UTMS 2009–11 June 18, 2009

The growth of the Nevanlinna proximity function

by

Atsushi Nitanda

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

The Growth of the Nevanlinna Proximity Function

Atsushi Nitanda

Abstract

Let f be a meromorphic mapping from $\mathbf{C}^n$ into a compact complex manifold M. In this paper we give some estimates of the growth of the proximity function $m_f(r, D)$ of f with respect to a divisor D. J.E. Littlewood [2] (cf. Hayman [1]) proved that a merormorphic function g on the complex plane $\mathbf{C}$ satisfies $\limsup_{r\to\infty} \frac{m_g(r,a)}{\log T(r,g)} \leq \frac{1}{2}$ for almost all point a of the Riemann sphere. We extend this result to the case of a meromrophic mapping $f : \mathbf{C}^n \to M$ and a linear system $P(E)$ on M. The main reuslt is an estimate of the following type: For almost all divisor $D \in P(E)$, $\limsup_{r\to\infty} \frac{m_f(r,D)-m_f(r,\mathcal{I}_{B(E)})}{\log T_{f_E}(r,H_E)} \leq \frac{1}{2}$.

1 Introduction.

J.E. Littlewood [2] (cf. [1]) proved that every non-constant meromorphic function g on $\mathbf{C}$ satisfies

$$\limsup_{r\to\infty} \frac{m_g(r,a)}{\log T(r,g)} \leq \frac{1}{2}$$

for almost all $a \in \mathbf{C}$, where $T(r, g)$ denotes the Nevanlinna characteristic function of g. Our main aim is to generalize this result to the case of several complex variables. Cf. A. Sadullaev [8], A. Sadullaev and P.V. Degtjar' [9], and S. Mori [2] for related results (see *Remark* at the end of §6).

Let $L \to M$ be a holomorphic line bundle over a compact complex manifold M. Let $\Gamma(M, L)$ be the vector space of all holomorphic sections of L over M, and $E \subset \Gamma(M, L)$ a vector subspace of dimension at least 2. Then we have a natural meromorphic mapping

$$\rho_E : M \to P(E^*),$$

where $P(E^*)$ is the projective space of the dual E^* of E. Let H_E be the hyperplane bundle over $P(E^*)$ and $B(E) \subset M$ the base of E. Let $f : \mathbf{C}^n \to M$ be a meromorphic mapping such that $f(\mathbf{C}^n) \not\subset B(E)$. Then we have the composite meromrophic mapping $f_E = \rho_E \circ f : \mathbf{C}^n \to P(E^*)$.

Our main result is as follows (cf. section 2 for more notation):

Main Theorem . *Let $f_E = \rho_E \circ f : \mathbf{C}^n \to P(E^*)$ be as above. If $T_{f_E}(r, H_E) \to \infty$ $(r \to \infty)$, then*

$$\limsup_{r\to\infty} \frac{m_f(r,D) - m_f(r, \mathcal{I}_{B(E)})}{\log T_{f_E}(r, H_E)} \leq \frac{1}{2}$$

for almost all divisor $D \in P(E)$.

In section 4 we first prove the Main Theorem in the case where $E = \Gamma(M, L)$ and $B(E) = \phi$. In section 5 we show an estimate of different type, In section 6 we deal with the general case.

Acknowledgement. The author would like to express his sincere gratitude to Professor Junjiro Noguchi for his valuable advice and encouragement.

2 Notation.

Let $z = (z^1, \ldots, z^n)$ be the natural coordinate system of $\mathbf{C}^n$. We set

$$\|z\|^2 = \sum_{j=1}^{n} |z^j|^2, \quad d^c = \frac{i}{4\pi}\left(\overline{\partial} - \partial\right),$$

$$\alpha = dd^c\|z\|^2, \quad \eta = d^c \log\|z\|^2 \wedge (dd^c \log\|z\|^2)^{n-1},$$
$$B(r) = \{z \in \mathbf{C}^n;\ \|z\| < r\}, \quad \Gamma(r) = \{z \in \mathbf{C}^n;\ \|z\| = r\}.$$

Let M be a compact complex manifold and (L, h) a Hermitian holomorphic line bundle over M. For a meromorphic mapping $f : \mathbf{C}^n \to M$ we define the order function of f with respect to the Chern form ω of (L, h) by

$$T_f(r, \omega) = \int_1^r \frac{dt}{t^{2n-1}} \int_{B(t)} f^*\omega \wedge \alpha^{n-1}$$

and we define the order function of f with respect to L by

$$T_f(r, L) = T_f(r, \omega).$$

$T_f(r, L)$ is well-defined up to a bounded term. We denote the space of holomorphic sections of L by $\Gamma(M, L)$. We have the natural identification

$$P(\Gamma(M, L)) = \{(\sigma);\ \sigma \in \Gamma(M, L) \setminus \{0\}\},$$

where the notation (σ) stands for the effective divisor of σ. Let $D \in P(\Gamma(M, L))$. Then we may take an element $\sigma \in \Gamma(M, L)$ which satisfies

$$D = (\sigma), \quad \|\sigma(x)\| = \sqrt{h(\sigma(x), \sigma(x))} \leq 1.$$

When $f(\mathbf{C}^n) \not\subset \text{supp } D$ (the support of D), the proximity function of f with respect to D is defined by

$$m_f(r, D) = \int_{z\in\Gamma(r)} \log \frac{1}{\|\sigma \circ f(z)\|}\eta(z)$$

and we define the counting function of f^*D by

$$N(r, f^*D) = \int_1^r \frac{dt}{t^{2n-1}} \int_{B(t)\cap f^*D} \alpha^{n-1},$$

where f^*D is the pullback of D by f. If L is non-negative, then we have the First Main Theorem

$$T_f(r, L) = N(r, f^*D) + m_f(r, D) + O(1). \tag{1}$$

3 Lemma.

Let M be a compact complex manifold and $L \to M$ a holomolphic line bundle. Set

$$V = \Gamma(M, L), \quad N + 1 = \dim\ M.$$

Here we assume that the set $B(V)$ of base points of V is empty, i.e.,

$$B(V) = \{x \in M;\ \sigma(x) = 0, \forall \sigma \in V\} = \phi.$$

We fix a Hermitian inner product $(\ ,\ )$ in V. Let $(\{U_\lambda\}, \{s_\lambda\})$ be a local trivialization covering of L and $\{\sigma_0, \dots, \sigma_N\}$ a orthonormal base of V. We identify $V^* = \mathbf{C}^{N+1}$ by the dual base of $\{\sigma_0, \dots, \sigma_N\}$. We define a holomorphic mapping Φ_L from M into $P(V^*) = \mathbf{P}^N(\mathbf{C})$ by

$$\Phi_L(x) = [\sigma_{0\lambda}(x) : \dots : \sigma_{N\lambda}(x)],\ x \in U_\lambda,$$

where $\sigma_{j\lambda}$ are holomorphic functions on U_λ with $\sigma_j|U_\lambda = \sigma_{j\lambda}s_\lambda$. Then it follows that $L = \Phi_L^* H_{V^*}$, where H_{V^*} is the hyperplane bundle over $P(V^*)$. Hence Fubini-Study metric in H_{V^*} induces a Hermitian metric h in L satisfying

$$h(s_\lambda(x), s_\lambda(x)) = \frac{1}{\sum_{j=0}^N |\sigma_{j\lambda}(x)|^2}. \tag{2}$$

We denote the Chern form of (L, h) by ω. Clearly, ω is non-negative. Hence L is non-negative. Let ω_V denote the Fubini-Study metric form on $P(V)$ induced by the Hermitian inner product $(\ ,\ )$. Since $\omega_V^N = \wedge^N \omega_V$ is a volume element on $P(V)$, it is considered as positive measure μ. We define a C^∞-function S_x on $P(V)$ by

$$S_x(D) = \frac{\sqrt{h(\sigma(x), \sigma(x))}}{\sqrt{(\sigma, \sigma)}}, \quad D = (\sigma) \in P(V).$$

We now prove the following key lemma.

Lemma 1. *Let the notation be as above and $X \subset P(V)$ a Lebesgue measurable subset with $\mu(X) > 0$. Then,*

$$\int_{D \in X} \log \frac{1}{S_x(D)} d\mu(D) \le \frac{\mu(X)}{2} \left(N + \log \frac{N}{\mu(X)} \right)$$

for all $x \in M$.

Proof. We identify $P(V) = \mathbf{P}^N(\mathbf{C})$ by the base $\{\sigma_0, \dots, \sigma_N\}$. For $x \in U_\lambda$ and $[z^0 : \dots : z^N] \in \mathbf{P}^N(\mathbf{C})$ it follows from (2) that

$$S_x([z^0 : \dots : z^N]) = \frac{\left| \sum_{j=0}^N z^j \sigma_{j\lambda}(x) \right|}{\left(\sum_{j=0}^N |\sigma_{j\lambda}(x)|^2 \right)^{1/2} \left(\sum_{j=0}^N |z^j|^2 \right)^{1/2}}. \tag{3}$$

Since $B(V) = \phi$, there exists a unitary matrix $G = (g_{ij})$ and a non-zero constant $a \in \mathbf{C}$ such that

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = a\ {}^tG \begin{pmatrix} \sigma_{0\lambda}(x) \\ \vdots \\ \sigma_{N\lambda}(x) \end{pmatrix}.$$

Let $\rho : \mathbf{C}^{N+1} \setminus \{0\} \to \mathbf{P}^N(\mathbf{C})$ be the Hopf fibering. We define a biholomorphic mapping G by $G(\rho(z)) = \rho(Gz)$, $z = {}^t(z^0, \dots, z^N) \in \mathbf{C}^{N+1}$. Since G is unitary, we easily see by (3) that

$$S_x(G([z^0 : \dots : z^N])) = \frac{|z^0|}{\left(\sum_{k=0}^N |z^k|^2 \right)^{1/2}}. \tag{4}$$

We denote the characteristic function of a subset $S \subset P(V)$ by χ_S. Since ω_V is unitary invariant, it follows from (4) that

$$\begin{aligned} &\int_{\rho(w) \in X} \log \frac{1}{S_x(\rho(w))} \omega_V^N \\ &= \int_{\rho(w) \in \mathbf{P}^N(\mathbf{C})} \chi_X(\rho(w)) \log \frac{1}{S_x(\rho(w))} \omega_V^N \\ &= \int_{\rho(z) \in \mathbf{P}^N(\mathbf{C})} G^* \left(\chi_X(\rho(w)) \log \frac{1}{S_x(\rho(w))} \omega_V^N \right) \\ &= \int_{\rho(z) \in \mathbf{P}^N(\mathbf{C})} \chi_{G^{-1}(X)}(\rho(z)) \log \frac{1}{S_x(G(\rho(z)))} \omega_V^N \end{aligned} \tag{5}$$

$$= \int_{\rho(z)\in G^{-1}(X)} \log \frac{\left(\sum_{k=0}^{N} |z^k|^2\right)^{1/2}}{|z^0|} \omega_V^N.$$

We put

$$V_0 = \{[z^0 : \ldots : z^N] \in \mathbf{P}^N(\mathbf{C});\ z^0 \neq 0\}$$

and we set an affine coordinate system on V_0 by

$$\zeta = (\zeta^1, \ldots, \zeta^N) = \left(\frac{z^1}{z^0}, \ldots, \frac{z^N}{z^0}\right).$$

Then by (5) we have

$$\int_{\rho(w)\in X} \log \frac{1}{S_x(\rho(w))} \omega_V^N$$

$$= \int_{\zeta\in\mathbf{C}^N} \frac{\chi_{G^{-1}(X)} N! \log\left(1 + \|\zeta\|^2\right)^{1/2}}{\left(1+\|\zeta\|^2\right)^{N+1}} \bigwedge_{k=1}^{N} \left(\frac{i}{2\pi} d\zeta^k \wedge d\overline{\zeta^k}\right)$$

$$= \int_{\zeta\in\mathbf{C}^N} \frac{\chi_{G^{-1}(X)} \log\left(1 + \|\zeta\|^2\right)^{1/2}}{\left(1+\|\zeta\|^2\right)^{N+1}} \alpha^N.$$

Furthermore, $\mu(X) = \mu(G^{-1}(X))$, so that it suffices to prove that

$$\int_{\zeta\in\mathbf{C}^N} \frac{\chi_X \log\left(1 + \|\zeta\|^2\right)^{1/2}}{\left(1+\|\zeta\|^2\right)^{N+1}} \alpha^N \leq \frac{\mu(X)}{2}\left(N + \log\frac{N}{\mu(X)}\right) \tag{6}$$

for a Lebesgue measurable set $X \subset \mathbf{C}^N$. Set

$$\Phi(r) = \int_{X\cap\{\zeta\in\mathbf{C}^N;\ \|\zeta\|>r\}} \omega_V^N.$$

Then, $\Phi(r)$ is a continuous decreasing function on $[0,\infty)$ and $0 \leq \Phi(r) \leq \mu(X) \leq 1$. Moreover,

$$\Phi(r) = \int_{\{\zeta\in\mathbf{C}^N;\ \|\zeta\|>r\}} \frac{\chi_X}{\left(1+\|\zeta\|^2\right)^{N+1}} \alpha^N \tag{7}$$

$$= \int_r^\infty \left\{\int_{\Gamma(t)} \frac{\chi_X 2N t^{2N-1}}{(1+t^2)^{N+1}} \eta\right\} dt,$$

so that $\Phi(r)$ is an absolutely continuous function on $[0,s]$ ($s \in [0,\infty)$).

Therefore it follows that

$$\int_0^s \log(1+r^2)^{1/2} d(-\Phi(r)) \tag{8}$$

$$= \int_0^s \log(1+r^2)^{1/2} \left\{ \int_{\Gamma(r)} \frac{\chi_X 2N r^{2N-1}}{(1+r^2)^{N+1}} \eta \right\} dr$$

$$= \int_{\zeta \in B(s)} \frac{\chi_X \log(1+\|\zeta\|^2)^{1/2}}{(1+\|\zeta\|^2)^{N+1}} \alpha^N.$$

On the other hand, we have

$$\int_0^s \log(1+r^2)^{1/2} d(-\Phi(r)) = \int_0^s \frac{r\Phi(r)}{1+r^2} dr - \Phi(s)\log(1+s^2)^{1/2}. \tag{9}$$

The following convergence will be proved later:

$$\Phi(s)\log(1+s^2)^{1/2} \to 0 \quad (s \to \infty). \tag{10}$$

Hence by (8), (9), (10) the left side of (6) is

$$\int_{\zeta \in \mathbf{C}^N} \frac{\chi_X \log(1+\|\zeta\|^2)^{1/2}}{(1+\|\zeta\|^2)^{N+1}} \alpha^N = \int_0^\infty \frac{r\Phi(r)}{1+r^2} dr. \tag{11}$$

To estimate (11), we put

$$\Psi(r) = \int_{\{\zeta \in \mathbf{C}^N;\ \|\zeta\| > r\}} \omega_V^N.$$

Then, $\Psi(r)$ is a strictly decreasing and continuous function on $[0, \infty)$ such that $0 \le \Phi(r) \le \Psi(r) \le 1$, $\Psi(0) = 1$, and $\lim_{r\to\infty} \Psi(r) = 0$.

We compute $\Psi(r)$ as follows.

$$\Psi(r) = \int_{\{\zeta \in \mathbf{C}^N;\ \|\zeta\| > r\}} \frac{1}{(1+\|\zeta\|^2)^{N+1}} \alpha^N$$

$$= \int_r^\infty \left\{ \int_{\Gamma(t)} \frac{2N t^{2N-1}}{(1+t^2)^{N+1}} \eta \right\} dt$$

$$= \int_r^\infty \frac{2N t^{2N-1}}{(1+t^2)^{N+1}} dt$$

$$= \sum_{j=1}^N \frac{r^{2(j-1)}}{(1+r^2)^j}.$$

Therefore we have

$$\frac{1}{1+r^2} \le \Psi(r) \le \frac{N}{1+r^2}. \tag{12}$$

We show (10) as follows.

$$0 \le \Phi(s)\log(1+s^2)^{1/2} \le \Psi(s)\log(1+s^2)^{1/2}$$
$$\le \frac{N}{1+s^2}\log(1+s^2)^{1/2} \to 0 \quad (s \to \infty).$$

Because of $\mu(X) > 0$ we can take a real number $r_1 \ge 0$ such that $\Psi(r_1) = \mu(X)$. By (12)

$$\frac{1}{\mu(X)} \le 1 + r_1^2 \le \frac{N}{\mu(X)}. \tag{13}$$

Note that $\Phi(0) = \mu(X)$, $\Phi(r)$ is decreasing, and that $\Phi(r) \le \min\{\Psi(r),\ \mu(X)\}$. Therefore, we get

$$\int_0^\infty \frac{r\Phi(r)}{1+r^2}dr \le \int_0^{r_1} \frac{r\mu(X)}{1+r^2}dr + \int_{r_1}^\infty \frac{r\Psi(r)}{1+r^2}dr$$
$$= \frac{\mu(X)}{2}\log(1+r_1^2) + \int_{r_1}^\infty \frac{r\Psi(r)}{1+r^2}dr.$$

Furthermore by (12) and (13) we see that

$$\int_0^\infty \frac{r\Phi(r)}{1+r^2}dr \le \frac{\mu(X)}{2}\log\frac{N}{\mu(X)} + \int_{r_1}^\infty \frac{rN}{(1+r^2)^2}dr$$
$$= \frac{\mu(X)}{2}\log\frac{N}{\mu(X)} + \frac{N}{2(1+r_1^2)} \le \frac{\mu(X)}{2}\left(N + \log\frac{N}{\mu(X)}\right).$$

Therefore, (6) follows from (11). □

4 Growth of the Nevanlinna proximity function 1.

We show the following theorem.

Theorem 2. *Let M be a compact complex manifold and $L \to M$ a holomorphic line bundle satisfying $B(\Gamma(M,L)) = \phi$. Let $f : \mathbf{C}^n \to M$ be a meromorphic mapping such that $T_f(r,L) \to \infty$ $(r \to \infty)$. Then we have that for almost all divisor $D \in P(\Gamma(M,L))$*

$$\limsup_{r\to\infty} \frac{m_f(r,D)}{\log T_f(r,L)} \le \frac{1}{2}.$$

Proof. Set $V = \Gamma(M,L)$. Let ω, ω_V and S_x be as in the section 3. Then

$$T_f(r,\omega) = T_f(r,L) + O(1).$$

Since $T_f(r, L) \to \infty \ \ (r \to \infty)$, for all positive integer $m \in \mathbf{N}$ we can choose real number $r_m \in (1, \infty)$ such that

$$T_f(r_m, \omega) = m.$$

Let $\beta > 1/2$ be an arbitrary real number and set

$$G(m, \beta) = \{D \in P(V);\ m_f(r_m, D) > \beta \log m\}.$$

We denote by $I(f)$ the indeterminacy locus of f. Because the codimension of $I(f)$ is greater than or equal to 2, it follows from lemma 1 that if $\mu(G(m, \beta)) > 0$, then

$$\begin{aligned}
\mu(G(m,\beta))\beta \log m &< \int_{D\in G(m,\beta)} m_f(r_m, D)\omega_V^N \\
&= \int_{D\in G(m,\beta)} \left\{ \int_{z\in\Gamma(r_m)\setminus I(f)} \log \frac{1}{S_{f(z)}(D)} \eta(z) \right\} \omega_V^N \\
&= \int_{z\in\Gamma(r_m)\setminus I(f)} \left\{ \int_{D\in G(m,\beta)} \log \frac{1}{S_{f(z)}(D)} \omega_V^N \right\} \eta(z) \\
&\leq \int_{z\in\Gamma(r_m)\setminus I(f)} \frac{\mu(G(m,\beta))}{2} \left(N + \log \frac{N}{\mu(G(m,\beta))} \right) \eta(z) \\
&= \frac{\mu(G(m,\beta))}{2} \left(N + \log \frac{N}{\mu(G(m,\beta))} \right).
\end{aligned}$$

Hence we deduce that

$$\mu(G(m, \beta)) < \frac{Ne^N}{m^{2\beta}}.$$

We set

$$G(\beta) = \bigcap_{m_0=1}^{\infty} \bigcup_{m=m_0}^{\infty} G(m, \beta).$$

Because of $\beta > 1/2$ it follows that

$$\mu(G(\beta)) \leq \lim_{m_0\to\infty} \sum_{m=m_0}^{\infty} \mu(G(m,\beta)) < \lim_{m_0\to\infty} \sum_{m=m_0}^{\infty} \frac{Ne^N}{m^{2\beta}} = 0. \tag{14}$$

Note that the set $X(f)$ defined by

$$X(f) = \{D \in P(V);\ \mathrm{supp}\ D \supset f(\mathbf{C}^n)\}$$

has zero measure. Let $D \notin G(\beta) \cup X(f)$. Then there exists an integer $m_D \in \mathbf{N}$ such that for all $m > m_D$

$$m_f(r_m, D) \leq \beta \log m. \tag{15}$$

We choose an arbitrary number $s \geq r_{m_D}$ and we take an integer $m_s \in \mathbf{N}$ satisfying $r_{m_s} \leq s < r_{m_s+1}$. Then $m_s \geq m_D$. Since $\omega \geq 0$ and $D \notin X(f)$, we have by the First Main Theorem (1) and (15)

$$\begin{aligned} m_f(s, D) &= T_f(s, \omega) - N(s, f^*D) + O(1) \\ &\leq T_f(r_{m_s+1}, \omega) - N(r_{m_s}, f^*D) + O(1) \\ &= T_f(r_{m_s}, \omega) - N(r_{m_s}, f^*D) + O(1) \\ &= m_f(r_{m_s}, D) + O(1) \leq \beta \log m_s + O(1) \\ &\leq \beta \log T_f(s, \omega) + O(1). \end{aligned}$$

Therefore it follows that for an arbitrary $D \notin G(\beta) \cup X(f)$

$$\limsup_{r\to\infty} \frac{m_f(r, D)}{\log T_f(r, \omega)} \leq \beta. \tag{16}$$

We set

$$G = \bigcup_{k=1}^{\infty} G\left(\frac{1}{2} + \frac{1}{k}\right) \cup X(f).$$

Then by (14), (16) we see that

$$\mu(G) \leq \sum_{k=1}^{\infty} \mu\left(G\left(\frac{1}{2} + \frac{1}{k}\right)\right) + \mu(X(f)) = 0$$

and that for $D \notin G$

$$\limsup_{r\to+\infty} \frac{m_f(r, D)}{\log T_f(r, \omega)} \leq \frac{1}{2}.$$

□

In general, let M be a compact complex manifold with a Hermitian metric form ω. Let $f : \mathbf{C}^n \to M$ be a meromorhic mapping. Then the order function of f with respect to ω is defined by

$$T_f(r, \omega) = \int_1^r \frac{dt}{t^{2n-1}} \int_{B(t)} f^*\omega \wedge \alpha^{n-1}.$$

We define the order of f by

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r, \omega)}{\log r},$$

which is independent of the choice of the Hermitian metric form ω.

We easily deduce the following corollary from Theorem 2.

Corollary 3. *Let M be a compact complex manifold and L a very ample holomorphic line bundle over M. Let $f : \mathbf{C}^n \to M$ be a meromorphic mapping. Assume that the order of f is finite and $T_f(r, L) \to \infty \quad (r \to \infty)$. Then,*

$$\limsup_{r\to\infty} \frac{m_f(r, D)}{\log r} \leq \frac{\rho_f}{2}$$

for almost all effective divisor $D \in P(\Gamma(M, L))$.

5 Growth of the Nevanlinna proximity function 2.

We now define the projective logarithmic capacity of a subset in the $\mathbf{P}^N(\mathbf{C})$ (See Molzon-Shiffman-Sibony [3]). Let K be a compact subset of $\mathbf{P}^N(\mathbf{C})$. We denote by $\mathcal{M}(K)$ the space of positive Borel measures on K with total mass 1. For $x = [x^0 : \ldots : x^N] \in \mathbf{P}^N(\mathbf{C})$ and $\nu \in \mathcal{M}(K)$ we set

$$u_\nu(x) = \int_{[w^0:\ldots:w^N]\in K} \log \frac{\left(\sum_{j=0}^N |x^j|^2\right)^{1/2} \left(\sum_{j=0}^N |w^j|^2\right)^{1/2}}{\left|\sum_{j=0}^N x^j w^j\right|} d\nu,$$

and

$$V(K) = \inf_{\nu\in\mathcal{M}(K)} \sup_{x\in\mathbf{P}^N(\mathbf{C})} u_\nu(x).$$

Define the projective logarithmic capacity of K by

$$C(K) = \frac{1}{V(K)}.$$

When $V(K) = \infty$, we set $C(K) = 0$. For an arbitrary subset E of $\mathbf{P}^N(\mathbf{C})$ we define the projective logarithmic capacity of E by

$$C(E) = \sup_{K\subset E} C(K),$$

where the supremum is taken over compact subsets K of E.

For real valued functions $A(r)$ and $B(r)$ on $[1, \infty)$ we write

$$A(r) \leq B(r)||$$

if there is a Borel subset $J \subset [1, \infty)$ with finite measure such that $A(r) \leq B(r)$ for $r \in [1, \infty) \setminus J$.

Let the notation be as in the previous section. We now show the following theorem.

Theorem 4. *Let M be a compact complex manifold, and $L \to M$ a holomorphic line bundle with $B(\Gamma(M,L)) = \phi$. Let $f : \mathbf{C}^n \to M$ be a meromorphic mapping. Let $\varphi(r) > 0$ be a Borel measurable function on $[1,\infty)$ which satisfies*

$$\int_1^\infty \frac{dr}{\varphi(r)} < \infty.$$

Then there exists a subset F of $P(\Gamma(M,L))$ such that $C(F) = 0$ and that

$$m_f(r,D) \le \varphi(r) + O(1)||$$

for an arbitrary divisor $D \in P(\Gamma(M,L)) \setminus F$.

Proof. We identify $P(\Gamma(M,L)) = \mathbf{P}^N(\mathbf{C})$ by the base $\{\sigma_0, \ldots, \sigma_N\}$. We set

$$F = \left\{ D \in P(\Gamma(M,L));\ \int_1^\infty \frac{m_f(r,D)}{\varphi(r)} dr = \infty \right\}.$$

Assume that $C(F) > 0$. Then there is a compact subset K of F with $C(K) > 0$. Therefore there exists a $\nu \in \mathcal{M}(K)$ such that

$$\sup_{x \in \mathbf{P}^N(\mathbf{C})} u_\nu(x) < \infty. \tag{17}$$

It follows from (3) and (17) that

$$\begin{aligned}
&\int_{[\zeta^0:\ldots:\zeta^N] \in K} \left\{ \int_1^\infty \frac{m_f(r, ([\zeta^0 : \ldots : \zeta^N]))}{\varphi(r)} dr \right\} d\nu \\
&= \int_1^\infty \frac{1}{\varphi(r)} \left\{ \int_{z \in \Gamma(r)} \left\{ \int_K \log \frac{1}{S_{f(z)}([\zeta^0 : \ldots : \zeta^N])} d\nu \right\} d\eta \right\} dr \\
&\le \int_1^\infty \frac{1}{\varphi(r)} \left\{ \int_{\Gamma(r)} \sup_{x \in \mathbf{P}^N(\mathbf{C})} u_\nu(x) \eta \right\} dr \\
&= \int_1^\infty \frac{1}{\varphi(r)} \sup_{x \in \mathbf{P}^N(\mathbf{C})} u_\nu(x) dr < \infty.
\end{aligned}$$

On the other hand, by the definition of F we have

$$\int_{[\zeta^0:\ldots:\zeta^N] \in K} \left\{ \int_1^\infty \frac{m_f(r, ([\zeta^0 : \ldots : \zeta^N]))}{\varphi(r)} dr \right\} d\nu = \infty.$$

This is a contradiction. Hence $C(F) = 0$. For an arbitrary divisor $D \in P(\Gamma(M,L))$ we set

$$J(D) = \left\{ r \in [1,\infty);\ \frac{m_f(r,D)}{\varphi(r)} > 1 \right\}.$$

If $D \notin F$, then we see

$$\int_{J(D)} dr < \int_{r\in J(D)} \frac{m_f(r,D)}{\varphi(r)}dr \le \int_1^\infty \frac{m_f(r,D)}{\varphi(r)}dr < \infty.$$

Therefore for $D \in P(\Gamma(M,L)) \setminus F$

$$m_f(r,D) \le \varphi(r) + O(1)||.$$

□

6 The general case.

In this section we deal with the growth of the proximity function with respect to an effective divisor $D \in P(E)$, where $L \to M$ be a holomorphic line bundle and E is a linear subspace of $\Gamma(M,L)$, and complete the proof of the Main Theorem.

Let M be a compact complex manifold and $\mathcal{I}$ a coherent ideal sheaf of the structure sheaf $\mathcal{O}_M$ over M. Let $\{V_\lambda\}$ be a finite open covering of M and $\eta_{\lambda j} \in \Gamma(V_\lambda, \mathcal{I})$, $j = 1, 2, \ldots$, be finitely many sections of which germs $\underline{\eta_{\lambda 1}}_x, \underline{\eta_{\lambda 2}}_x, \ldots$, generate the fiber $\mathcal{I}_x$ for all $x \in V_\lambda$. Following to [5], Chap. 2 or [7], §2, we let $\{\rho_\lambda\}$ be a partition of unity associated with $\{V_\lambda\}$ and set

$$d_{\mathcal{I}}(x) = \sum_\lambda \rho_\lambda(x) \left(\sum_j |\eta_{\lambda j}(x)|^2 \right)^{1/2}, \quad x \in M.$$

Let f be a meromorphic mapping from $\mathbf{C}^n$ into M such that

$$f(\mathbf{C}^n) \not\subset \operatorname{supp} \mathcal{O}_M/\mathcal{I}.$$

We define the proximity function of f for $\mathcal{I}$ by

$$m_f(r,\mathcal{I}) = \int_{z\in\Gamma(r)} -\log d_{\mathcal{I}} \circ f(z)\eta(z).$$

Next let $L \to M$ be a holomorphic line bundle and $\dim \Gamma(M,L) = N+1$. Let E be an $(l+1)$-dimensional linear subspace of $\Gamma(M,L)$. We take a base $\{\sigma_0, \ldots, \sigma_N\}$ of $\Gamma(M,L)$ and we identify $\Gamma(M,L) \cong \mathbf{C}^{N+1}$ by $\{\sigma_0, \ldots, \sigma_N\}$. Moreover we assume that E is spanned by $\{\sigma_0, \ldots, \sigma_l\}$. Let $\mathcal{I}$ denote the coherent ideal sheaf of $\mathcal{O}_M$ of which fiber over $x \in M$ is generated by $\{\underline{\sigma}_x;\ \sigma \in E\}$. Then the base of E is defined by $B(E) = \mathcal{O}_M/\mathcal{I}$. Thus we write $\mathcal{I} = \mathcal{I}_{B(E)}$.

Let $f : \mathbf{C}^n \to M$ be a meromorphic mapping. Suppose that

$$f(\mathbf{C}^n) \not\subset \operatorname{supp} B(E).$$

Let $(\{U_\lambda\}, \{s_\lambda\})$ be a local trivialization covering of L. We define a meromorphic mapping $\Phi_L : M \to \mathbf{P}^N(\mathbf{C})$ by

$$\Phi_L(x) = [\sigma_{0\lambda}(x) : \ldots : \sigma_{N\lambda}(x)], \quad x \in U_\lambda,$$

where $\sigma_{j\lambda}$ is a holomorphic function on U_λ such that $\sigma_j|U_\lambda = \sigma_{j\lambda}s_\lambda$. Let $(f^0, \ldots, f^N)$ be a reduced representation of $\Phi_L \circ f$. We denote by f_E the meromorphic mapping from $\mathbf{C}^n$ into $\mathbf{P}^l(\mathbf{C})$ represented by $(f^0, \ldots, f^l)$. For $z \in f|(\mathbf{C}^n \setminus I(f))^{-1}(U_\lambda \setminus \operatorname{supp} B(E))$

$$f_E(z) = [\sigma_{0\lambda} \circ f(z) : \ldots : \sigma_{l\lambda} \circ f(z)].$$

We denote by H_l hyperplane bundle over $\mathbf{P}^l(\mathbf{C})$. The following is known.

Proposition 5. *Let the notation be as above. We have the following.*
(i) *If $B(\Gamma(M, L)) = \phi$, then*

$$T_f(r, L) \geq T_{f_E}(r, H_l) + O(1).$$

(ii) (Cf. Noguchi [5].) *For $[\zeta^0 : \ldots : \zeta^l] \in P(E)$*

$$m_f\left(r, \left(\textstyle\sum_{j=0}^l \zeta^j \sigma_j\right)\right) - m_f(r, \mathcal{I}_{B(E)}) = m_{f_E}(r, ([\zeta^0 : \ldots : \zeta^l])) + O(1).$$

Proof. (i) We assume that $B(\Gamma(M, L)) = \phi$. Let $(g^0, \ldots, g^l)$ be a reduced representation of f_E. Then there is a holomorphic function h on $\mathbf{C}^n$ such that $(f^0, \ldots, f^l) = (hg^0, \ldots, hg^l)$. Since $L = \Phi_L^* H_N$ it follows that

$$\begin{aligned} T_f(r, L) &= \int_{z \in \Gamma(r)} \log \left(\sum_{j=0}^N |f^j(z)|^2 \right)^{1/2} \eta + O(1) \\ &\geq \int_{z \in \Gamma(r)} \log \left(\sum_{j=0}^l |f^j(z)|^2 \right)^{1/2} \eta + O(1) \\ &\geq \int_{z \in \Gamma(r)} \log \left(\sum_{j=0}^l |g^j(z)|^2 \right)^{1/2} + \int_{z \in \Gamma(1)} \log |h| \; \eta + O(1) \\ &\geq T_{f_E}(r, H_l) + O(1). \end{aligned}$$

(ii) Let h be a Hermitian metric in L and $|| \cdot ||$ denote the norms on L. Let $\{\tau_\lambda\}$ be a partition of unity associated with $\{U_\lambda\}$. For $x \in U_\nu$ we set

$$k(x) = \log \frac{\left(\sum_{j=0}^l |\zeta^j|^2\right)^{1/2}}{|| \sum_{j=0}^l \zeta^j \sigma_j(x) ||} - \log \frac{\left(\sum_{j=0}^l |\sigma_{j\nu}(x)|^2\right)^{1/2} \left(\sum_{j=0}^l |\zeta^j|^2\right)^{1/2}}{| \sum_{j=0}^l \sigma_{j\nu}(x) \zeta^j |}$$

$$+\log \sum_{\lambda} \tau_\lambda(x) \left(\sum_{j=0}^{l} |\sigma_{j\lambda}(x)|^2 \right)^{1/2} .$$

Since

$$|| \textstyle\sum_{j=0}^{l} \zeta^j \sigma_j(x) || = | \sum_{j=0}^{l} \sigma_{j\nu}(x) \zeta^j ||| s_\nu(x) ||,$$

we see

$$k(x) = \log \frac{\sum_\lambda \tau_\lambda(x) \left(\sum_{j=0}^{l} |\sigma_{j\lambda}(x)|^2 \right)^{1/2}}{||s_\nu(x)|| \left(\sum_{j=0}^{l} |\sigma_{j\nu}(x)|^2 \right)^{1/2}}.$$

We take an arbitrary point $y \in M$ and ν such that $\tau_\nu(y) > 0$. Then there are a relatively compact neighborhood $V \subset U_\nu$ of y and positive constant $C_1, C_2, C_3 > 0$ such that for $x \in V$

$$k(x) \leq \log \frac{\sum_\lambda C_1 \tau_\lambda(x) \left(\sum_{j=0}^{l} |\sigma_{j\nu}(x)|^2 \right)^{1/2}}{||s_\nu(x)|| \left(\sum_{j=0}^{l} |\sigma_{j\nu}(x)|^2 \right)^{1/2}} = \log \frac{C_1}{||s_\nu(x)||} \leq \log C_2,$$

and

$$k(x) \geq \log \frac{\tau_\nu(x)}{||s_\nu(x)||} \geq \log C_3.$$

Since M is compact there exists a positive constant C such that for an arbitrary $x \in M$

$$|k(x)| < C.$$

This finishes the proof of (ii).

□

Let μ_E denote the positive measure induced by Fubini-Study metric on $P(E) = \mathbf{P}^l(\mathbf{C})$.

Theorem 6. *Let M be a compact complex manifold and $L \to M$ a holomorphic line bundle. Let $1 \leq l \leq N$ be an integer and E an (l+1)-dimensional linear subspace of $\Gamma(M, L)$. Let $f : \mathbf{C}^n \to M$ be a meromorphic mapping such that $f(\mathbf{C}^n) \not\subset \operatorname{supp} B(E)$. If $T_{f_E}(r, H_l) \to \infty \quad (r \to \infty)$, then for almost all divisor $D \in P(E)$*

$$\limsup_{r \to \infty} \frac{m_f(r, D) - m_f(r, \mathcal{I}_{B(E)})}{\log T_{f_E}(r, H_l)} \leq \frac{1}{2}.$$

Otherwise for almost all divisor $D \in P(E)$

$$m_f(r, D) - m_f(r, \mathcal{I}_{B(E)}) = O(1).$$

Proof. Set

$$I = \left\{ [\zeta^0 : \dots : \zeta^l] \in P(E);\ \limsup_{r\to\infty} \frac{m_f(r, (\sum_{j=0}^{l} \zeta^j \sigma_j)) - m_f(r, \mathcal{I}_{B(E)})}{\log T_{f_E}(r, H_l)} > \frac{1}{2} \right\}.$$

Because of Proposition 5 we have that for $[\zeta^0 : \dots : \zeta^l] \in I$

$$\frac{1}{2} < \limsup_{r\to\infty} \frac{m_{f_E}(r, ([\zeta^0 : \dots : \zeta^l]))}{\log T_{f_E}(r, H_l)}.$$

Hence, if $T_{f_E}(r, H_l) \to \infty \ \ (r \to \infty)$, then we have $\mu_E(I) = 0$ by Theorem 2. We assume that $T_{f_E}(r, H_l) = O(1)$. Then f_E is a constant mapping. Hence by Proposition 5 (ii)

$$m_f(r, D) - m_f(r, \mathcal{I}_{B(E)}) = O(1).$$

□

By making use of the methods in the proofs of Proposition 5 and Theorem 4 one may aslo deduce the following:

Theorem 7. *Let M be a compact complex manifold and $L \to M$ a holomorphic line bundle. Let $1 \leq l \leq N$ be an integer and E an (l+1)-dimensional linear subspace of $\Gamma(M, L)$. Let $f : \mathbf{C}^n \to M$ be a meromorphic mapping. Let $\varphi(r) > 0$ be a Borel measurable function on $[1, \infty)$ which satisfies*

$$\int_1^\infty \frac{dr}{\varphi(r)} < \infty.$$

Then there exists a subset F of $P(E)$ such that $C(F) = 0$ and that for all $D \in P(E) \setminus F$

$$m_f(r, D) - m_f(r, \mathcal{I}_{B(E)}) \leq \varphi(r) + O(1)||.$$

Remark. S. Mori [4] proved that for a non-constant meromprphic mapping $f : \mathbf{C}^n \to \mathbf{P}^N(\mathbf{C})$, the set

$$\left\{ H \in \mathbf{P}^N(\mathbf{C})^*;\ \limsup_{r\to\infty} \frac{m_f(r, D)}{\sqrt{T_f(r, H_N)} \log T_f(r, H_N)} > 0 \right\}$$

is of projective logarithmic capacity zero. Moreover, A. Sadullaev [8] showed that this set forms a polar set.

Note the differences between these results and our Theorems 2 and 7.

References

[1] Hayman, W.K., Value Distribution and Exceptional Sets, in Lectures on Approximation and Value Distribution, pp. 79-147, Univ. Montreal Press, 1982, Montrreal.
[2] Littlewood, J.E., Mathematical notes (11): On exceptional values of power series, J. London Math. Soc. **5** (1930), 82–89.
[3] Molzon, R.E., Shiffman, B., Sibony, N., Average growth estimates for hyperplane sections of entire analytic sets, Math. Ann. **257** (1981), 43–59.
[4] Mori, S., Elimination of defects of meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$, Ann. Acad. Sci. Fenn. Math. **24** (1999), 89–104.
[5] Noguchi, J., Nevanlinna Theory in Several Complex Variables and Diophantine Approximation (in Japanese), Kyoritsu Publ. Co., Tokyo, 2003.
[6] Noguchi, J., Ochiai, T., Geometric Function Theory in Several Complex Variables, Transl. Math. Monogr. Vol. 80, Amer. Math. Soc., 1990.
[7] Noguchi, J., Winkelmann, J. and Yamanoi, K., The second main theorem for holomorphic curves into semi-abelian varieties II, Forum Math.**20** (2008), 469–503.
[8] Sadullaev, A., Defect divisors in the sense of Valiron, Mat. Sb. **108 (150)** (1979), 567–580; English transl. in Math. USSR Sb. **36** (1980), 535–547.
[9] Sadullaev, A. and Degtjar', P.V., Approximation divisors of a holomorphic mapping and defects of meromorphic functions of several variables (Russian), Ukrain. Mat. Zh. **33** (1981), 620–625, 716; English transl. Ukrainian Math. J. **33** (1981), no. 5, 473–477 (1982).

Atsushi Nitanda
Graduate School of Mathematical Sciences
University of Tokyo
Komaba, Meguro,Tokyo 153-8914
Japan
Current Address:
Mathematical Systems, Inc.
10F Four Seasons Bldg.
2-4-3 Shinjuku, Shinjuku-ku
Tokyo 160-0022, JAPAN
e-mail: nitanda_atsushi@msi.co.jp

UTMS

2008–33 Keiichi Sakai: *Configuration space integrals for embedding spaces and the Haefliger invariant.*

2009–1 Tadayuki Watanabe: *On Kontsevich's characteristic classes for higher dimensional sphere bundles II : higher classes.*

2009–2 Takashi Tsuboi: *On the uniform perfectness of the groups of diffeomorphisms of even-dimensional manifolds.*

2009–3 Hitoshi Kitada: *An implication of Gödel's incompleteness theorem.*

2009–4 Jin Cheng, Junichi Nakagawa, Masahiro Yamamoto and Tomohiro Yamazaki: *Uniqueness in an inverse problem for one-dimensional fractional diffusion equation.*

2009–5 Y. B. Wang, J. Cheng, J. Nakagawa, and M. Yamamoto : *A numerical method for solving the inverse heat conduction problem without initial value.*

2009–6 Dietmar Hömberg, Nataliya Togobytska, Masahiro Yamamoto: *On the evaluation of dilatometer experiments.*

2009–7 Toshio Oshima and Nobukazu Shimeno: *Heckman-Opdam hypergeometric functions and their specializations.*

2009–8 Atsushi Yamashita: *Compactification of the homeomorphism group of a graph.*

2009–9 Jingzhi Li, Masahiro Yamamoto, and Jun Zou: *Conditional stability and numerical reconstruction of initial temperature.*

2009–10 Taku Ishii and Takayuki Oda: *Calculus of principal series Whittaker functions on $SL(n, \mathbb{R})$.*

2009–11 Atsushi Nitanda: *The growth of the Nevanlinna proximity function.*

The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001   FAX +81-3-5465-7012