UTMS 2008–29 October 28, 2008

Classification of Fuchsian systems and their connection problem

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§ 1. Introduction

Middle convolutions introduced by Katz [Kz] and extensions and restrictions introduced by Yokoyama [Yo2] give interesting transformations between Fuchsian systems on the Riemannian sphere. The transformations are invertible, the solutions of the systems are transformed by integrable transformations and the correspondence of their monodromy groups can be concretely described (cf. [Ko], [Ha], [HY], [DG2], [HF] etc.).

In this note we review the Deligne-Simpson problem, a combinatorial structure of middle convolutions and their relation to a Kac-Moody root system pointed out by Crawley-Boevey [CB]. We show with examples that the Fuchsian systems with a fixed number of accessory parameters are transformed into finite number of basic systems by middle convolutions. In the last section we give an explicit connection formula for solutions of Fuchsian differential equations without moduli.

The author thanks Y. Haraoka, A. Kato, H. Ochiai, K. Okamoto, H. Sakai and T. Yokoyama since the discussions with them enabled the author to write this note.

§ 2. Tuples of partitions

Let $\mathbf{m} = \big(m_{j,\nu}\big)_{\substack{j=0,1,\ldots\\ \nu=1,2,\ldots}}$ be an ordered set of infinite number of non-negative integers indexed by non-negative integers j and positive integers ν. Then $\mathbf{m}$ is called a $(k+1)$-*tuple of partitions of* n if the following two conditions are satisfied.

$$\sum_{\nu=1}^{\infty} m_{j,\nu} = n \qquad (j = 0, 1, \ldots), \tag{2.1}$$

2000 Mathematics Subject Classification(s): Primary 34M35; Secondary 34M40, 34M15
Key Words: *Fuchsian systems, middle convolution, connection problem*
Supported by Grant-in-Aid for Scientific Researches (A), No. 20244008, Japan Society of Promotion of Science
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$$m_{j,1} = n \qquad (j = k+1, k+2, \dots). \tag{2.2}$$

The totality of $(k+1)$-tuples of partitions of n are denoted by $\mathcal{P}_{k+1}^{(n)}$ and we put

$$\mathcal{P}_{k+1} := \bigcup_{n=0}^{\infty} \mathcal{P}_{k+1}^{(n)}, \quad \mathcal{P}^{(n)} := \bigcup_{k=0}^{\infty} \mathcal{P}_{k+1}^{(n)}, \quad \mathcal{P} := \bigcup_{k=0}^{\infty} \mathcal{P}_{k+1}, \tag{2.3}$$

$$\operatorname{ord} \mathbf{m} := n \quad \text{if } \ \mathbf{m} \in \mathcal{P}^{(n)}, \tag{2.4}$$

$$\mathbf{1} := (1, 1, \dots) = \bigl(m_{j,\nu} = \delta_{\nu,1}\bigr)_{\substack{j=0,1,\dots\\ \nu=1,2,\dots}} \in \mathcal{P}^{(1)}, \tag{2.5}$$

$$\operatorname{idx}(\mathbf{m}, \mathbf{m}') := \sum_{j=0}^{k} \sum_{\nu=1}^{\infty} m_{j,\nu} m'_{j,\nu} - (k-1) \operatorname{ord} \mathbf{m} \cdot \operatorname{ord} \mathbf{m}' \quad (\mathbf{m},\ \mathbf{m}' \in \mathcal{P}_{k+1}). \tag{2.6}$$

Here $\operatorname{ord}\mathbf{m}$ is called the order of $\mathbf{m}$. For $\mathbf{m}$, $\mathbf{m}' \in \mathcal{P}$ and a non-negative integer p, $p\mathbf{m}$ and $\mathbf{m} + \mathbf{m}' \in \mathcal{P}$ are naturally defined. For $\mathbf{m} \in \mathcal{P}_{k+1}^{(n)}$ we choose integers $n_0, \dots, n_k$ so that $m_{j,\nu} = 0$ for $\nu > n_j$ and $j = 0, \dots, k$ and we will sometimes express $\mathbf{m}$ as

$$\begin{aligned}
\mathbf{m} &= (\mathbf{m}_0, \mathbf{m}_1, \dots, \mathbf{m}_k) \\
&= m_{0,1}, \dots, m_{0,n_0}; \dots; m_{k,1}, \dots, m_{k,n_k} \\
&= m_{0,1} \cdots m_{0,n_0}, m_{1,1} \cdots m_{1,n_1}, \dots, m_{k,1} \cdots m_{k,n_k}
\end{aligned}$$

if there is no confusion. Similarly $\mathbf{m} = (m_{0,1}, \dots, m_{0,n_0})$ if $\mathbf{m} \in \mathcal{P}_1$. Here

$$\mathbf{m}_j = (m_{j,1}, \dots, m_{j,n_j}) \ \text{ and } \ \operatorname{ord}\mathbf{m} = m_{j,1} + \cdots + m_{j,n_j} \quad (0 \le j \le k).$$

For example $\mathbf{m} = (m_{j,\nu}) \in \mathcal{P}_3^{(4)}$ with $m_{1,1} = 3$ and $m_{0,\nu} = m_{2,\nu} = m_{1,2} = 1$ for $\nu = 1, \dots, 4$ will be expressed by

$$\mathbf{m} = 1,1,1,1; 3,1; 1,1,1,1 = 1111, 31, 1111 = 1^4, 31, 1^4. \tag{2.7}$$

Definition 2.1. A tuple of partition $\mathbf{m} \in \mathcal{P}$ is called *monotone* if

$$m_{j,\nu} \ge m_{j,\nu+1} \quad (j = 0, 1, \dots,\ \nu = 1, 2, \dots) \tag{2.8}$$

and $\mathbf{m}$ is called *indivisible* if the greatest common divisor of $\{m_{j,\nu}\}$ equals 1.

Let $\mathfrak{S}_\infty$ be the restricted permutation group of the set of indices $\{0, 1, 2, 3, \dots\} = \mathbb{Z}_{\ge 0}$, which is generated by the transpositions $(j, j+1)$ with $j \in \mathbb{Z}_{\ge 0}$. Put $\mathfrak{S}'_\infty := \{\sigma \in \mathfrak{S}_\infty \,;\, \sigma(0) = 0\}$, which is isomorphic to $\mathfrak{S}_\infty$.

Definition 2.2. The transformation groups S_∞ and S'_∞ of $\mathcal{P}$ are defined by

$$\begin{gathered}
S_\infty := H \ltimes S'_\infty, \quad S'_\infty := \prod_{j=0}^{\infty} G_i, \quad G_i \simeq \mathfrak{S}'_\infty, \quad H \simeq \mathfrak{S}_\infty, \\
m'_{j,\nu} = m_{\sigma(j),\sigma_j(\nu)} \qquad (j = 0, 1, \dots,\ \nu = 1, 2, \dots)
\end{gathered} \tag{2.9}$$

for $g = (\sigma, \sigma_1, \dots) \in S_\infty$, $\mathbf{m} = (m_{j,\nu}) \in \mathcal{P}$ and $\mathbf{m}' = g\mathbf{m}$.

§ 3. Conjugacy classes of matrices

For $\mathbf{m} = (m_1, \ldots, m_N) \in \mathcal{P}_1^{(n)}$ and $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ we define a matrix $L(\mathbf{m}; \lambda) \in M(n, \mathbb{C})$ as follows, which is introduced and effectively used by [Os]:

If $\mathbf{m}$ is monotone, then

$$
\begin{aligned}
&L(\mathbf{m};\lambda) := \Bigl(A_{ij}\Bigr)_{\substack{1\le i\le N\\ 1\le j\le N}}, \quad A_{ij} \in M(m_i, m_j, \mathbb{C}),\\
&A_{ij} = \begin{cases} \lambda_i I_{m_i} & (i=j)\\ I_{m_i,m_j} := \Bigl(\delta_{\mu\nu}\Bigr)_{\substack{1\le\mu\le n_i\\ 1\le\nu\le n_j}} = \begin{pmatrix} I_{m_j}\\ 0\end{pmatrix} & (i=j-1)\\ 0 & (i\ne j,\ j-1)\end{cases}.
\end{aligned}
\tag{3.1}
$$

Here I_{m_i} denote the identity matrix of size m_i and $M(m_i, m_j, \mathbb{C})$ means the set of matrices of size $m_i \times m_j$ with components in $\mathbb{C}$ and $M(m, \mathbb{C}) := M(m, m, \mathbb{C})$.

For example

$$
L(2,1,1;\lambda_1,\lambda_2,\lambda_3) = \begin{pmatrix} \lambda_1 & 0 & 1 & 0\\ 0 & \lambda_1 & 0 & 0\\ 0 & 0 & \lambda_2 & 1\\ 0 & 0 & 0 & \lambda_3\end{pmatrix}.
\tag{3.2}
$$

If $\mathbf{m}$ is not monotone, fix a permutation σ of $\{1, \ldots, N\}$ so that $(m_{\sigma(1)}, \ldots, m_{\sigma(N)})$ is monotone and put $L(\mathbf{m}; \lambda) = L(m_{\sigma(1)}, \ldots, m_{\sigma(N)}; \lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(N)})$.

When $\lambda_1 = \cdots = \lambda_N = \mu$, $L(\mathbf{m}; \lambda)$ may be simply denoted by $L(\mathbf{m}, \mu)$.

We denote $A \sim B$ for A, $B \in M(n, \mathbb{C})$ if and only if there exists $g \in GL(n, \mathbb{C})$ with $B = gAg^{-1}$. If $A \sim L(\mathbf{m}; \lambda)$, $\mathbf{m}$ is called the *spectral type* of A and denoted by $\operatorname{spt} A$.

Remark. i) If $\mathbf{m} = (m_1, \ldots, m_N) \in \mathcal{P}_1^{(n)}$ is monotone, we have

$$
A \sim L(\mathbf{m};\lambda) \ \Leftrightarrow\ \operatorname{rank} \prod_{\nu=1}^{k} (A - \lambda_\nu) = n - (m_1 + \cdots + m_k) \quad (k = 0, 1, \ldots, N).
\tag{3.3}
$$

ii) For $\mu \in \mathbb{C}$ put

$$
(\mathbf{m};\lambda)_\mu = (m_{i_1}, \ldots, m_{i_K}, \mu) \ \text{ with } \ \{i_1, \ldots, i_K\} = \{i\,;\, \lambda_i = \mu\}.
\tag{3.4}
$$

Then we have

$$
L(\mathbf{m};\lambda) \sim \bigoplus_{\mu\in\mathbb{C}} L\bigl((\mathbf{m};\lambda)_\mu\bigr).
\tag{3.5}
$$

iii) Suppose $\mathbf{m}$ is monotone. Then for $\mu \in \mathbb{C}$

$$
\begin{aligned}
L(\mathbf{m}, \mu) &\sim \bigoplus_{j=1}^{m_1} J\bigl(\max\{\nu\,;\, m_\nu \ge j\}, \mu\bigr), \\
J(k, \mu) &:= L(1^k, \mu) \in M(k, \mathbb{C}). \qquad\qquad \text{(Jordan cell)}
\end{aligned}
\tag{3.6}
$$

iv) For $A \in M(n, \mathbb{C})$ we put $Z_{M(n,\mathbb{C})}(A) := \{X \in M(n, \mathbb{C})\,;\, AX = XA\}$. Then

$$
\dim Z_{M(n,\mathbb{C})}\bigl(L(\mathbf{m}; \lambda)\bigr) = m_1^2 + m_2^2 + \cdots . \tag{3.7}
$$

Note that the Jordan canonical form of $L(\mathbf{m}; \lambda)$ is easily obtained by (3.5) and (3.6). For example $L(2, 1, 1, \mu) \sim J(3, \mu) \oplus J(1, \mu)$.

Lemma 3.1. *Let $A(t)$ be a continuous map of $[0, 1)$ to $M(n, \mathbb{C})$. Suppose there exists a partition $\mathbf{m} = (m_1, \ldots, m_N)$ of n and continuous function $\lambda(t)$ of $(0, 1)$ to $\mathbb{C}^N$ so that $A(t) \sim L\bigl(\mathbf{m}; \lambda(t)\bigr)$ for any $t \in (0, 1)$. If $\dim Z_{M(n,\mathbb{C})}\bigl(A(t)\bigr)$ is constant for $t \in [0, 1)$, then $A(0) \sim L\bigl(\mathbf{m}; \lim_{t\to 0} \lambda(t)\bigr)$.*

Proof. The proof is reduced to the result (cf. Remark 20) in [Os] but a more elementary proof will be given. First note that $\lim_{t\to 0} \lambda(t)$ exists.

We may assume that $\mathbf{m}$ is monotone. Fix $\mu \in \mathbb{C}$ and put $\{i_1, \ldots, i_K\} = \{i;\, \lambda_i(0) = \mu\}$ with $1 \le i_1 < i_2 < \cdots < i_K \le N$. Then

$$
\operatorname{rank}\bigl(A(0) - \mu\bigr)^k \le \operatorname{rank} \prod_{\nu=1}^{k} \bigl(A(t) - \lambda_{i_\nu}(t)\bigr) = n - (m_{i_1} + \cdots + m_{i_k}).
$$

Putting $m'_{i_k} = \operatorname{rank}\bigl(A(0) - \mu\bigr)^{k-1} - \operatorname{rank}\bigl(A(0) - \mu\bigr)^k$, we have

$$
\begin{gathered}
m_{i_1} \ge m_{i_2} \ge \cdots \ge m_{i_K} > 0, \quad m'_{i_1} \ge m'_{i_2} \ge \cdots \ge m'_{i_K} \ge 0, \\
m_{i_1} + \cdots + m_{i_k} \le m'_{i_1} + \cdots + m'_{i_k} \quad (k = 1, \ldots, K).
\end{gathered}
$$

Then the following lemma and the equality $\sum m_i^2 = \sum (m'_i)^2$ imply $m_i = m'_i$. $\square$

Lemma 3.2. *Let $\mathbf{m}$ and $\mathbf{m}' \in \mathcal{P}_1$ are monotone and satisfy*

$$
m_1 + \cdots + m_j \le m'_1 + \cdots + m'_j \quad (j = 1, 2, \ldots). \tag{3.8}
$$

If $\mathbf{m} \ne \mathbf{m}'$, then

$$
\sum_{j=1}^{\infty} m_j^2 < \sum_{j=1}^{\infty} (m'_j)^2.
$$

Proof. Let K be the largest integer with $m_K \neq 0$. Let p be the smallest integer j such that the inequality in (3.8) holds. Note that the lemma is clear if $p \geq K$.

Suppose $p < K$. Then $m'_p > 1$. Let q and r be the smallest integers satisfying $m'_p > m'_{q+1}$ and $m'_p - 1 > m'_r$. Then $m_p < m'_q$ and the inequality in (3.8) holds for $k = p, \ldots, r-1$ because $m_k \leq m_p \leq m'_{r-1}$.

$$\begin{array}{cccccccccccc}
m'_1, & \ldots, & m'_{p-1}, & m'_p, & \ldots, & m'_q, & m'_{q+1}, & \ldots, & m'_{r-1}, & m'_r \\
\shortparallel & & \shortparallel & \vee & & \vee & \text{IV} & & \text{IV} & \\
m_1, & \ldots, & m_{p-1}, & m_p, & \ldots, & m_q, & m_{q+1}, & \ldots, & m_{r-1}, & m_r
\end{array}$$

Here $p \leq q < r \leq K+1$. Put

$$m''_j = m'_j - \delta_{j,q} + \delta_{j,r}.$$

Then $\mathbf{m}''$ is monotone, $\sum (m''_j)^2 < (\sum m'_j)^2$ and $m_1 + \cdots + m_j \leq m''_1 + \cdots + m''_j \quad (j = 1, 2, \ldots)$. Thus we have the lemma by the induction on the lexicographic order of the triplet $(K-p, m'_p, q)$ for a fixed $\mathbf{m}$. □

Proposition 3.3. *Let $A(t)$ be a real analytic function of $(-1,1)$ to $M(n,\mathbb{C})$ such that $\dim Z_{\mathfrak{g}}\big(A(t)\big)$ doesn't depend on t. Then there exist a partition $\mathbf{m} = (m_1, \ldots, m_N)$ of n and a continuo functions $\lambda(t) = (\lambda_1(t), \ldots, \lambda_N(t))$ of $(-1,1)$ satisfying*

$$A(t) \sim L\big(\mathbf{m}; \lambda(t)\big). \tag{3.9}$$

Proof. We find $c_j \in (-1,1)$, monotone $\mathbf{m}^{(j)} \in \mathcal{P}_1^{(n)}$ and real analytic functions $\lambda^{(j)}(t) = (\lambda_1^{(j)}(t), \ldots)$ on (c_j, c_{j+1}) such that

$$c_{j-1} < c_j < c_{j+1}, \ \lim_{\pm j \to \infty} c_j = \pm 1, \ A(t) \sim L\big(\mathbf{m}^{(j)}; \lambda^{(j)}(t)\big) \quad (\forall t \in I_j).$$

Lemma 3.1 assures that we may assume $\lambda^{(j)}(t)$ is continuous on the closure $\bar{I}_j$ of I_j and $A(t) \sim L\big(\mathbf{m}^{(j)}; \lambda^{(j)}(t)\big)$ for $t \in \bar{I}_j$. Hence $\mathbf{m}^{(j)}$ doesn't depend on j, which we denoted by $\mathbf{m}$. We can inductively define transformations $\sigma_{\pm j}$ of indices $\{1, \ldots, N\}$ for $j = 1, 2, \ldots$ so that $\sigma_0 = id$, $m_{\sigma_{\pm j}(p)} = m_p$ for $p = 1, \ldots, N$ and moreover that $\big(\lambda^{(\nu)}_{\sigma_\nu(1)}(t), \ldots, \lambda^{(\nu)}_{\sigma_\nu(N)}(t)\big)$ for $-j \leq \nu \leq j$ define a continuous function on (c_{-j}, c_{j+1}). □

Remark. Suppose that $\dim Z_{M(n,\mathbb{C})}\big(A(t)\big)$ is constant for a continuous map $A(t)$ of $(-1,1)$ to $M(n,\mathbb{C})$. For $c \in (-1,1)$ we can find $t_j \in (-1,1)$ and $\mathbf{m} \in \mathcal{P}^{(1)}$ such that $\lim_{j\to\infty} t_j = c$ and $\operatorname{spt} A(t_j) = \mathbf{m}$. The proof of Lemma 3.1 shows $\operatorname{spt} A(c) = \mathbf{m}$. Hence

$$\operatorname{spt} A(t) \text{ doesn't depend on } t \ \Leftrightarrow\ \dim Z_{M(n,\mathbb{C})}(A) \text{ doesn't depend on } t. \tag{3.10}$$

It is easy to show that Proposition 3.3 is valid even if "real analytic" is replaced by "continuous" but it is not true if "real analytic" and "$(-1,1)$" are replaced by "holomorphic" and "$\{t \in \mathbb{C}\,; |t| < 1\}$", respectively. The matrix $A(t) = \left(\begin{smallmatrix} 0 & 1 \\ t & 0 \end{smallmatrix}\right)$ is a counter example.

§ 4. Deligne-Simpson problem

For simplicity we put $\mathfrak{g} = M(n,\mathbb{C})$ and $G = GL(n,\mathbb{C})$ only in this section.

Let $\mathbf{A} = (A_0,\ldots,A_k) \in \mathfrak{g}^{k+1}$. Put

$$M(n,\mathbb{C})_0^{k+1} := \{(C_0,\ldots,C_k) \in \mathfrak{g}^{k+1}\,;\, C_0+\cdots+C_k = 0\}, \tag{4.1}$$

$$Z_{\mathfrak{g}}(\mathbf{A}) := \{X \in \mathfrak{g}\,;\, [A_j, X] = 0 \quad (j = 0,\ldots,k)\}. \tag{4.2}$$

A tuple of matrices $\mathbf{A} \in \mathfrak{g}^{k+1}$ is called *irreducible* if any subspace $V \subset \mathbb{C}^n$ satisfying $A_j V \subset V$ for $j = 0,\ldots,k$ is $\{0\}$ or $\mathbb{C}^n$.

Suppose $\operatorname{trace}(A_0+\cdots+A_k) = 0$. The *additive Deligne-Simpson problem* is to determine the condition to $\mathbf{A}$ for the existence of an irreducible $\mathbf{B} = (B_0,\ldots,B_k) \in M(n,\mathbb{C})_0^{k+1}$ satisfying $A_j \sim B_j$ for $j = 0,\ldots,k$. The condition is concretely given by [CB] (cf. [Ko]).

Suppose $\mathbf{A} \in M(n,\mathbb{C})_0^{k+1}$. Then $\mathbf{A}$ is called *rigid* if $\mathbf{A} \sim \mathbf{B}$ for any element $\mathbf{B} = (B_0,\ldots,B_k) \in M(n,\mathbb{C})_0^{k+1}$ satisfying $B_j \sim A_j$ for $j = 0,\ldots,k$. Here we denote $\mathbf{A} \sim \mathbf{B}$ if there exists $g \in G$ with $(B_0,\ldots,B_k) = (gA_0g^{-1},\ldots,gA_kg^{-1})$.

Remark. Note that the local monodromy at ∞ of the Fuchsian system

$$\frac{du}{dz} = \sum_{j=1}^{k} \frac{A_j}{z-z_j} u \tag{4.3}$$

on a Riemannian sphere corresponds to A_0 with $\mathbf{A} = (A_0,\ldots,A_k) \in M(n,\mathbb{C})_0^{k+1}$. Then the quotient $M(n,\mathbb{C})_0^{k+1}/\!\sim$ classifies the Fuchsian systems.

Under the identification of $\mathfrak{g}$ with its dual space by the symmetric bilinear form $\langle X,Y\rangle = \operatorname{trace} XY$ for $(X,Y) \in \mathfrak{g}^2$, the dual map of $\operatorname{ad}_A : X \mapsto [A,X]$ of $\mathfrak{g}$ equals $-\operatorname{ad}_A$ and therefore $\operatorname{ad}_A(\mathfrak{g})$ is the orthogonal compliment of $\ker \operatorname{ad}_A$ under the bilinear form:

$$\operatorname{ad}_A(\mathfrak{g}) := \{[A,X]\,;\, X \in \mathfrak{g}\} = \{X \in \mathfrak{g}\,;\, \operatorname{trace} XY = 0 \quad (\forall Y \in Z_{\mathfrak{g}}(A))\}. \tag{4.4}$$

For $\mathbf{A} = (A_0,\ldots,A_k) \in \mathfrak{g}^{k+1}$ we put

$$\begin{array}{cccc} \pi_{\mathbf{A}}: & G^{k+1} & \to & \mathfrak{g} \\ & \cup\!\!\!\!\cup & & \cup\!\!\!\!\cup \\ & (g_0,\ldots,g_k) & \mapsto & \sum_{j=0}^k g_jA_jg_j^{-1} \end{array}$$

The image of $\pi_{\mathbf{A}}$ is a homogeneous space G^{k+1}/H of G^{k+1} with

$$H := \{(g_0,\ldots,g_k) \in G^{k+1}\,;\, \sum_{j=0}^k g_jA_jg_j^{-1} = \sum_{j=0}^k A_j\}$$

and the tangent space of the image at $A_0 + \cdots + A_k$ is isomorphic to

$$\sum_{j=0}^{k} \mathrm{ad}_{A_j}(\mathfrak{g}) = \bigl\{X \in \mathfrak{g}\,;\, \mathrm{trace}\, XY = 0 \quad \bigl(\forall Y \in Z_{\mathfrak{g}}(\mathbf{A}) := \bigcap_{j=0}^{k} Z_{\mathfrak{g}}(A_j)\bigr)\bigr\}.$$

Hence the dimension of the manifold $\sum_{j=0}^{k} \mathrm{ad}_{A_j}(\mathfrak{g})$ equals $n^2 - \dim Z_{\mathfrak{g}}(\mathbf{A})$ and therefore the dimension of H equals $kn^2 + \dim Z_{\mathfrak{g}}(\mathbf{A})$. Since the manifold

$$\widetilde{O}_{\mathbf{A}} := \{(C_0, \ldots, C_k) \in \mathfrak{g}^{k+1}\,;\, C_j \sim A_j \text{ and } \sum C_j = \sum A_j\} \tag{4.5}$$

is naturally isomorphic to $H/\bigl(Z_G(A_0) \times \cdots \times Z_G(A_k)\bigr)$, its dimension equals $kn^2 + \dim Z_{\mathfrak{g}}(\mathbf{A}) - \sum_{j=0}^{k} \dim Z_{\mathfrak{g}}(\mathbf{A}_j)$. Note that the dimension of the manifold

$$O_{\mathbf{A}} := \bigcup_{g \in G} (gA_0g^{-1}, \ldots, gA_kg^{-1}) \subset \mathfrak{g}^{k+1} \tag{4.6}$$

equals $n^2 - \dim Z_{\mathfrak{g}}(\mathbf{A})$.

Suppose $\mathbf{A} \in M(n, \mathbb{C})_0^{k+1}$. Then $\widetilde{O}_{\mathbf{A}} \supset O_{\mathbf{A}}$ and we have

Proposition 4.1. $\dim \widetilde{O}_{\mathbf{A}} - \dim O_{\mathbf{A}} = (k-1)n^2 - \sum_{j=0}^{k} \dim Z_{\mathfrak{g}}(A_j) + 2 \dim Z_{\mathfrak{g}}(\mathbf{A})$.

Definition 4.2. The index of rigidity $\mathrm{idx}\, \mathbf{A}$ of $\mathbf{A}$ is introduced by [Kz]:

$$\mathrm{idx}\, \mathbf{A} := \sum_{j=0}^{k} \dim Z_{\mathfrak{g}}(A_j) - (k-1)n^2 = 2n^2 - \sum_{j=0}^{k} \dim\{gA_jg^{-1}\,;\, g \in G\},$$

$$\mathrm{Pidx}\, \mathbf{A} := \dim Z_{\mathfrak{g}}(\mathbf{A}) + \tfrac{1}{2}(k-1)n^2 - \tfrac{1}{2}\sum_{j=0}^{k} \dim Z_{\mathfrak{g}}(A_j) = \dim Z_{\mathfrak{g}}(\mathbf{A}) - \tfrac{1}{2}\, \mathrm{idx}\, \mathbf{m}.$$

Note that $\mathrm{Pidx}\, \mathbf{A} \geq 0$ and $\dim\{gA_jg^{-1}\,;\, g \in G\}$ are even.

Corollary 4.3. $\dim \widetilde{O}_{\mathbf{A}} - \dim O_{\mathbf{A}}$ *and* $\mathrm{idx}\, \mathbf{A}$ *are even and* $\mathrm{idx}\, \mathbf{A} \leq 2 \dim Z_{\mathfrak{g}}(\mathbf{A})$.

Note that if $\mathbf{A}$ is irreducible, $\dim Z_{\mathfrak{g}}(\mathbf{A}) = 1$.

The following result is fundamental.

Theorem 4.4 ([Kz]). *Suppose* $\mathbf{A} \in M(n, \mathbb{C})_0^{k+1}$ *is irreducible. Then* $\mathrm{idx}\, \mathbf{A} = 2$ *if and only if* $\mathbf{A}$ *is rigid, namely,* $\widetilde{O}_{\mathbf{A}} = O_{\mathbf{A}}$.

§ 5. Middle convolutions

We will review the additive middle convolutions in the way interpreted by [DG] and [DG2].

Definition 5.1 ([DG]). Fix $\mathbf{A} = (A_0, \ldots, A_k) \in M(n, \mathbb{C})_0^{k+1}$. The *addition* $M'_\mu(\mathbf{A}) \in \mathfrak{g}^{k+1}$ of $\mathbf{A}$ with respect to $\mu' \in \mathbb{C}^k$ is $(A_0 - \mu'_1 - \cdots - \mu'_k, A_1 + \mu'_1, \ldots, A_k + \mu'_k)$. The *convolution* $(G_0. \ldots, G_k) \in M(kn, \mathbb{C})_0^{k+1}$ of $\mathbf{A}$ with respect to $\lambda \in \mathbb{C}$ is defined by

$$G_j = \Bigl(\delta_{p,j}(A_q + \delta_{p,q}\lambda)\Bigr)_{\substack{1\le p\le k\\ 1\le q\le k}} \qquad (j = 1, \ldots, k) \tag{5.1}$$

$$= \begin{array}{rc} & \begin{array}{ccccccc} & & & \overset{j}{\smile} & \phantom{A_{j+1}} & & \end{array} \\ j) & \begin{pmatrix} \\ A_1 & A_2 & \cdots & A_j + \lambda & A_{j+1} & \cdots & A_k \\ \\ \end{pmatrix} \end{array},$$

$$G_0 = -(G_1 + \cdots + G_k). \tag{5.2}$$

Put $\mathcal{K} = \{{}^t(v_1, \ldots, v_k)\,;\, v_j \in \ker A_j \quad (j = 1, \ldots, k)\}$ and $\mathcal{L} = \ker G_0$. Then $\mathcal{K}$ and $\mathcal{L}$ are G_j-invariant subspaces of $\mathbb{C}^{kn}$ and we define $\bar{G}_j := G_j|_{\mathbb{C}^{kn}/(\mathcal{K}+\mathcal{L})} \in \operatorname{End}(\mathbb{C}^{n'}) \simeq M(n', \mathbb{C})$ with $n' = kn - \dim(\mathcal{K} + \mathcal{L})$. The *middle convolution* $mc_\lambda(\mathbf{A}) \in M(n', \mathbb{C})_0^{k+1}$ of $\mathbf{A}$ with respect to λ is defined by $mc_\lambda(\mathbf{A}) := (\bar{G}_0, \ldots, \bar{G}_k)$.

The conjugacy classes of $\bar{G}_j$ in the above definition are given in [DG2], which is simply described using the normal form in §3 (cf. Proposition 3.3):

Theorem 5.2 ([DG], [DG2]). *For $\mathbf{A} = (A_0, A_1, \ldots, A_k) \in M(n, \mathbb{C})_0^{k+1}$ and $\mu = (\mu_0, \ldots, \mu_k) \in \mathbb{C}^{k+1}$ put*

$$\begin{aligned} mc_\mu &:= M_{-\mu'} \circ mc_{|\mu|} \circ M_{-\mu'}, \\ \mu' &:= (\mu_1, \ldots, \mu_k), \quad |\mu| := \mu_0 + \mu_1 + \cdots + \mu_k. \end{aligned} \tag{5.3}$$

Assume the following conditions (which are satisfied if $n > 1$ and $\mathbf{A}$ is irreducible):

$$\bigcap_{\substack{1\le j\le k\\ j\ne i}} \ker(A_j - \mu_j) \cap \ker(A_0 - \tau) = \{0\} \qquad (i = 1, \ldots, k,\ \forall\tau \in \mathbb{C}) \tag{5.4}$$

$$\sum_{\substack{1\le j\le k\\ j\ne i}} \operatorname{Im}(A_j - \mu_j) + \operatorname{Im}(A_0 - \tau) = \mathbb{C}^n \qquad (i = 1, \ldots, k,\ \forall\tau \in \mathbb{C}) \tag{5.5}$$

Then $\mathbf{A}' := mc_\mu(\mathbf{A})$ satisfies (5.4) and (5.5) with replacing $-\mu_j$ by $+\mu_j$ and

$$\operatorname{idx} \mathbf{A}' = \operatorname{idx} \mathbf{A}. \tag{5.6}$$

If $\mathbf{A}$ is irreducible, so is $\mathbf{A}'$. If $|\mu| = 0$, then $\mathbf{A}' \sim \mathbf{A}$. If $\mathbf{A} \sim \mathbf{B}$, then $mc_\mu(\mathbf{A}) \sim mc_\mu(\mathbf{B})$. Moreover we have

$$mc_{(-\bar{\mu}_0,\, -\mu')} \circ mc_{(\mu_0,\, \mu')}(\mathbf{A}) \sim M_{2\mu'} \circ mc_{(2\mu_0 - \bar{\mu}_0 - |\mu|,\, \mu')}(\mathbf{A}), \tag{5.7}$$

$$mc_{-\mu} \circ mc_\mu(\mathbf{A}) \sim \mathbf{A}. \tag{5.8}$$

Choose $\mathbf{m} \in \mathcal{P}_{k+1}^{(n)}$ *and* $\lambda_{j,\nu} \in \mathbb{C}$ *so that*

$$A_j \sim L(\mathbf{m}_j;\lambda_j) \text{ with } \mathbf{m}_j := (m_{j,1},\dots,m_{j,n_j}) \text{ and } \lambda_j := (\lambda_{j,1},\dots,\lambda_{j,n_j}). \tag{5.9}$$

Denoting $I_j := \{\nu\,;\,\lambda_{j,\nu} = \mu_j\}$ *and putting*

$$\ell_j = \begin{cases} \min\{p \in I_j\,;\, m_p = \max\{m_\nu\,;\,\nu \in I_j\}\} & (I_j \neq \emptyset) \\ n_j + 1 & (I_j = \emptyset) \end{cases}, \tag{5.10}$$

$$d_\ell(\mathbf{m}) := m_{0,\ell_0} + m_{1,\ell_1} + \cdots + m_{k,\ell_k} - (k-1)n, \tag{5.11}$$

$$m'_{j,\nu} := m_{j,\nu} - \delta_{\ell_j,\nu}\cdot d_\ell(\mathbf{m}), \tag{5.12}$$

$$\lambda'_{j,\nu} := \begin{cases} \lambda_{j,\nu} + |\mu| - 2\mu_j & (\nu \neq \ell_j) \\ -\mu_j & (\nu = \ell_j) \end{cases}, \tag{5.13}$$

we have $A'_j \sim L(\mathbf{m}'_j;\lambda'_j)$ $(j = 0,\dots,k)$ *if* $|\mu| \neq 0$.

Example 5.3. Suppose λ_i, μ_j and τ_k are generic. Starting from $\mathbf{A} = (-\lambda_1 - \lambda_2, \lambda_1, \lambda_2) \in M(1,\mathbb{C})_0^3$, we have the following list of eigenvalues of the matrices under the application of middle convolutions to $\mathbf{A}$ (cf. hypergeometric family in Example 6.1):

$$1,1,1\ (H_1) \longleftrightarrow 11,11,11\ (H_2 : {}_2F_1) \longleftrightarrow 111,111,12\ (H_3 : {}_3F_2)$$

$$\begin{Bmatrix} -\lambda_1-\lambda_2 & \lambda_1 & \lambda_2 \end{Bmatrix} \xrightarrow{mc_{\mu_0,\mu_1,\mu_2}}$$

$$\begin{Bmatrix} -\lambda_1-\lambda_2-\mu_0+\mu_1+\mu_2 & \lambda_1+\mu_0-\mu_1+\mu_2 & \lambda_2+\mu_0+\mu_1-\mu_2 \\ -\mu_0 & -\mu_1 & -\mu_2 \end{Bmatrix} \xrightarrow{mc_{\tau_0,\tau_1,-\mu_2}}$$

$$\begin{Bmatrix} -\lambda_1-\lambda_2-\mu_0+\mu_1-\tau_0+\tau_1 & \lambda_1+\mu_0-\mu_1+\tau_0-\tau_1 & \lambda_2+\mu_0+\mu_1+\tau_0+\tau_1 \\ -\mu_0-\tau_0+\tau_1-\mu_2 & -\mu_1+\tau_0-\tau_1-\mu_2 & \mu_2 \\ -\tau_0 & -\tau_1 & \mu_2 \end{Bmatrix}$$

Here the eigenvalues are vertically written. Note that the matrices are semisimple if the parameters are generic. Denoting $\mathbf{A}' = (A'_0, A'_1, A'_2) = mc_{\mu_0,\mu_1,\mu_2}(\mathbf{A})$ and $\mathbf{A}'' = (A''_0, A''_1, A''_2) = mc_{\tau_0,\tau_1,-\mu_2}(\mathbf{A}')$, we have

$$\begin{aligned} &A'_0 \sim L(1,1;-\lambda_1-\lambda_2-\mu_0+\mu_1+\mu_2,-\mu_0), \\ &A'_j \sim L(1,1;\lambda_j+\mu_0+\mu_1+\mu_2-2\mu_j,-\mu_j) \quad (j=1,\,2), \end{aligned} \tag{5.14}$$

$$A''_2 \sim L(1,2;\lambda_2+\mu_0+\mu_1+\tau_0+\tau_1,\mu_2), \text{ etc.} \tag{5.15}$$

Then Theorem 5.2 implies that the irreducible rigid $\mathbf{A} = (A'_0, A'_1, A'_2) \in M(2,\mathbb{C})_0^3$ satisfying (5.14) exists if and only if $\lambda_1 \neq \mu_1$, $\lambda_2 \neq \mu_2$, $\lambda_1+\lambda_2+\mu_0 \neq 0$ and $\mu_0+\mu_1+\mu_2 \neq 0$. Moreover all the irreducible rigid $\mathbf{A} \in M(2,\mathbb{C})_0^3$ are obtained in this way.

Definition 5.4. Under the notation in Theorem 5.2 the tuple of partitions $\mathbf{m} \in \mathcal{P}_{k+1}^{(n)}$ is called the *spectral type* of $\mathbf{A}$ and denotes by $\mathrm{spt}(\mathbf{A}) = \mathbf{m}$.

A tuple of partitions $\mathbf{m} \in \mathcal{P}_{k+1}^{(n)}$ is called *realizable* if there exists $\mathbf{A} \in M(n,\mathbb{C})_0^{k+1}$ such that (5.9) holds for a generic $\lambda_{j,\nu}$ satisfying the condition

$$\sum_{j=0}^{k}\sum_{\nu=1}^{n_j} m_{j,\nu}\lambda_{j,\nu} = 0. \tag{5.16}$$

A realizable $\mathbf{m}$ is called *irreducibly realizable* if for a generic $\lambda_{j,\nu}$ satisfying (5.16) there exists an irreducible $\mathbf{A} \in M(n,\mathbb{C})_0^{k+1}$ with (5.9). An irreducibly realizable $\mathbf{m}$ is called *rigid* if $\mathrm{idx}\,\mathbf{m} := \mathrm{idx}(\mathbf{m},\mathbf{m}) = 2$, namely, the corresponding irreducible $\mathbf{A}$ is rigid.

For $\mathbf{m} \in \mathcal{P}$ and $\ell = (\ell_0,\dots,\ell_k) \in \mathbb{Z}_{\ge 1}^{k+1}$ we define $\partial_\ell(\mathbf{m}) = \mathbf{m}'$ by (5.11) and (5.12) and define $s(\mathbf{m})$ the unique monotone element in $S'_\infty\mathbf{m}$ and moreover

$$\partial(\mathbf{m}) := \partial_{(1,1,\dots)}(\mathbf{m}) = \partial_{\mathbf{1}}(\mathbf{m}), \tag{5.17}$$

$$\partial_{max}(\mathbf{m}) := \partial_\ell(\mathbf{m}) \ \text{ with } \ \ell_j = \min\bigl\{\nu\,;\, m_{j,\nu} = \max\{m_{j,1}, m_{j,2},\dots\}\bigr\}. \tag{5.18}$$

Under the notation (5.18) and (5.9) we put

$$mc_{max}(\mathbf{A}) := mc_{\lambda_{\ell_0},\lambda_{\ell_1},\dots}(\mathbf{A}). \tag{5.19}$$

Remark. i) If $\mathbf{m} \in \mathcal{P}$ is irreducibly realizable, $\mathbf{m}$ is indivisible ([Ko], [CB]).

ii) Suppose $\mathbf{m}$ is irreducibly realizable. Then $mc_\ell(\mathbf{m}) \in \mathcal{P}_{k+1}$ if $\#\{(j,\nu)\,;\, m_{j,\nu} > 0 \text{ and } \nu \ne \ell_j\} > 1$. Moreover if $\mathbf{A} \in M(n,\mathbb{C})_0^{k+1}$ is a generic element satisfying (5.9) and μ is generic under the condition $\mu_j = \lambda_{j,\ell_j}$ for any ℓ_j satisfying $m_{j,\ell_j} > 0$, $mc_\mu(\mathbf{A}) \in M(n,\mathbb{C})^{k+1}$ is a generic element with the spectral type $\partial_\ell(\mathbf{m})$.

iii) Let $\mathbf{A} \in M(n,\mathbb{C})_0^{k+1}$ with spectral type $\mathbf{m}$. Let $(\ell_0,\ell_1,\dots)$ with $\ell_j \in \mathbf{Z}_{>0}$ and $\ell_\nu = 1$ for $\nu > k$. Define $\mathbf{1}_\ell = (m'_{j,\nu}) \in \mathcal{P}^{(1)}$ by $m'_{j,\nu} = \delta_{j,\ell_j}$. Then

$$\mathrm{idx}\,\mathbf{A} = \mathrm{idx}\,\mathbf{m} := \mathrm{idx}(\mathbf{m},\mathbf{m}), \tag{5.20}$$

$$d_\ell(\mathbf{m}) = \mathrm{idx}(\mathbf{m},\mathbf{1}_\ell). \tag{5.21}$$

Theorem 5.5. i) ([Kz]) *Let $\mathbf{A} \in M(n,\mathbb{C})_0^{k+1}$ and put $\mathbf{m} = \mathrm{spt}\,\mathbf{A}$. Then $\mathbf{A}$ is irreducible and rigid if and only if $n = 1$ or $mc_{\max}(\mathbf{A})$ is irreducible and rigid and* $\mathrm{ord}\,\partial_{max}(\mathbf{m}) < n$. *Hence if $\mathbf{A}$ is irreducible and rigid, $\mathbf{A}$ is constructed by successive applications of suitable middle convolutions mc_μ in Theorem 5.2 to an element of $M(1,\mathbb{C})_0^{k+1}$.*

ii) ([Ko], [CB]) *An indivisible $\mathbf{m} \in \mathcal{P}$ is irreducibly realizable if and only if one of* *the flowing three conditions holds.*

(5.22) $\qquad \operatorname{ord}\mathbf{m} = 1$

(5.23) $\qquad$ $\mathbf{m}$ *is basic, namely,* $\mathbf{m}$ *is indivisible and* $\operatorname{ord}\partial_{max}(\mathbf{m}) \ge \operatorname{ord}\mathbf{m}$

(5.24) $\qquad$ $\partial_{max}(\mathbf{m}) \in \mathcal{P}$ *is well-defined and irreducibly realizable.*

Note that $\partial_\ell(\mathbf{m}) \in \mathcal{P}$ *is well-defined if and only if* $m_{j,\ell_j} \ge d_\ell(\mathbf{m})$ *for* $j = 0, 1, \ldots$.

Example 5.6. Successive applications of $s \circ \partial$ to monotone elements of $\mathcal{P}$:
$\underline{4}11, \underline{4}11, \underline{4}2, \underline{3}3 \xrightarrow{15-2\cdot 6=3} \underline{1}11, \underline{1}11, \underline{2}1 \xrightarrow{4-3=1} \underline{1}1, \underline{1}1, \underline{1}1 \xrightarrow{3-2=1} 1, 1, 1$ (rigid)
$\underline{2}11, \underline{2}11, \underline{1}111 \xrightarrow{5-4=1} \underline{1}11, \underline{1}11, \underline{1}11 \xrightarrow{3-3=0} 111, 111, 111$ (realizable, not rigid)
$\underline{2}11, \underline{2}11, \underline{2}11, \underline{3}1 \xrightarrow{9-8=1} \underline{1}11, \underline{1}11, \underline{1}11, \underline{2}1 \xrightarrow{5-6=-1} 211, 211, 211, 31$ (realizable, not rigid)
$\underline{2}2, \underline{2}2, \underline{1}111 \xrightarrow{5-4=1} \underline{2}1, \underline{2}1, \underline{1}11 \xrightarrow{5-3=2} \times$ (not realizable)
The numbers on the above arrows are $d_{(1,1,\ldots)}(\mathbf{m}) = m_{0,1} + \cdots + m_{k,1} - (k-1)\cdot \operatorname{ord}\mathbf{m}$.

§ 6. Rigid tuples

Let $\mathcal{R}_k^{(n)}$ denotes the totality of rigid tuples in $\mathcal{P}_k^{(n)}$ (cf. Definition 5.4). Put $\mathcal{R}_k = \bigcup_{n=1}^\infty \mathcal{R}_k^{(n)}$, $\mathcal{R}^{(n)} = \bigcup_{k=1}^\infty \mathcal{R}_k^{(n)}$ and $\mathcal{R} = \bigcup_{n=1}^\infty \mathcal{R}_k$. We will identify elements of $\mathcal{R}$ if they are in the same S_∞-orbit (cf. Definition 2.2) and then $\bar{\mathcal{R}}$ denotes the set of elements of $\mathcal{R}$ under this identification. Similarly we denote $\bar{\mathcal{R}}_k$ and $\bar{\mathcal{R}}^{(n)}$ for $\mathcal{R}_k$ and $\mathcal{R}^{(n)}$, respectively, with this identification.

Example 6.1. i) The list of $\mathbf{m} \in \bar{\mathcal{R}}^{(n)}$ with $\mathbf{m}_0 = 1^n$ is given by Simpson [Si]:

$1^n, 1^n, n-11$ (hypergeometric family)	$1^{2m}, mm, mm-11$ (even family)
$1^{2m+1}, m+1m, mm1$ (odd family)	$111111, 222, 42$ (extra case)

ii) We show other examples and the numbers of elements of $\bar{\mathcal{R}}^{(n)}$.

Table $\bar{\mathcal{R}}^{(n)}$ $(2 \le n \le 7)$

2:11,11,11	3:111,111,21	3:21,21,21,21
4:1111,1111,31	4:1111,211,22	4:211,211,211
4:211,22,31,31	4:22,22,22,31	4:31,31,31,31,31
5:11111,11111,41	5:11111,221,32	5:2111,2111,32
5:2111,221,311	5:221,221,221	5:221,221,41,41
5:221,32,32,41	5:311,311,32,41	5:32,32,32,32
5:32,32,41,41,41	5:41,41,41,41,41,41	6:111111,111111,51
6:111111,222,42	6:111111,321,33	6:21111,2211,42
6:21111,222,33	6:21111,222,411	6:21111,3111,33
6:2211,2211,33	6:2211,2211,411	6:2211,222,51,51
6:2211,321,321	6:2211,33,42,51	6:222,222,321
6:222,3111,321	6:222,33,33,51	6:222,33,411,51

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6:3111,3111,321
6:3111,33,411,51
6:321,321,42,51
6:321,33,51,51,51
6:321,42,42,42
6:33,33,33,42
6:33,33,411,42
6:33,411,411,42
6:33,42,42,51,51
6:411,411,411,42
6:411,42,42,51,51
6:51,51,51,51,51,51,51
7:1111111,1111111,61
7:1111111,331,43
7:211111,2221,52
7:211111,322,43
7:22111,22111,52
7:22111,2221,511
7:22111,3211,43
7:22111,331,421
7:2221,2221,43
7:2221,2221,61,61
7:2221,31111,43
7:2221,322,421
7:2221,331,331
7:2221,331,4111
7:2221,43,43,61
7:31111,31111,43
7:31111,322,421
7:31111,331,4111
7:3211,3211,421
7:3211,322,331
7:3211,322,4111
7:3211,331,52,61
7:322,322,322
7:322,322,52,61
7:322,331,511,61
7:322,421,43,61
7:322,43,52,52
7:331,331,43,61
7:331,331,61,61,61
7:331,43,511,52
7:4111,4111,43,61
7:4111,43,511,52
7:421,421,421,61
7:421,421,52,52
7:421,43,43,52
7:421,43,511,511
7:421,43,52,61,61
7:43,43,43,43
7:43,43,43,61,61
7:43,43,61,61,61,61
7:43,52,52,52,61
7:511,511,52,52,61
7:52,52,52,61,61,61
7:61,61,61,61,61,61,61,61
```

$\mathcal{R}_k^{(n)}$: rigid k-tuples of partitions with order n

ord	$\#\bar{\mathcal{R}}_3$	$\#\bar{\mathcal{R}}$	ord	$\#\bar{\mathcal{R}}_3$	$\#\bar{\mathcal{R}}$	ord	$\#\bar{\mathcal{R}}_3$	$\#\bar{\mathcal{R}}$
2	1	1	15	1481	2841	28	114600	190465
3	1	2	16	2388	4644	29	143075	230110
4	3	6	17	3276	6128	30	190766	310804
5	5	11	18	5186	9790	31	235543	371773
6	13	28	19	6954	12595	32	309156	493620
7	20	44	20	10517	19269	33	378063	588359
8	45	96	21	14040	24748	34	487081	763126
9	74	157	22	20210	36078	35	591733	903597
10	142	306	23	26432	45391	36	756752	1170966
11	212	441	24	37815	65814	37	907150	1365027
12	421	857	25	48103	80690	38	1143180	1734857
13	588	1177	26	66409	112636	39	1365511	2031018
14	1004	2032	27	84644	139350	40	1704287	2554015

§ 7. A Kac-Moody root system

We will review the relation between a Kac-Moody root system and the middle convolution which is clarified by [CB].

Let $\mathfrak{h}$ be an infinite dimensional real vector space with the set of basis Π, where

$$\Pi = \{\alpha_0, \alpha_{j,\nu}\,;\, j = 0, 1, 2, \ldots,\ \nu = 1, 2, \ldots\}. \tag{7.1}$$

Put

$$Q := \sum_{\alpha\in\Pi} \mathbb{Z}\alpha \ \supset\ Q_+ := \sum_{\alpha\in\Pi} \mathbb{Z}_{\geq 0}\alpha. \tag{7.2}$$

We define an indefinite inner product on $\mathfrak{h}$ by

$$\begin{aligned}(\alpha|\alpha) &= 2 \qquad (\alpha\in\Pi),\\ (\alpha_0|\alpha_{j,\nu}) &= -\delta_{\nu,1},\\ (\alpha_{i,\mu}|\alpha_{j,\nu}) &= \begin{cases}0 & (i\neq j \ \text{ or } \ |\mu-\nu|>1)\\ -1 & (i=j \ \text{ and } \ |\mu-\nu|=1)\end{cases}.\end{aligned} \tag{7.3}$$

Let $\mathfrak{g}_\infty$ denote the Kac-Moody Lie algebra associated to the Cartan matrix

$$A := \left(\frac{2(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)}\right)_{i,j\in I}, \tag{7.4}$$

$$I := \{0,\ (j,\nu)\,;\, j=0,1,\ldots,\ \nu=1,2,\ldots\}. \tag{7.5}$$

We introduce linearly independent vectors e_0 and $e_{j,\nu}$ $(j=0,1,\ldots,\ \nu=1,2,\ldots)$ with

$$(e_0|e_0)=2,\ (e_0|e_{j,\nu})=-\delta_{\nu,1} \ \text{ and } \ (e_{j,\nu}|e_{j',\nu'})=\delta_{j,j'}\delta_{\nu,\nu'}. \tag{7.6}$$

For a sufficiently large positive integer k let $\mathfrak{h}^k$ be a subspace of $\mathfrak{h}$ spanned by $\{\alpha_0,\ \alpha_{j,\nu}\,;\, j=0,1,\ldots,k,\ \nu=0,1,\ldots\}$. Putting $e_0^k = e_0+e_{0,1}+\cdots+e_{k,1}$, we have $(e_0^k|e_0^k) = 2+(k+1)-2(k+1) = 1-k$. For a sufficiently large k we have an orthogonal basis $\{e_0^k,\ e_{j,\nu}\,;\, j=0,\ldots,k,\ \nu=1,2,\ldots\}$ with

$$\begin{aligned}(e_0^k|e_0^k) &= 1-k, \quad (e_{j,\nu}|e_{j',\nu'})=\delta_{j,j'}\delta_{\nu,\nu'},\\ (e_0^k|e_{j,\nu}) &= 0 \qquad (j=0,\ldots,k,\ \nu=1,2,\ldots)\end{aligned} \tag{7.7}$$

and therefore we may put

$$\begin{aligned}\alpha_0 &= e_0 = e_0^k - e_{0,1} - e_{1,1} - \cdots - e_{k,1},\\ \alpha_{j,\nu} &= e_{j,\nu} - e_{j,\nu+1} \qquad (j=0,\ldots,k,\ \nu=1,2,\ldots).\end{aligned} \tag{7.8}$$

The element

$$\alpha_0(\ell_0,\ldots,\ell_k) := e_0^k - \sum_{j=0}^{k}\sum_{\nu=1}^{\ell_j+1} \frac{e_{j,\nu}}{\ell_j+1} \tag{7.9}$$

is in the space spanned by α_0 and $\alpha_{j,\nu}$ $(j=0,\ldots,k,\ \nu=1,\ldots,\ell_j)$ and it is orthogonal to any $\alpha_{j,\nu}$ for $\nu=1,\ldots,\ell_j$ and $j=0,\ldots,k$.

Remark. We may assume $\ell_0 \geq \ell_1 \geq \cdots \geq \ell_k \geq 1$. It is easy to have

$$\big(\alpha_0(\ell_0,\ldots,\ell_k)|\alpha_0(\ell_0,\ldots,\ell_k)\big) = 1 - k + \sum_{j=0}^{k} \frac{1}{\ell_j+1}$$

$$\begin{cases} >0 & (k=1) \\ >0 & (k=2: \ell_1=\ell_2=1 \text{ or } (\ell_0,\ell_1,\ell_2)=(2,2,1),\ (3,2,1) \text{ or } (4,2,1)) \\ =0 & (k=2: (\ell_0,\ell_1,\ell_2)=(2,2,2),\ (3,3,1) \text{ or } (5,2,1)) \\ <0 & (k=2: \ell_1\geq 2 \text{ and } \ \ell_0+2\ell_1+3\ell_2>12) \\ =0 & (k=3: \ell_0=\ell_1=\ell_2=\ell_3=1) \\ <0 & (k=3: \ell_0>1) \\ <0 & (k\geq 4) \end{cases}$$

The Weyl group W_∞ of $\mathfrak{g}_\infty$ is the subgroup of $O(\mathfrak{h}) \subset GL(\mathfrak{h})$ generated by the simple reflections

$$r_i(x) := x - 2\frac{(x|\alpha_i)}{(\alpha_i|\alpha_i)}\alpha_i = x - (x|\alpha_i)\alpha_i \qquad (x \in \mathfrak{h},\ i \in I). \tag{7.10}$$

A subgroup of W_∞ generated by r_i for $i \in I\setminus\{0\}$ is denoted by W'_∞. Putting $\sigma(\alpha_0) = \alpha_0$ and $\sigma(\alpha_{j,\nu}) = \alpha_{\sigma(j),\nu}$ for $\sigma \in \mathfrak{S}_\infty$, we define a subgroup of $O(\mathfrak{h})$:

$$\widetilde{W}_\infty := \mathfrak{S}_\infty \ltimes W_\infty. \tag{7.11}$$

For a tuple of partitions $\mathbf{m} = \big(m_{j,\nu}\big)_{j\geq 0,\ \nu\geq 1} \in \mathcal{P}_{k+1}^{(n)}$ of n, we define

$$\begin{aligned} &n_{j,\nu} := m_{j,\nu+1} + m_{j,\nu+2} + \cdots, \\ &\alpha_{\mathbf{m}} := n\alpha_0 + \sum_{j=0}^{\infty}\sum_{\nu=1}^{\infty} n_{j,\nu}\alpha_{j,\nu} = ne_0^k - \sum_{j=0}^{\infty}\sum_{\nu=1}^{\infty} m_{j,\nu}e_{j,\nu} \in Q_+. \end{aligned} \tag{7.12}$$

Proposition 7.1. i) $\qquad \operatorname{idx}(\mathbf{m},\mathbf{m}') = (\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}'}).$

ii) *Given $i \in I$, we have $\alpha_{\mathbf{m}'} = r_i(\alpha_{\mathbf{m}})$ with*

$$\mathbf{m}' = \begin{cases} \partial\mathbf{m} & (i=0), \\ (m_{0,1}\ldots, m_{j,1}\ldots \overset{\nu}{\breve{m}}_{j,\nu+1}\overset{\nu+1}{\breve{m}}_{j,\nu}\ldots,\ldots) & \big(i=(j,\nu)\big). \end{cases}$$

Moreover for $\ell = (\ell_0,\ell_1,\ldots) \in \mathbb{Z}_{>0}^{\infty}$ satisfying $\ell_\nu = 1$ for $\nu \gg 1$ we have

$$\alpha_\ell := \alpha_{\mathbf{1}_\ell} = \alpha_0 + \sum_{j=0}^{\infty}\sum_{\nu=1}^{\ell_j-1}\alpha_{j,\nu} = \Big(\prod_{j\geq 0} r_{j,\ell_j-1}\cdots r_{j,2}r_{j,1}\Big)(\alpha_0), \tag{7.13}$$

$$\alpha_{\partial_\ell(\mathbf{m})} = \alpha_{\mathbf{m}} - 2\frac{(\alpha_{\mathbf{m}}|\alpha_\ell)}{(\alpha_\ell|\alpha_\ell)}\alpha_\ell = \alpha_{\mathbf{m}} - (\alpha_m|\alpha_\ell)\alpha_\ell. \tag{7.14}$$

Proof. i) For a sufficiently large positive integer k we have

$$\begin{aligned}
\operatorname{idx}(\mathbf{m},\mathbf{m}') &= \sum_{j=0}^{\infty}\sum_{\nu=1}^{\infty} m_{j,\nu}m'_{j,\nu} - (k-1)\operatorname{ord}\mathbf{m}\cdot\operatorname{ord}\mathbf{m}'\\
&= \sum_{j=1}^{k}(n-n_{j,1})(n'-n'_{j,1}) + \sum_{j=0}^{k}\sum_{\nu=1}^{\infty}(n_{j,\nu}-n_{j,\nu+1})(n'_{j,\nu}-n'_{j,\nu+1}) - (k-1)nn'\\
&= 2nn' + 2\sum_{j=0}^{k} n_{j,\nu}n'_{j,\nu} - \sum_{j=0}^{k}(nn'_{j,1}+n'n_{j,1}) - \sum_{j=0}^{k}\sum_{\nu=1}^{\infty}(n_{j,\nu}n'_{j,\nu+1}+n'_{j,\nu}n_{j,\nu+1})\\
&= (\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}'}).
\end{aligned}$$

The claim ii) easily follows from i). ☐

Remark ([Kc]). The set Δ^{re} of real roots of the Kac-Moody Lie algebra is the W_∞-orbit of Π. Denoting $K := \{\beta\in Q_+\,;\,\operatorname{supp}\beta$ is connected and $(\beta,\alpha)\le 0 \quad (\forall\alpha\in\Pi)\}$, the set of positive imaginary roots Δ_+^{im} equals $W_\infty K$. The set Δ of roots equals $\Delta^{re}\cup\Delta^{im}$ by denoting $\Delta_-^{im} = -\Delta_+^{im}$ and $\Delta^{im} = \Delta_+^{im}\cup\Delta_-^{im}$. Put $\Delta_+ = \Delta\cap Q_+$, $\Delta_- = -\Delta_+$. Then $\Delta = \Delta_+\cup\Delta_-$. The root in Δ is called positive if and only if $\alpha\in Q_+$. Here $\operatorname{supp}\beta = \{\alpha\,;\,n_\alpha\ne 0\}$ if $\beta = \sum_{\alpha\in\Pi} n_\alpha\alpha$. A subset $L\subset\Pi$ is called connected if the decomposition $L_1\cup L_2 = L$ with $L_1\ne\emptyset$ and $L_2\ne\emptyset$ always implies the existence of $v_j\in L_j$ satisfying $(v_1|v_2)\ne 0$.

Lemma 7.2. i) *Let* $\alpha = n\alpha_0 + \sum_{j=0}^{\infty}\sum_{\nu=1}^{\infty} n_{j,\nu}\alpha_{j,\nu}\in\Delta_+$ *with* $\operatorname{supp}\alpha\supsetneqq\{\alpha_0\}$. *Then*

$$n\ge n_{j,1}\ge n_{j,2}\ge n_{j,3}\ge\cdots \qquad (j=0,1,\ldots), \tag{7.15}$$

$$n\le\sum n_{j,1} - \max\{n_{j,1},n_{j,2},\ldots\}. \tag{7.16}$$

ii) *Let* $\alpha = n\alpha_0 + \sum_{j=0}^{\infty}\sum_{\nu=1}^{\infty} n_{j,\nu}\alpha_{j,\nu}\in Q_+$. *Suppose* α *is indivisible, that is,* $\frac{1}{k}\alpha\notin Q$ *for* $k=2,3,\ldots$. *Then* α *corresponds to a basic tuple if and only if*

$$\begin{cases} 2n_{j,\nu}\le n_{j,\nu-1}+n_{j,\nu+1} \quad (n_{j,0}=n,\ j=0,1,\ldots,\ \nu=1,2,\ldots),\\ \quad 2n\le n_{0,1}+n_{1,1}+n_{2,1}+\cdots.\end{cases} \tag{7.17}$$

Proof. The lemma is clear from the following for $\alpha = n\alpha_0+\sum n_{j,\nu}\alpha_{j,\nu}\in\Delta_+$:

$$r_{i,\mu}(\alpha) = n\alpha_0 - \sum\bigl(n_{j,\nu}-\delta_{i,j}\delta_{\mu,\nu}(2n_{j,\mu}-n_{j,\mu-1}-n_{j,\mu+1})\bigr)\alpha_{j,\nu}\in\Delta, \tag{7.18}$$

$$r_0(\alpha) = \bigl(\sum n_{j,1}-n\bigr)\alpha_0 + \sum n_{j,\nu}\alpha_{j,\nu}\in\Delta. \tag{7.19}$$

For example, putting $n_{j,0} = n > 0$ and $r_{i,N} \cdots r_{i,\mu+1} r_{i,\mu}\alpha = n\alpha_0 + \sum n'_{j,\nu}\alpha_{j,\nu} \in \Delta_+$ for a sufficiently large N, we have $n'_{j,N} = n_{j,N} + n_{j,\mu-1} - n_{j,\mu} = n_{j,\mu-1} - n_{j,\mu} \geq 0$ for $\mu = 1, 2, \dots$ and moreover (7.16) by $r_0\alpha \in \Delta_+$. □

Remark. i) It follows from (7.14) that Katz' middle convolution corresponds to the reflection with respect to the root α_ℓ under the identification $\mathcal{P} \subset Q_+$ with (7.12).

Moreover there is a natural correspondence between the set of irreducibly realizable tuples of partitions and the set of positive indivisible roots of $\mathfrak{g}_\infty$ with support containing α_0. Then the rigid (resp. irreducibly realizable non-rigid) tuple of partitions corresponds to the positive real root whose support contains α_0 (resp. indivisible positive imaginary root). The corresponding objects with this identification are as follows.

$\mathcal{P}$	Kac-Moody root system
$\mathbf{m}$	$\alpha_{\mathbf{m}}$ (cf. (7.12))
$\mathbf{m}$: rigid	$\alpha \in \Delta_+^{re}$: $\operatorname{supp}\alpha \ni \alpha_0$
$\mathbf{m}$: basic (cf. (5.23))	$\alpha \in Q_+$: $(\alpha\|\beta) \leq 0$ $(\forall\beta \in \Pi)$ indivisible and $\operatorname{supp}\alpha$ is connected
$\mathbf{m}$: irreducibly realizable	$\alpha \in \Delta_+$: indivisible and $\operatorname{supp}\alpha \ni \alpha_0$
$\operatorname{ord}\mathbf{m}$	n_0 : $\alpha = n_0\alpha_0 + \sum_{j,\nu} n_{j,\nu}\alpha_{j,\nu}$
$\operatorname{idx}(\mathbf{m}, \mathbf{m}')$	$(\alpha_{\mathbf{m}}\|\alpha_{\mathbf{m}'})$
$\operatorname{Pidx}(\mathbf{m}) + \operatorname{Pidx}(\mathbf{m}') = \operatorname{Pidx}(\mathbf{m}+\mathbf{m}')$	$(\alpha_{\mathbf{m}}\|\alpha_{\mathbf{m}'}) = -1$
$(\nu, \nu+1) \in G_j \subset S'_\infty$ (cf. (2.9))	$s_{j,\nu} \in W'_\infty$ (cf. (7.10))
∂ in (5.17)	r_0 in (7.19)
$H \simeq \mathfrak{S}_\infty$ (cf. (2.9))	$\mathfrak{S}_\infty$ in (7.11)
$\langle \partial, S_\infty \rangle$ (cf. Definition 2.2)	$\widetilde{W}_\infty$ in (7.11)

Here we define $\operatorname{Pidx}(\mathbf{m}) := \frac{1}{2} - \operatorname{idx}(\mathbf{m})$ as in Definition 4.2.

ii) For an irreducibly realizable $\mathbf{m} \in \mathcal{P}$, $\partial(\mathbf{m})$ is defined if and only if $\operatorname{ord}\mathbf{m} > 1$ or $\sum_{j=0}^{\infty} m_{j,2} > 1$, which corresponds to (5.4).

iii) Suppose $\mathbf{m} \in \mathcal{P}$ is basic. The subgroup of W_∞ generated by reflections with respect to α_ℓ (cf. (7.13)) satisfying $(\alpha_{\mathbf{m}}|\alpha_\ell) = 0$, is infinite if and only if $\operatorname{idx}\mathbf{m} = 0$.

Note that the condition $(\alpha_{\mathbf{m}}|\alpha_\ell) = 0$ means the corresponding middle convolution of $\mathbf{A}$ with $\operatorname{spt}\mathbf{A} = \mathbf{m}$ doesn't change the partition type.

Proposition 7.3. *For irreducibly realizable* $\mathbf{m} \in \mathcal{P}$ *and* $\mathbf{m}' \in \mathcal{R}$ *satisfying*

$$\operatorname{ord}\mathbf{m} > \operatorname{idx}(\mathbf{m}, \mathbf{m}') \cdot \operatorname{ord}\mathbf{m}', \tag{7.20}$$

we have

$$\mathbf{m}'' := \mathbf{m} - \operatorname{idx}(\mathbf{m}, \mathbf{m}')\mathbf{m}' \text{ is irreducibly realizable,} \tag{7.21}$$

$$\operatorname{idx}\mathbf{m}'' = \operatorname{idx}\mathbf{m}. \tag{7.22}$$

Here (7.20) *is always valid if* $\mathbf{m}$ *is not rigid.*

Proof. The claim corresponds to the fact that an indivisible root transforms into an indivisible root by the reflection with respect to a real root. □

§ 8. A classification of tuples of partitions

In this section we assume that a $(k+1)$-tuple $\mathbf{m} = \big(m_{j,\nu}\big)_{\substack{0\le j\le k\\ 1\le \nu\le n_j}}$ of partitions of a positive integer satisfies

$$m_{j,1} \ge m_{j,2} \ge \cdots \ge m_{j,n_j} \ge 1 \quad \text{and} \quad n_j \ge 2 \quad (j = 0, 1, \ldots, k). \tag{8.1}$$

Note that

$$m_{j,1} + m_{j,2} + \cdots + m_{j,n_j} = \operatorname{ord}\mathbf{m} \ge 2 \quad (j = 0, 1, \ldots, k).$$

Proposition 8.1. *Let* $\mathcal{K}$ *denote the totality of basic elements of* $\mathcal{P}$ *defined in* (5.23) *and for an even integer* p *put*

$$\mathcal{K}(p) := \{\mathbf{m} \in \mathcal{K}\,;\, \operatorname{idx}\mathbf{m} = p\}.$$

Then $\#\mathcal{K}(p) < \infty$*. In particular* $\mathcal{K}(p) = \emptyset$ *if* $p > 0$ *and*

$$\bar{\mathcal{K}}(0) = \{11,11,11,11 \;\; 111,111,111 \;\; 22,1111,1111 \;\; 33,222,111111\}. \tag{8.2}$$

Here $\bar{\mathcal{K}}(p)$ *denotes the quotient of* $\mathcal{K}(p)$ *under the action of the group* S_∞*.*

Proof. It follows from the previous section that $\mathcal{K}$ corresponds to the set of indivisible roots in K under the notation in the remark preceding to Lemma 7.2 and the middle convolution corresponds to an element of W_∞. Since K is the set of complete representatives of Δ_+^{im}, we have the last claim of the proposition.

Let $\mathbf{m} \in \mathcal{K} \cap \mathcal{P}_{k+1}$. We may assume that $\mathbf{m}$ is monotone and indivisible. Since

$$\operatorname{idx}\mathbf{m} + \sum_{j=0}^{k}\sum_{\nu=2}^{n_j}(m_{j,1} - m_{j,\nu})\cdot m_{j,\nu} = \Big(\sum_{j=0}^{k} m_{j,1} - (k-1)\operatorname{ord}\mathbf{m}\Big)\cdot \operatorname{ord}\mathbf{m}, \tag{8.3}$$

the assumption $\mathbf{m} \in \mathcal{K}$ is equivalent to

$$\sum_{j=0}^{k}\sum_{\nu=2}^{n_j}(m_{j,1} - m_{j,\nu})\cdot m_{j,\nu} \le -\operatorname{idx}\mathbf{m}. \tag{8.4}$$

Hence $\operatorname{idx}\mathbf{m} \le 0$.

First suppose $\operatorname{idx}\mathbf{m} = 0$. Then $m_{j,1} = m_{j,2} = \cdots = m_{j,n_j}$ and the identity

$$\sum_{j=0}^{k}\sum_{\nu=1}^{n_j}\frac{(m_{j,1}-m_{j,\nu})m_{j,\nu}}{(\operatorname{ord}\mathbf{m})^2} + \sum_{j=0}^{k}\frac{m_{j,1}}{\operatorname{ord}\mathbf{m}} = k-1+\frac{\operatorname{idx}\mathbf{m}}{(\operatorname{ord}\mathbf{m})^2} \tag{8.5}$$

implies $\sum_{j=0}^{k}\frac{1}{n_j} = k-1$. Since $\sum_{j=0}^{k}\frac{1}{n_j} \le \frac{k+1}{2}$, we have $k \le 3$. When $k=3$, we have $n_0 = n_1 = n_2 = n_3 = 2$. When $k=2$, $\frac{1}{n_0}+\frac{1}{n_1}+\frac{1}{n_2} = 1$ and we easily conclude that $\{n_0, n_1, n_2\}$ equals $\{3,3,3\}$ or $\{2,4,4\}$ or $\{2,3,6\}$.

Since $\operatorname{idx}\mathbf{m} = 2(\operatorname{ord}\mathbf{m})^2 - \sum_{j=0}^{k} N_j$ with $N_j = (\operatorname{ord}\mathbf{m})^2 - \sum_{\nu=0}^{n_j} m_{j,\nu}^2 > 0$, there exist finite number of $\mathbf{m} \in \mathcal{P}$ with a fixed $\operatorname{ord}\mathbf{m}$ and $\operatorname{idx}\mathbf{A}$ because k is bounded. Therefore to prove the remaining part of the lemma we may assume

$$\operatorname{idx}\mathbf{m} \le -2 \quad\text{and}\quad \operatorname{ord}\mathbf{m} \ge -7\operatorname{idx}\mathbf{m} + 7. \tag{8.6}$$

Then

$$\operatorname{ord}\mathbf{m} \ge 21 \quad\text{and}\quad (\operatorname{ord}\mathbf{m})^2 > -147\operatorname{idx}\mathbf{m}. \tag{8.7}$$

If $m_{j,1} > m_{j,n_j} > 0$, (8.4) implies $m_{j,1} - 1 \le -\operatorname{idx}\mathbf{m} \le \frac{1}{7}\operatorname{ord}\mathbf{m} - 1$ and therefore

$$m_{j,1} \le \frac{1}{7}\operatorname{ord}\mathbf{m}, \tag{8.8}$$

$$\sum_{\nu=1}^{n_j} m_{j,\nu}^2 \le m_{j,1}\cdot\operatorname{ord}\mathbf{m} \le \frac{1}{7}(\operatorname{ord}\mathbf{m})^2. \tag{8.9}$$

Hence $2m_{j,1} \le \operatorname{ord}\mathbf{m}$ for $j = 0,\dots,k$,

$$\operatorname{idx}\mathbf{m} + (k-1)\cdot(\operatorname{ord}\mathbf{m})^2 = \sum_{j=0}^{k}\sum_{\nu=1}^{n_j} m_{j,\nu}^2 \le \sum_{j=0}^{k}\frac{1}{2}(\operatorname{ord}\mathbf{m})^2 = \frac{k+1}{2}(\operatorname{ord}\mathbf{m})^2$$

and $\frac{k-3}{2}(\operatorname{ord}\mathbf{m})^2 \le -\operatorname{idx}\mathbf{m} < \frac{1}{7}\operatorname{ord}\mathbf{m}$, which proves $k \le 3$.

Suppose $k=3$. Since $\mathbf{m} \ne 11,11,11,11$, we have $m_{j,1} \le \frac{1}{3}\operatorname{ord}\mathbf{m}$ with a suitable j,

$$\begin{aligned}\operatorname{idx}\mathbf{m} &= \sum_{j=0}^{3}\sum_{\nu=1}^{n_j} m_{j,\nu}^2 - 2\cdot(\operatorname{ord}\mathbf{m})^2 \le \sum_{j=0}^{3} m_{j,1}\operatorname{ord}\mathbf{m} - 2(\operatorname{ord}\mathbf{m})^2\\ &\le (\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{3}-2)(\operatorname{ord}\mathbf{m})^2 = -\tfrac{1}{6}(\operatorname{ord}\mathbf{m})^2\end{aligned}$$

and $\operatorname{ord}\mathbf{m} \le -\frac{6\operatorname{idx}\mathbf{m}}{\operatorname{ord}\mathbf{m}} \le -\frac{2}{7}\operatorname{idx}\mathbf{m}$, which contradicts to (8.6).

Suppose $k=2$ and put $J = \{j\,;\, m_{j,1} \ne m_{j,n_j} \quad (j = 0,1,2)\}$. Then

$$1 + \frac{\operatorname{idx}\mathbf{m}}{(\operatorname{ord}\mathbf{m})^2} = \frac{\sum_{\nu=1}^{n_0} m_{0,\nu}^2}{(\operatorname{ord}\mathbf{m})^2} + \frac{\sum_{\nu=1}^{n_1} m_{1,\nu}^2}{(\operatorname{ord}\mathbf{m})^2} + \frac{\sum_{\nu=1}^{n_2} m_{2,\nu}^2}{(\operatorname{ord}\mathbf{m})^2}$$

and therefore

$$1-\frac{1}{147}-\frac{\#J}{7}<\sum_{j\in\{0,1,2\}\setminus J}\frac{1}{n_j}<1$$

because of (8.9) for $j\in J$. Lemma 8.2 assures that this never holds. Here we note that $1-\frac{1}{147}-\frac{3}{7}>0$, $1-\frac{1}{147}-\frac{2}{7}>\frac{1}{2}$, $1-\frac{1}{147}-\frac{1}{7}>\frac{5}{6}$ and $1-\frac{1}{147}>\frac{41}{42}$ according to $\#J=3,2,1,0$, respectively. $\square$

Lemma 8.2. *Put* $I_{k+1}=\Big\{\sum_{j=0}^k\frac{1}{n_j}\,;\,n_j\in\{2,3,4,\ldots\}\Big\}\cap[0,1)$. *Then*

$$I_1\subset(0,\tfrac{1}{2}],\ I_2\subset(0,\tfrac{5}{6}]\ \text{ and }\ I_3\subset(0,\tfrac{41}{42}].$$

Proof. Let $r\in I_{k+1}$. It is clear that $r\le\frac{1}{2}$ for $r\in I_1$.
Let $r=\frac{1}{n_0}+\frac{1}{n_1}\in I_2$. If $n_0=2$, then $n_1\ge3$ and $r\le\frac{5}{6}$. If $n_0\ge3$, then $r\le\frac{2}{3}$.
Let $r=\frac{1}{n_0}+\frac{1}{n_1}+\frac{1}{n_2}\in I_3$. We may assume $n_0\le n_1\le n_2$.
If $n_0\le4$, then $r\le\frac{3}{4}$.
Suppose $n_0=3$. If $n_1\ge4$, $r\le\frac{5}{6}$. If $n_1=3$, then $n_2\ge4$ and $r\le\frac{11}{12}$.
Suppose $n_0=2$. Then $n_1\ge3$. If $n_1=3$, then $n_2>6$ and $r\le\frac{41}{42}$. If $n_1\ge4$, then $n_2>4$ and $r\le\frac{19}{20}$. $\square$

Remark. i) $\bar{\mathcal{K}}(0)$ is given in [Ko2] and its elements correspond to the indivisible positive null-roots α of the affine root systems $\tilde{D}_4$, $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ (cf. Remark after (7.9), Proposition 7.1 and Table $\bar{\mathcal{K}}(0)$).

ii) In the proof we showed $\operatorname{ord}\mathbf{m}+7\operatorname{idx}\mathbf{m}\le6$ for $\mathbf{m}\in\mathcal{K}$ but we can prove

$$\operatorname{ord}\mathbf{m}+3\operatorname{idx}\mathbf{m}\le6\quad\text{for}\quad\mathbf{m}\in\mathcal{K},\tag{8.10}$$

$$\operatorname{ord}\mathbf{m}+\operatorname{idx}\mathbf{m}\le2\quad\text{for}\quad\mathbf{m}\in\mathcal{K}\setminus\mathcal{P}_3.\tag{8.11}$$

Example 8.3. For a positive integer m we have special 4 elements

$$\begin{aligned}&mm-11,m^2,m^2,m^2 &&m^2m-11,m^3,m^3\\ &m^3m-11,m^4,(2m)^2 &&m^5m-11,(2m)^3,(3m)^2\end{aligned}\tag{8.12}$$

in $\bar{\mathcal{K}}(2-2m)$ with orders $2m$, $3m$, $4m$ and $6m$, respectively.

Proposition 8.4. *We have*

$$\begin{aligned}\bar{\mathcal{K}}(-2)=\big\{&11,11,11,11,11\ \ 21,21,111,111\ \ 31,22,22,1111\ \ 22,22,22,211\\ &211,1111,1111\ \ 221,221,11111\ \ 32,11111,11111\ \ 222,222,2211\\ &33,2211,111111\ \ 44,2222,22211\ \ 44,332,11111111\ \ 55,3331,22222\\ &66,444,2222211\big\}.\end{aligned}$$

Proof. Let $\mathbf{m} \in \mathcal{K}(-2) \cap \mathcal{P}_{k+1}$ be monotone. Then (8.4) and (8.3) with $\operatorname{idx} \mathbf{m} = -2$ implies $\sum (m_{j,1} - m_{j,\nu}) m_{j,\nu} = 0$ or 2 and we have the following 5 possibilities.

(A) $m_{0,1} \ldots m_{0,n_0} = 2 \ldots 211$ and $m_{j,1} = m_{j,n_j}$ for $1 \le j \le k$.

(B) $m_{0,1} \ldots m_{0,n_0} = 3 \ldots 31$ and $m_{j,1} = m_{j,n_j}$ for $1 \le j \le k$.

(C) $m_{0,1} \ldots m_{0,n_0} = 3 \ldots 32$ and $m_{j,1} = m_{j,n_j}$ for $1 \le j \le k$.

(D) $m_{i,1} \ldots m_{i,n_0} = 2 \ldots 21$ and $m_{j,1} = m_{j,n_j}$ for $0 \le i \le 1 < j \le k$.

(E) $m_{j,1} = m_{j,n_j}$ for $0 \le j \le k$ and $\operatorname{ord} \mathbf{m} = 2$.

Case (A). If $2 \cdots 211$ is replaced by $2 \cdots 22$, $\mathbf{m}$ is transformed into $\mathbf{m}'$ with $\operatorname{idx} \mathbf{m}' = 0$. If $\mathbf{m}'$ is indivisible, $\mathbf{m}' \in \mathcal{K}(0)$ and $\mathbf{m}$ is $211, 1^4, 1^4$ or $33, 2211, 1^6$. If $\mathbf{m}'$ is not indivisible, $\frac{1}{2}\mathbf{m}' \in \mathcal{K}(0)$ and $\mathbf{m}$ is one of the tuples given in (8.12) with $m = 2$.

Put $m = n_0 - 1$ and examine the identity (8.5).

Case (B). $\frac{9m+1}{(3m+1)^2} + \frac{1}{n_1} + \cdots + \frac{1}{n_k} = k - 1 - \frac{2}{(3m+1)^2}$. Since $n_j \ge 2$, $\frac{1}{2}k - 1 \le \frac{9m+1+2}{(3m+1)^2} = \frac{3}{3m+1} < 1$ and $k \le 3$.

If $k = 3$, we have $m = 1$, $\operatorname{ord} \mathbf{m} = 4$, $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = \frac{5}{4}$, $\{n_1, n_2, n_3\} = \{2, 2, 4\}$ and $\mathbf{m} = 31, 22, 22, 1111$.

Assume $k = 2$. Then $\frac{1}{n_1} + \frac{1}{n_2} = 1 - \frac{3}{3m+1}$ and Lemma 8.2 implies $m \le 5$. We have $1 - \frac{3}{3m+1} = \frac{13}{16}, \frac{10}{13}, \frac{7}{10}, \frac{4}{7}$ and $\frac{1}{4}$ according to $m = 5$, 4, 3, 2 and 1, respectively. Hence we have $m = 3$, $\{n_1, n_2\} = \{2, 5\}$ and $\mathbf{m} = 3331, 55, 22222$.

Case (C). $\frac{9m+4}{(3m+2)^2} + \frac{1}{n_1} + \cdots + \frac{1}{n_k} = k - 1 - \frac{2}{(3m+2)^2}$. Since $n_j \ge 2$, $\frac{1}{2}k - 1 \le \frac{9m+4+2}{(3m+2)^2} = \frac{3}{3m+2} < 1$ and $k \le 3$. If $k = 3$, then $m = 1$, $\operatorname{ord} \mathbf{m} = 5$ and $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = \frac{7}{5}$, which never occurs.

Thus we have $k = 2$, $\frac{1}{n_1} + \frac{1}{n_2} = 1 - \frac{3}{3m+2}$ and Lemma 8.2 implies $m \le 5$. We have $\frac{14}{17}, \frac{11}{14}, \frac{8}{11}, \frac{5}{8}$ and $\frac{2}{5}$ according to $m = 5$, 4, 3, 2 and 1, respectively. Hence we have $m = 1$ and $n_1 = n_2 = 5$ and $\mathbf{m} = 32, 11111, 11111$ or $m = 2$ and $n_1 = 2$ and $n_2 = 8$ and $\mathbf{m} = 332, 44, 11111111$.

Case (D). $\frac{2(4m+1)}{(2m+1)^2} + \frac{1}{n_2} + \cdots + \frac{1}{n_k} = k - 1 - \frac{2}{(2m+1)^2}$. Since $n_j \ge 3$ for $j \ge 2$, we have $k - 1 \le \frac{3}{2} \frac{2(4m+2)}{(2m+1)^2} = \frac{6}{2m+1}$ and $m \le 2$. If $m = 1$, then $k \le 3$ and $\frac{1}{n_2} + \frac{1}{n_3} = 2 - \frac{4}{3} = \frac{2}{3}$ and we have $\mathbf{m} = 21, 21, 111, 111$. If $m = 2$, then $k = 2$, $\frac{1}{n_2} = 1 - \frac{4}{5}$ and $\mathbf{m} = 221, 221, 11111$.

Case (E). Since $\sum_{j=0}^{k} 2m_{j,1} - 4(k-1) = -2$, we have $m_{j,1} = 1$, $k = 4$ and $\mathbf{m} = 11, 11, 11, 11, 11$. $\square$

By the aid of a computer we have the following tables.

Table of $\#\bar{\mathcal{K}}(p)$ for the rigidity indices p.

index	0	-2	-4	-6	-8	-10	-12	-14	-16	-18	-20
$\#\bar{\mathcal{K}}(p)$	4	13	36	67	90	162	243	305	420	565	720
# triplets	3	9	24	44	56	97	144	163	223	291	342
# 4-tuples	1	3	9	17	24	45	68	95	128	169	239

Table of (ord $\mathbf{m}$: $\mathbf{m}$) of $\bar{\mathcal{K}}(-4)$ ($*$ corresponds to (8.12) and + means $\partial_{max}(\mathbf{m}) \neq \mathbf{m}$))

+2:11,11,11,11,11,11	3:111,21,21,21,21	4:22,22,22,31,31
+3:111,111,111,21	+4:1111,22,22,22	4:1111,1111,31,31
4:211,211,22,22	4:1111,211,22,31	*6:321,33,33,33
6:222,222,33,51	+4:1111,1111,1111	5:11111,11111,311
5:11111,2111,221	6:111111,222,321	6:111111,21111,33
6:21111,222,222	6:111111,111111,42	6:222,33,33,42
6:111111,33,33,51	6:2211,2211,222	7:1111111,2221,43
7:1111111,331,331	7:2221,2221,331	8:11111111,3311,44
8:221111,2222,44	8:22211,22211,44	*9:3321,333,333
9:111111111,333,54	9:22221,333,441	10:1111111111,442,55
10:22222,3322,55	10:222211,3331,55	12:22221111,444,66
*12:33321,3333,66	14:2222222,554,77	*18:3333321,666,99

We write the root $\alpha_{\mathbf{m}}$ for $\mathbf{m} \in \bar{\mathcal{K}}(0)$ and $\bar{\mathcal{K}}(-2)$ using Dynkin diagram.

Table $\bar{\mathcal{K}}(0)$

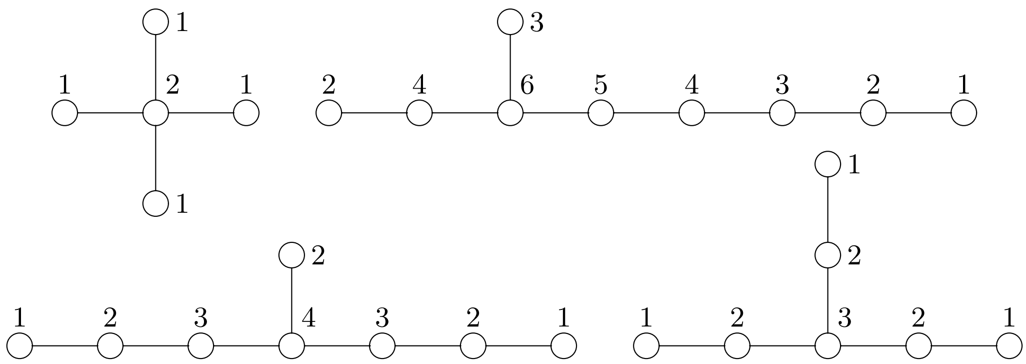
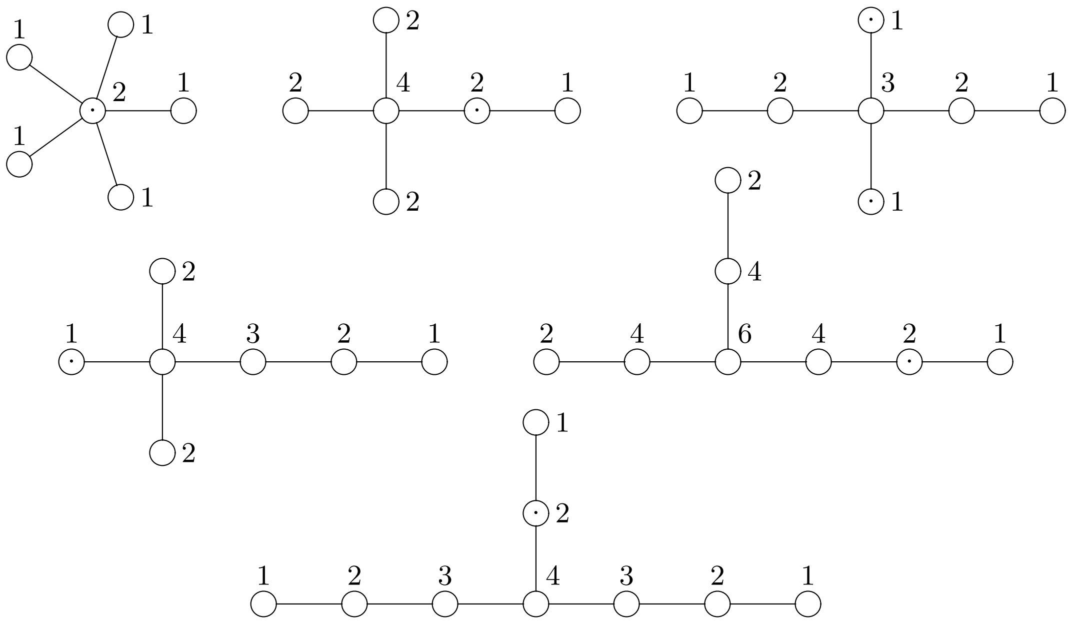
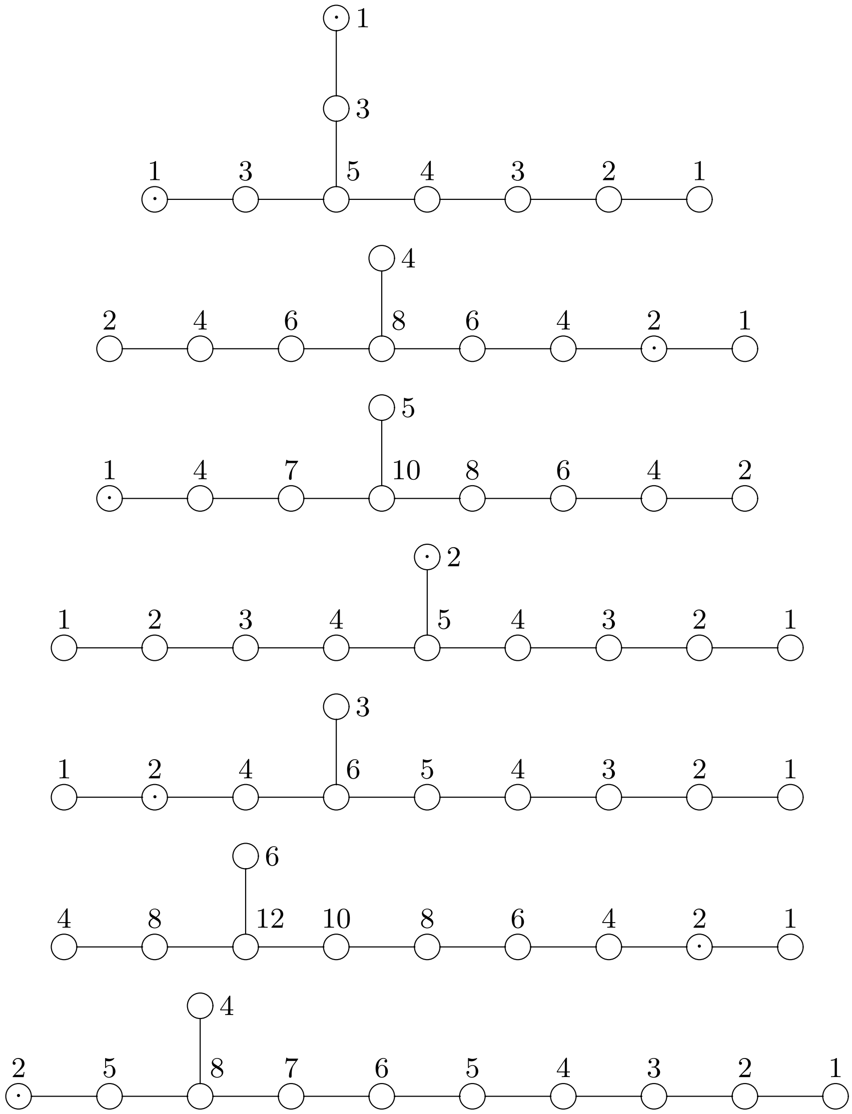


Table $\bar{\mathcal{K}}(-2)$

Dotted circles mean simple roots which are not orthogonal to the root.





§ 9. Connection problems

Fix $\mathbf{m} = \bigl(m_{j,\nu}\bigr)_{\substack{j=0,\dots,k\\ \nu=1,\dots,n_j}} \in \mathcal{P}_{k+1}^{(n)}$ in this section. For $\lambda_{j,\nu} \in \mathbb{C}$ and $\mu \in \mathbb{C}$ we put

$$\{\lambda_{\mathbf{m}}\} := \left\{\begin{matrix} [\lambda_{0,1}]_{(m_{0,1})} & \cdots & [\lambda_{k,1}]_{(m_{k,1})} \\ \vdots & \vdots & \vdots \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & \cdots & [\lambda_{k,n_k}]_{(m_{k,n_k})} \end{matrix}\right\}, \quad [\mu]_{(p)} := \begin{pmatrix} \mu \\ \mu+1 \\ \vdots \\ \mu+p-1 \end{pmatrix}.$$

We may identify $\{\lambda_{\mathbf{m}}\}$ with an element of $M(n, k+1, \mathbb{C})$.

Definition 9.1. A tuple $\mathbf{m} \in \mathcal{R}_{k+1}$ is a *rigid sum* of $\mathbf{m}'$ and $\mathbf{m}''$ if

$$\mathbf{m} = \mathbf{m}' + \mathbf{m}'' \quad \text{and} \quad \mathbf{m}',\ \mathbf{m}'' \in \mathcal{R}_{k+1} \tag{9.1}$$

and we express this by $\mathbf{m} = \mathbf{m}' \oplus \mathbf{m}''$, which we call a *rigid decomposition* of $\mathbf{m}$.

Theorem 9.2. i) *Fix $k+1$ points $\{z_0, \ldots, z_k\} \subset \mathbb{C} \cup \{\infty\}$ and $\mathbf{m} \in \mathcal{R}_{k+1}$. Assume $\lambda_{j,\nu} \in \mathbb{C}$ are generic under the Fuchs relation $|\{\lambda_{\mathbf{m}}\}| = 0$ with*

$$|\{\lambda_{\mathbf{m}}\}| := \sum_{j=0}^{k} \sum_{\nu=0}^{n_j} m_{j,\nu}\lambda_{j,\nu} - \operatorname{ord}\mathbf{m} + 1. \tag{9.2}$$

Then there uniquely exists a single Fuchsian differential equation $Pu = 0$ of order n with regular singularities at $\{z_0, \ldots, z_k\}$ such that the set of exponents at z_j is equal to that of components of the $(j+1)$-th column of $\{\lambda_{\mathbf{m}}\}$ and moreover that the local monodromies are semisimple at z_j for $j = 0, \ldots, k$.

ii) *Assume $k = 2$, $m_{0,n_0} = m_{1,n_1} = 1$ and $m_{j,\nu} > 0$ for $\nu = 1, \ldots, n_j$ and $j = 0, 1, 2$. Let $c(\lambda_{0,n_0} \rightsquigarrow \lambda_{1,n_1})$ denote the connection coefficient from the normalized local solution of $Pu = 0$ in i) corresponding to the exponent λ_{0,n_0} at z_0 to the normalized local solution corresponding to the exponent λ_{1,n_1} at z_1. Then*

$$c(\lambda_{0,n_0} \rightsquigarrow \lambda_{1,n_1}) = \frac{\displaystyle\prod_{\nu=1}^{n_0-1} \Gamma\bigl(\lambda_{0,n_0} - \lambda_{0,\nu} + 1\bigr) \cdot \prod_{\nu=1}^{n_1-1} \Gamma\bigl(\lambda_{1,\nu} - \lambda_{1,n_1}\bigr)}{\displaystyle\prod_{\substack{\mathbf{m}'\oplus\mathbf{m}''=\mathbf{m}\\ m'_{0,n_0}=m''_{1,n_1}=1}} \Gamma\bigl(|\{\lambda_{\mathbf{m}'}\}|\bigr)}, \tag{9.3}$$

$$\sum_{\substack{\mathbf{m}'\oplus\mathbf{m}''=\mathbf{m}\\ m'_{0,n_0}=m''_{1,n_1}=1}} m'_{j,\nu} = (n_1 - 1)m_{j,\nu} - \delta_{j,0}(1 - n_0\delta_{\nu,n_0}) + \delta_{j,1}(1 - n_1\delta_{\nu,n_1}) \qquad (0 \le j \le 2,\ 1 \le \nu \le n_j). \tag{9.4}$$

Remark. i) Putting $(j,\nu) = (0, n_0)$ in (9.4) or considering the sum $\sum_\nu$ for (9.4) with $j = 1$, we have

$$\#\{\mathbf{m}'\,;\, \mathbf{m}' \oplus \mathbf{m}'' = \mathbf{m} \ \text{ with } \ m'_{0,n_0} = m''_{0,n_1} = 1\} = n_0 + n_1 - 2, \tag{9.5}$$

$$\sum_{\substack{\mathbf{m}'\oplus\mathbf{m}''=\mathbf{m}\\ m'_{0,n_0}=m''_{1,n_1}=1}} \operatorname{ord}\mathbf{m}' = (n_1 - 1)\operatorname{ord}\mathbf{m}. \tag{9.6}$$

ii) We may consider $\{\lambda_{\mathbf{m}}\}$ as a Riemann scheme of the Fuchsian equation with the condition that the local monodromy at the singular point is semisimple for generic $\lambda_{j,\nu}$ under the Fuchs condition. The equation for a non-generic $\lambda_{j,\nu}$ is defined by the analytic continuation. The corresponding Riemann scheme will be denoted by $P\{\lambda_{\mathbf{m}}\}$.

iii) A proof of this theorem and related results will be given in another paper. The proof is a generalization of that of Gauss summation formula for Gauss hypergeometric series due to Gauss, which doesn't use integral representations of the solutions.

iv) In the theorem the condition $k = 2$ means that there exists no geometric moduli in the Fuchsian equation and we may assume $(z_0, z_1, z_2) = (0, 1, \infty)$. By the transformation of the solutions $u \mapsto z^{-\lambda_{0,n_0}}(1-z)^{-\lambda_{1,n_1}}u$ we may moreover assume $\lambda_{0,n_0} = \lambda_{1,n_1} = 0$. Then the meaning of "normalized local solution" is clear under the condition $m_{0,n_0} = m_{1,n_1} = 1$.

v) By the aid of a computer the author obtained the table of the concrete connection coefficients (9.3) for $\operatorname{ord}\mathbf{m} \le 40$ together with checking (9.4), which contains 4,111,704 cases.

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