UTMS 2009–10 May 20, 2009

Calculus of principal series Whittaker functions on $SL(n, \mathbb{R})$

by

Taku Ishii and Takayuki Oda

UNIVERSITY OF TOKYO
GRADUATE SCHOOL OF MATHEMATICAL SCIENCES
KOMABA, TOKYO, JAPAN

CALCULUS OF PRINCIPAL SERIES WHITTAKER FUNCTIONS ON $SL(n,\mathbf{R})$

TAKU ISHII AND TAKAYUKI ODA

Abstract. We study Whittaker functions for the principal series representation of $SL(n,\mathbf{R})$. We derive a system of partial differential equations characterizing our Whittaker functions. We give explicitly power series solutions at the regular singularity of the system, and integral representations of unique moderate growth Whittaker function.

Introduction

In this paper we explicitly determine the radial parts of Whittaker functions for the principal series representations of $SL(n,\mathbf{R})$, by extending our previous works for $n = 3$, 4 ([MIO], [HnIO]).

According to the celebrated result of Shalika [Sha], Fourier expansions (along a maximal parabolic subgroup) of cusp forms on $GL(n)$ can be described in terms of Whittaker functions. Therefore precise information of local Whittaker functions plays important roles in various aspects of automorphic forms on $GL(n)$, for example, analysis of local zeta integrals. The well-known formula of Shintani [Shi] asserts that the class one Whittaker functions over non-archimedean fields can be written by using the character of finite dimensional representations of $GL(n,\mathbf{C})$. Combined with some facts on representation theory of $GL(n,\mathbf{C})$, one can perform unramified computation for non-archimedean zeta integrals. This is a first step for a study of automorphic L-functions via Rankin-Selberg method.

Our interest here is archimedean Whittaker functions. Let $G = SL(n,\mathbf{R})$ and N the standard maximal unipotent subgroup of G consisting of upper triangular matrices. We fix a nondegenerate unitary character η of N and consider the induced representation $C^\infty\text{-}\mathrm{Ind}_N^G(\eta)$, whose representation space is

$$C^\infty_\eta(N\backslash G) = \{\varphi \in C^\infty(G,\mathbf{C}) \mid \varphi(xg) = \eta(x)\varphi(g) \text{ for all } (x,g) \in N \times G\}.$$

For an irreducible admissible representation π of G, a realization of π in $C^\infty_\eta(N\backslash G)$ is called a *Whittaker model* of π.

When π is the principal series representation $\pi_{I,\nu}$ (see Definition 2.1), Kostant [Ko, §5] proved that the dimension of the intertwing space

$$\mathcal{W}(\pi_{I,\nu},\eta) = \mathrm{Hom}_{(\ \mathbf{C},K)}(\pi_{I,\nu}, C^\infty_\eta(N\backslash G))$$

is $n!$, the order of the Weyl group $\mathfrak{S}_n$. Here $K = SO(n)$ is a maximal compact subgroup of G and $\mathfrak{g} = \mathfrak{gl}(n,\mathbf{R})$. For a nonzero vector v in $\pi_{I,\nu}$ and a nonzero intertwiner Φ in $\mathcal{W}(\pi_{I,\nu},\eta)$, we call the function $\Phi(v)$ the *Whittaker function*. We will give explicit formula of $\Phi(v)$ for the vectors v belonging to minimal K-types of $\pi_{I,\nu}$.

Date: April 27, 2009.

The local multiplicity one theorem of Shalika [Sha] or, more specifically, its variant by Wallach [Wa, §8] says the dimension of the space of moderate growth Whittaker functions is at most one. We refer to the unique moderate growth Whittaker functions as *primary Whittaker functions*. Jacquet [Ja1] introduced an integral expression of the primary Whittaker function:

$$\int_N f(n)\eta^{-1}(n)dn.$$

By means of the Jacquet integral, Jacquet, Shalika and Piatetski-Shapiro (cf. [Ja2]) developed the archimedean theory of automorphic L-functions on $GL(n)$.

On the other hand, the Jacquet integral is not adequate for direct computation of archimedean zeta integrals. For that purpose, more explicit formulas of Whittaker functions on $GL(n, \mathbf{R})$ have been developed by Vinogradov, Tahtajan [VT], Bump [Bu] ($n = 3$) and Stade [St1], [St2] for the class one principal series $\pi_{\emptyset,\nu}$. Starting from the Jacquet integral, Stade [St1] reaches a recursive formula between the primary Whittaker functions on $GL(n, \mathbf{R})$ and $GL(n-2, \mathbf{R})$. We call such recursive formulas *propagation formulas.* Based on the result of [St2], Stade and Ishii [IS] obtain a propagation formula between $GL(n, \mathbf{R})$ and $GL(n-1, \mathbf{R})$ (Theorem 4.5).

Let us recall the simplest case $G = SL(2, \mathbf{R})$. The radial part of the primary Whittaker functions for the class one principal series is essentially the K-Bessel function $K_\nu(y)$. Up to elementary factors, $K_\nu(2\pi y)$ has integral representations

(K) $\displaystyle\int_0^\infty \exp\{-\pi y(t + t^{-1})\} t^{\nu-1} dt,$

(J) $\displaystyle y^\nu \int_{-\infty}^\infty (x^2 + y^2)^{-\nu+1/2} \exp(2\pi i x) dx,$

(B) $\displaystyle\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma\Big(\frac{s+\nu}{2}\Big)\Gamma\Big(\frac{s-\nu}{2}\Big)(\pi y)^{-s} ds.$

The first expression (K) is the definition and, the second one (J) is the Jacquet integral. The expressions (K) and (B) are easily transformed each other, and they are suitable for computation of some archimedean zeta integrals. Actually by utilizing formulas in [St1] and [St2], Stade [St2], [St3] proves the coincidence of the appropriate archimedean L-factors and the archimedean zeta integrals for the convolution L-functions on $GL(n) \times GL(n)$ and $GL(n) \times GL(n+1)$, and the exterior square L-function on $GL(n)$. For the exterior square L-functions on $GL(n)$, Miller and Schmid [MS] gives a new approach for archimedean theory via automorphic distributions.

We mention a work of Hashizume [Ha]. Keep π being the class one principal series $\pi_{\emptyset,\nu}$. Let v be a spherical vector in π. The Whittaker function $w = \Phi(v)$ ($\Phi \in \mathcal{W}(\pi_{\emptyset,\nu}, \eta)$) is characterized by the properties

(i) $w(xgk) = \eta(x)w(g)$ for all $(x, g, k) \in N \times G \times K$,
(ii) $Zw = \chi_\nu(Z)w$ for all $Z \in \mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$.

Here $\mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$ is the center of universal enveloping algebra of $\mathfrak{g}_{\mathbf{C}}$, and χ_ν the infinitesimal character of $\pi_{\emptyset,\nu}$. In view of the Iwasawa decomposition $G = NAK$ and the condition (i), w is determined by its (A-)radial part, and can be regarded as a function on $(\mathbf{R}_+)^{n-1}$. Then the second condition leads to a system of partial differential equations of $(n-1)$ variables. In a similar way to Harish-Chanda's study for spherical functions, Hashizume gives a way of construction of the power series solution at the regular singularities of the system of partial differential equations. We call this basis functions *secondary Whittaker functions,* and they mutually transformed by the Weyl group $\mathfrak{S}_n$. The important fact is that Whittaker functions are essentially controlled by the Casimir operator. This is a consequence of right K-invariance of Whittaker functions. Hashizume also gives a *factorization formula*, that is, a decomposition of the primary Whittaker function (=the Jacquet integral) as a linear combination of the secondary Whittaker functions.

The aim of this paper is to study the non class one principal series Whittaker functions and establish explicit formulas of types (K) and (B). For some properties of Jacquet integrals such as analytic continuations and asymptotic behaviors, see [GW] and [Ca], for example. In the case of $SL(2,\mathbf{R})$, our Whittaker functions are nothing but classical Whittaker's functions. But for higher rank cases, they take values in a representation space of K. To handle these vector-valued Whittaker functions, we need more precise description of the principal series. As far as we know, explicit formulas have established in the case of $SL(3,\mathbf{R})$ ([MIO]), $SL(4,\mathbf{R})$ ([HnIO]), $Sp(2,\mathbf{R})$ ([I2]), and $GL(3,\mathbf{C})$ ([HO]).

Let us explain our strategy toward explicit formulas of Whittaker functions.

Step 1. Determine the $(\mathfrak{g}_{\mathbf{C}},K)$-module structure of the principal series representation around its minimal K-types, and give the system of partial differential equations for the principal series Whittaker functions (Theorem 3.3). To characterize Whittaker functions, in addition to generators of $\mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$ (Capelli elements), we need first-ordered differential operators called *Dirac-Schmid operators.* Historically speaking this kind variants of the Dirac operators were used in the papers of Schmid, Okamoto-Narasimhan, Hotta-Parthasarathy from late 60's to early 70's in the construction of discrete series. See Sections 2 and 3.

Step 2. Construct a propagation formula for the secondary Whittaker functions. The difficult part is to find a right answer, though the dirty remains of this procedure are under the carpet. Once we can arrive at a desired formula, our task is only to check the compatibility with the Dirac-Schmid and the Casimir equations established in step 1. See Sections 5 and 8.

Step 3. Find a working hypothesis for explicit formulas of the primary Whittaker functions. A point is an analogy of propagation formulas for the secondary Whittaker functions established in step 2 and Mellin-Barnes integral representations of the primary Whittaker functions. We prove the conjectured formulas by utilizing factorization formulas recursively.

Our explicit formulas given in steps 2 and 3 are propagation formula with respect to n, and *helicity* $h = \sharp(I)$ which represents the complexity of principal series. Our main results (Theorems 8.7, 9.1 and 9.2) are natural extension of the class one case, though the statements and their proofs look like quite complicated at the first sight. At now we can not avoid these long calculation for the non class one principal series. To lighten readers' burden, we first explain our inductive argument in the case of helicity 1 in Sections 5 and 6. We note that our proof does not rely on Jacquet integrals and thus it gives a new proof for the propagation formulas of [IS].

Acknowledgment: We thank Tatsuo Hina who is coauthor of our previous work [HnIO] for $SL(4,\mathbf{R})$-Whittaker functions.

Contents

1. Preliminaries

For a Lie group G with the Lie algebra $\mathfrak{g}$ and a maximal compact subgroup K, the algebraic structure of a quasi-simple representation of G is known by its associated $(\mathfrak{g}, K)$-module structure. This section is devoted to recall some basic results on K-modules and the generators of the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$.

Throught this paper we denote by $G = SL(n, \mathbf{R})$ and fix a maximal compact subgroup $K = SO(n)$. We denote the Lie algebra of a Lie group by the correpsonding german letter: $\mathfrak{g} = \mathfrak{sl}(n, \mathbf{R})$, $\mathfrak{k} = \mathfrak{so}(n)$. We set $\mathfrak{p} = \{X \in \mathfrak{g} \mid {}^tX = X\}$. We denote by E_{ij} the matrix unit of size n with 1 at the (i, j)-th entry and 0 at the other entries. For $1 \leq i, j \leq n$ we put

$$K_{ij} = E_{ij} - E_{ji}, \quad X_{ij} = \begin{cases} E_{ij} + E_{ji} & \text{if } i \neq j, \\ 2E_{ii} - (2/n)E_n & \text{if } i = j, \end{cases}$$

where E_n is the unit matrix of size n. Then $\mathfrak{k} = \bigoplus_{1 \leq i < j \leq n} \mathbf{R}K_{ij}$ and $\mathfrak{p} = \bigoplus_{1 \leq i \leq j \leq n} \mathbf{R}X_{ij}$.

For our combinatorical arugements we often use a notion of *colors*.

Definition 1.1. For a pair of positive integers (n, h) with $1 \leq h \leq n$, a sequence $X = \{x_1, \ldots, x_h\}$ of cardinality h satisfying a condition

$$1 \leq x_1 < x_2 < \cdots < x_h \leq n$$

is called a *color of type* (n, h). Or equivalently, for a strictly monotonous map $x : T_h = \{1, 2, \ldots, h\} \to T_n = \{1, 2, \ldots, n\}$, the image $x(T_h) = \{x(1), \ldots, x(h)\}$ is a color of type (n, h). We denote by ${}_n\mathcal{C}_h$ the set of colors of type (n, h).

In this paper we use the same bracket $\{\ \ \}$ for a color and a set, and we sometimes understand the set T_h as a color of type (n, h).

Notation 1.2. For a pair of positive integers (n, h) with $1 \leq h \leq n$, let $A = \{a_1, \ldots, a_h\}$ be a sequence satisfying $1 \leq a_i \leq n$ (not necessary a color). For $1 \leq k, l, c, d \leq n$ we define sequences $A_{k,c}$ and $A_{(k,l),(c,d)}$ by

$$A_{k,c} = \{a_1, \ldots, a_{k-1}, c, a_{k+1}, \ldots, a_h\}$$

and

$$A_{(k,l),(c,d)} = (A_{k,c})_{l,d} = \{a_1, \ldots, a_{k-1}, c, a_{k+1}, \ldots, a_{l-1}, d, a_{l+1}, \ldots, a_h\}.$$

1.1. The structure of K-modules.

We regard $\mathbf{R}^n$ as a $\mathbf{R}$-vector space consisting of column vectors on which G and $\mathfrak{g}$ act on V_n via left multiplications. By restriction we can consider the action of K and $\mathfrak{k}$ on the same space. Let $V_n = \mathbf{R}^n \otimes \mathbf{C}$ and $\{v_a \mid 1 \leq a \leq n\}$ be the standard basis of V_n. Then the action of K_{ij} on V_n is given by

$$K_{ij}v_a = \delta_{ja}v_i - \delta_{ia}v_j.$$

Here δ_{ij} is the Kronecker symbol.

For $1 \leq h \leq n$, we consider a K-module $\wedge^h V_n$. The standard basis of $\wedge^h V_n$ is given by

$$\{v_A := v_{a_1} \wedge \cdots \wedge v_{a_h} \mid A = \{a_1, \ldots, a_h\} \in {}_n\mathcal{C}_h\}$$

with the $\mathfrak{k}$-action

$$K_{ij}v_A = \sum_{k=1}^{h} (\delta_{ja_k} v_{A_{k,i}} - \delta_{ia_k} v_{A_{k,j}}).$$

Let $\mathfrak{p}_{\mathbf{C}} = \{X \in M(n, \mathbf{C}) \mid {}^tX = X, \operatorname{tr}(X) = 0\}$ be the complexification of $\mathfrak{p}$. By the adjoint action Ad , the space $\mathfrak{p}_{\mathbf{C}}$ becomes a K-module:

$$\text{Ad } (K_{ij})X_{ab} = \delta_{ja}X_{ib} - \delta_{ia}X_{jb} + \delta_{jb}X_{ia} - \delta_{ib}X_{ja}. \tag{1.1}$$

We study the tensor product $\mathfrak{p}_{\mathbf{C}} \otimes \wedge^h V_n$ and describe a subspace isomorphic to $\wedge^h V_n$ as K-module.

Proposition 1.3. *For $A = \{a_1, \ldots, a_h\}$ $(1 \leq a_j \leq n)$, we set*

$$v'_A := \sum_{k=1}^{h} \sum_{c \notin A \setminus \{a_k\}} X_{a_k c} \otimes v_{A_{k,c}} = \sum_{k=1}^{h} X_{a_k a_k} \otimes v_A + \sum_{k=1}^{h} \sum_{c \notin A} X_{a_k c} \otimes v_{A_{k,c}} \in \mathfrak{p}_{\mathbf{C}} \otimes \wedge^h V_n.$$

Then the set $\{v'_A \mid A \in {}_n\mathcal{C}_h\}$ is the standard basis of K-submodule $\wedge^h V_n$ in the tensor product $\mathfrak{p}_{\mathbf{C}} \otimes \wedge^h V_n$.

Proof. Our task is to show

$$K_{ij} v'_A = \sum_{k=1}^{h} (\delta_{ja_k} v'_{A_{k,i}} - \delta_{ia_k} v'_{A_{k,j}}). \tag{1.2}$$

By (1.1), we have

$$\begin{aligned}
K_{ij} v'_A &= \sum_{k=1}^{h} \sum_{c \notin A \setminus \{a_k\}} (\text{Ad } (K_{ij}) X_{a_k c} \otimes v_{A_{k,c}} + X_{a_k c} \otimes K_{ij} v_{A_{k,c}}) \\
&= \sum_{k=1}^{h} \sum_{c \notin A \setminus \{a_k\}} (\delta_{ja_k} X_{ic} - \delta_{ia_k} X_{jc} + \delta_{jc} X_{ia_k} - \delta_{ic} X_{ja_k}) \otimes v_{A_{k,c}} \\
&+ \sum_{k=1}^{h} \sum_{c \notin A \setminus \{a_k\}} X_{a_k c} \otimes \left\{ \delta_{jc} v_{A_{k,i}} - \delta_{ic} v_{A_{k,j}} + \sum_{1 \leq l \leq h, l \neq k} (\delta_{ja_l} v_{A_{(k,l),(c,i)}} - \delta_{ia_l} v_{A_{(k,l),(c,j)}}) \right\} \\
&= \sum_{k=1}^{h} \sum_{c \notin A \setminus \{a_k\}} (\delta_{ja_k} X_{ic} - \delta_{ia_k} X_{jc}) \otimes v_{A_{k,c}} \\
&+ \sum_{k=1}^{h} \sum_{c \notin A \setminus \{a_k\}} \sum_{1 \leq l \leq h, l \neq k} X_{a_k c} \otimes (\delta_{ja_l} v_{A_{(k,l),(c,i)}} - \delta_{ia_l} v_{A_{(k,l),(c,j)}}) \\
&+ \begin{cases} X_{ia_l} \otimes v_A + X_{a_l a_l} \otimes v_{A_{l,i}} & \text{if } j = a_l \in A, \\ \sum_{k=1}^{h} (X_{ia_k} \otimes v_{A_{k,j}} + X_{ja_k} \otimes v_{A_{k,i}}) & \text{if } j \notin A \end{cases} \\
&- \begin{cases} X_{ja_l} \otimes v_A + X_{a_l a_l} \otimes v_{A_{l,j}} & \text{if } i = a_l \in A, \\ \sum_{k=1}^{h} (X_{ja_k} \otimes v_{A_{k,i}} + X_{ia_k} \otimes v_{A_{k,j}}) & \text{if } i \notin A. \end{cases}
\end{aligned}$$

Then we have

$$
\begin{aligned}
K_{ij}v'_A &= \sum_{k=1}^{h} \sum_{c\notin A\setminus\{a_k\}} (\delta_{ja_k}X_{ic} - \delta_{ia_k}X_{jc}) \otimes v_{A_{k,c}} \\
&\quad + \sum_{k=1}^{h} \sum_{c\notin A\setminus\{a_k\}} \sum_{1\le l\le h, l\ne k} X_{a_kc} \otimes (\delta_{ja_l}v_{A_{(k,l),(c,i)}} - \delta_{ia_l}v_{A_{(k,l),(c,j)}}) \\
&\quad + \begin{cases} \sum_{1\le k\le h, k\ne l} X_{a_ka_l} \otimes v_{A_{k,j}} & \text{if } i = a_l \text{ and } j \notin A, \\ -\sum_{1\le k\le h, k\ne l} X_{a_ka_l} \otimes v_{A_{k,i}} & \text{if } i \notin A \text{ and } j = a_l, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{1.3}
$$

Let us consider the right hand side of (1.2). For $i \notin A\setminus\{a_k\}$ we can see that

$$
\begin{aligned}
v'_{A_{k,i}} &= \sum_{1\le l\le h, l\ne k} \sum_{c\notin A_{k,i}\setminus\{a_l\}} X_{a_lc} \otimes v_{A_{(k,l),(i,c)}} + \sum_{c\notin A_{k,i}\setminus\{i\}} X_{ic} \otimes v_{A_{k,c}} \\
&= \sum_{1\le l\le h, l\ne k} \sum_{c\notin A\setminus\{a_k,a_l\}} X_{a_lc} \otimes v_{A_{(k,l),(i,c)}} + \sum_{c\notin A\setminus\{a_k\}} X_{ic} \otimes v_{A_{k,c}} \\
&= \sum_{1\le l\le h, l\ne k} \left\{ X_{a_la_k} \otimes v_{A_{(k,l),(i,a_k)}} + \sum_{c\notin A\setminus\{a_l\}} X_{a_lc} \otimes v_{A_{(k,l),(i,c)}} \right\} + \sum_{c\notin A\setminus\{a_k\}} X_{ic} \otimes v_{A_{k,c}} \\
&= \sum_{1\le l\le h, l\ne k} \left\{ -X_{a_la_k} \otimes v_{A_{l,i}} + \sum_{c\notin A\setminus\{a_l\}} X_{a_lc} \otimes v_{A_{(k,l),(i,c)}} \right\} + \sum_{c\notin A\setminus\{a_k\}} X_{ic} \otimes v_{A_{k,c}}.
\end{aligned}
$$

Then the right hand side of (1.2) is sum of

$$
\begin{aligned}
&\sum_{1\le k\le h} \sum_{1\le l\le h, l\ne k} \sum_{c\notin A\setminus\{a_l\}} \delta_{ja_k}X_{a_lc} \otimes v_{A_{(k,l),(i,c)}} + \sum_{1\le k\le h} \sum_{c\notin A\setminus\{a_k\}} \delta_{ja_k}X_{ic} \otimes v_{A_{k,c}} \\
&- \sum_{1\le k\le h} \sum_{1\le l\le h, l\ne k} \sum_{c\notin A\setminus\{a_l\}} \delta_{ia_k}X_{a_lc} \otimes v_{A_{(k,l),(j,c)}} - \sum_{1\le k\le h} \sum_{c\notin A\setminus\{a_k\}} \delta_{ia_k}X_{jc} \otimes v_{A_{k,c}}
\end{aligned}
$$

and

$$
- \sum_{\substack{1\le k,l\le h \\ k\ne l}} \delta_{ja_k}X_{a_ka_l} \otimes v_{A_{l,i}} + \sum_{\substack{1\le k,l\le h \\ k\ne l}} \delta_{ia_k}X_{a_ka_l} \otimes v_{A_{l,j}}.
$$

We can check that the latter is equal to the third term of (1.3), and thus we get (1.2). □

1.2. **Capelli elements.** The Capelli elements are known to give generators of the center $Z(\mathfrak{g}_{\mathbf{C}})$ of the universal enveloping algebra $U(\mathfrak{g}_{\mathbf{C}})$ of $\mathfrak{g}_{\mathbf{C}}$ ([HU, §11]). We set $E'_{ii} = E_{ii} - (1/n)\sum_{j=1}^{n} E_{jj}$ and define an element $\mathcal{E} = (\mathcal{E}_{ij})_{1\le i,j\le n}$ of $M(n, U(\mathfrak{g}_{\mathbf{C}}))$ by

$$
\mathcal{E}_{ij} = \begin{cases} E'_{ii} - (n-2i+1)/2 & \text{if } i = j, \\ E_{ij} & \text{if } i \ne j. \end{cases}
$$

Then the *Capelli element* C_k $(2 \le k \le n)$ of degree k is defined by the following identity in a variable x:

$$
\operatorname{Det}(x \cdot E_n + \mathcal{E}) = x^n + \sum_{k=2}^{n} C_k x^{n-k}.
$$

Here Det means the vertical determinant:

$$\mathrm{Det}X = \sum_{\tau \in \ \ _n} \mathrm{sgn}(\tau) \prod_{i=1}^{n} x_{i,\tau(i)}, \quad X = (x_{ij})_{1 \le i,j \le n}.$$

2. The $(\mathfrak{g}_{\mathbf{C}}, K)$-module structure of the principal series

In this section we describe the $(\mathfrak{g}_{\mathbf{C}}, K)$-module structures around the minimal K-types of the principal series representation of G.

2.1. **The principal series representations.** We recall the definition of the principal series representations of G. Let P be a minimal parabolic subgroup of G consisting of the upper triangular matrices in G and $P = MAN$ the Langlands decomposition of P with

$$\begin{aligned} N &= \{(n_{ij}) \in G \mid n_{ii} = 1,\, n_{ij} = 0\,(i > j)\}, \\ A &= \{\mathrm{diag}(a_1, \ldots, a_n) \mid a_i > 0,\, \textstyle\prod_{i=1}^{n} a_i = 1\}, \\ M &= Z_K(A) = \{\mathrm{diag}(m_1, \ldots, m_n) \in A \mid m_i \in \{\pm 1\}\}. \end{aligned}$$

We firstly define a character of σ_I of M parametrized by a color $I = \{i_1, \ldots, i_h\}$ of type (n, h) as

$$\sigma_I(m) = \prod_{i \in I} m_i, \quad m = \mathrm{diag}(m_1, \ldots, m_n) \in M.$$

Obviously for the complement I^c of I we have $\sigma_I = \sigma_{I^c}$. Then we may assume $h \le [n/2]$. Here $[x]$ is the largest integer which does not exceed x.

Let $\mathfrak{a}$ be the Lie algebras of A. For a linear form $\nu \in \mathfrak{a}^* \otimes_{\mathbf{R}} \mathbf{C} = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}, \mathbf{C})$, we set

$$\nu_i = \nu(E_{ii}) \ \ (1 \le i \le n).$$

Then ν is identified with n-tuple of complex numbers $(\nu_1, \ldots, \nu_n)$ satisfying the condition

$$\sum_{i=1}^{n} \nu_i = 0.$$

Let ρ be the half-sum of (standard) positive roots of $(\mathfrak{g}, \mathfrak{a})$:

$$a^{\rho} = e^{\rho(\log a)} = \prod_{1 \le i < j \le n} \left(\frac{a_i}{a_j}\right)^{1/2} = \prod_{i=1}^{n-1} a_i^{n-i}, \quad a = \mathrm{diag}(a_1, \ldots, a_n) \in A.$$

Definition 2.1. Under the data above, we call the induced representation

$$\pi_{I,\nu} = \mathrm{Ind}_P^G(\sigma_I \otimes \exp(\nu + \rho) \otimes 1_N)$$

the *principal series representation* of G, and the cardinality $h = \#(I)$ the *helicity* of the principal series $\pi_{I,\nu}$.

The minimal K-type τ_{min} of the principal series $\pi_{I,\nu}$ of helicity h is $\wedge^h V_n$ if $n \ne 2h$, and two irreducible representations whose direct sum is $\wedge^h V_n$ if $n = 2h$. We describe the generators of the minimal K-type in the representation space $L^2_I(K)$, where

$$L^2_I(K) = \{f \in L^2(K) \mid f(mk) = \sigma_I(m) f(k) \text{ for } m \in M, k \in K\}.$$

This means to determine explicitly the image of the unique K-homomorphism

$$\iota : \wedge^h V_n \hookrightarrow L^2_I(K)$$

as functions on K.

Lemma 2.2. *Fix a color $I = \{i_1, \dots, i_h\}$ of type (n, h). For a sequence $A = \{a_1, \dots, a_h\}$ $(1 \le a_i \le n)$, we define a function f^I_A on K by*

$$f^I_A(k) = \det(k_{i_p,a_q})_{1\le p,q\le h}, \quad \textit{for } k = (k_{ia})_{1\le i,a\le n}.$$

Then under the K-homomorphism ι, the standard basis $\{v_A \mid A \in {}_n\mathcal{C}_h\}$ is mapped to $\{f^I_A \mid A \in {}_n\mathcal{C}_h\}$, that is, $\iota(v_A) = f^I_A$.

Proof. We have to check the compatibility of the action of K_{ij}. It is enough to show

$$\frac{d}{dt}\Big|_{t=0} f^I_A(k\exp(tK_{ij})) = \sum_{l=1}^{h}\big(\delta_{ja_l} f^I_{A_{l,i}}(k) - \delta_{ia_l} f^I_{A_{l,j}}(k)\big). \tag{2.1}$$

Since the (i_p, a_q)-component of $k\exp(tK_{ij})$ is

$$(1-\delta_{ia_q}-\delta_{ja_q})k_{i_p,a_q} + (\delta_{ia_q}+\delta_{ja_q})k_{i_p,a_q}\cos t + (-\delta_{ia_q}k_{i_p,j} + \delta_{ja_q}k_{i_p,i})\sin t,$$

the right left hand side of (2.1) is

$$\begin{aligned}&\frac{d}{dt}\Big|_{t=0}\det\big(k_{i_p,a_q}(1-\delta_{ia_q}-\delta_{ja_q}) + (-\delta_{ia_q}k_{i_p,j}+\delta_{ja_q}k_{i_p,i})t\big)_{1\le p,q\le h}\\ &= \begin{cases} f^I_{A_{l,i}}(k) & \text{if } i \notin A, j = a_l,\\ -f^I_{A_{l,j}}(k) & \text{if } i = a_l, j \notin A,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Then we can get the claim. □

2.2. **The eigenvalues of the Capelli elements.** Let $\chi_{\pi_{I,\nu}} : Z(\mathfrak{g}_{\mathbf{C}}) \to \mathbf{C}$ be the infinitesimal character of the principal series. We compute the eigenvalues $\chi_{\pi_{I,\nu}}(C_k)$ at the Capelli elements C_k.

Notation 2.3. For $x = (x_1, \dots, x_n) \in \mathbf{C}^n$ and $0 \le k \le n$, we denote by $S_k(x) = S_k(x_1, \dots, x_n)$ the elementary symmetric functions of degree k, which are defined by the identity in a variable t:

$$\prod_{i=1}^{n}(t + x_i) = \sum_{k=0}^{n} S_k(x)t^{n-k}.$$

We extend a map $\mathfrak{a}_{\mathbf{C}} \ni H \mapsto H + \rho(H) \in U(\mathfrak{a}_{\mathbf{C}})$ to the homomorphism $\gamma : Z(\mathfrak{g}_{\mathbf{C}}) \to U(\mathfrak{a}_{\mathbf{C}})$. From the definition of Capelli elements, we have

$$\begin{aligned}x^n + \sum_{k=2}^{n}\gamma(C_k)x^{n-k} &= \gamma(\mathrm{Det}(\mathrm{diag}(x + E'_{11}, \dots, x + E'_{nn})))\\ &= x^n + \sum_{k=2}^{n} S_k(E'_{11}, \dots, E'_{nn})x^{n-k}.\end{aligned}$$

Since $\nu(E'_{ii}) = \nu_i$, we can get the following.

Lemma 2.4. *We have $\chi_{\pi_{I,\nu}}(C_k) = S_k(\nu)$ for $2 \le k \le n$.*

2.3. **The Dirac-Schmid equations.** In Proposition 1.3 we have described the injection $\tau_{min} \ni v_A \mapsto v'_A \in \mathfrak{p}_{\mathbf{C}} \otimes \tau_{min}$. Consider the composite of this injection and a canonical surjection

$$\tau_{min} \hookrightarrow \mathfrak{p}_{\mathbf{C}} \otimes \tau_{min} \to \pi_{I,\nu}(\mathfrak{p}_{\mathbf{C}})\tau_{min} \subset L^2_I(K).$$

Since $\tau_{\min}$ occurs with multiplicity one in $\pi_{I,\nu}|_K$, this is a scalar multiple of $\tau_{min} \hookrightarrow \pi_{I,\nu}|_K$. Therefore Proposition 1.3 implies that

$$\sum_{k=1}^{h} \pi_{I,\nu}(X_{a_k a_k}) f^I_A + \sum_{k=1}^{h} \sum_{c \notin A} \pi_{I,\nu}(X_{a_k c}) f^I_{A_{k,c}} = \lambda f^I_A$$

with some constant λ. We call this equation *Dirac-Schmid equation*. Let us determine the constant λ by comparing the values at the identity e in G (cf. [MIO, §3.2]).

Lemma 2.5. *We have the Dirac-Schmid equation*

$$\sum_{k=1}^{h} \pi_{I,\nu}(X_{a_k a_k}) f^I_A + \sum_{k=1}^{h} \sum_{c \notin A} \pi_{I,\nu}(X_{a_k c}) f^I_{A_{k,c}} = 2\Big(\sum_{i \in I} \nu_i\Big) f^I_A.$$

Proof. Since $f^I_A(e) = 1$ (if $A = I$), 0 (if $A \neq I$), we have

$$\begin{aligned} \pi_{I,\nu}(X_{a_k a_k}) f^I_A(e) &= 2\big(\pi_{I,\nu}(E_{a_k a_k} - \tfrac{1}{n} E_{nn})\big) f^I_A(e) \\ &= 2\Big(\nu_{a_k} + n - a_k - \frac{1}{n}\sum_{i=1}^{n}(n-i)\Big) f^I_A(e) \\ &= \begin{cases} 2(\nu_{i_k} - i_k) + (n+1) & \text{if } A = I, \\ 0 & \text{if } A \neq I, \end{cases} \end{aligned}$$

and, for $c \neq a_k$,

$$\begin{aligned} \pi_{I,\nu}(X_{a_k c}) f^I_{A_{k,c}}(e) &= \begin{cases} \pi_{I,\nu}(K_{c a_k}) f^I_{A_{k,c}}(e) & \text{if } a_k < c, \\ \pi_{I,\nu}(K_{a_k c}) f^I_{A_{k,c}}(e) & \text{if } a_k > c \end{cases} \\ &= \begin{cases} -1 & \text{if } A = I \text{ and } a_k < c, \\ 1 & \text{if } A = I \text{ and } a_k > c, \\ 0 & \text{if } A \neq I. \end{cases} \end{aligned}$$

Then we get

$$\begin{aligned} \lambda &= 2\sum_{k=1}^{h}(\nu_{i_k} - i_k) + h(n+1) + \sum_{c \notin I}\sum_{k=1}^{h}\Big(\sum_{i_k < c}(-1) + \sum_{i_k > c} 1\Big) \\ &= 2\sum_{k=1}^{h}(\nu_{i_k} - i_k) + h(n+1) + \sum_{k=1}^{h}\{(-1)(n - i_k - h + k) + (i_k - k)\} \\ &= 2\sum_{k=1}^{h} \nu_{i_k} \end{aligned}$$

as desired. □

3. The holonomic system for Whittaker functions

In this section we derive a holonomic system for the radial parts of Whittaker functions.

3.1. **Whittaker models and Whittaker functions.** Let us recall the notion of Whittaker models. A unitary character η of the maximal unipotent subgroup N of G is written as

$$\eta(x) = \exp\Big(2\pi\sqrt{-1}\sum_{i=1}^{n-1} c_i x_{i,i+1}\Big), \quad x = (x_{ij})_{1\le i,j\le n} \in N,$$

with real numbers c_i $(1 \le i \le n-1)$. In this paper we assume that the character η is non-degenerate, that is, $c_i \neq 0$ for $1 \le i \le n-1$.

The *Whittaker model* of an irreducible smooth admissible representation π of G is a realization of π in the C^∞-induced module

$$C_\eta^\infty(N\backslash G) = \{\varphi : G \to \mathbf{C},\, C^\infty \mid \varphi(xg) = \eta(x)\varphi(g) \text{ for all } (x,g) \in N \times G\},$$

on which G acts by right translation. We put

$$\mathcal{W}(\pi,\eta) = \mathrm{Hom}_{(\ \mathbf{C},K)}(\pi, C_\eta^\infty(N\backslash G)),$$

and $\mathcal{W}(\pi,\eta)^{\mathrm{mod}}$ the subspace of $\mathcal{W}(\pi,\eta)$ consisting of the intertwining operators whose images are moderate growth functions ([Wa, §8]).

For $\Phi \in \mathcal{W}(\pi,\eta)$ and a smooth vector $v \in \pi$, we call the image $\Phi(v)$ the *Whittaker function* (corresponding to v). Especially, for $\Phi \in \mathcal{W}(\pi,\eta)^{\mathrm{mod}}$, we call $\Phi(v)$ the *primary Whittaker function.* From now on we consider Whittaker functions corresponding to the vectors belonging to minimal K-types of the principal series.

Definition 3.1. Let Φ be a nonzero intertwiner in $\mathcal{W}(\pi_{I,\nu},\eta)$. Under the notation as in Lemma 2.2, we denote by W_A $(A \in {}_n\mathcal{C}_h)$ the Whittaker function corresponding to the function f_A^I:

$$W_A = \Phi(f_A^I) = \Phi(\iota(v_A)), \quad A \in {}_n\mathcal{C}_h.$$

Because of the Iwasawa decomposition $G = NAK$, Whittaker function is determined by its restriction to the vector subgroup A of G, which we call the *radial part* of Whittaker functions.

3.2. **The holonomic system for the Whittaker functions.** To write differential equations satisfied by the radial parts of Whittaker functions, we introduce new coordinates $y = (y_1,\ldots,y_{n-1})$ on A by $y_i = a_i/a_{i+1}$. Then we can see that $a^\rho = y^\rho = \prod_{i=1}^{n-1} y_i^{i(n-i)/2}$. By using this coordinates we write the actions of E'_{ii}, E_{ij} and K_{ij} on Whittaker functions.

Lemma 3.2. (i) *Let ∂_i be the Euler operator $y_i\frac{\partial}{\partial y_i}$ for each i $(1 \le i \le n-1)$. Then for $f \in C^\infty(A)$, we have*

$$E'_{ii} f = (-\partial_{i-1} + \partial_i) f.$$

Here we understand $\partial_0 = \partial_n = 0$.

(ii) *For $f \in C_\eta^\infty(N\backslash G)$, we have*

$$E_{ij} f(y) = \begin{cases} 0 & \text{if } j-i>1, \\ 2\pi\sqrt{-1}c_i y_i f(y) & \text{if } j = i+1. \end{cases}$$

(iii) *For Whittaker function W_A corresponding to f_A^I, we have*

$$K_{ij} W_A = \sum_{t=1}^{h} (\delta_{ja_t} W_{A_{t,i}} - \delta_{ia_t} W_{A_{t,j}}).$$

Proof. See [MIO, §4.1] for (i) and (ii). The last claim (iii) immediately follows from $K_{ij}(\mathcal{W}(f_A^I)) = \mathcal{W}(\iota(K_{ij}v_A))$. □

Theorem 3.3. *Let $\{W_A(y) = y^\rho \widetilde{W}_A(y)\}_{A\in{}_n\mathcal{C}_h}$ be the radial part of Whittaker function for $\pi_{I,\nu}$. Then the set $\{\widetilde{W}_A(y)\}_{A\in{}_n\mathcal{C}_h}$ satisfies the following system of difference-differential equations.*

(i) *(Dirac-Schmid equation) If we denote by $A_t^{\pm} = A_{t,a_t\pm 1}$ then we have*

$$\sum_{t=1}^{h}(\partial_{a_t} - \partial_{a_t-1})\widetilde{W}_A + \sum_{t=1}^{h}\Big(2\pi\sqrt{-1}c_{a_t}y_{a_t}\widetilde{W}_{A_t^+} + 2\pi\sqrt{-1}c_{a_t-1}y_{a_t-1}\widetilde{W}_{A_t^-}\Big) = \Big(\sum_{i\in I}\nu_i\Big)\widetilde{W}_A.$$

Here we understand $\widetilde{W}_{A_t^{\pm}} = 0$ if $a_t \pm 1 = a_{t\pm 1}$.

(ii) *(Capelli equation) For a variable x, we have*

$$\mathrm{Det}(x\cdot E_n + \mathcal{C})\cdot \widetilde{W}_A(y) = \prod_{i=1}^{n}(x+\nu_i)\cdot \widetilde{W}_A(y).$$

Here $\mathcal{C} = (\mathcal{C}_{ij})_{1\le i,j\le n}$ is a operator matrix where each $\mathcal{C}_{ij}$ acts on a set of smooth functions $\{F_A(y)\}_{A\in{}_n\mathcal{C}_h}$ on $\mathbf{R}_+^{n-1}$ by

$$\mathcal{C}_{ij}F_A(y) = \begin{cases} 0 & \text{if } j\ge i+2,\\ 2\pi\sqrt{-1}c_iy_iF_A(y) & \text{if } j = i+1,\\ (-\partial_{i-1}+\partial_i)F_A(y) & \text{if } j=i,\\ 2\pi\sqrt{-1}c_{i-1}y_{i-1}F_A(y) - \mathcal{K}_{i-1,i}F_A(y) & \text{if } j = i-1,\\ -\mathcal{K}_{ji}F_A(y) & \text{if } j\le i-2. \end{cases}$$

Here $\mathcal{K}_{ij}$ $(1\le i,j\le n)$ is a operator defined by

$$\mathcal{K}_{ij}F_A = \sum_{t=1}^{h}\big(\delta_{ja_t}F_{A_{t,i}} - \delta_{ia_t}F_{A_{t,j}}\big).$$

Proof. In view of

$$X_{a_kc} = \begin{cases} 2E'_{a_ka_k} & \text{if } c = a_k,\\ 2E_{a_kc} + K_{a_kc} & \text{if } a_k < c,\\ 2E_{ca_k} + K_{a_kc} & \text{if } a_k > c, \end{cases}$$

we can get the Dirac-Schmid equation from Lemmas 2.5 and 3.2. By expanding the $\mathrm{Det}(x\cdot E_n + \mathcal{E})$ with respect to the first row we reach the Capelli equation by indunction on n. □

To write the Capelli equations more explicitly, let us compute $\mathrm{Det}(x\cdot E_n + \mathcal{C})$.

Notation 3.4. For $y = (y_1,\dots,y_{n-1})$, $z = (z_1,\dots,z_n)$ and $\kappa = (\kappa_{ij})_{1\le i<j\le n}$, we define a polynomial $\Delta_{n,k}(y,z,\kappa)$ by the identity

$$\det\begin{pmatrix} x+z_1 & y_1 & & & \\ y_1-\kappa_{12} & x+z_2 & y_2 & & \\ -\kappa_{13} & y_2-\kappa_{23} & x+z_3 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\kappa_{1n} & -\kappa_{2n} & \cdots & y_{n-1}-\kappa_{n,n-1} & x+z_n \end{pmatrix} = \sum_{k=0}^{n} x^{n-k}\Delta_{n,k}(y,z,\kappa).$$

Here is the explicit formulas of $\Delta_{n,k}$.

Lemma 3.5. (i) *We have*

$$\Delta_{n,k}(y,z,0)=\sum_{q=0}^{[k/2]}\Big\{\sum_{\{l_1,\dots,l_q\}}\Big(\prod_{t=1}^{q}(-y_{l_t}^2)\Big)S_{k-2q}(z_{l'_1},\dots,z_{l'_{n-2q}})\Big\},$$

where $\{l_1,\dots,l_q\}$ *ranges all the colors of type* $(n-1,q)$ *satisfying the condition* $l_{t+1}-l_t>1$ *for all* $1\le t\le q-1$, *and* $\{l'_1\dots,l'_{n-2q}\}$ *is the complement of* $\{l_1,l_1+1,l_2,l_2+1,\dots,l_q,l_q+1\}$ *in* T_n.

(ii) *We have*

$$\Delta_{n,k}(y,z,\kappa)=\sum_{r=0}^{[n/2]}\Delta_{n,k,r}(y,z,\kappa),$$

where the polynomial $\Delta_{n,k,r}(y,z,\kappa)$ *is written by using* $\Delta_{n',k'}(y,z,0)$ *as*

$$\sum_{\substack{\{i_1,\dots,i_r\}\\ \{j_1,\dots,j_r\}}}\Bigg[\prod_{t=1}^{r}\kappa_{i_tj_t}\cdot\prod_{t=1}^{r}(-1)^{i_t+j_t+1}(y_{i_t}y_{i_t+1}\cdots y_{j_t-1})$$
$$\cdot\sum_{\{k_1,\dots,k_{r+1}\}}\prod_{t=1}^{r+1}\Delta_{i_t-j_{t-1}-1,k_t}\big((y_{j_{t-1}+1},\dots,y_{i_t-2}),(z_{j_{t-1}+1},\dots,z_{i_t-1}),0\big)\Bigg].$$

Here the sets of positive integers $\{i_1,\dots,i_r\}$, $\{j_1,\dots,j_r\}$ *and* $\{k_1,\dots,k_{r+1}\}$ *range with the conditions* $1\le i_1<j_1<i_2<j_2<\cdots<i_r<j_r\le n$ *and* $\sum_{t=1}^{r+1}k_t=k-r+\sum_{t=1}^{r}(i_t-j_t)$. *Note that we understand* $j_0=0$ *and* $i_{r+1}=n+1$.

Proof. Exercise of linear algebra. □

By using $\Delta_{n,k,r}$, we can rewrite the Capelli equations in Theorem 3.3.

Proposition 3.6. *For* $2\le k\le n$, *we have*

$$\Bigg[\sum_{r=0}^{h}\Delta_{n,k,r}\big((2\pi\sqrt{-1}c_1y_1,\dots,2\pi\sqrt{-1}c_{n-1}y_{n-1}),$$
$$(\partial_1,-\partial_1+\partial_2,\dots,-\partial_{n-2}+\partial_{n-1},-\partial_{n-1}),(\mathcal{K}_{ij})_{1\le i<j\le n}\big)\Bigg]\widetilde{W}_A(y)=S_k(\nu)\widetilde{W}_A(y).$$

Especially we have the Casimir equation:

$$\Bigg[-\sum_{p=1}^{n-1}\partial_p^2+\sum_{p=1}^{n-2}\partial_p\partial_{p+1}-\sum_{p=1}^{n-1}(2\pi\sqrt{-1}c_py_p)^2\Bigg]\widetilde{W}_A(y)$$
$$-\sum_{t=1}^{h}(2\pi\sqrt{-1}c_{a_t}y_{a_t})\widetilde{W}_{A_t^+}(y)+\sum_{t=1}^{h}(2\pi\sqrt{-1}c_{a_t-1}y_{a_t-1})\widetilde{W}_{A_t^-}(y)=S_2(\nu)\widetilde{W}_A(y).$$

Proof. We note that $\mathcal{K}_{ij}$ commutes $\mathcal{K}_{i'j'}$ for $1\le i<j<i'<j'\le n$, and moreover $(\mathcal{K}_{i_1j_1}\cdots\mathcal{K}_{i_rj_r})\widetilde{W}_A(y)=0$ for $h+1\le r\le[n/2]$. □

Remark 1. We remark a rank of holonomic system in Theorem 3.3. The system consisting of the Capelli equations (ii), or Proposition 3.6 gives a holonomic system of rank $n!\cdot\binom{n}{h}$. For $h\ge1$, this system is divided into $\binom{n}{h}$ subsystems by using the Dirac-Schmid equations and so that each subsystem is of rank $n!$, the order of the Weyl group $\mathfrak{S}_n$.

4. Review of the case of helicity 0

The rest of this paper we will give explicit formulas for the *secondary Whittaker functions*, that is, the power series solutions of the system in Theorem 3.3 at the regular singularity $y = (0, \dots, 0)$, and the *primary Whittaker functions*, that is, the unique moderate growth Whittaker functions. Throughout this paper we assume $c_i = 1$ for the character η for simplicity. We also assume that the parameter ν is in a general position, more precisely,

$$\nu_i - \nu_j \notin \mathbf{Z}, \quad (i \neq j).$$

Since our explicit formulas will be recursive formulas with respect to the rank n as in [IS] (we call such formulas *propagation formulas*), we relate principal series representations of $SL(n, \mathbf{R})$ and $SL(n-1, \mathbf{R})$.

Definition 4.1. For the principal series $\pi_{I,\nu}$ of $SL(n, \mathbf{R})$ with $I = \{i_1, \dots, i_h\} \in {}_n\mathcal{C}_h$ and $\nu = (\nu_1, \dots, \nu_n) \in \mathbf{C}^n$ ($\sum_{p=1}^n \nu_p = 0$), we associate a principal series $\pi_{I_0, \tilde{\nu}}$ of $SL(n-1, \mathbf{R})$ by

- $I_0 = \{i_{0,1}, \dots, i_{0,h-1}\} \in {}_{n-1}\mathcal{C}_{h-1}$ with $i_{0,p} = i_{p+1} - 1$ $(1 \leq p \leq n-1)$,
- $\tilde{\nu} = (\tilde{\nu}_1, \dots, \tilde{\nu}_{n-1})$ with

$$\tilde{\nu}_p = \begin{cases} \nu_p + \frac{\nu_{i_1}}{n-1} & \text{if } 1 \leq p \leq i_1 - 1, \\ \nu_{p+1} + \frac{\nu_{i_1}}{n-1} & \text{if } i_1 \leq p \leq n-1. \end{cases}$$

In the case of $I = \emptyset$ $(h = 0)$, we set $I_0 = \emptyset$ and $\tilde{\nu}_p = \nu_{p+1} + \nu_1/(n-1)$ $(1 \leq p \leq n)$.

Whittaker functions have symmetries under the action of the Weyl group. The Weyl group of the restricted root system of $(\mathfrak{g}, \mathfrak{a})$ is the symmetric group $\mathfrak{S}_n$, and each element τ in $\mathfrak{S}_n$ acts on $\nu = (\nu_1, \dots, \nu_n)$ as a permutation of ν_i:

$$\tau\nu = (\nu_{\tau(1)}, \dots, \nu_{\tau(n)}).$$

Notation 4.2. For $I \in {}_n\mathcal{C}_h$, let us define a subgroup $\mathfrak{S}_I$ of the Weyl group $\mathfrak{S}_n$ by

$$\mathfrak{S}_I = \{\tau \in \mathfrak{S}_n \mid \tau(I) = I\} \cong \mathfrak{S}_h \times \mathfrak{S}_{n-h}.$$

Here we regard I as a subset of T_n, that is, $\tau(I) = I$ means that for each $i_t \in I$, there exists $i_{t'} \in I$ such that $\tau(i_t) = i_{t'}$.

We collect some conventions which will be used hereafter.

Notation 4.3.

- For $a \in \mathbf{C}$ and $n \in \mathbf{Z}$, we denote by $(a)_n = \Gamma(a+n)/\Gamma(a)$ (Pochhammer symbol).
- We use symbols $\mathbf{m} = (m_1, \dots, m_{n-1}) \in \mathbf{N}^{n-1}$ and $\mathbf{k} = (k_1, \dots, k_{n-2}) \in \mathbf{N}^{n-2}$.
- We denote by $\mathbf{e}_p$ (resp. $\mathbf{d}_p$) the p-th standard basis in $\mathbf{R}^{n-1}$ (resp. $\mathbf{R}^{n-2}$).
- For $(x_1, \dots, x_{n-1}) \in \mathbf{C}^{n-1}$, $\mathbf{m} \in \mathbf{N}^{n-1}$, $\mathbf{k} \in \mathbf{N}^{n-2}$ and $\kappa \in \mathbf{C}$, we set

$$Q(x_1, \dots, x_{n-1}) = \sum_{p=1}^{n-1} x_p^2 - \sum_{p=1}^{n-2} x_p x_{p+1},$$

$$Q_{\mathbf{m}}(x_1, \dots, x_{n-1}; \kappa) = Q(m_1, \dots, m_{n-1}) + \sum_{p=1}^{n-1} x_p m_p + \kappa,$$

$$Q_{\mathbf{k}}(x_1, \dots, x_{n-2}; \kappa) = Q(k_1, \dots, k_{n-2}) + \sum_{p=1}^{n-2} x_p k_p + \kappa.$$

- For a subset X of $T_n = \{1, \ldots, n\}$ we denote by χ_X the characteristic function of X. If X is empty set then χ_X is identically equal to zero.
- We agree that the interval $[a, b]$ means its intersection with the set of nonnegative integers: $[a, b] = \{x \in \mathbf{N} \mid a \leq x \leq b\}$.
- For a subset (or a color) X of T_n we write $X[\pm 1] := \{x \pm 1 \mid x \in X\}$.
- For a color $X = \{x_1, \ldots, x_h\}$ of type (n, h), we denote by $X^c = \{x'_1, \ldots, x'_{n-h}\}$ $(x'_1 < \cdots < x'_{n-h})$ the complement color of X in T_n.
- For a color $X = \{x_1, \ldots, x_h\}$ of type (n, h), we use a abbreviation $X_t^{\pm} = X_{t, x_t \pm 1} = \{x_1, \ldots, x_{t-1}, x_t \pm 1, x_{t+1}, \ldots, x_h\}$ (cf. Notation 1.2).

In this section we recall the explicit formulas for Whittaker functions for the class one principal series representations obtained by Stade and Ishii [IS].

4.1. **Secondary Whittaker functions.** Let us start with the secondary Whittaker functions. We have no Dirac-Schmid equations. The k-th $(2 \leq k \leq n)$ Capelli equation in Proposition 3.6 is

$$\begin{aligned}
&\Big[S_k(\partial_1, -\partial_1 + \partial_2, \ldots, -\partial_{n-2} + \partial_{n-1}, -\partial_{n-1}) \\
&+ \sum_{q=1}^{[k/2]} \sum_{\{l_1, \ldots, l_q\}} \Big\{ \prod_{t=1}^{q} (2\pi y_{l_t})^2 \cdot \sum_{\{l'_1, \ldots, l'_{k-2q}\}} \prod_{s=1}^{k-2q} (-\partial_{l'_s - 1} + \partial_{l'_s}) \Big\} \Big] \widetilde{W}(y) = S_k(\nu) \widetilde{W}(y).
\end{aligned} \tag{4.1}$$

Here $\{l_1, \ldots, l_q\}$ ranges all the colors of type $(n-1, q)$ satisfying $l_{t+1} - l_t > 1$ for all $1 \leq t \leq q-1$, and $\{l'_1, \ldots, l'_{k-2q}\}$ runs through all the colors of type $(n, k-2q)$ such that $l'_s \in T_n \backslash \{l_t, l_t + 1 \mid 1 \leq t \leq q\}$ for all $1 \leq s \leq k - 2q$.

Here is the propagation formula for the secondary Whittaker functions.

Theorem 4.4. [IS, Theorem 15] *We introduce a power series $M_\nu^n(y) = y^\rho \widetilde{M}_\nu^n(y)$ with*

$$\widetilde{M}_\nu^n(y) = \sum_{\mathbf{m} = (m_1, \ldots, m_{n-1}) \in \mathbf{N}^{n-1}} C_{\mathbf{m}}^n(\nu) \prod_{p=1}^{n-1} (\pi y_p)^{2m_p + \nu_1 + \cdots + \nu_p},$$

where the coefficients are recursively determined by the relation

$$C_{\mathbf{m}}^n(\nu) = \sum_{\mathbf{k} = (k_1, \ldots, k_{n-2})} \frac{C_{\mathbf{k}}^{n-1}(\tilde{\nu})}{\prod_{p=1}^{n-1} (m_p - k_{p-1})! (\frac{\nu_1 - \nu_{p+1}}{2} + 1)_{m_p - k_p}}$$

and $C_{m_1}^2(\nu) = 1/\{m_1! (\frac{\nu_1 - \nu_2}{2} + 1)_{m_1}\}$. Here we set $k_0 = k_{n-1} = 0$ and $\sum_{\mathbf{k}}$ means $\mathbf{k} = (k_1, \ldots, k_{n-2})$ runs through such that $0 \leq k_p \leq m_{p+1}$ for $1 \leq p \leq n-2$. Then the set of power series

$$\{\widetilde{M}_{\tau\nu}^n(y) \mid \tau \in \mathfrak{S}_n\}$$

forms a basis of the solutions of the system (4.1) at $y = 0$.

Remark 2. According to the result of Hashizume, secondary Whittaker functions are essentially determined by the Casimir equation. Actually Theorem 4.4 can be shown by the recurrence relation

$$\Big(\sum_{p=1}^{n-1} m_p^2 - \sum_{p=1}^{n-2} m_p m_{p+1} + \sum_{p=1}^{n-1} \frac{\nu_p - \nu_{p+1}}{2} m_p \Big) C_{\mathbf{m}}^n(\nu) = \sum_{p=1}^{n-1} C_{\mathbf{m} - \mathbf{e}_p}^n(\nu). \tag{4.2}$$

4.2. **Primary Whittaker functions.** Here is the propagation formula for the primary Whittaker functions.

Theorem 4.5. [IS, Theorems 12, 14] *We inductively define a function* $W_\nu^n(y) = y^\rho \widetilde{W}_\nu^n(y)$ *by*

$$
\begin{aligned}
\widetilde{W}_\nu^n(y) = &\int_{(\mathbf{R}_+)^{n-1}} \widetilde{W}_{\tilde{\nu}}^{n-1}\left(y_2\sqrt{\frac{t_2}{t_1}}, \dots, y_{n-1}\sqrt{\frac{t_{n-1}}{t_{n-2}}}\right) \\
&\cdot \prod_{p=1}^{n-1} \exp\Big\{-(\pi y_p)^2 t_p - \frac{1}{t_p}\Big\} \prod_{p=1}^{n-1} (\pi y_p)^{\frac{(n-p)\nu_1}{n-1}} \prod_{p=1}^{n-1} t_p^{\frac{n\nu_1}{2(n-1)}} \frac{dt_p}{t_p},
\end{aligned}
\tag{4.3}
$$

and $\widetilde{W}_\nu^2(y_1) = 2K_{(\nu_1-\nu_2)/2}(2\pi y_1)$. *Here* K_ν *is the modified Bessel function:*

$$
K_\nu(2\pi y) = \frac{1}{2}(\pi y)^\nu \int_0^\infty \exp\Big\{-(\pi y)^2 t - \frac{1}{t}\Big\} t^\nu \frac{dt}{t}.
$$

Then, up to a constant multiple, $W_\nu^n(y)$ *is the radial part of the primary Whittaker function of helicity* 0. *Moreover, let*

$$
V_\nu^n(s) \equiv V_\nu^n(s_1, \dots, s_{n-1}) = \int_{(\mathbf{R}_+)^{n-1}} \widetilde{W}_\nu^n(y) \prod_{p=1}^{n-1} (\pi y_p)^{s_p} \frac{dy_p}{y_p}
$$

be the Mellin transform of $\widetilde{W}_\nu^n(y)$. *Then we have*

$$
\begin{aligned}
V_\nu^n(s) = &\frac{2^{1-n}}{(2\pi i)^{n-2}} \int_{z_1,\dots,z_{n-2}} V_{\tilde{\nu}}^{n-1}(z_1, \dots, z_{n-2}) \\
&\cdot \prod_{p=1}^{n-1} \Big\{ \Gamma\Big(\frac{s_p - z_{p-1}}{2} + \frac{(n-p)\nu_1}{2(n-1)}\Big) \Gamma\Big(\frac{s_p - z_p}{2} - \frac{p\nu_1}{2(n-1)}\Big) \Big\} \prod_{p=1}^{n-2} dz_p
\end{aligned}
\tag{4.4}
$$

and $V_\nu^2(s_1) = 2^{-1}\Gamma((s_1+\nu_1)/2)\Gamma((s_1-\nu_1)/2)$. *Here the paths of integration in each* z_p *is a vertical line in the complex plane with sufficiently large real part to keep the poles of integrand on its left, and* $z_0 = z_{n-1} = 0$.

The proof of this theorem rely on Stade's evaluation of the Jacquet integral ([St1]) and a recursive formula between V_ν^n and V_ν^{n-2} ([St2]). Actually we first find (4.4) and then arrive at (4.3) by using the Mellin-Barnes integral representation of the modified Bessel function.

Our argument in this paper is different. In the later section, we give new proof based on the *factorization formula* for the class one principal series Whittaker functions ([Ha]). In the case of $SL(n, \mathbf{R})$, the explicit factorization formula is

$$
W_\nu^n(y) = \sum_{\tau \in \ _n} \tau \Big[\prod_{1 \le p < q \le n} \Gamma\Big(\frac{-\nu_p + \nu_q}{2}\Big) \cdot M_\nu^n(y) \Big].
\tag{4.5}
$$

We note that another approach to similar propagation formula for the primary Whittaker functions of helicity 0 is discussed in [GKLO] by using stationary phase integrals called Givental integrals ([Gi]).

5. Secondary Whittaker functions (1) -the case of helicity 1-

Before to discuss the case of general helicity h, to make our idea more understandable we explain the case of $I = \{i\}$ $(h = 1)$ in this section. We express the secondary Whittaker functions on $SL(n, \mathbf{R})$ with helicity 1 in terms of those on $SL(n-1, \mathbf{R})$ with helicity 0 (Definition 5.6). Our main result is Theorem 5.7, which tells that the power series defined in Definition 5.6 forms a basis of the solution of Theorem 3.3. To prove this, we will confirm the power series is compatible with the Dirac-Schmid equations and the Casimir equations in Subsections 5.5 and 5.6, respectively.

5.1. **The recurrence relations.** We first rewrite the system in Theorem 3.3. For $A = \{a\}$ $(1 \le a \le n)$ we denote by $A^{\pm} = A[\pm 1] = \{a \pm 1\}$. If we put $\widetilde{W}_A(y) = (\sqrt{-1})^{-a}\Phi_A(y)$, then the Dirac-Schmid equation in Theorem 3.3 (i) and the Casimir equation in Proposition 3.6 read

$$(-\partial_{a-1} + \partial_a - \nu_i)\Phi_A(y) - 2\pi y_{a-1}\Phi_{A^-}(y) + 2\pi y_a \Phi_{A^+}(y) = 0$$

and

$$\begin{aligned}&\Big\{-\sum_{p=1}^{n-1}\partial_p^2 + \sum_{p=1}^{n-2}\partial_p\partial_{p+1} + \sum_{p=1}^{n-1}(2\pi y_p)^2 - S_2(\nu)\Big\}\Phi_A(y)\\ &- 2\pi y_{a-1}\Phi_{A^-}(y) - 2\pi y_a\Phi_{A^+}(y) = 0,\end{aligned}$$

respectively. As in the case of helicity 0, we consider a vector-valued power series solution of the form

$$\Phi(y) = \{\Phi_{\{a\}}(y)\}_{1\le a\le n} = \sum_{\mathbf{m}=(m_1,\dots,m_{n-1})} C_{\mathbf{m}}(\Phi)\prod_{p=1}^{n-1}(\pi y_p)^{m_p+l_p},$$

where the coefficient $C_{\mathbf{m}}(\Phi) = \{C_{\{1\};\mathbf{m}}(\Phi), \dots, C_{\{n\};\mathbf{m}}(\Phi)\}$ satisfies the non-vanishing condition $C_{\mathbf{m}}(\Phi) \ne \mathbf{0} = (0, \dots, 0)$. We first write recurrence relations for $C_{\mathbf{m}}(\Phi)$.

Lemma 5.1. (i) *The Dirac-Schmid equation implies*

$$\{-(m_{a-1} + l_{a-1}) + (m_a + l_a) - \nu_i\}C_{A;\mathbf{m}}(\Phi) - 2C_{A^-;\mathbf{m}-\mathbf{e}_{a-1}}(\Phi) + 2C_{A^+;\mathbf{m}-\mathbf{e}_a}(\Phi) = 0.$$

(ii) *The Casimir equation implies*

$$\begin{aligned}&\Big\{\sum_{p=1}^{n-1}(m_p + l_p)^2 - \sum_{p=1}^{n-2}(m_p + l_p)(m_{p+1} + l_{p+1}) + S_2(\nu)\Big\}C_{A;\mathbf{m}}(\Phi)\\ &+ 2C_{A^-;\mathbf{m}-\mathbf{e}_{a-1}}(\Phi) + 2C_{A^+;\mathbf{m}-\mathbf{e}_a}(\Phi) = 4\sum_{p=1}^{n-1}C_{A;\mathbf{m}-2\mathbf{e}_p}(\Phi).\end{aligned}$$

5.2. **The coefficients with total degrees at most one.** We want to determine the coefficients $C_{\mathbf{m}}(\Phi)$ with the total degree $|\mathbf{m}| := |\sum_{p=1}^{n-1} m_p|$ at most one. From the recurrence relations in Lemma 5.1 we can prove the following:

Proposition 5.2. *The characteristic indices $l = (l_1, \dots, l_{n-1})$ at $y = (0, \dots, 0)$ are given by*

$$l_p = \nu_{\tau(1)} + \cdots + \nu_{\tau(p)} \quad (1 \le p \le n-1)$$

for some $\tau \in \mathfrak{S}_n$. *If we put* $j = \tau^{-1}(i)$ *then we may choose* $C_{\{a\};\mathbf{0}}(\Phi) = \delta_{a,j}$ *and*

$$C_{\{a\};\mathbf{e}_a}(\Phi) = -\frac{2}{\nu_{\tau(a)} - \nu_{\tau(a+1)} + 1} \cdot \delta_{a,j-1},$$
$$C_{\{a\};\mathbf{e}_{a-1}}(\Phi) = -\frac{2}{\nu_{\tau(a-1)} - \nu_{\tau(a)} + 1} \cdot \delta_{a,j+1},$$
$$C_{\{a\};\mathbf{e}_p}(\Phi) = 0 \quad \text{if } p \neq a-1, a.$$

Proof. Let us substitute $\mathbf{m} = \mathbf{0}$ in Lemma 5.1. The Dirac-Schmid equation gives

$$(-l_{p-1} + l_p - \nu_i)C_{\{p\};\mathbf{0}}(\Phi) = 0 \quad (1 \le p \le n).$$

Here $l_0 = l_n = 0$. From the Capelli equations, we have

$$S_k(l_1, -l_1 + l_2, \dots, -l_{n-1} + l_n, -l_n)C_{\{p\};\mathbf{0}}(\Phi) = S_k(\nu)C_{\{p\};\mathbf{0}}(\Phi) \quad (1 \le p \le n)$$

for $2 \le k \le n$. Since $C_{\{p\};\mathbf{0}}(\Phi)$ does not vanish for some p, we have

$$-l_{p-1} + l_p - \nu_i = 0,$$
$$S_k(l_1, -l_1 + l_2, \dots, -l_{n-1} + l_n, -l_n) = S_k(\nu).$$

Then we have

$$l_p - l_{p-1} = \nu_{\tau(p)}, \quad 1 \le p \le n$$

for some $\tau \in \mathfrak{S}_n$. Hence $(-l_{a-1} + l_a - \nu_i)C_{\{a\};\mathbf{0}}(\Phi) = 0$, that is, $(\nu_{\tau(a)} - \nu_{\tau(j)})C_{\{a\};\mathbf{0}}(\Phi) = 0$. Then we may set $C_{\{a\};\mathbf{0}}(\Phi) = \delta_{a,j}$.

Similarly we substitute $\mathbf{m} = \mathbf{e}_p$ to find

$$(1 + \nu_{\tau(a)} - \nu_{\tau(j)})C_{\{a\};\mathbf{e}_a}(\Phi) + 2C_{\{a+1\};\mathbf{0}}(\Phi) = 0;$$
$$(-1 + \nu_{\tau(a)} - \nu_{\tau(j)})C_{\{a\};\mathbf{e}_{a-1}}(\Phi) - 2C_{\{a-1\};\mathbf{0}}(\Phi) = 0;$$
$$(\nu_{\tau(a)} - \nu_{\tau(j)})C_{\{a\};\mathbf{e}_p}(\Phi) = 0 \ (p \neq a, a-1).$$

Hence we have the assertion for the coefficients with total degrees 1 except for the proof of $C_{\{a\};\mathbf{e}_p}(\Phi) = 0$ for $a = j$ and $p \neq a-1, a$. To see this we substitute $\mathbf{m} = \mathbf{e}_p$ $(p \neq a-1, a)$ into Lemma 5.1 (ii):

$$(2l_p - l_{p-1} - l_{p+1} + 1)C_{\{a\};\mathbf{e}_p}(\Phi) = 0.$$

□

Corollary 5.3. *For two colors* $A = \{a\}$ *and* $J = \{j\}$ *of types* $(n, 1)$, *we define a function* ε_A^J *on* T_{n-1} *by*

$$\varepsilon_A^J = \chi_{[1,j-1]} \cdot \chi_{[a,n-1]} + \chi_{[j,n-1]} \cdot \chi_{[1,a-1]}.$$

Fix $\tau \in \mathfrak{S}_n$. *The power series solution corresponding to the characteristic indices* $l = (l_1, \dots, l_{n-1})$ *with* $l_p = \nu_{\tau(1)} + \cdots + \nu_{\tau(p)}$ *is of the form*

$$\Phi_{\{a\}}^{\{\tau^{-1}(i)\}}(y) = \sum_{\mathbf{m}} C_{\{a\};\mathbf{m}} \prod_{p=1}^{n-1} (\pi y_p)^{2m_p + \nu_{\tau(1)} + \cdots + \nu_{\tau(p)} + \varepsilon_{\{a\}}^{\{\tau^{-1}(i)\}}(p)}.$$

Proof. From Proposition 5.2, the coefficients $C_{\{a\};\mathbf{m}}(\Phi)$ with $|\mathbf{m}| = 1$ vanish unless $\mathbf{m} = (\varepsilon_{\{a\}}^{\{\tau^{-1}(i)\}}(1), \dots, \varepsilon_{\{a\}}^{\{\tau^{-1}(i)\}}(n-1))$. Then the recurrence relation in Lemma 5.1 implies that non-vanishing coefficients $C_{\{a\};\mathbf{m}}(\Phi)$ should satisfy the congruence condition $\mathbf{m} \equiv (\varepsilon_{\{a\}}^{\{\tau^{-1}(i)\}}(1), \dots, \varepsilon_{\{a\}}^{\{\tau^{-1}(i)\}}(n-1)) \pmod 2$. □

For our later use we give some identities for the function ε_A^J.

Lemma 5.4. *For $1 \le p \le n$, we have the following.*

(i) $\varepsilon_A^J(p) - \varepsilon_A^J(p-1) = (\chi_{[1,a-1]}(p) - \chi_{[a+1,n]}(p))\delta_{j,p} + (\chi_{[1,j-1]}(p) - \chi_{[j+1,n]}(p))\delta_{a,p}$.

(ii) $(2\varepsilon_A^J(p) - 1)\chi_{[1,j-1]}(p) = \varepsilon_A^J(p) - \chi_{[1,a-1]}(p)$.

(iii) $(2\varepsilon_A^J(p) - 1)\chi_{[j,n-1]}(p) = \varepsilon_A^J(p) - \chi_{[a,n-1]}(p)$.

Proof. (i) From the definition the left hand side gives

$$\begin{aligned}
&\chi_{[1,j-1]} \cdot \chi_{[a,n-1]} + \chi_{[j,n-1]} \cdot \chi_{[1,a-1]} - \chi_{[2,j]} \cdot \chi_{[a+1,n]} - \chi_{[j+1,n]} \cdot \chi_{[2,a]} \\
&= \chi_{[a+1,n-1]}(\chi_{[1,j-1]} - \chi_{[2,j]}) + \chi_{\{a\}} \cdot \chi_{[1,j-1]} - \chi_{[2,j]} \cdot \chi_{\{n\}} \\
&+ \chi_{[1,a-1]}(\chi_{[j,n-1]} - \chi_{[j+1,n]}) - \chi_{\{a\}} \cdot \chi_{[j+1,n]} \\
&= (\chi_{[1,a-1]} - \chi_{[a+1,n]})\chi_{\{j\}} + (\chi_{[1,j-1]} - \chi_{[j+1,n]})\chi_{\{a\}}
\end{aligned}$$

as desired. Here we omitted (p) from $\chi_*(p)$. (ii) The left hand side can be written as

$$\begin{aligned}
2\chi_{[a,n-1]} \cdot \chi_{[1,j-1]} - \chi_{[1,j-1]} &= \chi_{[a,n-1]} \cdot \chi_{[1,j-1]} + (\chi_{[a,n-1]} - 1)\chi_{[1,j-1]} \\
&= \chi_{[a,n-1]} \cdot \chi_{[1,j-1]} - \chi_{[1,a-1]}(1 - \chi_{[j,n-1]}) \\
&= \varepsilon_A^J - \chi_{[1,a-1]}.
\end{aligned}$$

Similarly we can show (iii). □

5.3. **Explicit formulas of the secondary Whittaker functions.** Now we give explicit formulas for the secondary Whittaker functions in the case of $I = \{i\}$. Since there are $(n-1)!$ possible choices of τ such that $\tau(j) = i$ for a given color $J = \{j\}$, we specify an element $\tau^{I,J}$.

Definition 5.5. For $I = \{i\}$, $J = \{j\}$ $(1 \le i, j \le n)$, we define an element $\tau^{I,J}$ of the Weyl group $\mathfrak{S}_n$ by

$$\tau^{I,J}(j) = i, \qquad \tau^{I,J}(p + \chi_{[j,n-1]}(p)) = p + \chi_{[i,n-1]}(p),$$

that is,

$$\tau^{I,J}(p) = \begin{cases} p + \chi_{[i,n-1]}(p) & \text{if } 1 \le p \le j-1, \\ i & \text{if } p = j, \\ p - 1 + \chi_{[i,n-1]}(p-1) & \text{if } j+1 \le p \le n-1. \end{cases}$$

Definition 5.6. Under the notation above, we introduce a power series $M_\nu^{n,I,J}(y) = y^\rho \widetilde{M}_\nu^{n,I,J} = \{y^\rho(\sqrt{-1})^{-a}\widetilde{M}^{n,I,J}_{A=\{a\},\nu}(y)\}_{1 \le a \le n}$ with

$$\widetilde{M}^{n,I,J}_{A,\nu}(y) = \sum_{\mathbf{m}} C^{n,I,J}_{A;\mathbf{m}}(\nu) \prod_{p=1}^{n-1} (\pi y_p)^{2m_p + \nu_{\tau^{I,J}(1)} + \cdots + \nu_{\tau^{I,J}(p)} + \varepsilon_A^J(p)}.$$

Here the coefficients are described by using the coefficients $C^{n-1}_{\mathbf{k}}(\tilde{\nu})$ for the secondary Whittaker functions on $SL(n-1, \mathbf{R})$ of helicity 0 as

$$C^{n,I,J}_{A;\mathbf{m}}(\nu) = \sum_{\mathbf{k}} \frac{C^{n-1}_{\mathbf{k}}(\tilde{\nu})}{P^{I,J}_{A;\mathbf{m},\mathbf{k}}},$$

where

$$P^{I,J}_{A;\mathbf{m},\mathbf{k}} = \prod_{p=1}^{j-1} (-1)^{\varepsilon_A^J(p)} (m_p - k_p)! \left(\frac{-\nu_i + \nu_{\tau^{I,J}(p)} + 1}{2}\right)_{m_p - k_{p-1} + \varepsilon_A^J(p)}$$

$$\cdot \prod_{p=j}^{n-1} (-1)^{\varepsilon_A^J(p)} (m_p - k_{p-1})! \Big(\frac{\nu_i - \nu_{\tau^{I,J}(p+1)} + 1}{2} \Big)_{m_p - k_p + \varepsilon_A^J(p)}$$

($k_0 = k_{n-1} = 0$), and $\sum_{\mathbf{k}}$ means $\mathbf{k} = (k_1, \dots, k_{n-2})$ runs through such that $P_{A;\mathbf{m},\mathbf{k}}^{I,J}$ does not vanish, that is,

$$\begin{cases} 0 \le k_p \le m_p & \text{if } 1 \le p \le j-2, \\ 0 \le k_p \le \min\{m_p, m_{p+1}\} & \text{if } p = j-1, \\ 0 \le k_p \le m_{p+1} & \text{if } j \le p \le n-2. \end{cases}$$

Theorem 5.7. *The set of power series*

$$\{\widetilde{M}_{\tau\nu}^{n,\{i\},\{j\}}(y) \mid 1 \le j \le n,\ \tau \in \mathfrak{S}_{\{i\}} \cong \mathfrak{S}_{n-1}\}$$

forms a basis of the space of solutions of the system in Theorem 3.3 with $I = \{i\}$ at $y = 0$.

From now on we show this Theorem. In view of Lemma 5.1, the Dirac-Schmid and the Casimir equations uniquely determine the coefficient $C_{A;\mathbf{m}}(\Phi)$ of the power series, once characteristic indices are given. Then it is enough to check that the power series $\widetilde{M}_{\nu}^{n,I,J}(y)$ satisfies the Dirac-Schmid equations and the Casimir equations (Proposition 5.8 below). Actually the Capelli elements C_k commutes with the Casimir element C_2, the compatibility with the Capelli equations follows as in [Ha].

5.4. **Recurrence relations for $C_{A;\mathbf{m}}^{I,J}(\nu)$.** In view of Lemma 5.1 and Corollary 5.3, the recurrence relations we should confirm are the following.

Proposition 5.8. *To abbreviate our notation we introduce new symbols $\mathbf{m}^{\pm}$ by*

$$\mathbf{m}^- = \mathbf{m} - \varepsilon_{A^-}^J(a-1) \cdot \mathbf{e}_{a-1} = \mathbf{m} - \chi_{[1,j-1]}(a-1) \cdot \mathbf{e}_{a-1};$$
$$\mathbf{m}^+ = \mathbf{m} - \varepsilon_{A^+}^J(a) \cdot \mathbf{e}_a = \mathbf{m} - \chi_{[j,n-1]}(a) \cdot \mathbf{e}_a.$$

Then $C_{A;\mathbf{m}}^{n,I,J} = C_{A;\mathbf{m}}^{n,I,J}(\nu)$ satisfies the recurrence relations:

(5.1) $$\big(-m_{a-1} + m_a + \mu_A^{I,J}(\nu)\big) C_{A;\mathbf{m}}^{n,I,J} - C_{A^-;\mathbf{m}^-}^{n,I,J} + C_{A^+;\mathbf{m}^+}^{n,I,J} = 0;$$

(5.2) $$Q_{\mathbf{m}}\big(\{\lambda_{A,p}^{I,J}(\nu)\}_{1 \le p \le n-1}; \kappa_A^{I,J}(\nu)\big) C_{A;\mathbf{m}}^{n,I,J} + \frac{1}{2}\big(C_{A^-;\mathbf{m}^-}^{n,I,J} + C_{A^+;\mathbf{m}^+}^{n,I,J}\big) = \sum_{p=1}^{n-1} C_{A;\mathbf{m}-\mathbf{e}_p}^{n,I,J},$$

for $1 \le a \le n$. Here

$$\mu_A^{I,J}(\nu) = \frac{1}{2}\big(\nu_{\tau^{I,J}(a)} - \nu_i - \varepsilon_A^J(a-1) + \varepsilon_A^J(a)\big),$$
$$\lambda_{A,p}^{I,J}(\nu) = \frac{1}{2}\big(\nu_{\tau^{I,J}(p)} - \nu_{\tau^{I,J}(p+1)} + 2\varepsilon_A^J(p) - \varepsilon_A^J(p-1) - \varepsilon_A^J(p+1)\big) \quad (1 \le p \le n-1),$$
$$\kappa_A^{I,J}(\nu) = \frac{1}{4}\Big\{\sum_{p=1}^{n-1} \varepsilon_A^J(p)(\nu_{\tau^{I,J}(p)} - \nu_{\tau^{I,J}(p+1)}) + Q(\varepsilon_A^J(1), \dots, \varepsilon_A^J(n-1))\Big\}.$$

Lemma 5.9. *A more explicit expression of $\kappa_A^{I,J}(\nu)$ is*

$$\kappa_A^{I,J}(\nu) = \frac{-\nu_i + \nu_{\tau^{I,J}(a)} + 1}{4} \cdot \chi_{[1,j-1]}(a) + \frac{\nu_i - \nu_{\tau^{I,J}(a)} + 1}{4} \cdot \chi_{[j,n-1]}(a-1).$$

Proof. By using Lemma 5.4 (i), we have

$$\begin{aligned}
&\sum_{p=1}^{n-1} \varepsilon_A^J(p)(\nu_{\tau^{I,J}(p)} - \nu_{\tau^{I,J}(p+1)}) \\
&= \sum_{p=1}^{n} \nu_{\tau^{I,J}(p)}(\varepsilon_A^J(p) - \varepsilon_A^J(p-1)) \\
&= \nu_{\tau^{I,J}(j)}(\chi_{[1,a-1]}(j) - \chi_{[a+1,n]}(j)) + \nu_{\tau^{I,J}(a)}(\chi_{[1,j-1]}(a) - \chi_{[j,n-1]}(a-1)) \\
&= \nu_i(\chi_{[j,n-1]}(a) - \chi_{[1,j-1]}(a)) + \nu_{\tau^{I,J}(a)}(\chi_{[1,j-1]}(a) - \chi_{[j,n-1]}(a-1)).
\end{aligned}$$

Let us evaluate $Q(\varepsilon_A^J(1), \ldots, \varepsilon_A^J(n-1))$. Since

$$\begin{aligned}
&\sum_{p=1}^{n}(\varepsilon_A^J(p) - \varepsilon_A^J(p-1))^2 \\
&= \sum_{p=1}^{n} \varepsilon_A^J(p)(\varepsilon_A^J(p) - \varepsilon_A^J(p-1)) + \sum_{p=1}^{n} \varepsilon_A^J(p-1)(\varepsilon_A^J(p-1) - \varepsilon_A^J(p)) \\
&= 2Q(\varepsilon_A^J(1), \ldots, \varepsilon_A^J(n-1)),
\end{aligned}$$

Lemma 5.4 (i) implies that $2Q(\varepsilon_A^J(1), \ldots, \varepsilon_A^J(n-1))$ is

$$\begin{aligned}
&\sum_{p=1}^{n}(\chi_{[1,a-1]}(p) + \chi_{[a+1,n]}(p))\delta_{j,p} + \sum_{p=1}^{n}(\chi_{[1,j-1]}(p) + \chi_{[j+1,n]}(p))\delta_{a,p} \\
&= \chi_{[1,a-1]}(j) + \chi_{[a+1,n]}(j) + \chi_{[1,j-1]}(a) + \chi_{[j+1,n]}(a) \\
&= 2(\chi_{[1,j-1]}(a) + \chi_{[j,n-1]}(a-1)).
\end{aligned}$$

Hence we get the desired expression. □

5.5. **Compatibility with the Dirac-Schmid equations.** We start with the proof of the recurrence relation (5.1).

Lemma 5.10. *We have*

$$\begin{aligned}
\frac{P^{I,J}_{A;\mathbf{m},\mathbf{k}}}{P^{I,J}_{A^-;\mathbf{m}^-,\mathbf{k}}} &= -m_{a-1} + k_{a-1} + \frac{-\nu_i + \nu_{\tau^{I,J}(a)} - 1}{2} \cdot \chi_{[j,n-1]}(a-1); \\
\frac{P^{I,J}_{A;\mathbf{m},\mathbf{k}}}{P^{I,J}_{A^+;\mathbf{m}^+,\mathbf{k}}} &= -m_a + k_{a-1} + \frac{\nu_i - \nu_{\tau^{I,J}(a)} - 1}{2} \cdot \chi_{[1,j-1]}(a),
\end{aligned}$$

and

$$\mu_A^{I,J}(\nu) - \frac{-\nu_i + \nu_{\tau^{I,J}(a)} - 1}{2} \cdot \chi_{[j,n-1]}(a-1) + \frac{\nu_i - \nu_{\tau^{I,J}(a)} - 1}{2} \cdot \chi_{[1,j-1]}(a) = 0.$$

Proof. It easily follows from the definitions of $P^{I,J}_{A;\mathbf{m},\mathbf{k}}$ and $\mu_A^{I,J}(\nu)$, and $\tau^{I,J}(j) = i$. □

Now the compatibility is immediate since we do not need to count the contribution from $C^{n-1}_{\mathbf{k}}(\tilde{\nu})$. Indeed Lemma 5.10 implies

$$(-m_{a-1} + m_a + \mu_A^{I,J}(\nu)) - \frac{P^{I,J}_{A;\mathbf{m},\mathbf{k}}}{P^{I,J}_{A^-;\mathbf{m}^-,\mathbf{k}}} + \frac{P^{I,J}_{A;\mathbf{m},\mathbf{k}}}{P^{I,J}_{A^+;\mathbf{m}^+,\mathbf{k}}} = 0.$$

5.6. **Compatibility with the Casimir equations.** Next we consider the Casimir equations. The idea of proof is similar to those of [I1, Theorem 4] and [IS, Theorem 15].

Lemma 5.11. *We have the key identity:*

$$\begin{aligned}\sum_{p=1}^{n-1}\frac{P^{I,J}_{A;\mathbf{m},\mathbf{k}}}{P^{I,J}_{A;\mathbf{m}-\mathbf{e}_p,\mathbf{k}}}-\sum_{p=1}^{n-2}\frac{P^{I,J}_{A;\mathbf{m},\mathbf{k}}}{P^{I,J}_{A;\mathbf{m},\mathbf{k}+\mathbf{d}_p}}-\frac{1}{2}\cdot\frac{P^{I,J}_{A;\mathbf{m},\mathbf{k}}}{P^{I,J}_{A^-;\mathbf{m}^-,\mathbf{k}}}-\frac{1}{2}\cdot\frac{P^{I,J}_{A;\mathbf{m},\mathbf{k}}}{P^{I,J}_{A^+;\mathbf{m}^+,\mathbf{k}}}\\ =Q_{\mathbf{m}}\big(\{\lambda^{I,J}_{A,p}(\nu)\}_{1\le p\le n-1};\kappa^{I,J}_A(\nu)\big)-Q_{\mathbf{k}}\Big(\Big\{\frac{\tilde\nu_p-\tilde\nu_{p+1}}{2}\Big\}_{1\le p\le n-2};0\Big).\end{aligned}\tag{5.3}$$

Proof. Since

$$\frac{P^{I,J}_{A;\mathbf{m},\mathbf{k}}}{P^{I,J}_{A;\mathbf{m}-\mathbf{e}_p,\mathbf{k}}}=\begin{cases}(m_p-k_p)\big(m_p-k_{p-1}+\varepsilon^J_A(p)+\frac{-\nu_i+\nu_{\tau^{I,J}(p)}-1}{2}\big) & \text{if } 1\le p\le j-1,\\ (m_p-k_{p-1})\big(m_p-k_p+\varepsilon^J_A(p)+\frac{\nu_i-\nu_{\tau^{I,J}(p+1)}-1}{2}\big) & \text{if } j\le p\le n-1,\end{cases}$$

and

$$\frac{P^{I,J}_{A;\mathbf{m},\mathbf{k}}}{P^{I,J}_{A;\mathbf{m},\mathbf{k}+\mathbf{d}_p}}=\begin{cases}(m_p-k_p)\big(m_{p+1}-k_p+\varepsilon^J_A(p+1)+\frac{-\nu_i+\nu_{\tau^{I,J}(p+1)}-1}{2}\big) & \text{if } 1\le p\le j-2,\\ (m_p-k_p)(m_{p+1}-k_p) & \text{if } p=j-1,\\ (m_{p+1}-k_p)\big(m_p-k_p+\varepsilon^J_A(p)+\frac{\nu_i-\nu_{\tau^{I,J}(p+1)}-1}{2}\big) & \text{if } j\le p\le n-2,\end{cases}$$

the left hand side of (5.3) equals

$$\begin{aligned}&\sum_{p=1}^{n-1}\Big\{m_p-k_p+\Big(\varepsilon^J_A(p)+\frac{\nu_i-\nu_{\tau^{I,J}(p+1)}-1}{2}\Big)\chi_{[j,n-1]}(p)\Big\}\\ &\quad\cdot\Big\{m_p-k_{p-1}+\Big(\varepsilon^J_A(p)+\frac{-\nu_i+\nu_{\tau^{I,J}(p)}-1}{2}\Big)\chi_{[1,j-1]}(p)\Big\}\\ &-\sum_{p=1}^{n-2}\Big\{m_p-k_p+\Big(\varepsilon^J_A(p)+\frac{\nu_i-\nu_{\tau^{I,J}(p+1)}-1}{2}\Big)\chi_{[j,n-1]}(p)\Big\}\\ &\quad\cdot\Big\{m_{p+1}-k_p+\Big(\varepsilon^J_A(p+1)+\frac{-\nu_i+\nu_{\tau^{I,J}(p+1)}-1}{2}\Big)\chi_{[1,j-1]}(p+1)\Big\}\\ &-\frac{1}{2}\Big(-m_{a-1}+k_{a-1}+\frac{-\nu_i+\nu_{\tau^{I,J}(a)}-1}{2}\cdot\chi_{[j,n-1]}(a-1)\Big)\\ &-\frac{1}{2}\Big(-m_a+k_{a-1}+\frac{\nu_i-\nu_{\tau^{I,J}(a)}-1}{2}\cdot\chi_{[1,j-1]}(a)\Big)\\ &=Q_{\mathbf{m}}\big(\{\lambda_p\}_{1\le p\le n-1};\kappa^{I,J}_A(\nu)\big)-Q_{\mathbf{k}}\big(\{\tilde\lambda_p\}_{1\le p\le n-2};0\big),\end{aligned}$$

(see Lemma 5.9) where

$$\begin{aligned}2\lambda_p&=(2\varepsilon^J_A(p)-1)\chi_{[1,j-1]}(p)+(2\varepsilon^J_A(p)-1)\chi_{[j,n-1]}(p)\\ &\quad-(2\varepsilon^J_A(p+1)-1)\chi_{[1,j-1]}(p+1)-(2\varepsilon^J_A(p-1)-1)\chi_{[j,n-1]}(p-1)\\ &\quad+(\nu_i-\nu_{\tau^{I,J}(p)})(-\chi_{[1,j-1]}(p)-\chi_{[j,n-1]}(p-1))\\ &\quad+(\nu_i-\nu_{\tau^{I,J}(p+1)})(\chi_{[j,n-1]}(p)+\chi_{[1,j-1]}(p+1))+\delta_{p,a-1}+\delta_{p,a}\end{aligned}$$

and

$$\begin{aligned}2\tilde\lambda_p&=(2\varepsilon^J_A(p)-1)(\chi_{[1,j-1]}(p)-\chi_{[j,n-1]}(p))\\ &\quad+(2\varepsilon^J_A(p+1)-1)(-\chi_{[1,j-1]}(p+1)+\chi_{[j,n-1]}(p+1))\\ &\quad+(-\nu_i+\nu_{\tau^{I,J}(p)})\chi_{[1,j-1]}(p)+(\nu_i-\nu_{\tau^{I,J}(p+2)})\chi_{[j,n-1]}(p+1)\end{aligned}$$

$$+ (\nu_i - \nu_{\tau^{I,J}(p+1)})(\chi_{[1,j-1]}(p+1) - \chi_{[j,n-1]}(p)) + \delta_{p,a-1}.$$

In view of Lemma 5.4 (ii), (iii) and the definition of $\tau^{I,J}$, we find that $\lambda_p = \lambda^{I,J}_{A,p}(\nu)$ and $\tilde{\lambda}_p = (\tilde{\nu}_p - \tilde{\nu}_{p+1})/2$. □

Returning to the proof of the recurrence relation (5.2), we compute

$$\begin{aligned} &\sum_{p=1}^{n-1} C^{n,I,J}_{A;\mathbf{m}-\mathbf{e}_p} - \frac{1}{2} C^{n,I,J}_{A^-;\mathbf{m}^-} - \frac{1}{2} C^{n,I,J}_{A^+;\mathbf{m}^+} \\ &= \sum_{p=1}^{n-1} \sum_{\mathbf{k}} \frac{P^{I,J}_{A;\mathbf{m},\mathbf{k}}}{P^{I,J}_{A;\mathbf{m}-\mathbf{e}_p,\mathbf{k}}} \cdot \frac{C^{n-1}_{\mathbf{k}}(\tilde{\nu})}{P^{I,J}_{A;\mathbf{m},\mathbf{k}}} \\ &- \frac{1}{2} \sum_{\mathbf{k}} \frac{P^{I,J}_{A;\mathbf{m},\mathbf{k}}}{P^{I,J}_{A^-;\mathbf{m}^-,\mathbf{k}}} \cdot \frac{C^{n-1}_{\mathbf{k}}(\tilde{\nu})}{P^{I,J}_{A;\mathbf{m},\mathbf{k}}} - \frac{1}{2} \sum_{\mathbf{k}} \frac{P^{I,J}_{A;\mathbf{m},\mathbf{k}}}{P^{I,J}_{A^+;\mathbf{m}^+,\mathbf{k}}} \cdot \frac{C^{n-1}_{\mathbf{k}}(\tilde{\nu})}{P^{I,J}_{A;\mathbf{m},\mathbf{k}}}. \end{aligned} \tag{5.4}$$

By using Lemma 5.11, we can see that (5.4) is a sum of the three terms:

(5.5)

$$Q_{\mathbf{m}}\big(\{\lambda^{I,J}_{A,p}(\nu)\}_{1\le p\le n-1}; \kappa^{I,J}_A(\nu)\big) \sum_{\mathbf{k}} \frac{C^{n-1}_{\mathbf{k}}(\tilde{\nu})}{P^{I,J}_{A;\mathbf{m},\mathbf{k}}} = Q_{\mathbf{m}}\big(\{\lambda^{I,J}_{A,p}(\nu)\}_{1\le p\le n-1}; \kappa^{I,J}_A(\nu)\big) C^{I,J}_{A;\mathbf{m}},$$

$$-\sum_{\mathbf{k}} \frac{Q_{\mathbf{k}}(\{\lambda_p(\tilde{\nu})\}_{1\le p\le n-2}; 0)\, C^{n-1}_{\mathbf{k}}(\tilde{\nu})}{P^{I,J}_{A;\mathbf{m},\mathbf{k}}}, \tag{5.6}$$

$$\sum_{\mathbf{k}} \sum_{p=1}^{n-2} \frac{P^{I,J}_{A;\mathbf{m},\mathbf{k}}}{P^{I,J}_{A;\mathbf{m},\mathbf{k}+\mathbf{d}_p}} \cdot \frac{C^{n-1}_{\mathbf{k}}(\tilde{\nu})}{P^{I,J}_{A;\mathbf{m},\mathbf{k}}} = \sum_{\mathbf{k}} \frac{\sum_{p=1}^{n-2} C^{n-1}_{\mathbf{k}-\mathbf{d}_p}(\tilde{\nu})}{P^{I,J}_{A;\mathbf{m},\mathbf{k}}}. \tag{5.7}$$

In view of the recurrence relation (4.2) for $C^{n-1}_{\mathbf{k}}(\tilde{\nu})$, the second term (5.6) cancels the third term (5.7) and thus the compatibility with the Casimir equations is confirmed. This settles the proof of Theorem 5.7. □

6. Primary Whittaker functions (1) -the case of helicity 1-

In this section we discuss integral representations of the primary Whittaker functions. Here is the main result for the case of helicity 1.

Theorem 6.1. *Define a function* $W^{n,I}_{\nu}(y) = y^{\rho}\widetilde{W}^{n,I}_{\nu}(y) = \big\{y^{\rho}(\sqrt{-1})^{-a}\widetilde{W}^{n,I}_{\{a\},\nu}(y)\big\}_{1\le a\le n}$ *by using the primary Whittaker function of helicity* 0 *on* $SL(n-1,\mathbf{R})$*:*

$$\begin{aligned} \widetilde{W}^{n,I}_{A,\nu}(y) = \int_{(\mathbf{R}_+)^{n-1}} &\widetilde{W}^{n-1}_{\tilde{\nu}}\left(y_2\sqrt{\frac{t_2}{t_1}}, \dots, y_{n-1}\sqrt{\frac{t_{n-1}}{t_{n-2}}}\right) \\ &\cdot \prod_{p=1}^{n-1} \exp\Big\{-(\pi y_p)^2 t_p - \frac{1}{t_p}\Big\} \prod_{p=1}^{n-1} (\pi y_p)^{\frac{(n-p)\nu_i}{n-1} + \chi_{[1,a-1]}(p)} \\ &\cdot \prod_{p=1}^{n-1} t_p^{\frac{n\nu_i}{2(n-1)} + \frac{1}{2}(\chi_{[1,a-1]}(p) - \chi_{[a,n-1]}(p))} \prod_{p=1}^{n-1} \frac{dt_p}{t_p}. \end{aligned}$$

Then $W^{n,I}_{\nu}(y)$ is the radial part of the primary Whittaker function of helicity 1*, and we have the factorization formula*

$$W^{n,I}_{\nu}(y)=\sum_{\tau\in\ \ I}\tau\Big[\sum_{1\le j\le n}\Gamma^{n,I,\{j\}}(\nu)\cdot M^{n,I,\{j\}}_{\nu}(y)\Big], \tag{6.1}$$

where

$$\Gamma^{n,I,\{j\}}(\nu)=\prod_{1\le p<q\le n-1}\Gamma\Big(\frac{-\nu_{i'_p}+\nu_{i'_q}}{2}\Big)\cdot\prod_{p=1}^{j-1}\Gamma\Big(\frac{\nu_i-\nu_{i'_p}+1}{2}\Big)\prod_{p=j}^{n-1}\Gamma\Big(\frac{-\nu_i+\nu_{i'_p}+1}{2}\Big).$$

Here $\{i'_1,\dots,i'_{n-1}\}$ $(i'_1<\cdots<i'_{n-1})$ is the complement of $I=\{i\}$ in T_n.

The rest of this section is devoted to the proof of Theorem 6.1 by a similar argument as in [HrIO, §7]. Since the function $W^{n,I}_{A,\nu}(y)$ is of moderate growth, it is enough to show the factorization formula (6.1). Indeed, thanks to the multiplicity one theorem of Shalika [Sha] and Wallach [Wa], we may conclude that the function $W^{n,I}_{\nu}(y)$ is the radial part of the primary Whittaker function.

In Subsection 6.1, we first apply the factorization formula for Whittaker functions of helicity 0 and derive a certain factorization formula (Lemma 6.2). As stated in Proposition 6.3, the factorization formula in Lemma 6.2 contains *ghost solutions*, that is, the power series which does not contribute to the factorization formula (6.1). In the successive subsections the proof of Proposition 6.3 is given.

6.1. **A certain factorization of $W^{n,I}_{A,\nu}(y)$.** By inserting the factorization formula (4.5) for $\widetilde{W}^{n-1}_{\tilde\nu}(y)$ and changing the order of integrations and infinite sums, which can be justified as in [HrIO, §7], we get

$$\begin{aligned}
\widetilde{W}^{n,I}_{A,\nu}(y)&=\sum_{\tau\in\ \ I}\tau\Big[\prod_{1\le p<q\le n-1}\Gamma\Big(\frac{-\nu_{i'_p}+\nu_{i'_q}}{2}\Big)\sum_{\mathbf{k}}C^{n-1}_{\mathbf{k}}(\tilde\nu)\\
&\quad\cdot\int_{(\mathbf{R}_+)^{n-1}}\prod_{p=1}^{n-1}\exp\Big\{-(\pi y_p)^2t_p-\frac{1}{t_p}\Big\}\cdot\prod_{p=1}^{n-2}\Big(\pi y_{p+1}\sqrt{\frac{t_{p+1}}{t_p}}\Big)^{2k_p+\ \ {}^{p}_{t=1}\nu_{i'_t}+\frac{p\nu_i}{n-1}}\\
&\quad\cdot\prod_{p=1}^{n-1}(\pi y_p)^{\frac{(n-p)\nu_i}{n-1}+\chi_{[1,a-1]}(p)}\prod_{p=1}^{n-1}t_p^{\frac{n\nu_i}{2(n-1)}+\frac12(\chi_{[1,a-1]}(p)-\chi_{[a,n-1]}(p))}\prod_{p=1}^{n-1}\frac{dt_p}{t_p}\Big]\\
&=\sum_{\tau\in\ \ I}\tau\Big[\prod_{1\le p<q\le n-1}\Gamma\Big(\frac{-\nu_{i'_p}+\nu_{i'_q}}{2}\Big)\cdot\sum_{\mathbf{k}}C^{n-1}_{\mathbf{k}}(\tilde\nu)\\
&\quad\cdot\prod_{p=1}^{n-1}(\pi y_p)^{2k_{p-1}+\nu_i+\ \ {}^{p-1}_{t=1}\nu_{i'_t}+\chi_{[1,a-1]}(p)}\\
&\quad\cdot\prod_{p=1}^{n-1}\int_0^\infty\exp\Big\{-(\pi y_p)^2t_p-\frac{1}{t_p}\Big\}\cdot t_p^{k_{p-1}-k_p+\frac12(\nu_i-\nu_{i'_p})+\frac12(\chi_{[1,a-1]}(p)-\chi_{[a,n]}(p))}\frac{dt_p}{t_p}\Big].
\end{aligned}$$

Here we understand $k_0=k_{n-1}=0$.

The integrations with respect to t_p are carried out by using the formula

$$\int_0^\infty\exp\Big\{-(\pi y)^2t-\frac{1}{t}\Big\}t^{k+\nu}\frac{dt}{t}$$

$$
\begin{aligned}
&= 2(\pi y)^{-k-\nu} K_{k+\nu}(2\pi y) \\
&= \frac{\pi}{\sin(k+\nu)\pi} \left(\sum_{m=0}^{\infty} \frac{(\pi y)^{2(m-k)-2\nu}}{m!\Gamma(m-k-\nu+1)} - \sum_{m=0}^{\infty} \frac{(\pi y)^{2m}}{m!\Gamma(m+k+\nu+1)} \right)
\end{aligned}
$$

$$
= (-1)^k \Big(\sum_{m=0}^{\infty} \frac{\Gamma(\nu)\cdot(\pi y)^{2(m-k)-2\nu}}{m!(-\nu+1)_{m-k}} + \sum_{m=0}^{\infty} \frac{\Gamma(-\nu)\cdot(\pi y)^{2m}}{m!(\nu+1)_{m+k}} \Big) \tag{6.2}
$$

$$
= (-1)^k \Big(\sum_{m=0}^{\infty} \frac{\Gamma(\nu)\cdot(\pi y)^{2(m-k)-2\nu}}{m!(-\nu+1)_{m-k}} - \sum_{m=0}^{\infty} \frac{\Gamma(-\nu+1)\cdot(\pi y)^{2m}}{m!(\nu)_{m+k+1}} \Big), \tag{6.3}
$$

for $k \in \mathbf{Z}$ and $\nu \notin \mathbf{Z}$. Then the expression (6.2) implies that

$$
\begin{aligned}
\widetilde{W}^{n,I}_{A,\nu}(y) = \sum_{\tau \in \ _I} \tau \Bigg[& \prod_{1 \le p < q \le n-1} \Gamma\Big(\frac{-\nu_{i'_p} + \nu_{i'_q}}{2}\Big) \cdot \sum_{\mathbf{k}} C^{n-1}_{\mathbf{k}}(\tilde{\nu}) \\
& \cdot \prod_{p=1}^{n-1} \Bigg\{ \sum_{m_p=0}^{\infty} \frac{\Gamma\Big(\frac{\nu_i - \nu_{i'_p} + \chi_{[1,a-1]}(p) - \chi_{[a,n-1]}(p)}{2}\Big) \cdot (\pi y_p)^{2(m_p+k_p) + \ {}^{p}_{t=1} \nu_{i'_t} + \chi_{[a,n-1]}(p)}}{m_p! \Big(\frac{-\nu_i + \nu_{i'_p} - \chi_{[1,a-1]}(p) + \chi_{[a,n-1]}(p)}{2} + 1\Big)_{m_p - k_{p-1} + k_p}} \\
& + \sum_{m_p=0}^{\infty} \frac{\Gamma\Big(\frac{-\nu_i + \nu_{i'_p} - \chi_{[1,a-1]}(p) + \chi_{[a,n-1]}(p)}{2}\Big) \cdot (\pi y_p)^{2(m_p+k_{p-1}) + \nu_i + \ {}^{p-1}_{t=1} \nu_{i'_t} + \chi_{[1,a-1]}(p)}}{m_p! \Big(\frac{\nu_i - \nu_{i'_p} + \chi_{[1,a-1]}(p) - \chi_{[a,n-1]}(p)}{2} + 1\Big)_{m_p + k_{p-1} - k_p}} \Bigg\} \Bigg].
\end{aligned}
$$

In view of $\chi_{[1,a-1]}(p) + \chi_{[a,n-1]}(p) = 1$, we find

$$
\begin{aligned}
\frac{\Gamma\Big(\frac{\nu_i - \nu_{i'_p} + \chi_{[1,a-1]}(p) - \chi_{[a,n-1]}(p)}{2}\Big)}{\Big(\frac{-\nu_i + \nu_{i'_p} - \chi_{[1,a-1]}(p) + \chi_{[a,n-1]}(p)}{2} + 1\Big)_{m_p - k_{p-1} + k_p}} &= \frac{\Gamma\Big(\frac{\nu_i - \nu_{i'_p} + 1}{2} - \chi_{[a,n-1]}(p)\Big)}{\Big(\frac{-\nu_i + \nu_{i'_p} + 1}{2} + \chi_{[a,n-1]}(p)\Big)_{m_p - k_{p-1} + k_p}} \\
&= \frac{(-1)^{\chi_{[a,n-1]}(p)} \cdot \Gamma\Big(\frac{\nu_i - \nu_{i'_p} + 1}{2}\Big)}{\Big(\frac{-\nu_i + \nu_{i'_p} + 1}{2}\Big)_{m_p - k_{p-1} + k_p + \chi_{[a,n-1]}(p)}},
\end{aligned}
$$

and similarly

$$
\frac{\Gamma\Big(\frac{-\nu_i + \nu_{i'_p} - \chi_{[1,a-1]}(p) + \chi_{[a,n-1]}(p)}{2}\Big)}{\Big(\frac{\nu_i - \nu_{i'_p} + \chi_{[1,a-1]}(p) - \chi_{[a,n-1]}(p)}{2} + 1\Big)_{m_p + k_{p-1} - k_p}} = \frac{(-1)^{\chi_{[1,a-1]}(p)} \cdot \Gamma\Big(\frac{-\nu_i + \nu_{i'_p} + 1}{2}\Big)}{\Big(\frac{\nu_i - \nu_{i'_p} + 1}{2}\Big)_{m_p + k_{p-1} - k_p + \chi_{[1,a-1]}(p)}}.
$$

By using them, we arrange the order of summation (and replace the indices $m_p \to m_p - k_p$ or $m_p - k_{p-1}$) to get the following expression.

Lemma 6.2. *We have*

$$
\widetilde{W}^{n,I}_{A,\nu}(y) = \sum_{\tau \in \ _I} \sum_{R \subset T_{n-1}} \tau \Big[\Gamma^{I,R}(\nu) \cdot \widetilde{M}^{I,R}_{A,\nu}(y) \Big]. \tag{6.4}
$$

Here R ranges all the subsets of T_{n-1} and

$$
\Gamma^{I,R}(\nu) = \prod_{1 \le p < q \le n-1} \Gamma\Big(\frac{-\nu_{i'_p} + \nu_{i'_q}}{2}\Big) \cdot \prod_{p \in R^c} \Gamma\Big(\frac{\nu_i - \nu_{i'_p} + 1}{2}\Big) \prod_{p \in R} \Gamma\Big(\frac{-\nu_i + \nu_{i'_p} + 1}{2}\Big),
$$

(R^c means the complement of R in T_{n-1}) and

$$\widetilde{M}^{I,R}_{A,\nu}(y) = \sum_{\mathbf{m}} C^{I,R}_{A;\mathbf{m}}(\nu) \prod_{p\in R^c} (\pi y_p)^{2m_p + \;{}_{t=1}^{p} \nu_{i'_t} + \chi_{[a,n-1]}(p)} \prod_{p\in R} (\pi y_p)^{2m_p+\nu_i+ \;{}_{t=1}^{p-1} \nu_{i'_t} + \chi_{[1,a-1]}(p)},$$

where the coefficients are given by

$$C^{I,R}_{A;\mathbf{m}}(\nu) = \sum_{\mathbf{k}} \frac{C^{n-1}_{\mathbf{k}}(\tilde{\nu})}{P^{I,R}_{A;\mathbf{m},\mathbf{k}}},$$

with

$$P^{I,R}_{A;\mathbf{m},\mathbf{k}} = \prod_{p\in R^c} (-1)^{\chi_{[a,n-1]}(p)} (m_p - k_p)! \Big(\frac{-\nu_i + \nu_{i'_p} + 1}{2} \Big)_{m_p - k_{p-1} + \chi_{[a,n-1]}(p)}$$
$$\cdot \prod_{p\in R} (-1)^{\chi_{[1,a-1]}(p)} (m_p - k_{p-1})! \Big(\frac{\nu_i - \nu_{i'_p} + 1}{2} \Big)_{m_p - k_p + \chi_{[1,a-1]}(p)}.$$

Here $\sum_{\mathbf{k}}$ means k_p runs through such that $0 \le k_p \le m_p$ if $p \in R^c$ and $0 \le k_p \le m_{p+1}$ if $p+1 \in R$. Note that if $p \in R$ and $p+1 \in R^c$, the index k_p ranges all nonnegative integers.

Remark 3. We define a function ε^R_A on T_{n-1} by

$$\varepsilon^R_A = \chi_{[a,n-1]} \cdot \chi_{R^c} + \chi_{[1,a-1]} \cdot \chi_R.$$

By using the function ε^R_A we have

$$\widetilde{M}^{I,R}_{A,\nu}(y) = \sum_{\mathbf{m}} C^{I,R}_{A;\mathbf{m}}(\nu) \prod_{p=1}^{n-1} (\pi y_p)^{2m_p + \nu_i \cdot \chi_R(p) + \;{}_{t=1}^{p-1} \nu_{i'_t} + \nu_{i'_p} \cdot \chi_{R^c}(p) + \varepsilon^R_A(p)}.$$

6.2. **Vanishing of ghost solutions.** The right hand side of (6.4) involves *ghost solutions*, that is, some power series $\widetilde{M}^{I,R}_{A,\nu}$ in (6.4) cancel with their suitable partners. The next proposition tells us that which subset R really contributes to the summation in (6.4), and thus leads the factorization formula (6.1).

Proposition 6.3. *For a subset R of T_{n-1} put*

$$E(R) = \{ r \in T_{n-2} \mid r \in R \text{ and } r+1 \in R^c \}.$$

Then only the subsets R satisfying $E(R) = \emptyset$ contribute to the summation in (6.4). More precisely we have the following.

(i) *If $E(R) = \emptyset$, then R is of the form $[r, n-1]$ with some $r \in T_n$ ($r = n \iff R = \emptyset$), and we have $\widetilde{M}^{I,R}_{A,\nu}(y) = \widetilde{M}^{n,I,\{r\}}_{A,\nu}(y)$ and $\Gamma^{I,R}(\nu) = \Gamma^{n,I,\{r\}}(\nu)$.*

(ii) *For $r \in E(R)$ we have a cancellation of power series:*

$$\Gamma^{I,R}(\nu) \cdot \widetilde{M}^{I,R}_{A,\nu}(y) + \Gamma^{I,R}(\tau'_r \nu) \cdot \widetilde{M}^{I,R}_{A,\tau'_r\nu}(y) = 0.$$

Here τ'_r is the permutation (i'_r, i'_{r+1}) in $\mathfrak{S}_n$.

The first claim (i) is obvious. Let us explain our idea of proof of (ii). Since we can see that the characteristic exponents of $\widetilde{M}^{I,R}_{A,\nu}(y)$ are invariant under the permutation of $\nu_{i'_r}$ and $\nu_{i'_{r+1}}$, we have only to show the cancellation of the coefficients:

$$\Gamma^{I,R}(\nu) \cdot C^{I,R}_{A;\mathbf{m}}(\nu) + \Gamma^{I,R}(\tau'_r \nu) \cdot C^{I,R}_{A;\mathbf{m}}(\tau'_r \nu) = 0, \text{ for all } \mathbf{m} \text{ and } A. \tag{6.5}$$

Our proof of (6.5) is a little bit indirect. We first derive recurrence relations for the coefficients $C^{I,R}_{A;\mathbf{m}}(\nu)$ in Lemma 6.4, which uniquely determine $C^{I,R}_{A;\mathbf{m}}(\nu)$ from the initial values $C^{I,R}_{A;\mathbf{0}}(\nu)$. In Lemma 6.5, we will check that these recurrence relations are invariant under the action of τ'_r. This means that $C^{I,R}_{A;\mathbf{m}}(\nu)$ and $C^{I,R}_{A;\mathbf{m}}(\tau'_r\nu)$ are characterized by the same recurrence relations. Therefore our task can be reduced to confirm the cancellation of the initial values for the *edge components* $\{1\},\{n\}$:

$$\Gamma^{I,R}(\nu)\cdot C^{I,R}_{A;\mathbf{0}}(\nu)+\Gamma^{I,R}(\tau'_r\nu)\cdot C^{I,R}_{A;\mathbf{0}}(\tau'_r\nu)=0,\ \text{for } A=\{1\},\{n\}. \tag{6.6}$$

Actually Lemma (ii) 6.4 and (6.6) imply the vanishing (6.5) for $A=\{1\},\{n\}$. By using the recurrence relations in Lemma 6.4 (i), we can find that (6.5) holds for all A.

6.3. Recurrence relations for $C^{I,R}_{A;\mathbf{m}}(\nu)$.

Lemma 6.4. *We set*

$$\mathbf{m}^{-,R}=\mathbf{m}-\chi_{R^c}(a-1)\cdot\mathbf{e}_{a-1},\qquad \mathbf{m}^{+,R}=\mathbf{m}-\chi_R(a)\cdot\mathbf{e}_a.$$

Then we have the following recurrence relations for $C^{I,R}_{A;\mathbf{m}}=C^{I,R}_{A;\mathbf{m}}(\nu)$.

(i) *For $1\le a\le n$, we have*

$$(-m_{a-1}+m_a+\mu^{I,R}_A(\nu))C^{I,R}_{A;\mathbf{m}}-C^{I,R}_{A^-;\mathbf{m}^{-,R}}+C^{I,R}_{A^+;\mathbf{m}^{+,R}}=0; \tag{6.7}$$

$$Q_{\mathbf{m}}\big(\{\lambda^{I,R}_{A,p}(\nu)\}_{1\le p\le n-1};\kappa^{I,R}_A(\nu)\big)C^{I,R}_{A;\mathbf{m}}+\frac{1}{2}\big(C^{I,R}_{A^-;\mathbf{m}^{-,R}}+C^{I,R}_{A^+;\mathbf{m}^{+,R}}\big)=\sum_{p=1}^{n-1}C^{I,R}_{A;\mathbf{m}-\mathbf{e}_p}. \tag{6.8}$$

Here

$$\begin{aligned}
\mu^{I,R}_A(\nu)&=\frac{-\nu_i+\nu_{i'_{a-1}}-1}{2}\cdot\chi_R(a-1)+\frac{-\nu_i+\nu_{i'_a}+1}{2}\cdot\chi_{R^c}(a),\\
\lambda^{I,R}_{A,p}(\nu)&=\frac{1}{2}\{\nu_i(2\chi_R(p)-\chi_R(p+1)-\chi_R(p-1))\\
&\quad+\nu_{i'_{p-1}}\cdot\chi_R(p-1)+\nu_{i'_p}(\chi_{R^c}(p)-\chi_R(p))-\nu_{i'_{p+1}}\cdot\chi_{R^c}(p+1)\\
&\quad+2\varepsilon^R_A(p)-\varepsilon^R_A(p-1)-\varepsilon^R_A(p+1)\}\qquad(1\le p\le n-1),\\
\kappa^{I,R}_A(\nu)&=\frac{\nu_i-\nu_{i'_{a-1}}+1}{4}\cdot\chi_R(a-1)+\frac{-\nu_i+\nu_{i'_a}+1}{4}\cdot\chi_{R^c}(a)\\
&\quad+\sum_{p\in E(R)}\Big(\frac{\nu_i-\nu_{i'_p}-1}{2}+\chi_{[1,a-1]}(p)\Big)\Big(\frac{\nu_i-\nu_{i'_{p+1}}-1}{2}+\chi_{[1,a-1]}(p+1)\Big).
\end{aligned}$$

(ii) *For $a=1$ and n, we have*

$$Q_{\mathbf{m}}\big(\{\lambda'^{I,R}_{A,p}(\nu)\}_{1\le p\le n-1};\kappa'^{I,R}_A(\nu)\big)C^{I,R}_{A;\mathbf{m}}=\sum_{p=1}^{n-1}C^{I,R}_{A;\mathbf{m}-\mathbf{e}_p}, \tag{6.9}$$

where

$$\lambda'^{I,R}_{A,p}(\nu)=\begin{cases}\lambda^{I,R}_{A,p}-\frac{1}{2} & \textit{if } (a,p)=(1,1),(n,n-1),\\ \lambda^{I,R}_{A,p} & \textit{otherwise},\end{cases}$$

$$\kappa'^{I,R}_A(\nu)=\frac{1}{4}\sum_{p\in E(R)}(\nu_i-\nu_{i'_p}-\delta_{a,1}+\delta_{a,n})(\nu_i-\nu_{i'_{p+1}}-\delta_{a,1}+\delta_{a,n}).$$

Proof. The proof is similar to that of Theorem 5.7. By changing the roles of $[j, n-1]$ and $[1, j-1]$ by R and R^c respectively, the same identities in Lemma 5.10 hold for $P^{I,R}_{A^{\pm};\mathbf{m}^{\pm,R},\mathbf{k}}$ and then we get the recurrence relation (6.7).

The relation (6.8) can be shown by using the key identity (cf. Lemma 5.11):

$$\begin{aligned}
&\sum_{p=1}^{n-1} \frac{P^{I,R}_{A;\mathbf{m},\mathbf{k}}}{P^{I,R}_{A;\mathbf{m}-\mathbf{e}_p,\mathbf{k}}} - \sum_{p=1}^{n-2} \frac{P^{I,R}_{A;\mathbf{m},\mathbf{k}}}{P^{I,R}_{A;\mathbf{m},\mathbf{k}+\mathbf{d}_p}} - \frac{1}{2}\frac{P^{I,R}_{A;\mathbf{m},\mathbf{k}}}{P^{I,R}_{A^-;\mathbf{m}^{-,R},\mathbf{k}}} - \frac{1}{2}\frac{P^{I,R}_{A;\mathbf{m}.\mathbf{k}}}{P^{I,R}_{A^+;\mathbf{m}^{+,R},\mathbf{k}}} \\
&= \sum_{p=1}^{n-1} \Big\{ m_p - k_p + \Big(\chi_{[1,a-1]}(p) + \frac{\nu_i - \nu_{i'_p} - 1}{2}\Big)\chi_R(p)\Big\} \\
&\qquad \cdot \Big\{ m_p - k_{p-1} + \Big(\chi_{[a,n-1]}(p) + \frac{-\nu_i + \nu_{i'_p} - 1}{2}\Big)\chi_{R^c}(p)\Big\} \\
&- \sum_{p=1}^{n-2} \Big\{ m_p - k_p + \Big(\chi_{[1,a-1]}(p) + \frac{\nu_i - \nu_{i'_p} - 1}{2}\Big)\chi_R(p)\Big\} \\
&\qquad \cdot \Big\{ m_{p+1} - k_p + \Big(\chi_{[a,n-1]}(p+1) + \frac{-\nu_i + \nu_{i'_{p+1}} - 1}{2}\Big)\chi_{R^c}(p+1)\Big\} \\
&- \frac{1}{2}\Big(-m_{a-1} + k_{a-1} + \frac{-\nu_i + \nu_{i'_{a-1}} - 1}{2} \cdot \chi_R(a-1)\Big) \\
&- \frac{1}{2}\Big(-m_a + k_{a-1} + \frac{\nu_i - \nu_{i'_a} - 1}{2} \cdot \chi_{R^c}(a)\Big) \\
&= Q_{\mathbf{m}}\big(\{\lambda^{I,R}_{A,p}(\nu)\}_{1\le p\le n-1}; \kappa^{I,R}_A(\nu)\big) - Q_{\mathbf{k}}\Big(\Big\{\frac{\nu_{i'_p} - \nu_{i'_{p+1}}}{2}\Big\}_{1\le p\le n-2}; 0\Big).
\end{aligned}$$

The relation (6.9) follows from (6.7) and (6.8) with $a = 1$ and n, by eliminating the terms $C^{I,R}_{\{2\},*}$ and $C^{I,R}_{\{n-1\},*}$, respectively. Or we can derive it directly by using the identity

$$\begin{aligned}
&\sum_{p=1}^{n-1} \frac{P^{I,R}_{A;\mathbf{m},\mathbf{k}}}{P^{I,R}_{A;\mathbf{m}-\mathbf{e}_p,\mathbf{k}}} - \sum_{p=1}^{n-2} \frac{P^{I,R}_{A;\mathbf{m},\mathbf{k}}}{P^{I,R}_{A;\mathbf{m},\mathbf{k}+\mathbf{d}_p}} \\
&= Q_{\mathbf{m}}\big(\{\lambda'^{I,R}_{A,p}(\nu)\}_{1\le p\le n-1}; \kappa'^{I,R}_A(\nu)\big) - Q_{\mathbf{k}}\Big(\Big\{\frac{\nu_{i'_p} - \nu_{i'_{p+1}}}{2}\Big\}_{1\le p\le n-2}; 0\Big).
\end{aligned} \tag{6.10}$$

Note that $\nu_{i'_p} - \nu_{i'_{p+1}} = \tilde{\nu}_p - \tilde{\nu}_{p+1}$. ☐

Lemma 6.5. *For $r \in E(R)$, the values $\mu^{I,R}_A(\nu)$, $\lambda^{I,R}_{A,p}(\nu)$, $\lambda'^{I,R}_{A,p}(\nu)$, $\kappa^{I,R}_A(\nu)$ and $\kappa'^{I,R}_A(\nu)$ are invariant under the permutation of $\nu_{i'_r}$ and $\nu_{i'_{r+1}}$. Then the coefficients $C^{I,R}_{A;\mathbf{m}}(\nu)$ and $C^{I,R}_{A;\mathbf{m}}(\tau'_r\nu)$ are characterized by the same recurrence relations.*

Proof. It is immediate for $\mu^{I,R}_A(\nu)$. The values $\lambda^{I,R}_{A,p}(\nu)$ contains $\nu_{i'_r}$ or $\nu_{i'_{r+1}}$ only when $p = r, r+1$ and we can see the invariance easily. Our claim for $\kappa^{I,R}_A(\nu)$ is obvious for $a \ne r+1$. When $a = r+1$ we have

$$\begin{aligned}
\kappa^{I,R}_{\{r+1\}} &= \frac{1}{4}(\nu_i - \nu_{i'_r})(\nu_i - \nu_{i'_{r+1}}) + \frac{1}{4} \\
&+ \sum_{p\in E(R), p\ne r} \Big(\frac{\nu_i - \nu_{i'_p} - 1}{2} + \chi_{[1,a-1]}(p)\Big)\Big(\frac{\nu_i - \nu_{i'_{p+1}} - 1}{2} + \chi_{[1,a-1]}(p+1)\Big),
\end{aligned}$$

and finish the proof. ☐

6.4. **Initial values $C^{I,R}_{A;\mathbf{0}}(\nu)$.**

Lemma 6.6. *We have*

$$C^{I,R}_{A;\mathbf{0}}(\nu)=\prod_{p\in R^c}\frac{(-1)^{\chi_{[a,n-1]}(p)}}{\left(\frac{-\nu_i+\nu_{i'_p}+1}{2}\right)_{\chi_{[a,n-1]}(p)}}\prod_{p\in R}\frac{(-1)^{\chi_{[1,a-1]}(p)}}{\left(\frac{\nu_i-\nu_{i'_p}+1}{2}\right)_{\chi_{[1,a-1]}(p)}}$$
$$\cdot\prod_{p\in E(R)}\frac{\Gamma\left(\frac{\nu_{i'_p}-\nu_{i'_{p+1}}+2}{2}\right)}{\Gamma\left(\frac{-\nu_i+\nu_{i'_p}+1}{2}+\chi_{[a,n-1]}(p+1)\right)\Gamma\left(\frac{\nu_i-\nu_{i'_{p+1}}+1}{2}+\chi_{[1,a-1]}(p)\right)}.$$

In particular,

$$C^{I,R}_{\{1\};\mathbf{0}}(\nu)=\prod_{p\in R^c}\frac{2}{\nu_i-\nu_{i'_p}-1}\prod_{p\in E(R)}\frac{\Gamma\left(\frac{\nu_{i'_p}-\nu_{i'_{p+1}}+2}{2}\right)}{\Gamma\left(\frac{-\nu_i+\nu_{i'_p}+3}{2}\right)\Gamma\left(\frac{\nu_i-\nu_{i'_{p+1}}+1}{2}\right)},$$
$$C^{I,R}_{\{n\};\mathbf{0}}(\nu)=\prod_{p\in R}\frac{2}{-\nu_i+\nu_{i'_p}-1}\prod_{p\in E(R)}\frac{\Gamma\left(\frac{\nu_{i'_p}-\nu_{i'_{p+1}}+2}{2}\right)}{\Gamma\left(\frac{-\nu_i+\nu_{i'_p}+1}{2}\right)\Gamma\left(\frac{\nu_i-\nu_{i'_{p+1}}+3}{2}\right)}.$$

Proof. From the definition of $P^{I,R}_{A;\mathbf{m},\mathbf{k}}$, we can see that $k_p=0$ if $p\in R^c$ or $p+1\in R$. Then, in case of $E(R)=\emptyset$, we have $C^R_{A;\mathbf{0}}(\nu)=1/P^{I,R}_{A;\mathbf{0},\mathbf{0}}$ and our claim follows. When $E(R)\neq\emptyset$ we find that

$$C^{I,R}_{A;\mathbf{0}}(\nu)=\sum_{\mathbf{k}}\frac{C^{n-1}_{\mathbf{k}}(\tilde\nu)}{P^{I,R}_{A;\mathbf{0},\mathbf{k}}}.\tag{6.11}$$

Here $\mathbf{k}=(k_1,\dots,k_{n-2})$ runs through such that $k_p=0$ if $p\notin E(R)$ and $k_p\geq 0$ if $p\in E(R)$. Then the coefficient $C^{n-1}_{\mathbf{k}}(\tilde\nu)$ in (6.11) satisfies the recurrence relation

$$\sum_{p\in E(R)}\left(k_p^2+\frac{\tilde\nu_p-\tilde\nu_{p+1}}{2}k_p\right)C^{n-1}_{\mathbf{k}}(\tilde\nu)=\sum_{p\in E(R)}C^{n-1}_{\mathbf{k}-\mathbf{d}_p}(\tilde\nu),$$

and which can be immediately solved:

$$C^{n-1}_{\mathbf{k}}(\tilde\nu)=\prod_{p\in E(R)}\frac{1}{k_p!\left(\frac{\nu_{i'_p}-\nu_{i'_{p+1}}+2}{2}\right)_{k_p}}\cdot C^{n-1}_{\mathbf{0}}(\tilde\nu)=\prod_{p\in E(R)}\frac{1}{k_p!\left(\frac{\nu_{i'_p}-\nu_{i'_{p+1}}+2}{2}\right)_{k_p}}.$$

In view of

$$P^{I,R}_{A;\mathbf{0},\mathbf{k}}=\prod_{p\in R^c}(-1)^{\chi_{[a,n-1]}(p)}\left(\frac{-\nu_i+\nu_{i'_p}+1}{2}\right)_{\chi_{[a,n-1]}(p)}\prod_{p\in R}(-1)^{\chi_{[1,a-1]}(p)}\left(\frac{\nu_i-\nu_{i'_p}+1}{2}\right)_{\chi_{[1,a-1]}(p)}$$
$$\cdot\prod_{p\in E(R)}\left(\frac{-\nu_i+\nu_{i'_{p+1}}+1}{2}+\chi_{[a,n-1]}(p+1)\right)_{-k_p}\left(\frac{\nu_i-\nu_{i'_p}+1}{2}+\chi_{[1,a-1]}(p)\right)_{-k_p},$$

we get

$$C^{I,R}_{A;\mathbf{0}}(\nu)=\prod_{p\in R^c}\frac{(-1)^{\chi_{[a,n-1]}(p)}}{\left(\frac{-\nu_i+\nu_{i'_p}+1}{2}\right)_{\chi_{[a,n-1]}(p)}}\prod_{p\in R}\frac{(-1)^{\chi_{[1,a-1]}(p)}}{\left(\frac{\nu_i-\nu_{i'_p}+1}{2}\right)_{\chi_{[1,a-1]}(p)}}$$

$$\cdot \prod_{p\in E(R)} \sum_{k_p=0}^{\infty} \frac{1}{k_p!\Big(\frac{\nu_{i'_p}-\nu_{i'_{p+1}}+2}{2}\Big)_{k_p}\Big(\frac{-\nu_i+\nu_{i'_{p+1}}+1}{2}+\chi_{[a,n-1]}(p+1)\Big)_{-k_p}}$$
$$\cdot \frac{1}{\Big(\frac{\nu_i-\nu_{i'_p}+1}{2}+\chi_{[1,a-1]}(p)\Big)_{-k_p}}.$$

By using the formula

$$\sum_{k=0}^{\infty} \frac{1}{k!(a+1)_k(b+1)_{-k}(c+1)_{-k}} = \frac{\Gamma(a+1)\Gamma(a+b+c+1)}{\Gamma(a+b+1)\Gamma(a+c+1)}, \tag{6.12}$$

which is a consequence of Gauss's formula, we reach the desired formula. □

6.5. **End of proof.** Now we show the cancellation (6.6). From Lemma 6.6, we have

$$\begin{aligned}
\Gamma^{I,R}(\nu)\cdot C^{I,R}_{\{1\};\mathbf{0}}(\nu) = &\prod_{1\le p<q\le n-1} \Gamma\Big(\frac{-\nu_{i'_p}+\nu_{i'_q}}{2}\Big) \\
&\cdot \prod_{p\in R^c} \Gamma\Big(\frac{\nu_i-\nu_{i'_p}-1}{2}\Big) \prod_{p\in R} \Gamma\Big(\frac{-\nu_i+\nu_{i'_p}+1}{2}\Big) \\
&\cdot \prod_{p\in E(R)} \frac{\Gamma\Big(\frac{\nu_{i'_p}-\nu_{i'_{p+1}}}{2}+1\Big)}{\Gamma\Big(\frac{-\nu_i+\nu_{i'_p}+3}{2}\Big)\Gamma\Big(\frac{\nu_i-\nu_{i'_{p+1}}+1}{2}\Big)}.
\end{aligned}$$

Then, for $r\in E(R)$ we can see that

$$\begin{aligned}
\frac{\Gamma^{I,R}(\tau_r\nu)}{\Gamma^{I,R}(\nu)}\cdot\frac{C^{I,R}_{\{1\};\mathbf{0}}(\tau_r\nu)}{C^{I,R}_{\{1\};\mathbf{0}}(\nu)} &= \frac{\Gamma\Big(\frac{-\nu_{i'_{r+1}}+\nu_{i'_r}}{2}\Big)}{\Gamma\Big(\frac{-\nu_{i'_r}+\nu_{i'_{r+1}}}{2}\Big)}\cdot\frac{\Gamma\Big(\frac{\nu_i-\nu_{i'_r}-1}{2}\Big)\Gamma\Big(\frac{-\nu_i+\nu_{i'_{r+1}}+1}{2}\Big)}{\Gamma\Big(\frac{\nu_i-\nu_{i'_{r+1}}-1}{2}\Big)\Gamma\Big(\frac{-\nu_i+\nu_{i'_r}+1}{2}\Big)} \\
&\quad\cdot\frac{\Gamma\Big(\frac{\nu_{i'_{r+1}}-\nu_{i'_r}+2}{2}\Big)}{\Gamma\Big(\frac{-\nu_i+\nu_{i'_{r+1}}+3}{2}\Big)\Gamma\Big(\frac{\nu_i-\nu_{i'_r}+1}{2}\Big)}\cdot\frac{\Gamma\Big(\frac{-\nu_i+\nu_{i'_r}+3}{2}\Big)\Gamma\Big(\frac{\nu_i-\nu_{i'_{r+1}}+1}{2}\Big)}{\Gamma\Big(\frac{\nu_{i'_r}-\nu_{i'_{r+1}}+2}{2}\Big)} \\
&= \frac{\sin\Big(\frac{\nu_{i'_{r+1}}-\nu_{i'_r}}{2}\Big)\pi}{\sin\Big(\frac{\nu_{i'_r}-\nu_{i'_{r+1}}}{2}\Big)\pi}\cdot\frac{\nu_i-\nu_{i'_{r+1}}-1}{\nu_i-\nu_{i'_r}-1}\cdot\frac{-\nu_i+\nu_{i'_r}+1}{-\nu_i+\nu_{i'_{r+1}}+1} \\
&= -1.
\end{aligned}$$

The proof for $A=\{n\}$ is similar. Hence we can finish the proof of Theorem 6.1. □

7. Basic notation for some combinatorics

From now on we consider the case of general helicity. We first prepare some notation to describe power series, such as subsets in $T_n=\{1,\dots,n\}$, characteristic functions of subsets and some elements of the Weyl group.

7.1. **Colors and palettes.** We have defined the notion of *color* of type (n,h) in section 1, and utilized three kinds of colors, I, J and A to describe the secondary Whittaker functions in the case of helicity 1. In addition, it is better to introduce a notion of *palette* to discuss the case of $h \geq 2$.

Definition 7.1. For a color X_0 of type $(n-1,h-1)$ and an integer x_n in T_n, we call the set $\Pi = (X_0, x_n)$ a *palette* of type (n,h).

We define some subsets of T_n attached to a color.

Notation 7.2. Let $X = \{x_1, \dots, x_h\}$ be a color of type (n,h).

(i) We denote by

$$X^e \equiv X^{even} = \{x_t \mid 1 \leq t \leq h,\ t : \text{even}\},$$
$$X^o \equiv X^{odd} = \{x_t \mid 1 \leq t \leq h,\ t : \text{odd}\}.$$

We also denote by T_n^e (resp. T_n^o) the set of even (resp. odd) integers in T_n.

(ii) Let X^c be the complement color (of type $(n, n-h)$) of X in T_n. We regard X^c as a union of intervals and denote it by $I(X)$:

$$X^c \equiv I(X) = [1, x_1 - 1] \cup [x_1 + 1, x_2 - 1] \cup \cdots = \bigcup_{0 \leq t \leq h} [x_t + 1, x_{t+1} - 1].$$

We divide $I(X)$ into two subsets $I^e(X) \equiv I^{even}(X)$ and $I^o(X) \equiv I^{odd}(X)$ with

$$I^e(X) = [1, x_1 - 1] \cup [x_2 + 1, x_3 - 1] \cup \cdots = \bigcup_{\substack{0 \leq t \leq h \\ t:even}} [x_t + 1, x_{t+1} - 1],$$
$$I^o(X) = [x_1 + 1, x_2 - 1] \cup [x_3 + 1, x_4 - 1] \cup \cdots = \bigcup_{\substack{0 \leq t \leq h \\ t:odd}} [x_t + 1, x_{t+1} - 1].$$

Here we understand $x_0 = 0$ and $x_{h+1} = n+1$. We also set

$$I_l^e(X) = I^e(X) \cup X^e, \qquad I_r^e(X) = I^e(X) \cup X^o,$$
$$I_l^o(X) = I^o(X) \cup X^o, \qquad I_r^o(X) = I^o(X) \cup X^e.$$

(iii) Let Y be a color of type (n', h'). We define characteristic functions $\chi_{X,Y}$, $\chi_{X,I(Y)}$ and $\chi_{I(X),I(Y)}$ on T_n by

$$\chi_{X,Y} = (\chi_{X^e} - \chi_{X^o})(\chi_{Y^e} - \chi_{Y^o}),$$
$$\chi_{X,I(Y)} = (\chi_{X^e} - \chi_{X^o})(\chi_{I^e(Y)} - \chi_{I^o(Y)}),$$
$$\chi_{I(X),I(Y)} = (\chi_{I^e(X)} - \chi_{I^o(X)})(\chi_{I^e(Y)} - \chi_{I^o(Y)}).$$

Definition 7.3. For $A = \{a_1, \dots, a_h\} \in {}_n\mathcal{C}_h$ and $B = \{b_1, \dots, b_{h-1}\} \in {}_{n-1}\mathcal{C}_{h-1}$, we say that A and B *interlace* if

$$(1 \leq) a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_{h-1} \leq b_{h-1} < a_h (\leq n).$$

We denote by $B \prec A$ when A and B interlace.

Notation 7.4. Assume that $A = \{a_1, \dots, a_h\} \in {}_n\mathcal{C}_h$ and $B = \{b_1, \dots, b_{h-1}\} \in {}_{n-1}\mathcal{C}_{h-1}$ interlace. We define subsets of T_n by

$$I^e(A, B) = [a_2, b_2 - 1] \cup [a_4, b_4 - 1] \cup \cdots = \bigcup_{\substack{1 \leq t \leq h \\ t:even}} [a_t, b_t - 1],$$

$$I^o(A,B) = [a_1, b_1 - 1] \cup [a_3, b_3 - 1] \cup \cdots = \bigcup_{\substack{1 \le t \le h \\ t: odd}} [a_t, b_t - 1],$$

$$I^e(B,A) = [1, a_1 - 1] \cup [b_2 + 1, a_3 - 1] \cup \cdots = \bigcup_{\substack{0 \le t \le h-1 \\ t: even}} [b_t + 1, a_{t+1} - 1],$$

$$I^o(B,A) = [b_1 + 1, a_2 - 1] \cup [b_3 + 1, a_4 - 1] \cup \cdots = \bigcup_{\substack{0 \le t \le h-1 \\ t: odd}} [b_t + 1, a_{t+1} - 1].$$

We further set

$$I(A,B) = I^e(A,B) \cup I^o(A,B) \quad \text{and} \quad I(B,A) = I^e(B,A) \cup I^o(B,A).$$

We note that $b_0 = 0$ and $b_h = n$.

Since we will sometimes switch the role of $I^e(X)$ and $I^o(X)$ depending on $t \in T_h$ modulo 2, we use the following notation.

Notation 7.5. For $1 \le t \le h$, we use the subscript $(\epsilon(t), \bar{\epsilon}(t))$ as (e, o) if t is even, and (o, e) if t is odd.

7.2. **Characteristic functions.** Let us define three kinds of characteristic functions ε, ζ and η attached to colors or palettes. In this subsection we use the following notation.

- $A \in {}_n\mathcal{C}_h$, $B \in {}_{n-1}\mathcal{C}_{h-1}$ such that $B \prec A$.
- $A^c \in {}_n\mathcal{C}_{n-h}$ and $B^c \in {}_{n-1}\mathcal{C}_{n-h}$ (= the complements of A and B).
- $J \in {}_n\mathcal{C}_h$ and $J^c \in {}_n\mathcal{C}_{n-h}$.
- $J_0 \in {}_{n-1}\mathcal{C}_{h-1}$, $J_0^c \in {}_{n-1}\mathcal{C}_{n-h}$, $j_h \in T_n$ and $\Pi = (J_0, j_h)$.
- $R \subset T_{n-1}$, R^c (= the complements of R in T_{n-1}).

We first define an integer t_Π and a color J_Π associated to a palette Π.

Definition 7.6. For $\Pi = (J_0, j_h) = (\{j_1, \ldots, j_{h-1}\}, j_h)$, let t_Π $(0 \le t_\Pi \le h-1)$ be the unique integer such that $j_{t_\Pi} < j_h \le j_{t_\Pi+1}$. We define a color J_Π of type (n,h) by

$$J_\Pi = \{j_1, \ldots, j_{t_\Pi}, j_h, j_{t_\Pi+1} + 1, \ldots, j_{h-1} + 1\}.$$

Definition 7.7. We define characteristic functions ε_A^J, $\varepsilon_A^{J_0,R}$ and ε_A^Π on T_{n-1}.

- $\varepsilon_A^J = \chi_{I_l^e(J)} \cdot \chi_{I_l^o(A)} + \chi_{I_l^o(J)} \cdot \chi_{I_l^e(A)}$.
- $\varepsilon_A^{J_0,R} = (\chi_{I_l^e(J_0)} \cdot \chi_{I_l^o(A)} + \chi_{I_l^o(J_0)} \cdot \chi_{I_l^e(A)})\chi_{R^c} + (\chi_{I_r^e(J_0)} \cdot \chi_{I_l^e(A)} + \chi_{I_r^o(J_0)} \cdot \chi_{I_l^o(A)})\chi_R$.
- $\varepsilon_A^\Pi \equiv \varepsilon_A^{(J_0,j_h)} = \varepsilon_A^{J_0,[j_h,n-1]}$.

Lemma 7.8. *We have the following.*

(i) $\varepsilon_A^\Pi = \varepsilon_A^{J_\Pi}$.

(ii) $\varepsilon_A^J(p) - \varepsilon_A^J(p-1) = -\chi_{A,J^c}(p) - \chi_{A^c,J}(p)$.

(iii) $\displaystyle\sum_{p=1}^n (\chi_{A,J^c}(p) + \chi_{A^c,J}(p)) = 0$.

Proof. It is immediate from the definition. The last claim (iii) follows from (ii) and $\sum_{p=1}^n (\varepsilon_A^J(p) - \varepsilon_A^J(p-1)) = 0$. □

Definition 7.9. We define characteristic functions $\zeta_{A,B}^J$, $\eta_{A,B}^J$, $\zeta_{A,B}^{J_0,R}$, $\eta_{A,B}^{J_0,R}$, $\zeta_{A,B}^\Pi$ and $\eta_{A,B}^\Pi$ on T_{n-1} as follows.

- $\zeta_{A,B}^{J} = -\chi_{I_l^e(J)} \cdot \chi_{I^e(A,B)} - \chi_{I_l^o(J)} \cdot \chi_{I^o(A,B)} + \chi_{[j_h,n-1]} \cdot \chi_{I_l^{\varepsilon(h)}(B)}$.
- $\eta_{A,B}^{J} = \chi_{I^e(J)} \cdot \chi_{I^o(A,B)\cup B^o} + \chi_{I^o(J)} \cdot \chi_{I^e(A,B)\cup B^e} - \chi_{[j_h,n-1]} \cdot \chi_{I_r^{\varepsilon(h)}(B)}$.
- $\zeta_{A,B}^{J_0,R} = \zeta^{R^c} \cdot \chi_{R^c} + \zeta^{R} \cdot \chi_R$ with

$$\zeta^{R^c} = -\chi_{I_l^e(J_0)} \cdot \chi_{I^e(A,B)} - \chi_{I_l^o(J_0)} \cdot \chi_{I^o(A,B)},$$
$$\zeta^{R} = \chi_{I^e(J_0)} \cdot \chi_{I^e(B,A)\cup B^e} + \chi_{I^o(J_0)} \cdot \chi_{I^o(B,A)\cup B^o} - \chi_{J_0^o} \cdot \chi_{I^o(A,B)} - \chi_{J_0^e} \cdot \chi_{I^e(A,B)}.$$

- $\eta_{A,B}^{J_0,R} = \eta^{R^c} \cdot \chi_{R^c} + \eta^{R} \cdot \chi_R$ with

$$\eta^{R^c} = \chi_{I^e(J_0)} \cdot \chi_{I^o(A,B)\cup B^o} + \chi_{I^o(J_0)} \cdot \chi_{I^e(A,B)\cup B^e} - \chi_{J_0^o} \cdot \chi_{I^o(B,A)} - \chi_{J_0^e} \cdot \chi_{I^e(B,A)},$$
$$\eta^{R} = -\chi_{I_r^e(J_0)} \cdot \chi_{I^o(B,A)} - \chi_{I_r^o(J_0)} \cdot \chi_{I^e(B,A)}.$$

- $\zeta_{A,B}^{\Pi} \equiv \zeta_{A,B}^{(J_0,j_h)} = \zeta_{A,B}^{J_0,[j_h,n-1]}$.
- $\eta_{A,B}^{\Pi} \equiv \eta_{A,B}^{(J_0,j_h)} = \eta_{A,B}^{J_0,[j_h,n-1]}$.

For our later use we prepare some relations.

Lemma 7.10. (i) *For R and its complement R^c, we have*

$$\zeta^{R} - \zeta^{R^c} = \chi_{I^e(J_0)} \cdot \chi_{I_l^e(A)} + \chi_{I^o(J_0)} \cdot \chi_{I_l^o(A)},$$
$$\eta^{R} - \eta^{R^c} = -\chi_{J_0^c} + \zeta^{R} - \zeta^{R^c}.$$

(ii) *For adjacent indices $p, p\pm 1$, we have*

$$\eta^R(p) - \eta^R(p+1) = \chi_{I^e(J_0)}(\chi_{B^o} - \chi_{A^e[-1]}) + \chi_{I^o(J_0)}(\chi_{B^e} - \chi_{A^o[-1]}) + \chi_{J_0^e}(\chi_{I^o(B,A)[-1]} - \chi_{I^e(B,A)}) + \chi_{J_0^o}(\chi_{I^e(B,A)[-1]} - \chi_{I^o(B,A)}),$$
$$\zeta^{R^c}(p) - \zeta^{R^c}(p-1) = \chi_{I^e(J_0)}(\chi_{B^e} - \chi_{A^e}) + \chi_{I^o(J_0)}(\chi_{B^o} - \chi_{A^o}) + \chi_{J_0^e}(\chi_{I^o(A,B)[+1]} - \chi_{I^e(A,B)}) + \chi_{J_0^o}(\chi_{I^e(A,B)[+1]} - \chi_{I^o(A,B)}).$$

Here we omit (p) from $\chi_*(p)$ in the light hand side.

(iii) *We have*

$$\varepsilon_A^{J_0,R}(p) - (2\zeta_{A,B}^{J_0,R}(p) - \chi_{J_0^c}(p) \cdot \chi_R(p)) = \chi_{I(A,B)}(p) + \varepsilon_{J_0}^{B}(p),$$
$$\varepsilon_A^{J_0,R}(p) - (2\eta_{A,B}^{J_0,R}(p) - \chi_{J_0^c}(p) \cdot \chi_R(p)) = \chi_{I(B,A)}(p) + \varepsilon_{J_0}^{B}(p-1).$$

Proof. We can directly check from the definition of characteristic functions. For example,

$$\begin{aligned}
&\varepsilon_A^{J_0,R} - (2\zeta_{A,B}^{J_0,R} - \chi_{J_0^c} \cdot \chi_R) \\
&= \chi_{R^c}(\chi_{I_l^e(J_0)} \cdot \chi_{I_l^o(A)} + \chi_{I_l^o(J_0)} \cdot \chi_{I_l^e(A)}) + \chi_R(\chi_{I_r^e(J_0)} \cdot \chi_{I_l^e(A)} + \chi_{I_r^o(J_0)} \cdot \chi_{I_l^o(A)}) \\
&+ 2\chi_{R^c}(\chi_{I_l^e(J_0)} \cdot \chi_{I^e(A,B)} + \chi_{I_l^o(J_0)} \cdot \chi_{I^o(A,B)}) \\
&+ 2\chi_R(-\chi_{I^e(J_0)} \cdot \chi_{I^e(B,A)\cup B^e} - \chi_{I^o(J_0)} \cdot \chi_{I^o(B,A)\cup B^o} + \chi_{J_0^e} \cdot \chi_{I^e(A,B)} + \chi_{J_0^o} \cdot \chi_{I^o(A,B)}) \\
&+ \chi_R(\chi_{I^e(J_0)} + \chi_{I^o(J_0)}) \\
&= \chi_{R^c}\{\chi_{I_l^e(J_0)}(\chi_{I_l^o(A)} + 2\chi_{I^e(A,B)}) + \chi_{I_l^0(J_0)}(\chi_{I_l^e(A)} + 2\chi_{I^o(A,B)})\} \\
&+ \chi_R\{\chi_{I^e(J_0)}(\chi_{I_l^o(A)} + 2\chi_{I^e(A,B)}) + \chi_{I^o(J_0)}(\chi_{I_l^e(A)} + 2\chi_{I^o(A,B)}) \\
&\qquad + \chi_{J_0^e}(\chi_{I_l^e(A)} + 2\chi_{I^o(A,B)}) + \chi_{J_0^o}(\chi_{I_l^o(A)} + 2\chi_{I^e(A,B)})\} \\
&= \chi_{I_l^e(J_0)}(\chi_{I_l^o(A)} + 2\chi_{I^e(A,B)}) + \chi_{I_l^o(J_0)}(\chi_{I_l^e(A)} + 2\chi_{I^o(A,B)}) \\
&= \chi_{I_l^e(J_0)}(\chi_{I^o(A,B)} + \chi_{I_l^o(B)} + \chi_{I^e(A,B)}) + \chi_{I_l^o(J_0)}(\chi_{I^e(A,B)} + \chi_{I_l^e(B)} + \chi_{I^o(A,B)}) \\
&= \chi_{I(A,B)} + \chi_{I_l^e(J_0)} \cdot \chi_{I_l^o(B)} + \chi_{I_l^o(J_0)} \cdot \chi_{I_l^e(B)} \\
&= \chi_{I(A,B)} + \varepsilon_B^{J_0}.
\end{aligned}$$

□

7.3. **Weyl group.** Let $I = \{i_1, \ldots, i_h\}$ be the color of type (n, h) defining the principal series $\pi_{I,\nu}$ of G. As in the case of helicity 1, we specify elements $\tau^{I,J}$ and $\tau^{I,\Pi}$ of the Weyl group $\mathfrak{S}_n$ satisfying $\tau^{I,J}(J) = I$ and $\tau^{I,\Pi}(J_\Pi) = I$, respectively. Recall that we have defined the color $I_0 = \{i_{0,1}, \ldots, i_{0,h-1}\}$ of type $(n-1, h-1)$ in Definition 4.1. For I and I_0, let

$$I^c = \{i'_1, \ldots, i'_{n-h}\} \;\; (i'_1 < \cdots < i'_{n-h}) \;\; \text{and} \;\; I_0^c = \{i'_{0,1}, \ldots, i'_{0,n-h}\} \;\; (i'_{0,1} < \cdots < i'_{0,n-h})$$

be the complements of I and I^c in T_n and T_{n-1}, respectively.

Definition 7.11. For a color $J = \{j_1, \ldots, j_h\}$ of type (n, h) and its complement $J^c = \{j'_1, \ldots, j'_{n-h}\}$ $(j'_1 < \cdots < j'_{n-h})$, we define an element $\tau^{I,J}$ of $\mathfrak{S}_n$ by

$$\tau^{I,J}(j_t) = i_{h+1-t} \; (1 \le t \le h), \quad \tau^{I,J}(j'_t) = i'_t \; (1 \le t \le n-h).$$

Remark 4. For $p \in J^c$, if we take an integer t $(0 \le t \le h)$ such that $p \in [j_t + 1, j_{t+1} - 1]$, then $\tau^{I,J}(p) = i'_{p-t}$.

Next we define an element $\tau^{I,\Pi}$ of $\mathfrak{S}_n$ related to a palette $\Pi = (J_0 = \{j_1, \ldots, j_{h-1}\}, j_h)$.

Definition 7.12. Under the same notation as in Definition 7.6, we set

$$\tau^{I,\Pi} = (i_1, i_2, \ldots, i_{h-t_\Pi}) \cdot \tau^{I,J_\Pi}.$$

Here $(i_1, i_2, \ldots, i_{h-t_\Pi})$ is the cyclic permutation of length $h - t_\Pi$.

Lemma 7.13. (i) *For $p = j_t \in J_0$ $(1 \le t \le h-1)$, we have $\tau^{I,\Pi}(p + \chi_{[j_h, n-1]}(p)) = i_{h+1-t}$, and $\tau^{I,\Pi}(j_h) = i_1$.*

(ii) *For $p \in J_0^c$, we take an integer t $(0 \le t \le h-1)$ such that $p \in [j_t + 1, j_{t+1} - 1]$ if $p < j_{h-1}$, and $t = h-1$ if $p > j_{h-1}$. Then we have $\tau^{I,\Pi}(p + \chi_{[j_h, n-1]}(p)) = i'_{p-t}$.*

Proof. It easily follows from the definition. □

Here is a relation between $\tau^{I_0,J_0} \in \mathfrak{S}_{n-1}$ and $\tau^{I,\Pi} \in \mathfrak{S}_n$.

Lemma 7.14. *We use the abbreviation*

$$\tilde{\tau}_0(p) = \tau^{I_0,J_0}(p) + \chi_{[i_1,n-1]}(\tau^{I_0,J_0}(p))$$

for $1 \le p \le n-1$. Then we have

$$\tilde{\tau}_0(p) = \tau^{I,\Pi}(p + \chi_{[j_h,n-1]}(p)).$$

Proof. When $p = j_t \in J_0$, we find $\tau^{I_0,J_0}(p) = i_{(h-1)+1-t+1} - 1$, and our claim follows from Lemma 7.13 (i). When $p \in J_0^c$, the identity $i'_{0,t} + \chi_{[i_1,n-1]}(i'_{0,t}) = i'_t$ $(1 \le t \le n-h)$ and Lemma 7.13 (ii) imply the assertion. □

8. Secondary Whittaker functions (2) -the case of general helicity h-

Similarly as the case of helicity 1, we express Whittaker functions on $SL(n, \mathbf{R})$ with helicity h in terms of (a sum of) those on $SL(n-1, \mathbf{R})$ with helicity $h-1$ (Definitions 8.4 and 8.5). Our argument is analogous to section 5, though we need more complicated combinatorics.

8.1. **The recurrence relations.** As in the case of helicity 1, we set

$$\widetilde{W}_A(y) = \sqrt{-1}^{-(a_1+\cdots+a_h)}\Phi_A(y),$$

and consider a power series solution of the form

$$\Phi_A(y) = \sum_{\mathbf{m}} C_{A;\mathbf{m}}(\Phi) \prod_{p=1}^{n-1} (\pi y_p)^{m_p+l_p}.$$

with $C_{A;\mathbf{0}}(\Phi) \neq 0$ for some component A. We can derive the following recurrence relations for $C_{A;\mathbf{m}}(\Phi)$:

Lemma 8.1. (i) *The Dirac-Schmid equation implies*

$$\sum_{t=1}^{h} \{-(m_{a_t-1} + l_{a_t-1}) + (m_{a_t} + l_{a_t}) - \nu_{i_t}\} C_{A;\mathbf{m}}(\Phi)$$
$$+ 2\sum_{t=1}^{h} \big(-C_{A_t^-;\mathbf{m}-\mathbf{e}_{a_t-1}}(\Phi) + C_{A_t^+;\mathbf{m}-\mathbf{e}_{a_t}}(\Phi)\big) = 0.$$

(ii) *The Capelli equation of degree two implies*

$$\Big\{\sum_{p=1}^{n-1}(m_p + l_p)^2 - \sum_{p=1}^{n-2}(m_p + l_p)(m_{p+1} + l_{p+1}) + S_2(\nu)\Big\} C_{A;\mathbf{m}}(\Phi)$$
$$+ 2\sum_{t=1}^{h} \big(C_{A_t^-;\mathbf{m}-\mathbf{e}_{a_t-1}}(\Phi) + C_{A_t^+;\mathbf{m}-\mathbf{e}_{a_t}}(\Phi)\big) = 4\sum_{p=1}^{n-1} C_{A;\mathbf{m}-2\mathbf{e}_p}(\Phi).$$

8.2. **The coefficients with total degrees at most one.** We can determine the coefficients $C_{A;\mathbf{m}}(\Phi)$ with total degrees 0 and 1.

Proposition 8.2. *The characteristic indices $l = (l_1, \ldots, l_{n-1})$ at $y = 0$ are given by*

$$l_p = \nu_{\tau(1)} + \cdots + \nu_{\tau(p)} \quad (1 \leq p \leq n-1)$$

for some $\tau \in \mathfrak{S}_n$. If we denote by J a color of type (n, h) satisfying $\tau(J) = I$, then we may impose the normalizing condition:

$$C_{A;\mathbf{0}}(\Phi) = \begin{cases} 1 & \text{if } A = J, \\ 0 & \text{if } A \neq J, \end{cases}$$

and have

$$C_{A;\mathbf{e}_p}(\Phi) = \begin{cases} \frac{-2}{\nu_{\tau(p)} - \nu_{\tau(p+1)} + 1} & \text{if } (J,p) = (A_t^+, a_t), (A_t^-, a_t - 1) \text{ for some } t, \\ 0 & \text{otherwise.} \end{cases}$$

In view of this proposition we can show the following:

Corollary 8.3. *We fix $\tau \in \mathfrak{S}_n$. The power series solution corresponding to the characteristic indices $l = (l_1, \ldots, l_{n-1})$ with $l_p = \nu_{\tau(1)} + \cdots + \nu_{\tau(p)}$ is of the form*

$$\Phi_A^J(y) = \sum_{\mathbf{m}} C_{A;\mathbf{m}}(\Phi) \prod_{p=1}^{n-1} (\pi y_p)^{2m_p + \nu_{\tau(1)} + \cdots + \nu_{\tau(p)} + \varepsilon_A^J(p)}.$$

8.3. **Explicit formulas of the secondary Whittaker functions.** Now we give explicit formulas of the power series $M_\nu^{n,I,J}$, which can be described in terms of $M_{\tilde\nu}^{n-1,I_0,J_0}$. Though we can construct the solution space only by the power series $M_{A,\nu}^{n,I,J}$ (Theorem 8.7), to make our proof more understandable it is better to introduce yet another power series solutions $M_{A,\nu}^{n,I,\Pi}$ parametrized by palettes, each of which are connected to the solutions $M_{A,\nu}^{n,I,J}$ by *palette symmetries* (Proposition 8.6).

Definition 8.4. We define a power series

$$M_\nu^{n,I,J}(y) = y^\rho \widetilde{M}_\nu^{n,I,J}(y) = \left\{ y^\rho \sqrt{-1}^{-\sum_{t=1}^{h} a_t} \widetilde{M}_{A,\nu}^{n,I,J}(y) \right\}_{A \in {}_n\mathcal{C}_h}$$

by

$$\widetilde{M}_{A,\nu}^{n,I,J}(y) = \sum_{\mathbf{m}} C_{A;\mathbf{m}}^{n,I,J}(\nu) \prod_{p=1}^{n-1} (\pi y_p)^{2m_p + \nu_{\tau^{I,J}(1)} + \cdots + \nu_{\tau^{I,J}(p)} + \varepsilon_A^J(p)}$$

where the coefficients are given by

$$C_{A;\mathbf{m}}^{n,I,J}(\nu) = \sum_{B \prec A} \sum_{\mathbf{k}} \frac{C_{B;\mathbf{k}}^{n-1,I_0,J_0}(\tilde\nu)}{P_{A,B;\mathbf{m},\mathbf{k}}^{I,J}},$$

with

$$\begin{aligned} P_{A,B;\mathbf{m},\mathbf{k}}^{I,J} = & \prod_{p=1}^{n-1} (-1)^{\zeta_{A,B}^J(p) + \eta_{A,B}^J(p)} \\ & \cdot \prod_{p=1}^{j_h-1} \left(m_p - k_p + \zeta_{A,B}^J(p)\right)! \left(\frac{-\nu_{i_1} + \nu_{\tau^{I,J}(p)} + 1 + \chi_{J_0}(p)}{2} \right)_{m_p - k_{p-1} + \eta_{A,B}^J(p)} \\ & \cdot \prod_{p=j_h}^{n-1} \left(m_p - k_{p-1} + \eta_{A,B}^J(p)\right)! \left(\frac{\nu_{i_1} - \nu_{\tau^{I,J}(p+1)} + 1 + \chi_{J_0}(p)}{2} \right)_{m_p - k_p + \zeta_{A,B}^J(p)}. \end{aligned}$$

Here in the sum $\sum_{B \prec A}$, B runs through colors of type $(n-1, h-1)$ with $B \prec A$, and in the sum $\sum_{\mathbf{k}}$, $\mathbf{k} = (k_1, \ldots, k_{n-2}) \in \mathbf{N}^{n-2}$ ranges over vectors of nonnegative integers with non-vanishing $P_{A,B;\mathbf{m},\mathbf{k}}^{I,J}$, that is,

$$\begin{cases} k_p \le m_p - \chi_{I^e(A,B)}(p) & \text{if } p \in I_l^e(J_0) \cap [1, j_h - 1], \\ k_p \le m_p - \chi_{I^o(A,B)}(p) & \text{if } p \in I_l^o(J_0) \cap [1, j_h - 1], \\ k_p \le m_{p+1} - \chi_{I^{\bar\varepsilon(h)}(B,A)}(p+1) & \text{if } p \in [j_h - 1, n-2]. \end{cases}$$

Definition 8.5. For a palette $\Pi = (J_0, j_h)$ of type (n, h), we define a power series

$$M_\nu^{n,I,\Pi}(y) = y^\rho \widetilde{M}_\nu^{n,I,\Pi}(y) = \left\{ y^\rho \sqrt{-1}^{-\sum_{t=1}^{h} a_t} \widetilde{M}_{A,\nu}^{n,I,\Pi}(y) \right\}_{A \in {}_n\mathcal{C}_h}$$

by

$$\widetilde{M}_{A,\nu}^{n,I,\Pi}(y) = \sum_{\mathbf{m}} C_{A;\mathbf{m}}^{n,I,\Pi}(\nu) \prod_{p=1}^{n-1} (\pi y_p)^{2m_p + \nu_{\tau^{I,\Pi}(1)} + \cdots + \nu_{\tau^{I,\Pi}(p)} + \varepsilon_A^\Pi(p)}$$

where the coefficients are given by

$$C_{A;\mathbf{m}}^{n,I,\Pi}(\nu) = \sum_{B \prec A} \sum_{\mathbf{k}} \frac{C_{B;\mathbf{k}}^{n-1,I_0,J_0}(\tilde\nu)}{P_{A,B;\mathbf{m},\mathbf{k}}^{I,\Pi}},$$

with

$$
\begin{aligned}
P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}} = &\prod_{p=1}^{n-1} (-1)^{\zeta^{\Pi}_{A,B}(p)+\eta^{\Pi}_{A,B}(p)} \\
&\cdot \prod_{p=1}^{j_h-1} \big(m_p - k_p + \zeta^{\Pi}_{A,B}(p)\big)! \left(\frac{-\nu_{i_1} + \nu_{\tau^{I,\Pi}(p)} + 1 + \chi_{J_0}(p)}{2}\right)_{m_p - k_{p-1} + \eta^{\Pi}_{A,B}(p)} \\
&\cdot \prod_{p=j_h}^{n-1} \big(m_p - k_{p-1} + \eta^{\Pi}_{A,B}(p)\big)! \left(\frac{\nu_{i_1} - \nu_{\tau^{I,\Pi}(p+1)} + 1 + \chi_{J_0}(p)}{2}\right)_{m_p - k_p + \zeta^{\Pi}_{A,B}(p)}.
\end{aligned}
$$

Here $\sum_{\mathbf{k}}$ means $\mathbf{k} = (k_1, \dots, k_{n-2}) \in \mathbf{N}^{n-2}$ runs through such that

$$
\begin{cases}
k_p \le m_p + \zeta^{\Pi}_{A,B}(p) & \text{if } 1 \le p \le j_h - 2, \\
k_p \le \min\{m_p + \zeta^{\Pi}_{A,B}(p), m_{p+1} + \eta^{\Pi}_{A,B}(p+1)\} & \text{if } p = j_h - 1, \\
k_p \le m_{p+1} + \eta^{\Pi}_{A,B}(p+1) & \text{if } j_h \le p \le n-2.
\end{cases}
$$

Here is a palette symmetry, that is a relation between $M^{n,I,\Pi}_{\nu}(y)$ and $M^{n,I,J}_{\nu}(y)$.

Proposition 8.6. *For a palette $\Pi = (J_0, j_h)$ of type (n, h), we have*

$$
M^{n,I,\Pi}_{\nu}(y) = M^{n,I,J_\Pi}_{\tau^{I,\Pi}_{cy}\nu}(y).
$$

Here we denote by $\tau^{I,\Pi}_{cy}$ the cyclic permutation $(i_1, i_2, \dots, i_{h-t_\Pi})$. In particular, if $t_\Pi = h$ then $\{j_1, \dots, j_h\}$ is a color of type (n, h) and we have

$$
M^{n,I,\Pi}_{\nu}(y) = M^{n,I,\{j_1,\dots,j_h\}}_{\nu}(y).
$$

In the case of $t_\Pi = h$, our claim is obvious from the definition of the power series. When $t_\Pi < h$, our proof is indirect. We will show that by checking the coincidence of recurrence relations characterizing the coefficients of both of the power series (Lemma 8.9). Note that at the level of the characteristic exponents, our assertion can be seen from the definition of $\tau^{I,\Pi}$ and Lemma 7.8 (i).

Now we state our main result in this section.

Theorem 8.7. *The set of power series*

$$
\{\widetilde{M}^{n,I,J}_{\tau\nu}(y) \mid J \in {}_n\mathcal{C}_h,\, \tau \in \mathfrak{S}_I \cong \mathfrak{S}_h \times \mathfrak{S}_{n-h}\}
$$

forms a basis of the space of solutions of the system in Theorem 3.3 at $y = 0$.

The rest of this section is devoted to the proof of Theorem 8.7. Our argument is an induction on n and h. The difference of the proof with the case of helicity 1 (Theorem 5.7) is use of the power series parametrized by palette. Indeed, assuming Theorem 8.7 for $I = I_0$, we firstly prove the set

$$
\{\widetilde{M}^{n,I,(J_0,j_h)}_{\tau\nu}(y) \mid J_0 \in {}_{n-1}\mathcal{C}_{h-1}, 1 \le j_h \le n,\, \tau \in \mathfrak{S}_{I_0}\}
$$

forms a basis of solutions, by showing the coefficients $C^{n,I,\Pi}_{A;\mathbf{m}}(\nu)$ satisfy the recurrence relations coming from the Dirac-Schmid and the Casimir equations (Proposition 8.8). Then we may also confirm Proposition 8.6 by Lemma 8.9. Hence we conclude our proof.

8.4. **Recurrence relations for $C^{n,I,\Pi}_{A;\mathbf{m}}(\nu)$.** By Lemma 8.1 and Corollary 8.3, the recurrence relations we should show is the following.

Proposition 8.8. *For abbreviation we denote by*

$$\mathbf{m}_t^- = \mathbf{m} - \varepsilon^{\Pi}_{A_t^-}(a_t-1)\cdot \mathbf{e}_{a_t-1}, \quad \mathbf{m}_t^+ = \mathbf{m} - \varepsilon^{\Pi}_{A_t^+}(a_t)\cdot \mathbf{e}_{a_t}.$$

Then the coefficients $C^{n,I,\Pi}_{A;\mathbf{m}} = C^{n,I,\Pi}_{A;\mathbf{m}}(\nu)$ satisfy the following:

$$\left\{\sum_{t=1}^{h}(-m_{a_t-1}+m_{a_t}) + \mu^{I,\Pi}_A(\nu)\right\}C^{n,I,\Pi}_{A;\mathbf{m}} + \sum_{t=1}^{h}\Big(-C^{n,I,\Pi}_{A_t^-;\mathbf{m}_t^-} + C^{n,I,\Pi}_{A_t^+;\mathbf{m}_t^+}\Big) = 0; \tag{8.1}$$

$$Q_{\mathbf{m}}\big(\{\lambda^{I,\Pi}_{A,p}(\nu)\}_{1\le p\le n-1};\kappa^{I,\Pi}_A(\nu)\big)C^{n,I,\Pi}_{A;\mathbf{m}} + \frac{1}{2}\sum_{t=1}^{h}\Big(C^{n,I,\Pi}_{A_t^-;\mathbf{m}_t^-} + C^{n,I,\Pi}_{A_t^+;\mathbf{m}_t^+}\Big) = \sum_{p=1}^{n-1} C^{n,I,\Pi}_{A;\mathbf{m}-\mathbf{e}_p}. \tag{8.2}$$

Here

$$\mu^{I,\Pi}_A(\nu) = \frac{1}{2}\left\{\sum_{t=1}^{h}(\nu_{\tau^{I,\Pi}(a_t)} - \nu_{i_t}) + \sum_{t=1}^{h}\big(\varepsilon^{\Pi}_A(a_t) - \varepsilon^{\Pi}_A(a_t-1)\big)\right\},$$

$$\lambda^{I,\Pi}_{A,p}(\nu) = \frac{1}{2}\big(\nu_{\tau^{I,\Pi}(p)} - \nu_{\tau^{I,\Pi}(p+1)} + 2\varepsilon^{\Pi}_A(p) - \varepsilon^{\Pi}_A(p-1) - \varepsilon^{\Pi}_A(p+1)\big) \qquad (1\le p\le n-1),$$

$$\kappa^{I,\Pi}_A(\nu) = \frac{1}{4}\left\{\sum_{p=1}^{n-1}\varepsilon^{\Pi}_A(p)(\nu_{\tau^{I,\Pi}(p)} - \nu_{\tau^{I,\Pi}(p+1)}) + Q(\varepsilon^{\Pi}_A(1),\dots,\varepsilon^{\Pi}_A(n-1))\right\}.$$

Proposition 8.6 is deduced from Proposition 8.8 and the following Lemma.

Lemma 8.9. *For a color $J=\{j_1,\dots,j_h\}$ of type (n,h), we define $\mu^{I,J}_A(\nu)$, $\lambda^{I,J}_{A,p}(\nu)$ and $\kappa^{I,J}_A(\nu)$ by replacing $J=(\{j_1,\dots,j_{h-1}\},j_h)$ in $\mu^{I,\Pi}_A(\nu)$, $\lambda^{I,J}_{A,p}(\nu)$ and $\kappa^{I,J}_A(\nu)$, respectively. Then we have $\mu^{I,\Pi}_A(\nu) = \mu^{I,J_\Pi}_A(\tau^{I,\Pi}_{cy}\nu)$, $\lambda^{I,\Pi}_{A,p}(\nu) = \lambda^{I,J_\Pi}_{A,p}(\tau^{I,\Pi}_{cy}\nu)$ and $\kappa^{I,\Pi}_A(\nu) = \kappa^{I,J_\Pi}_A(\tau^{I,\Pi}_{cy}\nu)$.*

Proof. It can be seen from Lemmas 7.8 (i) and 7.13. □

Before to start the proof of Proposition 8.8, we remark that Proposition 8.8 is a special case of Lemma 9.8. In our proof of Proposition 8.8, the key identities in Lemmas 8.15 and 8.16 will be shown in Lemmas 9.11 and 9.12 (ii), respectively. Based on these identities and some relations for $P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}$ given in Subsection 8.5, we prove the recurrence relations (8.1) and (8.2) in Subsections 8.6 and 8.7, respectively.

For our later use we write $\kappa^{I,\Pi}_A(\nu)$ and $\kappa^{I,J}_A(\nu)$ explicitly (cf. Lemma 5.9).

Lemma 8.10. *We have*

$$\begin{aligned}
4\kappa^{I,\Pi}_A(\nu) &= \sum_{p=1}^{j_h-1}(\chi_{A,J_0^c}(p) + \chi_{A^c,J_0}(p))(\nu_{i_1} - \nu_{\tau^{I,\Pi}(p)})\\
&\quad + \sum_{p=j_h}^{n-1}(-\chi_{A[-1],J_0^c}(p) - \chi_{(A[-1])^c,J_0}(p))(\nu_{i_1} - \nu_{\tau^{I,\Pi}(p+1)})\\
&\quad + h - \sharp(J_\Pi\cap A).
\end{aligned}$$

Proof. We denote by $\varepsilon^{\Pi}_A = \varepsilon^{\Pi}_{A,1}\cdot\chi_{[1,j_h-1]} + \varepsilon^{\Pi}_{A,2}\cdot\chi_{[j_h,n-1]}$, that is,

$$\varepsilon^{\Pi}_{A,1} = \chi_{I_l^e(J_0)}\cdot\chi_{I_l^o(A)} + \chi_{I_l^o(J_0)}\cdot\chi_{I_l^e(A)},$$

$$\varepsilon^{\Pi}_{A,2} = \chi_{I^e_r(J_0)} \cdot \chi_{I^e_l(A)} + \chi_{I^o_r(J_0)} \cdot \chi_{I^o_l(A)}.$$

Then we have

$$\begin{aligned}
&\varepsilon^{\Pi}_{A,1}(p) - \varepsilon^{\Pi}_{A,1}(p-1) = -\chi_{A,J_0^c}(p) - \chi_{A^c,J_0}(p),\\
&\varepsilon^{\Pi}_{A,2}(p) - \varepsilon^{\Pi}_{A,2}(p+1) = -\chi_{A[-1],J_0^c}(p) - \chi_{(A[-1])^c,J_0}(p),\\
&\varepsilon^{\Pi}_{A,2}(p) - \varepsilon^{\Pi}_{A,1}(p-1) = (\chi_{I^e(A)}(p) - \chi_{I^o(A)}(p))(\chi_{I^e_r(J_0)}(p) - \chi_{I^o_r(J_0)}(p))
\end{aligned} \tag{8.3}$$

(cf. Lemma 7.8 (ii)), and

$$\begin{aligned}
\sum_{p=1}^{n-1} \varepsilon^{\Pi}_A(p)(\nu_{\tau^{I,\Pi}(p)} - \nu_{\tau^{I,\Pi}(p+1)}) &= \sum_{p=1}^{n-1} \varepsilon^{\Pi}_A(p)\{(-\nu_{i_1} + \nu_{\tau^{I,\Pi}(p)}) + (\nu_{i_1} - \nu_{\tau^{I,\Pi}(p+1)})\}\\
&= \sum_{p=1}^{j_h-1} (\varepsilon^{\Pi}_{A,1}(p) - \varepsilon^{\Pi}_{A,1}(p-1))(-\nu_{i_1} + \nu_{\tau^{I,\Pi}(p)})\\
&\quad + \sum_{p=j_h}^{n-1} (\varepsilon^{\Pi}_{A,2}(p) - \varepsilon^{\Pi}_{A,2}(p+1))(\nu_{i_1} - \nu_{\tau^{I,\Pi}(p+1)}).
\end{aligned}$$

Here we used $\tau^{I,\Pi}(j_h) = i_1$.

Let us consider the term $Q(\varepsilon^{\Pi}_A(1), \ldots, \varepsilon^{\Pi}_A(n-1))$. Since

$$\begin{aligned}
&\sum_{p=1}^{n} (\varepsilon^{\Pi}_A(p) - \varepsilon^{\Pi}_A(p-1))^2\\
&= \sum_{p=1}^{n} \varepsilon^{\Pi}_A(p)(\varepsilon^{\Pi}_A(p) - \varepsilon^{\Pi}_A(p-1)) + \sum_{p=1}^{n} \varepsilon^{\Pi}_A(p-1)(\varepsilon^{\Pi}_A(p-1) - \varepsilon^{\Pi}_A(p))\\
&= 2Q(\varepsilon^{\Pi}_A(1), \ldots, \varepsilon^{\Pi}_A(n-1)),
\end{aligned}$$

we get

$$\begin{aligned}
Q(\varepsilon^{\Pi}_A(1), \ldots, \varepsilon^{\Pi}_A(n-1)) = \frac{1}{2}\Bigg\{ &\sum_{p=1}^{j_h-1} (\varepsilon^{\Pi}_{A,1}(p) - \varepsilon^{\Pi}_{A,1}(p-1))^2\\
&+ (\varepsilon^{\Pi}_{A,2}(j_h) - \varepsilon^{\Pi}_{A,1}(j_h-1))^2 + \sum_{p=j_h}^{n} (\varepsilon^{\Pi}_{A,2}(p+1) - \varepsilon^{\Pi}_{A,2}(p))^2 \Bigg\}.
\end{aligned}$$

By using (8.3) we have

$$\begin{aligned}
(\varepsilon^{\Pi}_{A,1}(p) - \varepsilon^{\Pi}_{A,1}(p-1))^2 &= (\chi_{A,J_0^c})^2 + (\chi_{A^c,J_0})^2\\
&= (\chi_{A^e} + \chi_{A^o})(\chi_{I^e(J_0)} + \chi_{I^o(J_0)}) + (\chi_{I^e(A)} + \chi_{I^o(A)})(\chi_{J_0^e} + \chi_{J_0^o})\\
&= \chi_A + \chi_{J_0} - 2\chi_A \cdot \chi_{J_0},\\
(\varepsilon^{\Pi}_{A,2}(p+1) - \varepsilon^{\Pi}_{A,2}(p))^2 &= \chi_{A[-1]} + \chi_{J_0} - 2\chi_{A[-1]} \cdot \chi_{J_0},
\end{aligned}$$

(we omitted (p) from $\chi_*(p)$ in the right hand sides), and

$$(\varepsilon^{\Pi}_{A,2}(j_h) - \varepsilon^{\Pi}_{A,1}(j_h-1))^2 = 1 - \chi_A(j_h).$$

Hence we get

$$Q(\varepsilon^{\Pi}_A(1), \ldots, \varepsilon^{\Pi}_A(n-1)) = h - \chi_A(j_h) - \sum_{p=1}^{j_h-1} \chi_A(p) \cdot \chi_{J_0}(p) - \sum_{p=j_h+1}^{n} \chi_A(p) \cdot \chi_{J_0[+1]}(p)$$

$$= h - \sharp(J_\Pi \cap A),$$

and complete the proof. $\square$

Corollary 8.11. *We have*

$$4\kappa_A^{I,J}(\nu) = \sum_{p=1}^{n} (\chi_{A,J^c}(p) + \chi_{A^c,J}(p))(\nu_{i_1} - \nu_{\tau^{I,J}(p)}) + h - \sharp(A \cap J).$$

For a palette $\Pi = (J_0, j_h)$ *of type* (n, h) *we have*

$$4\kappa_B^{I_0,J_0}(\tilde{\nu}) = \sum_{p=1}^{n-1} (\chi_{B,J_0^c}(p) + \chi_{B^c,J_0}(p))(\nu_{i_1} - \nu_{\tau^{I,\Pi}(p+[j_h,n-1](p))}) + (h-1) - \sharp(B \cap J_0).$$

Proof. The formula for $\kappa_A^{I,J}(\nu)$ is immediate from Lemma 8.10 and then

$$4\kappa_B^{I_0,J_0}(\tilde{\nu}) = \sum_{p=1}^{n-1} (\chi_{B,J_0^c}(p) + \chi_{B^c,J_0}(p))(\tilde{\nu}_{i_2-1} - \tilde{\nu}_{\tau^{I_0,J_0}(p)}) + (h-1) - \sharp(B \cap J_0).$$

In view of the definition of $\tilde{\nu}_p$ and Lemma 7.14, we have

$$\tilde{\nu}_{i_2-1} - \tilde{\nu}_{\tau^{I_0,J_0}(p)} = \nu_{i_2} - \nu_{\tilde{\tau}_0(p)} = (-\nu_{i_1} + \nu_{i_2}) + (\nu_{i_1} - \nu_{\tau^{I,\Pi}(p+\chi_{[j_h,n-1]}(p))}).$$

Hence we can get the formula of $\kappa_B^{I_0,J_0}(\tilde{\nu})$ by using Lemma 7.8 (iii). $\square$

8.5. **Identities between** $P_{A,B;\mathbf{m},\mathbf{k}}^{I,\Pi}$**.** We prepare some identities between $P_{A,B;\mathbf{m},\mathbf{k}}^{I,\Pi}$, which we will be utilized in the proof of compatibilities with the Dirac-Schmid and the Casimir equations.

Definition 8.12. We define functions $\alpha \equiv \alpha_A^{I,\Pi}$ and $\beta \equiv \beta_A^{I,\Pi}$ on T_{n-1} by

$$\alpha(p) = \frac{1}{2}\{\nu_{i_1} - \nu_{\tau^{I,\Pi}(p+1)} + (\chi_{I^e(J_0)}(p) - \chi_{I^o(J_0)}(p))(\chi_{I_l^e(A)}(p) - \chi_{I_l^o(A)}(p))\}\chi_{[j_h,n-1]}(p),$$

$$\beta(p) = \frac{1}{2}\{-\nu_{i_1} + \nu_{\tau^{I,\Pi}(p)} - (\chi_{I^e(J_0)}(p) - \chi_{I^o(J_0)}(p))(\chi_{I_l^e(A)}(p) - \chi_{I_l^o(A)}(p))\}\chi_{[1,j_h-1]}(p).$$

Here is an analogue of the former part of Lemma 5.10.

Lemma 8.13. *For each* $1 \le t \le h$, *we have the following relations.*

(i) *Let* $B \in {}_{n-1}\mathcal{C}_{h-1}$ *be a color of type* $(n-1, h-1)$ *satisfying* $B \prec A_t^-$.
- *If* B *also satisfies* $B \prec A$, *that is,* $b_{t-1} \in [a_{t-1}, a_t - 2]$ *and* $b_{t'} \in [a_{t'}, a_{t'+1} - 1]$ $(t' \neq t-1)$, *then we have*

$$\frac{P_{A,B;\mathbf{m},\mathbf{k}}^{I,\Pi}}{P_{A_t^-,B;\mathbf{m}_t^-,\mathbf{k}}^{I,\Pi}} = -(m_{a_t-1} - k_{a_t-1} + \alpha(a_t - 1)).$$

- *If* B *does not satisfy* $B \prec A$, *then we have* $B = B_{t,a_t-1}$ $(1 \le t \le h-1)$ *and*

$$P_{A_t^-,B;\mathbf{m}_t^-,\mathbf{k}}^{I,\Pi} = P_{A,B_t^+;\mathbf{m},\mathbf{k}+\varepsilon_B^{J_0}(b_t)\cdot\mathbf{d}_{b_t}}^{I,\Pi} = P_{A,B_{t,a_t};\mathbf{m},\mathbf{k}+\chi_{I_l^{\bar{\varepsilon}(t)}(J_0)}(a_t-1)\cdot\mathbf{d}_{a_t-1}}^{I,\Pi}.$$

(ii) *Let* $B \in {}_{n-1}\mathcal{C}_{h-1}$ *be a color of type* $(n-1, h-1)$ *satisfying* $B \prec A_t^+$.

- *If B also satisfies $B \prec A$, that is, $b_t \in [a_t+1, a_{t+1}-1]$ and $b_{t'} \in [a_{t'}, a_{t'+1}-1]$ $(t' \neq t)$, then we have*
$$\frac{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}}{P^{I,\Pi}_{A_t^+,B;\mathbf{m}_t^+,\mathbf{k}}} = -(m_{a_t} - k_{a_t-1} + \beta(a_t)).$$
- *If B does not satisfy $B \prec A$, then we have $B = B_{t-1,a_t}$ $(2 \le t \le h)$ and*
$$P^{I,\Pi}_{A_t^+,B;\mathbf{m}_t^+,\mathbf{k}} = P^{I,\Pi}_{A,B_{t-1}^-;\mathbf{m},\mathbf{k}+\varepsilon_B^{J_0}(b_{t-1}-1)\cdot\mathbf{d}_{b_{t-1}-1}} = P^{I,\Pi}_{A,B_{t-1,a_t-1};\mathbf{m},\mathbf{k}+\chi_{I_l^{\bar\epsilon(t)}(J_0)}(a_t-1)\cdot\mathbf{d}_{a_t-1}}.$$

Proof. (i) From the definition, the ratio $P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}/P^{I,\Pi}_{A_t^-,B;\mathbf{m}_t^-,\mathbf{k}}$ is

$$(8.4)\qquad -\frac{(m_{a_t-1}-k_{a_t-1}+\zeta^{\Pi}_{A,B}(a_t-1))!}{(m_{a_t-1}-k_{a_t-1}-\varepsilon^{\Pi}_{A_t^-}(a_t-1)+\zeta^{\Pi}_{A_t^-,B}(a_t-1))!} \cdot \frac{\left(\frac{-\nu_{i_1}+\nu_{\tau^{I,\Pi}(a_t-1)}+1+\chi_{J_0}(a_t-1)}{2}\right)_{m_{a_t-1}-k_{a_t-2}+\eta^{\Pi}_{A,B}(a_t-1)}}{\left(\frac{-\nu_{i_1}+\nu_{\tau^{I,\Pi}(a_t-1)}+1+\chi_{J_0}(a_t-1)}{2}\right)_{m_{a_t-1}-k_{a_t-2}-\varepsilon^{\Pi}_{A_t^-}(a_t-1)+\eta^{\Pi}_{A_t^-,B}(a_t-1)}},$$

if $a_t-1 \in [1, j_h-1]$ and

$$(8.5)\qquad -\frac{(m_{a_t-1}-k_{a_t-2}+\eta^{\Pi}_{A,B}(a_t-1))!}{(m_{a_t-1}-k_{a_t-2}-\varepsilon^{\Pi}_{A_t^-}(a_t-1)+\eta^{\Pi}_{A_t^-,B}(a_t-1))!} \cdot \frac{\left(\frac{\nu_{i_1}-\nu_{\tau^{I,\Pi}(a_t)}+1+\chi_{J_0}(a_t-1)}{2}\right)_{m_{a_t-1}-k_{a_t-1}+\zeta^{\Pi}_{A,B}(a_t-1)}}{\left(\frac{\nu_{i_1}-\nu_{\tau^{I,\Pi}(a_t)}+1+\chi_{J_0}(a_t-1)}{2}\right)_{m_{a_t-1}-k_{a_t-1}-\varepsilon^{\Pi}_{A_t^-}(a_t-1)+\zeta^{\Pi}_{A_t^-,B}(a_t-1)}},$$

if $a_t-1 \in [j_h, n-1]$.

For $a_t-1 \in [1, j_h-1]$, we have $\zeta^{\Pi}_{A,B}(a_t-1) = 0$, $-\varepsilon^{\Pi}_{A_t^-}(a_t-1)+\zeta^{\Pi}_{A_t^-,B}(a_t-1) = -\chi_{I_l^{\bar\epsilon(t)}(J_0)}(a_t-1)-\chi_{I_l^{\epsilon(t)}(J_0)}(a_t-1) = -1$ and $\eta^{\Pi}_{A,B}(a_t-1) = -\varepsilon^{\Pi}_{A_t^-}(a_t-1)+\eta^{\Pi}_{A_t^-,B}(a_t-1) = -\chi_{J_0^{\bar\epsilon(t)}}(a_t-1)$. Then we find that (8.4) is equal to $-(m_{a_t-1}-k_{a_t-1})$.

Similarly, in the case of $a_t-1 \in [j_h, n-1]$ we have $\eta^{\Pi}_{A,B}(a_t-1) = -\varepsilon^{\Pi}_{A_t^-}(a_t-1)+\eta^{\Pi}_{A_t^-,B}(a_t-1) = -\chi_{I_r^{\epsilon(t)}(J_0)}(a_t-1)$, $\zeta^{\Pi}_{A,B}(a_t-1) = \chi_{I^{\bar\epsilon(t)}(J_0)}(a_t-1)$, and $-\varepsilon^{\Pi}_{A_t^-}(a_t-1)+\zeta^{\Pi}_{A_t^-,B}(a_t-1) = -\chi_{I^{\epsilon(t)}(J_0)\cup J_0}(a_t-1)$. Then we have $\zeta^{\Pi}_{A,B}(a_t-1)+\varepsilon^{\Pi}_{A_t^-}(a_t-1)-\zeta^{\Pi}_{A_t^-,B}(a_t-1) = 1$. In view of

$$\frac{\nu_{i_1}-\nu_{\tau^{I,\Pi}(a_t)}-\chi_{J_0^c}(a_t-1)}{2}+\zeta^{\Pi}_{A,B}(a_t-1) = \alpha^{I,\Pi}_A(a_t-1),$$

(8.5) is equal to $-(m_{a_t-1}-k_{a_t-1}+\alpha^{I,\Pi}_A(a_t-1))$ and we have confirmed the first part of (i). The others can be similarly checked. ☐

Lemma 8.14. *Let $B = \{b_1, \dots, b_{h-1}\}$ be a color of type $(n-1, h-1)$ satisfying $B \prec A$. For $1 \le t \le h-1$ we have the following.*

(i) *If B also satisfies $B_t^+ \prec A$ then we have*
$$\frac{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}}{P^{I,\Pi}_{A,B_t^+;\mathbf{m},\mathbf{k}+\varepsilon_B^{J_0}(b_t)\cdot\mathbf{d}_{b_t}}} = -(m_{b_t}-k_{b_t}+\alpha(b_t)).$$

(ii) *If B also satisfies $B_t^- \prec A$ then we have*

$$\frac{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}}{P^{I,\Pi}_{A,B_t^-;\mathbf{m},\mathbf{k}+\varepsilon_B^{J_0}(b_t-1)\cdot\mathbf{d}_{b_t-1}}} = -(m_{b_t} - k_{b_t-1} + \beta(b_t)).$$

Proof. In the same way as in Lemma 8.13, we can check these relations. □

8.6. **Compatibility with the Dirac-Schmid equations.** To prove the Dirac-Schmid relation (8.1), we connect the coefficients $C^{n,I,J}_{A;\mathbf{m}}(\nu)$ with $C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde{\nu})$. The following is the key identity in the proof of (8.1), which will be settled more generally in the next section.

Lemma 8.15. *For $A \in {}_n\mathcal{C}_h$ and $B \in {}_{n-1}\mathcal{C}_{h-1}$ satisfying $B \prec A$, we have*

$$\mu_A^{I,\Pi}(\nu) + \sum_{t=1}^{h}(\alpha(a_t - 1) - \beta(a_t)) = \sum_{t=1}^{h} \frac{\nu_{i_1} - \nu_{i_t}}{2} = \mu_B^{I_0,J_0}(\tilde{\nu}) + \sum_{t=1}^{h-1}(\alpha(b_t) - \beta(b_t)).$$

Proof. This is a special case of Lemma 9.11 ($R = [j_h, n-1]$). □

Now let us prove (8.1) by induction on (n, h). Our task is to confirm that

$$\begin{aligned}(8.6)\qquad &\left\{\sum_{t=1}^{h}(-m_{a_t-1} + m_{a_t}) + \mu_A^{I,\Pi}(\nu)\right\} \sum_{B\prec A}\sum_{\mathbf{k}} \frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde{\nu})}{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}} \\ &+ \sum_{t=1}^{h}\left\{-\sum_{B\prec A_t^-}\sum_{\mathbf{k}} \frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde{\nu})}{P^{I,\Pi}_{A_t^-,B;\mathbf{m}_t^-,\mathbf{k}}} + \sum_{B\prec A_t^+}\sum_{\mathbf{k}} \frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde{\nu})}{P^{I,\Pi}_{A_t^+,B;\mathbf{m}_t^+,\mathbf{k}}}\right\} = 0,\end{aligned}$$

by using the induction hypothesis

$$\begin{aligned}(8.7)\qquad &\left\{\sum_{t=1}^{h-1}(-k_{b_t-1} + k_{b_t}) + \mu_B^{I_0,J_0}(\tilde{\nu})\right\} C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde{\nu}) \\ &+ \sum_{t=1}^{h-1}\left\{-C^{n-1,I_0,J_0}_{B_t^-;\mathbf{k}-\varepsilon^{J_0}_{B_t^-}(b_t-1)\cdot\mathbf{d}_{b_t-1}}(\tilde{\nu}) + C^{n-1,I_0,J_0}_{B_t^+;\mathbf{k}-\varepsilon^{J_0}_{B_t^+}(b_t)\cdot\mathbf{d}_{b_t}}(\tilde{\nu})\right\} = 0, \quad (B \in {}_{n-1}\mathcal{C}_{h-1}).\end{aligned}$$

We divide the summations over B in (8.6) as $\sum_{B\prec A_t^\pm} = \sum_{B\prec A, A_t^\pm} + \sum_{B\ \ A, B\prec A_t^\pm}$. In view of the relations in Lemma 8.13, the left hand side of (8.6) can be written as the sum of the following five terms:

$$(8.8)\qquad \left\{\sum_{t=1}^{h}(-m_{a_t-1} + m_{a_t}) + \mu_A^{I,\Pi}(\nu)\right\} \sum_{B\prec A}\sum_{\mathbf{k}} \frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde{\nu})}{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}};$$

$$(8.9)\qquad \sum_{t=1}^{h}\sum_{B\prec A,A_t^-}\sum_{\mathbf{k}} \frac{(m_{a_t-1} - k_{a_t-1} + \alpha(a_t-1))C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde{\nu})}{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}};$$

$$(8.10)\qquad -\sum_{t=1}^{h-1}\sum_{B\ \ A,B\prec A_t^-}\sum_{\mathbf{k}} \frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde{\nu})}{P^{I,\Pi}_{A,B_t^+;\mathbf{m},\mathbf{k}+\varepsilon_B^{J_0}(b_t)\cdot\mathbf{d}_{b_t}}};$$

$$(8.11)\qquad -\sum_{t=1}^{h}\sum_{B\prec A,A_t^+}\sum_{\mathbf{k}} \frac{(m_{a_t} - k_{a_t-1} + \beta(a_t))C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde{\nu})}{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}};$$

$$(8.12)\qquad \sum_{t=2}^{h}\sum_{B}\sum_{A,B\prec A_t^+}\sum_{\mathbf{k}}\frac{C_{B;\mathbf{k}}^{n-1,I_0,J_0}(\tilde{\nu})}{P^{I,\Pi}_{A,B_{t-1}^-;\mathbf{m},\mathbf{k}+\varepsilon_B^{J_0}(b_{t-1}-1)\cdot\mathbf{d}_{b_{t-1}-1}}}$$

Let us compute (8.8) + (8.9) + (8.11). Note that, for $B=\{b_1,\dots,b_{h-1}\}\in {}_{n-1}\mathcal{C}_{h-1}$,

- $B\prec A, A_t^- \iff a_{t-1}\le b_{t-1}\le a_t-2$ and $a_{t'}\le b_{t'}\le a_{t'+1}-1$ for $t'\ne t-1$,
- $B\prec A, A_t^+ \iff a_t+1\le b_t\le a_{t+1}-1$ and $a_{t'}\le b_{t'}\le a_{t'+1}-1$ for $t'\ne t$.

Now we divide the summations $\sum_{B\prec A}$ and $\sum_{t=1}^{h}\sum_{B\prec A,A_t^\pm}$ for given A. Let

$$S_0=\{t\in T_{h-1}\mid a_t=a_{t+1}-1\}.$$

If $t_0\in S_0$ and $B\prec A$, then $b_{t_0}=a_{t_0}=a_{t_0+1}-1$ and there is no color B of type $(n-1,h-1)$ such that $B\prec A_{t_0+1}^-$ or $B\prec A_{t_0}^+$. That is, if $B\prec A_t^-$ (resp. $B\prec A_t^+$) then $t-1\notin S_0$ (resp. $t\notin S_0$). Let us consider a decomposition of the set $T_{h-1}\backslash S_0$ into three disjoint subsets S_i $(i=1,2,3)$ and put

$$\mathcal{S}=\{S=\{S_1,S_2,S_3\}\mid T_{h-1}=S_0\amalg S_1\amalg S_2\amalg S_3\}.$$

For each $S\in\mathcal{S}$ we define a set $D(S)$ of colors of types $(n-1,h-1)$ by

$$D(S)=D(\{S_1,S_2,S_3\})=\{B=\{b_1,\dots,b_{h-1}\}\in {}_{n-1}\mathcal{C}_{h-1}\mid B \text{ satisfies } (\sharp)\},$$

with

$$(\sharp)\begin{cases} b_t=a_t & \text{if } t\in S_0\cup S_1,\\ b_t\in[a_t+1,a_{t+1}-2] & \text{if } t\in S_2,\\ b_t=a_{t+1}-1 & \text{if } t\in S_0\cup S_3.\end{cases}$$

Then we find

$$\sum_{B\prec A}=\sum_{S\in\mathcal{S}}\sum_{B\in D(S)},$$

$$\sum_{1\le t\le h}\sum_{B\prec A,A_t^-}=\sum_{S=\{S_1,S_2,S_3\}\in\mathcal{S}}\sum_{\substack{1\le t\le h\\ t-1\notin S_0\cup S_3}}\sum_{B\in D(S)}=\sum_{S\in\mathcal{S}}\sum_{B\in D(S)}\Bigg[\sum_{1\le t\le h}-\sum_{t-1\in S_0\cup S_3}\Bigg],$$

$$\sum_{1\le t\le h}\sum_{B\prec A,A_t^+}=\sum_{S=\{S_1,S_2,S_3\}\in\mathcal{S}}\sum_{\substack{1\le t\le h\\ t\notin S_0\cup S_1}}\sum_{B\in D(S)}=\sum_{S\in\mathcal{S}}\sum_{B\in D(S)}\Bigg[\sum_{1\le t\le h}-\sum_{t-1\in S_0\cup S_1}\Bigg].$$

Hence the sum (8.8) + (8.9) + (8.11) is

$$\sum_{S\in\mathcal{S}}\sum_{B\in D(S)}\sum_{\mathbf{k}}\Bigg[\sum_{t=1}^{h}\{(-m_{a_{t-1}}+m_{a_t})+(m_{a_t-1}-k_{a_t-1}+\alpha(a_t-1))$$
$$-(m_{a_t}-k_{a_t-1}+\beta(a_t))\}+\mu_A^{I,\Pi}(\nu)$$
$$-\sum_{t-1\in S_0\cup S_3}(m_{a_t-1}-k_{a_t-1}+\alpha(a_t-1))+\sum_{t\in S_0\cup S_1}(m_{a_t}-k_{a_t-1}+\beta(a_t))\Bigg]\frac{C_{B;\mathbf{k}}^{n-1,I_0,J_0}(\tilde{\nu})}{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}}.$$

By using the identity in Lemma 8.15 and

$$\begin{cases} m_{a_t-1}-k_{a_t-1}+\alpha(a_t-1)=m_{b_{t-1}}-k_{b_{t-1}}+\alpha(b_{t-1}) & t-1\in S_0\cup S_3 \text{ and } B\in D(S),\\ m_{a_t}-k_{a_t-1}+\beta(a_t)=m_{b_t}-k_{b_t-1}+\beta(b_t) & t\in S_0\cup S_1 \text{ and } B\in D(S),\end{cases}$$

we find that the contents in the bracket [ ] in the above yields

$$\begin{aligned}
&\sum_{1\le t\le h-1}(\alpha(b_t)-\beta(b_t))+\mu_B^{I_0,J_0}(\tilde\nu)\\
&\quad-\sum_{\substack{1\le t\le h-1\\ t\in S_0\cup S_3}}(m_{b_t}-k_{b_t}+\alpha(b_t))+\sum_{\substack{1\le t\le h-1\\ t\in S_0\cup S_1}}(m_{b_t}-k_{b_t-1}+\beta(b_t))\\
&=\sum_{1\le t\le h-1}(-k_{b_t-1}+k_{b_t})+\mu_B^{I_0,J_0}(\tilde\nu)\\
&\quad+\sum_{\substack{1\le t\le h-1\\ t\notin S_0\cup S_3}}(m_{b_t}-k_{b_t}+\alpha(b_t))-\sum_{\substack{1\le t\le h-1\\ t\notin S_0\cup S_1}}(m_{b_t}-k_{b_t-1}+\beta(b_t))
\end{aligned}$$

for $B\in D(S)$.

Hence, in view of Lemma 8.14, we can find that (8.8) + (8.9) + (8.11) is the sum of the following three terms:

$$\begin{aligned}
&\sum_S\sum_{B\in D(S)}\sum_{\mathbf{k}}\frac{\{\sum_{t=1}^{h-1}(-k_{b_t-1}+k_{b_t})+\mu_B^{I_0,J_0}(\tilde\nu)\}C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde\nu)}{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}}\\
&=\sum_{B\prec A}\sum_{\mathbf{k}}\frac{\{\sum_{t=1}^{h-1}(-k_{b_t-1}+k_{b_t})+\mu_B^{I_0,J_0}(\tilde\nu)\}C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde\nu)}{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}};
\end{aligned}\tag{8.13}$$

(8.14)

$$-\sum_S\sum_{B\in D(S)}\sum_{\mathbf{k}}\sum_{t\notin S_0\cup S_3}\frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde\nu)}{P^{I,\Pi}_{A,B_t^+;\mathbf{m},\mathbf{k}+\varepsilon_B^{J_0}(b_t)\cdot\mathbf{d}_{b_t}}}=-\sum_{t=1}^{h-1}\sum_{B\prec A,A_{t+1}^-}\sum_{\mathbf{k}}\frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde\nu)}{P^{I,\Pi}_{A,B_t^+;\mathbf{m},\mathbf{k}+\varepsilon_B^{J_0}(b_t)\cdot\mathbf{d}_{b_t}}};$$

(8.15)

$$\sum_S\sum_{B\in D(S)}\sum_{\mathbf{k}}\sum_{t\notin S_0\cup S_1}\frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde\nu)}{P^{I,\Pi}_{A,B_t^-;\mathbf{m},\mathbf{k}+\varepsilon_B^{J_0}(b_t-1)\cdot\mathbf{d}_{b_t-1}}}=\sum_{t=1}^{h-1}\sum_{B\prec A,A_t^+}\sum_{\mathbf{k}}\frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde\nu)}{P^{I,\Pi}_{A,B_t^-;\mathbf{m},\mathbf{k}+\varepsilon_B^{J_0}(b_t-1)\cdot\mathbf{d}_{b_t-1}}}.$$

Let us consider the sum (8.10)+(8.14). In view of

- $B\prec A,A_{t+1}^-\iff b_t\in[a_t,a_{t+1}-2]$ and $b_{t'}\in[a_{t'},a_{t'+1}-1]$ $(t'\neq t)$,
- $B\not\prec A,B\prec A_t^-\iff b_t=a_t-1$ and $b_{t'}\in[a_{t'},a_{t'+1}-1]$ $(t'\neq t)$,

we have

$$\begin{aligned}
(8.10)+(8.14)&=-\sum_{t=1}^{h-1}\sum_{B_t^+\prec A}\sum_{\mathbf{k}}\frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde\nu)}{P^{I,\Pi}_{A,B_t^+;\mathbf{m},\mathbf{k}+\varepsilon_B^{J_0}(b_t)\cdot\mathbf{d}_{b_t}}}\\
&=-\sum_{B\prec A}\sum_{t=1}^{h-1}\sum_{\mathbf{k}}\frac{C^{n-1,I_0,J_0}_{B_t^-;\mathbf{k}-\varepsilon^{J_0}_{B_t^-}(b_t-1)\cdot\mathbf{d}_{b_t-1}}(\tilde\nu)}{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}}.
\end{aligned}$$

Similarly we can find that

$$(8.12)+(8.15)=\sum_{t=1}^{h-1}\sum_{B_t^-\prec A}\sum_{\mathbf{k}}\frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}-\varepsilon_B^{J_0}(b_t-1)\cdot\mathbf{d}_{b_t-1}}(\tilde\nu)}{P^{I,\Pi}_{A,B_t^-;\mathbf{m},\mathbf{k}}}$$

$$= \sum_{B \prec A} \sum_{t=1}^{h-1} \sum_{\mathbf{k}} \frac{C^{n-1,I_0,J_0}_{B_t^+;\mathbf{k}-\varepsilon^{J_0}_{B_t^+}(b_t)\cdot\mathbf{d}_{b_t}}(\tilde{\nu})}{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}}.$$

Thus, the hypothesis (8.7) implies that

$$\begin{aligned}&(8.8) + (8.9) + (8.10) + (8.11) + (8.12)\\ &= (8.13) + \{(8.10) + (8.14)\} + \{(8.12) + (8.15)\} = 0,\end{aligned}$$

and we can finish the proof of (8.1).

8.7. **Compatibility with the Casimir equations.** As is the case of Dirac-Schmid equations, we show the recurrence relation (8.2) by induction on (n, h). Here is the key relation, which will be proved in the next section.

Lemma 8.16. *We set*

$$X^{I,\Pi}_{A,B} = \sum_{p=1}^{n-1} \frac{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}}{P^{I,\Pi}_{A,B;\mathbf{m}-\mathbf{e}_p,\mathbf{k}}} - \sum_{p=1}^{n-2} \frac{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}}{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}+\mathbf{d}_p}}$$

for $B \prec A$. Then we have

$$\begin{aligned}&Q_{\mathbf{m}}\big(\{\lambda^{I,\Pi}_{A,p}(\nu)\}_{1\le p\le n-1}; \kappa^{I,\Pi}_A(\nu)\big)\\ &- \frac{1}{2}\sum_{t=1}^{h}(m_{a_t-1} - k_{a_t-1} + \alpha(a_t-1)) - \frac{1}{2}\sum_{t=1}^{h}(m_{a_t} - k_{a_t-1} + \beta(a_t))\\ &= X^{I,\Pi}_{A,B} + Q_{\mathbf{k}}\big(\{\lambda^{I_0,J_0}_{B,p}(\tilde{\nu})\}_{1\le p\le n-2}; \kappa^{I_0,J_0}_B(\tilde{\nu})\big)\\ &- \frac{1}{2}\sum_{t=1}^{h-1}(m_{b_t} - k_{b_t} + \alpha(b_t)) - \frac{1}{2}\sum_{t=1}^{h-1}(m_{b_t} - k_{b_t-1} + \beta(b_t)).\end{aligned}$$

Proof. This is a special case of Lemma 9.12 (ii) ($R = [j_h, n-1]$). □

Let us start our proof of (8.2). For abbreviation we denote by

$$Q_{\mathbf{m}} = Q_{\mathbf{m}}\big(\{\lambda^{I,\Pi}_{A,p}(\nu)\}_{1\le p\le n-1}; \kappa^{I,\Pi}_A(\nu)\big), \quad Q_{\mathbf{k}} = Q_{\mathbf{k}}\big(\{\lambda^{I_0,J_0}_{B,p}(\tilde{\nu})\}_{1\le p\le n-2}; \kappa^{I_0,J_0}_B(\tilde{\nu})\big).$$

Then our task is to show

$$\begin{aligned}&Q_{\mathbf{m}} \sum_{B\prec A}\sum_{\mathbf{k}} \frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde{\nu})}{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}}\\ &+ \frac{1}{2}\sum_{1\le t\le h}\left\{\sum_{B\prec A_t^-}\sum_{\mathbf{k}} \frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde{\nu})}{P^{I,\Pi}_{A_t^-,B;\mathbf{m}_t^-,\mathbf{k}}} + \sum_{B\prec A_t^+}\sum_{\mathbf{k}} \frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde{\nu})}{P^{I,\Pi}_{A_t^+,B;\mathbf{m}_t^+,\mathbf{k}}}\right\} = 0.\end{aligned} \tag{8.16}$$

We divide the summations over B and use the relations in Lemma 8.13, the left hand side of (8.16) can be written as the sum of the following five terms:

$$Q_{\mathbf{m}} \sum_{B\prec A}\sum_{\mathbf{k}} \frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde{\nu})}{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}}; \tag{8.17}$$

$$-\frac{1}{2}\sum_{t=1}^{h}\sum_{B\prec A,A_t^-}\sum_{\mathbf{k}} \frac{(m_{a_t-1} - k_{a_t-1} + \alpha(a_t-1))C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde{\nu})}{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}}; \tag{8.18}$$

$$\tag{8.19} \frac{1}{2}\sum_{t=1}^{h-1}\sum_{B=B_{t,a_t-1}\prec A_t^-}\sum_{\mathbf{k}}\frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}-\varepsilon_B^{J_0}(b_t)\cdot\mathbf{d}_{b_t}}(\tilde{\nu})}{P^{I,\Pi}_{A,B_t^+;\mathbf{m},\mathbf{k}}};$$

$$\tag{8.20} -\frac{1}{2}\sum_{t=1}^{h}\sum_{B\prec A,A_t^+}\sum_{\mathbf{k}}\frac{(m_{a_t}-k_{a_t-1}+\beta(a_t))C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde{\nu})}{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}};$$

$$\tag{8.21} \frac{1}{2}\sum_{t=1}^{h-1}\sum_{B=B_{t,a_{t+1}}\prec A_{t+1}^+}\sum_{\mathbf{k}}\frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}-\varepsilon_B^{J_0}(b_t-1)\cdot\mathbf{d}_{b_t-1}}(\tilde{\nu})}{P^{I,\Pi}_{A,B_t^-;\mathbf{m},\mathbf{k}}}.$$

In the similar way to the proof of (8.1), we can see that (8.17) + (8.18) + (8.20) equals

(8.22)

$$\begin{aligned}
&\sum_{S\in\mathcal{S}}\sum_{B\in D(S)}\sum_{\mathbf{k}}\Bigg\{Q_{\mathbf{m}}-\frac{1}{2}\sum_{t-1\notin S_0\cup S_3}(m_{a_t-1}-k_{a_t-1}+\alpha(a_t-1))\\
&\quad-\frac{1}{2}\sum_{t\notin S_0\cup S_1}(m_{a_t}-k_{a_t-1}+\beta(a_t))\Bigg\}\frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde{\nu})}{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}}\\
&=\sum_{S\in\mathcal{S}}\sum_{B\in D(S)}\sum_{\mathbf{k}}\Bigg\{Q_{\mathbf{m}}-\frac{1}{2}\sum_{t=1}^{h}\{(m_{a_t-1}-k_{a_t-1}+\alpha(a_t-1))+(m_{a_t}-k_{a_t-1}+\beta(a_t))\}\\
&\quad+\frac{1}{2}\sum_{t-1\in S_0\cup S_3}(m_{a_t-1}-k_{a_t-1}+\alpha(a_t-1))\\
&\quad+\frac{1}{2}\sum_{t\in S_0\cup S_1}(m_{a_t}-k_{a_t-1}+\beta(a_t))\Bigg\}\frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde{\nu})}{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}}.
\end{aligned}$$

By using the identity in Lemma 8.16, (8.22) equals

$$\begin{aligned}
&\sum_{S\in\mathcal{S}}\sum_{B\in D(S)}\sum_{\mathbf{k}}\Bigg\{\sum_{p=1}^{n-1}\frac{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}}{P^{I,\Pi}_{A,B;\mathbf{m}-\mathbf{e}_p,\mathbf{k}}}-\sum_{p=1}^{n-2}\frac{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}}{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}+\mathbf{d}_p}}+Q_{\mathbf{k}}\\
&\quad-\frac{1}{2}\sum_{t\notin S_0\cup S_3}(m_{b_t}-k_{b_t}+\alpha(b_t))-\frac{1}{2}\sum_{t\notin S_0\cup S_1}(m_{b_t}-k_{b_t-1}+\beta(b_t))\Bigg\}\frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde{\nu})}{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}}\\
&=\sum_{p=1}^{n-1}\sum_{B\prec A}\sum_{\mathbf{k}}\frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde{\nu})}{P^{I,\Pi}_{A,B;\mathbf{m}-\mathbf{e}_p,\mathbf{k}}}-\sum_{p=1}^{n-2}\sum_{B\prec A}\sum_{\mathbf{k}}\frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}-\mathbf{d}_p}(\tilde{\nu})}{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}}+\sum_{B\prec A}\sum_{\mathbf{k}}\frac{Q_{\mathbf{k}}\cdot C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde{\nu})}{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}}\\
&\quad+\frac{1}{2}\sum_{S\in\mathcal{S}}\sum_{B\in D(S)}\sum_{\mathbf{k}}\Bigg\{\sum_{t\notin S_0\cup S_3}\frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}-\varepsilon_B^{J_0}(b_t)\cdot\mathbf{d}_{b_t}}(\tilde{\nu})}{P^{I,\Pi}_{A,B_t^+;\mathbf{m},\mathbf{k}}}+\sum_{t\notin S_0\cup S_1}\frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}-\varepsilon_B^{J_0}(b_t-1)\cdot\mathbf{d}_{b_t-1}}(\tilde{\nu})}{P^{I,\Pi}_{A,B_t^-;\mathbf{m},\mathbf{k}}}\Bigg\}.
\end{aligned}$$

Here we used Lemma 8.14. By adding (8.19) and (8.21) to the above, we find that the left hand side of (8.16) can be written as

$$\sum_{p=1}^{n-1}C^{n,I,\Pi}_{A;\mathbf{m}-\mathbf{e}_p}(\nu)+\sum_{B\prec A}\sum_{\mathbf{k}}\frac{Q_{\mathbf{k}}\cdot C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde{\nu})-\sum_{p=1}^{n-2}C^{n-1,I_0,J_0}_{B;\mathbf{k}-\mathbf{d}_p}(\tilde{\nu})}{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}}$$

$$+\frac{1}{2}\sum_{t=1}^{h-1}\sum_{B_t^+\prec A}\frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}-\varepsilon_B^{J_0}(b_t)\cdot\mathbf{d}_{b_t}}(\tilde{\nu})}{P^{I,\Pi}_{A,B_t^+;\mathbf{m},\mathbf{k}}}+\frac{1}{2}\sum_{t=1}^{h-1}\sum_{B_t^-\prec A}\frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}-\varepsilon_B^{J_0}(b_t-1)\cdot\mathbf{d}_{b_t-1}}(\tilde{\nu})}{P^{I,\Pi}_{A,B_t^-;\mathbf{m},\mathbf{k}}}$$

$$=\sum_{p=1}^{n-1}C^{n,I,\Pi}_{A;\mathbf{m}-\mathbf{e}_p}(\nu)+\sum_{B\prec A}\sum_{\mathbf{k}}\frac{1}{P^{I,\Pi}_{A,B;\mathbf{m},\mathbf{k}}}\Big\{Q_{\mathbf{k}}\cdot C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde{\nu})-\sum_{p=1}^{n-2}C^{n-1,I_0,J_0}_{B;\mathbf{k}-\mathbf{d}_p}(\tilde{\nu})$$

$$+\frac{1}{2}\sum_{t=1}^{h-1}\Big(C^{n-1,I_0,J_0}_{B_t^-;\mathbf{k}-\varepsilon^{J_0}_{B_t^-}(b_t-1)\cdot\mathbf{d}_{b_t-1}}(\tilde{\nu})+C^{n-1,I_0,J_0}_{B_t^+;\mathbf{k}-\varepsilon^{J_0}_{B_t^+}(b_t)\cdot\mathbf{d}_{b_t}}(\tilde{\nu})\Big)\Big\}.$$

From the recurrence relation coming from Casimir equation for $C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde{\nu})$, the bracket $\{\ \}$ is equal to zero and hence we conclude the proof of Proposition 8.8. □

8.8. **The recurrence relations for the edge components.** For our later use we derive auxiliary recurrence relations for $C^{n,I,J}_{A;\mathbf{m}}(\nu)$ with a fixed A where A is eigher the *edge components* A_i $(i=1,2)$:

$$A_1=\{1,2,\dots,h\},\quad A_2=\{n-h+1,\dots,n\}.$$

When $A=A_1$, the components $A_t^\pm$ do not exist except for A_h^+. By eliminating the coefficients $C^{n,I,J}_{A_h^+;\mathbf{m}}(\nu)$ from the recurrence relations in Proposition 8.8, we can get

$$Q_{\mathbf{m}}\big(\{\lambda^{I,J}_{A_1,p}(\nu)-\tfrac{1}{2}\delta_{p,h}\}_{1\le p\le n-1};\kappa^{I,J}_{A_1}(\nu)-\tfrac{1}{2}\mu^{I,J}_{A_1}(\nu)\big)C^{n,I,J}_{A_1;\mathbf{m}}(\nu)=\sum_{p=1}^{n-1}C^{n,I,J}_{A_1;\mathbf{m}-\mathbf{e}_p}(\nu),$$

where $\lambda^{I,J}_{A_1,p}(\nu)$, $\kappa^{I,J}_{A_1}(\nu)$ and $\mu^{I,J}_{A_1}(\nu)$ are defined by $\lambda^{I,\Pi}_{A_1,p}(\nu)$, $\kappa^{I,\Pi}_{A_1}(\nu)$ and $\mu^{I,\Pi}_{A_1}(\nu)$ with $\Pi=(J_0,j_h)$, respectively. Similarly we have

$$Q_{\mathbf{m}}\big(\{\lambda^{I,J}_{A_2,p}(\nu)-\tfrac{1}{2}\delta_{p,n-h}\}_{1\le p\le n-1};\kappa^{I,J}_{A_2}(\nu)+\tfrac{1}{2}\mu^{I,J}_{A_2}(\nu)\big)C^{n,I,J}_{A_2;\mathbf{m}}(\nu)=\sum_{p=1}^{n-1}C^{n,I,J}_{A_2;\mathbf{m}-\mathbf{e}_p}(\nu).$$

From now on we want to rewrite these recurrence relations by shifting the indices $\mathbf{m}$ suitably to eliminate the constant terms $\kappa^{I,J}_{A_1}(\nu)-\frac{1}{2}\mu^{I,J}_{A_1}(\nu)$ and $\kappa^{I,J}_{A_2}(\nu)+\frac{1}{2}\mu^{I,J}_{A_2}(\nu)$. Here is the desired expression and this is a special case of Lemma 9.8 (ii) (cf. Lemma 6.4 (ii)).

Proposition 8.17. *Let us define functions $\delta^J_{A_i}$ on T_{n-1} $(i=1,2)$ by*

$$\delta^J_{A_1}(p)=\Big[\frac{\min\{p,h\}-t}{2}\Big],\quad \delta^J_{A_2}(p)=\Big[\frac{t+\min\{n-h-p,0\}}{2}\Big],$$

where t $(0\le t\le h)$ is an integer satisfying $j_t\le p\le j_{t+1}-1$. If we set $\mathbf{m}'=(m'_1,\dots,m'_{n-1})$ with $m'_p=m_p-\delta^J_{A_i}(p)$, then we have the recurrence relations

$$Q_{\mathbf{m}'}\big(\{\lambda'^{I,J}_{A_i,p}(\nu)\}_{1\le p\le n-1};0\big)C^{n,I,J}_{A_i;\mathbf{m}}(\nu)=\sum_{p=1}^{n-1}C^{n,I,J}_{A_i;\mathbf{m}-\mathbf{e}_p}(\nu)\quad(i=1,2).$$

Here

$$2\lambda'^{I,J}_{A_i,p}(\nu)=\begin{cases}\nu_{\tau^{I,J}(p)}-\nu_{\tau^{I,J}(p+1)}-\chi_J(p)+\chi_J(p+1) & \textit{if } i=1,\\ \nu_{\tau^{I,J}(p)}-\nu_{\tau^{I,J}(p+1)}+\chi_J(p)-\chi_J(p+1) & \textit{if } i=2.\end{cases}$$

Remark 5. We note that $\delta^J_{A_1}=\sum_{t=1}^h\chi_{[t,j_t-1]}\chi_{I_l^{\epsilon(t)}(J)}$ and $\delta^J_{A_2}=\sum_{t=1}^h\chi_{[j_t,n-t]}\chi_{I_l^{\epsilon(t)}(J)}$.

To prove this proposition, we start with some identities for $\delta^J_{A_i}$.

Lemma 8.18. (i) *We have*

$$\delta^J_{A_i}(p) - \delta^J_{A_i}(p-1) = \begin{cases} \chi_{I^e(J)}(p)\cdot\chi_{A_1^e}(p) + \chi_{I^o(J)}(p)\cdot\chi_{A_1^o}(p) - \chi_{J^{\bar{\epsilon}(h)}}(p)\cdot\chi_{[h+1,n]}(p) & \text{if } i=1,\\ \chi_{J^e}(p)\cdot\chi_{[1,n-h]}(p) - \chi_{I^e(J)}(p)\cdot\chi_{A_2^o}(p) - \chi_{I^o(J)}(p)\cdot\chi_{A_2^e}(p) & \text{if } i=2.\end{cases}$$

(ii) *We have*

$$2\delta^J_{A_i}(p) - \delta^J_{A_i}(p-1) - \delta^J_{A_i}(p+1) + \frac{1}{2}(2\varepsilon^J_{A_i}(p) - \varepsilon^J_{A_i}(p-1) - \varepsilon^J_{A_i}(p+1)) = \begin{cases} \frac{1}{2}(-\chi_J(p) + \chi_{J[-1]}(p) + \delta_{p,h}) & \text{if } i=1,\\ \frac{1}{2}(\chi_J(p) - \chi_{J[-1]}(p) + \delta_{p,n-h}) & \text{if } i=2.\end{cases}$$

(iii) *We have*

$$2Q(\delta^J_{A_i}(1),\dots,\delta^J_{A_i}(n-1)) = \begin{cases} \sum_{p=1}^{h}(\chi_{I^e(J)}(p)\cdot\chi_{A_1^e}(p) + \chi_{I^o(J)}(p)\cdot\chi_{A_1^o}(p)) + \sum_{p=h+1}^{n}\chi_{J^{\bar{\epsilon}(h)}}(p) & \text{if } i=1,\\ \sum_{p=1}^{n-h}\chi_{J^e}(p) + \sum_{p=n-h+1}^{n}(\chi_{I^e(J)}(p)\cdot\chi_{A_2^o}(p) + \chi_{I^o(J)}(p)\cdot\chi_{A_2^e}(p)) & \text{if } i=2.\end{cases}$$

Proof. We can check (i) from the definition. By using (i) and Lemma 7.8 (i) we get $2(\delta^J_{A_i}(p) - \delta^J_{A_i}(p-1)) + (\varepsilon^J_{A_i}(p) - \varepsilon^J_{A_i}(p-1)) = (-1)^i(\chi_J(p) - \chi_{A_i}(p))$ and then (ii) follows. The evaluation of $Q(\delta^J_{A_i}(1),\dots,\delta^J_{A_i}(n-1))$ can be done as in the proof of Lemma 8.10 by using (i). □

Now let us show Proposition 8.17. Our claim for A_1 follows from

$$2\delta^J_{A_1}(p) - \delta^J_{A_1}(p-1) - \delta^J_{A_1}(p+1) + \lambda^{I,J}_{A_1,p}(\nu) - \frac{1}{2}\delta_{p,h} = \lambda'^{I,J}_{A_1,p}(\nu) \tag{8.23}$$

for $1 \le p \le n-1$, and

$$Q(\delta^J_{A_1}(1),\dots,\delta^J_{A_1}(n-1)) + \sum_{p=1}^{n-1}\Big(\lambda^{I,J}_{A_1,p}(\nu) - \frac{1}{2}\delta_{p,h}\Big)\delta^J_{A_1}(p) + \kappa^{I,J}_{A_1}(\nu) - \frac{1}{2}\mu^{I,J}_{A_1}(\nu) = 0. \tag{8.24}$$

By means of Lemma 8.18 (ii), we get (8.23). To prove the identity (8.24), it is enough to show

$$\begin{aligned}&\frac{1}{2}\sum_{p=1}^{n-1}(\nu_{\tau^{I,J}(p)} - \nu_{\tau^{I,J}(p+1)})\delta^J_{A_1}(p) + \frac{1}{4}\sum_{p=1}^{h}\chi_{A_1,J^c}(p)(\nu_{i_1} - \nu_{\tau^{I,J}(p)})\\ &+\frac{1}{4}\sum_{p=h+1}^{n}(\chi_{J^{\epsilon(h)}}(p) - \chi_{J^{\bar{\epsilon}(h)}}(p))(\nu_{i_1} - \nu_{\tau^{I,J}(p)}) - \frac{1}{4}\sum_{p=1}^{h}\nu_{\tau^{I,J}(p)} + \frac{1}{4}\sum_{p\in J}\nu_{\tau^{I,J}(p)} = 0,\end{aligned} \tag{8.25}$$

and

$$\begin{aligned}&-Q(\delta^J_{A_1}(1),\dots,\delta^J_{A_1}(n-1)) + \frac{1}{2}\sum_{p=1}^{n}(-\chi_J(p) + \chi_{J[-1]}(p))\delta^J_{A_1}(p)\\ &+\frac{1}{4}(h - \sharp(A_1\cap J)) + \frac{1}{4}\sum_{p=1}^{h}\chi_{A_1,J^c}(p) = 0.\end{aligned} \tag{8.26}$$

Here we applied Lemmas 8.18 (ii) and 7.8 (i) for $\lambda^{I,J}_{A_1,p}(\nu)$ and $\mu^{I,J}_{A_1}(\nu)$, respectively. In viwe of

$$\sum_{p=1}^{n-1}(\nu_{\tau^{I,J}(p)}-\nu_{\tau^{I,J}(p+1)})\delta^J_{A_1}(p)=\sum_{p=1}^{n}(\delta^J_{A_1}(p)-\delta^J_{A_1}(p-1))(-\nu_{i_1}+\nu_{\tau^{I,J}(p)}),$$

$$-\sum_{p=1}^{h}\nu_{\tau^{I,J}(p)}+\sum_{p\in J}\nu_{\tau^{I,J}(p)}=\sum_{p=1}^{h}\chi_{J^c}(p)(\nu_{i_1}-\nu_{\tau^{I,J}(p)})-\sum_{p=h+1}^{n}\chi_J(p)(\nu_{i_1}-\nu_{\tau^{I,J}(p)}),$$

and Lemma 8.18 (i), we can get (8.25).

Finally let us confirm (8.26). By Lemma 8.18 (i), we have

$$\begin{aligned}&\frac{1}{2}\sum_{p=1}^{n}(-\chi_J(p)+\chi_{J[-1]}(p))\delta^J_{A_1}(p)\\ &=-\frac{1}{2}\sum_{p=1}^{n}\chi_J(p)(\chi_{I^e(J)}(p)\cdot\chi_{A_1^e}(p)+\chi_{I^o(J)}(p)\cdot\chi_{A_1^o}(p)-\chi_{J^{\bar{\epsilon}(h)}}(p)\cdot\chi_{[h+1,n]}(p))\\ &=\frac{1}{2}\sum_{p=h+1}^{n}\chi_{J^{\bar{\epsilon}(h)}}(p)\end{aligned}$$

Hence, in view of Lemma 8.18 (iii) and $h-\sharp(A_1\cap J)=\sum_{p=1}^{h}(\chi_{A_1^e}(p)+\chi_{A_1^o}(p))(\chi_{I^e(J)}(p)+\chi_{I^o(J)}(p))$, we obtain (8.26).

The case of $A=A_2$ is treated similarly and thus we finish the proof of Proposition 8.17. □

9. Primary Whittaker functions (2) -the case of general h-

Here is main result for the primary Whittaker functions. Similarly to the secondary Whittaker functions our explicit formula is a recursive formula with respect to n and h.

Theorem 9.1. *We define a function* $W^{n,I}_{\nu}(y)=\{W^{n,I}_{A,\nu}(y)\}_{A\in{}_n\mathcal{C}_h}=\{y^{\rho}\widetilde{W}^{n,I}_{A,\nu}(y)\}_{A\in{}_n\mathcal{C}_h}$ *by the following relation:*

$$\begin{aligned}\widetilde{W}^{n,I}_{A,\nu}(y)=\sum_{B\prec A}\int_{(\mathbf{R}_+)^{n-1}}&\widetilde{W}^{n-1,I_0}_{B,\tilde{\nu}}\left(y_2\sqrt{\frac{t_2}{t_1}},\ldots,y_{n-1}\sqrt{\frac{t_{n-1}}{t_{n-2}}}\right)\\ &\cdot\prod_{p=1}^{n-1}\exp\Big\{-(\pi y_p)^2t_p-\frac{1}{t_p}\Big\}\prod_{p=1}^{n-1}(\pi y_p)^{\frac{n-p}{n-1}\nu_{i_1}+\chi_{I(B,A)}(p)}\\ &\cdot\prod_{p=1}^{n-1}t_p^{\frac{n}{2(n-1)}\nu_{i_1}+\frac{1}{2}(\chi_{I(B,A)}(p)-\chi_{I(A,B)}(p))}\prod_{p=1}^{n-1}\frac{dt_p}{t_p}.\end{aligned}$$

Then $W^{n,I}_{\nu}(y)$ *is the radial part of the primary Whittaker function and we have the factorization formula*

$$W^{n,I}_{A,\nu}(y)=\sum_{\tau\in\ \ _I}\tau\Bigg[\sum_{J\in{}_n\mathcal{C}_h}\Gamma^{n,I,J}(\nu)\cdot M^{n,I,J}_{A,\nu}(y)\Bigg], \tag{9.1}$$

where

$$\Gamma^{n,I,J}(\nu) = \prod_{1\le p<q\le h} \Gamma\Big(\frac{\nu_{i_p}-\nu_{i_q}}{2}\Big) \prod_{1\le p<q\le n-h} \Gamma\Big(\frac{-\nu_{i'_p}+\nu_{i'_q}}{2}\Big)$$
$$\cdot \prod_{t=0}^{h} \prod_{p=j_t+1}^{j_{t+1}-1} \Big\{ \prod_{s=1}^{h-t} \Gamma\Big(\frac{\nu_{i_s}-\nu_{i'_{p-t}}+1}{2}\Big) \prod_{s=h-t+1}^{h} \Gamma\Big(\frac{-\nu_{i_s}+\nu_{i'_{p-t}}+1}{2}\Big)\Big\}.$$

Here is a propagation formula of the Mellin transform of the primary Whittaker functions.

Theorem 9.2. *Let*

$$V^{n,I}_{A,\nu}(s) \equiv V^{n,I}_{A,\nu}(s_1,\dots,s_{n-1}) = \int_{(\mathbf{R}_+)^{n-1}} \widetilde{W}^{n,I}_{A,\nu}(y) \prod_{p=1}^{n-1} (\pi y_p)^{s_p}$$

be the Mellin transform of the radial part $\widetilde{W}^{n,I}_{A,\nu}$. *Then we have a propagation formula*

$$V^{n,I}_{A,\nu}(s) = \sum_{B\prec A} \frac{2^{1-n}}{(2\pi i)^{n-2}} \int_{z_1,\dots,z_{n-2}} V^{n-1,I_0}_{B,\tilde\nu}(z_1,\dots,z_{n-2})$$
$$\cdot \prod_{p=1}^{n-1} \Gamma\Big(\frac{s_p - z_{p-1} + \chi_{I(B,A)}(p)}{2} + \frac{(n-p)\nu_{i_1}}{2(n-1)}\Big)$$
$$\cdot \prod_{p=1}^{n-1} \Gamma\Big(\frac{s_p - z_p + \chi_{I(A,B)}(p)}{2} - \frac{p\nu_{i_1}}{2(n-1)}\Big) \prod_{p=1}^{n-2} dz_p.$$

From now on we will prove Theorems 9.1 and 9.2. Similarly as the case of helicity 1 our task is to show the factorization formula (9.1). Our argument is an induction on n and h.

9.1. **A certain factorization of formula for $\widetilde{W}^{n,I}_{A,\nu}$.** By inserting the factorization formula (9.1) for $\widetilde{W}^{n-1,I_0}_{\tilde\nu}(y)$ and changing the order of integrations and infinite sums, which can be justified as in [HrIO, §7], we get

$$\widetilde{W}^{n,I}_{A,\nu}(y) = \sum_{\tau\in\ I_0} \sum_{J_0\in{}_{n-1}\mathcal{C}_{h-1}} \tau\Big[\Gamma^{n-1,I_0,J_0}(\tilde\nu) \sum_{B\prec A}\sum_{\mathbf{k}} C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde\nu)$$
$$\cdot \int_{(\mathbf{R}_+)^{n-1}} \prod_{p=1}^{n-1} \exp\Big\{-(\pi y_p)^2 t_p - \frac{1}{t_p}\Big\}$$
$$\cdot \prod_{p=1}^{n-2} \Big(\pi y_{p+1}\sqrt{\frac{t_{p+1}}{t_p}}\Big)^{2k_p+\tilde\nu_{\tau^{I_0,J_0}(1)}+\cdots+\tilde\nu_{\tau^{I_0,J_0}(p)}+\varepsilon^{J_0}_B(p)}$$
$$\cdot \prod_{p=1}^{n-1} (\pi y_p)^{\frac{n-p}{n-1}\nu_{i_1}+\chi_{I(B,A)}(p)} \prod_{p=1}^{n-1} t_p^{\frac{n}{2(n-1)}\nu_{i_1}+\frac{1}{2}(\chi_{I(B,A)}(p)-\chi_{I(A,B)}(p))} \prod_{p=1}^{n-1}\frac{dt_p}{t_p}\Big]$$

$$= \sum_{\tau\in\ I_0} \sum_{J_0\in{}_{n-1}\mathcal{C}_{h-1}} \tau\Big[\Gamma^{n-1,I_0,J_0}(\tilde\nu) \sum_{B\prec A}\sum_{\mathbf{k}} C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde\nu) \prod_{p=1}^{n-1} U_p\Big], \tag{9.2}$$

where

$$U_p = (\pi y_p)^{2k_{p-1}+\nu_{i_1}+\sum_{q=1}^{p-1}\nu_{\tilde{\tau}_0(q)}+\varepsilon_B^{J_0}(p-1)+\chi_{I(B,A)}(p)}$$
$$\cdot \int_0^\infty \exp\Big\{-(\pi y_p)^2 t_p - \frac{1}{t_p}\Big\} t_p^{k_{p-1}-k_p+\frac{1}{2}(\nu_{i_1}-\nu_{\tilde{\tau}_0(p)}+\chi_{I(B,A)}(p)-\chi_{I(A,B)}(p)+\varepsilon_B^{J_0}(p-1)-\varepsilon_B^{J_0}(p))} \frac{dt_p}{t_p}.$$

Recall that $\tilde{\tau}_0(p) = \tau^{I_0,J_0}(p) + \chi_{[i_1,n-1]}(\tau^{I_0,J_0}(p))$. We can see that

$$\chi_{I(B,A)}(p) - \chi_{I(A,B)}(p) + \varepsilon_B^{J_0}(p-1) - \varepsilon_B^{J_0}(p)$$
$$= \begin{cases} 2(\chi_{I^e(B,A)}(p) - \chi_{I^e(A,B)}(p)) & \text{if } p \in J_0^e, \\ 2(\chi_{I^o(B,A)}(p) - \chi_{I^o(A,B)}(p)) & \text{if } p \in J_0^o, \\ \chi_{I(B,A)\cup B^e}(p) - \chi_{I(A,B)\cup B^o}(p) & \text{if } p \in I^e(J_0), \\ \chi_{I(B,A)\cup B^o}(p) - \chi_{I(A,B)\cup B^e}(p) & \text{if } p \in I^o(J_0). \end{cases}$$

Especially the above integer is odd if $p \in J_o^c = I^e(J_0) \cup I^o(J_0)$. Let us evaluate the integrals U_p by applying (6.2) for $p \in J_0$, and (6.3) for $p \in J_0^c$.

When $p \in J_0^e$ we use the formula (6.2) with $k = k_{p-1} - k_p + \chi_{I^e(B,A)}(p) - \chi_{I^e(A,B)}(p)$ and $\nu = (\nu_{i_1} - \nu_{\tilde{\tau}_0(p)})/2$ to find that

$$U_p = (\pi y_p)^{2k_{p-1}+\nu_{i_1}+\sum_{q=1}^{p-1}\nu_{\tilde{\tau}_0(q)}+\varepsilon_B^{J_0}(p-1)+\chi_{I(B,A)}(p)} \cdot (-1)^{k_{p-1}+k_p+\chi_{I^e(B,A)}(p)-\chi_{I^e(A,B)}(p)}$$
$$\cdot \Bigg\{ \Gamma\Big(\frac{\nu_{i_1}-\nu_{\tilde{\tau}_0(p)}}{2}\Big) \sum_{m_p=0}^\infty \frac{(\pi y_p)^{2(m_p-k_{p-1}+k_p+\chi_{I^e(A,B)}(p)-\chi_{I^e(B,A)}(p))-\nu_{i_1}+\nu_{\tilde{\tau}_0(p)}}}{m_p!\big(\frac{-\nu_{i_1}+\nu_{\tilde{\tau}_0(p)}+2}{2}\big)_{m_p+k_p+\chi_{I^e(A,B)}(p)-k_{p-1}-\chi_{I^e(B,A)}(p)}}$$
$$+ \Gamma\Big(\frac{-\nu_{i_1}+\nu_{\tilde{\tau}_0(p)}}{2}\Big) \sum_{m_p=0}^\infty \frac{(\pi y_p)^{2m_p}}{m_p!\big(\frac{\nu_{i_1}-\nu_{\tilde{\tau}_0(p)}+2}{2}\big)_{m_p+k_{p-1}+\chi_{I^e(B,A)}(p)-k_p-\chi_{I^e(A,B)}(p)}} \Bigg\}.$$

In view of $\varepsilon_B^{J_0}(j_t - 1) = \chi_{I_r^e(B)}(j_t)$ we get

$$U_p = (-1)^{k_{p-1}+k_p+\chi_{I^e(B,A)}(p)-\chi_{I^e(A,B)}(p)}$$
$$\cdot \Bigg\{ \Gamma\Big(\frac{\nu_{i_1}-\nu_{\tilde{\tau}_0(p)}}{2}\Big) \sum_{m_p=0}^\infty \frac{(\pi y_p)^{2(m_p+k_p+\chi_{I^e(A,B)}(p))+\sum_{q=1}^{p}\nu_{\tilde{\tau}_0(q)}+\chi_{I_l^o(A)}(p)}}{m_p!\big(\frac{-\nu_{i_1}+\nu_{\tilde{\tau}_0(p)}+2}{2}\big)_{m_p+k_p+\chi_{I^e(A,B)}(p)-k_{p-1}-\chi_{I^e(B,A)}(p)}}$$
$$+ \Gamma\Big(\frac{-\nu_{i_1}+\nu_{\tilde{\tau}_0(p)}}{2}\Big) \sum_{m_p=0}^\infty \frac{(\pi y_p)^{2(m_p+k_{p-1}+\chi_{I^e(B,A)}(p))+\nu_{i_1}+\sum_{q=1}^{p-1}\nu_{\tilde{\tau}_0(q)}+\chi_{I_l^o(A)}(p)}}{m_p!\big(\frac{\nu_{i_1}-\nu_{\tilde{\tau}_0(p)}+2}{2}\big)_{m_p+k_{p-1}+\chi_{I^e(B,A)}(p)-k_p-\chi_{I^e(A,B)}(p)}} \Bigg\}$$

for $p \in J_0^e$. Similarly for $p \in J_0^o$, we have

$$U_p = (-1)^{k_{p-1}+k_p+\chi_{I^o(B,A)}(p)-\chi_{I^o(A,B)}(p)}$$
$$\cdot \Bigg\{ \Gamma\Big(\frac{\nu_{i_1}-\nu_{\tilde{\tau}_0(p)}}{2}\Big) \sum_{m_p=0}^\infty \frac{(\pi y_p)^{2(m_p+k_p+\chi_{I^o(A,B)}(p))+\sum_{q=1}^{p}\nu_{\tilde{\tau}_0(q)}+\chi_{I_l^e(A)}(p)}}{m_p!\big(\frac{-\nu_{i_1}+\nu_{\tilde{\tau}_0(p)}+2}{2}\big)_{m_p+k_p+\chi_{I^o(A,B)}(p)-k_{p-1}-\chi_{I^o(B,A)}(p)}}$$
$$+ \Gamma\Big(\frac{-\nu_{i_1}+\nu_{\tilde{\tau}_0(p)}}{2}\Big) \sum_{m_p=0}^\infty \frac{(\pi y_p)^{2(m_p+k_{p-1}+\chi_{I^o(B,A)}(p))+\nu_{i_1}+\sum_{q=1}^{p-1}\nu_{\tilde{\tau}_0(q)}+\chi_{I_l^e(A)}(p)}}{m_p!\big(\frac{\nu_{i_1}-\nu_{\tilde{\tau}_0(p)}+2}{2}\big)_{m_p+k_{p-1}+\chi_{I^o(B,A)}(p)-k_p-\chi_{I^o(A,B)}(p)}} \Bigg\}.$$

In the case of $p \in I^e(J_0)$, we get

$$U_p = (-1)^{k_{p-1}+k_p+\frac{1}{2}(\chi_{I(B,A)\cup B^e}(p)-\chi_{I(A,B)\cup B^o}(p)-1)}$$

$$\cdot\Bigg\{\Gamma\Big(\frac{\nu_{i_1}-\nu_{\tilde\tau_0(p)}+1}{2}\Big)\sum_{m_p=0}^{\infty}\frac{(\pi y_p)^{2(m_p+k_p+\chi_{I^e(A,B)}(p))+\sum_{q=1}^{p}\nu_{\tilde\tau_0(q)}+\chi_{I_l^o(A)}(p)}}{m_p!\big(\frac{-\nu_{i_1}+\nu_{\tilde\tau_0(p)}+1}{2}\big)_{m_p+k_p+\chi_{I^e(A,B)}(p)-k_{p-1}+\chi_{I^o(A,B)\cup B^o}(p)}}$$

$$-\Gamma\Big(\frac{-\nu_{i_1}+\nu_{\tilde\tau_0(p)}+1}{2}\Big)\sum_{m_p=0}^{\infty}\frac{(\pi y_p)^{2(m_p+k_{p-1}+\chi_{I^o(B,A)}(p))+\nu_{i_1}+\sum_{q=1}^{p-1}\nu_{\tilde\tau_0(q)}+\chi_{I_l^e(A)}(p)}}{m_p!\big(\frac{\nu_{i_1}-\nu_{\tilde\tau_0(p)}+1}{2}\big)_{m_p+k_{p-1}+\chi_{I^o(B,A)}(p)-k_p+\chi_{I^e(B,A)\cup B^e}(p)}}\Bigg\}.$$

When $p\in I^o(J_0)$ we switch the subscript e and o in the above.

By inserting the expressions above into (9.2) and replacing $m_p\to m_p-k_p-\chi_*$ and $m_p\to m_p-k_{p-1}-\chi_*$, we can find the following:

Lemma 9.3. *We have*

$$\widetilde{W}^{n,I}_{A,\nu}(y)=\sum_{\tau\in\ {}_{I_0}}\ \sum_{J_0\in{}_{n-1}\mathcal{C}_{h-1}}\ \sum_{R\subset T_{n-1}}\tau\Big[\Gamma^{I,J_0,R}(\nu)\cdot\widetilde{M}^{I,J_0,R}_{A,\nu}(y)\Big], \tag{9.3}$$

where

(9.4)

$$\Gamma^{I,J_0,R}(\nu)=\Gamma^{n-1,I_0,J_0}(\tilde\nu)\prod_{p\in R^c}\Gamma\Big(\frac{\nu_{i_1}-\nu_{\tilde\tau_0(p)}+\chi_{J_0^c}(p)}{2}\Big)\prod_{p\in R}\Gamma\Big(\frac{-\nu_{i_1}+\nu_{\tilde\tau_0(p)}+\chi_{J_0^c}(p)}{2}\Big),$$

and

$$\widetilde{M}^{I,J_0,R}_{A,\nu}(y)=\sum_{\mathbf{m}}C^{I,J_0,R}_{A;\mathbf{m}}(\nu)\prod_{p=1}^{n-1}(\pi y_p)^{2m_p+\chi_R(p)\cdot\nu_{i_1}+\chi_{R^c}(p)\cdot\nu_{\tilde\tau_0(p)}+\sum_{q=1}^{p-1}\nu_{\tilde\tau_0(q)}+\varepsilon_A^{J_0,R}(p)}.$$

Here the coefficients $C^{I,J_0,R}_{A;\mathbf{m}}(\nu)$ are given by

$$C^{I,J_0,R}_{A;\mathbf{m}}(\nu)=\sum_{B\prec A}\sum_{\mathbf{k}}\frac{C^{n-1,I_0,J_0}_{B;\mathbf{k}}(\tilde\nu)}{P^{I,J_0,R}_{A,B;\mathbf{m},\mathbf{k}}},$$

where

$$\begin{aligned}P^{I,J_0,R}_{A,B;\mathbf{m},\mathbf{k}}=&\prod_{p=1}^{n-1}(-1)^{\zeta^{J_0,R}_{A,B}(p)+\eta^{J_0,R}_{A,B}(p)}\\&\cdot\prod_{p\in R^c}(m_p-k_p+\zeta^{J_0,R}_{A,B}(p))!\Big(\frac{-\nu_{i_1}+\nu_{\tilde\tau_0(p)}+1+\chi_{J_0}(p)}{2}\Big)_{m_p-k_{p-1}+\eta^{J_0,R}_{A,B}(p)}\\&\cdot\prod_{p\in R}(m_p-k_{p-1}+\eta^{J_0,R}_{A,B}(p))!\Big(\frac{\nu_{i_1}-\nu_{\tilde\tau_0(p)}+1+\chi_{J_0}(p)}{2}\Big)_{m_p-k_p+\zeta^{J_0,R}_{A,B}(p)}.\end{aligned}$$

For our later use we write $\Gamma^{I,J_0,R}(\nu)$ more explicitly.

Lemma 9.4. *We have*

$$\begin{aligned}\Gamma^{I,J_0,R}(\nu)=&\prod_{2\le p<q\le h}\Gamma\Big(\frac{\nu_{i_p}-\nu_{i_q}}{2}\Big)\prod_{1\le p<q\le n-h}\Gamma\Big(\frac{-\nu_{i'_p}+\nu_{i'_q}}{2}\Big)\\&\cdot\prod_{t=0}^{h-1}\prod_{p=j_t+1}^{j_{t+1}-1}\Big\{\prod_{s=2}^{h-t}\Gamma\Big(\frac{\nu_{i_s}-\nu_{i'_{p-t}}+1}{2}\Big)\prod_{s=h-t+1}^{h}\Gamma\Big(\frac{-\nu_{i_s}+\nu_{i'_{p-t}}+1}{2}\Big)\Big\}\end{aligned}$$

$$\cdot \prod_{t=1}^{h-1} \Big\{ \chi_{R^c}(j_t) \cdot \Gamma\Big(\frac{\nu_{i_1} - \nu_{i_{h+1-t}}}{2} \Big) + \chi_R(j_t) \cdot \Gamma\Big(\frac{-\nu_{i_1} + \nu_{i_{h+1-t}}}{2} \Big) \Big\}$$
$$\cdot \prod_{t=0}^{h-1} \prod_{p=j_t+1}^{j_{t+1}-1} \Big\{ \chi_{R^c}(p) \cdot \Gamma\Big(\frac{\nu_{i_1} - \nu_{i'_{p-t}} + 1}{2} \Big) + \chi_R(p) \cdot \Gamma\Big(\frac{-\nu_{i_1} + \nu_{i'_{p-t}} + 1}{2} \Big) \Big\}.$$

Here we understand $j_h = n$.

Proof. Let us compute $\Gamma^{n-1,I_0,J_0}(\tilde{\nu})$. In view of the relations $\tilde{\nu}_p - \tilde{\nu}_q = \nu_{p+\chi_{[i_1,n-1]}(p)} - \nu_{q+\chi_{[i_1,n-1]}(q)}$, and

$$i_{0,p} + \chi_{[i_1,n-1]}(i_{0,p}) = i_{p+1}, \quad i'_{0,p} + \chi_{[i_1,n-1]}(i'_{0,p}) = i'_p$$

for $I_0 = \{i_{0,1}, \ldots, i_{0,h-1}\}$ and $I_0^c = \{i'_{0,1}, \ldots, i'_{0,n-h}\}$, we can get

$$\Gamma^{n-1,I_0,J_0}(\tilde{\nu}) = \prod_{1 \le p < q \le h-1} \Gamma\Big(\frac{\nu_{i_{p+1}} - \nu_{i_{q+1}}}{2} \Big) \prod_{1 \le p < q \le n-h} \Gamma\Big(\frac{-\nu_{i'_p} + \nu_{i'_q}}{2} \Big)$$
$$\cdot \prod_{t=0}^{h-1} \prod_{p=j_t+1}^{j_{t+1}-1} \Big\{ \prod_{s=1}^{t} \Gamma\Big(\frac{-\nu_{i_{h-s+1}} + \nu_{i'_{p-t}} + 1}{2} \Big) \prod_{s=t+1}^{h-1} \Gamma\Big(\frac{\nu_{i_{h-s+1}} - \nu_{i'_{p-t}} + 1}{2} \Big) \Big\}.$$

Because of $\tilde{\tau}_0(j_t) = i_{h+1-t}$ $(1 \le t \le h-1)$ and $\tilde{\tau}_0(p) = i'_{p-t}$ $(j_t + 1 \le p \le j_{t+1} - 1)$, we have the desired formula. □

In Proposition 9.6, we will see that only the terms $R = [j_h, n-1]$ contribute to the summation in (9.3). Let us compute the $\Gamma^{I,J_0,R}(\nu)$ for such R.

Corollary 9.5. *For a palette $\Pi = (J_0, j_h)$ of type (n,h) and the integer t_Π and the color J_Π of type (n,h), we have*

$$\Gamma^{I,J_0,[j_h,n-1]}(\nu) = \Gamma^{n,I,J_\Pi}(\tau_{cy}^{I,\Pi}\nu),$$

where $\tau_{cy}^{I,\Pi}$ is the cyclic permutation $(i_1, \ldots, i_{h-t_\Pi})$.

Proof. From Lemma 9.4 we have

$$\Gamma^{I,J_0,[j_h,n-1]}(\nu) = \gamma_1 \cdot \gamma_2 \cdot \gamma_3$$

where

$$\gamma_1 = \prod_{2 \le p < q \le h} \Gamma\Big(\frac{\nu_{i_p} - \nu_{i_q}}{2} \Big) \prod_{1 \le t \le t_\Pi} \Gamma\Big(\frac{\nu_{i_1} - \nu_{i_{h+1-t}}}{2} \Big) \prod_{t_\Pi + 1 \le t \le h-1} \Gamma\Big(\frac{-\nu_{i_1} + \nu_{i_{h+1-t}}}{2} \Big),$$

$$\gamma_2 = \prod_{1 \le p < q \le n-h} \Gamma\Big(\frac{-\nu_{i'_p} + \nu_{i'_q}}{2} \Big),$$

$$\gamma_3 = \prod_{t=0}^{h-2} \prod_{p=j_t+1}^{j_{t+1}-1} \Big\{ \prod_{s=2}^{h-t} \Gamma\Big(\frac{\nu_{i_s} - \nu_{i'_{p-t}} + 1}{2} \Big) \prod_{s=h-t+1}^{h} \Gamma\Big(\frac{-\nu_{i_s} + \nu_{i'_{p-t}} + 1}{2} \Big) \Big\}$$
$$\cdot \prod_{p=j_{h-1}+1}^{n-1} \prod_{s=2}^{h} \Gamma\Big(\frac{-\nu_{i_s} + \nu_{i'_{p-h+1}} + 1}{2} \Big) \cdot \gamma_4.$$

Here

$$\gamma_4 = \prod_{t=0}^{t_\Pi - 1} \prod_{p=j_t+1}^{j_{t+1}-1} \Gamma\Big(\frac{\nu_{i_1} - \nu_{i'_{p-t}} + 1}{2} \Big)$$

$$\cdot \prod_{p=j_{t_\Pi}+1}^{j_h-1} \Gamma\Big(\frac{\nu_{i_1} - \nu_{i'_{p-t_\Pi}} + 1}{2}\Big) \cdot \prod_{p=j_h}^{j_{t_\Pi+1}-1} \Gamma\Big(\frac{-\nu_{i_1} + \nu_{i'_{p-t_\Pi}} + 1}{2}\Big)$$

$$\cdot \prod_{t=t_\Pi+1}^{h-2} \prod_{p=j_t+1}^{j_{t+1}-1} \Gamma\Big(\frac{-\nu_{i_1} + \nu_{i'_{p-t}} + 1}{2}\Big) \cdot \prod_{p=j_{h-1}+1}^{n-1} \Gamma\Big(\frac{-\nu_{i_1} + \nu_{i'_{p-h+1}} + 1}{2}\Big)$$

for $0 \le t_\Pi \le h-2$, and

$$\gamma_4 = \prod_{t=0}^{h-1} \prod_{p=j_t+1}^{j_{t+1}-1} \Gamma\Big(\frac{\nu_{i_1} - \nu_{i'_{p-t}} + 1}{2}\Big) \cdot \prod_{p=j_h+1}^{n} \Gamma\Big(\frac{-\nu_{i_1} + \nu_{i'_{p-h}} + 1}{2}\Big)$$

for $t_\Pi = h-1$.

When $t_\Pi = h-1$ we can readily find that $\Gamma^{I,J_0,[j_h,n-1]}(\nu) = \Gamma^{n,I,J}(\nu)$. Let us consider the case $0 \le t_\Pi \le h-2$. We have $\gamma_3 = \prod_{i=1}^5 \gamma_{3,i}$ with

$$\gamma_{3,1} = \prod_{t=0}^{t_\Pi-1} \prod_{p=j_t+1}^{j_{t+1}-1} \Big\{ \prod_{s=1}^{h-t} \Gamma\Big(\frac{\nu_{i_s} - \nu_{i'_{p-t}} + 1}{2}\Big) \prod_{s=h-t+1}^{h} \Gamma\Big(\frac{-\nu_{i_s} + \nu_{i'_{p-t}} + 1}{2}\Big)\Big\},$$

$$\gamma_{3,2} = \prod_{p=j_{t_\Pi}+1}^{j_h-1} \Big\{ \prod_{s=1}^{h-t_\Pi} \Gamma\Big(\frac{\nu_{i_s} - \nu_{i'_{p-t_\Pi}} + 1}{2}\Big) \prod_{s=h-t_\Pi+1}^{h} \Gamma\Big(\frac{-\nu_{i_s} + \nu_{i'_{p-t_\Pi}} + 1}{2}\Big)\Big\},$$

$$\gamma_{3,3} = \prod_{p=j_h}^{j_{t_\Pi+1}-1} \Big\{ \prod_{s=2}^{h-t_\Pi} \Gamma\Big(\frac{\nu_{i_s} - \nu_{i'_{p-t}} + 1}{2}\Big) \cdot \Gamma\Big(\frac{-\nu_{i_1} + \nu_{i'_{p-t_\Pi}} + 1}{2}\Big)$$
$$\cdot \prod_{s=h-t_\Pi+1}^{h} \Gamma\Big(\frac{-\nu_{i_s} + \nu_{i'_{p-t_\Pi}} + 1}{2}\Big)\Big\},$$

$$\gamma_{3,4} = \prod_{p=t_\Pi+1}^{h-2} \prod_{p=j_t+1}^{j_{t+1}-1} \Big\{ \prod_{s=2}^{h-t} \Gamma\Big(\frac{\nu_{i_s} - \nu_{i'_{p-t}} + 1}{2}\Big) \cdot \Gamma\Big(\frac{-\nu_{i_1} + \nu_{i'_{p-t}} + 1}{2}\Big)$$
$$\cdot \prod_{s=h-t+1}^{h} \Gamma\Big(\frac{-\nu_{i_s} + \nu_{i'_{p-t}} + 1}{2}\Big)\Big\},$$

$$\gamma_{3,5} = \prod_{p=j_{h-1}+1}^{n-1} \prod_{s=1}^{h} \Gamma\Big(\frac{-\nu_{i_s} + \nu_{i'_{p-h+1}} + 1}{2}\Big).$$

Now we compute $\Gamma^{n,I,J_\Pi}(\tau_{cy}^{I,\Pi}\nu)$. Since $\tau_{cy}^{I,\Pi}(i'_p) = i'_p$, we have

$$\Gamma^{n,I,J_\Pi}(\tau_{cy}^{I,\Pi}\nu) = \gamma'_1 \cdot \gamma'_2 \cdot \gamma'_3,$$

where

$$\gamma'_1 = \prod_{1 \le p < q \le h} \Gamma\Big(\frac{\nu_{\tau_{cy}^{I,\Pi}(i_p)} - \nu_{\tau_{cy}^{I,\Pi}(i_q)}}{2}\Big), \quad \gamma'_2 = \gamma_2$$

and $\gamma'_3 = \prod_{i=1}^5 \gamma'_{3,i}$ with

$$\gamma'_{3,1} = \prod_{t=0}^{t_\Pi-1} \prod_{p=j_t+1}^{j_{t+1}-1} \Big\{ \prod_{s=1}^{h-t} \Gamma\Big(\frac{\nu_{\tau_{cy}^{I,\Pi}(i_s)} - \nu_{i'_{p-t}} + 1}{2}\Big) \prod_{s=h-t+1}^{h} \Gamma\Big(\frac{-\nu_{\tau_{cy}^{I,\Pi}(i_s)} + \nu_{i'_{p-t}} + 1}{2}\Big)\Big\},$$

$$\gamma'_{3,2} = \prod_{p=j_{t_\Pi}+1}^{j_h-1} \Big\{ \prod_{s=1}^{h-t_\Pi} \Gamma\Big(\frac{\nu_{\tau_{cy}^{I,\Pi}(i_s)} - \nu_{i'_{p-t}} + 1}{2}\Big) \prod_{s=h-t_\Pi+1}^{h} \Gamma\Big(\frac{-\nu_{\tau_{cy}^{I,\Pi}(i_s)} + \nu_{i'_{p-t}} + 1}{2}\Big)\Big\},$$

$$\gamma'_{3,3} = \prod_{p=j_h+1}^{j_{t_\Pi+1}} \Big\{ \prod_{s=1}^{h-t_\Pi-1} \Gamma\Big(\frac{\nu_{\tau_{cy}^{I,\Pi}(i_s)} - \nu_{i'_{p-t_\Pi-1}} + 1}{2}\Big) \prod_{s=h-t_\Pi}^{h} \Gamma\Big(\frac{-\nu_{\tau_{cy}^{I,\Pi}(i_s)} + \nu_{i'_{p-t_\Pi-1}} + 1}{2}\Big)\Big\},$$

$$\gamma'_{3,4} = \prod_{t=t_\Pi+1}^{h-2} \prod_{p=j_t+2}^{j_{t+1}} \Big\{ \prod_{s=1}^{h-t-1} \Gamma\Big(\frac{\nu_{\tau_{cy}^{I,\Pi}(i_s)} - \nu_{i'_{p-t-1}} + 1}{2}\Big) \prod_{p=h-t}^{h} \Gamma\Big(\frac{-\nu_{\tau_{cy}^{I,\Pi}(i_s)} + \nu_{i'_{p-t-1}} + 1}{2}\Big)\Big\},$$

$$\gamma'_{3,5} = \prod_{p=j_{h-1}+2}^{n} \prod_{s=1}^{h} \Gamma\Big(\frac{-\nu_{\tau_{cy}^{I,\Pi}(i_s)} + \nu_{i'_{p-h}} + 1}{2}\Big).$$

In view of

$$\tau_{cy}^{I,\Pi}(i_p) = \begin{cases} i_{p+1} & \text{if } 1 \le p \le h - t_\Pi - 1, \\ i_1 & \text{if } p = h - t_\Pi, \\ i_p & \text{if } h - t_\Pi + 1 \le p \le h - t_\Pi + 1 \le h, \end{cases}$$

we have

$$\gamma'_1 = \prod_{1 \le p < q \le h-t_\Pi-1} \Gamma\Big(\frac{\nu_{i_{p+1}} - \nu_{i_{q+1}}}{2}\Big) \prod_{h-t_\Pi < q \le h} \Gamma\Big(\frac{\nu_{i_1} - \nu_{i_q}}{2}\Big) \prod_{1 \le p < h-t_\Pi} \Gamma\Big(\frac{\nu_{i_{p+1}} - \nu_{i_1}}{2}\Big)$$
$$\cdot \prod_{\substack{1 \le p \le h-t_\Pi-1 \\ h-t_\Pi+1 \le q \le h}} \Gamma\Big(\frac{\nu_{i_{p+1}} - \nu_{i_q}}{2}\Big) \prod_{h-t_\Pi+1 \le p < q \le h} \Gamma\Big(\frac{\nu_{i_p} - \nu_{i_q}}{2}\Big)$$
$$= \gamma_1$$

Similarly we can check that $\gamma_{3,i} = \gamma'_{3,i}$ for $1 \le i \le 5$. □

9.2. **Vanishing of ghost solutions.** As in the case of helicity 1 (Proposition 6.3), we delete those power series $\widetilde{M}_{A,\nu}^{I,J_0,R}(y)$ which do not contribute to the summation (9.3). Our vanishing process is more complicated than the case of helicity 1.

Proposition 9.6. *For a subset R of T_{n-1} we set*

$$E(R) = \{r \in T_{n-2} \mid r \in R \text{ and } r + 1 \in R^c\}.$$

Then only the subsets R satisfying $E(R) = \emptyset$ contribute to the summation in (6.4). More precisely we have the following.

(1) *When $E(R) = \emptyset$, the set R is of the form $[j_h, n-1]$ with some $j_h \in T_n$ ($j_h = n \iff R = \emptyset$), and we have $\widetilde{M}_{A,\nu}^{I,J_0,R}(y) = \widetilde{M}_{A,\nu}^{n,I,\Pi=(J_0,j_h)}(y)$ and $\Gamma^{I,J_0,R}(\nu) = \Gamma^{n,I,J_\Pi}(\tau_{cy}^{I,\Pi}\nu)$. Then we get*

$$\sum_{\tau \in \sigma_{I_0}} \sum_{J_0 \in {}_{n-1}\mathcal{C}_{h-1}} \sum_{1 \le j_h \le n} \tau\Big[\Gamma^{I,J_0,[j_h,n-1]}(\nu) \cdot \widetilde{M}_{A,\nu}^{I,J_0,[j_h,n-1]}(y)\Big]$$
$$= \sum_{\tau \in \sigma_I} \sum_{J \in {}_n\mathcal{C}_h} \tau\Big[\Gamma^{n,I,J}(\nu) \cdot \widetilde{M}_{A,\nu}^{n,I,J}(y)\Big].$$

(2) *When $E(R) \ne \emptyset$, we define subsets $E_i(J_0, R)$ ($i = 1, 2, 3$) of $E(R)$ for a given $J_0 = \{j_1, \ldots, j_{h-1}\} \in {}_{n-1}\mathcal{C}_{h-1}$ by*

$$E_1(J_0, R) = E(R) \cap J_0,$$

$E_2(J_0,R)=E(R)\cap J_0[-1]$,
$E_3(J_0,R)=E(R)\backslash(E_1(J_0,R)\cup E_2(J_0,R))=\cup_{t=0}^{h-1}(E(R)\cap[j_t+1,j_{t+1}-2])$.

(a) *The case of* $E_3(J_0,R)\neq\emptyset$. *For* $r\in E_3(J_0,R)$ *we take an integer* t $(0\le t\le h-1)$ *such that* $r\in[j_t+1,j_{t+1}-2]$. *Then we have*

$$\Gamma^{I,J_0,R}(\nu)\cdot\widetilde{M}^{I,J_0,R}_{A,\nu}(y)+\Gamma^{I,J_0,R}(\tau'_r\nu)\cdot\widetilde{M}^{I,J_0,R}_{A,\tau'_r\nu}(y)=0.$$

Here τ'_r *is the permutation* (i'_{r-t},i'_{r+1-t}).

(b) *The case of* $E_3(J_0,R)=\emptyset$.

(i) *If* $E_1(J_0,R)\cap E_2(J_0,R)$ *is not empty, there exists an integer* $t\in T_{n-1}$ *such that* $j_t\in E(R)$ *and* $j_t=j_{t+1}-1$. *Then we have*

$$\Gamma^{I,J_0,R}(\nu)\cdot\widetilde{M}^{I,J_0,R}_{A,\nu}(y)+\Gamma^{I,J_0,R}(\tau_t\nu)\cdot\widetilde{M}^{I,J_0,R}_{A,\tau_t\nu}(y)=0.$$

Here τ_t *is the permutation* (i_{h-t},i_{h-t+1}).

(ii) *We denote by* $\mathcal{J}(R)$ *a set of colors* J_0 *satisfying* $E_1(J_0,R)\cap E_2(J_0,R)=\emptyset$, *that is,*

$$\mathcal{J}(R)=\{J_0\in{}_{n-1}\mathcal{C}_{h-1}\mid E(R)=E_1(J_0,R)\sqcup E_2(J_0,R)\}.$$

For $J_0\in\mathcal{J}(R)$, *we set*

$$T_1(J_0,R)=\{t\in T_{h-1}\mid j_t\in E_1(J_0,R)\},$$
$$T_2(J_0,R)=\{t\in T_{h-1}\mid j_t-1\in E_2(J_0,R)\},$$

and define a color $J_0^\vee=\{j_1^\vee,\dots,j_{h-1}^\vee\}$ *of type* $(n-1,h-1)$ *as follows. Let* t_0 *be the smallest element of* $T_1(J_0,R)\cap T_2(J_0,R)$. *We put* $j_t^\vee=j_t+\delta_{t,t_0}$ *if* $t_0\in T_1(J_0,R)$, *and* $j_t^\vee=j_t-\delta_{t,t_0}$ *if* $t_0\in T_2(J_0,R)$. *Then we have* $J_0^\vee\in\mathcal{J}(R)$, $(J_0^\vee)^\vee=J_0$ *and*

$$\Gamma^{I,J_0,R}(\nu)\cdot\widetilde{M}^{I,J_0,R}_{A,\nu}(y)+\Gamma^{I,J_0^\vee,R}(\nu)\cdot\widetilde{M}^{I,J_0^\vee,R}_{A,\nu}(y)=0.$$

Example Let us see the cancellation in the case of $(n,h)=(6,3)$, $I=\{1,3,5\}$ and $R=\{1,2,5\}$. Then $I_0=\{2,4\}$, $I_0^c=\{1,3,5\}$ and $E(R)=\{2\}$.

We write the *principal exponents* of y_p in $\widetilde{M}^{I,J_0,R}_{A,\nu}$, that is, $\nu_{i_1}\cdot\chi_R(p)+\nu_{\tilde{\tau}_0(p)}\cdot\chi_{R^c}(p)+\sum_{q=1}^{p-1}\nu_{\tilde{\tau}_0(q)}$ in the following tabular.

J_0	y_1	y_2	y_3	y_4	y_5
$\{1,2\}$	ν_1	$\nu_1+\nu_5$	$\nu_5+\nu_3+\nu_2$	$\nu_5+\nu_3+\nu_2+\nu_4$	$\nu_5+\nu_3+\nu_2+\nu_4+\nu_6$
$\{1,3\}$	ν_1	$\nu_1+\nu_5$	$\nu_5+\nu_3+\nu_2$	$\nu_5+\nu_3+\nu_2+\nu_4$	$\nu_5+\nu_3+\nu_2+\nu_4+\nu_6$
$\{1,4\}$	ν_1	$\nu_1+\nu_5$	$\nu_5+\nu_2+\nu_4$	$\nu_5+\nu_2+\nu_4+\nu_3$	$\nu_5+\nu_2+\nu_4+\nu_3+\nu_6$
$\{1,5\}$	ν_1	$\nu_1+\nu_5$	$\nu_5+\nu_2+\nu_4$	$\nu_5+\nu_2+\nu_4+\nu_6$	$\nu_5+\nu_2+\nu_4+\nu_6+\nu_3$
$\{2,3\}$	ν_1	$\nu_1+\nu_2$	$\nu_2+\nu_5+\nu_3$	$\nu_2+\nu_5+\nu_3+\nu_4$	$\nu_2+\nu_5+\nu_3+\nu_4+\nu_6$
$\{2,4\}$	ν_1	$\nu_1+\nu_2$	$\nu_2+\nu_5+\nu_4$	$\nu_2+\nu_5+\nu_4+\nu_3$	$\nu_2+\nu_5+\nu_4+\nu_3+\nu_6$
$\{2,5\}$	ν_1	$\nu_1+\nu_2$	$\nu_2+\nu_5+\nu_4$	$\nu_2+\nu_5+\nu_4+\nu_6$	$\nu_2+\nu_5+\nu_4+\nu_6+\nu_3$
$\{3,4\}$	ν_1	$\nu_1+\nu_2$	$\nu_2+\nu_4+\nu_5$	$\nu_2+\nu_4+\nu_5+\nu_3$	$\nu_2+\nu_4+\nu_5+\nu_3+\nu_6$
$\{3,5\}$	ν_1	$\nu_1+\nu_2$	$\nu_2+\nu_4+\nu_5$	$\nu_2+\nu_4+\nu_5+\nu_6$	$\nu_2+\nu_4+\nu_5+\nu_6+\nu_3$
$\{4,5\}$	ν_1	$\nu_1+\nu_2$	$\nu_2+\nu_4+\nu_6$	$\nu_2+\nu_4+\nu_6+\nu_5$	$\nu_2+\nu_4+\nu_6+\nu_5+\nu_3$

Here is the list of the symbols in Proposition 9.6.

J_0	$E_1(R,J_0)$	$E_2(R,J_0)$	$E_3(R,J_0)$	(r,t)	τ'_r	τ_t	$J_0^\vee$
$\{1,2\}$	$\{j_2=2\}$	$\emptyset$	$\emptyset$	$-$	$-$	$-$	$\{1,3\}$
$\{1,3\}$	$\emptyset$	$\{j_2-1=2\}$	$\emptyset$	$-$	$-$	$-$	$\{1,2\}$
$\{1,4\}$	$\emptyset$	$\emptyset$	$\{2\}$	$(2,1)$	$(2,4)$	$-$	$-$
$\{1,5\}$	$\emptyset$	$\emptyset$	$\{2\}$	$(2,1)$	$(2,4)$	$-$	$-$
$\{2,3\}$	$\{j_1=2\}$	$\{j_2-1=2\}$	$\emptyset$	$-$	$-$	$(3,5)$	$-$
$\{2,4\}$	$\{j_1=2\}$	$\emptyset$	$\emptyset$	$-$	$-$	$-$	$\{3,4\}$
$\{2,5\}$	$\{j_1=2\}$	$\emptyset$	$\emptyset$	$-$	$-$	$-$	$\{3,5\}$
$\{3,4\}$	$\emptyset$	$\{j_1-1=2\}$	$\emptyset$	$-$	$-$	$-$	$\{2,4\}$
$\{3,5\}$	$\emptyset$	$\{j_1-1=2\}$	$\emptyset$	$-$	$-$	$-$	$\{2,5\}$
$\{4,5\}$	$\emptyset$	$\emptyset$	$\{2\}$	$(2,0)$	$(4,6)$	$-$	$-$

From now on we will prove Proposition 9.6. It is obvious that $\widetilde{M}_{A,\nu}^{I,J_0,[j_h,n-1]}(y) = \widetilde{M}_{A,\nu}^{n,I,(J_0,j_h)}(y)$ from the definition and then claim (1) follows from Proposition 8.6 and Corollary 9.5. To prove the cancellations stated in Proposition 9.6 (ii), we give recurrence relations for $C_{A;\mathbf{m}}^{I,J_0,R}(\nu)$ in the next subsection. As a result of these recurrence relations, for example, we can find that $C_{A;\mathbf{m}}^{I,J_0,R}(\nu)$ and $C_{A;\mathbf{m}}^{I,J_0,R}(\tau'_r\nu)$ satisfy the same recurrence relations from Lemma 9.9 (cf. Lemma 6.5).

Thus our proof is reduced to show the vanishing of initial values for the edge components A_i $(i=1,2)$:

$$\Gamma^{I,J_0,R}(\nu)\cdot C_{A_i;\mathbf{m}_0}^{I,J_0,R}(\nu)+\Gamma^{I,J_0,R}(\tau'_r\nu)\cdot C_{A_i;\mathbf{m}_0}^{I,J_0,R}(\tau'_r\nu)=0; \tag{9.5}$$

$$\Gamma^{I,J_0,R}(\nu)\cdot C_{A_i;\mathbf{m}_0}^{I,J_0,R}(\nu)+\Gamma^{I,J_0,R}(\tau_t\nu)\cdot C_{A_i;\mathbf{m}_0}^{I,J_0,R}(\tau_t\nu)=0; \tag{9.6}$$

$$\Gamma^{I,J_0,R}(\nu)\cdot C_{A_i;\mathbf{m}_0}^{I,J_0,R}(\nu)+\Gamma^{I,J_0^\vee,R}(\nu)\cdot C_{A_i;\mathbf{m}_0}^{I,J_0^\vee,R}(\nu)=0. \tag{9.7}$$

Here is the definition of the vector $\mathbf{m}_0$ in $\mathbf{R}^{n-1}$.

Definition 9.7. For $i=1,2$ we define $\mathbf{m}_0=(\delta_{A_i}^{J_0,R}(p))_{1\le p\le n-1}$ by

$$\delta_{A_i}^{J_0,R}(p)=(\delta_{B_i}^{J_0}(p-1)-\zeta^{R^c}(p))\chi_{R^c}(p)+(\delta_{B_i}^{J_0}(p)-\eta^R(p))\chi_R(p).$$

Here B_i is the color of type $(n-1,h-1)$ such that $B_i\prec A_i$, that is, $B_1=\{1,\dots,h-1\}$ and $B_2=\{n-h+1,\dots,n-1\}$.

9.3. **The recurrence relations for $C_{A;\mathbf{m}}^{I,J_0,R}(\nu)$.** The aim of this subsection is to prove the following recurrence relations, which include Propositions 8.8 and 8.17 as special cases.

Lemma 9.8. (i) *For $1\le t\le h$ we set*

$$\mathbf{m}_t^{-,R}=\mathbf{m}-\varepsilon_{A_t^-}^{J_0,R}(a_t-1)\cdot\mathbf{e}_{a_t-1},\quad \mathbf{m}_t^{+,R}=\mathbf{m}-\varepsilon_{A_t^+}^{J_0,R}(a_t)\cdot\mathbf{e}_{a_t}.$$

Then the following recurrence relations for $C_{A;\mathbf{m}}^{I,J_0,R}=C_{A;\mathbf{m}}^{I,J_0,R}(\nu)$ hold.

$$\left\{\sum_{t=1}^{h}(-m_{a_t-1}+m_{a_t})+\mu_A^{I,J_0,R}(\nu)\right\}C_{A;\mathbf{m}}^{I,J_0,R}+\sum_{t=1}^{h}\Big(-C_{A_t^-;\mathbf{m}_t^{-,R}}^{I,J_0,R}+C_{A_t^+;\mathbf{m}_t^{+,R}}^{I,J_0,R}\Big)=0; \tag{9.8}$$

$$\begin{aligned}&Q_{\mathbf{m}}\big(\{\lambda_{A,p}^{I,J_0,R}(\nu)\}_{1\le p\le n-1};\kappa_A^{I,J_0,R}(\nu)\big)C_{A;\mathbf{m}}^{I,J_0,R}\\&+\frac{1}{2}\sum_{t=1}^{h}\Big(C_{A_t^-;\mathbf{m}_t^{-,R}}^{I,J_0,R}+C_{A_t^+;\mathbf{m}_t^{+,R}}^{I,J_0,R}\Big)=\sum_{p=1}^{n-1}C_{A;\mathbf{m}-\mathbf{e}_p}^{I,J_0,R},\end{aligned} \tag{9.9}$$

where

$$\mu_A^{I,J_0,R}(\nu) = \frac{1}{2}\sum_{t=1}^{h}\{(\nu_{i_1}-\nu_{i_t}) - (\nu_{i_1}-\nu_{\tilde{\tau}_0(a_t)})\chi_{R^c}(a_t) - (\nu_{i_1}-\nu_{\tilde{\tau}_0(a_t-1)})\chi_R(a_t-1)\}$$
$$+ \frac{1}{2}\sum_{t=1}^{h}(\varepsilon_A^{J_0,R}(a_t) - \varepsilon_A^{J_0,R}(a_t-1)),$$
$$\lambda_{A,p}^{I,J_0,R}(\nu) = \frac{1}{2}\big\{\nu_{\tilde{\tau}_0(p-1)}\chi_R(p-1) + \nu_{\tilde{\tau}_0(p)}(\chi_{R^c}(p)-\chi_R(p)) - \nu_{\tilde{\tau}_0(p+1)}\chi_{R^c}(p+1)$$
$$+ \nu_{i_1}(2\chi_R(p) - \chi_R(p-1) - \chi_R(p+1))$$
$$+ 2\varepsilon_A^{J_0,R}(p) - \varepsilon_A^{J_0,R}(p-1) - \varepsilon_A^{J_0,R}(p+1)\big\}$$

for $1 \le p \le n-1$*, and*

$$\kappa_A^{I,J_0,R}(\nu) = \frac{1}{4}\sum_{p=1}^{n-1}\{(\chi_{A,J_0^c}(p) + \chi_{A^c,J_0}(p))\chi_{R^c}(p)$$
$$+ (-\chi_{A[-1],J_0^c}(p) - \chi_{(A[-1])^c,J_0}(p))\chi_R(p)\}(\nu_{i_1}-\nu_{\tilde{\tau}_0(p)})$$
$$+ \frac{1}{4}\sum_{p=1}^{n-1}(\chi_A(p)\cdot\chi_{R^c}(p) + \chi_{A[-1]}(p)\cdot\chi_R(p))\chi_{J_0^c}(p) + \sum_{p\in E(R)}\xi(p)\xi(p+1),$$

with

$$\xi(p) = \frac{\nu_{i_1}-\nu_{\tilde{\tau}_0(p)} - \chi_{J_0^c}(p)}{2} + \chi_{I^e(J_0)}(p)\cdot\chi_{I_l^e(A)}(p) + \chi_{I^o(J_0)}(p)\cdot\chi_{I_l^o(A)}(p).$$

(ii) *For the the edge components* A_i *(*$i=1,2$*), we have*

$$Q_{\mathbf{m}''}\big(\{\lambda'^{I,J_0,R}_{A_i,p}(\nu)\}_{1\le p\le n-1};\kappa'^{I,J_0,R}_{A_i}(\nu)\big)C^{I,J_0,R}_{A_i;\mathbf{m}}(\nu) = \sum_{p=1}^{n-1}C^{I,J_0,R}_{A_i;\mathbf{m}-\mathbf{e}_p}(\nu), \tag{9.10}$$

where $\mathbf{m}'' = \mathbf{m}-\mathbf{m}_0$ *and*

$$\lambda'^{I,J_0,R}_{A_i,p}(\nu) = \frac{\nu_{i_1}-\nu_{\tilde{\tau}_0(p)} \pm \chi_{J_0^c}(p)}{2}(\chi_R(p)-\chi_{R^c}(p))$$
$$+ \frac{\nu_{i_1}-\nu_{\tilde{\tau}_0(p+1)} \pm \chi_{J_0^c}(p+1)}{2}\cdot\chi_{R^c}(p+1)$$
$$+ \frac{-\nu_{i_1}+\nu_{\tilde{\tau}_0(p+1)} \mp \chi_{J_0^c}(p-1)}{2}\cdot\chi_R(p-1),$$
$$\kappa'^{I,J_0,R}_{A_i}(\nu) = \sum_{p\in E(R)}\frac{\nu_{i_1}-\nu_{\tilde{\tau}_0(p)} \pm \chi_{J_0^c}(p)}{2}\cdot\frac{\nu_{i_1}-\nu_{\tilde{\tau}_0(p+1)} \pm \chi_{J_0^c}(p+1)}{2}.$$

Here $\pm$ *means* $+$ *if* $i=1$ *and* $-$ *if* $i=2$.

Lemma 9.9. *Let* $X^{J_0}(\nu)$ *be one of the symbols* $\mu_A^{I,J_0,R}(\nu)$, $\lambda_{A,p}^{I,J_0,R}(\nu)$, $\kappa_A^{I,J_0,R}(\nu)$, $\lambda'^{I,J_0,R}_{A_i,p}(\nu)$ *or* $\kappa'^{I,J_0,R}_{A_i}(\nu)$*. Under the notation in Proposition 9.6 (2), we have* $X^{J_0}(\nu) = X^{J_0}(\tau_t'\nu)$, $X^{J_0}(\nu) = X^{J_0}(\tau_t\nu)$ *and* $X^{J_0}(\nu) = X^{J_0^\vee}(\nu)$*.*

Proof. We can check from the definition of $X^{J_0}(\nu)$. □

Let us start the proof of Lemma 9.8.

9.3.1. *Proof of the recurrence relation (9.8).*

Definition 9.10. We define functions $\alpha^R \equiv \alpha_A^{I,J_0,R}$ and $\beta^R \equiv \beta_A^{I,J_0,R}$ on T_{n-1} by

$$\alpha^R(p) = \frac{1}{2}\{\nu_{i_1} - \nu_{\tilde{\tau}_0(p)} + (\chi_{I^e(J_0)}(p) - \chi_{I^o(J_0)}(p))(\chi_{I_l^e(A)}(p) - \chi_{I_l^o(A)}(p))\}\chi_R(p),$$

$$\beta^R(p) = \frac{1}{2}\{-\nu_{i_1} + \nu_{\tilde{\tau}_0(p)} + (\chi_{I^o(J_0)}(p) - \chi_{I^e(J_0)}(p))(\chi_{I_l^e(A)}(p) - \chi_{I_l^o(A)}(p))\}\chi_{R^c}(p).$$

We can obtain the same identities for $P_{A,B;\mathbf{m},\mathbf{k}}^{I,J_0,R}$ as in Lemmas 8.13 and 8.14 by replacing $P_{A,B;\mathbf{m},\mathbf{k}}^{I,\Pi}$, $\alpha(p)$ and $\beta(p)$ by $P_{A,B;\mathbf{m},\mathbf{k}}^{I,J_0,R}$, $\alpha^R(p)$ and $\beta^R(p)$, respectively. Hence the proof of Lemma 9.8 (i) is reduced to the following key relation (cf. Lemma 8.15). Actually, by using Lemma 9.11, we can finish our proof of the recurrence relation (9.8) via the same argument as in Subsection 8.6.

Lemma 9.11. *We have*

$$\mu_A^{I,J_0,R}(\nu) + \sum_{t=1}^{h}(\alpha^R(a_t-1) - \beta^R(a_t)) = \sum_{t=1}^{h}\frac{\nu_{i_1} - \nu_{i_t}}{2} = \mu_B^{I_0,J_0}(\tilde{\nu}) + \sum_{t=1}^{h-1}(\alpha^R(b_t) - \beta^R(b_t)).$$

Proof. From the definitions of $\mu_A^{I,J_0,R}(\nu)$, α^R and β^R, we have

$$\begin{aligned}
&2\mu_A^{I,J_0,R}(\nu) + 2\sum_{t=1}^{h}(\alpha^R(a_t-1) - \beta^R(a_t)) - \sum_{t=1}^{h}(\nu_{i_1} - \nu_{i_t})\\
&= \sum_{t=1}^{h}(\varepsilon_A^{J_0,R}(a_t) - \varepsilon_A^{J_0,R}(a_t-1))\\
&+ \sum_{t=1}^{h}(\chi_{I^e(J_0)}(a_t-1) - \chi_{I^o(J_0)}(a_t-1))(\chi_{I_l^e(A)}(a_t-1) - \chi_{I_l^o(A)}(a_t-1))\chi_R(a_t-1)\\
&- \sum_{t=1}^{h}(\chi_{I^o(J_0)}(a_t) - \chi_{I^e(J_0)}(a_t))(\chi_{I_l^e(A)}(a_t) - \chi_{I_l^o(A)}(a_t))\chi_{R^c}(a_t)\\
&= \sum_{p=1}^{n}\varepsilon_A^{J_0,R}(p)(\chi_A(p) - \chi_{A[-1]}(p)) - \sum_{p=1}^{n}\chi_{A[-1],J_0^c}(p)\cdot\chi_R(p) + \sum_{p=1}^{n}\chi_{A,J_0^c}(p)\cdot\chi_{R^c}(p)\\
&= \sum_{p=1}^{n}(\chi_R(p) + \chi_{R^c}(p))\{(\chi_{I_r^e(J_0)}(p)\cdot\chi_{A^e}(p) + \chi_{I_r^o(J_0)}(p)\cdot\chi_{A^o}(p))\\
&\qquad - (\chi_{I_l^e(J_0)}(p)\cdot\chi_{A^e[-1]}(p) + \chi_{I_l^o(J_0)}(p)\cdot\chi_{A^o[-1]}(p)\}\\
&= 0.
\end{aligned}$$

Let us consider the right hand side. We have

$$\sum_{t=1}^{h-1}(\varepsilon_B^{J_0}(b_t) - \varepsilon_B^{J_0}(b_t-1)) = -\sum_{p=1}^{n-1}\chi_{B,J_0^c}(p)$$

from Lemma 7.8 (iii), and

$$\sum_{t=1}^{h-1}(\alpha^R(b_t) - \beta^R(b_t)) = \frac{1}{2}\sum_{p=1}^{n-1}\{(\nu_{i_1} - \nu_{\tilde{\tau}_0(p)})\chi_B(p) + \chi_{B,J_0^c}(p)\}$$

from Definition 9.10. Then we can get the assertion. □

9.3.2. *Proof of the recurrence relation (9.9).* Similarly to the case of the Dirac-Schmid relations, to prove the recurrence relation (9.9) it is enough to show the following key identity (9.13).

Lemma 9.12. *We set*

$$X^{I,J_0,R}_{A,B} = \sum_{p=1}^{n-1} \frac{P^{I,J_0,R}_{A,B;\mathbf{m},\mathbf{k}}}{P^{I,J_0,R}_{A,B;\mathbf{m}-\mathbf{e}_p,\mathbf{k}}} - \sum_{p=1}^{n-2} \frac{P^{I,J_0,R}_{A,B;\mathbf{m},\mathbf{k}}}{P^{I,J_0,R}_{A,B;\mathbf{m},\mathbf{k}+\mathbf{d}_p}}$$

for $B \prec A$.
(i) *If we put*

$$\tilde{\zeta}^{J_0,R}_{A,B}(p) = \zeta^{J_0,R}_{A,B}(p) - \frac{1}{2}\chi_{J_0^c}(p)\cdot\chi_R(p), \quad \tilde{\eta}^{J_0,R}_{A,B}(p) = \eta^{J_0,R}_{A,B}(p) - \frac{1}{2}\chi_{J_0^c}(p)\cdot\chi_R(p),$$

then we have

$$X^{I,J_0,R}_{A,B} = Q_{\mathbf{m}}\big(\{\lambda_{X,p}(\nu)\}_{1\le p\le n-1}; 0\big) - Q_{\mathbf{k}}\big(\{\tilde{\lambda}_{X,p}(\nu)\}_{1\le p\le n-2}; 0\big) + \kappa_X(\nu), \tag{9.11}$$

where

$$\begin{aligned}\lambda_{X,p}(\nu) = {} & \frac{1}{2}\big\{\nu_{\tilde{\tau}_0(p-1)}\chi_R(p-1) + \nu_{\tilde{\tau}_0(p)}(\chi_{R^c}(p)-\chi_R(p)) - \nu_{\tilde{\tau}_0(p+1)}\chi_{R^c}(p+1) \\ & + \nu_{i_1}(2\chi_R(p) - \chi_R(p-1) - \chi_R(p+1))\big\} \\ & + \tilde{\zeta}^{J_0,R}_{A,B}(p) + \tilde{\eta}^{J_0,R}_{A,B}(p) - \tilde{\zeta}^{J_0,R}_{A,B}(p-1) - \tilde{\eta}^{J_0,R}_{A,B}(p+1), \\ \tilde{\lambda}_{X,p}(\nu) = {} & \frac{1}{2}(\nu_{\tilde{\tau}_0(p)} - \nu_{\tilde{\tau}_0(p+1)}) - \tilde{\zeta}^{J_0,R}_{A,B}(p) + \tilde{\eta}^{J_0,R}_{A,B}(p) + \tilde{\zeta}^{J_0,R}_{A,B}(p+1) - \tilde{\eta}^{J_0,R}_{A,B}(p+1),\end{aligned}$$

and

$$\begin{aligned}\kappa_X(\nu) = {} & \frac{1}{2}\sum_{p\in R}(\eta^R(p) - \eta^R(p+1))(\nu_{i_1} - \nu_{\tilde{\tau}_0(p)} - \chi_{J_0^c}(p)) \\ & + \frac{1}{2}\sum_{p\in R^c}(\zeta^{R^c}(p) - \zeta^{R^c}(p-1))(-\nu_{i_1} + \nu_{\tilde{\tau}_0(p)} - \chi_{J_0^c}(p)) \\ & + \sum_{p\in E(R)} \xi(p)\xi(p+1).\end{aligned} \tag{9.12}$$

(ii) *We have the key identity*

$$\begin{aligned} & Q_{\mathbf{m}}\big(\{\lambda^{I,J_0,R}_{A,p}(\nu)\}_{1\le p\le n-1}; \kappa^{I,J_0,R}_A(\nu)\big) \\ & - \frac{1}{2}\sum_{t=1}^{h}(m_{a_t-1} - k_{a_t-1} + \alpha^R(a_t-1)) - \frac{1}{2}\sum_{t=1}^{h}(m_{a_t} - k_{a_t-1} + \beta^R(a_t)) \\ & = X^{I,J_0,R}_{A,B} + Q_{\mathbf{k}}\big(\{\lambda^{I_0,J_0}_{B,p}(\tilde{\nu})\}_{1\le p\le n-2}; \kappa^{I_0,J_0}_B(\tilde{\nu})\big) \\ & - \frac{1}{2}\sum_{t=1}^{h-1}(m_{b_t} - k_{b_t} + \alpha^R(b_t)) - \frac{1}{2}\sum_{t=1}^{h-1}(m_{b_t} - k_{b_t-1} + \beta^R(b_t)).\end{aligned} \tag{9.13}$$

Proof. (i) The formulas for $\lambda_{X,p}(\nu)$ and $\tilde{\lambda}_{X,p}(\nu)$ are immediate from

$$X^{I,J_0,R}_{A,B} = \sum_{p=1}^{n-1}\Big(m_p - k_p + \frac{\nu_{i_1} - \nu_{\tilde{\tau}_0(p)}}{2}\cdot\chi_R(p) + \tilde{\zeta}^{J_0,R}_{A,B}(p)\Big)$$

$$\cdot\Big(m_p - k_{p-1} + \frac{-\nu_{i_1} + \nu_{\tilde\tau_0(p)}}{2}\cdot\chi_{R^c}(p) + \tilde\eta^{J_0,R}_{A,B}(p)\Big)$$
$$-\sum_{p=1}^{n-2}\Big(m_p - k_p + \frac{\nu_{i_1} - \nu_{\tilde\tau_0(p)}}{2}\cdot\chi_R(p) + \tilde\zeta^{J_0,R}_{A,B}(p)\Big)$$
$$\cdot\Big(m_{p+1} - k_p + \frac{-\nu_{i_1} + \nu_{\tilde\tau_0(p+1)}}{2}\cdot\chi_{R^c}(p+1) + \tilde\eta^{J_0,R}_{A,B}(p+1)\Big).$$

Then we have only to confirm that

$$\kappa_X(\nu) = \sum_{p=1}^{n-1}\Big(\frac{\nu_{i_1} - \nu_{\tilde\tau_0(p)} - \chi_{J_0^c}(p)}{2}\cdot\chi_R(p) + \zeta^{J_0,R}_{A,B}(p)\Big)$$
$$\cdot\Big(\frac{-\nu_{i_1} + \nu_{\tilde\tau_0(p)} - \chi_{J_0^c}(p)}{2}\cdot\chi_{R^c}(p) + \eta^{J_0,R}_{A,B}(p)\Big)$$
$$-\sum_{p=1}^{n-2}\Big(\frac{\nu_{i_1} - \nu_{\tilde\tau_0(p)} - \chi_{J_0^c}(p)}{2}\cdot\chi_R(p) + \zeta^{J_0,R}_{A,B}(p)\Big) \tag{9.14}$$
$$\cdot\Big(\frac{-\nu_{i_1} + \nu_{\tilde\tau_0(p+1)} - \chi_{J_0^c}(p+1)}{2}\cdot\chi_{R^c}(p+1) + \eta^{J_0,R}_{A,B}(p+1)\Big).$$

In view of the relations $\zeta^R(p)\eta^R(p) = \zeta^{R^c}(p)\eta^{R^c}(p) = 0$ and $\zeta^R(p)\eta^R(p+1) = \zeta^{R^c}(p)\eta^R(p+1) = \zeta^{R^c}(p)\eta^{R^c}(p+1) = 0$, the right hand side of (9.14) is a sum of the following:

$$\frac{1}{2}\sum_{p=1}^{n-1}\chi_R(p)\cdot\eta^R(p)(\nu_{i_1} - \nu_{\tilde\tau_0(p)} - \chi_{J_0^c}(p)), \tag{9.15}$$

$$\frac{1}{2}\sum_{p=1}^{n-1}\chi_{R^c}(p)\cdot\zeta^{R^c}(p)(-\nu_{i_1} + \nu_{\tilde\tau_0(p)} - \chi_{J_0^c}(p)), \tag{9.16}$$

$$\frac{1}{2}\sum_{p=1}^{n-2}\chi_R(p)\chi_R(p+1)\cdot\eta^R(p+1)(-\nu_{i_1} + \nu_{\tilde\tau_0(p)} - \chi_{J_0^c}(p)), \tag{9.17}$$

$$\frac{1}{2}\sum_{p\in E(R)}\eta^{R^c}(p+1)(-\nu_{i_1} + \nu_{\tilde\tau_0(p)} - \chi_{J_0^c}(p)), \tag{9.18}$$

$$\frac{1}{2}\sum_{p\in E(R)}\zeta^R(p)(\nu_{i_1} - \nu_{\tilde\tau_0(p+1)} - \chi_{J_0^c}(p+1)), \tag{9.19}$$

$$\frac{1}{2}\sum_{p=1}^{n-2}\chi_{R^c}(p)\chi_{R^c}(p+1)\cdot\zeta^{R^c}(p)(\nu_{i_1} - \nu_{\tilde\tau_0(p+1)} - \chi_{J_0^c}(p+1)), \tag{9.20}$$

$$\frac{1}{4}\sum_{p\in E(R)}\{(\nu_{i_1} - \nu_{\tilde\tau_0(p)} - \chi_{J_0^c}(p))(\nu_{i_1} - \nu_{\tilde\tau_0(p+1)} + \chi_{J_0^c}(p+1)) + \zeta^R(p)\eta^{R^c}(p+1)\}. \tag{9.21}$$

We divide the terms (9.17) and (9.20) by using $\chi_R(p)\cdot\chi_R(p+1) = \chi_R(p) - \chi_R(p)\cdot\chi_{R^c}(p+1)$ and $\chi_{R^c}(p)\cdot\chi_{R^c}(p+1) = \chi_{R^c}(p+1) - \chi_R(p)\cdot\chi_{R^c}(p+1)$, respectively, we can see that

$$(9.15) + (9.17) = \sum_{p\in R}(\eta^R(p) - \eta^R(p+1))(\nu_{i_1} - \nu_{\tilde\tau_0(p)} - \chi_{J_0^c}(p))$$

$$+\sum_{p\in E(R)} \eta^R(p+1)(\nu_{i_1}-\nu_{\tilde\tau_0(p)}-\chi_{J_0^c}(p))$$

and

$$(9.16)+(9.20)=\sum_{p\in R^c}(\zeta^{R^c}(p)-\zeta^{R^c}(p-1))(-\nu_{i_1}+\nu_{\tilde\tau_0(p)}-\chi_{J_0^c}(p)) + \sum_{p\in E(R)} \zeta^{R^c}(p+1)(-\nu_{i_1}+\nu_{\tilde\tau_0(p+1)}-\chi_{J_0^c}(p+1)).$$

Hence the terms $\sum_{p\in E(R)}$ are collected as

$$\sum_{p\in E(R)}\Big(\frac{\nu_{i_1}-\nu_{\tilde\tau_0(p)}-\chi_{J_0^c}(p)}{2}+\zeta^R(p)-\zeta^{R^c}(p)\Big) \cdot\Big(\frac{\nu_{i_1}-\nu_{\tilde\tau_0(p+1)}-\chi_{J_0^c}(p+1)}{2}+\eta^R(p+1)-\eta^{R^c}(p+1)\Big).$$

Here we used $(\zeta^R(p)-\zeta^{R^c}(p))(\eta^R(p+1)-\eta^{R^c}(p+1)) = -\zeta^R(p)\cdot\eta^{R^c}(p+1)$. By using Lemma 7.10 (i), we can get the relation (9.11).

(ii) To conclude the proof of Lemma 9.12 (ii) we have to verify the following:

$$\lambda^{I,J_0,R}_{A,p}(\nu)-\lambda_{X,p}(\nu)=\frac{1}{2}(\chi_A(p)+\chi_{A[-1]}(p))-\chi_B(p), \tag{9.22}$$

$$\lambda^{I_0,J_0}_{B,p}(\tilde\nu)-\tilde\lambda_{X,p}(\nu)=\chi_{A[-1]}(p)-\frac{1}{2}(\chi_B(p)+\chi_{B[-1]}(p)), \tag{9.23}$$

and

(9.24)
$$\kappa^{I,J_0,R}_A(\nu)-\kappa_X(\nu)=\kappa^{I_0,J_0}_B(\tilde\nu)+\frac{1}{2}\sum_{t=1}^{h}(\alpha^R(a_t-1)+\beta^R(a_t))-\frac{1}{2}\sum_{t=1}^{h-1}(\alpha^R(b_t)+\beta^R(b_t)).$$

From the definition of $\lambda^{I,J_0,R}_{A,p}(\nu)$ and $\lambda_{X,p}(\nu)$, we have

$$2(\lambda^{I,J_0,R}_{A,p}(\nu)-\lambda_{X,p}(\nu))=(\varepsilon^{J_0,R}_A(p)-2\tilde\zeta^{J_0,R}_{A,B}(p))-(\varepsilon^{J_0,R}_A(p-1)-2\tilde\zeta^{J_0,R}_{A,B}(p-1)) + (\varepsilon^{J_0,R}_A(p)-2\tilde\eta^{J_0,R}_{A,B}(p))-(\varepsilon^{J_0,R}_A(p+1)-2\tilde\eta^{J_0,R}_{A,B}(p+1)).$$

In view of Lemma 7.10 (iii) and

$$\chi_{I(A,B)}(p)-\chi_{I(A,B)}(p-1)=\chi_A(p)-\chi_B(p),$$
$$\chi_{I(B,A)}(p)-\chi_{I(B,A)}(p+1)=\chi_{A[-1]}(p)-\chi_B(p),$$

we can get the identity (9.22).

Now we consider (9.23). Since

$$2\lambda^{I_0,J_0}_{B,p}(\tilde\nu)=\nu_{\tilde\tau_0(p)}-\nu_{\tilde\tau_0(p+1)}+2\varepsilon^{J_0}_B(p)-\varepsilon^{J_0}_B(p-1)+\varepsilon^{J_0}_B(p+1),$$

we find

$$2(\lambda^{I_0,J_0}_{B,p}(\tilde\nu)-\tilde\lambda_{X,p})=2(\tilde\zeta^{J_0,R}_{A,B}(p)-\tilde\eta^{J_0,R}_{A,B}(p))+\varepsilon^{J_0}_B(p)-\varepsilon^{J_0}_B(p-1) - \{2(\tilde\zeta^{J_0,R}_{A,B}(p+1)-\tilde\eta^{J_0,R}_{A,B}(p+1))+\varepsilon^{J_0}_B(p+1)-\varepsilon^{J_0}_B(p)\}.$$

By considering the difference of two identities in Lemma 7.10 (iii), we obtain (9.23) .

Finally we show (9.24). By (9.11) and Corollary 8.11, we can find that

$$4\Big(\kappa_X(\nu)+\kappa_B^{I_0,J_0}(\tilde\nu)+\frac{1}{2}\sum_{t=1}^{h}(\alpha^R(a_t-1)+\beta^R(a_t))-\frac{1}{2}\sum_{t=1}^{h-1}(\alpha^R(b_t)+\beta^R(b_t))\Big)$$

can be written as a sum of the following:

(9.25)
$$\sum_{p\in R^c}\{-2(\zeta^{R^c}(p)-\zeta^{R^c}(p-1))+\chi_{B,J_0^c}(p)+\chi_{B^c,J_0}(p)-\chi_A(p)+\chi_B(p)\}(\nu_{i_1}-\nu_{\tilde\tau_0(p)}),$$

(9.26)
$$\sum_{p\in R}\{2(\eta^R(p)-\eta^R(p+1))+\chi_{B,J_0^c}(p)+\chi_{B^c,J_0}(p)+\chi_{A[-1]}(p)-\chi_B(p)\}(\nu_{i_1}-\nu_{\tilde\tau_0(p)}),$$

$$4\sum_{p\in E(R)}\xi(p)\xi(p+1)\cdot(\nu_{i_1}-\nu_{\tilde\tau_0(p)}),$$

and

$$\begin{aligned}&\sum_{p\in R^c}\{2(\zeta^{R^c}(p)-\zeta^{R^c}(p-1))\chi_{J_0^c}(p)\\&\qquad+(\chi_{I^e(J_0)}(p)-\chi_{I^o(J_0)}(p))(\chi_{I_l^e(A)}(p)-\chi_{I_l^o(A)}(p))(-\chi_A(p)+\chi_B(p))\}\\&+\sum_{p\in R}\{2(\eta^R(p)-\eta^R(p+1))\chi_{J_0^c}(p)\\&\qquad+(\chi_{I^e(J_0)}(p)-\chi_{I^o(J_0)}(p))(\chi_{I_l^e(A)}(p)-\chi_{I_l^o(A)}(p))(\chi_{A[-1]}(p)-\chi_B(p))\}\\&+(h-1)-\sum_{p=1}^{n-1}\chi_B(p)\cdot\chi_{J_0}(p).\end{aligned}\tag{9.27}$$

By using Lemma 7.10 (i), we can see that

$$\begin{aligned}(9.25)&=\sum_{p\in R^c}(\chi_{A,J_0^c}(p)+\chi_{A^c,J_0}(p))(\nu_{i_1}-\nu_{\tilde\nu_0(p)})\\(9.26)&=\sum_{p\in R}(-\chi_{A[-1],J_0^c}(p)-\chi_{(A[-1])^c,J_0}(p))(\nu_{i_1}-\nu_{\tilde\nu_0(p)})\end{aligned}$$

and

$$(9.27)=\sum_{p\in R^c}\chi_{J_0^c}(p)(\chi_A(p)-\chi_B(p))+\sum_{p\in R}\chi_{J_0^c}(p)(\chi_{A[-1]}(p)-\chi_B(p))+\sum_{p=1}^{n-1}\chi_{J_0^c}(p)\cdot\chi_B(p).$$

Hence we have done the proof of (9.24) and we finish the proof of Lemma 9.12. □

9.3.3. *Proof of the recurrence relation (9.10).* In the case of helicity 1 (Lemma 6.4 (ii)), we remark that the key identity is (6.10). Our task here is to show the identity (9.30) given below.

Let us compute $X_{A_i,B_i}^{I,J_0,R}$. If we denote by $\mathbf{m}''=(m''_p)_{1\le p\le n-1}$ and $\mathbf{k}'=(k'_p)_{1\le p\le n-2}$, that is, $m''_p=m_p-\delta_{A_i}^{J_0,R}(p)$ and $k'_p=k_p-\delta_{B_i}^{J_0}(p)$, then we have

$$X_{A_i,B_i}^{I,J_0,R}=\sum_{p=1}^{n-1}\Big(m''_p-k'_p+\frac{\nu_{i_1}-\nu_{\tilde\tau_0(p)}\pm\chi_{J_0^c}(p)}{2}\cdot\chi_R(p)\Big)$$

$$
\cdot \Big(m''_p - k'_{p-1} + \frac{-\nu_{i_1} + \nu_{\tilde\tau_0(p)} \mp \chi_{J_0^c}(p)}{2} \cdot \chi_{R^c}(p)\Big)
$$

$$
- \sum_{p=1}^{n-2} \Big(m''_p - k'_p + \frac{\nu_{i_1} - \nu_{\tilde\tau_0(p)} \pm \chi_{J_0^c}(p)}{2} \cdot \chi_R(p)\Big)
$$

$$
\cdot \Big(m''_{p+1} - k'_p + \frac{-\nu_{i_1} + \nu_{\tilde\tau_0(p+1)} \mp \chi_{J_0^c}(p+1)}{2} \cdot \chi_{R^c}(p+1)\Big)
$$

(see the proof of (9.11)). Here we used

$$
\zeta^R(p) - \eta^R(p) - \delta^{J_0}_{B_i}(p) + \delta^{J_0}_{B_i}(p-1) = \begin{cases} 0 & \text{if } i = 1, \\ \chi_{J_0^c}(p) & \text{if } i = 2, \end{cases} \tag{9.28}
$$

and

$$
\eta^{R^c}(p) - \zeta^{R^c}(p) + \delta^{J_0}_{B_i}(p) - \delta^{J_0}_{B_i}(p-1) = \begin{cases} \chi_{J_0^c}(p) & \text{if } i = 1, \\ 0 & \text{if } i = 2. \end{cases} \tag{9.29}
$$

Hence we can check that

$$
\begin{aligned}
X^{I,J_0,R}_{A_i,B_i} &= Q_{\mathbf{m}''}\big(\{\lambda'^{I,J_0,R}_{A_i,p}(\nu)\}_{1\le p\le n-1}; \kappa'^{I,J_0,R}_{A_i}(\nu)\big) \\
&\quad - Q_{\mathbf{k}'}\Big(\Big\{\frac{\nu_{\tilde\tau_0(p)} - \nu_{\tilde\tau_0(p+1)} \mp (\chi_{J_0}(p) - \chi_{J_0}(p+1))}{2}\Big\}_{1\le p\le n-2}; 0\Big),
\end{aligned} \tag{9.30}
$$

and thus we get the assertion from Proposition 8.17 (note that $\nu_{\tilde\tau_0(p)} - \nu_{\tilde\tau_0(p+1)} = \tilde\nu_{\tau^{I_0,J_0}(p)} - \tilde\nu_{\tau^{I_0,J_0}(p+1)}$). □

9.4. **Initial values.** We compute the initial values $C^{I,J_0,R}_{A;\mathbf{m}_0}(\nu)$ for the edge component $A = A_i$ $(i = 1, 2)$.

Lemma 9.13. *We have*

$$
C^{I,J_0,R}_{A_1;\mathbf{m}_0}(\nu) = \prod_{p\in R^c\cap J_0^c} \frac{-2}{-\nu_{i_1} + \nu_{\tilde\tau_0(p)} + 1} \cdot \prod_{t=1}^{h-1}\prod_{p=1}^{j_t - t} \frac{-2}{-\nu_{i_{h+1-t}} + \nu_{i'_p} + 1}
$$

$$
\cdot \prod_{p\in E(R)} \frac{\Gamma\Big(\frac{\nu_{\tilde\tau_0(p)} - \nu_{\tilde\tau_0(p+1)} - \chi_{J_0}(p) + \chi_{J_0}(p+1)}{2} + 1\Big)}{\Gamma\Big(\frac{-\nu_{i_1} + \nu_{\tilde\tau_0(p)} + 3 - \chi_{J_0}(p)}{2}\Big)\Gamma\Big(\frac{\nu_{i_1} - \nu_{\tilde\tau_0(p+1)} + 1 - \chi_{J_0}(p+1)}{2}\Big)}
$$

and

$$
C^{I,J_0,R}_{A_2;\mathbf{m}_0}(\nu) = \prod_{p\in R\cap J_0^c} \frac{2}{\nu_{i_1} - \nu_{\tilde\tau_0(p)} + 1} \cdot \prod_{t=1}^{h-1}\prod_{p=j_t-t+1}^{n-h} \frac{2}{\nu_{i_{h+1-t}} - \nu_{i'_p} + 1}
$$

$$
\cdot \prod_{p\in E(R)} \frac{\Gamma\Big(\frac{\nu_{\tilde\tau_0(p)} - \nu_{\tilde\tau_0(p+1)} + \chi_{J_0}(p) - \chi_{J_0}(p+1)}{2} + 1\Big)}{\Gamma\Big(\frac{-\nu_{i_1} + \nu_{\tilde\tau_0(p)} + 1 + \chi_{J_0}(p)}{2}\Big)\Gamma\Big(\frac{\nu_{i_1} - \nu_{\tilde\tau_0(p+1)} + 3 - \chi_{J_0}(p+1)}{2}\Big)}.
$$

Proof. From the definition (Lemma 9.3) we have

$$
C^{I,J_0,R}_{A_i;\mathbf{m}_0}(\nu) = \sum_{\mathbf{k}} \frac{C^{n-1,I_0,J_0}_{B_i;\mathbf{k}}(\tilde\nu)}{P^{I,J_0,R}_{A_i,B_i;\mathbf{m}_0;\mathbf{k}}}
$$

with

$$P^{I,J_0,R}_{A_i,B_i;\mathbf{m}_0,\mathbf{k}} = \prod_{p\in R^c}(-1)^{\zeta^{R^c}(p)+\eta^{R^c}(p)}\Big(\frac{-\nu_{i_1}+\nu_{\tilde{\tau}_0(p)}+1+\chi_{J_0}(p)}{2}\Big)_{-k_{p-1}-\zeta^{R^c}(p)+\eta^{R^c}(p)+\delta^{J_0}_{B_i}(p)}$$

$$\cdot\prod_{p\in R}(-1)^{\zeta^{R}(p)+\eta^{R}(p)}\Big(\frac{\nu_{i_1}-\nu_{\tilde{\tau}_0(p)}+1+\chi_{J_0}(p)}{2}\Big)_{-k_p+\zeta^R(p)-\eta^R(p)+\delta^{J_0}_{B_i}(p-1)}.$$

Here $\mathbf{k}=(k_1,\dots,k_{p-1})$ runs through such that

$$k_p=\delta^{J_0}_{B_i}(p)\ \ (p\notin E(R)),\quad k_p\geq\delta^{J_0}_{B_i}(p)\ \ (p\in E(R)). \tag{9.31}$$

Let us rewrite $P^{I,J_0,R}_{A_1,B_1;\mathbf{m}_0,\mathbf{k}}$ by using the relations (9.28) and (9.29). We can find that

$$\begin{aligned}\prod_{p\in R^c}(-1)^{\zeta^{R^c}(p)+\eta^{R^c}(p)}\prod_{p\in R}(-1)^{\zeta^{R}(p)+\eta^{R}(p)} &= \prod_{p\in R^c}(-1)^{\chi_{J_0^c}(p)}\prod_{p=1}^{n-1}(-1)^{\delta^{J_0}_{B_1}(p-1)-\delta^{J_0}_{B_1}(p)}\\ &= \prod_{p\in R^c\cap J_0^c}(-1),\end{aligned}$$

and

$$P^{I,J_0,R}_{A_1,B_1;\mathbf{m}_0,\mathbf{k}} = \prod_{p\in R^c\cap J_0^c}(-1)\prod_{p\in R^c}\Big(\frac{-\nu_{i_1}+\nu_{\tilde{\tau}_0(p)}+1+\chi_{J_0}(p)}{2}\Big)_{-k_{p-1}+\delta^{J_0}_{B_1}(p-1)+\chi_{J_0^c}(p)}$$

$$\cdot\prod_{p\in R}\Big(\frac{\nu_{i_1}-\nu_{\tilde{\tau}_0(p)}+1+\chi_{J_0}(p)}{2}\Big)_{-k_p+\delta^{J_0}_{B_1}(p)}.$$

In view of $(\alpha)_{-k_p+1}=\alpha(\alpha+1)_{-k_p}$, we have

$$C^{I,J_0,R}_{A_1;\mathbf{m}_0}(\nu)=\frac{1}{\prod_{p\in R^c\cap J_0^c}\frac{\nu_{i_1}-\nu_{\tilde{\tau}_0(p)}-1}{2}}$$

$$\cdot\sum_{\mathbf{k}}\frac{C^{n-1,I_0,J_0}_{B_1;\mathbf{k}}(\tilde{\nu})}{\prod_{p\in E(R)}\big(\frac{-\nu_{i_1}+\nu_{\tilde{\tau}_0(p+1)}+3-\chi_{J_0}(p+1)}{2}\big)_{-(k_p-\delta^{J_0}_{B_1}(p))}\big(\frac{\nu_{i_1}-\nu_{\tilde{\tau}_0(p)}+1+\chi_{J_0}(p)}{2}\big)_{-(k_p-\delta^{J_0}_{B_1}(p))}}.$$

Similarly we can get

$$C^{I,J_0,R}_{A_2;\mathbf{m}_0}(\nu)=\frac{1}{\prod_{p\in R,p\in J_0^c}\frac{-\nu_{i_1}+\nu_{\tilde{\tau}_0(p)}-1}{2}}$$

$$\cdot\sum_{\mathbf{k}}\frac{C^{n-1,I_0,J_0}_{B_2;\mathbf{k}}(\tilde{\nu})}{\prod_{p\in E(R)}\big(\frac{-\nu_{i_1}+\nu_{\tilde{\tau}_0(p+1)}+1+\chi_{J_0}(p+1)}{2}\big)_{-(k_p-\delta^{J_0}_{B_2}(p))}\big(\frac{\nu_{i_1}-\nu_{\tilde{\tau}_0(p)}+3-\chi_{J_0}(p)}{2}\big)_{-(k_p-\delta^{J_0}_{B_2}(p))}}.$$

Next we compute the values $C^{n-1,I_0,J_0}_{B_i;\mathbf{k}}(\tilde{\nu})$ under the condition (9.31). By Proposition 8.17, the recurrence relation

$$\Big(\sum_{p\in E(R)}k_p^2-\sum_{p\in E(R)}\lambda'^{I_0,J_0}_{B_i,p}(\tilde{\nu})k_p\Big)C^{n-1,I_0,J_0}_{B_i;\mathbf{k}}(\tilde{\nu})=\sum_{p\in E(R)}C^{n-1,I_0,J_0}_{B_i;\mathbf{k}-\mathbf{d}_p}(\tilde{\nu})$$

holds. Then we have

$$C^{n-1,I_0,J_0}_{B_i;\mathbf{k}}(\tilde{\nu})=\frac{C^{n-1,I_0,J_0}_{B_i;\mathbf{k}_0}(\tilde{\nu})}{\prod_{p\in E(R)}(k_p-\delta^{J_0}_{B_i}(p))!\big(\frac{\nu_{\tilde{\tau}_0(p)}-\nu_{\tilde{\tau}_0(p+1)}\mp(\chi_{J_0}(p)-\chi_{J_0}(p+1))}{2}+1\big)_{k_p-\delta^{J_0}_{B_i}(p)}},$$

with $\mathbf{k}_0 = (\delta^{J_0}_{B_i}(1), \dots, \delta^{J_0}_{B_i}(n-2))$. To find the initial values $C^{n-1,I_0,J_0}_{B_i;\mathbf{k}_0}(\tilde{\nu})$, we consider $C^{n,I,J}_{A_i;\mathbf{m}_0}(\nu)$. From the definition (Definition 8.4) we have a recursive relation

$$C^{n,I,J}_{A_i;\mathbf{m}_0}(\nu) = \frac{C^{n-1,I_0,J_0}_{B_i;\mathbf{k}}(\tilde{\nu})}{P^{I,J}_{A_i,B_i;\mathbf{m}_0,\mathbf{k}_0}}$$

Since

$$P^{I,J}_{A_i,B_i;\mathbf{m}_0,\mathbf{k}_0} = \begin{cases} \prod_{p\in[1,j_h-1],p\in J_0^c} (-1)\dfrac{-\nu_{i_1}+\nu_{\tilde{\tau}_0(p)}+1}{2} & \text{if } i=1, \\ \prod_{p\in[j_h,n-1]} (-1)\dfrac{\nu_{i_1}-\nu_{\tilde{\tau}_0(p)}+1}{2} & \text{if } i=2, \end{cases}$$

we get

$$C^{n,I,J}_{A_i;\mathbf{m}_0}(\nu) = \begin{cases} \prod_{t=1}^{h}\prod_{p=1}^{j_t-t} \dfrac{2}{\nu_{i_{h+1-t}}-\nu_{i'_p}-1} & \text{if } i=1, \\ \prod_{t=1}^{h}\prod_{p=j_t-t+1}^{n-h} \dfrac{2}{-\nu_{i_{h+1-t}}+\nu_{i'_p}-1} & \text{if } i=2. \end{cases}$$

Hence we have

$$C^{n-1,I_0,J_0}_{B_1;\mathbf{k}}(\tilde{\nu}) = \frac{\prod_{t=1}^{h-1}\prod_{p=1}^{j_t-t}\frac{2}{\nu_{i_{h+1-t}}-\nu_{i'_p}-1}}{\prod_{p\in E(R)}(k_p-\delta^{J_0}_{B_1}(p))!\left(\frac{\nu_{\tilde{\tau}_0(p)}-\nu_{\tilde{\tau}_0(p+1)}-\chi_{J_0}(p)+\chi_{J_0}(p+1)}{2}+1\right)_{k_p-\delta^{J_0}_{B_1}(p)}},$$

and

$$C^{n-1,I_0,J_0}_{B_2;\mathbf{k}}(\tilde{\nu}) = \frac{\prod_{t=1}^{h-1}\prod_{p=j_t-t+1}^{n-h}\frac{2}{\nu_{i_{h+1-t}}-\nu_{i'_p}-1}}{\prod_{p\in E(R)}(k_p-\delta^{J_0}_{B_2}(p))!\left(\frac{\nu_{\tilde{\tau}_0(p)}-\nu_{\tilde{\tau}_0(p+1)}+\chi_{J_0}(p)-\chi_{J_0}(p+1)}{2}+1\right)_{k_p-\delta^{J_0}_{B_2}(p)}}.$$

By using the formula (6.12) we can reach the desired formulas for $C^{I,J_0,R}_{A_i;\mathbf{m}_0}(\nu)$. □

9.5. **Proof of Proposition 9.6 (ii).** To conclude our proof of Proposition 9.6 (ii), we show the vanishing relations (9.5), (9.6) and (9.7).

9.5.1. *The case (a).* Since $\tilde{\tau}_0(r) = i'_{r-t}$ and $\tilde{\tau}_0(r+1) = i'_{r-t+1}$, Lemmas 9.4 and 9.13 imply

$$\frac{\Gamma^{I,J_0,R}(\nu)}{\Gamma^{I,J_0,R}(\tau'_r\nu)} = \frac{\Gamma\left(\frac{-\nu_{i'_{r-t}}+\nu_{i'_{r-t+1}}}{2}\right)}{\Gamma\left(\frac{-\nu_{i'_{r-t+1}}+\nu_{i'_{r-t}}}{2}\right)} \cdot \frac{\Gamma\left(\frac{\nu_{i_1}-\nu_{i'_{r-t+1}}+1}{2}\right)\Gamma\left(\frac{-\nu_{i_1}+\nu_{i'_{r-t}}+1}{2}\right)}{\Gamma\left(\frac{\nu_{i_1}+\nu_{i'_{r-t}}+1}{2}\right)\Gamma\left(\frac{-\nu_{i_1}+\nu_{i'_{r-t+1}}+1}{2}\right)},$$

and

$$\frac{C^{I,J_0,R}_{A_1;\mathbf{m}_0}(\nu)}{C^{I,J_0,R}_{A_1;\mathbf{m}_0}(\tau'_r\nu)} = \frac{\frac{-\nu_{i_1}+\nu_{i'_{r-t}}+1}{2}}{\frac{-\nu_{i_1}+\nu_{i'_{r-t+1}}+1}{2}} \cdot \frac{\Gamma\left(\frac{\nu_{i'_{r-t}}-\nu_{i'_{r-t+1}}+2}{2}\right)}{\Gamma\left(\frac{-\nu_{i_1}+\nu_{i'_{r-t}}+3}{2}\right)\Gamma\left(\frac{\nu_{i_1}-\nu_{i'_{r-t+1}}+1}{2}\right)}$$
$$\cdot \frac{\Gamma\left(\frac{-\nu_{i_1}+\nu_{i'_{r-t+1}}+3}{2}\right)\Gamma\left(\frac{\nu_{i_1}-\nu_{i'_{r-t}}+1}{2}\right)}{\Gamma\left(\frac{\nu_{i'_{r-t+1}}-\nu_{i'_{r-t}}+2}{2}\right)},$$

respectively. By using the formula $\Gamma(x)\Gamma(1-x) = \pi/\sin\pi x$, we can see that

$$\frac{\Gamma^{I,J_0,R}(\nu)}{\Gamma^{I,J_0,R}(\tau'_r\nu)} \cdot \frac{C^{I,J_0,R}_{A_1;\mathbf{m}_0}(\nu)}{C^{I,J_0,R}_{A_1;\mathbf{m}_0}(\tau'_r\nu)} = -1.$$

The case of $A = A_2$ is treated similarly.

9.5.2. *The case of (b)-(i).* Form Lemmas 9.4 and 9.13,

$$\frac{\Gamma^{I,J_0,R}(\nu)}{\Gamma^{I,J_0,R}(\tau_t\nu)} = \frac{\Gamma\left(\frac{\nu_{i_{h-t}}-\nu_{i_{h-t+1}}}{2}\right)}{\Gamma\left(\frac{\nu_{i_{h-t+1}}-\nu_{i_{h-t}}}{2}\right)} \cdot \frac{\Gamma\left(\frac{\nu_{i_1}-\nu_{i_{h-t}}}{2}\right)\Gamma\left(\frac{-\nu_{i_1}+\nu_{i_{h-t+1}}}{2}\right)}{\Gamma\left(\frac{\nu_{i_1}-\nu_{i_{h-t+1}}}{2}\right)\Gamma\left(\frac{-\nu_{i_1}+\nu_{i_{h-t}}}{2}\right)},$$

and

$$\begin{aligned}\frac{C^{I,J_0,R}_{A_1;\mathbf{m}_0}(\nu)}{C^{I,J_0,R}_{A_1;\mathbf{m}_0}(\tau_t\nu)} &= \frac{C^{I,J_0,R}_{A_2;\mathbf{m}_0}(\nu)}{C^{I,J_0,R}_{A_2;\mathbf{m}_0}(\tau_t\nu)} \\ &= \frac{\Gamma\left(\frac{\nu_{i_{h-t+1}}-\nu_{i_{h-t}}+2}{2}\right)}{\Gamma\left(\frac{-\nu_{i_1}+\nu_{i_{h-t+1}}+2}{2}\right)\Gamma\left(\frac{\nu_{i_1}-\nu_{i_{h-t}}+2}{2}\right)} \cdot \frac{\Gamma\left(\frac{-\nu_{i_1}+\nu_{i_{h-t}}+2}{2}\right)\Gamma\left(\frac{\nu_{i_1}-\nu_{i_{h-t+1}}+2}{2}\right)}{\Gamma\left(\frac{\nu_{i_{h-t}}-\nu_{i_{h-t+1}}+2}{2}\right)}.\end{aligned}$$

Hence we can show (9.6) similarly to the case (a).

9.5.3. *The case of (b)-(ii).* Since $t_0 \in T_2(J_0,R)$ implies $t_0 \in T_1(J_0^\vee,R)$, we may assume $t_0 \in T_1(J_0,R)$. If we denote by $\tilde{\tau}_0^\vee(p) = \tau^{I_0,J_0^\vee}(p) + \chi_{[i_1,n-1]}(\tau^{I_0,J_0^\vee}(p))$, then we can see that

$$\tilde{\tau}_0^\vee(j_{t_0}+1) = i_{h-t_0+1} = \tilde{\tau}_0(j_{t_0}), \quad \tilde{\tau}_0^\vee(j_{t_0}) = i'_{j_{t_0}-t_0+1} = \tilde{\tau}(j_{t_0}+1),$$
$$\tilde{\tau}_0^\vee(p) = \tilde{\tau}_0(p), \quad \text{if } p \neq j_{t_0}, j_{t_0+1}.$$

To compute the ratio $\Gamma^{I,J_0,R}(\nu)/\Gamma^{I,J_0^\vee,R}(\nu)$ we use the expression (9.4). In view of $\tilde{\nu}_{i_{0,s}} - \tilde{\nu}_{i'_{0,p-t}} = \nu_{i_{s+1}} - \nu_{\tilde{\tau}_0(p)}$, then we have

$$\begin{aligned}\frac{\Gamma^{n-1,I_0,J_0}(\tilde{\nu})}{\Gamma^{n-1,I_0,J_0^\vee}(\tilde{\nu})} &= \frac{\prod_{p=j_{t_0-1}+1}^{j_{t_0}-1}\prod_{s=1}^{h-t_0}\Gamma\left(\frac{\nu_{i_{s+1}}-\nu_{\tilde{\tau}_0(p)}+1}{2}\right)\prod_{s=h-t_0+1}^{h-1}\Gamma\left(\frac{-\nu_{i_{s+1}}+\nu_{\tilde{\tau}_0(p)}+1}{2}\right)}{\prod_{p=j^\vee_{t_0-1}+1}^{j^\vee_{t_0}-1}\prod_{s=1}^{h-t_0}\Gamma\left(\frac{\nu_{i_{s+1}}-\nu_{\tilde{\tau}_0^\vee(p)}+1}{2}\right)\prod_{s=h-t_0+1}^{h-1}\Gamma\left(\frac{-\nu_{i_{s+1}}+\nu_{\tilde{\tau}_0^\vee(p)}+1}{2}\right)} \\ &\quad\cdot \frac{\prod_{p=j_{t_0}+1}^{j_{t_0+1}-1}\prod_{s=1}^{h-t_0-1}\Gamma\left(\frac{\nu_{i_{s+1}}-\nu_{\tilde{\tau}_0(p)}+1}{2}\right)\prod_{s=h-t_0}^{h-1}\Gamma\left(\frac{-\nu_{i_{s+1}}+\nu_{\tilde{\tau}_0(p)}+1}{2}\right)}{\prod_{p=j^\vee_{t_0}+1}^{j^\vee_{t_0+1}-1}\prod_{s=1}^{h-t_0-1}\Gamma\left(\frac{\nu_{i_{s+1}}-\nu_{\tilde{\tau}_0^\vee(p)}+1}{2}\right)\prod_{s=h-s_1}^{h-1}\Gamma\left(\frac{-\nu_{i_{s+1}}+\nu_{\tilde{\tau}_0^\vee(p)}+1}{2}\right)} \\ &= \frac{\prod_{s=1}^{h-t_0-1}\Gamma\left(\frac{\nu_{i_{s+1}}-\nu_{\tilde{\tau}_0(j_{t_0}+1)}+1}{2}\right)\prod_{s=h-t_0}^{h-1}\Gamma\left(\frac{-\nu_{i_{s+1}}+\nu_{\tilde{\tau}_0(j_{t_0}+1)}+1}{2}\right)}{\prod_{s=1}^{h-t_0}\Gamma\left(\frac{\nu_{i_{s+1}}-\nu_{\tilde{\tau}_0^\vee(j_{t_0})}+1}{2}\right)\prod_{s=h-t_0+1}^{h-1}\Gamma\left(\frac{-\nu_{i_{s+1}}+\nu_{\tilde{\tau}_0^\vee(j_{t_0})}+1}{2}\right)} \\ &= \frac{\Gamma\left(\frac{-\nu_{i_{h-t_0+1}}+\nu_{i'_{j_{t_0}-t_0+1}}+1}{2}\right)}{\Gamma\left(\frac{\nu_{i_{h-t_0+1}}-\nu_{i'_{j_{t_0}-t_0+1}}+1}{2}\right)}.\end{aligned}$$

Hence we can obtain

$$\frac{\Gamma^{I,J_0,R}(\nu)}{\Gamma^{I,J_0^\vee,R}(\nu)} = \frac{\Gamma\Big(\frac{-\nu_{i_{h-t_0+1}}+\nu_{i'_{j_{t_0}-t_0+1}}+1}{2}\Big)}{\Gamma\Big(\frac{\nu_{i_{h-t_0+1}}-\nu_{i'_{j_{t_0}-t_0+1}}+1}{2}\Big)} \cdot \frac{\Gamma\Big(\frac{-\nu_{i_1}+\nu_{i_{h-t_0+1}}}{2}\Big)}{\Gamma\Big(\frac{\nu_{i_1}-\nu_{i_{h-t_0+1}}}{2}\Big)} \cdot \frac{\Gamma\Big(\frac{\nu_{i_1}-\nu_{i'_{j_{t_0}-t_0+1}}+1}{2}\Big)}{\Gamma\Big(\frac{-\nu_{i_1}+\nu_{i'_{j_{t_0}-t_0+1}}+1}{2}\Big)}.$$

By Lemma 9.13 we find

$$\frac{C^{I,J_0,R}_{A_1;\mathbf{m}_0}(\nu)}{C^{I,J_0^\vee,R}_{A_1;\mathbf{m}_0}(\nu)} = \frac{\frac{-\nu_{i_{h-t_0+1}}+\nu_{i'_{j_{t_0}-t_0+1}}+1}{2}}{\frac{-\nu_{i_1}+\nu_{i'_{j_{t_0}-t_0+1}}+1}{2}} \cdot \frac{\Gamma\Big(\frac{\nu_{i_{h-t_0+1}}-\nu_{i'_{j_{t_0}-t_0+1}}+1}{2}\Big)}{\Gamma\Big(\frac{-\nu_{i_1}+\nu_{i_{h-t_0+1}}+2}{2}\Big)\Gamma\Big(\frac{\nu_{i_1}-\nu_{i'_{j_{t_0}-t_0+1}}+1}{2}\Big)}$$
$$\cdot \frac{\Gamma\Big(\frac{-\nu_{i_1}+\nu_{i'_{j_{t_0}-t_0+1}}+3}{2}\Big)\Gamma\Big(\frac{\nu_{i_1}-\nu_{i_{h-t_0+1}}+2}{2}\Big)}{\Gamma\Big(\frac{\nu_{i'_{j_{t_0}-t_0+1}}-\nu_{i_{h-t_0+1}}+1}{2}\Big)}.$$

Then we can check that

$$\frac{\Gamma^{I,J_0,R}(\nu)}{\Gamma^{I,J_0^\vee,R}(\nu)} \cdot \frac{C^{I,J_0,R}_{A_1;\mathbf{m}_0}(\nu)}{C^{I,J_0^\vee,R}_{A_1;\mathbf{m}_0}(\nu)} = -1.$$

The case of $A = A_2$ can be discussed similarly.

Hence we can finish the proof of Proposition 9.6 (ii) and thus we conclude Theorem 9.1. □

9.6. **Proof of Theorem 9.2.** We can prove the propagation formulas of Mellin-Barnes type as in [St2]. By Theorem 9.1 and the Mellin inversion implies

$$\begin{aligned}
\widetilde{W}^{n,I}_{A,\nu}(y) = &\sum_{B\prec A}\int_{(\mathbf{R}_+)^{n-1}} \frac{1}{(2\pi i)^{n-2}}\int_{z_1,\ldots,z_{n-2}} V^{n-1,I_0}_{B,\tilde{\nu}}(z_1,\ldots,z_{n-2}) \\
&\cdot \Big(\pi y_2\sqrt{\frac{t_2}{t_1}}\Big)^{-z_1}\cdots\Big(\pi y_{n-1}\sqrt{\frac{t_{n-1}}{t_{n-2}}}\Big)^{-z_{n-2}} \\
&\cdot \prod_{p=1}^{n-1}\exp\Big\{-(\pi y_p)^2 t_p - \frac{1}{t_p}\Big\}(\pi y_p)^{\frac{n-p}{n-1}\nu_{i_1}+\chi_{I(B,A)}(p)} \\
&\cdot \prod_{p=1}^{n-1} t_p^{\frac{n}{2(n-1)}\nu_{i_1}+\frac{1}{2}(\chi_{I(B,A)}(p)-\chi_{I(B,A)}(p))}\prod_{p=1}^{n-1}\frac{dt_p}{t_p}\prod_{p=1}^{n-2}dz_p \\
= &\sum_{B\prec A}\frac{1}{(2\pi i)^{n-2}}\int_{z_1,\ldots,z_{n-2}} V^{n-1,I_0}_{B,\tilde{\nu}}(z_1,\ldots,z_{n-2})\prod_{p=1}^{n-1}(\pi y_p)^{\frac{n-p}{n-1}\nu_{i_1}+\chi_{I(B,A)}(p)-z_{p-1}} \\
&\cdot \prod_{p=1}^{n-1}\Big[\int_0^\infty \exp\Big\{-(\pi y_p)^2 t_p - \frac{1}{t_p}\Big\} t_p^{\frac{n}{2(n-1)}\nu_{i_1}+\frac{1}{2}(\chi_{I(B,A)}(p)-\chi_{I(A,B)}(p))+\frac{1}{2}(z_p-z_{p-1})}\frac{dt_p}{t_p}\Big] \\
&\cdot \prod_{p=1}^{n-2}dz_p.
\end{aligned}$$

By using the formula

$$(\pi y)^{\nu}\int_0^{\infty}\exp\Big\{-(\pi y)^2 t-\frac{1}{t}\Big\}t^{\nu}\frac{dt}{t}=2K_{\nu}(2\pi y)$$
$$=\frac{2^{-1}}{2\pi i}\int_s \Gamma\Big(\frac{s+\nu}{2}\Big)\Gamma\Big(\frac{s-\nu}{2}\Big)(\pi y)^{-s}ds,$$

the integration over t_p in the above can be written as

$$\frac{2^{-1}}{2\pi i}\int_{s_p}\Gamma\Big(\frac{s_p-z_{p-1}}{2}+\frac{n-p}{2(n-1)}\nu_{i_1}+\frac{\chi_{I(B,A)}(p)}{2}\Big)$$
$$\cdot\Gamma\Big(\frac{s_p-z_p}{2}-\frac{p}{2(n-1)}\nu_{i_1}+\frac{\chi_{I(A,B)}(p)}{2}\Big)(\pi y_p)^{-s_p+z_{p-1}-\frac{n-j}{n-1}\nu_{i_1}-\chi_{I(B,A)(p)}}ds_p.$$

Then we reach the expression

$$\widetilde{W}^{n,I}_{A,\nu}(y)=\sum_{B\prec A}\frac{2^{1-n}}{(2\pi i)^{n-1}}\int_{s_1,\dots,s_{n-1}}\frac{1}{(2\pi i)^{n-2}}\int_{z_1,\dots,z_{n-2}}V^{n-1,I_0}_{B,\tilde{\nu}}(z_1,\dots,z_{n-2})$$
$$\cdot\prod_{p=1}^{n-1}\Big[\Gamma\Big(\frac{s_p-z_{p-1}}{2}+\frac{n-p}{2(n-1)}\nu_{i_1}+\frac{\chi_{I(B,A)}(p)}{2}\Big)$$
$$\cdot\Gamma\Big(\frac{s_p-z_p}{2}-\frac{p}{2(n-1)}\nu_{i_1}+\frac{\chi_{I(A,B)}(p)}{2}\Big)\Big]\prod_{p=1}^{n-1}(\pi y_p)^{-s_p}\prod_{p=1}^{n-1}ds_p\prod_{p=1}^{n-2}dz_p,$$

to get our assertion. □

References

[Bu] D. Bump, Automorphic forms on $GL(3,\mathbf{R})$, Lect. Note Math., **1083**, Springer, 1984.

[Ca] B. Casselman, H. Hecht and D. Miličić, Bruhat filtration and Whittaker vectors for real groups, Proc. Symp. in Pure Math. **68** (2000), 151–190.

[Gi] A. Givental, Stationary phase integrals, quantum Toda lattices, flag manifolds and the mirror conjecture, Topics in singularity theory, A.M.S. Transl. (2) **180** (1997), 103-115.

[GKLO] A. Gerasimov, S. Kharchev, D. Lededev, S. Oblezin, On a Gauss-Givental representation for quantum Toda chain wave function, Int. Math. Res. Not., **2006**, Art. ID 96489, 23 pp.

[GW] R. Goodman and N. Wallach, Whittaker vectors and conical vectors, J. Funct. Anal. **39** (1980), 199–279.

[Ha] M. Hashizume, Whittaker functions on semisimple Lie groups, Hiroshima Math. J. **12** (1982), 259–293.

[HnIO] T. Hina, T. Ishii and T. Oda, Calculus of principal series Whittaker functions on $SL(4,\mathbf{R})$, preprint.

[HrIO] M. Hirano, T. Ishii and T. Oda, Whittaker functions for P_J-principal series representations of $Sp(3,\mathbf{R})$, Adv. Math. **215** (2007), 734–765.

[HO] M. Hirano and T. Oda, Calculus of principal series Whittaker functions on $GL(3,\mathbf{C})$, J. Funct. Anal. (to appear).

[HU] R. Howe and T. Umeda, The Capelli identity, the double commutant theorem, and multiplicity-free actions, Math. Ann. **290** (1991), 565–619.

[I1] T. Ishii, A remark on Whittaker functions on $SL(n,\mathbf{R})$, Ann. Inst. Fourier **55** (2005), 483–492.

[I2] T. Ishii, On principal series Whittaker functions on $Sp(2,\mathbf{R})$, J. Funct. Anal. **225** (2005), 1–32.

[IS] T. Ishii and E. Stade, New formulas for Whittaker functions on $GL(n,\mathbf{R})$, J. Funct. Anal. **244** (2007), 289–314.

[Ja1] H. Jacquet, Fonctions de Whittaker associées aux groupes de Chevalley, Bull. Soc. Math. France **95** (1967), 243–309.

[Ja2] H. Jacquet, Archimedean Rankin-Selberg integrals, Contemporary Math (to appear).

[Ko] B. Kostant On Whittaker vectors and representation theory, Invent. Math. **48** (1978), 101–184.
[MIO] H. Manabe, T. Ishii and T. Oda, Principal series Whittaker functions on $SL(3,\mathbf{R})$, Japan. J. Math. **30** (2004), 183–226.
[MS] S. D. Miller and W. Schmid, The Rankin-Selberg method for automorphic distributions, Representation theory and automorphic forms, 111–150, Prog. Math., 255, Birkhäuser Boston, MA, 2008.
[Sha] J. Shalika The multiplicity one theorem for GL_n, Ann. Math. (2) **100** (1974), 171–193.
[Shi] T. Shintani On an explicit formula for class-1 "Whittaker functions" on GL_n over P-adic fields. Proc. Japan Acad. **52** (1976), no. 4, 180–182.
[St1] E. Stade, On explicit integral formulas for $GL(n,\mathbf{R})$-Whittaker functions, Duke Math. J. **60** (1990), 313–362.
[St2] E. Stade, Mellin transforms of $GL(n,\mathbf{R})$ Whittaker functions, Amer. J. Math. **123** (2001), 121–161.
[St3] E. Stade, Archimedean L-factors on $GL(n)\times GL(n)$ and generalized Barnes integrals, Isr. J. Math. **127** (2002), 201–219.
[VT] I. Vinogradov and L. Tahtajan, Theory of the Eisenstein series for the group $SL(3,\mathbf{R})$ and its application to a binary problem, J. of Soviet Math. vol. **18**, number 3 (1982), 293–324.
[Wa] N. Wallach, Asymptotic expansions of generalized matrix entries of representations of real reductive groups, Lect. Notes in Math., **1024**, (1983), 287–369.

Faculty of Science and Technology, Seikei University, 3-3-1 Kichijoji-Kitamachi, Musashino, Tokyo, 180-8633, Japan
E-mail address: `ishii@st.seikei.ac.jp`

Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo, 153-8914, Japan
E-mail address: `takayuki@ms.u-tokyo.ac.jp`

UTMS

2008–33 Keiichi Sakai: *Configuration space integrals for embedding spaces and the Haefliger invariant.*

2009–1 Tadayuki Watanabe: *On Kontsevich's characteristic classes for higher dimensional sphere bundles II : higher classes.*

2009–2 Takashi Tsuboi: *On the uniform perfectness of the groups of diffeomorphisms of even-dimensional manifolds.*

2009–3 Hitoshi Kitada: *An implication of Gödel's incompleteness theorem.*

2009–4 Jin Cheng, Junichi Nakagawa, Masahiro Yamamoto and Tomohiro Yamazaki: *Uniqueness in an inverse problem for one-dimensional fractional diffusion equation.*

2009–5 Y. B. Wang, J. Cheng, J. Nakagawa, and M. Yamamoto : *A numerical method for solving the inverse heat conduction problem without initial value.*

2009–6 Dietmar Hömberg, Nataliya Togobytska, Masahiro Yamamoto: *On the evaluation of dilatometer experiments.*

2009–7 Toshio Oshima and Nobukazu Shimeno: *Heckman-Opdam hypergeometric functions and their specializations.*

2009–8 Atsushi Yamashita: *Compactification of the homeomorphism group of a graph.*

2009–9 Jingzhi Li, Masahiro Yamamoto, and Jun Zou: *Conditional stability and numerical reconstruction of initial temperature.*

2009–10 Taku Ishii and Takayuki Oda: *Calculus of principal series Whittaker functions on $SL(n, \mathbb{R})$.*

The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012