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On Kontsevich's characteristic classes for higher dimensional sphere bundles II : higher classes

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ON KONTSEVICH'S CHARACTERISTIC CLASSES FOR HIGHER DIMENSIONAL SPHERE BUNDLES II: HIGHER CLASSES

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Abstract. This paper studies Kontsevich's characteristic classes of smooth bundles with fiber a 'singularly' framed odd-dimensional homology sphere which are defined through his graph complex and configuration space integral. We will give a systematic construction of smooth bundles parametrized by trivalent graphs and will show that our smooth bundles are nontrivially detected by Kontsevich's characteristic classes. It turns out that there are many nontrivial elements of the rational homotopy groups of the diffeomorphism groups of spheres which are not in K. Igusa's stable range. In particular, the homotopy groups of the diffeomorphism groups in some non-stable dimension range are not finite.

Contents

1. Introduction

This paper is a continuation of [W]. See [Kon, W] for introduction and backgrounds. In this paper M denotes a homology sphere of dimension d, i.e., a closed smooth d-manifold with $H_*(M;\mathbb{Z}) \cong H_*(S^d;\mathbb{Z})$, together with a fixed point $\infty \in M$. Let $D_M := M \setminus U_\infty$ where $U_\infty \subset M$ is a small open ball around ∞. Our principal object is to study the rational homotopy groups of $B\mathrm{Diff}(D_M, \partial D_M)$, the classifying space of the group of relative diffeomorphisms that restrict to the identity on ∂D_M. The main result of the present paper (Theorem 3.1, Corollary 3.2) says that the rational homotopy groups $\pi_i(B\mathrm{Diff}(D_M, \partial D_M)) \otimes \mathbb{Q}$ for some (i, d) in non-stable range include many linearly independent elements whose ranks are given by the dimensions of some spaces of 3-valent graphs quotiented by some diagramatic Lie relations.

The main point of the present paper is to provide a technique of surgery on fibered objects such as smooth fiber bundles or families of embeddings for making complicated ones, that is strongly motivated by Goussarov–Habiro theory in 3-dimension [Hab] and that is somewhat reminiscent of Cattaneo–Cotta-Ramusino–Longoni's work [CCL] on 1-knot spaces. The surgery construction of fiber bundles together with a correct generalization of Kuperberg–Thurston–Lescop's computation technique of configuration space integrals [KT, Les2] to higher-dimensions leads us to find many nontrivial elements of the rational homotopy groups.

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So we again have a partial answer to the problem of Burghelea [Bur], which asks whether $\pi_i(B\mathrm{Diff}(D^d, \partial D^d))$ is finite or not, for other pairs (i, d) than those discussed in [W]. The proof needs only elementary cut-and-paste arguments with a little homology theory.

The organization of the present paper is as follows. §2 and §A explain the definition of Kontsevich's configuration space characteristic classes of smooth bundles. The main theorem and its sketch proof are reviewed in §3. Proof of the main theorem will be given in §4, 5, 6, 7.

2. Review of Kontsevich's characteristic classes

In this section and in Appendix A we review the definition of Kontsevich's characteristic classes, originally developed in [Kon].

2.1. **Classifying space for smooth (framed) fiber bundles.** For a smooth compact manifold pair (W, X), a (W, X)-*bundle* denotes a smooth fiber bundle with fiber diffeomorphic to W such that its fiberwise restriction to X has a trivialization. If W has a trivialization of the tangent bundle $\tau : TW \xrightarrow{\cong} \mathbb{R}^d \times W$, one can consider a framed (W, X)-bundle, that is a (W, X)-bundle with its vertical tangent bundle framed such that the restriction of the framing to the sub X-bundle coincides with (the pullback of) the standard behavior on a single fiber (under the X-bundle trivialization). For a given (W, X)-bundle $\pi : E \to B$, we will often denote by (W_t, X_t) the fiber over a point $t \in B$ when the bundle π is understood. We implicitly assume that all framed bundles are pointed (see [W, §2.4]).

The classifying space $\widetilde{B\mathrm{Diff}}(W, X; \tau)$ for framed (W, X)-bundles* is given as follows. We will denote by $\mathrm{Diff}(W, X)$ the (topological) group of self-diffeomorphisms of W which restrict to the identity on X, equipped with the Whitney C^∞-topology. Thus a (W, X)-bundle is nothing but a smooth W-bundle with the structural group $\mathrm{Diff}(W, X)$. If W has a tangent framing, we define $\mathrm{Fr}(W, X; \tau)$ to be the space of framings on W having a fixed behavior $\tau|X$ on X, with the C^0-topology. $\mathrm{Fr}(W, X; \tau)$ is a $\mathrm{Diff}(W, X)$-space with respect to the pullback action and one can define

$$\widetilde{B\mathrm{Diff}}(W, X; \tau) := E\mathrm{Diff}(W, X) \times_{\mathrm{Diff}(W,X)} \mathrm{Fr}(W, X; \tau) \tag{2.1}$$

(Borel construction) where $E\mathrm{Diff}(W, X)$ is the total space of a universal principal $\mathrm{Diff}(W, X)$-bundle. The projection $\widetilde{B\mathrm{Diff}}(W, X; \tau) \to B\mathrm{Diff}(W, X)$ is a fibration with fiber $\mathrm{Fr}(W, X; \tau)$. It is indeed the classifying space for framed (W, X)-bundles (see e.g., [W]). When the standard behavior τ is obviously understood, we will simply write $\widetilde{B\mathrm{Diff}}(W, X)$ and $\mathrm{Fr}(W, X)$ for $\widetilde{B\mathrm{Diff}}(W, X; \tau)$ and $\mathrm{Fr}(W, X; \tau)$ respectively.

In this paper we are especially interested in the case $(W, X) = (D_M, \partial D_M)$, in which case one has the following

Proposition 2.1 (see [W]). *The homotopy fiber of the fibration $\widetilde{B\mathrm{Diff}}(D_M, \partial) \to B\mathrm{Diff}(D_M, \partial)$ is $\Omega^d SO_d$. If d is odd ≥ 3 and i is even ≥ 0, then the sequence*

$$\pi_i(\Omega^d SO_d) \otimes \mathbb{Q} \to \pi_i(\widetilde{B\mathrm{Diff}}(D_M, \partial)) \otimes \mathbb{Q} \to \pi_i(B\mathrm{Diff}(D_M, \partial)) \otimes \mathbb{Q} \to 0$$

is exact.

Because $\pi_*(SO_{2k+1}) \otimes \mathbb{Q} \cong \pi_*(S^3 \times S^7 \times \cdots \times S^{4k-1}) \otimes \mathbb{Q}$, the projection induces an isomorphism

$$\pi_i(\widetilde{B\mathrm{Diff}}(D_M, \partial)) \otimes \mathbb{Q} \cong \pi_i(B\mathrm{Diff}(D_M, \partial)) \otimes \mathbb{Q} \tag{2.2}$$

if d is odd and $i > d - 3$ even.

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*In [Igu] the same symbol $\widetilde{B\mathrm{Diff}}$ has been used in a different meaning.

2.2. **Graph complex.** By a *graph* we mean a finite connected graph with valence at least 3. For a graph Γ with n vertices and m edges, we will consider a label, that is a choice of bijections $\rho : \{\text{vertices of } \Gamma\} \to \{1, 2, \ldots, n\}$ and $\mu : \{\text{edges of } \Gamma\} \to \{1, 2, \ldots, m\}$. An orientation of Γ is a choice of an orientation of the real vector space

$$\mathbb{R}^{\{\text{vertices of } \Gamma\}} \oplus \bigoplus_e \mathbb{R}^{H(e)} \tag{2.3}$$

where $H(e)$ is the set $\{e^+, e^-\}$ of half-edges of e. Thus we may represent an orientation of a graph by the total order on $\{\text{vertices of } \Gamma\}$ given by ρ and the edge orientations.

Let $\mathcal{G}_{n,m}$ be the vector space over $\mathbb{Q}$ generated by oriented labeled graphs (Γ, o) with n vertices and m edges, quotiented by the following relation:

$$(\Gamma, -o) \sim -(\Gamma, o).$$

We will often suppress o and write as Γ for (Γ, o). The sequence $\{\mathcal{G}_{n,m}\}$ can be made into a chain complex by considering the boundary operator $d : \mathcal{G}_{n,m} \to \mathcal{G}_{n-1,m-1}$ defined by

$$d(\Gamma, o) := \sum_{\substack{e:\ \text{edge} \\ \text{of } \Gamma}} (\Gamma/e, \text{induced ori})$$

where for $e_i = (u, v)$ (u, v vertices) the induced orientation of Γ/e_i is formally given by

$$\iota\Big(\frac{\partial}{\partial v}\Big)(dv_1 \wedge \cdots \wedge dv_n) \wedge (de_1^+ \wedge de_1^-) \wedge \overset{\overset{i}{\wedge}}{\cdots} \wedge (de_m^+ \wedge de_m^-).$$

It follows from the property $\iota(X)\iota(Y) + \iota(Y)\iota(X) = 0$ that $d \circ d = 0$. So $(\{\mathcal{G}_{n,m}\}, d)$ indeed forms a chain complex, so called a graph complex.

Remark 2.2. In [CV] following [Th], it is shown that there is a canonical isomorphism

$$\det\big[\mathbb{R}^{\{\text{vertices of } \Gamma\}} \oplus \bigoplus_{\substack{e:\ \text{edge} \\ \text{of } \Gamma}} \mathbb{R}^{H(e)}\big] \cong \det\big[\mathbb{R}^{\{\text{edges of } \Gamma\}} \oplus H^1(\Gamma; \mathbb{R})\big] \tag{2.4}$$

which is natural with respect to d. The orientation of the latter was used in [Kon]. Moreover, if Γ is trivalent and a total order of the vertices is given, then fixing of an orientation of (2.4) is equivalent to fixing of an orientation of $\bigoplus_v \mathbb{R}^{H(v)}$ where $H(v)$ is the set of the (three) half-edges incident to v.

It will be seen later that the condition for a given linear sum $\gamma = \sum_{(\Gamma,o)} a_\Gamma\, \Gamma \in \mathcal{G}_{n,m}$ over all graphs to be a cycle in the graph complex formally describes the cancelling rule that will be necessary to get genuine characteristic classes of smooth fiber bundles. One may see that the cycle condition $d\gamma = 0$ is equivalent to a system of linear relations among the coefficients a_Γ. We will call this condition the $(*)$-*relation*. Clearly, cycles span a subspace of $\mathcal{G}_{n,m}$ with dimension equal to the dimension of the vector space of solutions of the $(*)$-relation.

In this paper, we will consider the universal cycle, including all the cycles at once, instead of considering an arbitrary single cycle. Namely, let $\mathcal{A}_{n,m}$ be the quotient space of $\mathcal{G}_{n,m}$ by the $(*)$-relation and the *label change relation*, which equates two labeled oriented graphs with isomorphic underlying graph when one is obtained from another by even swappings of labels or of edge orientations. We will denote by $[\Gamma]$ the equivalence class in $\mathcal{A}_{n,m}$ represented by $\Gamma \in \mathcal{G}_{n,m}$. The universal cycle $\widetilde{\gamma}_{n,m}$ is then defined by

$$\widetilde{\gamma}_{n,m} := \frac{1}{n!m!} \sum_{\Gamma} [\Gamma] \otimes \Gamma \in \mathcal{A}_{n,m} \otimes \mathcal{G}_{n,m}$$

where the sum is over labeled graphs with an arbitrary choice of orientations. Note that $[-\Gamma] \otimes (-\Gamma) = [\Gamma] \otimes \Gamma$. It is obvious from the definition that $\widetilde{\gamma}_{n,m}$ is a $(1 \otimes d)$-cycle. It is universal in the sense that any cycle of $\mathcal{G}_{n,m}$ can be written as $(W \otimes 1)(\widetilde{\gamma}_{n,m})$ for some linear functional $W : \mathcal{A}_{n,m} \to \mathbb{Q}$.

Example 2.3. (1) In $\mathcal{G}_{2n,3n}$, i.e., in the space of trivalent graphs, the relations for $\mathcal{A}_{2n,3n}$ are precisely the IHX relation of [BN1], which is a diagramatic Jacobi identity.

(2) Dror Bar-Natan has made vast amounts of computations [BN3] and found many cycles in the graph complex.

2.3. **Fulton–MacPherson–Kontsevich compactification of configuration space** [FM, Kon]**.** In this paper, the configuration space of n points on M is the space

$$C_n(M,\infty) := \{(x_1,\ldots,x_n)\in (M\setminus\infty)^{\times n}\,|\,x_i\neq x_j \text{ if } i\neq j\}.$$

This can be considered as a smooth (noncompact) submanifold of $M^{\times n}$ that is the complement of the closed subset

$$\Sigma = \{(x_1,\ldots,x_n)\in M^{\times n}\,|\,x_i = x_j \text{ for some } i\neq j, \text{ or } x_i=\infty \text{ for some } i\}\subset M^{\times n}.$$

Σ has a natural filtration $\Sigma=\Sigma_n\supset\Sigma_{n-1}\supset\cdots\supset\Sigma_1\supset\Sigma_0$ with

$$\Sigma_j = \{(x_1,\ldots,x_n)\in M^{\times n}\,|\,|\{x_1,\ldots,x_n,\infty\}| = j+1\}.$$

It has the property that $\Sigma_{i+1}\setminus\Sigma_i$ is a disjoint union of smooth submanifolds of $M^{\times n}\setminus\Sigma_i$. So one can iterate (real) blow-ups along the filtration: First one can apply blow-up $B\ell(M^{\times n},\Sigma_0)$ along the 0-dimensional submanifold $\Sigma_0=\{(\infty,\ldots,\infty)\}$. Recall that a blow-up replaces a submanifold with its normal sphere bundle. Since the closure of $\Sigma_1\setminus\Sigma_0$ in $B\ell(M^{\times n},\Sigma_0)$ is also a disjoint union of smooth submanifolds, one can apply another blow-up along it, and so on. After the blow-ups along all the strata of Σ, one obtains a smooth compact manifold with corners, which we denote by $\overline{C}_n(M,\infty)$.

It will be necessary to know a precise description of the codimension one strata of $\overline{C}_n(M,\infty)$, i.e., the boundary strata, in the proof of well-definedness of Kontsevich's characteristic classes, so we shall briefly recall it here. The boundary of $\overline{C}_n(M,\infty)$ has a stratification corresponding to bracketings of the $n+1$ letters $1,2,\ldots,n,\infty$, e.g., $((137)(25))46\infty$ (see [FM]). Then the codimension one strata correspond to bracketings of the form $(\cdots)\cdots$, with only one pair of brackets. There are two cases: $(\cdots)\cdots\infty$ or $(\cdots\infty)\cdots$.

$\underline{(\cdots)\cdots\infty \text{ case}}$: We consider only the bracketing $(12\cdots j)j+1\cdots n\infty$ for simplicity. The stratum $\mathcal{S}_j$ of $\partial\overline{C}_n(M;\infty)$ corresponding to the bracketing $(1\cdots j)j+1\cdots n\infty$ is the face created by the blow-up along the closure of the following submanifold of $(M\setminus\infty)^{\times n}$:

$$\Delta_j := \{(x_1,\ldots,x_n)\in(M\setminus\infty)^{\times n}\,|\,x_1=\cdots=x_j;\ \text{otherwise distinct}\}$$

in the result of the previous blow-ups. More precisely, $\mathcal{S}_j$ can be identified with the blow-ups of the total space of the normal $S^{(j-1)d-1}$-bundle of $\Delta_j\subset(M\setminus\infty)^{\times n}$ along the intersection with (closures of) deeper diagonals that correspond to deeper bracketings. The fiber of the normal $S^{(j-1)d-1}$-bundle over a point $(x_j,\ldots,x_n)\in\Delta_j$ is as follows:

$$\big(\{(y_1,\ldots,y_j)\in(\mathbb{R}^d)^{\times j}\,|\,y_1=0\}-0\big)/(\text{dilation})\cong S^{(j-1)d-1}$$

where the Euclidean coordinate y_i corresponds to x_i-x_1 (where it makes sense) via a framing on $T(M\setminus\infty)$. Thus the fiber $\overline{C}_j^{\text{local}}(\mathbb{R}^d)$ of $\mathcal{S}_j$ over a point of Δ_j is the Fulton–MacPherson–Kontsevich compactification of $C_j(\mathbb{R}^d)/(\text{translation},\text{dilation})$. Clearly, the base space Δ_j can be canonically identified with $\overline{C}_{n-j+1}(M,\infty)$ and there is a natural projection $\mathrm{pr}_j:\Delta_j\to M$ defined by $(x_j,\ldots,x_n)\mapsto x_j$. Now $\mathcal{S}_j$ is described as the pullback of the associated $\overline{C}_j^{\text{local}}(\mathbb{R}^d)$-bundle of TM ($\overline{C}_j^{\text{local}}(\mathbb{R}^d)$ is an SO_d-space) under pr_j.

$\underline{(\cdots\infty)\cdots \text{ case}}$: This case is similar to the previous case. Instead of Δ_j one should consider

$$\Delta_j^\infty := \{(x_1,\ldots,x_n)\in M^{\times n}\,|\,x_1=\cdots=x_j=\infty;\ \text{otherwise distinct}\}.$$

2.4. **Propagator on family of configuration spaces.** Given a (D_M, ∂)-bundle $\pi : E \to B$ one has the associated $\mathrm{Diff}(D_M, \partial)$-bundle with fiber $\overline{C}_n(M, \infty)$:

$$\overline{C}_n(\pi) : E\overline{C}_n(\pi) \to B$$

by considering $\mathrm{Diff}(D_M, \partial)$ as a subgroup of $\mathrm{Diff}(M, U_\infty)$ and by the natural action of $\mathrm{Diff}(D_M, \partial)$ on $\overline{C}_n(M, \infty)$. This can be given more explicitly by the Borel construction

$$E\overline{C}_n(\pi) = P \times_{\mathrm{Diff}(D_M,\partial)} \overline{C}_n(M, \infty)$$

(see [W, §2]) where $P \to B$ is the principal bundle that is the pullback of the universal $\mathrm{Diff}(D_M, \partial)$-bundle by the classifying map $B \to B\mathrm{Diff}(D_M, \partial)$ for π. Its fiberwise restriction to the boundaries of the fibers is then given by

$$\partial^{\mathrm{fib}} E\overline{C}_n(\pi) = P \times_{\mathrm{Diff}(D_M,\partial)} \partial\overline{C}_n(M, \infty).$$

According to the description of the boundary strata as given in the last subsection, a framing τ_E on π defines a map

$$p(\tau_E) : \partial^{\mathrm{fib}} E\overline{C}_2(\pi) \to S^{d-1} \tag{2.5}$$

by using the Gauss map with respect to the infinitesimal Euclidean coordinate which is defined by the framing near the point where the two points come close to each other (see [W, §2]). Then one can prove the following lemma by restating [W, Lemma 3] in a de Rham theoretic term.

Lemma 2.4. *Let $\pi : E \to B$ be a (D_M, ∂)-bundle with a vertical framing τ_E and let $\mathrm{Vol}_{S^{d-1}} \in \Omega^{d-1}_{\mathrm{dR}}(S^{d-1})$ be the SO_d-invariant volume form on S^{d-1} with $\int_{S^{d-1}} \mathrm{Vol}_{S^{d-1}} = 1$. Then there exists a closed form $\omega \in \Omega^{d-1}_{\mathrm{dR}}(E\overline{C}_2(\pi))$ such that*

$$\omega|_{\partial^{\mathrm{fib}} E\overline{C}_2(\pi)} = p(\tau_E)^*\mathrm{Vol}_{S^{d-1}}.$$

We will call a closed form with the property given in Lemma 2.4 a *propagator*. Of course the choice of a propagator is not at all unique.

2.5. **Configuration space integral.** Now let us assume that d is odd. For a given oriented graph $\Gamma = (\Gamma, o) \in \mathcal{G}_{n,m}$, choose an orientation of $\mathbb{R}^{H(e)}$, i.e., an order of the half edges, for each edge e so that the wedge over all e together with the total order ρ of the vertices gives the orientation o. This choice determines the projection

$$\phi_e : E\overline{C}_n(\pi) \to E\overline{C}_2(\pi)$$

by picking up two points labeled by the same labels (given by the bijection ρ) as the boundary vertices of e. Then fix a propagator ω and let

$$\begin{aligned} \omega(\Gamma) &:= \bigwedge_{\substack{e:\text{ edge}\\ \text{of }\Gamma}} \phi_e^*\omega \in \Omega^{m(d-1)}_{\mathrm{dR}}(E\overline{C}_2(\pi)) \\ I(\Gamma) = I(\Gamma)(\pi; \tau_E) &:= \int_{\mathrm{Fib}(\overline{C}_n(\pi))} \omega(\Gamma) \in \Omega^{m(d-1)-nd}_{\mathrm{dR}}(B) \end{aligned} \tag{2.6}$$

where $\int_{\mathrm{Fib}(\overline{C}_n(\pi))}$ denotes the integration along the fiber of $\overline{C}_n(\pi)$. Extending linearly, we have a linear map

$$I : \mathcal{G}_{n,m} \to \Omega^{m(d-1)-nd}_{\mathrm{dR}}(B).$$

Theorem 2.5 (Kontsevich [Kon])**.** *Let γ be a cycle in $\mathcal{G}_{n,m}$.*

(1) *I is a chain map: $dI(\Gamma) = (-1)^{\deg I(d\Gamma)} I(d\Gamma)$. In particular, $dI(\gamma) = 0$.*
(2) *$[I(\gamma)] \in H^{m(d-1)-nd}(B; \mathbb{R})$ is independent of the choice of the propagator ω. (But may depend on the choice of the framing.)*
(3) *$[I(\gamma)] \in H^{m(d-1)-nd}(B; \mathbb{R})$ is natural with respect to bundle morphisms of framed bundles. Hence it is a well-defined characteristic class of framed (D_M, ∂)-bundles.*

(4) *Evaluation gives a homomorphism*

$$\Omega_{m(d-1)-nd}(\widetilde{B\mathrm{Diff}}(D_M,\partial))\to\mathbb{R}$$

where $\Omega_*(\cdot)$ *is the oriented bordism group functor.*

Following [Kon], we will explain a self-contained proof of Theorem 2.5 in Appendix A, which one can easily guess from [Kon]. The same statements hold also for (trivial) $\mathcal{A}_{n,m}\otimes\mathbb{R}$-coefficients and one has a characteristic class

$$\zeta_{n,m}(\pi;\tau_E):=[I(\widetilde{\gamma}_{n,m})]\in H^{m(d-1)-nd}(B;\mathcal{A}_{n,m}\otimes\mathbb{R}).$$

Because there are $\frac{n!m!}{|\mathrm{Aut}\,\Gamma|}$ different labellings on an unlabeled graph, one has

$$I(\widetilde{\gamma}_{n,m})=\frac{1}{n!m!}\sum_{\substack{\Gamma\\ \text{labeled}}}[\Gamma]\,I(\Gamma)=\sum_{\substack{\Gamma\\ \text{unlabeled}}}\frac{[\Gamma]}{|\mathrm{Aut}\,\Gamma|}\,I(\Gamma)$$

where $\mathrm{Aut}\,\Gamma$ denotes the group of automorphisms of Γ.

3. Main results – Nontriviality of higher classes

Recall that we are assuming that the fiber dimension is odd: $d=2k+1$. From now on, we focus on the 3-valent graphs, namely on $\mathcal{G}_{2n,3n}$ and related constructions. In the next section, we will define a linear map

$$\psi_{2n}:\mathcal{G}_{2n,3n}\to\Omega_{2n(k-1)}(\widetilde{B\mathrm{Diff}}(D_M,\partial))\otimes\mathbb{Q},$$

which is not necessarily unique. Note that the degree $2n(k-1)$ agrees with the degree of $\zeta_{2n,3n}$. The main theorem of the present paper is the following

Theorem 3.1. *Let k be an odd integer ≥ 3 and let $n\geq 1$. Let $r_k:=2o(\pi_{3k-1}(SO_{2k+1}))$ where $o(G):=\min\{d\in\mathbb{Z}_{>0}\,|\,dx=0$ for all $x\in G\}$ for a finite abelian group G. Then for a suitably defined ψ_{2n} the following holds.*

(1) *The diagram*

$$\begin{array}{ccc}\mathcal{G}_{2n,3n} & \xrightarrow{\psi_{2n}} & \Omega_{2n(k-1)}(\widetilde{B\mathrm{Diff}}(D_M,\partial))\otimes\mathbb{Q}\\ {\scriptstyle[\cdot]\otimes 1}\downarrow & & \downarrow{\scriptstyle\langle\zeta_{2n,3n},\cdot\rangle}\\ \mathcal{A}_{2n,3n}\otimes\mathbb{R} & \xrightarrow{\times r_k^{2n}} & \mathcal{A}_{2n,3n}\otimes\mathbb{R}\end{array}$$

is commutative.

(2) $\mathrm{Im}\,\psi_{2n}$ *is included in the image of the Hurewicz homomorphism* $\pi_{2n(k-1)}(\widetilde{B\mathrm{Diff}}(D_M,\partial))\otimes\mathbb{Q}\to\Omega_{2n(k-1)}(\widetilde{B\mathrm{Diff}}(D_M,\partial))\otimes\mathbb{Q}$.

Proof of Theorem 3.1 will be given from the next section. Theorem 3.1 together with the isomorphism (2.2) implies the following

Corollary 3.2. *For $n\geq 2$ and $k\geq 3$ odd, the following inequality holds:*

$$\dim\pi_{2n(k-1)}(B\mathrm{Diff}(D_M,\partial))\otimes\mathbb{Q}\geq\dim\mathcal{A}_{2n,3n}.$$

Remark 3.3. The dimensions of the spaces $\mathcal{A}_{2n,3n}$ for $n\leq 11$ are computed in [BN2] as follows:

n	1	2	3	4	5	6	7	8	9	10	11
$\dim\mathcal{A}_{2n,3n}$	1	1	1	2	2	3	4	5	6	8	9

At present, there is no general formula explaining the behavior of $\dim\mathcal{A}_{2n}$. Of course, weight systems coming from semi-simple Lie algebras (e.g., [BN1]) or from the Rozansky–Witten invariant of hyper-Kähler manifolds (e.g., [RW]) give lower bounds.

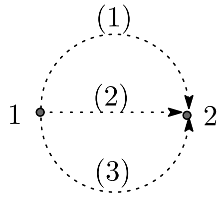

FIGURE 1. The theta-graph Θ, generator of the one-dimensional space $\mathcal{A}_{2,3}$.

The total space of a smooth S^{2k+1}-bundle over a sphere $S^{2n(k-1)}$ that is standard near ∞-section is diffeomorphic to the connected sum of $S^{2k+1} \times S^{2n(k-1)}$ with a homotopy sphere ([W]). Because the h-cobordism group of homotopy spheres is finite [KM], we have the following corollary.

Corollary 3.4. *If $n \geq 2$ and $k \geq 3$ odd, the subgroup of $\pi_{2n(k-1)}(B\mathrm{Diff}(S^{2k+1}))$ consisting of smooth S^{2k+1}-bundles over $S^{2n(k-1)}$ having the total space diffeomorphic to that of the trivial one $S^{2k+1} \times S^{2n(k-1)}$ has a free abelian subgroup with rank at least $\dim \mathcal{A}_{2n,3n}$. Hence for $n \geq 2$ the diffeomorphism type of the total space misses the information of the rational subspaces found in Corollary 3.2.*

For $n = 1$, we have a partial result. In this case the isomorphism (2.2) is not true. Instead we recall the following result from the part I.

Theorem 3.5 ([W]). *Let $k \geq 2$ and let $\pi : E \to S^{2k-2}$ be a (D^{2k+1}, ∂)-bundle over S^{2k-2} with a vertical framing τ_E. Then the number*

$$\hat{Z}_2(\pi) := \langle W_\Theta(\zeta_{2,3}(\pi; \tau_E)), [S^{2k-2}] \rangle - \frac{(2k)!}{2^{2k+2}(2^{2k-1}-1)B_k} \Delta_k(\pi; \tau_E) \in \mathbb{Q},$$

where B_k is the k-th Bernoulli number, Δ_k is the signature defect and $W_\Theta : \mathcal{A}_{2,3} \to \mathbb{Q}$ is the linear map which takes the theta-graph $[\Theta]$ (Figure 1) to 12, does not depend on the choice of the framing τ_E, and hence gives rise to a group homomorphism $\hat{Z}_2 : \pi_{2k-2}(B\mathrm{Diff}(D^{2k+1}, \partial)) \to \mathbb{Q}$.

See [W, §3.3] for the precise definition of Δ_k. Let π_ℓ^S denote the stable homotopy group $\pi_{n+\ell}(S^n)$ $(n > \ell + 1)$.

Corollary 3.6. *Let $k \geq 3$ odd. If the rational number*

$$\frac{3 \cdot 2^{2k+7}(2^{2k-1}-1)B_k}{(2k)!} |\pi_{4k-1}^S| \prod_{\ell=1}^{k-1} o(\pi_\ell^S)^2 \tag{3.1}$$

is not integral, then $\hat{Z}_2$ for such a k is non-trivial. Hence the inequality of Corollary 3.2 holds also for $n = 1$, $D_M = D^{2k+1}$ and $\psi_2(\Theta)$ gives a generator of the 1-dimensional subspace of $\pi_{2k-2}(B\mathrm{Diff}(D^{2k+1}, \partial)) \otimes \mathbb{Q}$ where ψ_2 is the one satisfying Theorem 3.1.

We have checked by using Maxima that the numbers (3.1) are non-integral for all odd k in $3 \leq k \leq 399$.

Proof of Corollary 3.6. We denote by $(\pi^\Gamma; \tau^\Gamma)$ the framed bundle corresponding to $\psi_{2n}(\Gamma)$. By Theorem 3.1(2), there is a framed (D^{2k+1}, ∂)-bundle $(H\pi^\Theta : H^\Theta \to S^{2k-2}, H\tau^\Theta)$ for which

$$[(H\pi^\Theta, H\tau^\Theta)] = [(\pi^\Theta, \tau^\Theta)] \text{ in } \Omega_{2k-2}(\widetilde{B\mathrm{Diff}}(D^{2k+1}, \partial)).$$

We compute $\hat{Z}_2(o(\Theta^{4k-1})[H\pi^\Theta])$ where Θ^{4k-1} is the group of h-cobordism classes of homotopy $(4k-1)$-spheres, which is finite of order

$$|\Theta^{4k-1}| = |\pi_{4k-1}^S|\, 2^{2k-4}(2^{2k-1}-1)\frac{B_k a_k}{k} \tag{3.2}$$

where $a_k = (3-(-1)^k)/2$ (see [KM]). Note that the closing (see §3.2 of [W]) of the total space of $o(\Theta^{4k-1})H\pi^\Theta$ is diffeomorphic to the standard $(4k-1)$-sphere. So the signature defect term vanishes for a suitable choice of a vertical framing. In general, by [W, (11)] the framing correction term contributes just by the relative Pontrjagin number $\frac{1}{4}\langle p_k(TD^{4k};\tau'_E),[D^{4k},\partial D^{4k}]\rangle$ for some stable framing τ'_E on ∂D^{4k}, which by [MK, Lemma 2] is an integer multiple of $a_k(2k-1)!/4$. By Theorem 3.1(1), we have

$$\hat{Z}_2(o(\Theta^{4k-1})[H\pi^\Theta]) = 12r_k^2 o(\Theta^{4k-1}) + \frac{a_k(2k-1)!}{4}N_k$$

for some $N_k\in\mathbb{Z}$. The right hand side is non-zero if

$$\frac{48r_k^2 o(\Theta^{4k-1})}{a_k(2k-1)!}\notin\mathbb{Z}.$$

By Lemma 3.8 below, we have $r_k|4\prod_{j=1}^{k-1}o(\pi_j^S)$. Replacing r_k with $4\prod_{j=1}^{k-1}o(\pi_j^S)$ and $o(\Theta^{4k-1})$ with $|\pi_{4k-1}^S|2^{2k-4}(2^{2k-1}-1)B_k a_k/k$, we obtain the number of the statement. □

Corollary 3.7. *Let $k\geq 3$ odd. If*

- *$2k-1$ is prime and*
- *$2k-1\nmid \mathrm{num}\,(B_k)$,*

then $\hat{Z}_2$ for such a k is non-trivial. In particular, if $2k-1$ is a regular prime, then $\hat{Z}_2$ is non-trivial.

Proof. We prove that the number (3.1) is not integral when k satisfies the two conditions of the statement. Since $2k-1$ is prime, $(2k)!$ has a prime factor $2k-1$. By Corollary 3.6 it is enough to prove that

$$2k-1\nmid (2^{2k-1}-1)|\pi_{4k-1}^S|\prod_{\ell=1}^{k-1}o(\pi_i^S)^2$$

since B_k does not have a factor $2k-1$ by the assumption. It is easy to see that $2k-1\nmid 2^{2k-1}-1$ (see Fact 1(1) of [W]). Further, by H. Toda's theorem (e.g., [To]) which says that for any odd prime p the p-primary component of π_ℓ^S is isomorphic to

$$\begin{cases}\mathbb{Z}_p & \text{for } \ell=2i(p-1)-1,\ i=1,2,\ldots,p-1,\\ 0 & \text{otherwise for } \ell<2p(p-1)-2,\end{cases}$$

we see that $|\pi_{4k-1}^S|\prod_{\ell=1}^{k-1}o(\pi_\ell^S)^2$ does not have the prime factor $p=2k-1$ because $2i(p-1)-1=4k-5,8k-9,\ldots$ and $k<4k-5<4k-1<8k-9$ for $k>2$. This completes the proof. □

The following lemma is shown in [W, Lemma 11].

Lemma 3.8. *If p is even and $\frac{p+3}{2}\leq j\leq p+1$, then $o(\pi_p(SO_j))|4\prod_{\ell=1}^{p-j+1}o(\pi_\ell^S)$.*

3.1. **Sketch of the proof of Theorem 3.1.**

Construction. Let V_0 be a $(2k+1)$-dimensional handlebody obtained by attaching three k-handles to a 0-handle in a trivial way, namely attaching handles along the trivial framed 3 component link in the boundary of the 0-handle. Thus V_0 is a smooth manifold with boundary and is homotopy equivalent to $S^k\vee S^k\vee S^k$. Fixing an embedding $\phi: V_0\hookrightarrow \mathrm{int}(D^{2k+1})$, a standard framing τ_ϕ on V_0 is defined as the pullback of the standard one on the unit disk $D^{2k+1}\subset\mathbb{R}^{2k+1}$ by ϕ. We will construct a nontrivial element

$$\alpha\in\pi_{k-1}(\widetilde{B\mathrm{Diff}}(V_0,\partial;\tau_\phi))$$

by taking a fibered representative of the long Borromean rings complement. Nontrivial elements of $\pi_{k-1}(\widetilde{B\mathrm{Diff}}(V_0,\partial;\tau_\phi))$ can be used to do 'surgery' on a framed (D^{2k+1},∂)-bundle over S^{k-1} to get another framed bundle.

There is a nice way of iterating such surgeries whose prototype is Goussarov–Habiro's (clasper) theory in 3-dimension [Hab]. In our case it is done as follows. Choose an embedding $\phi = \phi_1 \cup \cdots \cup \phi_{2n} : V_1 \cup \cdots \cup V_{2n} \hookrightarrow \mathrm{int}(D^{2k+1})$ of a disjoint union of $2n$ copies of V_0 into $\mathrm{int}(D^{2k+1})$. Then we define a multilinear form

$$\beta : \prod_{i=1}^{2n} \pi_{k-1}(\widetilde{B\mathrm{Diff}}(V_i, \partial; \tau_{\phi_i})) \to \Omega_{2n(k-1)}(\widetilde{B\mathrm{Diff}}(D^{2k+1}, \partial))$$

by the iterated surgery on a trivial framed (D^{2k+1}, ∂)-bundle over $(S^{k-1})^{\times 2n}$. Moreover, there is a systematic way to associate to a given trivalent graph an embedding $\phi_\Gamma : V_1 \cup \cdots \cup V_{2n} \hookrightarrow \mathrm{int}(D^{2k+1})$ (Figure 3). Given such embeddings $\{\phi_\Gamma\}_\Gamma$ we define a linear map

$$\psi_{2n} : \mathcal{G}_{2n,3n} \to \Omega_{2n(k-1)}(\widetilde{B\mathrm{Diff}}(D_M, \partial)) \otimes \mathbb{Q}$$

by $\beta(\alpha \times \cdots \times \alpha)$ followed by the map $\Omega_*(\widetilde{B\mathrm{Diff}}(D^{2k+1}, \partial)) \to \Omega_*(\widetilde{B\mathrm{Diff}}(D_M, \partial))$ induced by a fixed inclusion $D^{2k+1} \subset D_M$.

Normalization of a propagator / evaluation. The proof of the assertion (1) of Theorem needs an explicit computation of the configuration space integral. But the definition of a propagator in Lemma 2.4 just claims its existence and so in general there seems no way to compute the integral explicitly so far. On the other hand, for 3-manifolds there is a normalization technique of propagator, which has been developed in [KT, Les2], to compute the configuration space integral for some surgery defined manifolds explicitly! Surprisingly, a similar technique can be applied to our higher-dimensional construction with a suitable modification (though the proof is rather long).

Now we shall give an informal explanation of how to compute the integral. Intuitively, the framed (D^{2k+1}, ∂)-bundle $\pi^\Gamma : E^\Gamma \to (S^{k-1})^{\times 2n}$ corresponding to the construction $\psi_{2n}(\Gamma)$ given above restricts to the trivial framed bundle on sub $X = D^{2k+1} \setminus \phi_\Gamma(V_1 \cup \cdots \cup V_{2n})$-bundle. So in the associated $\overline{C}_{2n}(S^{2k+1}, \infty)$-bundle the information which are captured by points on a fiber X is the same as those on the fiber over the base point if the propagator is normalized nicely enough. Thus the integral along the subset of $E\overline{C}_{2n}(\pi^\Gamma)$ consisting of configurations which include points on X vanishes by a dimensional reason. Thus it is enough to integrate along the fiber restricted to configurations on $\phi_\Gamma(V_1 \cup \cdots \cup V_{2n})$. Similarly, let $X_i' := D^{2k+1} \setminus \phi_\Gamma(V_i)$ and consider the integral along the subset of $E\overline{C}_{2n}(\pi^\Gamma)$ consisting of configurations on X_i'. The sub X_i'-bundle of π^Γ is trivial in the direction of some $(k-1)$-parameters in $(S^{k-1})^{\times 2n}$ and the integral vanishes again by a dimensional reason. Hence nontrivial integral must come from the integral along the fiber of the form $\phi_\Gamma(V_1) \times \cdots \times \phi_\Gamma(V_{2n})$. One can see that such an integral is rewritten as a direct product $W_1 \times \cdots \times W_{2n}$ of some manifolds W_i and thus can be interpreted in terms of the de Rham cohomology. This reduces the computation of the integral to an easy algebraic problem. Namely, cohomologically a propagator can be written as a sum of products of some k-forms corresponding to half-edges: $[\omega|_{W_i \times W_j}] \in H^k(W_i) \otimes H^k(W_j)$. So we have that

$$\begin{aligned}\Big\langle [\omega(\Gamma')], \prod_i [W_i, \partial W_i] \Big\rangle &= \Big\langle \bigcup_{\substack{\text{e: edge}\\ \text{of } \Gamma'}} [\phi_e^* \omega], \prod_i [W_i, \partial W_i] \Big\rangle \\ &= \Big\langle N_{\Gamma'} \bigcup_i (\eta_{e(i,1)} \cup \eta_{e(i,2)} \cup \eta_{e(i,3)}), \prod_i [W_i, \partial W_i] \Big\rangle\end{aligned}$$

for each Γ', for some integer $N_{\Gamma'}$ and for some k-cocycles $\eta_{e(i,1)}, \eta_{e(i,2)}, \eta_{e(i,3)}$ on W_i corresponding to the three half-edges $e(i,1)$, $e(i,2)$, $e(i,3)$ adjacent to the vertex i. The construction of α from the Borromean rings implies that $N_{\Gamma'}$ is non-zero if and only if $\Gamma' \cong \pm\Gamma$, and that the triple-product $\eta_{e(i,1)} \cup \eta_{e(i,2)} \cup \eta_{e(i,3)}$ is a compact support volume form of W_i. □

4. Surgery construction

4.1. The special element $\alpha \in \pi_{k-1}(\widetilde{B\mathrm{Diff}}(V_0, \partial; \phi))$.

Borromean rings. The Borromean rings is a three-component link

$$S^p \cup S^q \cup S^r \hookrightarrow \mathbb{R}^n$$

which is defined only when $0 < p, q, r < n$, $p + q + r = 2n - 3$. It is given by the submanifold of $\mathbb{R}^n = \mathbb{R}^{n-p-1} \times \mathbb{R}^{n-q-1} \times \mathbb{R}^{n-r-1}$ consisting of points $(x, y, z) \in \mathbb{R}^n$ such that

$$\begin{aligned} &\frac{|y|^2}{4} + |z|^2 = 1,\ x = 0 \text{ or} \\ &\frac{|z|^2}{4} + |x|^2 = 1,\ y = 0 \text{ or} \\ &\frac{|x|^2}{4} + |y|^2 = 1,\ z = 0. \end{aligned} \tag{4.1}$$

We orient each component by the outward normal first convention in the planes $\{x = 0\} = \mathbb{R}^{p+1}, \{y = 0\} = \mathbb{R}^{q+1}, \{z = 0\} = \mathbb{R}^{r+1}$. By replacing "$= 1$" in (4.1) by "$\leq 1$" we obtain three spanning disks bounded by the Borromean rings, with the triple intersection at the origin.

There is a long version of the Borromean rings. Let us give an explicit form of the long Borromean rings of dimensions $(2k - 1, k, k)$ in $(2k + 1)$-dimensional space for it will be necessary later. First fix a standard inclusions $\mathrm{incl} : \mathbb{R}^{2k-1} \cup \mathbb{R}^k \cup \mathbb{R}^k \hookrightarrow \mathbb{R}^{2k+1}$ as follows:

$$\begin{aligned} &(\text{1st component}) : (x_1, \ldots, x_{2k-1}) \mapsto (x_1, x_2, \ldots, x_{2k-1}, -10, 0) \\ &(\text{2nd component}) : (x_1, \ldots, x_k) \mapsto (x_1, x_2, \ldots, x_k, \overbrace{0, \ldots, 0}^{k-1}, -10, 10) \\ &(\text{3rd component}) : (x_1, \ldots, x_k) \mapsto (x_1, x_2, \ldots, x_k, 0, \ldots, 0, -10, 20) \end{aligned} \tag{4.2}$$

One may see that the image of this inclusion is disjoint from the Borromean rings given as (4.1). Then choose a path in $[-50, 50]^{\times 2k+1}$ from the origin of each component of the standard plane to a point on the corresponding component of the Borromean rings so that the paths are disjoint from the components and disjoint from each other. Then make connected sums along the paths so that the orientations are consistently extended. The result is represented by an embedding

$$\phi_B : \mathbb{R}^{2k-1} \cup \mathbb{R}^k \cup \mathbb{R}^k \hookrightarrow \mathbb{R}^{2k+1}$$

which is standard outside a ball. Moreover, by restricting ϕ_B to $\phi_B^{-1}[-100, 100]^{\times 2k+1}$, one obtains an embedding

$$\phi_B^{\square} : D^{2k-1} \cup D^k \cup D^k \hookrightarrow D^{2k+1}$$

which has a standard behavior near the boundary.

From a family of embeddings to a fiber bundle. In the following, all embedding space $\mathrm{Emb}(X, W)$ is to be considered as the space of embeddings from a smooth manifold X with boundary to a smooth manifold W with boundary which embeds ∂X into ∂W and $\mathrm{int}(X)$ into $\mathrm{int}(W)$, and which agrees with a fixed standard embedding $X \hookrightarrow W$ near ∂X that intersects ∂W transversely. Let $\mathrm{Emb}^{\mathrm{f}}(X, W)$ denote the space of *normally* framed embeddings with the fixed behavior near the boundary as above.

First we define a map

$$\delta : \pi_{k-1}(\mathrm{Emb}^{\mathrm{f}}(D^k \cup D^k \cup D^k, D^{2k+1})) \to \pi_{k-1}(B\mathrm{Diff}(V_0, \partial))$$

as follows. Given a $(k - 1)$-dimensional family σ of framed embeddings of three k-disks in a $(2k + 1)$-disk, one can consider the family of complements of a fiberwise open tubular neighborhood of the family σ (it is known that a family of embeddings as above can be extended to a family of embeddings of a tubular neighborhood). The normal framing of an

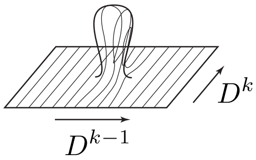

FIGURE 2. From an embedding $D^{2k-1} \hookrightarrow D^{2k+1}$ to a family of embeddings $\varphi_t : D^k \hookrightarrow D^{2k+1}$, $t \in D^{k-1}$.

embedding gives a trivialization of the boundary of the complement. Thus the total space forms a smooth V_0-bundle that is given trivialization on the boundary, i.e., a (V_0, ∂)-bundle over D^{k-1}.

Next we use the map δ to construct the element $\alpha \in \pi_{k-1}(\widetilde{B\mathrm{Diff}}(V_0, \partial; \phi))$ from a nontrivial element of $\pi_{k-1}(\mathrm{Emb}^{\mathrm{f}}(D^k \cup D^k \cup D^k, D^{2k+1}))$. We will choose a nontrivial element of $\pi_{k-1}(\mathrm{Emb}^{\mathrm{f}}(D^k, D^{2k+1} \setminus (D^k \cup D^k)))$ and will take the image of it under the natural map

$$i_1 : \pi_{k-1}(\mathrm{Emb}^{\mathrm{f}}(D^k, D^{2k+1} \setminus (D^k \cup D^k))) \to \pi_{k-1}(\mathrm{Emb}^{\mathrm{f}}(D^k \cup D^k \cup D^k, D^{2k+1})).$$

Namely, there is an element of $\pi_{k-1}(\mathrm{Emb}^{\mathrm{f}}(D^k, D^{2k+1} \setminus (D^k \cup D^k)))$ corresponding to the long Borromean rings $\phi_B^{\square}$ of dimensions $(2k-1, k, k)$ in D^{2k+1} with the canonical framing given (note that the Borromean rings is regularly homotopic to the trivial inclusion). Indeed, by the identification $D^{2k-1} = D^k \times D^{k-1}$ and by the fact that the (long) Borromean rings with one of its components removed is isotopically trivial, the long Borromean rings can be considered as a $(k-1)$-dimensional family of framed embeddings

$$\phi_t : D^k \hookrightarrow D^{2k+1} \setminus (D^k \cup D^k), \quad t \in D^{k-1}, \tag{4.3}$$

after a suitable modification of the second and the third components (see Figure 2). However, these embeddings do not satisfy the standardness condition on the boundary because the image of ∂D^k shifts when the parameter varies. We use the following lemma to deform ϕ_t to make it standard near the boundary.

Lemma 4.1. *There exists a fiberwise isotopy in $D^{2k+1} \setminus (D^k \cup D^k)$ from ϕ_t, $t \in D^{k-1}$ to another family which takes $\phi_t|_{\partial D^k}$ to $\mathrm{incl}|_{\partial D^k}$.*

Proof. It is sufficient to prove the claim for linear embeddings (without Borromean part) because we can then interpolate the fiberwise isotopy near the boundary and the identity on a disk including the Borromean part. Namely, we construct a family of isotopies taking the shifted linear inclusions

$$\begin{aligned} &\phi_t^0 : D^k \hookrightarrow D^{2k+1} \setminus (D^k \cup D^k) \\ &\phi_t^0(x_1, \ldots, x_k) = (x_1, \ldots, x_k, t_1, \ldots, t_{k-1}, -10, 0), \quad t = (t_1, \ldots, t_{k-1}) \end{aligned}$$

to the standard inclusion

$$\begin{aligned} &\mathrm{incl} : D^k \hookrightarrow D^{2k+1} \setminus (D^k \cup D^k) \\ &\mathrm{incl}(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0, \ldots, 0, -10, 0). \end{aligned}$$

The explicit isotopy is given by

$$(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, s \cdot t_1, \ldots, s \cdot t_{k-1}, -10, 0), \quad 0 \le s \le 1.$$

It is obvious that this does not intersect the second and the third component of (4.2). $\square$

We denote by $\alpha' \in \pi_{k-1}(\mathrm{Emb}^{\mathrm{f}}(D^k \cup D^k \cup D^k, D^{2k+1}))$ the image under i_1 of the resulting family of the deformation of Lemma 4.1 and then we have an element

$$\delta(\alpha') \in \pi_{k-1}(B\mathrm{Diff}(V_0, \partial)).$$

Framing.

Proposition 4.2. *Let k be an odd integer ≥ 3 and let $r_k := 2o(\pi_{3k}(SO_{2k+1}))$. Then for any $x \in \pi_{k-1}(B\mathrm{Diff}(V_0,\partial))$ and for any embedding $\phi : V_0 \hookrightarrow \mathrm{int}(D^{2k+1})$, there is a lift $y \in \pi_{k-1}(\widetilde{B\mathrm{Diff}}(V_0,\partial;\phi))$ of $r_k x$.*

In other words, the proposition says that for any given (V_0,∂)-bundle $\widetilde{V}_0 \to S^{k-1}$ and a fixed standard framing on $\partial\widetilde{V}_0$, there exists a framing on the r_k times iteration of $\widetilde{V}_0$ extending the given one on the boundary. Here is the point where the assumption that k is odd in Theorem 3.1 is necessary.

Proof of Proposition 4.2. Any (V_0,∂)-bundle over S^{k-1} can be obtained by fiberwise gluing of two trivial V_0-bundles $(V_0 \times D^{k-1}) \sqcup (V_0 \times D^{k-1})$ along the fibers over $\partial D^{k-1} = S^{k-2}$ by using an S^{k-2}-parameter family of relative diffeomorphisms of $\mathrm{Diff}(V_0,\partial)$. Moreover, we may assume that the family of diffeomorphisms are identities outside a small $(k-2)$-disk in S^{k-2}. Denote this family by $\varphi_t \in \mathrm{Diff}(V_0,\partial)$, $t \in D^{k-2}$.

For a given embedding $\phi : V_0 \hookrightarrow \mathrm{int}(D^{2k+1})$, we can take a (vertical) framing on each piece $V_0 \times D^{k-1}$ which is the pullback of τ_ϕ under the projection $V_0 \times D^{k-1} \to V_0$. Let τ_{std} be the restriction of this framing on $V_0 \times \partial D^{k-1}$. Now the problem is to show that the obstruction to homotoping the section of the principal SO_{2k+1}-product bundle $SO_{2k+1} \times D^{k-2}$ given by the fiberwise transformations $\varphi_t^*\tau_{\mathrm{std}} \circ \tau_{\mathrm{std}}^{-1}$ into the identity section is torsion.

The primary obstruction to extending a given homotopy on the $(p-1)$-skeleton of $V_0 \times D^{k-2}$ to the p-skeleton lies in $C^p(V_0 \times D^{k-2}, \partial(V_0 \times D^{k-2}); \pi_p(SO_{2k+1}))$. By taking a direct product of D^{k-2} with the natural cellular decomposition $V_0 = \partial V_0 \cup (D^{k+1} \cup D^{k+1} \cup D^{k+1}) \cup D^{2k+1}$, we have a relative cellular decomposition

$$V_0 \times D^{k-2} = \partial(V_0 \times D^{k-2}) \cup (D^{k+1} \times D^{k-2})^{\cup 3} \cup (D^{2k+1} \times D^{k-2})$$

and thus the relative cellular chain complex $C_*(V_0 \times D^{k-2}, \partial(V_0 \times D^{k-2}))$ is given as follows:

$$C_p(V_0 \times D^{k-2}, \partial(V_0 \times D^{k-2})) = \begin{cases} \mathbb{Z} & p = 3k-1 \\ \mathbb{Z}^{\oplus 3} & p = 2k-1 \\ 0 & \text{otherwise} \end{cases}$$

Extension over the $(2k-1)$*-skeleton*: The first obstruction lies in $\mathrm{Hom}(\mathbb{Z}^{\oplus 3}, \pi_{2k-1}(SO_{2k+1}))$ that is isomorphic to $\mathbb{Z}_2^{\oplus 3}$ or 0 when k is odd. So a two times iteration of $V_0 \times D^{k-2}$ kills this obstruction. This can be done by a multiplication by 2 to the given (V_0,∂)-bundle $\widetilde{V}_0 \to S^{k-1}$.

Extension over the $(3k-1)$*-skeleton*: Assume that the homotopy is already extended over the $(2k-1)$-skeleton by a multiplication by 2 if necessary. Then the next obstruction lies in $\mathrm{Hom}(\mathbb{Z}, \pi_{3k-1}(SO_{2k+1}))$. Since k is odd, the group $\pi_{3k-1}(SO_{2k+1})$ is finite. Therefore, a multiplication by r_k to the given (V_0,∂)-bundle turns all the obstruction into 0. $\square$

Remark 4.3. We have another proof of the existence of the lifting using the based version of the Federer spectral sequence [Smi] computing $\pi_*\big(\mathrm{Map}_*((V_0,\partial V_0),(SO_{2k+1},1))\big) \otimes \mathbb{Q}$, which is shorter than (but essentially equivalent to) that given above. Namely, the sum of the E_2-terms converging to $\pi_{k-2}\big(\mathrm{Map}_*((V_0,\partial V_0),(SO_{2k+1},1))\big) \otimes \mathbb{Q}$ is isomorphic to

$$\Big[(\pi_{2k-1}(SO_{2k+1}))^{\oplus 3} \oplus \pi_{3k-1}(SO_{2k+1})\Big] \otimes \mathbb{Q} = 0$$

for k odd. This is the π_{k-2} of the homotopy fiber of the fibration $\widetilde{B\mathrm{Diff}}(V_0,\partial) \to B\mathrm{Diff}(V_0,\partial)$. Hence the homotopy sequence implies the surjectivity of the map

$$\pi_{k-1}\big(\widetilde{B\mathrm{Diff}}(V_0,\partial)\big) \otimes \mathbb{Q} \to \pi_{k-1}\big(B\mathrm{Diff}(V_0,\partial)\big) \otimes \mathbb{Q}.$$

$\square$

Applying Proposition 4.2 to $x = \delta(\alpha')$, we get an element $\alpha \in \pi_{k-1}(\widetilde{B\mathrm{Diff}}(V_0, \partial; \phi))$. We will see later in Corollary 5.4 that α is indeed nontrivial.

4.2. **Surgery.** Let A, B be compact oriented smooth manifolds with boundary such that $\partial A = -\partial B \sqcup P$ for some components P (assume that an embedding $-\partial B \hookrightarrow \partial A$ is fixed). Let $C := A \cup_{\partial B} B$ (gluing map is given by the embedding of ∂B) and assume that C has a tangent framing. We will define a map

$$m : \widetilde{B\mathrm{Diff}}(A, \partial; \tau_C|_A) \times \widetilde{B\mathrm{Diff}}(B, \partial; \tau_C|_B) \to \widetilde{B\mathrm{Diff}}(C, \partial; \tau_C)$$

for some framing $\tau_C : TC \to \mathbb{R}^{\dim C} \times C$. m will be used for cut-and-paste construction of a framed (C, ∂)-bundle out of a framed (A, ∂)-bundle and a framed (B, ∂)-bundle.

Considering ∂B as a subspace of C, we have the natural homeomorphisms:

$$\begin{aligned} D &: \mathrm{Diff}(C, \partial C \cup \partial B) \cong \mathrm{Diff}(A, \partial) \times \mathrm{Diff}(B, \partial), \\ F &: \mathrm{Fr}(C, \partial C \cup \partial B; \tau_C) \cong \mathrm{Fr}(A, \partial; \tau_C|_A) \times \mathrm{Fr}(B, \partial; \tau_C|_B). \end{aligned}$$

Let us write as G_A, G_B, G'_C for the groups $\mathrm{Diff}(A, \partial)$, $\mathrm{Diff}(B, \partial)$, $\mathrm{Diff}(C, \partial C \cup \partial B)$ respectively and write as Fr_A, Fr_B, Fr'_C for $\mathrm{Fr}(A, \partial; \tau_C|_A)$, $\mathrm{Fr}(B, \partial; \tau_C|_B)$, $\mathrm{Fr}(C, \partial C \cup \partial B; \tau_C)$ respectively. The homeomorphism F is G'_C-equivariant where the G'_C-action on $\mathrm{Fr}_A \times \mathrm{Fr}_B$ is defined through D. So F induces a homeomorphism

$$\widetilde{B\mathrm{Diff}}(C, \partial C \cup \partial B; \tau_C) = EG'_C \times_{G'_C} \mathrm{Fr}'_C \xrightarrow{\cong} EG'_C \times_{G'_C} (\mathrm{Fr}_A \times \mathrm{Fr}_B).$$

Moreover, D induces a G'_C-equivariant homotopy equivalence $EG'_C \simeq EG_A \times EG_B$ and hence D also induces a homotopy equivalence on the associated bundles

$$\begin{aligned} EG'_C \times_{G'_C} (\mathrm{Fr}_A \times \mathrm{Fr}_B) &\simeq (EG_A \times EG_B) \times_{(G_A \times G_B)} (\mathrm{Fr}_A \times \mathrm{Fr}_B) \\ &= \widetilde{B\mathrm{Diff}}(A, \partial; \tau_C|_A) \times \widetilde{B\mathrm{Diff}}(B, \partial; \tau_C|_B). \end{aligned}$$

Therefore we get a homotopy equivalence

$$m' : \widetilde{B\mathrm{Diff}}(A, \partial; \tau_C|_A) \times \widetilde{B\mathrm{Diff}}(B, \partial; \tau_C|_B) \simeq \widetilde{B\mathrm{Diff}}(C, \partial C \cup \partial B; \tau_C).$$

Composition of m' with the natural map $i : \widetilde{B\mathrm{Diff}}(C, \partial C \cup \partial B; \tau_C) \to \widetilde{B\mathrm{Diff}}(C, \partial C; \tau_C)$ gives the desired map m.

4.3. **Multilinear construction β.** Multiple applications of the surgery map m at once will yield interesting framed bundles. This is done as follows. Suppose given an embedding

$$\phi = \phi_1 \cup \cdots \cup \phi_N : V_1 \cup \cdots \cup V_N \hookrightarrow \mathrm{int}(C)$$

where $V_1 \cup \ldots \cup V_N$ is a disjoint union of copies of V_0. Then one can apply the map m for $B = \bigcup_{i=1}^N \phi_i(V_i)$, $A = C \setminus \mathrm{int}(B)$ to get a map

$$m(\phi) : \widetilde{B\mathrm{Diff}}(A, \partial; \tau_C|_A) \times \prod_{i=1}^N \widetilde{B\mathrm{Diff}}(V_i, \partial; \tau_{\phi_i}) \to \widetilde{B\mathrm{Diff}}(C, \partial; \tau_C)$$

where $\tau_{\phi_i} = \phi_i^* \tau_C$.

Now let us assume that C is the unit $(2k+1)$-disk $D^{2k+1} \subset \mathbb{R}^{2k+1}$ and that τ_C is the standard framing, which is induced from the standard framing on $\mathbb{R}^{2k+1}$. For an odd integer $k \geq 3$ and for any maps $\alpha^{(i)} : S^{k-1} \to \widetilde{B\mathrm{Diff}}(V_i, \partial; \tau_{\phi_i})$, we define a map

$$f(\phi) : (S^{k-1})^{\times 2n} \to \widetilde{B\mathrm{Diff}}(D^{2k+1}, \partial)$$

by

$$f(\phi)(t_1, \ldots, t_{2n}) := m(\phi)(* \times \alpha^{(1)}(t_1) \times \cdots \times \alpha^{(2n)}(t_{2n})).$$

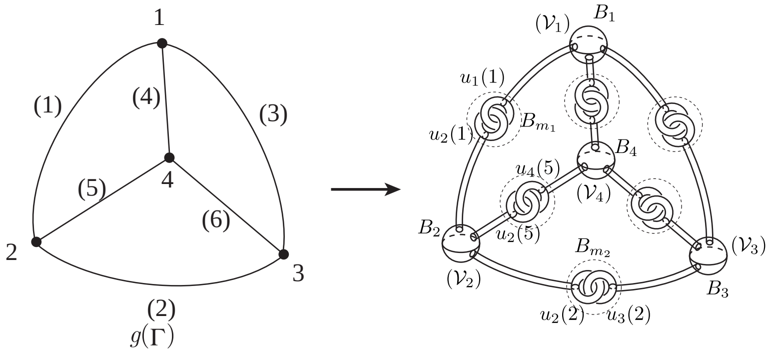

FIGURE 3. Associating an embedding $\phi_\Gamma : V_1\cup\cdots\cup V_{2n} \hookrightarrow D^{2k+1}$ to a trivalent graph Γ.

Then the correspondence $\alpha^{(1)} \times \cdots \times \alpha^{(2n)} \mapsto [f(\phi)]$ defines a (multilinear) map

$$\beta : \prod_{i=1}^{2n} \pi_{k-1}(\widetilde{B\mathrm{Diff}}(V_i, \partial; \tau_{\phi_i})) \to \pi_0\Big(\mathrm{Map}_*\big((S^{k-1})^{\times 2n}, \widetilde{B\mathrm{Diff}}(D^{2k+1}, \partial)\big)\Big).$$

Note that the right hand side is a group because $\widetilde{B\mathrm{Diff}}(D^{2k+1}, \partial)$ has the homotopy type of an H-space.

4.4. **Special embedding ϕ_Γ from graphs.** For each labeled 3-valent graph $\Gamma \in \mathcal{G}_{2n,3n}$, we define an embedding

$$\phi_\Gamma : V_1 \cup \cdots \cup V_{2n} \hookrightarrow \mathrm{int}(D^{2k+1})$$

where $V_1, \ldots, V_{2n}$ are diffeomorphic copies of V_0, as follows.

Step 1: Fix an embedding $g : \Gamma \hookrightarrow \mathrm{int}(D^{2k+1})$.

Step 2: For each vertex i, let $B_i \subset \mathrm{int}(D^{2k+1})$ be a small $(2k+1)$-disk around $g(i)$. For each edge e and its middle point m_e (with respect to a fixed metric on edges), let $B_{m_e} \subset \mathrm{int}(D^{2k+1})$ be a small $(2k+1)$-disk around $g(m_e)$. Inside B_{m_e}, consider a closed tubular neighborhood of a Hopf link:

$$u_i(e) \cup u_j(e) : (D^{k+1} \times S^k) \cup (D^{k+1} \times S^k) \hookrightarrow \mathrm{int}(B_{m_e}), \quad e = (i, j)$$

which is small enough. We orient the Hopf link so that the linking number is $+1$.

Step 3: Connect these objects B_i, $u_i(e) = \mathrm{Im}(u_i(e))$ by thin bands of the form $I \times D^{2k}$ along the embedded graph. More precisely, if a vertex i belongs to an edge e, then connect B_i and $u_i(e)$ by a band $\ell_{i,e} \cong I \times D^{2k}$. Here, $\{0\} \times D^{2k}$ will be attached to ∂B_i and $\{1\} \times D^{2k}$ will be attached to $\partial u_i(e)$. Assume that two different bands do not intersect each other and that the band $\ell_{i,e}$ intersects $\bigcup_i B_i \cup \bigcup_{e,i} u_i(e)$ only at $\partial B_i \cup \partial u_i(e)$. The result of all these attachments forms a disjoint union of $2n$ submanifolds $\mathcal{V}_1 \cup \cdots \cup \mathcal{V}_{2n}$ of D^{2k+1} each diffeomorphic to V_0.

Step 4: Recall from Remark 2.2 that an orientation of a labeled trivalent graph is canonically determined if the cyclic order of the three half-edges around each vertex is given. Fix cyclic orders of the half-edges around vertices which are compatible with the graph orientation and fix a diffeomorphism $h_i : V_0 \to \mathcal{V}_i$ for each i so that the cyclically ordered basis (b_1^0, b_2^0, b_3^0) of $H_k(V_0)$ which corresponds to the small meridianal k-spheres to the three components of the standard inclusions $D^k \cup D^k \cup D^k \hookrightarrow D^{2k+1}$, each of which links to a (oriented) component with linking number $+1$, is taken by

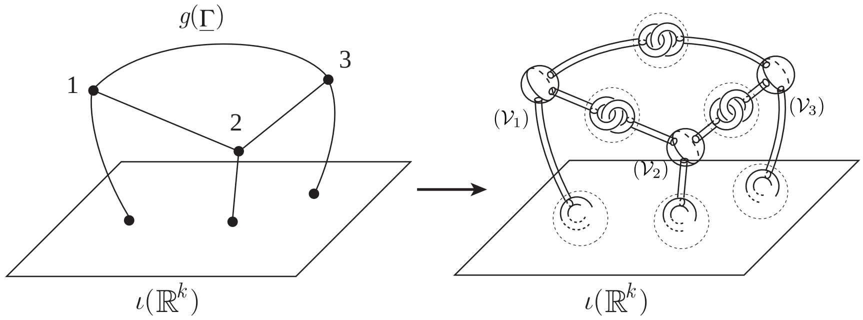

FIGURE 4. $\phi_{\underline{\Gamma}}$ for a uni-trivalent graph $\underline{\Gamma}$.

h_* to the cyclically ordered basis (b_1^i, b_2^i, b_3^i) of $H_k(\mathcal{V}_i)$ represented by the (oriented) components of the Hopf links appeared in Step 2. This gives an embedding

$$\phi_\Gamma : V_1 \cup \cdots \cup V_{2n} \hookrightarrow \mathrm{int}(D^{2k+1}).$$

Definition 4.4. By using the element $\alpha \in \pi_{k-1}(\widetilde{B\mathrm{Diff}}(V_0, \partial; \phi))$ given in §4.1 for any ϕ, we define a linear map

$$\psi_{2n} : \mathcal{G}_{2n,3n} \to \Omega_{2n(k-1)}(\widetilde{B\mathrm{Diff}}(D^{2k+1}, \partial)) \otimes \mathbb{Q}$$

by $\psi_{2n}(\Gamma) := \big[\beta\big((h_1)_*\alpha \times \cdots \times (h_{2n})_*\alpha\big)\big] \otimes 1$. The target can be replaced by $\Omega_{2n(k-1)}(\widetilde{B\mathrm{Diff}}(D_M, \partial)) \otimes \mathbb{Q}$ by composing the inclusion induced map. Let $(\pi^\Gamma : E^\Gamma \to (S^{k-1})^{\times 2n}, \tau^\Gamma)$ denote the framed (D_M, ∂)-bundle over $(S^{k-1})^{\times 2n}$ corresponding to $\mathrm{incl}_*\beta\big((h_1)_*\alpha \times \cdots \times (h_{2n})_*\alpha\big)$. $\square$

Because the lifting $\delta(\alpha') \mapsto \alpha$ is not necessarily unique, the definition of ψ_{2n} may not be unique as well.

4.5. **Application to long embedding spaces.** We shall briefly sketch an application of the surgery construction above to the long embedding spaces. This part is not necessary for the proof of the main theorem so can be skipped.

By considering uni-trivalent graphs $\underline{\Gamma}$ instead of trivalent graphs, we can apply a similar construction for (long) embedding spaces $\mathrm{Emb}(\mathbb{R}^k, \mathbb{R}^{2k+1})$. Namely, remove some components from $\mathcal{V}_1 \cup \cdots \cup \mathcal{V}_{2n} \subset D^{2k+1}$ (and rename as $\mathcal{V}_1 \cup \cdots \cup \mathcal{V}_{n'}$), unlink some of the Hopf links in the definition of ϕ_Γ and make them link meridianally with the standardly included plane $\iota : \mathbb{R}^k \subset \mathbb{R}^{2k+1}$. See Figure 4. Then the fibered surgery along $\mathcal{V}_1 \cup \cdots \cup \mathcal{V}_{n'}$ as above yields an $\mathbb{R}^{2k+1}$-bundle

$$\pi^{\underline{\Gamma}} : E(\pi^{\underline{\Gamma}}) \to (S^{k-1})^{\times n'}$$

over $(S^{k-1})^{\times n'}$ which is standard outside a ball $B^{2k+1}(R)$ of some large constant radius $R > 0$ around the origin, with the plane $\iota(\mathbb{R}^k)$ included fiberwise.

Lemma 4.5. *The $(\mathbb{R}^{2k+1}, \mathbb{R}^{2k+1} \setminus B^{2k+1}(R))$-bundle $\pi^{\underline{\Gamma}}$ is fiberwise isotopic to the trivial one when $\underline{\Gamma}$ is a connected uni-trivalent graph with at least one univalent vertex.*

Proof. The proof is done by induction on the number n'.

$n' = 1$ *case*: Consider the two step extension $\mathcal{V}_1 \overset{i_1}{\subset} D^{2k+1} \overset{i_2}{\subset} \mathbb{R}^{2k+1}$ of the fibers. Recall that the (V_0, ∂)-bundle over S^{k-1} corresponding to α was defined as the family of complements of a family of framed embeddings $\phi_t : D^k \hookrightarrow D^{2k+1} \setminus (D^k \cup D^k)$. The first extension i_1 corresponds under the trivialization on $\partial\mathcal{V}_1$ to gluing back of a trivial D^{k+1} bundle of $D^k \cup D^k \cup D^k$ to the complement of the framed embeddings ϕ_t (see (4.3)). This is same as just to forget the framed embeddings from the ambient space D^{2k+1}. So this already gives a trivial unframed (D^{2k+1}, ∂)-bundle. Then one gets a trivial $(\mathbb{R}^{2k+1}, \mathbb{R}^{2k+1} \setminus B^{2k+1}(R))$-bundle after the second extension, i.e., the bundle $\pi^{\underline{\Gamma}} : E(\pi^{\underline{\Gamma}}) \to S^{k-1}$, which corresponds to $\beta((h_1)_*\alpha)$, is trivial.

$n' > 1$ *general case*: Let us assume that the claim is true for $n'-1$, namely, for any uni-trivalent graph $\underline{\Gamma}$ with $n'-1$ trivalent vertices the bundle

$$\pi^{\underline{\Gamma}} : E(\pi^{\underline{\Gamma}}) \to (S^{k-1})^{\times n'-1}$$

is fiberwise isotopic to the trivial. Now consider the bundle $\pi^{\underline{\Gamma'}}$ for a uni-trivalent graph $\underline{\Gamma'}$ with n' trivalent vertex. Choose a univalent vertex u which is next to a trivalent vertex v and remove from $\underline{\Gamma'}$ the (half-)edge (u,v) and the half-edges adjacent to v. The result is another uni-trivalent graph with $n'-1$ trivalent vertices. This corresponds to removing of the component $\mathcal{V}_u$ from $\mathcal{V}_1 \cup \cdots \cup \mathcal{V}_{n'}$. So $\pi^{\underline{\Gamma'}}$ restricted to $(S^{k-1})^{\times n'-1} \subset (S^{k-1})^{\times n'}$ is fiberwise isotopic to the trivial by the hypothesis. In the complemental directions, we use an extended version of the $n'=1$ case. Namely, surgery along the sub $\mathcal{V}_u$-bundle on $\mathbb{R}^{2k+1} \setminus (\mathcal{V}_1 \cup \cdots \cup \mathcal{V}_{n'} \setminus \mathcal{V}_u)$ is fiberwise isotopic to the previous state, i.e., the surgery changes nothing, when one of the three holes of $\mathcal{V}_u$ is *free*, i.e., the interior of one of the $(k+1)$-disk $S(b^u_\ell)$ is disjoint from other handlebodies or from $\mathcal{V}_u$ itself. This follows from the fact that the removing of the free hole corresponds to the removing of one component from a framed embedding of $\mathrm{Emb}^{\mathrm{f}}(D^k \cup D^k \cup D^k, D^{2k+1})$, and the fact that the Borromean rings has the property that if one of the components is removed then the result is isotopic to the trivial one. □

Applying the fiberwise isotopy of Lemma 4.5 to the pair $(\pi^{\underline{\Gamma}}; \iota)$, we obtain a family of embeddings $\mathbb{R}^k \hookrightarrow \mathbb{R}^{2k+1}$ parametrized by $(S^{k-1})^{\times n'}$, which gives a cycle on $\mathrm{Emb}(\mathbb{R}^k, \mathbb{R}^{2k+1})$.

5. Normalization of propagator

The goal of this section is to state Proposition 5.2 about the existence of a propagator of a convenient form for the computation of configuration space integrals.

5.1. **Thom class.** We will use differential form representatives of the Thom classes to give an explicit description of a cohomology class. Recall that the Thom class of the normal bundle $E(\nu_A) \to A$ of a codimension i oriented submanifold $A \subset N$ of an oriented manifold N is a cohomology class

$$\eta_A \in H^i_{cv}(E(\nu_A))$$

of the de Rham cohomology of vertically compact support i-forms. This depends on the orientation of the normal direction, which is determined by the orientation of A and of N by the rule (tangent ori) $\wedge$ (normal ori) $=$ (ori of N). We will denote a differential form representative of η_A again by η_A. By the canonical identification of $E(\nu_A)$ with the tubular neighborhood $\mathrm{Tub}_\varepsilon(A)$ of small radius ε with respect to a fixed Riemannian metric on N, the Thom class of ν_A can also be considered as a cohomology class of $H^i_{cv}(\mathrm{Tub}_\varepsilon(A))$ and moreover as a cohomology class of $H^i(N)$. One of the significant properties of the Thom class is that $\eta_A = \mathrm{PD}([A]) \in H^i(N)$. A basic reference is [BT, Ch. I §6] for a detail.

5.2. **Partial explicit formula for normalized** ω**.** Let $\mathcal{V}_i := \phi_\Gamma(V_i) \subset \mathrm{int}(D^{2k+1})$, $i = 1, \ldots, 2n$. There is a standard system of generating cycles $a^i_k, b^i_k \cong S^k$, $k = 1,2,3$, of $H_k(\partial\mathcal{V}_i;\mathbb{Z}) \cong H_k((S^k \times S^k)^{\#3};\mathbb{Z})$ such that

- $a^i_k \cdot a^i_\ell = 0, \quad b^i_k \cdot b^i_\ell = 0, \quad a^i_k \cdot b^i_\ell = \delta_{k\ell}.$
- b^i_k, $k = 1,2,3$ corresponds to the standard ordered basis of $H_k(V_0)$ under $(h_i)_*$.

Each cycle a^i_k spans a $(k+1)$-disk $S(a^i_k)$ inside $\mathcal{V}_i$ and each cycle b^i_k spans a $(k+1)$-disk $S(b^i_k)$ outside $\mathcal{V}_i$. See Figure 5. We assume that the k-handles of $\mathcal{V}_j$ $(j \neq i)$ intersects $S(b^i_k)$ transversely at finitely many (in fact, at most one is sufficient) $(k+1)$-disks in $\mathrm{int}(S(b^i_k))$. We put $\eta(a^i_k) := \eta_{S(a^i_k)} \in \Omega^k_{\mathrm{dR}}(\mathcal{V}_i)$. We will also write a^i_k, b^i_ℓ, $S(a^i_k)$, $S(b^i_\ell)$, $\eta(a^i_k)$ etc. for $h_i^{-1}(a^i_k)$, $h_i^{-1}(b^i_\ell)$, $h_i^{-1}(S(a^i_k))$, $h_i^{-1}(S(b^i_\ell))$, $h_i^*(\eta(a^i_k))$ etc. in V_i.

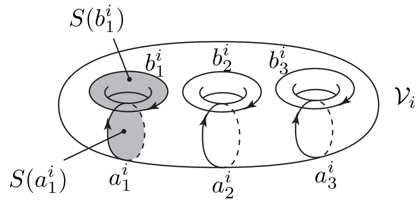

FIGURE 5. Standard generators of $H_k(\partial\mathcal{V}_i)$.

We write $V = V_i$ and denote by $\pi_\alpha : \widetilde{V} \to S^{k-1}$ the framed (V,∂)-bundle corresponding to $\alpha \in \pi_{k-1}(\widetilde{B\mathrm{Diff}}(V,\partial;\phi))$. For $a = a^i_k$, let

$$\widetilde{a} := a \times S^{k-1} \subset \partial\widetilde{V} = \partial V \times S^{k-1}.$$

Lemma 5.1. *There exists a compact oriented smooth submanifold $S(\widetilde{a})$ of $\widetilde{V}$ with boundary such that*

(1) $\partial S(\widetilde{a}) = \widetilde{a} \subset \partial\widetilde{V}$.
(2) $S(\widetilde{a}) \cap \partial\widetilde{V} = \widetilde{a}$. *The intersection is transversal.*
(3) $S(\widetilde{a}) \cap (\pi_\alpha)^{-1}(t^0) = S(a)$ *over the base point* $t^0 \in S^{k-1}$.

Proof. Recall that α was constructed from the $(k-1)$-parameter family of embeddings

$$\phi_t : D^k \hookrightarrow D^{2k+1} \setminus (D^k \cup D^k), \quad t \in D^{k-1}$$

that are standard where t is outside a ball $U_0 \subset D^{k-1}$. Here we fix the standard family of embeddings explicitly as follows. The fixed inclusion $D^k \cup D^k \subset D^{2k+1}$ (2nd and 3rd components) is the same as those of (4.2). The standard family of embeddings of the first component is ϕ^0_t in the proof of Lemma 4.1, i.e., given by

$$\text{(1st component)}: (x_1,\ldots,x_k) \mapsto (x_1,\ldots,x_k,y_1,\ldots,y_{k-1},-10,0)$$

where $(y_1,\ldots,y_{k-1}) \in D^{k-1}$. Namely, we say that the family $\{\phi_t\}_t$ is standard outside $U_0 \subset D^{k-1}$ if it agrees with the standard family there. By Lemma 4.1 this is indeed the trivial one up to a fiberwise isotopy.

We consider the fibered representation of the family $\phi_* = \{\phi_t\}$, i.e, a graph over the parameter space D^{k-1}:

$$\begin{aligned} G(\phi_*) : D^k \times D^{k-1} &\hookrightarrow (D^{2k+1} \setminus (D^k \cup D^k)) \times D^{k-1} = D^{3k} \setminus (D^{2k-1} \times D^{2k-1}) \\ (x_1,\ldots,x_k;y_1,\ldots,y_{k-1}) &\mapsto (\phi_{t=(y_1,\ldots,y_{k-1})}(x_1,\ldots,x_k);y_1,\ldots,y_{k-1}). \end{aligned}$$

Let $K \subset D^{2k+1} \times D^{k-1} = D^{3k} = [-100,100]^{\times 3k}$ be the image of $G(\phi_*)$. Let

$$\begin{aligned} S &:= \{(x,y,-10,-100) \mid x \in D^k,\ y \in D^{k-1}\}, \\ T &:= \{(x',y,-10,h) \mid x' \in \partial D^k,\ y \in D^{k-1},\ -100 \le h \le 0\} \\ &\quad \cup \{(x,y',-10,h) \mid x \in D^k,\ y' \in \partial D^{k-1},\ -100 \le h \le 0\}, \\ K_0 &:= \{(x,y,-10,0) \mid x \in D^k,\ y \in D^{k-1}\}, \end{aligned}$$

which are defined independently of ϕ_*. Note that K_0 is the image of the first component of the standard family of embeddings. To complete the proof, it is enough to find a compact oriented submanifold $W \subset D^{3k} \setminus (D^{2k-1} \cup D^{2k-1})$ with

$$\partial W = K \cup S \cup T. \tag{5.1}$$

To find W, we deform $G(\phi_*)$ by an isotopy to a map $H(\phi_*)$ which is not necessarily fiber-preserving, with the following property:

- The image L of $H(\phi_*)$ is standard, i.e., agrees with K_0, on $(D^{2k+1} \setminus (D^k \cup D^k)) \times (D^{k-1} \setminus \{0\})$ and

- L forms a long Borromean rings of dimensions $(2k-1,k,k)$ on $(D^{2k+1}\setminus(D^k\cup D^k))\times\{0\}$.

Recall that ϕ_t is nonstandard only for $t\in U_0\subset D^{k-1}$. Choose a closed $(k-1)$-disk $U_0'\subset D^{k-1}$ including U_0 inside and take a 1-parameter family of maps $u_s:U_0'\to U_0'$, $0\le s\le 1$ such that (i) $u_0=\mathrm{id}$, (ii) $u_s|_{\partial U_0'}=\mathrm{id}$, (iii) $u_1|_{U_0}$ is the constant map to 0, and extend to $u_s:D^{k-1}\to D^{k-1}$ by the identity on $D^{k-1}\setminus U_0'$. Then let

$$H(\phi_*):D^k\times D^{k-1}\hookrightarrow D^{3k}\setminus(D^{2k-1}\cup D^{2k-1})$$

be the resulting embedding (at $s=1$) of the 1-parameter deformation:

$$(x_1,\ldots,x_k;y_1,\ldots,y_{k-1})\mapsto(\phi_{(y_1,\ldots,y_{k-1})}(x_1,\ldots,x_k);u_s(y_1,\ldots,y_{k-1})),\ 0\le s\le 1.$$

By the definition of ϕ_* this is indeed a 1-parameter family of embeddings. So $H(\phi_*)$ is as required above.

It is known (e.g., [Tak, §3.3]) that each one of the three components $P\cup Q\cup R\subset\mathbb{R}^n$ in a Borromean rings, say P, has a manifold $W'\subset\mathbb{R}^n\setminus(Q\cup R)$ with $\partial W'=P$. By the observation above, a trivial enclosing in ∂D^{3k} of the three components $(L\cup S\cup T)\cup D^{2k-1}\cup D^{2k-1}$ forms a Borromean rings of dimensions $(2k-1,2k-1,2k-1)$ in $\mathbb{R}^{3k}$ after a suitable roundings of corners. So there is a compact oriented manifold $W'\subset D^{3k}\setminus(D^{2k-1}\cup D^{2k-1})$ such that $\partial W'=L\cup S\cup T$. Hence we have obtained W satisfying (5.1). ☐

Notice that $S(\widetilde{a})$ is not a subbundle of π_α, but just a submanifold of the manifold $\widetilde{V}$. In the following $S(\widetilde{a})$ for $a=a_k^i$ will denote the manifold W chosen in the proof of Lemma 5.1. We put $\eta(a,t):=\eta_{S(\widetilde{a})}|_t$ for $t\in S^{k-1}$.

Proposition 5.2 (Normalization)**.** *There is a propagator $\omega\in\Omega^{2k}_{\mathrm{dR}}(E\overline{C}_2(\pi^\Gamma))$ satisfying the following conditions:*

(1) *Let $t=(t_1,\ldots,t_{2n}),t'=(t_1',\ldots,t_{2n}')\in(S^{k-1})^{\times 2n}$ be such that $t_j=t_j'$ for all $j\in J\subset\{1,2,\ldots,2n\}$. Then $\omega(t_1,\ldots,t_{2n})=\omega(t_1',\ldots,t_{2n}')$ where it makes sense, namely on*

$$\overline{C}_2\Big((M\setminus\bigcup_{i=1}^{2n}\mathrm{int}(\mathcal{V}_i))\cup\bigcup_{j\in J}(\mathcal{V}_j)_t,\infty\Big)\qquad(\widetilde{\mathcal{V}}_j=\bigcup_t(\mathcal{V}_j)_t).$$

(2) *On $(\mathcal{V}_i\times\mathcal{V}_j)_t$,*

$$\omega(t)=\sum_{1\le k,\ell\le 3}\mathrm{Lk}(b_k^i,b_\ell^j)\,\mathrm{pr}_1^*\eta(a_k^i,t_i)\wedge\mathrm{pr}_2^*\eta(a_\ell^j,t_j)$$

where $\mathrm{pr}_i:\overline{C}_2(M_t,\infty)\to\overline{C}_1(M_t,\infty)$ is the i-th projection.

Proof of Proposition 5.2 will be given in §7.

5.3. Non-triviality of α.

Lemma 5.3. *Let $\pi_\alpha:\widetilde{V}_0\to S^{k-1}$ be the framed (V_0,∂)-bundle corresponding to $\alpha\in\pi_{k-1}(\widetilde{B\mathrm{Diff}}(V_0,\partial))$. Then*

$$\int_{\widetilde{V}_0}\eta_{S(\widetilde{a}_1)}\wedge\eta_{S(\widetilde{a}_2)}\wedge\eta_{S(\widetilde{a}_3)}=r_k. \tag{5.2}$$

Proof. For a (closed) Borromean rings $P_1\cup P_2\cup P_3\subset\mathbb{R}^n$, there are compact oriented manifolds $W_j\subset\mathbb{R}^n\setminus(\bigcup_{k\neq j}P_k)$ with $\partial W_j=P_j$ as in the proof of Lemma 5.1, such that the triple intersection $W_1\cap W_2\cap W_3$ is a transversal single point (see [Tak, §3.3]). Thus the integral of the lemma is equal to the number of the triple intersection points of $S(\widetilde{a}_1)$, $S(\widetilde{a}_2)$ and $S(\widetilde{a}_3)$. Since we have chosen (in §4.4, Step 4) canonical orientations for a_1,a_2,a_3, it follows that the sign is $+1$. (This agrees with the sign of the triple intersection of the three spanning disks, considered in §4.1, by the construction of W_j of [Tak].) Recall that in the definition of α in

§4.1 (in particular, in Proposition 4.2) we made a multiple $r_k\delta(\alpha')$. So α is made from r_k copies of Borromean rings. □

Corollary 5.4. *α is nontrivial.*

Because the vertical framing was not necessary to define $\eta_{S(\widetilde{a})}$, α has nontrivial image in $\pi_{k-1}(B\mathrm{Diff}(V_0,\partial))$.

Proof of Corollary 5.4. A change $\eta_{S(\widetilde{a}_j)}\mapsto\eta_{S(\widetilde{a}_j)}+d\mu$ on the complement E of an open collar neighborhood of $\partial\widetilde{V}\subset\widetilde{V}$ by an exact form $d\mu$ with μ closed on ∂E, does not change the integral of the LHS in (5.2). It is proved as follows. Note that the closed form $\eta_{S(\widetilde{a}_2)}\wedge\eta_{S(\widetilde{a}_3)}$ vanishes on ∂E and hence represents a relative cohomology class of $H^{2k}(E,\partial E)$. Hence

$$\int_{\widetilde{V}}(\eta_{S(\widetilde{a}_3)}+d\mu)\wedge\eta_{S(\widetilde{a}_2)}\wedge\eta_{S(\widetilde{a}_3)}-\int_{\widetilde{V}}\eta_{S(\widetilde{a}_3)}\wedge\eta_{S(\widetilde{a}_2)}\wedge\eta_{S(\widetilde{a}_3)}=\int_E d\mu\wedge\eta_{S(\widetilde{a}_2)}\wedge\eta_{S(\widetilde{a}_3)}=0$$

because the integral computes the cup product $H^k(E)\otimes H^{2k}(E,\partial E)\to H^{3k}(E,\partial E)$.

If α were trivial, then $\eta_{S(\widetilde{a}_i)}|_E$ must be related to $\eta_{S(a_i)\times S^{k-1}}|_E$ by additions of exact forms as above. Because $S(a_1)\cap S(a_2)\cap S(a_3)=\emptyset$, $\eta_{S(a_1)\times S^{k-1}}\wedge\eta_{S(a_2)\times S^{k-1}}\wedge\eta_{S(a_3)\times S^{k-1}}=0$. This contradicts Lemma 5.3. □

6. Evaluation

We state the main theorem here again.

Theorem 6.1. *Let k be an odd integer ≥ 3 and let $n\geq 1$. Let $r_k:=2o(\pi_{3k-1}(SO_{2k+1}))$. Then for a suitably defined ψ_{2n} the following holds.*

(1) *The diagram*

$$\begin{array}{ccc}\mathcal{G}_{2n,3n} & \xrightarrow{\psi_{2n}} & \Omega_{2n(k-1)}(\widetilde{B\mathrm{Diff}}(D_M,\partial))\otimes\mathbb{Q}\\ {\scriptstyle[\cdot]\otimes 1}\downarrow & & \downarrow{\scriptstyle\langle\zeta_{2n,3n},\cdot\rangle}\\ \mathcal{A}_{2n,3n}\otimes\mathbb{R} & \xrightarrow{\times r_k^{2n}} & \mathcal{A}_{2n,3n}\otimes\mathbb{R}\end{array}$$

is commutative.

(2) $\mathrm{Im}\,\psi_{2n}$ *is included in the image of the Hurewicz homomorphism* $\pi_{2n(k-1)}(\widetilde{B\mathrm{Diff}}(D_M,\partial))\otimes\mathbb{Q}\to\Omega_{2n(k-1)}(\widetilde{B\mathrm{Diff}}(D_M,\partial))\otimes\mathbb{Q}$.

To complete the proof of Theorem 6.1(1), we need to compute the integral

$$\int_{(S^{k-1})^{\times 2n}}I(\Gamma')(\pi^\Gamma;\tau^\Gamma)=\int_{(S^{k-1})^{\times 2n}}\int_{\mathrm{Fib}(\overline{C}_{2n}(\pi^\Gamma))}\omega(\Gamma')$$

for each $\Gamma'\in\mathcal{G}_{2n,3n}$.

Let $(U_i)_t\subset\overline{C}_{2n}(M_t,\infty)$, $t\in(S^{k-1})^{\times 2n}$ be the submanifold consisting of configurations such that all points avoid $(\mathcal{V}_i)_t$. We can also consider the family $\pi^\Gamma_{U_i}:\widetilde{U}_i\to(S^{k-1})^{\times 2n}$, $\widetilde{U}_i:=\bigcup_t(U_i)_t\subset E\overline{C}_{2n}(\pi^\Gamma)$.

Lemma 6.2. *For a propagator ω normalized as in Proposition 5.2 and for any $\Gamma'\in\mathcal{G}_{2n,3n}$, we have*

$$\int_{(t_1,\ldots,t_{2n})\in(S^{k-1})^{\times 2n}}\int_{\mathrm{Fib}(\pi^\Gamma_{U_i})}\omega(\Gamma')(t_1,\ldots,t_{2n})=0.$$

Proof. It is enough to consider only the case of $i=1$. By Proposition 5.2(1), we have

$$\omega(\Gamma')(t_1,t_2,\ldots,t_{2n})=\hat{\pi}_1^*\omega(\Gamma')(t_1^0,t_2,\ldots,t_{2n})$$

on $\widetilde{U}_1$ where t_i^0 denotes the base point of i-th S^{k-1} and $\hat{\pi}_1:S^{k-1}\times S^{k-1}\times\cdots\times S^{k-1}\to\{t_1^0\}\times S^{k-1}\times\cdots\times S^{k-1}$ is the projection. This is the pullback of a $6nk$-form on a codimension $k-1$ submanifold. Hence it vanishes. □

Thus it is enough to compute the integral over $Y := E\overline{C}_{2n}(\pi^\Gamma)\setminus\bigcup_i \widetilde{U}_i$. Because Y consists of configurations such that all $\mathcal{V}_i$ has one of the $2n$ points, it is a disjoint union of subbundles with fibers of the form

$$\mathcal{V}_1\times\cdots\times\mathcal{V}_{2n}. \tag{6.1}$$

There are $(2n)!$ such components corresponding to permutations of the $2n$ points. Let $\Pi\pi_\alpha : \widetilde{\Pi\mathcal{V}}\to(S^{k-1})^{\times 2n}$ be the component of Y with fiber (6.1).

Lemma 6.3. *For a propagator ω normalized as in Proposition 5.2,*

$$\int_{(S^{k-1})^{\times 2n}}\int_{\mathrm{Fib}(\Pi\pi_\alpha)}\omega(\Gamma')=\begin{cases}\pm|\mathrm{Aut}_e\Gamma|\,r_k^{2n} & \text{if } \Gamma'\cong\pm\Gamma\\ 0 & \text{otherwise}\end{cases} \tag{6.2}$$

Here $\mathrm{Aut}_e\Gamma$ *denotes the group of automorphisms of* Γ *fixing all vertices.*

Proof. According to Proposition 5.2(2), $\omega(\Gamma')$ restricted to $(\mathcal{V}_1\times\cdots\times\mathcal{V}_{2n})_t$, $t\in(S^{k-1})^{\times 2n}$ has the explicit form:

$$\omega(\Gamma')(t)=\bigwedge_{\substack{(i,j)\\ \text{edge of }\Gamma}}\Big(\sum_{1\le k,\ell\le 3}\ell^{ij}_{k\ell}\eta^i_k(t_i)\eta^j_\ell(t_j)\Big)$$

where $\ell^{ij}_{k\ell}=\mathrm{Lk}(b^i_k,b^j_\ell)$, $\eta^i_k(t_i)=\mathrm{pr}_i^*\eta(a^i_k,t_i)$. This is a linear sum of $6nk$-forms which are products of k-forms $\eta^i_k(t_i)$. Since we have assumed that k is odd, a non-vanishing product must be a multiple of the wedge of all $6n$ different $\eta^i_k(t_i)$'s, namely,

$$\omega(\Gamma')(t)=\pm\prod_{(i,j)}\Big(\sum_{(k_i,\ell_j)\in P_{ij}}\ell^{ij}_{k_i\ell_j}\Big)\bigwedge_{p=1}^{2n}(\eta^p_1\eta^p_2\eta^p_3)(t_p)$$

where $P_{ij}=\{1\le k_i,\ell_j\le 3\,|\,\ell^{ij}_{k_i\ell_j}\neq 0\}$. If Γ' is not isomorphic to Γ, then the coefficient $\prod_{(i,j)}\Big(\sum_{(k_i,\ell_j)\in P_{ij}}\ell^{ij}_{k_i\ell_j}\Big)$ must be zero because there is (i,j) such that $\ell^{ij}_{k\ell}=0$ for all k,ℓ. Hence we have

$$\omega(\Gamma')(t)=\begin{cases}\pm|\mathrm{Aut}_e\Gamma|\displaystyle\bigwedge_{p=1}^{2n}(\eta^p_1\eta^p_2\eta^p_3)(t_p) & \text{if } \Gamma'\cong\pm\Gamma\\ 0 & \text{otherwise}\end{cases}$$

Here the sign is explicitly determined because we have fixed cyclic orders of edges around each vertex so that it is compatible with the graph orientation and have used it in §4.4 to define ϕ_Γ.

Let $\pi_{\alpha,p}:\widetilde{\mathcal{V}}_p\to S^{k-1}$ be a copy of π_α corresponding to $\mathcal{V}_p$. Let $A_p(t_p):=\eta(a^p_1,t_p)\eta(a^p_2,t_p)\eta(a^p_3,t_p)\in\Omega^{3k}_{\mathrm{dR}}(\widetilde{\mathcal{V}}_p)$ so that $(\eta^p_1\eta^p_2\eta^p_3)(t_p)=\widehat{\mathrm{pr}}_p^*A_p(t_p)$ where $\widehat{\mathrm{pr}}_p:\widetilde{\Pi\mathcal{V}}\to\widetilde{\mathcal{V}}_p$ is the natural projection which makes the following diagram

$$\begin{tikzcd}
\prod_q\mathcal{V}_q \arrow[r] \arrow[d, "\mathrm{pr}_p"'] & \widetilde{\Pi\mathcal{V}} \arrow[r, "\Pi\pi_\alpha"] \arrow[d, "\widehat{\mathrm{pr}}_p"'] & (S^{k-1})^{\times 2n} \arrow[d, "\mathrm{pr}_p"] \\
\mathcal{V}_p \arrow[r] & \widetilde{\mathcal{V}}_p \arrow[r, "\pi_{\alpha,p}"] & S^{k-1}
\end{tikzcd}$$

commutative. *Key observation*: It is easy to check that the product of $\widehat{\mathrm{pr}}_p$'s gives a canonical diffeomorphism

$$\widehat{\mathrm{pr}}_1\times\cdots\times\widehat{\mathrm{pr}}_{2n}:\widetilde{\Pi\mathcal{V}}\xrightarrow{\cong}\widetilde{\mathcal{V}}_1\times\cdots\times\widetilde{\mathcal{V}}_{2n}$$

which is natural with respect to the orientation. Hence the integral of the LHS of (6.2) in the case $\Gamma'\cong\Gamma$ becomes

$$|\mathrm{Aut}_e\Gamma|\int_{\widetilde{\Pi\mathcal{V}}}\bigwedge_{p=1}^{2n}\widehat{\mathrm{pr}}_p^*A_p=|\mathrm{Aut}_e\Gamma|\int_{\widetilde{\mathcal{V}}_1\times\cdots\times\widetilde{\mathcal{V}}_{2n}}\bigwedge_{p=1}^{2n}A_p=|\mathrm{Aut}_e\Gamma|\prod_{p=1}^{2n}\int_{\widetilde{\mathcal{V}}_p}A_p=|\mathrm{Aut}_e\Gamma|\,r_k^{2n}$$

by Lemma 5.3. □

Proof of Theorem 6.1. Let us fix a propagator ω normalized as in Proposition 5.2. If $\Gamma' \cong \Gamma$ as an oriented graph and if Γ does not have an orientation reversing automorphism, then the same integral as in (6.2) is counted $|\mathrm{Aut}\,\Gamma|/|\mathrm{Aut}_e\Gamma|$ times (with the same sign). Hence by Lemma 6.3 and by Lemma 5.3 the integral of $\omega(\Gamma')$ over Y is

$$\frac{[\Gamma]}{|\mathrm{Aut}\,\Gamma|}\frac{|\mathrm{Aut}\,\Gamma|}{|\mathrm{Aut}_e\Gamma|}|\mathrm{Aut}_e\Gamma| r_k^{2n} = r_k^{2n}\,[\Gamma].$$

If Γ has an orientation reversing automorphism, then the sum of the integrals over the $(2n)!$ components cancels in pairs. But in this case $[\Gamma]=0$. As was shown in Lemma 6.3, $\int_{(S^{k-1})^{\times 2n}} I(\Gamma')(\pi^\Gamma;\tau^\Gamma)=0$ when Γ' is not isomorphic to Γ. Therefore in any case we have

$$\left\langle \zeta_{2n}(\pi^\Gamma;\tau^\Gamma), [(S^{k-1})^{\times 2n}]\right\rangle = r_k^{2n}\,[\Gamma].$$

The assertion (1) is proved.

For the assertion (2) we first claim that the pullback framed bundle of π^Γ

$$\pi^\Gamma(i): E^\Gamma(i) = \mathrm{incl}_i^* E^\Gamma \to S^{k-1}$$

by the inclusion $\mathrm{incl}_i : S^{k-1} \subset (S^{k-1})^{\times 2n}$ into the i-th factor is a trivial (D_M,∂)-bundle as an unframed one. The reason is the same as Lemma 4.5. In other words, the classifying map $\tilde{f}(\pi^\Gamma(i)) : S^{k-1} \to \widetilde{B\mathrm{Diff}}(D_M,\partial)$ for $\pi^\Gamma(i)$ projects to a nullhomotopic map $f(\pi^\Gamma(i)) : S^{k-1} \to B\mathrm{Diff}(D_M,\partial)$. Then we would like to find a lift of the nullhomotopy of $f(\pi^\Gamma(i))$ to a nullhomotopy of $\tilde{f}(\pi^\Gamma(i))$. The obstruction to the lifting of the nullhomotopy lies in

$$H^{k-1}(D^{k-1},\partial D^{k-1};\pi_{k-1}(\Omega^{2k+1}SO_{2k+1})) \cong \pi_{3k}(SO_{2k+1})$$

by Proposition 2.1. The obstructing element of $\pi_{k-1}(\widetilde{B\mathrm{Diff}}(D_M,\partial))$ can be obtained from the trivial framed (D_M,∂)-bundle by twisting the vertical framing on a trivial sub (D^{2k+1},∂)-bundle which is included in $\widetilde{\mathcal{V}}_i$. So one can change the vertical framing on $\widetilde{V}_0$ in the definition of α by twisting by a suitable element of $\pi_{3k}(SO_{2k+1})$ so that the obstruction vanishes, without affecting the assertion (1). After this modification $\tilde{f}(\pi^\Gamma(i))$ becomes nullhomotopic and hence the classifying map $\tilde{f}(\pi^\Gamma) : (S^{k-1})^{\times 2n} \to \widetilde{B\mathrm{Diff}}(D_M,\partial)$ for π^Γ becomes bordant to a map from $S^{2n(k-1)} = (S^{k-1})^{\times 2n}/S^{k-1}\vee\cdots\vee S^{k-1}$. This completes the proof of the assertion (2). □

7. Proof of normalization Proposition 5.2

In this section we give a smooth bundle analogue of Kuperberg–Thurston–Lescop's normalization of a propagator, in many parts following Lescop's arguments of [Les2].

7.1. **Normalization, unparametrized case.**

- Fix a point p^i on $\partial\mathcal{V}_i$ which is disjoint from all a_k^i's and b_k^i's. Let $[p^i,\infty] \subset M$ be a smoothly embedded path from p^i to ∞.
- Let $\omega(p^i)$ be a closed $2k$-form with support a tubular neighborhood of $[p^i,\infty] \cup \partial\overline{C}_1(M,\infty)) \subset \overline{C}_2(M,\infty)$ which restricts to an SO_{2k+1}-invariant unit volume form on $\partial\overline{C}_1(M,\infty) \cong S^{2k}$ and which is disjoint from supports of all the $\eta(a),\eta(b)$'s.
- Identify a small tubular neighborhood of $\partial\mathcal{V}_i \subset M$ with $[-4,4]\times\partial\mathcal{V}_i$ to fix a coordinate. For $h \in [-4,4]$ let

$$\mathcal{V}_i[h] := \left\{ \begin{array}{ll} \mathcal{V}_i \cup ([0,h]\times\partial\mathcal{V}_i) & \text{if } h \geq 0 \\ \mathcal{V}_i \setminus ((h,0]\times\partial\mathcal{V}_i) & \text{if } h \leq 0 \end{array} \right.$$

- We denote the fiber M_{t^0} of π^Γ over the base point $t^0 = (t_1^0,\ldots,t_{2n}^0) \in (S^{k-1})^{\times 2n}$ simply by M.

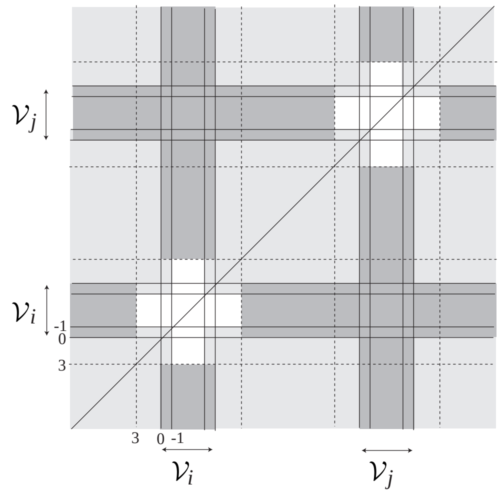

FIGURE 6. The subset $D_I(\omega^0_{M_t}) \subset \overline{C}_2(M_t,\infty)$ is shadowed. On the light-gray area the propagator is prescribed by ω_M (Proposition 7.1) while on the dark-gray area the propagator is prescribed by $\omega^0_{M_t}$. The extension over the white area will be discussed in Lemma 7.3.

Proposition 7.1. *For any subset $N \subset \{1,\ldots,2n\}$, we can choose a propagator ω_M on $\overline{C}_2(M,\infty)$ so that for all $i \in N$,*

$$\omega_M|_{\mathcal{V}_i\times(\overline{C}_1(M,\infty)\setminus\mathcal{V}_i[3])} = \sum_{1\leq k,\ell\leq 3} \mathrm{Lk}(b^i_k, a^i_\ell[4])\,\mathrm{pr}_1^*\eta(a^i_k)\wedge \mathrm{pr}_2^*\eta(b^i_\ell) + \mathrm{pr}_2^*\omega(p^i)$$

where $a^i_\ell[4] = 4\times a^i_\ell \subset [-4,4]\times\partial\mathcal{V}_i$.

Let us assume Proposition 7.1 for the moment. The proof will be given later in §7.3.

7.2. **Partial explicit formula in a family and its extension.** For a subset $I\subset\{1,\ldots,2n\}$ and a parameter $t\in(S^{k-1})^{\times 2n}$ such that $t_j\neq t^0_j \Rightarrow j\in I$, we define a closed $2k$-form $\omega^0_{M_t}$ on the subset

$$D_I(\omega^0_{M_t}) := \Big[\overline{C}_2(M_t,\infty)\setminus\bigcup_{i\in I}(\mathcal{V}_i[-1]_t\times\mathcal{V}_i[3]_t)\cup(\mathcal{V}_i[3]_t\times\mathcal{V}_i[-1]_t)\Big]\cup\mathrm{pr}_{12}^{-1}\Delta_{M_t\setminus\infty}\subset\overline{C}_2(M_t,\infty)$$

(see Figure 6) where $\mathrm{pr}_{12}:\overline{C}_2(M_t,\infty)\to M_t\times M_t$ is the projection of the blow-up, by

(1) $\omega^0_{M_t} = \omega_M$
on $\overline{C}_2(M_t\setminus\bigcup_{i\in I}\mathcal{V}_i[-1]_t,\infty) = \overline{C}_2(M\setminus\bigcup_{i\in I}\mathcal{V}_i[-1],\infty)$ (canonical identification with respect to the trivialization of π^Γ on the sub $M\setminus\bigcup_{i\in I}\mathcal{V}_i[-1]$-bundle).

(2) $\omega^0_{M_t} = \displaystyle\sum_{1\leq k,\ell\leq 3}\mathrm{Lk}(b^i_k, a^i_\ell[4])\,\mathrm{pr}_1^*\eta(a^i_k,t_i)\wedge\mathrm{pr}_2^*\eta(b^i_\ell,t_i) + \mathrm{pr}_2^*\omega(p^i)$
on $\mathrm{pr}_{12}^{-1}((\mathcal{V}_i)_t\times(M_t\setminus\mathcal{V}_i[3]_t))$, $i\in I$.

Here $\eta(b^i_\ell,t_i)$ is defined as follows. Because $S(b^i_\ell)$ transversely intersects $\mathcal{V}_j$ for one $j\neq i$ in a $(k+1)$-disk, we may assume that the intersection agrees exactly with the $S(a^j_p)$ for some p. Then in a family, the replacement of $S(b^i_\ell)\cap\mathcal{V}_j = S(a^j_p)$ by $S(\widetilde{a}^j_p)$ replaces $S(b^i_\ell)$ by a manifold $S(\widetilde{b}^i_\ell)$. We put $\eta(b^i_\ell,t_i) := \eta_{S(\widetilde{b}^i_\ell)}|_{t_i}$.

$\omega^0_{M_t}$ gives a partial solution to Proposition 5.2 on $D_I(\omega^0_{M_t})$. So it remains to prove that $\omega^0_{M_t}$ can be extended to a propagator on $E\overline{C}_2(\pi^\Gamma)$. Now let

$$C(i) := \bigcup_{t=(t_1^0,\ldots,t_i,\ldots,t_{2n}^0)} \overline{C}_2(M_t,\infty) \subset E\overline{C}_{2n}(\pi^\Gamma),$$
$$D(i) := \bigcup_{t=(t_1^0,\ldots,t_i,\ldots,t_{2n}^0)} D_{\{i\}}(\omega^0_{M_t}).$$

The problem is to prove that the closed form $\omega^0_{M_t}$ on $D(i)$ can be extended over $C(i)$.

Lemma 7.2. *The Leray–Serre spectral sequence for the fibration $D(i) \to S^{k-1}$:*

$$E^2_{p,q} \cong H_p(S^{k-1}; H_q(D_{\{i\}}(\omega^0_M);\mathbb{R})) \Rightarrow H_{p+q}(D(i);\mathbb{R})$$

satisties the following:

- $E^2_{p,q} = E^\infty_{p,q}$ *if* $p+q \le 2k$,
- $E^\infty_{p,q} = 0$ *if* $p+q \le 2k$ *and* $(p,q) \ne (0,0),(k-1,0),(0,k+1),(k-1,k+1),(0,2k)$,
- $H_{2k}(D(i);\mathbb{R}) = E^\infty_{0,2k} \oplus E^\infty_{k-1,k+1} = E^2_{0,2k} \oplus E^2_{k-1,k+1}$.

Lemma 7.3. (1) $E^\infty_{k-1,k+1} \cong \mathrm{Ker}(\mathrm{incl}_* : H_{2k}(D(i);\mathbb{R}) \to H_{2k}(C(i);\mathbb{R}))$.
(2) $\omega^0_{M_t}$ *evaluated on* $E^\infty_{k-1,k+1}$ *vanishes.*

We remark that Lemma 7.2 is indeed necessary in the proof of Lemma 7.3. Let us assume these lemmas for a moment. Lemma 7.3 implies that there is a closed extension $\omega(i) \in \Omega^{2k}_{\mathrm{dR}}(C(i))$ of $\omega^0_{M_t} \in \Omega^{2k}_{\mathrm{dR}}(D(i))$. Using this we put

$$\omega_{M_t} := \begin{cases} \omega^0_{M_t} & \text{on } D_I(\omega^0_{M_t}) \\ \omega(i)_t & \text{on } \overline{C}_2(\mathcal{V}_i[4]_t) \text{ for } i \in I \end{cases}$$

This gives a full solution to Proposition 5.2.

7.3. Proof of Proposition 7.1, Lemma 7.2.

Proof of Proposition 7.1. We give a proof by induction on the size of N.

$\underline{N=\{1\}}$: Let ω_0 be a propagator on $\overline{C}_2(M,\infty)$ and let ω be the closed $2k$-form on

$$A := \mathcal{V}_1[1] \times (\overline{C}_1(M,\infty) \setminus \mathrm{Int}\,\mathcal{V}_1[2])$$

defined by the statement. Because both ω_0 and ω evaluated on cycles of $H_{2k}(A;\mathbb{R})$ coincide, there exists a $(2k-1)$-form μ on A such that $\omega = \omega_0 + d\mu$. We may assume that $\mu = 0$ on $\mathcal{V}_1[1] \times \partial\overline{C}_1(M,\infty)$ because μ is closed on $\mathcal{V}_1[1] \times \partial\overline{C}_1(M,\infty)$ and hence exact there.

Let $\chi : \overline{C}_2(M,\infty) \to [0,1]$ be a smooth function such that

- $\mathrm{Supp}\,\chi = A$,
- $\chi = 1$ on $\mathcal{V}_1 \times (\overline{C}_1(M,\infty) \setminus \mathcal{V}_1[3])$ $(\subset A)$.

Then let

$$\omega_a := \omega_0 + d(\chi\mu).$$

ω_a is as required on $\mathcal{V}_1 \times (\overline{C}_1(M,\infty) \setminus \mathcal{V}_1[3])$ and coincides with ω_0 on $\partial\overline{C}_2(M,\infty)$ because $d(\chi\mu) = 0$ there. A similar modification of ω_a on $(\overline{C}_1(M,\infty) \setminus \mathcal{V}_1[3]) \times \mathcal{V}_1$, that can be done disjointly from the previous ones, yields another $2k$-form ω_b that is as required on

$$\partial\overline{C}_2(M,\infty) \cup (\mathcal{V}_1 \times (\overline{C}_1(M,\infty) \setminus \mathcal{V}_1[3])) \cup ((\overline{C}_1(M,\infty) \setminus \mathcal{V}_1[3]) \times \mathcal{V}_1).$$

Thus $\omega_M := \omega_b$ is of the required form for $N = \{1\}$.

$\underline{N=\{1,\ldots,i\}}$: Let ω_0 be a propagator on $\overline{C}_2(M,\infty)$ satisfying all the hypotheses for $N = \{1,\ldots,i-1\}$ and let ω be a propagator on $\overline{C}_2(M,\infty)$ satisfying the hypotheses for $\{i\}$ which is obtained from ω_0 by the first step, with $\mathcal{V}_i$ replaced by $\mathcal{V}_i[1]$. Then there exists a $(2k-1)$-form μ on $\overline{C}_2(M,\infty)$ such that $\omega = \omega_0 + d\mu$ where we may assume $\mu = 0$ on $\partial\overline{C}_2(M,\infty)$ because $H^{2k-1}(\partial\overline{C}_2(M,\infty);\mathbb{R}) = 0$.

Let $\chi : \overline{C}_2(M,\infty) \to [0,1]$ be a smooth function such that

- $\mathrm{Supp}\,\chi = A_i := \mathcal{V}_i[1] \times (\overline{C}_1(M,\infty) \setminus \mathrm{Int}\,\mathcal{V}_i[2])$,
- $\chi = 1$ on $\mathcal{V}_i \times (\overline{C}_1(M,\infty) \setminus \mathcal{V}_i[3]) \subset A_i$.

Let $\omega_a := \omega_0 + d(\chi\mu)$. Then ω_a is as required on

$$\partial\overline{C}_2(M,\infty) \cup \bigcup_{k\in N} (\mathcal{V}_k \times (\overline{C}_1(M,\infty) \setminus \mathcal{V}_k[3])) \cup \bigcup_{k\in N\setminus\{i\}} ((\overline{C}_1(M,\infty) \setminus \mathcal{V}_k[3]) \times \mathcal{V}_k).$$

It remains to prove that ω_a is as required on

$$\mathcal{V}_i[1] \times (\partial\overline{C}_1(M,\infty) \cup \bigcup_{k=1}^{i-1} \mathcal{V}_k),$$

where $\mathrm{Supp}\,\chi$ intersects the previous changes for ω_0. By the assumptions, μ may be assumed to vanish on $\mathcal{V}_i[1]\times\partial\overline{C}_1(M,\infty)$ and is closed on $\mathcal{V}_i[1]\times\mathcal{V}_k$ for $i \neq k$. Further by $H^{2k-1}(\mathcal{V}_i[1]\times\mathcal{V}_k;\mathbb{R}) = 0$, we may assume that $\mu = 0$ on $\mathcal{V}_i[1] \times \mathcal{V}_k$. By a similar modifications as in the first step, we can modify ω_a into ω_b which is as required. $\square$

Proof of Lemma 7.2. The assertion follows from a computation of $H_q(D_{\{i\}}(\omega_M^0);\mathbb{R})$ for $0 \leq q \leq 2k$. From now on we show that $H_q(D_{\{i\}}(\omega_M^0);\mathbb{R}) = 0$ for $1 \leq q \leq k$, $k+2 \leq q \leq 2k-1$.

We write $\overline{M} = \overline{C}_1(M,\infty)$ for simplicity. For a submanifold $X \subset \overline{M}$, we denote by STX the face of $\partial\overline{C}_2(X)$ or $\partial\overline{C}_2(X,\infty)$ corresponding to the blow-up along the diagonal $\Delta_X \subset X^{\times 2}$. By using the homotopy equivalence

$$\begin{aligned} D_{\{i\}}(\omega_M^0) &\simeq (\overline{C}_2(M,\infty) \setminus \overline{C}_2(\mathcal{V})) \cup ST\mathcal{V} \\ &= (\overline{C}_2(M,\infty) \setminus \overline{C}_2(\mathcal{V})) \cup_{ST(\overline{M}\setminus\mathcal{V})} ST\overline{M} \end{aligned}$$

where $\mathcal{V} = \mathcal{V}_i$, it is enough to compute the homology of the latter space. We consider the Mayer–Vietoris sequence for the latter. By using the diffeomorphisms $ST\overline{M} \cong \overline{M} \times S^{2k}$, $ST(\overline{M} \setminus \mathcal{V}) \cong (\overline{M} \setminus \mathcal{V}) \times S^{2k}$ and the computation of $H_*(\overline{C}_2(M,\infty) \setminus \overline{C}_2(\mathcal{V});\mathbb{R})$ given in Lemma 7.4 below, the Mayer–Vietoris sequence for the pair $(\overline{C}_2(M,\infty) \setminus \overline{C}_2(\mathcal{V}), ST\overline{M})$ is as follows:

$$\begin{array}{c|ccccccc} & & (\overline{M}\setminus\mathcal{V})\times S^{2k} & & \begin{array}{c}\overline{M}\times S^{2k}\\ +\overline{C}_2(M,\infty)\setminus\overline{C}_2(\mathcal{V})\end{array} & & \overline{C}_2(M,\infty)\setminus\overline{C}_2(\mathcal{V})\cup ST\overline{M} \\ \hline H_{2k-1\sim k+2} & \to & 0 & \to & 0+0 & \to & ? \\ H_{k+1} & \to & 0 & \to & 0+0 & \to & ? \\ H_k & \to & \sum_j \mathbb{R}[\check{a}_j^i\otimes 1] & \to & 0+0 & \to & ? \\ H_{k-1\sim 1} & \to & 0 & \to & 0+0 & \to & ? \\ H_0 & \to & \mathbb{R} & \to & \mathbb{R}+\mathbb{R} & \to & \mathbb{R} \end{array} \tag{7.1}$$

Here $\check{a}_j^i := a_j^i[4]$. The homology of $\overline{M}\setminus\mathcal{V}$ is given in (7.3) below. Then it follows immediately that $H_q(D_{\{i\}}(\omega_M^0);\mathbb{R})$ vanishes for $1 \leq q \leq k$, $k+2 \leq q \leq 2k-1$ and is three dimensional for $q = k+1$.

Therefore $E^2_{p,q}$ in $0 \leq q \leq 2k$, $p \geq 0$ is zero unless

$$(p,q) = (0,0), (0,k+1), (0,2k), (k-1,0), (k-1,k+1), (k-1,2k).$$

This implies that all the differentials $d^r : E^r_{*,*} \to E^r_{*-r,*+r-1}$, $r \geq 1$ involving terms $E^r_{p,q}$ for $p+q \leq 2k$ are 0. Hence $E^2_{p,q} = E^\infty_{p,q}$ there. $\square$

Lemma 7.4. *$H_q(\overline{C}_2(M,\infty) \setminus \overline{C}_2(\mathcal{V});\mathbb{R}) = 0$ for $1 \leq q \leq 2k-1$.*

Proof. There is a homotopy equivalence $\overline{C}_2(M,\infty)\setminus\overline{C}_2(\mathcal{V})\simeq C_2(\overline{M})\setminus C_2(\mathcal{V})$. So we compute the homology of the latter. The homology of $C_2(\overline{M})\setminus C_2(\mathcal{V})$ is computed by the exact sequence of the pair $(\overline{M}^{\times 2}\setminus\mathcal{V}^{\times 2}, C_2(\overline{M})\setminus C_2(\mathcal{V}))$:

$$
\begin{aligned}
&\to H_*(C_2(\overline{M})\setminus C_2(\mathcal{V}))\to H_*(\overline{M}^{\times 2}\setminus\mathcal{V}^{\times 2})\\
&\qquad\to H_*(\overline{M}^{\times 2}\setminus\mathcal{V}^{\times 2}, C_2(\overline{M})\setminus C_2(\mathcal{V}))\to\cdots
\end{aligned}
\tag{7.2}
$$

(coefficients are in $\mathbb{R}$.)

(I) $H_*(\overline{M}^{\times 2}\setminus\mathcal{V}^{\times 2})$: We apply the Mayer–Vietoris sequence for

$$\overline{M}^{\times 2}\setminus\mathcal{V}^{\times 2}=(\overline{M}\times(\overline{M}\setminus\mathcal{V}))\cup_{(\overline{M}\setminus\mathcal{V})^{\times 2}}((\overline{M}\setminus\mathcal{V})\times\overline{M}).$$

Here we have

$$
H_*(\overline{M}\setminus\mathcal{V})=\begin{cases}
\mathbb{R}[\partial\overline{M}] & \text{if } *=2k\\
\mathbb{R}[a_1^i[4]]\oplus\mathbb{R}[a_2^i[4]]\oplus\mathbb{R}[a_3^i[4]] & \text{if } *=k\\
\mathbb{R}[\mathrm{pt}] & \text{if } *=0\\
0 & \text{otherwise}
\end{cases}
\tag{7.3}
$$

Hence the Mayer–Vietoris sequence is as follows:

		$(\overline{M}\setminus\mathcal{V})^{\times 2}$		$\overline{M}\times(\overline{M}\setminus\mathcal{V})+(\overline{M}\setminus\mathcal{V})\times\overline{M}$		$\overline{M}^{\times 2}\setminus\mathcal{V}^{\times 2}$
H_{2k}	$\to$	$\mathbb{R}[\partial\overline{M}\otimes 1]+\mathbb{R}[1\otimes\partial\overline{M}]+\sum_{j,k}\mathbb{R}[\check{a}_j^i\otimes\check{a}_k^i]$	$\twoheadrightarrow$	$\mathbb{R}[\partial\overline{M}\otimes 1]+\mathbb{R}[1\otimes\partial\overline{M}]$	$\xrightarrow{0}$	?
$H_{2k-1\sim k+1}$	$\to$	0	$\to$	0	$\to$	?
H_k	$\to$	$\sum_j(\mathbb{R}[1\otimes\check{a}_j^i]+\mathbb{R}[\check{a}_j^i\otimes 1])$	$\hookrightarrow$	$\sum_j(\mathbb{R}[1\otimes\check{a}_j^i]+\mathbb{R}[\check{a}_j^i\otimes 1])$	$\xrightarrow{0}$	?
$H_{k-1\sim 1}$	$\to$	0	$\to$	0	$\to$	?
H_0	$\to$	$\mathbb{R}$	$\to$	$\mathbb{R}+\mathbb{R}$	$\to$	$\mathbb{R}$

It follows that

$$
H_*(\overline{M}^{\times 2}\setminus\mathcal{V}^{\times 2})=\begin{cases}
0 & \text{if } 1\leq *\leq 2k\\
\mathbb{R} & \text{if } *=0
\end{cases}
\tag{7.4}
$$

(II) $H_*(\overline{M}^{\times 2}\setminus\mathcal{V}^{\times 2}, C_2(\overline{M})\setminus C_2(\mathcal{V}))$: By excision we have

$$
\begin{aligned}
&H_*(\overline{M}^{\times 2}\setminus\mathcal{V}^{\times 2}, C_2(\overline{M})\setminus C_2(\mathcal{V}))\cong H_*((\overline{M}\setminus\mathcal{V})\times\mathbb{R}^{2k+1},(\overline{M}\setminus\mathcal{V})\times(\mathbb{R}^{2k+1}\setminus\{0\}))\\
&\cong H_{*-(2k+1)}(\overline{M}\setminus\mathcal{V})\otimes H_{2k}(S^{2k}).
\end{aligned}
$$

In particular,

$$H_*(\overline{M}^{\times 2}\setminus\mathcal{V}^{\times 2}, C_2(\overline{M})\setminus C_2(\mathcal{V}))=0 \text{ for } 0\leq *\leq 2k. \tag{7.5}$$

Finally, substituting (7.4), (7.5) into (7.2), we get the following sequence:

		$C_2(\overline{M})\setminus C_2(\mathcal{V})$		$\overline{M}^{\times 2}\setminus\mathcal{V}^{\times 2}$		$(\overline{M}^{\times 2}\setminus\mathcal{V}^{\times 2},C_2(\overline{M})\setminus C_2(\mathcal{V}))$
H_{2k}	$\to$	?	$\to$	0	$\to$	0
$H_{2k-1\sim 1}$	$\to$	0	$\to$	0	$\to$	0
H_0	$\to$	$\mathbb{R}$	$\to$	$\mathbb{R}$	$\to$	0

which determines $H_q(C_2(\overline{M})\setminus C_2(\mathcal{V}))$ in $0\leq q\leq 2k-1$. $\square$

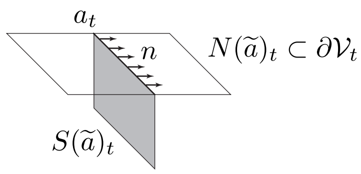

FIGURE 7

7.4. **Lescop's cycles $F(a) \subset D(i)$ and proof of Lemma 7.3.** Now we give a proof of Lemma 7.3 by giving a higher dimensional analogue of Lescop's clever construction of cycles in a configuration space, which were constructed in [Les2] for 3-manifolds. The higher dimensional Lescop cycles will give an explicit basis of $E^{\infty}_{k-1,k+1}(D(i))$.

For each $a = a^i_k$, the higher dimensional Lescop cycle $F(a) \subset D(i)$ is defined to be the $2k$-cycle of the form:

$$\begin{aligned}F(a) :=&(C(a) \times S^{k-1})\\ &\cup -(S(\widetilde{a})\widetilde{\times}(4 \times p(a))) \cup -((4 \times p(a))\widetilde{\times}S(\widetilde{a}))\\ &\cup \mathrm{diag}(n)(S(\widetilde{a})) \qquad (p(a)\text{: base point of } a)\end{aligned}$$

(see Lemma 5.1 for the definition of $S(\widetilde{a})$) where

$$\begin{aligned}S(\widetilde{a})\widetilde{\times}(4 \times p(a)) &:= \cup_t\{x_t \times (4 \times p(a)_t) \mid x_t \in S(\widetilde{a}) \cap \mathcal{V}_t\},\\ (4 \times p(a))\widetilde{\times}S(\widetilde{a}) &:= \cup_t\{(4 \times p(a)_t) \times x_t \mid x_t \in S(\widetilde{a}) \cap \mathcal{V}_t\},\end{aligned}$$

and the other chains $C(a), \mathrm{diag}(n)(S(\widetilde{a}))$ will be defined below.

Vertical vector field n on $S(\widetilde{a})$. Recall that $\widetilde{\mathcal{V}}$ has a framing that agrees with the standard one τ_ϕ ($\phi = \phi_\Gamma|\mathcal{V}_i$) on the boundary. Let us denote this framing by $\tau_{\alpha,\phi}$ and let $q(\tau_{\alpha,\phi}) : T^{\mathrm{fib}}\widetilde{\mathcal{V}} \to \mathbb{R}^{2k+1}$ be the composition

$$T^{\mathrm{fib}}\widetilde{\mathcal{V}} \xrightarrow{\tau_{\alpha,\phi}} \mathbb{R}^{2k+1} \times \widetilde{\mathcal{V}} \xrightarrow{\mathrm{pr}_1} \mathbb{R}^{2k+1}.$$

By a suitable deformation of ϕ, we may assume that $q(\tau_{\alpha,\phi})^{-1}(1,0,\ldots,0)$ gives a smooth (nowhere zero) section $n : \widetilde{\mathcal{V}} \to T^{\mathrm{fib}}\widetilde{\mathcal{V}}$ such that its restriction to a neighborhood $N(\widetilde{a})$ of $\widetilde{a} \subset \partial\widetilde{\mathcal{V}}$ is tangent to $\partial\mathcal{V}_t$ in each fiber and normal to $S(\widetilde{a})_t$ (Figure 7). Let $Sn : \widetilde{\mathcal{V}} \to ST^{\mathrm{fib}}\widetilde{\mathcal{V}}$ be the section of the associated (unit) tangent sphere bundle. Clearly the composite map

$$S(\widetilde{a}) \xrightarrow{\mathrm{incl}} \widetilde{\mathcal{V}} \xrightarrow{Sn} ST^{\mathrm{fib}}\widetilde{\mathcal{V}} \xrightarrow{\tau_{\alpha,\phi}} S^{2k} \times \widetilde{\mathcal{V}} \xrightarrow{\mathrm{pr}_1} S^{2k} \tag{7.6}$$

is the constant map to the point $(1,0,\ldots,0)$. This fact will be necessary to show that $F(a)$ represents an element of $E^{\infty}_{k-1,k+1}$.

The section n also defines a local coordinate $a_t \times [0,1] \subset \partial\mathcal{V}_t$ all of whose points are very close to a_t such that

- $\widetilde{a} \times [0,1] \subset N(\widetilde{a})$,
- for any point $x \in a_t$, the curve $x \times [0,1]$ agrees with the integral curve in a fiber $\partial\mathcal{V}_t$ of the vector field n (extended to the normal bundle by the pullback) which starts from x.

$C(a)$ and $\mathrm{diag}(n)(S(\widetilde{a}))$.

<u>$C(a)$</u>**:** $C(a)$ is the $(k+1)$-chain in $\overline{C}_2([0,4] \times a \times [0,1]) \subset \overline{C}_2(M,\infty) \setminus \overline{C}_2(\mathcal{V}[-1]) \cup ST\mathcal{V}$ (in a single fiber) defined as the sum of the following chains:

- $T(0 \times a \times 0, 0 \times a \times 1)$
- $A(0,1)$
- $(0 \times a \times 0) \times \left[-([0,4] \times p(a) \times 1) \cup (4 \times p(a) \times [0,1])\right]$

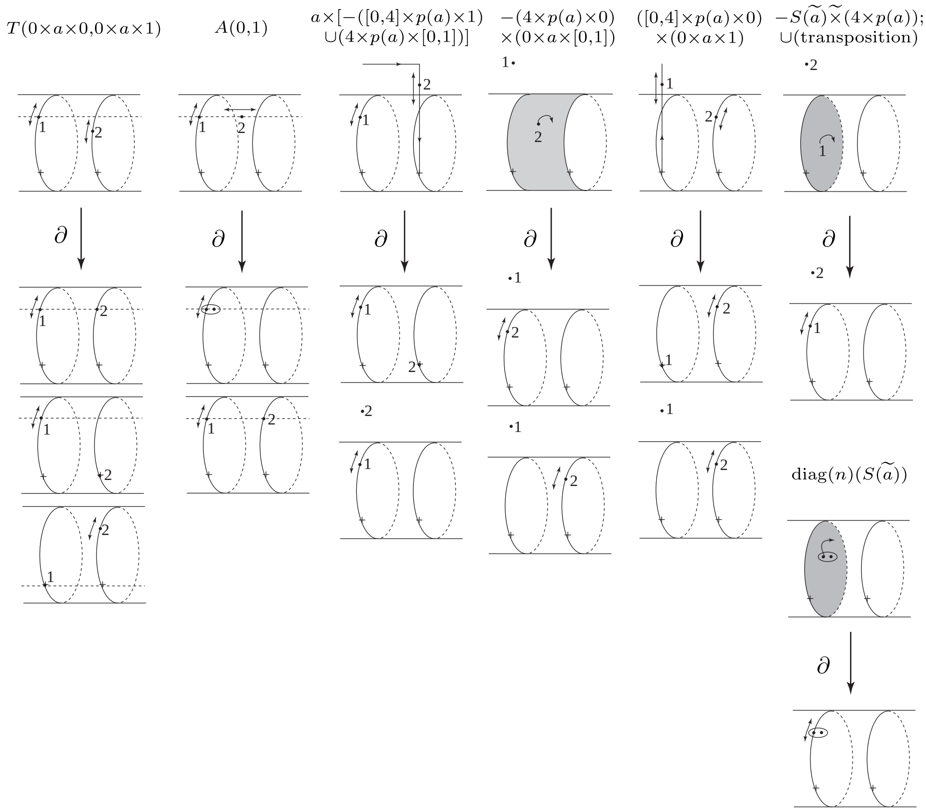

FIGURE 8. Lescop's cycle $F(a)$

- $\big(-(4\times p(a)\times 0)\times(0\times a\times[0,1])\big)\cup\big(([0,4]\times p(a)\times 0)\times(0\times a\times 1)\big)$

where $T(0\times a\times 0, 0\times a\times 1)$ and $A(0,1)$ will be defined from now on. Fix an identification $S^k = SS^{k-1} = (S^{k-1}\times I)/(S^{k-1}\times\{0,1\}\cup\{\infty\}\times I)$, $\quad I=[-1,1]$ where SX denotes the reduced suspension of X. This introduces a corresponding coordinate on S^k given by $(x,z)\in S^{k-1}\times I$. Consider the $(k+1)$-dimensional submanifold $T\subset(S^{k-1}\times I)\times(S^{k-1}\times I)$ given by

$$T := \{(x,z)\times(x,z')\,|\,x\in S^{k-1}, z,z'\in I,\ z\geq z'\}\subset(S^{k-1}\times I)\times(S^{k-1}\times I)$$

with boundary $\partial T = \{(x,z)\times(x,z)\}\cup\{(x,z)\times(x,-1)\}\cup\{(x,1)\times(x,z)\}$. Fix pointed diffeomorphisms

$$\varphi_0 : (0\times a\times 0, 0\times\{p(a)\}\times 0)\xrightarrow{\cong}(S^k,\{\infty\}),$$
$$\varphi_1 : (0\times a\times 1, 0\times\{p(a)\}\times 1)\xrightarrow{\cong}(S^k,\{\infty\}).$$

Then we define

$$\begin{aligned} T(0\times a\times 0, 0\times a\times 1) &:= (\varphi_0\times\varphi_1)^{-1}\mathrm{pr}(T)\\ A(0,1) &:= \overline{\{(0\times x\times 0)\times(0\times x\times s)\,|\,x\in a, s\in(0,1]\}}\\ &\subset \overline{C}_2(M,\infty)\end{aligned}$$

where $\mathrm{pr} : (S^{k-1}\times I)\times(S^{k-1}\times I)\to S^k\times S^k$ is the quotient map for the suspensions. Note that $\mathrm{pr}(T)$ has the boundary (as a k-chain) of the form

$$\Delta_{S^k}\cup(S^k\times\{\infty\})\cup(\{\infty\}\times S^k)\subset S^k\times S^k.$$

$\underline{\mathrm{diag}(n)(S(\widetilde{a}))}$**:** The section $Sn|_{S(\widetilde{a})} : S(\widetilde{a}) \to ST^{\mathrm{fib}}\widetilde{\mathcal{V}}$ canonically defines a map

$$\overline{Sn} : S(\widetilde{a}) \to \bigcup_t \mathrm{pr}_{12}^{-1}\Delta_{M_t\setminus\infty} \subset D(i)$$

as a section of the normal sphere bundle over the diagonal. Let

$$\mathrm{diag}(n)(S(\widetilde{a})) := \overline{Sn}(S(\widetilde{a})).$$

See Figure 8 for a picture of $F(a)$.

Proof of Lemma 7.3. Lemma 7.3 is divided into the following Lemmas 7.5 and 7.6.

Lemma 7.5. (1) $[F(a_\ell^i)]$, $\ell = 1,2,3$, *spans* $E^\infty_{k-1,k+1}(D(i))$.
(2) $F(a)$ *is null in* $H_{2k}(C(i);\mathbb{R})$.

Proof. By Lemma 7.2, the $2k$-cycle $F(a) \subset D(i)$ represents an element of $E^\infty_{0,2k} \oplus E^\infty_{k-1,k+1}$. Let us show that $F(a)$ belongs to $E^\infty_{k-1,k+1}$. If $F(a)$ had nontrivial summand on $E^\infty_{0,2k}$, then the evaluation of the restriction of $F(a)$ to $D_{\{i\}}(\omega^0_M)$ (a single fiber on t^0) by any propagator on the fiber $\overline{C}_2(M,\infty)$ (over the base point $t^0 \in S^{k-1}$) must be nontrivial. But it is easily seen by the definition of $F(a)$ and by the property Lemma 5.1(3) that such an evaluation vanishes. Hence $[F(a)] \in E^\infty_{k-1,k+1}$.

Then recall from the proof of Lemma 7.2 that in the Mayer–Vietoris sequence (7.1) the connecting homomorphism

$$\partial_{MV} : H_{k+1}(\overline{C}_2(M,\infty) \setminus \overline{C}_2(\mathcal{V}) \cup ST\overline{M}) \to H_k(ST(\overline{M}\setminus\mathcal{V})) \cong \bigoplus_{j=1}^3 \mathbb{R}[\check{a}_j^i \otimes 1]$$

is an isomorphism. So it is enough to show that $\partial_{MV}[F(a)_{t^0}]$ is equal to a nontrivial multiple of $[\check{a}_{t^0} \otimes 1]$. But it is obvious from the definition of $F(a)$. The assertion (1) is proved.

The second assertion follows from the naturality of the Leray–Serre spectral sequences (see e.g., [HatSS]). Namely, the naturality together with Lemma 7.2 implies that there are homomorphisms between $E^\infty_{*,*}$'s induced by the inclusion

$$E^\infty_{0,2k}(D(i)) \to E^\infty_{0,2k}(C(i))$$
$$E^\infty_{k-1,k+1}(D(i)) \to E^\infty_{k-1,k+1}(C(i)) = 0$$

which is isomorphism on $E^\infty_{0,2k}$ and is zero on $E^\infty_{k-1,k+1}$. □

Lemma 7.6. *The $2k$-form $\omega^0_{M_t}$ on $D(i)$ evaluated on any cycle of $E^\infty_{k-1,k+1}(D(i))$ vanishes.*

Proof. By Lemma 7.5(1), it is enough to prove

$$\int_{F(a)} \omega^0_{M_t} = 0.$$

First we extend the form ω_M given on $\overline{C}_2(M,\infty)$ of Proposition 7.1 naturally to a propagator on the trivial bundle $\overline{C}_2(M,\infty) \times S^{k-1}$ and denote this extension again by ω_M. Then we must have

$$\int_{C(a)\times S^{k-1}} \omega^0_{M_t} = \int_{C(a)\times S^{k-1}} \omega_M = 0$$

since $C(a) \times \{t\}$ lives inside $C_2([0,4]\times a_t \times [0,1]) \subset C_2(M_t)$ where $\omega^0_{M_t}$ and ω_M coincide.

Next the normalization of ω_M in Proposition 7.1 and the partial extension which followed imply that the integral vanish on $-(S(\widetilde{a})\widetilde{\times}(4\times p(a))) \cup -((4\times p(a))\widetilde{\times}S(\widetilde{a}))$.

Since $F(a)$ is null homologous in $C(i)$ by Lemma 7.5, the evaluation of ω_M on $F(a)$ in $C(i)$ vanishes. It is then enough to prove that

$$\int_{\mathrm{diag}(n)(S(\widetilde{a}))} \omega^0_{M_t} = \int_{\mathrm{diag}(n)(S(\widetilde{a}))} \omega_M = 0. \tag{7.7}$$

This is equivalent to

$$\int_{S(\widetilde{a})} \overline{Sn}^* p(\tau^\Gamma)^* \mathrm{Vol}_{S^{2k}} = 0$$

($p(\cdot)$ is the map (2.5)) which is obvious because the map (7.6) is relatively nullhomotopic. □

Appendix A. Well-definedness of the characteristic class

A.1. **Generalized Stokes theorem.** The following identity is well-known:

$$d\int_{\mathrm{Fib}(\pi)} \alpha = \int_{\mathrm{Fib}(\pi)} d\alpha + J\int_{\partial\mathrm{Fib}(\pi)} \alpha \tag{A.1}$$

where $J\gamma = (-1)^{\deg\gamma}\gamma$ and the orientation $o(\partial\mathrm{Fib}(\pi))$ on the boundary of the fiber of π is given by $\iota(n)o(\mathrm{Fib}(\pi))$ for an inward normal vector field n on $\partial\mathrm{Fib}(\pi)$, namely, the one given by the inward-normal-first convention (this may be opposite to the usual Stokes theorem).

A.2. **Codimension one strata of $\partial^{\mathrm{fib}} E\overline{C}_n(\pi)$.** Recall that each codimension one stratum of the boundary of the fiber of $\overline{C}_n(M,\infty)$ is associated to a collapse of all points of a subset $A \subset \{1,\ldots,n\}$ such that $|A| \geq 2$. We denote the face of $\partial\overline{C}_n(M,\infty)$ corresponding to the collapse of points of A by $\mathcal{S}_A$. Let $\pi_A : E\mathcal{S}_A(\pi) \to B$ be the associated $\mathcal{S}_A$-bundle of a given (D_M,∂)-bundle $\pi : E \to B$. Then there is a natural diffeomorphism

$$f_A : E\mathcal{S}_A(\pi) \xrightarrow{\cong} \overline{C}_j^{\mathrm{local}}(\mathbb{R}^d) \times E\overline{C}_{n,A}(\pi)$$

where $j = |A|$ and $E\overline{C}_{n,A}(\pi)$ is the associated bundle to π with fiber the Fulton–MacPherson–Kontsevich compactification of the configuration space

$$C_{n,A}(M,\infty) = \{(x_1,\ldots,x_n) \in (M\setminus\infty)^{\times n} \mid x_k = x_\ell \; (k\neq\ell) \text{ if and only if } k,\ell\in A\}$$

with multiple point. $\overline{C}_j^{\mathrm{local}}(\mathbb{R}^d)$ corresponds to the (unit) normal direction from $C_{n,A}(M,\infty) \subset (M\setminus\infty)^{\times n}$. Moreover, there is a natural diffeomorphism

$$c : E\overline{C}_{n,A}(\pi) \xrightarrow{\cong} E\overline{C}_{n-j+1}(\pi)$$

sending $(x_1,x_2,\ldots,x_n)$ consisting of $n-j+1$ distinct points on a fiber to the coordinate with the extra multiple values removed. The differential forms $\omega(\Gamma_A) \in \Omega^*_{\mathrm{dR}}(\overline{C}_j^{\mathrm{local}}(\mathbb{R}^d))$ and $\omega(\Gamma/A) \in \Omega^*_{\mathrm{dR}}(E\overline{C}_{n-j+1}(\pi))$ are also defined similarly as (2.6) and we have

$$\omega(\Gamma)|_{E\mathcal{S}_A(\pi)} = f_A^*\Big[\mathrm{pr}_1^*\omega(\Gamma_A) \wedge \mathrm{pr}_2^* c^*\omega(\Gamma/A)\Big]. \tag{A.2}$$

A.3. **Proof of well-definedness.**

Proof of Theorem 2.5. By the generalized Stokes theorem (A.1), we have

$$dI(\Gamma) = J\int_{\partial\mathrm{Fib}(\overline{C}_n(\pi))} \omega(\Gamma) = J\sum_{\substack{A\subset\{1,\ldots,n\}\\ |A|\geq 2}} \int_{\mathrm{Fib}(\pi_A)} \omega(\Gamma).$$

By Lemmas A.1 and A.2 below, we have

$$dI(\Gamma) = JI(d\Gamma).$$

The assertion (1) is proved.

For the assertion (2), consider the fiberwise cylinder $I \times E\overline{C}_n(\pi)$ and suppose that on $\{0,1\} \times E\overline{C}_n(\pi)$ propagators ω_0 and ω_1 which correspond to the same homotopy class of the framing, are given. Then by [W, Lemma 3], there is an extension $\widetilde{\omega}$ of ω_0 and ω_1 on $I \times E\overline{C}_n(\pi)$ and by the generalized Stokes theorem (A.1), we have

$$\begin{aligned} d\int_{I\times \mathrm{Fib}(\overline{C}_n(\pi))} \widetilde{\omega}(\Gamma) &= J\int_{\partial(I\times \mathrm{Fib}(\overline{C}_n(\pi)))} \widetilde{\omega}(\Gamma) \\ &= J\left[\int_{\mathrm{Fib}(\overline{C}_n(\pi))} \omega_0(\Gamma) - \int_{\mathrm{Fib}(\overline{C}_n(\pi))} \omega_1(\Gamma) + \int_{I\times\partial \mathrm{Fib}(\overline{C}_n(\pi))} \widetilde{\omega}(\Gamma)\right]. \end{aligned}$$

By making a linear sum corresponding to a cycle γ in $\mathcal{G}_{n,m}$ and by a similar argument as above, the third term vanishes. Therefore the difference of $I(\gamma)$ for ω_0 and ω_1 is an exact form and so the second assertion is proved. The third and the fourth assertions are obvious. □

Lemma A.1. *When* $|A| \geq 3$

$$\int_{\mathrm{Fib}(\pi_A)} \omega(\Gamma) = 0.$$

Proof. There are two cases:

(1) All vertex of Γ_A is at least 3-valent.
(2) Γ_A has a vertex with valence at most 2.

The case (1): suppose that Γ_A has n' vertices and m' edges. Then the condition that Γ_A is at least 3-valent implies the inequality

$$2m' - 3n' \geq 0. \tag{A.3}$$

Regarding the splitting (A.2), we can integrate $\omega(\Gamma_A)$ over $\overline{C}_j^{\mathrm{local}}(\mathbb{R}^d)$ first and the integral is nontrivial only if the dimension of $\overline{C}_j^{\mathrm{local}}(\mathbb{R}^d)$ is equal to $\deg\omega(\Gamma_A)$, i.e.,

$$(d-1)m' = dn' - d - 1. \tag{A.4}$$

(A.3) and (A.4) implies $(d-3)n' + 2d + 2 \leq 0$, which is a contradiction.

The case (2): if Γ_A has a bivalent vertex, say a, then there are two edges of Γ_A with the boundary vertices $\{a, b\}$ and $\{a, c\}$ respectively. Consider the automorphism $\iota : E\mathcal{S}_A(\pi) \to E\mathcal{S}_A(\pi)$ which sends x_a to $x_b + x_c - x_a$ and fixes other variables. Then $\iota^*\omega(\Gamma) = \omega(\Gamma)$ and ι reverses the orientation of the fiber $\overline{C}_j^{\mathrm{local}}(\mathbb{R}^d)$. Hence the integral vanishes.

If Γ_A has a univalent vertex, say b, then there is an edge e of Γ_A with the boundary vertices $\{a, b\}$. Because $|A| \geq 3$, the group $\mathbb{R}_{>0}$ acts freely on $\overline{C}_j^{\mathrm{local}}(\mathbb{R}^d)$ by the dilation of the edge e centered at x_b and we can consider the quotient $q : \overline{C}_j^{\mathrm{local}}(\mathbb{R}^d) \to \overline{C}_j^{\mathrm{local}}(\mathbb{R}^d)/\mathbb{R}_{>0}$. Because the map (2.5) is invariant under the action of $\mathbb{R}_{>0}$, the form $\omega(\Gamma_A)$ restricted to $\overline{C}_j^{\mathrm{local}}(\mathbb{R}^d)$ is basic with respect to q, i.e., can be written as $q^*\omega'(\Gamma_A)$ for some $\omega'(\Gamma_A) \in \Omega_{\mathrm{dR}}^{(d-1)m'}(\overline{C}_j^{\mathrm{local}}(\mathbb{R}^d)/\mathbb{R}_{>0})$. But the dimension of $\overline{C}_j^{\mathrm{local}}(\mathbb{R}^d)/\mathbb{R}_{>0}$ is one dimension less than (A.4). Hence the integral vanishes. □

Lemma A.2. *When* $|A| = 2$,

$$\int_{\mathrm{Fib}(\pi_A)} \omega(\Gamma) = I(\Gamma/A, \text{induced ori}).$$

Proof. The vertical framing τ_E on $E\overline{C}_n(\pi)$ defines a Euclidean coordinate on the vertical tangent space $T^{\mathrm{fib}}_{(x_1,\ldots,x_n)}E\overline{C}_n(\pi) \cong (\mathbb{R}^d)^{\oplus n}$ with respect to the basis

$$\frac{\partial}{\partial x_p} = \Big(\frac{\partial}{\partial x_p^{(1)}}, \ldots, \frac{\partial}{\partial x_p^{(d)}}\Big), \quad p = 1, \ldots, n.$$

Then the induced orientation on $E\mathcal{S}_A(\pi) \cong S^{d-1} \times E\overline{C}_{n-j+1}(\pi)$ is given by a positive scalar multiple of

$$\mathrm{Vol}_{S^{d-1}} \wedge \iota\Big(\frac{\partial}{\partial x_j}\Big) dx_1 \wedge \cdots \wedge dx_n$$

where $dx_p = dx_p^{(1)} \wedge \cdots \wedge dx_p^{(d)}$ and $\iota\big(\frac{\partial}{\partial x_j}\big) = \iota\big(\frac{\partial}{\partial x_j^{(n)}}\big) \cdots \iota\big(\frac{\partial}{\partial x_j^{(1)}}\big)$ (so that $\iota\big(\frac{\partial}{\partial x_j}\big) dx_i = \delta_{ij}$). Hence the induced orientation on the fiber of $\overline{C}_{n-1}(\pi)$ agrees with the one from the induced orientation on Γ/A. Therefore we have

$$\begin{aligned}\int_{\mathrm{Fib}(\pi_A)} \omega(\Gamma) &= \int_{\overline{C}_2^{\mathrm{local}}(\mathbb{R}^d)} \omega(\Gamma_A) \int_{\mathrm{Fib}(\overline{C}_{n-1}(\pi))} \omega(\Gamma/A) \\ &= \int_{S^{d-1}} \mathrm{Vol}_{S^{d-1}} \int_{\mathrm{Fib}(\overline{C}_{n-1}(\pi))} \omega(\Gamma/A) = \int_{\mathrm{Fib}(\overline{C}_{n-1}(\pi))} \omega(\Gamma/A) = I(\Gamma/A).\end{aligned}$$

□

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