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Uniqueness by Dirichlet-to-Neumann map on an arbitrary part of boundary in two dimensions

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UNIQUENESS BY DIRICHLET-TO-NEUMANN MAP ON AN ARBITRARY PART OF BOUNDARY IN TWO DIMENSIONS

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Abstract. We prove for a two dimensional bounded simply connected domain that the Cauchy data for the Schrödinger equation measured on an arbitrary open subset of the boundary determines uniquely the potential. This implies, for the conductivity equation, that if we measure the current fluxes at the boundary on an arbitrary open subset of the boundary produced by voltage potentials supported in the same subset, we can determine uniquely the conductivity. We use Carleman estimates with degenerate weight functions to construct appropriate complex geometrical optics solutions to prove the results.

1. Introduction

We consider the problem of determining a complex-valued potential q in a bounded simply connected two dimensional domain from the Cauchy data measured on an arbitrary open subset of the boundary for the associated Schrödinger equation $\Delta + q$. A motivation comes from the classical inverse problem of electrical impedance tomography problem. In this inverse problem one attempts to determine the electrical conductivity of a body by measurements of voltage and current on the boundary of the body. This problem was proposed by Calderón [7] and is also known as Calderón's problem. In dimensions $n \geq 3$, the first global uniqueness result for C^2-conductivities was proven in [23]. In [21] the global uniqueness result was extended to less regular conductivities. Also see [12] as for the determination of more singular conormal conductivities. In dimension $n \geq 3$ global uniqueness was shown for the Schrödinger equation with bounded potentials in [23]. The case of more singular potentials was considered in [3], [21]. The case of more singular conormal potentials was studied in [12].

In two dimensions the first global uniqueness result for Calderón's problem was obtained in [20] for C^2-conductivities. Later the regularity assumptions were relaxed in [4], [3] and [1]. In particular, the paper [1] proves uniqueness for L^∞- conductivities. In two dimensions a recent result of Bukgheim [5] gives unique identifiability of the potential from the Cauchy data for the associated Schrödinger equation. As for the uniqueness in determining two coefficients, see [8], [16].

In all the above mentioned articles, the measurements are made on the whole boundary. The purpose of this paper is to show the global uniqueness in two dimensions, both for the Schrödinger and conductivity equation, by measuring all the Neumann data on an arbitrary open subset $\widetilde{\Gamma}$ of the boundary produced by inputs of Dirichlet data supported on $\widetilde{\Gamma}$. We

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formulate this inverse problem more precisely below. Let $\Omega \subset \mathbf{R}^2$ be a simply connected bounded domain with smooth boundary, and let ν be the unit outward normal vector to $\partial\Omega$. We denote $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$. A bounded and non-zero function $\gamma(x)$ (possibly complex-valued) models the electrical conductivity of Ω. Then a potential $u \in H^1(\Omega)$ satisfies the Dirichlet problem

$$\begin{aligned} \mathrm{div}(\gamma\nabla u) &= 0 \text{ in } \Omega, \\ u\big|_{\partial\Omega} &= f, \end{aligned} \tag{1.1}$$

where $f \in H^{\frac{1}{2}}(\partial\Omega)$ is a given boundary voltage potential. The Dirichlet-to-Neumann (DN) map is defined by

$$\Lambda_\gamma(f) = \gamma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}. \tag{1.2}$$

This problem can be reduced to studying the set of Cauchy data for the Schrödinger equation with the potential q given by:

$$q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}. \tag{1.3}$$

More generally we define the set of Cauchy data for a bounded potential q by:

$$\widehat{C_q} = \left\{ \left(u|_{\partial\Omega}, \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} \right) \mid (\Delta + q)u = 0 \text{ on } \Omega, \quad u \in H^1(\Omega) \right\}. \tag{1.4}$$

We have $\widehat{C_q} \subset H^{\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$.

Let $\widetilde{\Gamma} \subset \partial\Omega$ be a non-empty open subset of the boundary. Denote $\Gamma_0 = \partial\Omega \setminus \widetilde{\Gamma}$.

Our main result gives global uniqueness by measuring the Cauchy data on $\widetilde{\Gamma}$. Let $q_j \in C^{1+\alpha}(\overline{\Omega})$, $j = 1, 2$ for some $\alpha > 0$ and let q_j be complex-valued. Consider the following sets of Cauchy data on an $\widetilde{\Gamma}$:

$$\mathcal{C}_{q_j} = \left\{ \left(u|_{\widetilde{\Gamma}}, \frac{\partial u}{\partial \nu}\Big|_{\widetilde{\Gamma}} \right) \mid (\Delta + q_j)u = 0 \text{ on } \Omega, \, u|_{\Gamma_0} = 0, \; u \in H^1(\Omega) \right\}, \quad j = 1, 2. \tag{1.5}$$

Theorem 1.1. *Assume $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$. Then $q_1 \equiv q_2$.*

Using Theorem 1.1 one concludes immediately as a corollary the following global identifiability result for the conductivity equation (1.1).

Corollary 1.1. *With some $\alpha > 0$, let $\gamma_j \in C^{3+\alpha}(\overline{\Omega})$, $j = 1, 2$, be non-vanishing functions. Assume that $\gamma_1 = \gamma_2$ on $\partial\Omega$ and*

$$\Lambda_{\gamma_1} u = \Lambda_{\gamma_2} u \text{ in } \widetilde{\Gamma} \text{ for all } u \in H^{\frac{1}{2}}(\Gamma), \; \mathrm{supp}\, u \subset \widetilde{\Gamma}.$$

Then $\gamma_1 = \gamma_2$.

To the authors' knowledge, there are no uniqueness results similar to Theorem 1.1 with Dirichlet data supported and Neumann data measured on the same arbitrary open subset of the boundary, even for smooth potentials or conductivities. In dimensions $n \geq 3$, [6] proves global uniqueness in determining a bounded potential on with Dirichlet data supported on the whole boundary and Neumann data measured in roughly half the boundary. The proof relies on a Carleman estimate with a linear weight function. This implies a similar result for the conductivity equation with C^2 conductivities. In [18] the regularity assumption on the conductivity was relaxed to $C^{3/2+\alpha}$ with some $\ell > 0$. The corresponding stability estimates are proved in [13]. As for the stability estimates for the magnetic Schrödinger equation with partial data, see [24]. In [17], the result in [6] was generalized to show that by all possible pairs of Dirichlet data on an arbitrary open subset Γ_+ of the boundary and Neumann data on a slightly larger open domain than $\partial\Omega \setminus \Gamma_+$, one can uniquely determine the potential. The method of the proof uses Carleman estimates with non-linear weights. The case of the magnetic Schrödinger equation was considered in [9] and an improvement on the regularity of the coefficients is done in [19]. Stability estimates for the magnetic Schrödinger equation with partial data were proven in [24].

In two dimensions the first general result was given by the authors in [15]. It is shown that the same global uniqueness result as [17] holds in this case. The two dimensional case has special features since one can construct a much larger set of complex geometrical optics solutions than in higher dimensions. On the other hand, the problem is not formally overdetermined and therefore more difficult. The proof of our main result here follows the same broad outline of [15] and is based on the construction of appropriate complex geometrical optics solutions by Carleman estimates with degenerate weight functions. However, we need a much more delicate analysis of the solutions.

This paper is composed of four sections. In Section 2, we establish our key Carleman estimates, and in Section 3, we construct complex geometrical optics solutions. In Section 4, we complete the proof of Theorem 1.1.

2. Carleman estimates with degenerate weights

Throughout the paper we use the following notations:

Notations

$i = \sqrt{-1}$, $x_1, x_2, \xi_1, \xi_2 \in \mathbf{R}$, $z = x_1 + ix_2$, $\zeta = \xi_1 + i\xi_2$, $\overline{z}$ denotes the complex conjugate of $z \in \mathbf{C}$. We identify $x = (x_1, x_2) \in \mathbf{R}^2$ with $z = x_1 + ix_2 \in \mathbf{C}$. $\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$, $\partial_{\overline{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$, $H^{1,\tau}(\Omega)$ denotes the space $H^1(\Omega)$ with norm $\|v\|^2_{H^{1,\tau}(\Omega)} = \|v\|^2_{H^1(\Omega)} + \tau^2\|v\|^2_{L^2(\Omega)}$. The tangential derivative on the boundary is given by $\partial_\tau = \nu_2\frac{\partial}{\partial x_1} - \nu_1\frac{\partial}{\partial x_2}$, with $\nu = (\nu_1, \nu_2)$ the unit outer normal to $\partial\Omega$, $B(\widehat{x}, \delta) = \{x \in \mathbf{R}^2 | |x - \widehat{x}| < \delta\}$, $f(x) : \mathbf{R}^2 \to \mathbf{R}^1$, f'' is the Hessian matrix with entries $\frac{\partial^2 f}{\partial x_i \partial x_j}$. $\mathcal{L}(X, Y)$ denotes the Banach space of all bounded linear operators from a Banach space X to another Banach space Y.

By using a conformal map, thanks to the Kellog-Warchawski theorem (see e.g. p 42, [22]), without loss of generality we assume that $\Omega = B(0, 1)$.

Let $\Phi(z) = \varphi(x_1, x_2) + i\psi(x_1, x_2) \in C^2(\overline{\Omega})$ be a holomorphic function in Ω with real-valued φ and ψ:

$$\partial_{\overline{z}}\Phi(z) = 0 \quad \text{in } \Omega. \tag{2.1}$$

Denote by $\mathcal{H}$ the set of critical points of a function Φ

$$\mathcal{H} = \{z \in \overline{\Omega} | \partial_z \Phi(z) = 0\}.$$

Assume that Φ has no critical points on the boundary, and that all the critical points are nondegenerate:

$$\mathcal{H} \cap \partial\Omega = \{\emptyset\}, \quad \partial_z^2\Phi(z) \neq 0, \quad \forall z \in \mathcal{H}. \tag{2.2}$$

Then we know that Φ has only a finite number of critical points and we can set:

$$\mathcal{H} = \{\widetilde{x}_1, ..., \widetilde{x}_\ell\}. \tag{2.3}$$

Consider the following problem

$$\Delta u + q_0 u = f \quad \text{in } \Omega, \quad u|_{\Gamma_0} = g, \tag{2.4}$$

where ν is the unit outward normal vector to $\partial\Omega$ and

$$\Gamma_0 = \{x \in \partial\Omega | (\nu, \nabla\varphi) = 0\}.$$

We have

Proposition 2.1. *Let $q_0 \in L^\infty(\Omega)$. There exists $\tau_0 > 0$ such that for all $|\tau| > \tau_0$ there exists a solution to problem (2.4) such that*

$$\|ue^{-\tau\varphi}\|_{L^2(\Omega)} \leq C(\|fe^{-\tau\varphi}\|_{L^2(\Omega)}/\tau + \|ge^{-\tau\varphi}\|_{L^2(\Gamma_0)}/\tau^{\frac{1}{4}}). \tag{2.5}$$

For the proof, see Proposition 2.2 in [15].

Let us introduce the operators:

$$\partial_{\overline{z}}^{-1} g = \frac{1}{2\pi i}\int_\Omega \frac{g(\xi_1,\xi_2)}{\zeta - z} d\zeta \wedge d\overline{\zeta} = -\frac{1}{\pi}\int_\Omega \frac{g(\xi_1,\xi_2)}{\zeta - z} d\xi_1 d\xi_2,$$

$$\partial_z^{-1} g = -\frac{1}{2\pi i}\overline{\int_\Omega \frac{\overline{g}(\xi_1,\xi_2)}{\zeta - z} d\zeta \wedge d\overline{\zeta}} = -\frac{1}{\pi}\int_\Omega \frac{g(\xi_1,\xi_2)}{\overline{\zeta} - \overline{z}} d\xi_1 d\xi_2 = \overline{\partial_{\overline{z}}^{-1}\overline{g}}.$$

See e.g., pp.28-31 in [26] where $\partial_{\overline{z}}^{-1}$ and ∂_z^{-1} are denoted by T and $\overline{T}$ respectively. Then we know (e.g., p.47 and p.56 in [26]):

Proposition 2.2. A) *Let $m \geq 0$ be an integer number and $\alpha \in (0,1)$. The operators $\partial_{\overline{z}}^{-1}, \partial_z^{-1} \in \mathcal{L}(C^{m+\alpha}(\overline{\Omega}), C^{m+\alpha+1}(\overline{\Omega}))$.*
B) *Let $1 \leq p \leq 2$ and $1 < \gamma < \frac{2p}{2-p}$. Then $\partial_{\overline{z}}^{-1}, \partial_z^{-1} \in \mathcal{L}(L^p(\Omega), L^\gamma(\Omega))$.*

We define two other operators:

$$R_{\Phi,\tau} g = e^{\tau(\overline{\Phi(z)} - \Phi(z))}\partial_{\overline{z}}^{-1}(ge^{\tau(\Phi(z) - \overline{\Phi(z)})}), \quad \widetilde{R}_{\Phi,\tau} g = e^{\tau(\overline{\Phi(z)} - \Phi(z))}\partial_z^{-1}(ge^{\tau(\Phi(z) - \overline{\Phi(z)})}). \tag{2.6}$$

Proposition 2.3. *Let $g \in C^\alpha(\overline{\Omega})$ for some positive α. The function $R_{\Phi,\tau}g$ is a solution to*

$$\partial_{\overline{z}} R_{\Phi,\tau} g - \tau(\overline{\partial_z\Phi(z)}) R_{\Phi,\tau} g = g \quad \text{in } \Omega. \tag{2.7}$$

The function $\widetilde{R}_{\Phi,\tau}g$ solves

$$\partial_z \widetilde{R}_{\Phi,\tau} g + \tau(\partial_z\Phi(z))\widetilde{R}_{\Phi,\tau} g = g \quad \text{in } \Omega. \tag{2.8}$$

The proof is done by direct computations (see the proof of Proposition 3.3 in [15]).

Denote

$$\mathcal{O}_\epsilon = \{x \in \Omega | dist(x, \partial\Omega) \leq \epsilon\} = \{x \in B(0,1) | \, 1 - \epsilon < |x| < 1\}.$$

Proposition 2.4. *Let $g \in C^1(\Omega)$ and $g|_{\mathcal{O}_\epsilon} = 0$, $g(x) \neq 0$ for all $x \in \mathcal{H}$. Then*

$$|R_{\Phi,\tau}g(x)| + |\widetilde{R}_{\Phi,\tau}g(x)| \le C \max_{x\in\mathcal{H}} |g(x)|/\tau \tag{2.9}$$

for all $x \in \mathcal{O}_{\epsilon/2}$. If $g \in C^2(\overline{\Omega})$ and $g|_{\mathcal{H}} = 0$, then

$$|R_{\Phi,\tau}g(x)| + |\widetilde{R}_{\Phi,\tau}g(x)| \le C/\tau^2 \tag{2.10}$$

for all $x \in \mathcal{O}_{\epsilon/2}$.

The proof uses the Cauchy-Riemann equations and stationary phase (e.g., Section 4.5.3 in [11], Chapter VII, §7.7 in [14]). See also the proof of Proposition 3.4 in [15].

Denote

$$r(x) = \Pi_{k=1}^{\ell}(x - \widetilde{x}_k) \text{ where } \mathcal{H} = \{\widetilde{x}_1, \dots, \widetilde{x}_\ell\}.$$

The following proposition can be proved and see Proposition 3.5 in [15]:

Proposition 2.5. *Let $g \in C^1(\overline{\Omega})$ and $g|_{\mathcal{O}_\epsilon} = 0$. Then for each $\delta \in (0,1)$, there exists a constant $C(\delta) > 0$ such that*

$$\|\widetilde{R}_{\Phi,\tau}(\overline{r(z)}g)\|_{L^2(\Omega)} \le C(\delta)\|g\|_{C^1(\overline{\Omega})}/|\tau|^{1-\delta}, \quad \|R_{\Phi,\tau}(r(z)g)\|_{L^2(\Omega)} \le C(\delta)\|g\|_{C^1(\overline{\Omega})}/|\tau|^{1-\delta}. \tag{2.11}$$

Henceforth we set $\psi_1 \equiv \mathrm{Re}\partial_z\Phi = \partial_{x_1}\varphi$ and $\psi_2 \equiv \mathrm{Im}\partial_z\Phi = \partial_{x_1}\psi$. We also need the following proposition, which we can prove by Proposition 2.1 in [15] and noting that

$$\partial_{x_1}(e^{-i\tau\psi}\widetilde{v})e^{i\tau\psi} = \partial_{x_1}\widetilde{v} - i\tau\psi_2\widetilde{v}$$

and

$$\partial_{x_2}(e^{-i\tau\psi}\widetilde{v})e^{i\tau\psi} = \partial_{x_2}\widetilde{v} - i\tau\psi_1\widetilde{v},$$

etc. which follow from the Cauchy-Riemann equations.

Proposition 2.6. *Let Φ satisfy (2.1) and (2.2). Let $\widetilde{f} \in L^2(\Omega)$ and $\widetilde{v}$ be solution to the problem*

$$2\partial_z\widetilde{v} - \tau(\partial_z\Phi)\widetilde{v} = \widetilde{f} \quad in\ \Omega \tag{2.12}$$

or $\widetilde{v}$ be solution to the problem

$$2\partial_{\overline{z}}v - \tau(\partial_{\overline{z}}\overline{\Phi})\widetilde{v} = \widetilde{f} \quad in\ \Omega. \tag{2.13}$$

In the case (2.12) we have

$$\|\partial_{x_1}(e^{-i\tau\psi}\widetilde{v})\|^2_{L^2(\Omega)} - \tau\int_{\partial\Omega}(\nabla\varphi,\nu)|\widetilde{v}|^2 d\sigma$$
$$+Re\int_{\partial\Omega} i\left(\left(\nu_2\frac{\partial}{\partial x_1} - \nu_1\frac{\partial}{\partial x_2}\right)\widetilde{v}\right)\overline{\widetilde{v}}d\sigma + \|\partial_{x_2}(e^{-i\tau\psi}\widetilde{v})\|^2_{L^2(\Omega)} = \|\widetilde{f}\|^2_{L^2(\Omega).} \tag{2.14}$$

In the case that $\widetilde{v}$ solves (2.13) we have

$$\|\partial_{x_1}(e^{i\tau\psi}\widetilde{v})\|_{L^2(\Omega)} - \tau\int_{\partial\Omega}(\nabla\varphi,\nu)|\widetilde{v}|^2 d\sigma + Re\int_{\partial\Omega} i\left(\left(-\nu_2\frac{\partial}{\partial x_1} + \nu_1\frac{\partial}{\partial x_2}\right)\widetilde{v}\right)\overline{\widetilde{v}}d\sigma$$
$$+\|\partial_{x_2}(e^{i\tau\psi}\widetilde{v})\|^2_{L^2(\Omega)} = \|\widetilde{f}\|^2_{L^2(\Omega)}. \tag{2.15}$$

We have

Proposition 2.7. *Let* $g \in C^2(\Omega), g|_{\mathcal{O}_\epsilon} = 0$ *and* $g|_{\mathcal{H}} = 0$. *Then*

$$\left\| R_{\Phi,\tau} g + \frac{g}{\tau \overline{\partial_z \Phi}} \right\|_{L^2(\Omega)} + \left\| \widetilde{R}_{\Phi,\tau} g - \frac{g}{\tau \partial_z \Phi} \right\|_{L^2(\Omega)} = o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \to +\infty. \tag{2.16}$$

Proof. By (2.2) and Proposition 2.4

$$\|\widetilde{R}_{\Phi,\tau} g\|_{C(\overline{\mathcal{O}_{\frac{\epsilon}{2}}})} + \|R_{\Phi,\tau} g\|_{C(\overline{\mathcal{O}_{\frac{\epsilon}{2}}})} = o\left(\frac{1}{\tau}\right). \tag{2.17}$$

Therefore instead of (2.16) it suffices to prove

$$\left\| \chi_1 R_{\Phi,\tau} g + \frac{g}{\tau \overline{\partial_z \Phi}} \right\|_{L^2(\Omega)} + \left\| \chi_1 \widetilde{R}_{\Phi,\tau} g - \frac{g}{\tau \partial_z \Phi} \right\|_{L^2(\Omega)} = o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \to +\infty, \tag{2.18}$$

where $\chi_1 \in C_0^\infty(\Omega)$ and $\chi_1|_{\Omega \setminus \mathcal{O}_{\epsilon/2}} \equiv 1$. Denote $w = \chi_1 \widetilde{R}_{\Phi,\tau} g - \frac{g}{\tau \partial_z \Phi}$. Here we note that $\frac{g}{\partial_z \Phi} \in C(\overline{\Omega})$. This follows from (2.2), $g \in C^1(\overline{\Omega})$ and $g|_{\mathcal{H}} = 0$. Then (2.8) and $g|_{\mathcal{O}_\varepsilon} = 0$ yield

$$\partial_z w + \tau(\partial_z \Phi) w = -\partial_z \left(\frac{g}{\tau \partial_z \Phi}\right) + (\partial_z \chi_1) \widetilde{R}_{\Phi,\tau} g \quad \text{in } \Omega, \quad w|_{\partial\Omega} = 0. \tag{2.19}$$

Note that by (2.2) and the fact that $g|_{\mathcal{H}} = 0$, we have:

$$\left| \partial_z \left(\frac{g}{\partial_z \Phi}\right) \right| = \left| \frac{\partial_z g}{\partial_z \Phi} - \frac{g}{\partial_z \Phi} \frac{\partial_z^2 \Phi}{\partial_z \Phi} \right| \le \frac{C}{\Pi_{k=1}^{\ell} |x - \widetilde{x}_k|}. \tag{2.20}$$

Consider the cut off function $\chi \in C_0^\infty(\Omega)$ such that

$$\chi \ge 0, \quad \chi|_{B(0,\frac{1}{2})} = 1.$$

By (2.20) and Proposition 2.2 B),

$$\widetilde{R}_{\Phi,\tau} \left(\sum_{k=1}^{\ell} \chi((x - \widetilde{x}_k) \ln |\tau|) \partial_z \left(\frac{g}{\partial_z \Phi}\right) \right) \to 0 \quad \text{in} \quad L^2(\Omega) \text{ as } |\tau| \to +\infty. \tag{2.21}$$

In fact, fixing large $|\tau|$, small $\delta > 0$ and $p > 1$ such that $p - 1$ is sufficiently small, we apply Proposition 2.2 B) and (2.20) to have

$$\begin{aligned}
&\left\| \widetilde{R}_{\Phi,\tau} \left(\sum_{k=1}^{\ell} \chi((x - \widetilde{x}_k) \ln |\tau|) \partial_z \left(\frac{g}{\partial_z \Phi}\right) \right) \right\|^2_{L^2(\Omega)} \\
\le\ & C \sum_{k=1}^{\ell} \int_{B(\widetilde{x}_k, \delta)} |\chi((x - \widetilde{x}_k) \ln |\tau|)|^p \left| \partial_z \left(\frac{g}{\partial_z \Phi}\right) \right|^p dx \\
\le\ & C' \sum_{k=1}^{\ell} \int_{B(\widetilde{x}_k, \delta)} |\chi((x - \widetilde{x}_k) \ln |\tau|)|^p \frac{1}{|x - \widetilde{x}_k|^p} dx \le C'' \int_0^\delta |\chi(\rho \ln |\tau|)|^p \rho^{1-p} d\rho.
\end{aligned}$$

Thus we see (2.21) by the Lebesgue theorem.

By Proposition 2.5, we obtain

$$\widetilde{R}_{\Phi,\tau}\left(\left(1-\sum_{k=1}^{\ell}\chi((x-\widetilde{x}_k)\ln|\tau|)\right)\partial_z\left(\frac{g}{\partial_z\Phi}\right)\right)\to 0 \quad \text{in} \quad L^2(\Omega) \text{ as } |\tau|\to+\infty. \tag{2.22}$$

Therefore (2.21) and (2.22) yield

$$\left\|\widetilde{R}_{\Phi,\tau}\partial_z\left(\frac{g}{\partial_z\Phi}\right)\right\|_{L^2(\Omega)} = o(1) \quad \text{as } \tau\to+\infty. \tag{2.23}$$

Denote $\widetilde{w} = w + \frac{1}{\tau}\chi_1\widetilde{R}_{\Phi,\tau}\partial_z(\frac{g}{\partial_z\Phi})$.

By (4.17), it suffices to prove

$$\|\widetilde{w}\|_{L^2(\Omega)} = o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau|\to+\infty. \tag{2.24}$$

In terms of (2.19) and (2.8), observe that

$$\partial_z\widetilde{w} + \tau(\partial_z\Phi)\widetilde{w} = f \quad \text{in } \Omega, \quad \widetilde{w}|_{\partial\Omega} = 0, \tag{2.25}$$

where $f = \frac{1}{\tau}(\partial_z\chi_1)\widetilde{R}_{\Phi,\tau}(\partial_z(\frac{g}{\partial_z\Phi})) + (\partial_z\chi_1)\widetilde{R}_{\Phi,\tau}g$. By (4.17) and (2.17),

$$\|f\|_{L^2(\Omega)} = o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau|\to+\infty. \tag{2.26}$$

Noting $\widetilde{w}|_{\partial\Omega} = 0$, applying Proposition 2.6 to equation (2.25) and using (2.26), we obtain (2.24). As for the first term in (2.18), we can argue similarly. The proof of the proposition is completed. □

3. Complex geometrical optics solutions

In this section, we construct complex geometrical optics solutions of the Schrödinger equation $\Delta + q_1$ with q_1 satisfying the conditions of Theorem 1.1. Consider

$$L_1u = \Delta u + q_1u = 0 \quad \text{in } \Omega. \tag{3.1}$$

We will construct solutions to (3.1) of the form

$$u_1(x) = e^{\tau\Phi(z)}a(z) + e^{\tau\overline{\Phi(z)}}\overline{a(z)} + e^{\tau\varphi}u_{11} + e^{\tau\varphi}u_{12}, \quad u_1|_{\Gamma_0} = 0. \tag{3.2}$$

The function Φ satisfies (2.1), (2.2) and

$$\operatorname{Im}\Phi|_{\Gamma_0} = 0. \tag{3.3}$$

The amplitude function $a(z)$ is not identically zero on $\overline{\Omega}$ and has the following properties:

$$a\in C^2(\overline{\Omega}), \quad \partial_{\overline{z}}a\equiv 0, \ \operatorname{Re}a|_{\Gamma_0} = 0. \tag{3.4}$$

and the function u_{11} is given by

$$\begin{aligned} u_{11} = & -\frac{1}{4}e^{i\tau\psi}\widetilde{R}_{\Phi,\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z))) - \frac{1}{4}e^{-i\tau\psi}R_{\Phi,-\tau}(e_1(\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\overline{z}))) \\ & -\frac{e^{i\tau\psi}}{\tau}\frac{e_2(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z))}{4\partial_z\Phi} - \frac{e^{-i\tau\psi}}{\tau}\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\overline{z}))}{4\overline{\partial_z\Phi}} \\ = & \ w_1e^{-\tau\varphi} + w_2e^{-\tau\varphi}, \end{aligned} \tag{3.5}$$

where the polynomials $M_1(z)$ and $M_3(z)$ satisfy

$$\partial_z^j(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z)) = 0, \quad x \in \mathcal{H},\ j = 0,1,2, \tag{3.6}$$

$$\partial_{\overline{z}}^j(\partial_z^{-1}(\overline{a}q_1)(z) - M_3(\overline{z})) = 0, \quad x \in \mathcal{H},\ j = 0,1,2, \tag{3.7}$$

$e_1, e_2 \in C^\infty(\Omega)$ are constructed such that $e_1 + e_2 \equiv 1$ on $\overline{\Omega}$, e_2 vanishes in some neighborhood of $\mathcal{H}$ and e_1 vanishes in a neighborhood of $\partial\Omega$, and we set

$$w_1 = -\frac{1}{4}e^{\tau\Phi}\widetilde{R}_{\Phi,\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z))) - \frac{1}{4}e^{\tau\overline{\Phi}}R_{\Phi,-\tau}(e_1(\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\overline{z}))$$

and

$$w_2 = -\frac{e^{\tau\Phi}}{\tau}\frac{e_2(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z))}{4\partial_z\Phi} - \frac{e^{\tau\overline{\Phi}}}{\tau}\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\overline{z}))}{4\overline{\partial_z\Phi}}.$$

Then, noting $\partial_{\overline{z}}\overline{\Phi} = \overline{\partial_z\Phi}$, (2.7) and (2.8), we have

$$\begin{aligned}
&\Delta w_1 = 4\partial_z\partial_{\overline{z}}w_1\\
=\ & -\partial_{\overline{z}}(e^{\tau\Phi}\partial_z\widetilde{R}_{\Phi,\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z))) + (\tau\partial_z\Phi)e^{\tau\Phi}\widetilde{R}_{\Phi,\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z)))\\
-\ & \partial_z(e^{\tau\overline{\Phi}}\partial_{\overline{z}}R_{\Phi,-\tau}(e_1(\partial_z^{-1}(\overline{a}q_1) - M_3(\overline{z}))) + (\tau\overline{\partial_z\Phi})e^{\tau\overline{\Phi}}R_{\Phi,-\tau}(e_1(\partial_z^{-1}(\overline{a}q_1) - M_3(\overline{z})))\\
=\ & -\partial_{\overline{z}}(e^{\tau\Phi}e_1(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z))) - \partial_z(e^{\tau\overline{\Phi}}e_1(\partial_z^{-1}(\overline{a}q_1) - M_3(\overline{z}))).
\end{aligned}$$

Moreover

$$\begin{aligned}
&\Delta w_2 = 4\partial_z\partial_{\overline{z}}w_2\\
=\ & -\partial_{\overline{z}}(e^{\tau\Phi}(e_2(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z))) - \partial_z(e^{\tau\overline{\Phi}}e_2(\partial_z^{-1}(\overline{a}q_1) - M_3(\overline{z})))\\
-\ & e^{\tau\Phi}\Delta\left(\frac{e_2(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z))}{4\tau\partial_z\Phi}\right) - e^{\tau\overline{\Phi}}\Delta\left(\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\overline{z}))}{4\tau\overline{\partial_z\Phi}}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\Delta(u_{11}e^{\tau\varphi}) = \Delta(w_1 + w_2) = -aq_1e^{\tau\Phi} - \overline{a}q_1e^{\tau\overline{\Phi}}\\
-e^{\tau\Phi}\Delta\left(\frac{e_2(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z))}{4\tau\partial_z\Phi}\right) - e^{\tau\overline{\Phi}}\Delta\left(\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\overline{z}))}{4\tau\overline{\partial_z\Phi}}\right).
\end{aligned} \tag{3.8}$$

By (3.4) and (3.3), observe that

$$(e^{\tau\Phi(z)}a(z) + e^{\tau\overline{\Phi(z)}}\overline{a(z)})|_{\Gamma_0} = 0. \tag{3.9}$$

By Proposition 2.1, we can define u_{12} as a solution to the inhomogeneous problem

$$\Delta(u_{12}e^{\tau\varphi}) + q_1u_{12}e^{\tau\varphi} = -q_1u_{11}e^{\tau\varphi} + h_1e^{\tau\varphi} \quad \text{in } \Omega, \tag{3.10}$$

$$u_{12} = -u_{11} \quad \text{on } \Gamma_0, \tag{3.11}$$

where

$$h_1 = e^{\tau i\psi}\Delta\left(\frac{e_2(\partial_z^{-1}(a(z)q_1) - M_1(z))}{4\tau\partial_z\Phi}\right) + e^{-\tau i\psi}\Delta\left(\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\overline{z}))}{4\tau\overline{\partial_z\Phi}}\right). \tag{3.12}$$

Then, by (3.4) and (3.8) - (3.12), we see that (3.1) is satisfied.

By Proposition 2.1 there exists a positive τ_0 such that for all $|\tau| > \tau_0$ there exists a solution to (3.10), (3.11) satisfying

$$\|u_{12}\|_{L^2(\Omega)} \le C/\tau^{\frac{5}{4}}. \tag{3.13}$$

This can be done because

$$\|q_1 u_{11} + h_1\|_{L^2(\Omega)} \le C(\delta)/\tau^{1-\delta} \quad \forall \delta \in (0,1); \ \|u_{11}\|_{L^2(\partial\Omega)} \le C/\tau$$

and $(\nabla\varphi, \nu) = 0$ on Γ_0. The latter is seen as follows: On $\partial\Omega = \{x \in \mathbb{R}^2 | \, |x| = 1\}$, the Cauchy-Riemann equations imply

$$(\nabla\varphi, \nu) = x_1\partial_{x_1}\varphi + x_2\partial_{x_2}\varphi = x_1\partial_{x_2}\psi - x_2\partial_{x_1}\psi,$$

which is the tangential derivative of $\psi = \operatorname{Im}\Phi$ on $\partial\Omega$. By (3.3) we see that the tangential derivative of ψ vanishes on Γ_0.

Consider the Schrödinger equation

$$L_2 v = \Delta v + q_2 v = 0 \quad \text{in } \Omega. \tag{3.14}$$

We will construct solutions to (3.14) of the form

$$v(x) = e^{-\tau\Phi(z)}a(z) + e^{-\tau\overline{\Phi(z)}}\overline{a(z)} + e^{-\tau\varphi}v_{11} + e^{-\tau\varphi}v_{12}, \quad v|_{\Gamma_0} = 0. \tag{3.15}$$

The construction of v repeats the corresponding steps of the construction of u_1. The only difference is that instead of q_1 and τ, we use q_2 and $-\tau$ respectively. We provide the details for the sake of completeness. The function v_{11} is given by

$$v_{11} = -\frac{1}{4}e^{-i\tau\psi}\widetilde{R}_{\Phi,-\tau}(e_1(\partial_{\overline{z}}^{-1}(q_2 a(z)) - M_2(z))) - \frac{1}{4}e^{i\tau\psi}R_{\Phi,\tau}(e_1(\partial_z^{-1}(q_2\overline{a(z)}) - M_4(\overline{z}))) \tag{3.16}$$
$$+\frac{e^{-i\tau\psi}}{\tau}\frac{e_2(\partial_{\overline{z}}^{-1}(aq_2) - M_2(z))}{4\partial_z\Phi} + \frac{e^{i\tau\psi}}{\tau}\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_2) - M_4(\overline{z}))}{4\overline{\partial_z\Phi}},$$

where

$$\partial_z^j(\partial_{\overline{z}}^{-1}(aq_2) - M_2(z)) = 0, \quad x \in \mathcal{H}, \ j = 0,1,2, \tag{3.17}$$

$$\partial_{\overline{z}}^j(\partial_z^{-1}(\overline{a}q_2)(z) - M_4(\overline{z})) = 0, \quad x \in \mathcal{H}, \ j = 0,1,2. \tag{3.18}$$

Denote

$$h_2 = e^{-\tau i\psi}\Delta\left(\frac{e_2(\partial_z^{-1}(a(z)q_2) - M_2(z))}{4\tau\partial_z\Phi}\right) + e^{\tau i\psi}\Delta\left(\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_2) - M_4(\overline{z}))}{4\tau\overline{\partial_z\Phi}}\right).$$

The function v_{12} is a solution to the problem:

$$\Delta(v_{12}e^{-\tau\varphi}) + q_2 v_{12}e^{-\tau\varphi} = -q_2 v_{11}e^{-\tau\varphi} - h_2 e^{-\tau\varphi} \quad \text{in } \Omega, \tag{3.19}$$

$$v_{12}|_{\Gamma_0} = -v_{11}|_{\Gamma_0}. \tag{3.20}$$

such that

$$\|v_{12}\|_{L^2(\Omega)} \le C/\tau^{\frac{5}{4}}. \tag{3.21}$$

4. **Proof of the theorem.**

Proposition 4.1. *Suppose that Φ satisfies (2.1),(2.2) and (3.3). Let $\{\widetilde{x}_1, \dots, \widetilde{x}_\ell\}$ be the set of critical points of the function $Im\Phi$. Then for any potentials $q_1, q_2 \in C^{1+\alpha}(\overline{\Omega})$, $\alpha > 0$ with the same Dirichlet-to-Neumann maps and for any holomorphic function a satisfying (3.4), we have*

$$2\sum_{k=1}^{\ell} \frac{\pi(q|a|^2)(\widetilde{x}_k) Re\, e^{2i\tau Im\Phi(\widetilde{x}_k)}}{|(det\, Im\Phi'')(\widetilde{x}_k)|^{\frac12}} \tag{4.1}$$

$$+ \frac14 \int_\Omega \left(qa\frac{\partial_{\overline{z}}^{-1}(aq_2) - M_2(z)}{\partial_z\Phi} + q\overline{a}\frac{\partial_z^{-1}(q_2\overline{a}) - M_4(\overline{z})}{\overline{\partial_z\Phi}}\right) dx$$
$$- \frac14 \int_\Omega \left(qa\frac{(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z))}{\partial_z\Phi} + q\overline{a}\frac{(\partial_z^{-1}(\overline{a}q_1) - M_3(\overline{z}))}{\overline{\partial_z\Phi}}\right) dx = 0, \quad \tau > 0,$$

where we set

$$q = q_1 - q_2.$$

Proof. We note by the Cauchy-Riemann equations that $\{\widetilde{x}_{1,1} + i\widetilde{x}_{1,2}, ..., \widetilde{x}_{\ell,1} + i\widetilde{x}_{\ell,2}\} = \{z \in \overline{\Omega} |\, \partial_z \mathrm{Im}\,\Phi(z) = 0\}$. Let u_1 be a solution to (3.1) and satisfy (3.2), and u_2 be a solution to the following equation

$$\Delta u_2 + q_2 u_2 = 0 \quad \text{in } \Omega, \quad u_2|_{\partial\Omega} = u_1|_{\partial\Omega}.$$

Since the Dirichlet-to-Neumann maps are equal, we have

$$\nabla u_2 = \nabla u_1 \quad \text{on } \widetilde{\Gamma}.$$

Denoting $u = u_1 - u_2$, we obtain

$$\Delta u + q_2 u = -qu_1 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = \frac{\partial u}{\partial\nu}|_{\widetilde{\Gamma}} = 0. \tag{4.2}$$

Let v satisfy (3.14) and (3.15). We multiply (4.2) by v, integrate over Ω and we use $v|_{\Gamma_0} = 0$ and $\frac{\partial u}{\partial \nu} = 0$ on $\widetilde{\Gamma}$ to obtain $\int_\Omega qu_1 v dx = 0$. By (3.2), (3.13), (3.15) and (3.21), we have

$$0 = \int_\Omega qu_1 v dx = \int_\Omega q(a^2 + \overline{a}^2 + |a|^2 e^{\tau(\Phi - \overline{\Phi})} + |a|^2 e^{\tau(\overline{\Phi} - \Phi)}$$
$$+ u_{11} e^{\tau\varphi}(ae^{-\tau\Phi} + \overline{a}e^{-\tau\overline{\Phi}})$$
$$+ (ae^{\tau\Phi} + \overline{a}e^{\tau\overline{\Phi}}) v_{11} e^{-\tau\varphi}) dx + o\left(\frac{1}{\tau}\right), \quad \tau > 0. \tag{4.3}$$

The first and second terms in the asymptotic expansion of (4.3) are independent of τ, so that

$$\int_\Omega q(a^2 + \overline{a}^2) dx = 0. \tag{4.4}$$

Using stationary phase (see p.215 in [11]. cf. [14]), we obtain

$$\int_\Omega q(|a|^2 e^{\tau(\Phi-\overline{\Phi})} + |a|^2 e^{\tau(\overline{\Phi}-\Phi)})dx = 2\sum_{k=1}^{\ell} \frac{\pi q|a|^2(\widetilde{x}_k)\mathrm{Re}\, e^{2\tau i \mathrm{Im}\,\Phi(\widetilde{x}_k)}}{\tau|(\det \mathrm{Im}\Phi'')(\widetilde{x}_k)|^{\frac{1}{2}}} + o\left(\frac{1}{\tau}\right). \tag{4.5}$$

Here by the Cauchy-Riemann equations, we see that $\mathrm{sgn}(\mathrm{Im}\,\Phi''(\widetilde{x}_k)) = 0$, where sgn A denotes signature of the matrix A, that is the number of positive eigenvalues of A minus the number of negative eigenvalues (e.g., [11], p.210). Moreover we use (2.2) and the Cauchy-Riemann equations to see that

$$\det \mathrm{Im}\,\Phi''(z) = -(\partial_{x_1}\partial_{x_2}\varphi)^2 - (\partial^2_{x_1}\varphi)^2 \neq 0$$

by $\partial_z^2\Phi = -\frac{1}{2}\partial^2_{x_1}\varphi - \frac{1}{2}i\partial_{x_1}\partial_{x_2}\varphi \neq 0$ in $\mathcal{H}$. We compute the two remaining terms in (4.3). We get:

$$\begin{aligned}
&(4.6)\quad \int_\Omega qu_{11}e^{\tau\varphi}(ae^{-\tau\Phi} + \overline{a}e^{-\tau\overline{\Phi}})dx\\
= & -\frac{1}{4}\int_\Omega q\left\{e^{\tau\Phi}\widetilde{R}_{\Phi,\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z))) + e^{\tau\overline{\Phi}}R_{\Phi,-\tau}(e_1(\partial_z^{-1}(\overline{a}q_1) - M_3(\overline{z})))\right\}(ae^{-\tau\Phi} + \overline{a}e^{-\tau\overline{\Phi}})dx\\
- & \int_\Omega\left(\frac{e^{\tau\Phi}}{\tau}\frac{e_2(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z))}{4\partial_z\Phi} + \frac{e^{\tau\overline{\Phi}}}{\tau}\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\overline{z}))}{4\overline{\partial_z\Phi}}\right)q(ae^{-\tau\Phi} + \overline{a}e^{-\tau\overline{\Phi}})dx\\
= & -\frac{1}{4}\int_\Omega(qa\widetilde{R}_{\Phi,\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z))) + q\overline{a}R_{\Phi,-\tau}(e_1(\partial_z^{-1}(\overline{a}q_1) - M_3(\overline{z}))))dx\\
- & \frac{1}{4}\int_\Omega(q\overline{a}\widetilde{R}_{\Phi,\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z)))e^{\tau(\Phi-\overline{\Phi})} + qaR_{\Phi,-\tau}(e_1(\partial_z^{-1}(\overline{a}q_1) - M_3(\overline{z})))e^{-\tau(\Phi-\overline{\Phi})})dx\\
- & \int_\Omega q\left(\frac{e^{\tau(\Phi-\overline{\Phi})}}{\tau}\frac{\overline{a}e_2(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z))}{4\partial_z\Phi} + \frac{e^{\tau(\overline{\Phi}-\Phi)}}{\tau}\frac{ae_2(\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\overline{z}))}{4\overline{\partial_z\Phi}}\right)dx\\
- & \int_\Omega q\left(\frac{a}{\tau}\frac{e_2(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z))}{4\partial_z\Phi} + \frac{\overline{a}}{\tau}\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\overline{z}))}{4\overline{\partial_z\Phi}}\right)dx\\
\equiv & \; I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

We compute I_1 and I_2 separately. By Proposition 2.7, (3.6) and stationary phase (e.g., p.215 in [11]), we obtain

$$\begin{aligned}
&(4.7)\qquad I_2 = -\frac{1}{4}\int_\Omega(q\overline{a}\widetilde{R}_{\Phi,\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z)))e^{\tau(\Phi-\overline{\Phi})}\\
+ & \; qaR_{\Phi,-\tau}(e_1(\partial_z^{-1}(\overline{a}q_1) - M_3(\overline{z})))e^{-\tau(\Phi-\overline{\Phi})})dx\\
= & -\frac{1}{4}\int_\Omega\left(e_1q\overline{a}\frac{1}{\tau\partial_z\Phi}(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z))e^{2i\tau\mathrm{Im}\Phi} + e_1qa\frac{1}{\tau\overline{\partial_z\Phi}}(\partial_z^{-1}(\overline{a}q_1) - M_3(\overline{z}))e^{-2i\tau\mathrm{Im}\Phi}\right)dx\\
+ & \; o\left(\frac{1}{\tau}\right) = o\left(\frac{1}{\tau}\right).
\end{aligned}$$

By Proposition 2.7, we obtain

$$I_1 = -\frac{1}{4\tau}\int_\Omega e_1\left(qa\frac{(\partial_{\overline{z}}^{-1}(aq_1)-M_1(z))}{\partial_z\Phi}+q\overline{a}\frac{(\partial_z^{-1}(\overline{a}q_1)-M_3(\overline{z}))}{\overline{\partial_z\Phi}}\right)dx + o\left(\frac{1}{\tau}\right). \tag{4.8}$$

Using stationary phase again and (3.6) we conclude that

$$I_3 = o\left(\frac{1}{\tau}\right). \tag{4.9}$$

Similarly

$$\begin{aligned}
&(4.10) \qquad \int_\Omega qv_{11}e^{-\tau\varphi}(ae^{\tau\Phi}+\overline{a}e^{\tau\overline{\Phi}})dx\\
= & -\frac{1}{4}\int_\Omega q\left\{e^{-\tau\Phi}\widetilde{R}_{\Phi,-\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_2)-M_2(z)))+e^{-\tau\overline{\Phi}}R_{\Phi,\tau}(e_1(\partial_z^{-1}(\overline{a}q_2)-M_4(\overline{z})))\right\}(ae^{\tau\Phi}+\overline{a}e^{\tau\overline{\Phi}})dx\\
+ & \int_\Omega q\left(\frac{e^{-\tau\Phi}}{\tau}\frac{e_2(\partial_{\overline{z}}^{-1}(aq_2)-M_2(z)))}{4\partial_z\Phi}+\frac{e^{-\tau\overline{\Phi}}}{\tau}\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_2)-M_4(\overline{z}))}{4\overline{\partial_z\Phi}}\right)(ae^{\tau\Phi}+\overline{a}e^{\tau\overline{\Phi}})dx\\
= & -\frac{1}{4}\int_\Omega(qa\widetilde{R}_{\Phi,-\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_2)-M_2(z)))+q\overline{a}R_{\Phi,\tau}(e_1(\partial_z^{-1}(\overline{a}q_2)-M_4(\overline{z}))))dx\\
- & \frac{1}{4}\int_\Omega[q\overline{a}e^{\tau(\overline{\Phi}-\Phi)}(\widetilde{R}_{\Phi,-\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_2)-M_2(z)))+qae^{\tau(\overline{\Phi}-\Phi)}R_{\Phi,\tau}(e_1(\partial_z^{-1}(\overline{a}q_2)-M_4(\overline{z})))]dx\\
+ & \int_\Omega q\left(\frac{e^{-\tau(\Phi-\overline{\Phi})}}{\tau}\frac{\overline{a}e_2(\partial_{\overline{z}}^{-1}(aq_2)-M_2(z)))}{4\partial_z\Phi}+\frac{e^{\tau(\Phi-\overline{\Phi})}}{\tau}\frac{ae_2(\partial_z^{-1}(\overline{a(z)}q_2)-M_4(\overline{z}))}{4\overline{\partial_z\Phi}}\right)dx\\
+ & \int_\Omega q\left(\frac{a}{\tau}\frac{e_2(\partial_{\overline{z}}^{-1}(aq_2)-M_2(z)))}{4\partial_z\Phi}+\frac{\overline{a}}{\tau}\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_2)-M_4(\overline{z}))}{4\overline{\partial_z\Phi}}\right)dx\\
\equiv & J_1+J_2+J_3+J_4.
\end{aligned}$$

By (3.17) and Proposition 2.7, we have

$$J_1 = \frac{1}{4\tau}\int_\Omega e_1\left(qa\frac{\partial_{\overline{z}}^{-1}(aq_2)-M_2(z)}{\partial_z\Phi}+q\overline{a}\frac{\partial_z^{-1}(\overline{a}q_2)-M_4(\overline{z})}{\overline{\partial_z\Phi}}\right)dx+o\left(\frac{1}{\tau}\right). \tag{4.11}$$

The stationary phase argument, (3.17) and Proposition 2.7 yield
(4.12)

$$J_2 = -\frac{1}{4}\int_\Omega[q\overline{a}e^{\tau(\overline{\Phi}-\Phi)}\widetilde{R}_{\Phi,-\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_2)-M_2(z)))+qae^{\tau(\overline{\Phi}-\Phi)}R_{\Phi,\tau}(e_1(\partial_z^{-1}(\overline{a}q_2)-M_4(\overline{z})))]dx = o\left(\frac{1}{\tau}\right).$$

By the stationary phase argument and (3.17), we see that

$$J_3 = o\left(\frac{1}{\tau}\right). \tag{4.13}$$

Therefore, applying (4.5), (4.7), (4.11), (4.12), (4.9) and (4.13) in (4.3), we obtain

$$\begin{aligned} &2\sum_{k=1}^{\ell}\frac{\pi(q|a|^2)(\widetilde{x}_k)\mathrm{Re}\, e^{2i\tau \mathrm{Im}\Phi(\widetilde{x}_k)}}{|(\det \mathrm{Im}\Phi'')(\widetilde{x}_k)|^{\frac12}} \\ +\ &\frac14\int_\Omega\left(qa\frac{\partial_{\overline{z}}^{-1}(a(z)q_2)-M_2(z)}{\partial_z\Phi}+q\overline{a}\frac{\partial_z^{-1}(q_2\overline{a(z)})-M_4(\overline{z})}{\overline{\partial_z\Phi}}\right)dx \\ &-\frac14\int_\Omega\left(qa\frac{\partial_{\overline{z}}^{-1}(q_1 a)-M_1(z)}{\partial_z\Phi}+q\overline{a}\frac{\partial_z^{-1}(q_1\overline{a})-M_3(\overline{z})}{\overline{\partial_z\Phi}}\right)dx=o(1). \end{aligned} \tag{4.14}$$

as $\tau\to+\infty$. Passing to the limit in this equality and applying Bohr's theorem (e.g., [2], p.393), we finish the proof of the proposition. □

Now we start the construction of the weight function Φ. Let $\widetilde{y}_1,\dots,\widetilde{y}_m\in\Omega$. Denote by $\mathcal{R}=(\mathcal{R}(\widetilde{y}_1),\dots,\mathcal{R}(\widetilde{y}_m))$ the following operator:

$$\mathcal{R}(\widetilde{y}_k)g=(u(\widetilde{y}_k),\partial_{x_1}u(\widetilde{y}_k),\partial_{x_2}u(\widetilde{y}_k),\partial_{x_1x_2}u(\widetilde{y}_k)),$$

where

$$\Delta u=0\quad\text{in }\Omega,\quad u|_{\Gamma_0}=0,\quad u|_{\widetilde{\Gamma}}=g. \tag{4.15}$$

We have

Proposition 4.2. $\mathcal{R}C_0^\infty(\widetilde{\Gamma})=\mathbb{R}^{4m}$.

Proof. We note that $\mathcal{R}C_0^\infty(\widetilde{\Gamma})=\mathbb{R}^{4m}$ if and only if the closure of $\mathcal{R}C_0^\infty(\widetilde{\Gamma})$ is equal to $\mathbb{R}^{4m}$. Our proof is by contradiction. Contrarily suppose that

$$\mathcal{R}C_0^\infty(\widetilde{\Gamma})\ne\mathbb{R}^{4m}.$$

Then there exists a nonzero vector $\vec{A}=(A_0^1,A_1^1,A_2^1,A_{12}^1,\dots,A_0^m,A_1^m,A_2^m,A_{12}^m)\in\mathbb{R}^{4m}$ from the orthogonal complement of $\mathcal{R}C_0^\infty(\widetilde{\Gamma})$. Let function p be a solution to the boundary value problem

$$\Delta p=\sum_{k=1}^m(A_0^k\delta(x-\widetilde{y}_k)-A_1^k\partial_{x_1}\delta(x-\widetilde{y}_k)-A_2^k\partial_{x_2}\delta(x-\widetilde{y}_k)+A_{12}^k\partial_{x_1}\partial_{x_2}\delta(x-\widetilde{y}_k)), \tag{4.16}$$

$$p|_{\partial\Omega}=0.$$

Let the function u be a solution to problem (4.15). Since

$$\int_{\partial\Omega}\frac{\partial p}{\partial\nu}gd\sigma=(\mathcal{R}g,\vec{A})=0,$$

we have $\frac{\partial p}{\partial\nu}|_{\widetilde{\Gamma}}=0$. By the Holmgren theorem, we have $\mathrm{supp}\, p\subset\{\widetilde{y}_1,\dots,\widetilde{y}_m\}$. Since p is the distribution, this means that $p=\sum_{k=1}^m\sum_{|\alpha|\le j(k)}C_{k,\alpha}D^\alpha\delta(x-\widetilde{y}_k)$.
Since ${}_{\mathcal{D}(\Omega)'}<\Delta p,\phi>_{\mathcal{D}(\Omega)}={}_{\mathcal{D}(\Omega)'}<p,\Delta\phi>_{\mathcal{D}(\Omega)}$ for any $\phi\in\mathcal{D}(\Omega)$, we see that equality (4.16) is possible only if $\vec{A}=0$ which is a contradiction. □

Now we finish the proof of Theorem 1.1.

Proof. Let $\widehat{x}$ be an arbitrary point from Ω. We will construct a complex geometric optic solutions of form (3.2) where Φ and a satisfy (2.1), (2.2), (3.3) and (3.4), $a(\widehat{x}) \neq 0$, $\widehat{x} \in \mathcal{G} \equiv \{x \in \overline{\Omega}| \, \partial_z \mathrm{Im}\, \Phi(x) = 0\}$, $\mathrm{Im}\, \Phi(\widehat{x}) \neq \mathrm{Im}\, \Phi(x)$ if $x \in \mathcal{G}$ and $x \neq \widehat{x}$. The construction of a is as follows: We choose $\mathrm{Re} a$ as a harmonic function in Ω with zero boundary condition on Γ_0 and some boundary condition on $\widetilde{\Gamma}$ such that $\mathrm{Re}\, a$ is smooth on $\partial\Omega$ and $\mathrm{Re}\, a(\widehat{x}) \neq 0$. Then, since Ω is simply connected, we construct the complex conjugate function to find a.

For the function Φ, we first construct its imaginary part. By Proposition 4.2 there exists a harmonic function u such that $u|_{\Gamma_0} = 0$ and $u(\widehat{x}) = \partial_z u(\widehat{x}) = 0$, and $\det u''(\widehat{x}) \neq 0$. In general the function u may have a critical points on the boundary. By Proposition 4.2 from [15] we can construct a harmonic function p such that $u + \epsilon p$ does not have a critical points on $\partial\Omega$ for all sufficiently small ϵ and $p|_{\Gamma_0} = 0$. Denote by $\mathcal{H}_\epsilon$ the set of critical points of the function $u + \epsilon p$ on $\overline{\Omega}$. By the implicit function theorem, there exists a neighborhood of $\widehat{x}$ such that for all small ϵ in this neighborhood the function $u + \epsilon p$ has only one critical point $\widehat{x}(\epsilon)$, this critical point is nondegenerate and

$$\widehat{x}(\epsilon) \rightarrow \widehat{x}. \tag{4.17}$$

Let us fix sufficiently small ϵ_1. Let $\mathcal{H}_\epsilon = \{x_{k,\epsilon}\}_{1 \le k \le N(\epsilon)}$. By Proposition 4.2, there exists a harmonic function w such that

$$w|_{\Gamma_0} = 0, \;\; w(x_{k,\epsilon_1}) \neq w(x_{j,\epsilon_1}) \quad \text{for } \; k \neq j, \quad \partial_z w|_{\mathcal{H}_{\epsilon_1}} = 0, \quad \partial_{x_1 x_2} w|_{\mathcal{H}_{\epsilon_1}} \neq 0.$$

Denote $\psi_\delta = u + \epsilon_1 p + \delta w$. For all sufficiently small positive δ

$$\mathcal{H}_{\epsilon_1} \subset \mathcal{G}_\delta = \{x \in \overline{\Omega} | \nabla \psi_\delta(x) = 0\}$$

and

$$\psi_\delta(x) \neq \psi_\delta(y) \quad \forall x, y \in \mathcal{H}_{\epsilon_1}, \; x \neq y.$$

Since Ω is a simply connected domain, we construct a function $\varphi_\delta(x)$ such that $\Phi_\delta = \varphi_\delta + i\psi_\delta$ is holomorphic in Ω. Let us show that for all small positive δ, critical points of the function Φ_δ are nondegenerate. Let $\widetilde{x}$ be a critical point of the function $u + \epsilon p$. If $\widetilde{x}$ is nondegenerate critical point, by the implicit function theorem, there exists a ball $B(\widetilde{x}, \delta_1)$ such that the function Φ_δ in this ball has only one nondegenerate critical point for all small δ. Let $\widetilde{x}$ be a degenerate critical point of $u + \epsilon p$. Without loss of generality we may assume that $\widetilde{x} = 0$. In some neighborhood of 0, we have $\partial_z \Phi_\delta = \sum_{k=1}^\infty c_k z^{k+p} - \delta \sum_{k=1}^\infty b_k z^k$ for some integer positive p and some $b_1 \neq 0$ and $c_1 \neq 0$. Let $z_\delta \in \mathcal{G}_\delta$ and $z_\delta \rightarrow 0$. Then either $z_\delta = 0$ or $z_\delta^p = \delta b_1/c_1 + o(\delta)$. Therefore $\partial_z^2 \Phi(z_\delta) \neq 0$ for all small δ. Hence we can apply Proposition 4.1. According to this proposition

$$\sum_{x \in \mathcal{G}_\delta} q(x) c(x) e^{2i\tau \mathrm{Im} \Phi_\delta(x)} = 0.$$

By (4.1) $c(\widehat{x}(\epsilon))$ is not equal to zero. Also $\mathrm{Im}\Phi_\delta(\widehat{x}(\epsilon)) \neq \mathrm{Im}\Phi_\delta(x)$ for all $x \in \mathcal{G}_\delta$ such that $\widehat{x}(\epsilon) \neq x$. Since the exponents are the linearly independent functions of τ, we have $q(\widehat{x}(\epsilon)) = 0$. Thus (4.17) implies $q(\widehat{x}) = 0$. $\square$

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UTMS

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