta}\Phi(\bar\zeta-\overline{\tilde z_k})}{\bar\zeta-\bar z}\\
&+ \frac{\frac{\partial_{\bar z}\partial_z\tilde g_3(\tilde x_k)}{\partial^2_z\Phi(\tilde x_k)}(\bar\zeta-\overline{\tilde z_k})\partial_\zeta\Phi + \frac12\frac{\partial^2_z\tilde g_3(\tilde x_k)}{\partial^2_z\Phi(\tilde x_k)}\partial_\zeta\Phi(\zeta-\tilde z_k)}{\bar\zeta-\bar z}\Bigg\} e^{-\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2 + o(\frac{1}{\tau^2}) =\\
&- \frac{1}{\pi\tau}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)} \tilde e_k \Bigg\{ \frac{\frac12\frac{\partial_{\bar z}\tilde g_3(\tilde x_k)}{\overline{\partial^2_z\Phi(\tilde x_k)}}\frac{\partial^3_{\bar z}\overline{\Phi}(\tilde x_k)}{\partial^2_{\bar z}\overline{\Phi}(\tilde x_k)} - \frac12\frac{\partial^2_{\bar z}\tilde g_3(\tilde x_k)}{\overline{\partial^2_z\Phi(\tilde x_k)}}}{\bar\zeta-\bar z} + \frac{\frac{\partial_{\bar z}\tilde g_3(\tilde x_k)}{\overline{\partial^2_z\Phi(\tilde x_k)}}(1-\frac12\frac{\partial^3_{\bar z}\overline{\Phi}(\tilde x_k)}{\partial^2_{\bar z}\overline{\Phi}(\tilde x_k)}(\bar\zeta-\overline{\tilde z_k}))}{(\bar\zeta-\bar z)^2}\\
&+ \frac{\frac12\frac{\partial^2_z\tilde g_3(\tilde x_k)}{\partial^2_z\Phi(\tilde x_k)}}{\bar\zeta-\bar z} + \frac12\frac{\frac{\partial^2_{\bar z} g_3(\tilde x_k)(\bar\zeta-\bar{\tilde z}_k)}{\partial^2_{\bar z}\overline{\Phi}(\tilde x_k)}}{(\bar\zeta-\bar z)^2}\Bigg\} e^{-\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2 + o(\frac{1}{\tau^2}).
\end{aligned}
\tag{7.10}
$$

Observe that by the stationary phase argument

$$
\int_{B(\tilde x_k,\delta)} \tilde e_k \frac{(\overline{\zeta}-\overline{\tilde z_k})}{(\bar\zeta-\overline{z})^2} e^{-\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2 = o(\frac{1}{\tau}) \quad \text{as } |\tau|\to+\infty.
$$

Finally we have

$$
\begin{aligned}
&- \frac{1}{\pi}\int_\Omega \frac{e_1\tilde g_3 e^{-\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}}{\bar\zeta-\bar z}d\xi_1 d\xi_2 =\\
&- \frac{1}{\pi\tau}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)} \tilde e_k \Bigg\{ \frac{\frac12\frac{\partial_{\bar z}\tilde g_3(\tilde x_k)}{\overline{\partial^2_z\Phi(\tilde x_k)}}\frac{\partial^3_{\bar z}\overline{\Phi}(\tilde x_k)}{\partial^2_{\bar z}\overline{\Phi}(\tilde x_k)} - \frac12\frac{\partial^2_{\bar z}\tilde g_3(\tilde x_k)}{\overline{\partial^2_z\Phi(\tilde x_k)}}}{\bar\zeta-\bar z} + \frac{\frac{\partial_{\bar z}\tilde g_3(\tilde x_k)}{\overline{\partial^2_z\Phi(\tilde x_k)}}}{(\bar\zeta-\bar z)^2}\\
&+ \frac{\frac12\frac{\partial^2_z\tilde g_3(\tilde x_k)}{\partial^2_z\Phi(\tilde x_k)}}{\bar\zeta-\bar z}\Bigg\} e^{-\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1 d\xi_2 + o(\frac{1}{\tau^2}) =\\
&- \frac{1}{4\tau^2}\sum_{k=1}^{\ell} \frac{\frac{\partial_{\bar z}\tilde g_3(\tilde x_k)}{\overline{\partial^2_z\Phi(\tilde x_k)}}\frac{\partial^3_{\bar z}\overline{\Phi}(\tilde x_k)}{\partial^2_{\bar z}\overline{\Phi}(\tilde x_k)} - \frac{\partial^2_{\bar z}\tilde g_3(\tilde x_k)}{\overline{\partial^2_z\Phi(\tilde x_k)}} + \frac{\partial^2_z\tilde g_3(\tilde x_k)}{\partial^2_z\Phi(\tilde x_k)}}{\overline{\tilde z_k}-\bar z}\frac{e^{-2i\tau\psi(\tilde x_k)}}{|\det \operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}}\\
&- \frac{1}{2\tau^2}\sum_{k=1}^{\ell} \frac{\frac{\partial_{\bar z}\tilde g_3(\tilde x_k)}{\overline{\partial^2_z\Phi(\tilde x_k)}}}{(\bar{\tilde z}_k-\bar z)^2}\frac{e^{-2i\tau\psi(\tilde x_k)}}{|\det \operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}} + o(\frac{1}{\tau^2}) \quad \text{as } |\tau|\to+\infty.
\end{aligned}
$$

□

Proposition 7.2. *For any x from the boundary of Ω, the following asymptotic formulae hold true as $|\tau|$ goes to $+\infty$:*

$$\mathfrak{G}_1(x,\tau) = -\frac{1}{\tau}\sum_{k=1}^{\ell}\frac{e^{-2i\tau\psi(\tilde x_k)}}{|det\, Im\, \Phi''(\tilde x_k)|^{\frac12}}\left(\frac{\frac18\frac{\partial(g_1e^{-\mathcal{A}_1})}{\partial z}(\tilde x_k)+\frac{\partial\Phi}{\partial z}m_{1,k}}{\tilde z_k - z}+\frac{q_{1,k}\frac{\partial\Phi}{\partial z}}{(\tilde z_k-z)^2}\right)+o(\frac{1}{\tau}), \tag{7.11}$$

$$\mathfrak{G}_2(x,\tau) = -\frac{1}{\tau}\sum_{k=1}^{\ell}\frac{e^{2i\tau\psi(\tilde x_k)}}{|det\, Im\, \Phi''(\tilde x_k)|^{\frac12}}\left(\frac{\frac18\frac{\partial(g_2e^{-\mathcal{B}_1})}{\partial \bar z}(\tilde x_k)+\frac{\partial\overline{\Phi}}{\partial \bar z}\tilde m_{1,k}}{\overline{\tilde z_k} - \bar z}+\frac{\tilde q_{1,k}\frac{\partial\overline{\Phi}}{\partial \bar z}}{(\overline{\tilde z_k}-\bar z)^2}\right)+o(\frac{1}{\tau}), \tag{7.12}$$

$$\mathfrak{G}_3(x,\tau) = \frac{1}{\tau}\sum_{k=1}^{\ell}\frac{e^{-2i\tau\psi(\tilde x_k)}}{|det\, Im\, \Phi''(\tilde x_k)|^{\frac12}}\left(\frac{\frac18\frac{\partial(g_4e^{-\mathcal{B}_2})}{\partial \bar z}(\tilde x_k)+\frac{\partial\overline{\Phi}}{\partial \bar z}t_{1,k}}{\overline{\tilde z_k} - \bar z}+\frac{r_{1,k}\frac{\partial\overline{\Phi}}{\partial \bar z}}{(\overline{\tilde z_k}-\bar z)^2}\right)+o(\frac{1}{\tau}), \tag{7.13}$$

$$\mathfrak{G}_4(x,\tau) = -\frac{1}{\tau}\sum_{k=1}^{\ell}\frac{e^{2i\tau\psi(\tilde x_k)}}{|det\, Im\, \Phi''(\tilde x_k)|^{\frac12}}\left(\frac{\frac18\frac{\partial(g_3e^{-\mathcal{A}_2})}{\partial z}(\tilde x_k)+\frac{\partial\Phi}{\partial z}\tilde t_{1,k}}{\tilde z_k - z}+\frac{\tilde r_{1,k}\frac{\partial\Phi}{\partial z}}{(\tilde z_k-z)^2}\right)+o(\frac{1}{\tau}). \tag{7.14}$$

Here $\tilde z_k = \tilde x_{k,1} + i\tilde x_{k,2}$. Moreover the following asymptotic formula holds true

$$\|\frac{\partial\mathfrak{G}_2(\cdot,\tau)}{\partial z}\|_{C(\overline{\Omega})}+\|\frac{\partial\mathfrak{G}_1(\cdot,\tau)}{\partial \bar z}\|_{C(\overline{\Omega})}+\|\frac{\partial\mathfrak{G}_3(\cdot,\tau)}{\partial \bar z}\|_{C(\overline{\Omega})}+\|\frac{\partial\mathfrak{G}_4(\cdot,\tau)}{\partial \bar z}\|_{C(\overline{\Omega})} = o(\frac{1}{\tau}) \quad as\ |\tau|\to+\infty. \tag{7.15}$$

Proof. Observe that the functions $\mathfrak{G}_1(x,\tau),\dots,\mathfrak{G}_4(x,\tau)$ are given by formulas

$$\mathfrak{G}_1(x,\tau) = -\frac{1}{2\pi}\int_\Omega \frac{(\frac{\partial\mathcal{A}_1}{\partial\zeta}(\zeta,\bar\zeta)+\tau\frac{\partial\Phi}{\partial\zeta}(\zeta))-(\frac{\partial\mathcal{A}_1}{\partial z}(z,\bar z)+\tau\frac{\partial\Phi}{\partial z}(z))}{\zeta-z}(e_1g_1e^{-\mathcal{A}_1})(\xi_1,\xi_2)e^{\tau(\overline{\Phi(\zeta)}-\Phi(\zeta))}d\xi_1d\xi_2,$$

$$\mathfrak{G}_2(x,\tau) = -\frac{1}{2\pi}\int_\Omega \frac{\frac{\partial\mathcal{B}_1}{\partial\bar\zeta}(\zeta,\bar\zeta)+\tau\frac{\partial\overline{\Phi}}{\partial\bar\zeta}(\bar\zeta)-(\frac{\partial\mathcal{B}_1}{\partial \bar z}(z,\bar z)+\tau\frac{\partial\overline{\Phi}}{\partial \bar z}(\bar z))}{\bar\zeta-\bar z}(e_1g_2e^{-\mathcal{B}_1})(\xi_1,\xi_2)e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1d\xi_2,$$

$$\mathfrak{G}_3(x,\tau) = \frac{1}{2\pi}\int_\Omega \frac{(\tau\frac{\partial\overline{\Phi}(\bar\zeta)}{\partial\bar\zeta}-\frac{\partial\mathcal{A}_2(\zeta,\bar\zeta)}{\partial\zeta})-(\tau\frac{\partial\overline{\Phi}(\bar z)}{\partial\bar z}-\frac{\partial\overline{\mathcal{A}_2}(z,\bar z)}{\partial\bar z})}{\bar\zeta-\bar z}(e_1g_4e^{-\mathcal{A}_2})(\xi_1,\xi_2)e^{\tau(\overline{\Phi(\zeta)}-\Phi(\zeta))}d\xi_1d\xi_2,$$

$$\mathfrak{G}_4(x,\tau) = \frac{1}{2\pi}\int_\Omega \frac{(\tau\frac{\partial\Phi(\zeta)}{\partial\zeta}-\frac{\partial\mathcal{B}_2(\zeta,\bar\zeta)}{\partial\zeta})-(\tau\frac{\partial\Phi(z)}{\partial z}-\frac{\partial\mathcal{B}_2(z,\bar z)}{\partial z})}{\zeta-z}(e_1g_3e^{-\mathcal{B}_2})(\xi_1.\xi_2)e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1d\xi_2.$$

These explicit formulae and the stationary phase argument imply (7.15). Let $z = x_1 + ix_2$ where $x = (x_1, x_2) \in \partial\Omega$. The following asymptotic holds true

$$\frac{\tau}{2\pi}\int_\Omega\frac{\partial_\zeta\Phi(\zeta)}{\zeta-z}(e_1g_1e^{-\mathcal{A}_1})e^{\tau(\overline{\Phi(\zeta)}-\Phi(\zeta))}d\xi_1d\xi_2 = -\frac{1}{2\pi}\int_\Omega\frac{1}{\zeta-z}(e_1g_1e^{-\mathcal{A}_1})\frac{\partial}{\partial\zeta}e^{\tau(\overline{\Phi(\zeta)}-\Phi(\zeta))}d\xi_1d\xi_2 =$$
$$\frac{1}{2\pi}\int_\Omega\frac{\partial}{\partial\zeta}\left(\frac{1}{\zeta-z}(e_1g_1e^{-\mathcal{A}_1})\right)e^{\tau(\overline{\Phi(\zeta)}-\Phi(\zeta))}d\xi_1d\xi_2 = \frac{1}{8\tau}\sum_{k=1}^{\ell}\frac{e^{-2i\tau\psi(\tilde x_k)}}{|\det\mathrm{Im}\,\Phi''(\tilde x_k)|^{\frac12}}\frac{\partial_z(g_1e^{-\mathcal{A}_1})(\tilde x_k)}{\tilde z_k-z}.$$

$$\frac{\tau}{2\pi}\int_\Omega \frac{\partial_{\bar\zeta}\overline{\Phi}}{\bar\zeta-\bar z}(e_1g_2e^{-\mathcal{B}_1})e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1d\xi_2=-\frac{1}{2\pi}\int_\Omega\frac{1}{\bar\zeta-\bar z}(e_1g_2e^{-\mathcal{B}_1})\frac{\partial}{\partial\bar\zeta}e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1d\xi_2=$$
$$\frac{1}{2\pi}\int_\Omega\frac{\partial}{\partial\bar\zeta}(\frac{1}{\bar\zeta-\bar z}(e_1g_2e^{-\mathcal{B}_1}))e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1d\xi_2=\frac{1}{8\tau}\sum_{k=1}^{\ell}\frac{e^{2i\tau\psi(\tilde x_k)}}{|\det\operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}}\frac{\partial_{\bar z}(g_2e^{-\mathcal{B}_1})(\tilde x_k)}{\overline{\tilde z_k}-\bar z}$$

$$\frac{\tau}{2\pi}\int_\Omega\frac{\partial_\zeta\Phi(\zeta)}{\zeta-z}(e_1g_3e^{-\mathcal{A}_2})e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1d\xi_2=\frac{1}{2\pi}\int_\Omega\frac{1}{\zeta-z}(e_1g_3e^{-\mathcal{A}_2})\partial_\zeta e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1d\xi_2=$$
$$-\frac{1}{2\pi}\int_\Omega\partial_\zeta\left(\frac{1}{\zeta-z}(e_1g_3e^{-\mathcal{A}_2})\right)e^{\tau(\Phi(\zeta)-\overline{\Phi(\zeta)})}d\xi_1d\xi_2=-\frac{1}{8\tau}\sum_{k=1}^{\ell}\frac{e^{2i\tau\psi(\tilde x_k)}}{|\det\operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}}\frac{\partial_z(g_3e^{-\mathcal{A}_2})(\tilde x_k)}{\tilde z_k-z}$$

and

$$\frac{\tau}{2\pi}\int_\Omega\frac{\partial_{\bar\zeta}\overline{\Phi}}{\bar\zeta-\bar z}(e_1g_4e^{-\mathcal{B}_2})e^{\tau(\overline{\Phi(\zeta)}-\Phi(\zeta))}d\xi_1d\xi_2=\frac{1}{2\pi}\int_\Omega\frac{1}{\bar\zeta-\bar z}(e_1g_4e^{-\mathcal{B}_2})\partial_{\bar\zeta}e^{\tau(\overline{\Phi(\zeta)}-\Phi(\zeta))}d\xi_1d\xi_2=$$
$$-\frac{1}{2\pi}\int_\Omega\partial_{\bar\zeta}\left(\frac{1}{\bar\zeta-\bar z}(e_1g_4e^{-\mathcal{B}_2})\right)e^{\tau(\overline{\Phi(\zeta)}-\Phi(\zeta))}d\xi_1d\xi_2=-\frac{1}{8\tau}\sum_{k=1}^{\ell}\frac{e^{-2i\tau\psi(\tilde x_k)}}{|\det\operatorname{Im}\Phi''(\tilde x_k)|^{\frac12}}\frac{\partial_{\bar z}(g_4e^{-\mathcal{B}_2})(\tilde x_k)}{\overline{\tilde z_k}-\bar z}.$$

Taking into account Proposition 7.1, we obtain asymptotic formulae (7.11)-(7.14) for the functions $\mathfrak{G}_1(x,\tau),...,\mathfrak{G}_4(x,\tau)$. ☐

Proof of Proposition 5.2. Using (5.14) and (5.16) we have

$$\begin{aligned}\mathfrak{L}_0\equiv\quad & (2(A_1-A_2)\frac{\partial U_1}{\partial z},b_\tau e^{\mathcal{B}_2-\tau\Phi}+c_\tau e^{\mathcal{A}_2-\tau\overline{\Phi}})_{L^2(\Omega)}\\ +\quad & (2(B_1-B_2)\frac{\partial U_1}{\partial\bar z},b_\tau e^{\mathcal{B}_2-\tau\Phi}+c_\tau e^{\mathcal{A}_2-\tau\overline{\Phi}})_{L^2(\Omega)}\\ =\quad & -2((A_1-A_2)e^{\tau\overline{\Phi}}\frac{\partial}{\partial z}\mathcal{R}_{-\tau,A_1}\{e_1g_1\},b_\tau e^{\mathcal{B}_2-\tau\Phi}+c_\tau e^{\mathcal{A}_2-\tau\overline{\Phi}})_{L^2(\Omega)}\\ +\quad & ((B_1-B_2)(-e_1g_1+A_1\mathcal{R}_{-\tau,A_1}\{e_1g_1\})e^{\tau\overline{\Phi}},b_\tau e^{\mathcal{B}_2-\tau\Phi}+c_\tau e^{\mathcal{A}_2-\tau\overline{\Phi}})_{L^2(\Omega)}\\ =\quad & -2((A_1-A_2)e^{\tau\overline{\Phi}}(\frac{\frac{\partial(e_1g_1)}{\partial z}}{2\tau\overline{\partial_z\Phi}}-e^{\mathcal{A}_1}e^{-\tau(\overline{\Phi}-\Phi)}\mathfrak{G}_1+o_{L^2(\Omega)}(\frac{1}{\tau})),b_\tau e^{\mathcal{B}_2-\tau\Phi}+c_\tau e^{\mathcal{A}_2-\tau\overline{\Phi}})_{L^2(\Omega)}\\ (7.16)\;+\quad & ((B_1-B_2)(-e_1g_1+A_1\mathcal{R}_{-\tau,A_1}\{e_1g_1\})e^{\tau\overline{\Phi}},b_\tau e^{\mathcal{B}_2-\tau\Phi}+c_\tau e^{\mathcal{A}_2-\tau\overline{\Phi}})_{L^2(\Omega)}.\end{aligned}$$

By Proposition 3.4

$$2((A_1-A_2)e^{\tau\overline{\Phi}}\frac{\frac{\partial(e_1g_1)}{\partial z}}{2\tau\overline{\partial_z\Phi}},c_\tau e^{\mathcal{A}_2-\tau\overline{\Phi}})_{L^2(\Omega)}=o(\frac{1}{\tau})\quad\text{as }|\tau|\to+\infty. \tag{7.17}$$

By (7.11) and Proposition 3.4

$$-2((A_1-A_2)e^{\tau\overline{\Phi}}e^{\mathcal{A}_1}e^{-\tau(\overline{\Phi}-\Phi)}\mathfrak{G}_1,b_\tau e^{\mathcal{B}_2-\tau\Phi})_{L^2(\Omega)}=o(\frac{1}{\tau})\quad\text{as }|\tau|\to+\infty. \tag{7.18}$$

Integrating by parts and using (7.15) and (3.2), we have

$$2((A_1-A_2)e^{\tau\overline{\Phi}}e^{\mathcal{A}_1}e^{-\tau(\overline{\Phi}-\Phi)}\mathfrak{G}_1, c_\tau e^{\mathcal{A}_2-\tau\overline{\Phi}})_{L^2(\Omega)} = 2((A_1-A_2)e^{\mathcal{A}_1}\mathfrak{G}_1, c_\tau e^{\mathcal{A}_2})_{L^2(\Omega)}$$
$$= \int_\Omega 2(A_1-A_2)e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}\overline{c_\tau(\overline{z})}\mathfrak{G}_1(x,\tau)dx = -4\int_\Omega \frac{\partial}{\partial\overline{z}}e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}\overline{c_\tau(\overline{z})}\mathfrak{G}_1(x,\tau)dx$$
$$= -2\int_{\partial\Omega}(\nu_1+i\nu_2)e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}\overline{c_\tau(\overline{z})}\mathfrak{G}_1(x,\tau)d\sigma + o(\frac{1}{\tau}) \quad \text{as } |\tau|\to+\infty. \tag{7.19}$$

By (7.11), (7.17)-(7.19) and Propositions 3.3 and 3.5, we obtain

$$\begin{aligned}
\mathfrak{L}_0 = \quad & (\quad (B_1-B_2)(-e_1g_1 + A_1\mathcal{R}_{-\tau,A_1}\{e_1g_1\}), b_\tau e^{\mathcal{B}_2})_{L^2(\Omega)} \\
& +2 \quad ((A_1-A_2)e^{\tau\overline{\Phi}}e^{\mathcal{A}_1}e^{-\tau(\overline{\Phi}-\Phi)}\mathfrak{G}_1, c_\tau e^{\mathcal{A}_2-\tau\overline{\Phi}})_{L^2(\Omega)} \\
& - \quad 2((A_1-A_2)(\frac{\frac{\partial(e_1g_1)}{\partial z}}{2\tau\overline{\partial_z\Phi}} - e^{\mathcal{A}_1}e^{-\tau(\overline{\Phi}-\Phi)}\mathfrak{G}_1), b_\tau e^{\mathcal{B}_2})_{L^2(\Omega)} + o(\frac{1}{\tau}) \\
& = \quad ((B_1-B_2)(-e_1g_1 + \frac{A_1e_1g_1}{2\tau\overline{\partial_z\Phi}}), b_\tau e^{\mathcal{B}_2})_{L^2(\Omega)} \\
& - \quad 2((A_1-A_2)\frac{\frac{\partial(e_1g_1)}{\partial z}}{2\tau\overline{\partial_z\Phi}}, b_\tau e^{\mathcal{B}_2})_{L^2(\Omega)} \\
& - \quad 2\int_{\partial\Omega}(\nu_1+i\nu_2)e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}\overline{c_\tau(\overline{z})}\mathfrak{G}_1(x,\tau)d\sigma + o(\frac{1}{\tau}) \quad \text{as } |\tau|\to+\infty.
\end{aligned} \tag{7.20}$$

Using (5.14) and (5.18) we obtain after simple computations

$$\begin{aligned}
\mathfrak{L}_1 \equiv \quad & (\quad 2(A_1-A_2)\frac{\partial U_2}{\partial z}, b_\tau e^{\mathcal{B}_2-\tau\Phi} + c_\tau e^{\mathcal{A}_2-\tau\overline{\Phi}})_{L^2(\Omega)} \\
& + \quad (2(B_1-B_2)\frac{\partial U_2}{\partial\overline{z}}, b_\tau e^{\mathcal{B}_2-\tau\Phi} + c_\tau e^{\mathcal{A}_2-\tau\overline{\Phi}})_{L^2(\Omega)} \\
& = \quad ((A_1-A_2)(-e_1g_2 + B_1\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1g_2\})e^{\tau\Phi}, b_\tau e^{\mathcal{B}_2-\tau\Phi} + c_\tau e^{\mathcal{A}_2-\tau\overline{\Phi}})_{L^2(\Omega)} \\
& - \quad 2((B_1-B_2)\frac{\partial}{\partial\overline{z}}\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1g_2\}e^{\tau\Phi}, b_\tau e^{\mathcal{B}_2-\tau\Phi} + c_\tau e^{\mathcal{A}_2-\tau\overline{\Phi}})_{L^2(\Omega)} \\
& = \quad ((A_1-A_2)(-e_1g_2 + B_1\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1g_2\})e^{\tau\Phi}, b_\tau e^{\mathcal{B}_2-\tau\Phi} + c_\tau e^{\mathcal{A}_2-\tau\overline{\Phi}})_{L^2(\Omega)} \\
& - \quad 2((B_1-B_2)(\frac{\frac{\partial(e_1g_2)}{\partial\overline{z}}}{2\tau\partial_z\Phi} - e^{\mathcal{B}_1}e^{\tau(\overline{\Phi}-\Phi)}\mathfrak{G}_2 + o_{L^2(\Omega)}(\frac{1}{\tau}))e^{\tau\Phi}, b_\tau e^{\mathcal{B}_2-\tau\Phi} + c_\tau e^{\mathcal{A}_2-\tau\overline{\Phi}})_{L^2(\Omega)}.
\end{aligned} \tag{7.21}$$

Then, by (7.15) we have the asymptotic formula

$$-2((B_1-B_2)(\frac{\frac{\partial(e_1g_2)}{\partial\overline{z}}}{2\tau\partial_z\Phi} - e^{\mathcal{B}_1}e^{\tau(\overline{\Phi}-\Phi)}\mathfrak{G}_2 + o_{L^2(\Omega)}(\frac{1}{\tau}))e^{\tau\Phi}, b_\tau e^{\mathcal{B}_2-\tau\Phi})_{L^2(\Omega)} \tag{7.22}$$
$$= 2((B_1-B_2)e^{\mathcal{B}_1}\mathfrak{G}_2, b_\tau e^{\mathcal{B}_2})_{L^2(\Omega)} + o(\frac{1}{\tau}) = 2\int_\Omega (B_1-B_2)e^{\mathcal{B}_1+\overline{\mathcal{B}}_2}\overline{b_\tau(z)}\mathfrak{G}_2(x,\tau)dx$$
$$= -4\int_\Omega \frac{\partial}{\partial z}e^{\mathcal{B}_1+\overline{\mathcal{B}}_2}\overline{b_\tau(z)}\mathfrak{G}_2(x,\tau)dx + o(\frac{1}{\tau}) = -2\int_{\partial\Omega}(\nu_1-i\nu_2)e^{\mathcal{B}_1+\overline{\mathcal{B}}_2}\overline{b_\tau(z)}\mathfrak{G}_2(x,\tau)d\sigma + o(\frac{1}{\tau}).$$

By (7.12) and Proposition 3.4 we obtain from (7.21):

$$-2((B_1-B_2)(\frac{\frac{\partial(e_1g_2)}{\partial\overline{z}}}{2\tau\partial_z\Phi}-e^{\mathcal{B}_1}e^{\tau(\overline{\Phi}-\Phi)}\mathfrak{G}_2)e^{\tau\Phi},c_\tau e^{\mathcal{A}_2-\tau\overline{\Phi}})_{L^2(\Omega)}$$
$$=-2((B_1-B_2)(\frac{\frac{\partial(e_1g_2)}{\partial\overline{z}}}{2\tau\partial_z\Phi}),c_\tau e^{\mathcal{A}_2-\tau\overline{\Phi}})_{L^2(\Omega)}+o(\frac{1}{\tau})+\frac{\chi}{\tau}. \tag{7.23}$$

Here χ is a constant which is independent of τ.

By (7.22) and (7.23) we have

$$\begin{aligned}\mathfrak{L}_1= &\ ((A_1-A_2)(-e_1g_2+B_1\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1g_2\}),c_\tau e^{\mathcal{A}_2})_{L^2(\Omega)}\\ &-2((B_1-B_2)(\frac{\frac{\partial(e_1g_2)}{\partial\overline{z}}}{2\tau\partial_z\Phi}-e^{\mathcal{B}_1}e^{\tau(\overline{\Phi}-\Phi)}\mathfrak{G}_2),c_\tau e^{\mathcal{A}_2})_{L^2(\Omega)}+o(\frac{1}{\tau})\\ &=((A_1-A_2)(-e_1g_2+\frac{B_1e_1g_2}{2\tau\partial_z\Phi}),c_\tau e^{\mathcal{A}_2})_{L^2(\Omega)}\\ &-2((B_1-B_2)(\frac{\partial\mathcal{B}_1}{\partial\overline{z}}\frac{e_1g_2}{2\tau\partial_z\Phi}+\frac{\frac{\partial(e_1g_2)}{\partial\overline{z}}}{2\tau\partial_z\Phi}),c_\tau e^{\mathcal{A}_2})_{L^2(\Omega)}\\ &-2\int_{\partial\Omega}(\nu_1-i\nu_2)e^{\mathcal{B}_1+\overline{\mathcal{B}}_2}\overline{b_\tau(z)}\mathfrak{G}_2(x,\tau)d\sigma+o(\frac{1}{\tau})\quad \text{as } |\tau|\to+\infty.\end{aligned} \tag{7.24}$$

Recall $V_1=-e^{-\tau\Phi}\widetilde{\mathcal{R}}_{-\tau,\overline{A_2}}\{e_1g_4\}$ and $V_2=-e^{-\tau\overline{\Phi}}\mathcal{R}_{\tau,-\overline{B_2}}\{e_1g_3\}$.

By Proposition 3.2 we conclude

$$2\frac{\partial V_1}{\partial z}=(-e_1g_4+\overline{A}_2\widetilde{\mathcal{R}}_{-\tau,\overline{A}_2}\{e_1g_4\})e^{-\tau\Phi} \tag{7.25}$$

and

$$2\frac{\partial V_2}{\partial\overline{z}}=(-e_1g_3-\overline{B_2}\mathcal{R}_{\tau,-\overline{B}_2}\{e_1g_3\})e^{-\tau\overline{\Phi}}. \tag{7.26}$$

Similarly to (5.15) and (5.17) we calculate $\frac{\partial V_1}{\partial\overline{z}}$ and $\frac{\partial V_2}{\partial z}$:

$$\frac{\partial V_1}{\partial\overline{z}}=-e^{-\tau\Phi}\widetilde{\mathcal{R}}_{-\tau,\overline{A}_2}\left\{\frac{\partial(e_1g_4)}{\partial\overline{z}}\right\}+e^{-\tau\overline{\Phi}+\mathcal{A}_2}\mathfrak{G}_3(\cdot,\tau) \tag{7.27}$$

and

$$\frac{\partial V_2}{\partial z}=-e^{-\tau\overline{\Phi}}\mathcal{R}_{\tau,-\overline{B}_2}\left\{\frac{\partial(e_1g_3)}{\partial z}\right\}+e^{-\tau\Phi+\mathcal{B}_2}\mathfrak{G}_4(\cdot,\tau). \tag{7.28}$$

Using (7.25) and (7.26) we obtain

$$\begin{aligned}\mathfrak{L}_2\equiv &\ (2(A_1-A_2)\frac{\partial}{\partial z}(a_\tau e^{\mathcal{A}_1+\tau\Phi}+d_\tau e^{\mathcal{B}_1+\tau\overline{\Phi}}),V_1+V_2)_{L^2(\Omega)}\\ &=-((A_1-A_2)d_\tau B_1e^{\mathcal{B}_1}e^{\tau\overline{\Phi}},V_1+V_2)_{L^2(\Omega)}\\ &+((A_1-A_2)(\nu_1-i\nu_2)a_\tau e^{\mathcal{A}_1+\tau\Phi},V_1+V_2)_{L^2(\partial\Omega)}\\ &-(2\frac{\partial}{\partial z}(A_1-A_2)a_\tau e^{\mathcal{A}_1+\tau\Phi},V_1+V_2)_{L^2(\Omega)}\\ &-((A_1-A_2)a_\tau e^{\mathcal{A}_1+\tau\Phi},2(\frac{\partial V_1}{\partial\overline{z}}+\frac{\partial V_2}{\partial\overline{z}}))_{L^2(\Omega)}.\end{aligned}$$

We observe that by (7.27), (7.15), Proposition 3.3 and Proposition 3.4

$$-((A_1-A_2)a_\tau e^{\mathcal{A}_1+\tau\Phi}, 2\frac{\partial V_1}{\partial \overline{z}})_{L^2(\Omega)} = -4\int_\Omega a_\tau(z)\frac{\partial}{\partial\overline{z}}e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}\overline{\mathfrak{G}_3(x,\tau)}dx + o(\frac{1}{\tau}) =$$

$$-2\int_{\partial\Omega} a_\tau(z)\overline{\mathfrak{G}_3(x,\tau)}(\nu_1+i\nu_2)e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}d\sigma + o(\frac{1}{\tau}) \quad \text{as } |\tau|\to+\infty. \tag{7.29}$$

Hence, using (7.26) we have

$$\begin{aligned}
\mathfrak{L}_2 = & \ ((A_1-A_2)d_\tau B_1 e^{\mathcal{B}_1}, \widetilde{\mathcal{R}}_{-\tau,\overline{A}_2}\{e_1 g_4\})_{L^2(\Omega)}\\
+ & \ ((A_1-A_2)(\nu_1-i\nu_2)a_\tau e^{\mathcal{A}_1+\tau\Phi}, V_1+V_2)_{L^2(\partial\Omega)}\\
+ & \ (2\frac{\partial}{\partial z}(A_1-A_2)a_\tau e^{\mathcal{A}_1}, \mathcal{R}_{\tau,-\overline{B}_2}\{e_1 g_3\})_{L^2(\Omega)}\\
+ & \ ((A_1-A_2)a_\tau e^{\mathcal{A}_1}, -e_1 g_3 - \overline{B_2}\mathcal{R}_{\tau,-\overline{B}_2}\{e_1 g_3\})_{L^2(\Omega)} + o(\frac{1}{\tau})\\
- & \ 2\int_{\partial\Omega} a_\tau(z)\overline{\mathfrak{G}_3(x,\tau)}(\nu_1+i\nu_2)e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}d\sigma.
\end{aligned}$$

By (4.34) and the stationary phase argument

$$((A_1-A_2)(\nu_1-i\nu_2)a_\tau e^{\mathcal{A}_1+\tau\Phi}, V_1+V_2)_{L^2(\partial\Omega)} = o(\frac{1}{\tau}) \quad \text{as } |\tau|\to+\infty.$$

Therefore, by Proposition 4.1

$$\begin{aligned}
\mathfrak{L}_2 = - & \ ((A_1-A_2)d_\tau B_1 e^{\mathcal{B}_1}, \frac{e_1 g_4}{\tau\partial_z\Phi})_{L^2(\Omega)}\\
+ & \ (\frac{\partial}{\partial z}(A_1-A_2)a_\tau e^{\mathcal{A}_1}, \frac{e_1 g_3}{\tau\overline{\partial_z\Phi}})_{L^2(\Omega)}\\
+ & \ ((A_1-A_2)a_\tau e^{\mathcal{A}_1}, -e_1 g_3 + \overline{B_2}\frac{e_1 g_3}{\tau\overline{\partial_z\Phi}})_{L^2(\Omega)}\\
- & \ 2\int_{\partial\Omega} a_\tau(z)\overline{\mathfrak{G}_3(x,\tau)}(\nu_1+i\nu_2)e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}d\sigma + o(\frac{1}{\tau}) \quad \text{as } |\tau|\to+\infty.
\end{aligned} \tag{7.30}$$

Integrating by parts we compute

$$\begin{aligned}
\mathfrak{L}_3 \equiv & \ (2(B_1-B_2)\frac{\partial}{\partial\overline{z}}(a_\tau e^{\mathcal{A}_1+\tau\Phi} + d_\tau e^{\mathcal{B}_1+\tau\overline{\Phi}}), V_1+V_2)_{L^2(\Omega)} =\\
- & \ (2\frac{\partial}{\partial\overline{z}}(B_1-B_2)d_\tau e^{\mathcal{B}_1+\tau\overline{\Phi}}, V_1+V_2)_{L^2(\Omega)}\\
- & \ ((B_1-B_2)A_1 a_\tau e^{\mathcal{A}_1+\tau\Phi}, V_1+V_2)_{L^2(\Omega)}\\
+ & \ ((\nu_1+i\nu_2)(B_1-B_2)d_\tau e^{\mathcal{B}_1+\tau\Phi}, V_1+V_2)_{L^2(\partial\Omega)}\\
- & \ ((B_1-B_2)d_\tau e^{\mathcal{B}_1+\tau\overline{\Phi}}, 2(\frac{\partial V_1}{\partial z}+\frac{\partial V_2}{\partial z}))_{L^2(\Omega)}.
\end{aligned}$$

We observe that by (7.15), (7.28), Proposition 3.3 and Proposition 3.4:

$$-2((B_1-B_2)d_\tau e^{\mathcal{B}_1+\tau\overline{\Phi}}, \frac{\partial V_2}{\partial z})_{L^2(\Omega)} = -4\int_\Omega \frac{\partial}{\partial z}e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}d_\tau(\overline{z})\overline{\mathfrak{G}_4(x,\tau)}dx =$$

$$-2\int_{\partial\Omega}(\nu_1-i\nu_2)e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}d_\tau(\overline{z})\overline{\mathfrak{G}_4(x,\tau)}d\sigma + o(\frac{1}{\tau}) \quad \text{as } |\tau|\to+\infty. \tag{7.31}$$

Hence

$$\begin{aligned}
\mathfrak{L}_3 &= (2\frac{\partial}{\partial \overline{z}}(B_1-B_2)d_\tau e^{\mathcal{B}_1}, \widetilde{\mathcal{R}}_{\tau,-\overline{A}_2}\{e_1g_4\})_{L^2(\Omega)}\\
&- ((B_1-B_2)A_1a_\tau e^{\mathcal{A}_1}, \mathcal{R}_{\tau,-\overline{B}_2}\{e_1g_3\})_{L^2(\Omega)}\\
&+ ((B_1-B_2)d_\tau e^{\mathcal{B}_1}, -e_1g_4+\overline{A}_2\widetilde{\mathcal{R}}_{-\tau,\overline{A}_2}\{e_1g_4\})_{L^2(\Omega)}\\
&+ ((\nu_1+i\nu_2)(B_1-B_2)d_\tau e^{\mathcal{B}_1+\tau\Phi}, V_1+V_2)_{L^2(\partial\Omega)}\\
&+ 2\int_{\partial\Omega}(\nu_1-i\nu_2)e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}d_\tau(\overline{z})\overline{\mathfrak{G}_4(x,\tau)}d\sigma + o(\frac{1}{\tau}) \quad \text{as } |\tau|\to+\infty.
\end{aligned} \tag{7.32}$$

By (4.34) and the stationary phase argument

$$((\nu_1+i\nu_2)(B_1-B_2)d_\tau e^{\mathcal{B}_1+\tau\Phi}, V_1+V_2)_{L^2(\partial\Omega)} = o(\frac{1}{\tau}) \quad \text{as } |\tau|\to+\infty.$$

Therefore, applying Proposition 3.3 we have

$$\begin{aligned}
\mathfrak{L}_3 = &- 2(\frac{\partial}{\partial \overline{z}}(B_1-B_2)d_\tau e^{\mathcal{B}_1}, \frac{e_1g_4}{\tau\partial_z\Phi})_{L^2(\Omega)}\\
&- ((B_1-B_2)A_1a_\tau e^{\mathcal{A}_1}, \frac{e_1g_3}{2\tau\overline{\partial_z\Phi}})_{L^2(\Omega)}\\
&+ ((B_1-B_2)d_\tau e^{\mathcal{B}_1}, -e_1g_4-\frac{\overline{A}_2e_1g_4}{2\tau\partial_z\Phi})_{L^2(\Omega)}\\
&+ 2\int_{\partial\Omega}(\nu_1-i\nu_2)e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}d_\tau(\overline{z})\overline{\mathfrak{G}_4(x,\tau)}d\sigma + o(\frac{1}{\tau}) \quad \text{as } |\tau|\to+\infty.
\end{aligned} \tag{7.33}$$

The sum $\sum_{k=0}^3 \mathfrak{L}_k$ equal to the left hand side of (5.19). By (7.20), (7.24), (7.30) and (7.33), there exist numbers κ, κ_0 such that the asymptotic formula (5.19) holds true. □

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