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Theory of Completeness for Logical Spaces

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Abstract

A logical space is a pair $(\mathsf{A}, \mathcal{B})$ of a non-empty set A and a subset $\mathcal{B}$ of $\mathcal{P}\mathsf{A}$, and a deduction pair on A is a pair (R, D) of a subset D of A and a relation R between A^* and A. It will be shown that we can understand the true nature of logical completeness via consideration of those simple notions.

Key words: abstract logic, completeness, deduction.

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Contents

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1 Logical spaces for logical systems

A **logical space** is a pair $(\mathsf{A}, \mathcal{B})$ of a non-empty set A and a subset $\mathcal{B}$ of the power set $\mathcal{P}\mathsf{A}$ of A. Since $\mathcal{P}\mathsf{A}$ is identified with $\{0, 1\}^{\mathsf{A}}$ and $\{0, 1\}$ is a typical lattice, a pair $(\mathsf{A}, \mathcal{F})$ of a non-empty set A and a subset $\mathcal{F}$ of $\mathbb{B}^{\mathsf{A}}$ for a lattice $\mathbb{B}$ with $\#\mathbb{B} > 1$ is also called a **functional** logical space. Also, a **deduction pair** on A is a pair (R, D) of a subset D of A and a relation R between $\mathsf{A}^* = \bigcup_{n=0}^{n} \mathsf{A}^n$ and A.

The purpose of this introductory section is to show that each **logical system** with a **truth** yields a logical space. The main purpose of the subsequent sections is to show that we can understand the true nature of the completeness in the logical systems via consideration of the simplest notions of logical spaces and deduction pairs.

In fact, this paper is an abridged translation of an impermanent aspect of the author's personal electronic publication *Mathematical Psychology* [4], where work in progress has been shown for more than decade by frequent revisions, and in particular, the theory of logical spaces has been developed more elaborately than here and applied. There are other abridged translations [2][3][5] of [4], and the present paper is also intended to be preliminaries to [2] and a continuation of [3].

Although mathematical psychology overlaps with mathematical logic, philosophy, linguistics, and so on, it is a new branch different from any one of them, and as such, free to use new terminologies and formulations. Therefore, this paper will be made as self-contained as tolerated.

However, our set-theoretical notation and terminology will be standard except that we denote the set of the finite subsets of a set X by $\mathcal{P}'\mathsf{X}$ and that we denote the set of the mappings of a set Y into a set Z by $\mathsf{Y} \to \mathsf{Z}$ instead of Z^{Y}. Thus $\mathsf{f} \in \mathsf{Y} \to \mathsf{Z}$ means $\mathsf{f} : \mathsf{Y} \to \mathsf{Z}$.

1.1 Sorted algebras

Here is given an account of algebras to the extent necessary for the definition of the notion of logical systems. For omitted proofs and further results, we refer the reader to [4][5].

For each set A and each natural number n, an n-ary **operation** on A is a mapping α of a subset D of A^n into A. The set D is called the **domain** of α and denoted by $\operatorname{Dom}\alpha$. The number n is called an **arity** of α, and so if $D = \emptyset$, every natural number is an arity of α. We say that α is **global** if $D = A^n$. A subset B of A is said to be **closed** under the operation α if $\alpha(a_1, \dots, a_n) \in B$ for each $(a_1, \dots, a_n) \in B^n \cap D$. If B is closed under α, the **restriction** $\alpha|_{B^n \cap D}$ of α to B becomes an operation on B.

An **algebra** is a set A equipped with a family $(\alpha_\lambda)_{\lambda\in\Lambda}$ of operations on A, which we call the **operation system** or **OS** of the algebra A. We often identify the operation α_λ with its index λ. The algebra $(A, (\alpha_\lambda)_{\lambda\in\Lambda})$ is said to be **global** if α_λ is global for every $\lambda \in \Lambda$.

The algebra $(A, (\alpha_\lambda)_{\lambda\in\Lambda})$ has two kinds of **subalgebras**. The first is an algebra $(A, (\alpha_\mu)_{\mu\in M})$ obtained by reducing the OS of A from $(\alpha_\lambda)_{\lambda\in\Lambda}$ to $(\alpha_\mu)_{\mu\in M}$ for a subset M of Λ. Such an algebra will be called an **operational subalgebra**. Also, if a subset B of A is closed under α_λ for each $\lambda \in \Lambda$, then B becomes an algebra equipped with the operation system $(\beta_\lambda)_{\lambda\in\Lambda}$ consisting of restrictions β_λ of α_λ to B. Such an algebra $(B, (\beta_\lambda)_{\lambda\in\Lambda})$ is called a **support subalgebra**.

Let A be an algebra. Then the intersection of support subalgebras of A is also a support subalgebra of A, and A itself is a support subalgebra of A. Therefore, for each subset S of A, the intersection of all support subalgebras of A which contain S is the smallest of the support subalgebras of A which contain S. We denote it by $[S]$.

Two algebras A and B are said to be **similar**, if they have operation systems $(\alpha_\lambda)_{\lambda\in\Lambda}$ and $(\beta_\lambda)_{\lambda\in\Lambda}$ indexed by the same set Λ, and α_λ and β_λ have a common arity for each $\lambda \in \Lambda$.

Let $(A, (\alpha_\lambda)_{\lambda\in\Lambda})$ and $(B, (\beta_\lambda)_{\lambda\in\Lambda})$ be similar algebras. Then a mapping f of A into B is called a **homomorphism** or a **Λ-homomorphism** if it satisfies the following two conditions for all $\lambda \in \Lambda$, where n_λ denotes an arity common to α_λ and β_λ.

- If $(a_1, \dots, a_{n_\lambda}) \in \operatorname{Dom}\alpha_\lambda$, then $(fa_1, \dots, fa_{n_\lambda}) \in \operatorname{Dom}\beta_\lambda$ and $f(\alpha_\lambda(a_1, \dots, a_{n_\lambda})) = \beta_\lambda(fa_1, \dots, fa_{n_\lambda})$.
- If $(a_1, \dots, a_{n_\lambda}) \in A^{n_\lambda}$ and $(fa_1, \dots, fa_{n_\lambda}) \in \operatorname{Dom}\beta_\lambda$, then $(a_1, \dots, a_{n_\lambda}) \in \operatorname{Dom}\alpha_\lambda$.

A bijective homomorphism is called an **isomorphism**. If both A and B are global algebras, a mapping f of A into B is a homomorphism iff it satisfies the following condition for all $\lambda \in \Lambda$ and all $(a_1, \dots, a_{n_\lambda}) \in A^{n_\lambda}$:

$$f(\alpha_\lambda(a_1, \dots, a_{n_\lambda})) = \beta_\lambda(fa_1, \dots, fa_{n_\lambda}).$$

A **sorted algebra** is an algebra A equipped with an algebra T similar to A and a homomorphism σ of A into T. We call T and σ the **type algebra** and the **sorting** of the sorted algebra A. For each subset S of A and each $t \in T$, we define the **t-part** S_t of S to be the inverse image $\{a \in S \mid \sigma a = t\}$ of t in S by σ.

Let (A, T, σ) and (B, T, τ) be sorted algebras with the same type algebra T. Then a mapping f of A into B is said to be **sort-consistent**, if it satisfies $\tau f = \sigma$, or equivalently $f(A_t) \subseteq B_t$ for all $t \in T$.

A sorted algebra (A, T, σ) is said to be **universal** or called a **USA** if A has a subset S which satisfies the following two conditions, the latter being called the **universality**.

- $A = [S]$.
- If (A', T, σ') is a sorted algebra and φ is a mapping of S into A' satisfying $\sigma'\varphi = \sigma|_S$, then there exists a sort-consistent homomorphism f of A into A' which extends φ.

It is known that f in the above condition is uniquely determined by φ.

The following theorem is known to hold.

Theorem 1.1 (Unique Existence of USA) Let S be a set, T be an algebra, and τ be a mapping of S into T. Then there exists a USA (A, T, σ, S) with $\sigma|_S = \tau$. If (A', T, σ', S) is also a USA with $\sigma'|_S = \tau$, then there exists a sort-consistent isomorphism of A onto A' extending id_S.

Let (A, T, σ) be a sorted algebra and V be a non-empty set. Define $A^V = \bigcup_{t \in T}(V \to A_t)$. Then we can construct a sorted algebra (A^V, T, ρ) as follows. First define the sorting ρ of A^V into T by $\rho b = t$ for each $b \in V \to A_t$ and each $t \in T$. Then

$$\rho b = \sigma(bv)$$

for each $b \in A^V$ and each $v \in V$. Let $(\alpha_\lambda)_{\lambda \in \Lambda}$ and $(\tau_\lambda)_{\lambda \in \Lambda}$ be the OS's of A and T respectively, and let n_λ be an arity of α_λ and τ_λ. For each $\lambda \in \Lambda$, define the operation β_λ on A^V as follows. First define the domain of β_λ to be

$$D_\lambda = \left\{(b_1, \ldots, b_{n_\lambda}) \in (A^V)^{n_\lambda} \mid (\rho b_1, \ldots, \rho b_{n_\lambda}) \in \mathrm{Dom}\,\tau_\lambda\right\}.$$

If $(b_1, \ldots, b_{n_\lambda}) \in D_\lambda$, then $\big(\sigma(b_1 v), \ldots, \sigma(b_{n_\lambda} v)\big) = (\rho b_1, \ldots, \rho b_{n_\lambda}) \in \mathrm{Dom}\,\tau_\lambda$ so $(b_1 v, \ldots, b_{n_\lambda} v) \in \mathrm{Dom}\,\alpha_\lambda$ for each $v \in V$, and we can define the mapping $\beta_\lambda(b_1, \ldots, b_{n_\lambda})$ of V into A by

$$\big(\beta_\lambda(b_1, \ldots, b_{n_\lambda})\big)v = \alpha_\lambda(b_1 v, \ldots, b_{n_\lambda} v)$$

for each $v \in V$. Furthermore

$$\sigma\big(\alpha_\lambda(b_1 v, \ldots, b_{n_\lambda} v)\big) = \tau_\lambda\big(\sigma(b_1 v), \ldots, \sigma(b_{n_\lambda} v)\big) = \tau_\lambda(\rho b_1, \ldots, \rho b_{n_\lambda}),$$

and $t = \tau_\lambda(\rho b_1, \ldots, \rho b_{n_\lambda})$ is not varied by $v \in V$, hence $\beta_\lambda(b_1, \ldots, b_{n_\lambda}) \in V \to A_t \subseteq A^V$. Thus β_λ certainly is an operation on A^V for each $\lambda \in \Lambda$, and so $(A^V, (\beta_\lambda)_{\lambda \in \Lambda})$ becomes an algebra. Furthermore

$$\begin{aligned}\rho(\beta_\lambda(b_1, \ldots, b_{n_\lambda})) &= \sigma\big(\big(\beta_\lambda(b_1, \ldots, b_{n_\lambda})\big)v\big) \\ &= \sigma\big(\alpha_\lambda(b_1 v, \ldots, b_{n_\lambda} v)\big) = \tau_\lambda(\rho b_1, \ldots, \rho b_{n_\lambda})\end{aligned}$$

with any element $\nu \in V$, and so ρ is a homomorphism of A^V into T. Thus we have constructed the sorted algebra (A^V, T, ρ), which we call the **power algebra** of A with **exponent** V.

1.2 From logical systems with a truth to logical spaces

By definition of moderate generality, a **formal language** is a universal sorted algebra (A, T, σ, S) equipped with subsets C and $X \neq \emptyset$ of S and a set Γ which satisfy the following three conditions.

- The prime set S is the direct sum $C \amalg X$ of C and X.
- Let $(\tau_\lambda)_{\lambda\in\Lambda}$ be the OS of the type algebra T. Then its index set Λ is contained in the direct sum $\Gamma \amalg \Gamma X$ of Γ and ΓX, where ΓX is the set of all formal products γx of $\gamma \in \Gamma$ and $x \in X$.
- The arity of each operation τ_λ with $\lambda \in \Lambda \cap \Gamma X$ is equal to 1.

We call C and X the sets of the **constants** and **variables** respectively. Henceforth, we identify each index $\lambda \in \Lambda \cap \Gamma X$ with the operation τ_λ, call it a **variable operation**, and denote its domain by T_λ.

Let $(A, T, \sigma, S, C, X, \Gamma)$ be a formal language. Define $\Lambda' = \Lambda \cap \Gamma$ and let T' be the operational subalgebra of T obtained by reducing the OS of T from $(\tau_\lambda)_{\lambda\in\Lambda}$ to $(\tau_\lambda)_{\lambda\in\Lambda'}$. Then, a sorted algebra W is called a **representable world** or a **cognizable world** for A, if it satisfies the following two conditions.

- The type algebra of W is equal to T'.
- $W_t \neq \emptyset$ for each $t \in \sigma S$.

Furthermore, an arbitrarily chosen non-empty collection $\mathcal{W}$ of representable worlds for A is called the domain of the **actual worlds** for A.

For each actual world $W \in \mathcal{W}$ for A, a C**-denotation** into W is a mapping Φ of C into W which satisfies $\Phi C_t \subseteq W_t$ for each $t \in T$. There is at least one C-denotation. If $C = \emptyset$, then since $\emptyset \to W = \{\emptyset\}$ by the set-theoretical definition of $Y \to Z$, $\emptyset$ is the unique C-denotation. Similarly, an X**-denotation** into W is a mapping ν of X into W which satisfies $\nu X_t \subseteq W_t$ for each $t \in T$. We denote the set of all X-denotations into W by $V_{X,W}$. Then $V_{X,W} \neq \emptyset$, and so we can construct the power algebra $(W^{V_{X,W}}, T', \rho)$ of W with exponent $V_{X,W}$. Let $(\beta_\lambda)_{\lambda\in\Lambda'}$ be its OS.

Suppose that, for an actual world $W \in \mathcal{W}$ for A and for each variable operation $\lambda \in \Lambda \cap \Gamma X$ and the variable x such that $\lambda \in \Gamma x$, we are given a mapping

$$\lambda_W \in \left(\bigcup_{t\in T_\lambda} (W_{\sigma x} \to W_t)\right) \to W$$

which satisfies

$$\lambda_W(W_{\sigma x} \to W_t) \subseteq W_{\lambda t}$$

for each $t \in T_\lambda$. Then we can define the unary operation β_λ on $W^{V_{X,W}}$ for each $\lambda \in \Lambda \cap \Gamma X$ as follows, and extending the OS of $W^{V_{X,W}}$ from $(\beta_\lambda)_{\lambda \in \Lambda'}$ to $(\beta_\lambda)_{\lambda \in \Lambda}$, we can construct the sorted algebra $(W^{V_{X,W}}, T, \rho)$. First we define, for each pair x, w of $x \in X$ and $w \in W_{\sigma x}$, the transformation $v \mapsto (x/w)v$ on $V_{X,W}$ by

$$\big((x/w)v\big)y = \begin{cases} vy & \text{when } X \ni y \neq x, \\ w & \text{when } y = x. \end{cases}$$

Next we define, for each quadruple t, φ, x, v consisting of $t \in T$, $\varphi \in V_{X,W} \to W_t$, $x \in X$ and $v \in V_{X,W}$, the mapping $\varphi\big((x/\square)v\big)$ of $W_{\sigma x}$ into W_t by

$$\Big(\varphi\big((x/\square)v\big)\Big)\, w = \varphi\big((x/w)v\big)$$

for each $w \in W_{\sigma x}$. We finally define for each $\lambda \in \Lambda \cap \Gamma X$ the unary operation β_λ on $W^{V_{X,W}}$ as follows. Suppose $\lambda \in \Gamma x$ with $x \in X$. First we define

$$\mathrm{Dom}\, \beta_\lambda = \bigcup_{t \in T_\lambda} (V_{X,W} \to W_t).$$

Next for each $t \in T_\lambda$ and each $\varphi \in V_{X,W} \to W_t$ we define $\beta_\lambda \varphi$ to be the element of $V_{X,W} \to W_{\lambda t}$ such that

$$(\beta_\lambda \varphi)v = \lambda_W\Big(\varphi\big((x/\square)v\big)\Big)$$

for each $v \in V_{X,W}$. Since $\varphi\big((x/\square)v\big) \in W_{\sigma x} \to W_t$ and $\lambda_W(W_{\sigma x} \to W_t) \subseteq W_{\lambda t}$, certainly $(\beta_\lambda \varphi)v \in W_{\lambda t}$. Since $V_{X,W} \to W_t$ is the t-part of $W^{V_{X,W}}$ for each $t \in T$, we have thus constructed the sorted algebra $(W^{V_{X,W}}, T, \rho)$. We call the mapping λ_W used above for $\lambda \in \Lambda \cap \Gamma X$ an **interpretation** of λ on W.

Now let Φ be a C-denotation into W. Then we can construct the sort-consistent homomorhism Φ^* of A into $W^{V_{X,W}}$ as follows. First we define the mapping φ of S into $V_{X,W} \to W$ so that

$$(\varphi a)v = \begin{cases} \Phi a & \text{when } a \in C, \\ va & \text{when } a \in X \end{cases}$$

for each $v \in V_{X,W}$. Then $\varphi S_t \subseteq V_{X,W} \to W_t$ for each $t \in T$ because $\Phi C_t \subseteq W_t$ and $vX_t \subseteq W_t$, and so φ maps S into $W^{V_{X,W}}$ and satisfies $\rho\varphi = \sigma|_S$. Therefore by the universality of A, there exists a unique sort-consistent homomorphism of A into $W^{V_{X,W}}$ which extends φ. We call it the **semantic mapping** determined by Φ and denote it by Φ^*.

By definition, a **logical system** is a triple $A, \mathcal{W}, (\lambda_W)_{\lambda,W}$ of a formal language $(A, T, \sigma, S, C, X, \Gamma)$, a domain $\mathcal{W}$ of actual worlds for A, and a family $(\lambda_W)_{\lambda,W}$ of interpretations λ_W of variable operations $\lambda \in \Lambda \cap \Gamma X$ on $W \in \mathcal{W}$.

Suppose the logical system $A, \mathcal{W}, (\lambda_W)_{\lambda,W}$ satisfies the following condition.

- For an element $\phi \in T$, the ϕ-part A_ϕ of A is non-empty, and the ϕ-part W_ϕ of each $W \in \mathcal{W}$ is equal to $\mathbb{T} = \{0, 1\}$.

Then we call ϕ a **truth** and call the elements of A_ϕ the **sentences**.

Suppose $A, \mathcal{W}, (\lambda_W)_{\lambda,W}$ is a logical system with a truth ϕ. Then we can construct a non-empty subset $\mathcal{F}_\mathcal{W}$ of $A_\phi \to \mathbb{T}$ as follows. Let $W \in \mathcal{W}$ be an actual world and Φ be a C-denotation into W. Then since the semantic mapping Φ^* is sort-consistent and the ϕ-part $V_{X,W} \to W_\phi$ of $W^{V_{X,W}}$ is equal to $V_{X,W} \to \mathbb{T}$ because $W_\phi = \mathbb{T}$, we have $\Phi^* A_\phi \subseteq V_{X,W} \to \mathbb{T}$, and so for each $\nu \in V_{X,W}$, we obtain the mappping $a \mapsto (\Phi^* a)\nu$ of A_ϕ into $\mathbb{T}$. We define $\mathcal{F}_\mathcal{W}$ to be the set of all those mappings obtained from all possible triples W, Φ, ν of actual worlds $W \in \mathcal{W}$ and C-denotations Φ into W and $\nu \in V_{X,W}$.

Thus we have seen above that each logical system $A, \mathcal{W}, (\lambda_W)_{\lambda,W}$ with a truth ϕ yields the pair $(A_\phi, \mathcal{F}_\mathcal{W})$ of A_ϕ and the subset $\mathcal{F}_\mathcal{W} \neq \emptyset$ of $A_\phi \to \mathbb{T}$. We call $(A_\phi, \mathcal{F}_\mathcal{W})$ the **sentence logical space**.

2 Preliminaries

Here is given an account of notions which underlie the theory of logical spaces.

2.1 Covers and properties of finite character

Definition 2.1 Let A be a set, $\mathcal{B}$ be a subset of $\mathcal{P}A$, and X be a subset of A. Then X is said to be **extra-covered** by $\mathcal{B}$, if for each $Y \in \mathcal{P}'X$ there exists an element $B \in \mathcal{B}$ such that $Y \subseteq B$. Also, X is said to be **super-covered** by $\mathcal{B}$, if for each $Y \in \mathcal{P}'X$ there exists an element $B \in \mathcal{B}$ such that $Y \subseteq B \subseteq X$. Also, X is said to be **ultra-covered** by $\mathcal{B}$, if $\mathcal{P}'X \subseteq \mathcal{B}$.

As an immediate consequence of Definition 2.1, we have that if X is ultra-covered by $\mathcal{B}$, then X is super-covered by $\mathcal{B}$. Also, if X is super-covered by $\mathcal{B}$, then X is extra-covered by $\mathcal{B}$. Also, if X is extra-covered by $\mathcal{B}$, then X is covered by $\mathcal{B}$ in the usual sense that $X \subseteq \bigcup_{B \in \mathcal{B}} B$ holds. Also, if $X \in \mathcal{B}$, then X is super-covered by $\mathcal{B}$. Furthermore, if $\mathcal{B}$ is **downward** in the sense that $\mathcal{P}B \subseteq \mathcal{B}$ for each $B \in \mathcal{B}$, then the notions of extra-cover, super-cover, and ultra-cover by $\mathcal{B}$ are identical.

Definition 2.2 Let A be a set and $\mathcal{B}$ be a subset of $\mathcal{P}A$. Then $\mathcal{B}$ is said to be **finitary**, if every subset of A which is extra-covered by $\mathcal{B}$ belongs to $\mathcal{B}$. Also, $\mathcal{B}$ is said to be **quasi-finitary**, if every subset of A which is super-covered by $\mathcal{B}$ belongs to $\mathcal{B}$. Also, $\mathcal{B}$ is said to be **semi-finitary**, if every subset of A which is ultra-covered by $\mathcal{B}$ belongs to $\mathcal{B}$.

As an immediate consequence of Definition 2.2 and the remark following Definition 2.1, we have that if $\mathcal{B}$ is finitary then $\mathcal{B}$ is quasi-finitary, and that if $\mathcal{B}$ is quasi-finitary then $\mathcal{B}$ is semi-finitary. Furthermore the following holds, which in particular shows that the above definition of "finitary" is identical with the usual one.

Theorem 2.1 Let A be a set and $\mathcal{B}$ be a subset of $\mathcal{P}A$. Then the following four conditions are equivalent.

(1) $\mathcal{B}$ is finitary.

(2) $\mathcal{B}$ is downward and quasi-finitary.

(3) $\mathcal{B}$ is downward and semi-finitary.

(4) A subset X of A belongs to $\mathcal{B}$ iff X is ultra-covered by $\mathcal{B}$.

Proof Since elements of $\mathcal{B}$ are extra-covered by $\mathcal{B}$, (1) is equivalent to the following condition.

(5) A subset X of A belongs to $\mathcal{B}$ iff X is extra-covered by $\mathcal{B}$.

Also, each of the conditions (1) - (5) implies that $\mathcal{B}$ is downward, and if $\mathcal{B}$ is downward, then extra-cover, super-cover, and ultra-cover by $\mathcal{B}$ are identical. Therefore, the conditions (1) - (5) are equivalent.

Theorem 2.2 Let A be a set and $\mathcal{B}$ be a subset of $\mathcal{P}A$. Assume that $\mathcal{B}$ is quasi-finitary. Then the ordered set $(\mathcal{B}, \subseteq)$ is inductive.

Proof Let $\{B_i \mid i \in I\}$ be a non-empty totally ordered subset of $\mathcal{B}$. Define $X = \bigcup_{i \in I} B_i$. If $Y \in \mathcal{P}'X$, then there exists an index $i \in I$ such that $Y \subseteq B_i$. Since $B_i \in \mathcal{B}$ and $B_i \subseteq X$, X is super-covered by $\mathcal{B}$, hence $X \in \mathcal{B}$. Therefore $(\mathcal{B}, \subseteq)$ is inductive.

Definition 2.3 Let A, B be sets and φ be a mapping of $\mathcal{P}A$ into $\mathcal{P}B$. Then φ is said to be **finitary**, if $\varphi X = \bigcup_{Y \in \mathcal{P}'X} \varphi Y$ for each $X \in \mathcal{P}A$.

Note that an increasing mapping $\varphi \in \mathcal{P}A \to \mathcal{P}B$ is finitary iff $\varphi X \subseteq \bigcup_{Y \in \mathcal{P}'X} \varphi Y$ for each $X \in \mathcal{P}A$. Note also that $\mathcal{P}A \to \mathcal{P}B$ is a complete lattice with respect to the order $\subseteq$ defined by

$$\varphi \subseteq \psi \iff \varphi X \subseteq \psi X \text{ for all } X \in \mathcal{P}A$$

for each $(\varphi, \psi) \in (\mathcal{P}A \to \mathcal{P}B)^2$. Incidentally, a mapping $\varphi \in \mathcal{P}A \to \mathcal{P}B$ is finitary iff, for each element $b \in B$, the subset $\mathcal{A}_b = \{X \in \mathcal{P}A \mid b \notin \varphi X\}$ of $\mathcal{P}A$ is finitary.

Theorem 2.3 Let A, B be sets. Then the following holds.

(1) If $\varphi \in \mathcal{P}A \to \mathcal{P}B$ is finitary, then φ is increasing.

(2) If $\varphi \in \mathcal{P}A \to \mathcal{P}B$ is finitary and $\psi \in \mathcal{P}A \to \mathcal{P}B$ is increasing, then the subset $\{X \in \mathcal{P}A \mid \varphi X \subseteq \psi X\}$ of $\mathcal{P}A$ is quasi-finitary, and $\varphi \subseteq \psi$ holds iff $\varphi Y \subseteq \psi Y$ for all $Y \in \mathcal{P}'A$.

(3) If $\varphi \in \mathcal{P}A \to \mathcal{P}B$ and $\psi \in \mathcal{P}B \to \mathcal{P}C$ are finitary for a set C, then the composite mapping $\psi \cdot \varphi \in \mathcal{P}A \to \mathcal{P}C$ is finitary.

(4) If $(\varphi_i)_{i \in I}$ is a family of finitary mappings $\varphi_i \in \mathcal{P}A \to \mathcal{P}B$, then their supremum $\bigcup_{i \in I} \varphi_i$ in $\mathcal{P}A \to \mathcal{P}B$ is also finitary.

(5) If $\varphi \in \mathcal{P}A \to \mathcal{P}B$ is finitary and $D \subseteq A$, then the mapping $X \mapsto \varphi(X \cup D)$ is also finitary.

Proof (1) If $X, Y \in \mathcal{P}A$ satisfy $X \subseteq Y$, then since $\mathcal{P}'X \subseteq \mathcal{P}'Y$, we have $\varphi X \subseteq \varphi Y$.

(2) Suppose $X \in \mathcal{P}A$ is super-covered by $\mathcal{B} = \{Z \in \mathcal{P}A \mid \varphi Z \subseteq \psi Z\}$. Then for each $Y \in \mathcal{P}'X$, there exists a set $Z \in \mathcal{B}$ such that $Y \subseteq Z \subseteq X$, so $\varphi Y \subseteq \varphi Z \subseteq \psi Z \subseteq \psi X$. Hence $\varphi X \subseteq \psi X$, that is, $X \in \mathcal{B}$. Therefore $\mathcal{B}$ is quasi-finitary. If $\varphi Y \subseteq \psi Y$ for all $Y \in \mathcal{P}'A$, every member of $\mathcal{P}A$ is ultra-covered by $\mathcal{B}$, hence $\mathcal{P}A \subseteq \mathcal{B}$, which implies $\varphi \subseteq \psi$.

(3) Let $X \in \mathcal{P}A$ and $Z \in \mathcal{P}'(\varphi X)$. Then $Z \subseteq \varphi X = \bigcup_{Y \in \mathcal{P}'X} \varphi Y$, and so for each element $z \in Z$, there exists a set $Y_z \in \mathcal{P}'X$ such that $z \in \varphi Y_z$. Define $Y_Z = \bigcup_{z \in Z} Y_z$. Then $Y_Z \in \mathcal{P}'X$ and $Z \subseteq \varphi(Y_Z)$, hence $\psi Z \subseteq \psi\big(\varphi(Y_Z)\big)$. Therefore, $\psi(\varphi X) = \bigcup_{Z \in \mathcal{P}'(\varphi X)} \psi Z \subseteq \bigcup_{Z \in \mathcal{P}'(\varphi X)} \psi\big(\varphi(Y_Z)\big) \subseteq \bigcup_{Y \in \mathcal{P}'X} \psi(\varphi Y)$. Since $\psi \cdot \varphi$ is increasing, we conclude that $\psi \cdot \varphi$ is finitary.

(4) Let $\varphi = \bigcup_{i \in I} \varphi_i$. Then the following holds for each $X \in \mathcal{P}A$, hence φ is finitary:

$$\varphi X = \bigcup_{i \in I} \varphi_i X = \bigcup_{i \in I} \left(\bigcup_{Y \in \mathcal{P}'X} \varphi_i Y \right) = \bigcup_{Y \in \mathcal{P}'X} \left(\bigcup_{i \in I} \varphi_i Y \right) = \bigcup_{Y \in \mathcal{P}'X} \varphi Y.$$

(5) Let δ be the constant mapping $X \mapsto D$ on $\mathcal{P}A$. Then, since $\varphi, \mathrm{id}_{\mathcal{P}A}$, and δ are finitary, so is $\varphi \cdot (\mathrm{id}_{\mathcal{P}A} \cup \delta)$, which is equal to $X \mapsto \varphi(X \cup D)$.

2.2 Closure operators

A **closure operator** on an ordered set $(A, \leq)$ is a mapping $\varphi \in A \to A$ which satisfies the following three conditions.

(1) φ is **extensive** in the sense that $x \leq \varphi x$ for all $x \in A$.

(2) φ is **idempotent** in the sense that $\varphi(\varphi x) = \varphi x$ for all $x \in A$.

(3) φ is increasing in the usual sense that if $x \leq y$ then $\varphi x \leq \varphi y$.

The set $\{x \in A \mid \varphi x = x\}$ of **φ-fixed** elements will be called the **fixture domain** of φ. It is equal to the image φA of φ and also to the set $\{x \in A \mid \varphi x \leq x\}$ of **φ-closed** elements. Note that the order $\leq$ on A induces the order $\leq$ on $A \to A$ defined by

$$\varphi \leq \psi \iff \varphi x \leq \psi x \text{ for all } x \in A$$

for each $(\varphi, \psi) \in (A \to A)^2$.

The following theorem is very important, but it is probably well-known and easily proved, so we omit its proof and sometimes use it without notices.

Theorem 2.4 Let $(A, \leq)$ be an ordered set. Then the following holds.

(1) If φ is a closure operator on $(A, \leq)$ and B is the fixture domain of φ, then $\varphi x = \min\{y \in B \mid x \leq y\}$.

(2) If B is a subset of A such that there exists $\min\{y \in B \mid x \leq y\}$ for each element $x \in A$, then the mapping $x \mapsto \min\{y \in B \mid x \leq y\}$ is a closure operator on $(A, \leq)$ and its fixture domain is equal to B.

(3) Two closure operators φ, ψ on $(A, \leq)$ satisfy $\varphi \leq \psi$ iff the fixture domain of ψ is contained in that of φ.

Theorem 2.5 Let $(A, \leq)$ be a complete lattice and, for each subset B of A, let $B^{\cap}$ denote the set $\{\inf X \mid X \subseteq B\}$. Let us say that B is **$\cap$-closed** in $(A, \leq)$ if $B^{\cap} = B$. Then $B^{\cap}$ is the smallest of the $\cap$-closed subsets of A which contain B (for this reason, we call $B^{\cap}$ the **$\cap$-closure** of B in $(A, \leq)$). Therefore, the mapping $B \mapsto B^{\cap}$ is a closure operator on $(\mathcal{P}A, \subseteq)$, and its fixture domain is equal to the set of the $\cap$-closed subsets of A.

In particular, if A is a set and $\mathcal{B}$ is a subset of $\mathcal{P}A$, then the following holds for the $\cap$-closure $\mathcal{B}^{\cap}$ of $\mathcal{B}$ in $(\mathcal{P}A, \subseteq)$.

(1) $\bigcap_{Y \subseteq B \in \mathcal{B}} B = \bigcap_{Y \subseteq X \in \mathcal{B}^{\cap}} X$ for each $Y \in \mathcal{P}A$.

(2) $\bigcup_{B \in \mathcal{B} - \{A\}} B = \bigcup_{X \in \mathcal{B}^{\cap} - \{A\}} X$.

Proof Since $\inf\{x\} = x$ for all $x \in A$, we have $B \subseteq B^{\cap}$. Next, if $(X_i)_{i \in I}$ is a family of subsets of A, then $\inf\{\inf X_i \mid i \in I\} = \inf\left(\bigcup_{i \in I} X_i\right)$, which implies that $B^{\cap}$ is $\cap$-closed. Also, if $B \subseteq B' \subseteq PA$ and B' is $\cap$-closed, then $B^{\cap} \subseteq {B'}^{\cap} = B'$. Thus the former assertion holds. As for the latter assertion, if $Y \in \mathcal{P}A$ and $Y \subseteq X \in \mathcal{B}^{\cap} - \{A\}$, then $Y \subseteq X = \bigcap_{X \subseteq B \in \mathcal{B} - \{A\}} B$. Therefore (1) and (2) hold.

Theorem 2.6 Let A be a set and, for each subset $\mathcal{B}$ of $\mathcal{P}A$, let $\overline{\mathcal{B}}$ be the set of the subsets of A which are super-covered by $\mathcal{B}$. Then the mappping $\mathcal{B} \mapsto \overline{\mathcal{B}}$ is a closure operator on $(\mathcal{P}(\mathcal{P}A), \subseteq)$ and its fixture domain is equal to the set of the quasi-finitary subsets of $\mathcal{P}A$. Therefore, $\overline{\mathcal{B}}$ is the smallest of the quasi-finitary subsets of $\mathcal{P}A$ which contain $\mathcal{B}$ (for this reason, we call $\overline{\mathcal{B}}$ the **quasi-finitary closure** of $\mathcal{B}$).

Furthermore the following holds.

(1) $\bigcap_{Y \subseteq B \in \mathcal{B}} B = \bigcap_{Y \subseteq X \in \overline{\mathcal{B}}} X$ for each $Y \in \mathcal{P}'A$.

(2) $\bigcup_{B \in \mathcal{B} - \{A\}} B = \bigcup_{X \in \overline{\mathcal{B}} - \{A\}} X$.

(3) $\overline{\mathcal{B}} - \mathcal{B}$ consists of infinite sets.

Proof Let $\mathcal{B} \subseteq \mathcal{P}A$. Then elements of $\mathcal{B}$ are super-covered by $\mathcal{B}$, hence $\mathcal{B} \subseteq \overline{\mathcal{B}}$. If $\mathcal{B} \subseteq \mathcal{B}' \subseteq \mathcal{P}A$, then elements of $\mathcal{P}A$ super-covered by $\mathcal{B}$ are also super-covered by $\mathcal{B}'$, hence $\overline{\mathcal{B}} \subseteq \overline{\mathcal{B}'}$. Suppose $X \in \mathcal{P}A$ is super-covered by $\overline{\mathcal{B}}$. If $Y \in \mathcal{P}'X$, then there exists $B' \in \overline{\mathcal{B}}$ such that $Y \subseteq B' \subseteq X$, and so there exists $B \in \mathcal{B}$ such that $Y \subseteq B \subseteq B' \subseteq X$. Thus X is super-covered by $\mathcal{B}$. Therefore $\overline{\overline{\mathcal{B}}} \subseteq \overline{\mathcal{B}}$. We have shown that the mappping $\mathcal{B} \mapsto \overline{\mathcal{B}}$ is a closure operator. By Definition 2.2, $\overline{\mathcal{B}} = \mathcal{B}$ iff $\mathcal{B}$ is quasi-finitary.

Suppose $Y \in \mathcal{P}'A$ and $Y \subseteq X \in \overline{\mathcal{B}}$. Then there exists a set $B \in \mathcal{B}$ such that $Y \subseteq B \subseteq X$. Therefore (1) holds. If $x \in X \in \overline{\mathcal{B}} - \{A\}$, then there exists a set $B \in \mathcal{B} - \{A\}$ such that $x \in B \subseteq X$. Therefore (2) holds. If X is a finite set in $\overline{\mathcal{B}}$, then since $X \in \mathcal{P}'X$, there exists a set $B \in \mathcal{B}$ such that $X \subseteq B \subseteq X$, hence $X = B \in \mathcal{B}$. Thus (3) holds.

Theorem 2.7 Let A be a set. Then the mapping $\mathcal{B} \mapsto \overline{\mathcal{B}^{\cap}}$ of $\mathcal{P}(\mathcal{P}A)$ into itself is a closure operator on $(\mathcal{P}(\mathcal{P}A), \subseteq)$, and its fixture domain is equal to the set of the subsets of $\mathcal{P}A$ which are $\cap$-closed in $(\mathcal{P}A, \subseteq)$ and quasi-finitary. Therefore, $\overline{\mathcal{B}^{\cap}}$ is the smallest of the subsets of $\mathcal{P}A$ which contain $\mathcal{B}$ and are $\cap$-closed in $(\mathcal{P}A, \subseteq)$ and quasi-finitary (for this reason, we call $\overline{\mathcal{B}^{\cap}}$ the **quasi-finitary $\cap$-closure** of $\mathcal{B}$).

Proof We only need to show that if $\mathcal{B}$ is a $\cap$-closed subset of $\mathcal{P}A$ then so is $\overline{\mathcal{B}}$. Let $(X_i)_{i \in I}$ be a family of subsets of $\overline{\mathcal{B}}$ and define $X = \bigcap_{i \in I} X_i$. If $Y \in \mathcal{P}'X$, then for each $i \in I$, there exists an element $B_i \in \mathcal{B}$ such that $Y \subseteq B_i \subseteq X_i$. Let $B = \bigcap_{i \in I} B_i$. Then $Y \subseteq B \subseteq X$, and $B \in \mathcal{B}$ because $\mathcal{B}$ is $\cap$-closed. Therefore $X \in \overline{\mathcal{B}}$ as desired.

Theorem 2.8 Let $(A, \leq)$ be an ordered set. For each element $a \in A$, let $(\leftarrow, a]$ denote the downward interval $\{x \in A \mid x \leq a\}$. Also, le us say that a subset B of A is **downward** if $(\leftarrow, b] \subseteq B$ for all $b \in B$. Furthermore, for each subset B of A, define

$$\overleftarrow{B} = \bigcup_{b \in B} (\leftarrow, b].$$

Then the mapping $B \mapsto \overleftarrow{B}$ is a closure operator on $(\mathcal{P}A, \subseteq)$ and its fixture domain is equal to the set of the downward subsets of A.

Proof This is because $\overleftarrow{B}$ is the smallest of the downward subsets of A which contain B (for this reason, we call $\overleftarrow{B}$ the **downward closure** of B).

Theorem 2.9 Let A be a set and $\mathcal{B}$ be a subset of $\mathcal{P}A$. Then the quasi-finitary closure $\overline{\overleftarrow{\mathcal{B}}}$ of the downward closure $\overleftarrow{\mathcal{B}}$ of $\mathcal{B}$ in $(\mathcal{P}A, \subseteq)$ is equal to the set of the elements of $\mathcal{P}A$ which are extra-covered by $\mathcal{B}$, and the mapping $\mathcal{B} \mapsto \overline{\overleftarrow{\mathcal{B}}}$ is a closure operator on $(\mathcal{P}(\mathcal{P}A), \subseteq)$ whose fixture domain is equal to the set of the finitary subsets of $\mathcal{P}A$. Therefore, $\overline{\overleftarrow{\mathcal{B}}}$ is the smallest of the finitary subsets of $\mathcal{P}A$ which contain $\mathcal{B}$ (for this reason, we call $\overline{\overleftarrow{\mathcal{B}}}$ the **finitary closure** of $\mathcal{B}$).

Proof First assume that $X \in \mathcal{P}A$ is extra-covered by $\mathcal{B}$. If $Y \in \mathcal{P}'X$, then there exists an element $B \in \mathcal{B}$ such that $Y \subseteq B$, hence $X \cap B \in \overleftarrow{\mathcal{B}}$ and $Y \subseteq X \cap B \subseteq X$. Thus $X \in \overleftarrow{\overleftarrow{\mathcal{B}}}$. Next assume $X \subseteq X' \in \overleftarrow{\overleftarrow{\mathcal{B}}}$. If $Y \in \mathcal{P}'X$, then there exists an element $Z \in \overleftarrow{\mathcal{B}}$ such that $Y \subseteq Z \subseteq X'$, and so there exists an element $B \in \mathcal{B}$ such that $Y \subseteq Z \subseteq B$. Therefore X is extra-covered by $\mathcal{B}$. We have shown that $\overleftarrow{\overleftarrow{\mathcal{B}}}$ is equal to the set of the elements of $\mathcal{P}A$ which are extra-covered by $\mathcal{B}$ and also that $\overleftarrow{\overleftarrow{\mathcal{B}}}$ is downward. Therefore, the mapping $\mathcal{B} \mapsto \overleftarrow{\overleftarrow{\mathcal{B}}}$ is a closure operator and its fixture domain is equal to the set of the finitary subsets of $\mathcal{P}A$.

Theorem 2.10 Let φ be a closure operator on a complete lattice $(A, \leq)$ and let B be the fixture domain of φ. Then the following holds.

(1) $\sup_B X = \varphi(\sup_A X)$ and $\inf_B X = \varphi(\inf_A X) = \inf_A X$ for all sets $X \subseteq B$.

(2) $\min B = \varphi(\min A)$ and $\max B = \max A$.

(3) $\varphi(\sup_A(\varphi Y)) = \varphi(\sup_A Y)$ for all sets $Y \subseteq A$, where $\varphi Y = \{\varphi y \mid y \in Y\}$.

Consequently, $(B, \leq)$ is also complete and B is $\cap$-closed in A.

Proof Suppose $X \subseteq B$. Then, for all elements $x \in X$, we have $x \leq \sup_A X$, hence $x = \varphi x \leq \varphi(\sup_A X) \in B$. Conversely, if an element $y \in B$ satisfies $x \leq y$ for all elements $x \in X$, then $\sup_A X \leq y$, hence $\varphi(\sup_A X) \leq \varphi y = y$. Thus $\varphi(\sup_A X) = \sup_B X$, and similarly $\varphi(\inf_A X) = \inf_B X$. Furthermore, since $\varphi(\inf_A X) \leq \varphi x = x$ for all elements $x \in X$, we have $\varphi(\inf_A X) \leq \inf_A X$, so $\inf_A X \in B$, hence $\inf_B X = \inf_A X$.

For all elements $x \in B$, we have $\min A \leq x$, hence $B \ni \varphi(\min A) \leq \varphi x = x$. Therefore $\min B = \varphi(\min A)$. Similarly $\max B = \varphi(\max A)$. Furthermore, since $\max A \leq \varphi(\max A)$, we have $\varphi(\max A) = \max A$.

Suppose $Y \subseteq A$. Then, since $\varphi y \geq y$ for all elements $y \in Y$, we have $\sup_A(\varphi Y) \geq \sup_A Y$, hence $\varphi(\sup_A(\varphi Y)) \geq \varphi(\sup_A Y)$. Also, since $y \leq \sup_A Y$ for all elements $y \in Y$, we have $\varphi y \leq \varphi(\sup_A Y)$, and so $\sup_A(\varphi Y) \leq \varphi(\sup_A Y)$, hence $\varphi(\sup_A(\varphi Y)) \leq \varphi(\sup_A Y)$. Therefore $\varphi(\sup_A(\varphi Y)) = \varphi(\sup_A Y)$.

Theorem 2.11 Let A be a set, φ be a closure operator on $(\mathcal{P}A, \subseteq)$, $\mathcal{B}$ be the fixture domain of φ, $D \subseteq A$, and define $\psi \in \mathcal{P}A \to \mathcal{P}A$ by $\psi X = \varphi(X \cup D)$. Then ψ is also a closure operator and its fixture domain is equal to $\{Y \in \mathcal{B} \mid D \subseteq Y\}$. If furthermore φ is finitary, so is ψ.

Proof Let $\mathcal{B}' = \{Y \in \mathcal{B} \mid D \subseteq Y\}$. Then $\varphi(X \cup D) = \min\{Y \in \mathcal{B}' \mid X \subseteq Y\}$ by Theorem 2.4, hence the former assertion. The latter assertion is a consequence of Theorem 2.3.

3 Latticed representations

This section is also of preliminary nature, but is much closer to logical spaces than the previous section.

Throughout this section, we let A be a set, and denote by A^* the set of all sequences $x_1 \cdots x_n$ of elements $x_1, \ldots, x_n$ of A of arbitrary finite length $n \geq 0$. In other words, A^* is the free monoid on A. We denote elements of A by $x, y, \ldots,$ while elements of A^* by $\alpha, \beta, \ldots,$ both with or without numerical subscripts. This **alphabetical convention** will be used throughout the remainder of this paper. In particular, the element of A^* of length 0 or the identity element of the monoid A^* will be denoted by ε. When $\alpha = x_1 \cdots x_n \in A^*$, we denote the subset $\{x_1, \ldots, x_n\}$ of A also by α, where if $n = 0$, then $\alpha = \varepsilon$ and $\{x_1, \ldots, x_n\} = \emptyset$. This **sequence convention** will also be used throughout the remainder of this paper.

Let S, T be sets. Then the relations between S and T may be regarded as the subsets of $S \times T$, and if we regard so, we may discuss the order $R \subseteq Q$ of relations R, Q between S and T, and if $R \subseteq Q$, we may say that R is contained in Q or that Q contains R. In particular, the term "largest" in Definition 3.3, Theorems 3.20, 6.1, and so on means "largest with respect to the order $\subseteq$," and similarly for the term "smallest."

3.1 Validity relations of latticed representations

Throughout this subsection, we let B be a lattice with respect to the order $\leq$, meet $\wedge$, and join $\vee$, and with the smallest element 0 and the largest element 1. Then a **latticed representation** of A on B is simply a mapping f of A into B.

Roughly speaking, each "non-trivial" latticed representation is regarded as a functional logical space, and conversely, each logical space is "equivalent" to a latticed representation. This is the reason why latticed representations are relevant to the theory of logical spaces.

For the latticed representation (A, B, f), we can define the relation $\preccurlyeq_f$ on A^* by

$$\alpha \preccurlyeq_f \beta \iff \inf f\alpha \leq \sup f\beta, \tag{3.1}$$

because every finite subset of B has its infimum and supremum in B. Thus if $\alpha = x_1 \cdots x_m$, $\beta = y_1 \cdots y_n$, then $f\alpha = f\{x_1, \ldots, x_m\} = \{fx_1, \ldots, fx_m\}$ and $f\beta = f\{y_1, \ldots, y_n\} = \{fy_1, \ldots, fy_n\}$ by the sequence convention, hence

$$\begin{aligned} x_1 \cdots x_m \preccurlyeq_f y_1 \cdots y_n &\iff \inf\{fx_1, \ldots, fx_m\} \leq \sup\{fy_1, \ldots, fy_n\} \\ &\iff fx_1 \wedge \ldots \wedge fx_m \leq fy_1 \vee \ldots \vee fy_n, \end{aligned}$$

where $\inf \emptyset = 1$ and $\sup \emptyset = 0$. We call $\preccurlyeq_f$ the **f-validity relation**.

Here we study the property of the f-validity relation under various algebraic additional conditions on the latticed representation (A, B, f).

Theorem 3.1 The f-validity relation $\preccurlyeq_{\mathsf{f}}$ satisfies the following four laws, where $\succcurlyeq_{\mathsf{f}}$ is the dual of $\preccurlyeq_{\mathsf{f}}$:

$$x \preccurlyeq_f x, \qquad \text{(repetition law)}$$

$$\left.\begin{array}{l} \alpha \preccurlyeq_f \beta \implies x\alpha \preccurlyeq_f \beta, \\ \alpha \succcurlyeq_f \beta \implies x\alpha \succcurlyeq_f \beta, \end{array}\right\} \qquad \text{(weakening law)}$$

$$\left.\begin{array}{l} xx\alpha \preccurlyeq_f \beta \implies x\alpha \preccurlyeq_f \beta, \\ xx\alpha \succcurlyeq_f \beta \implies x\alpha \succcurlyeq_f \beta, \end{array}\right\} \qquad \text{(contraction law)}$$

$$\left.\begin{array}{l} \alpha xy\beta \preccurlyeq_f \gamma \implies \alpha yx\beta \preccurlyeq_f \gamma, \\ \alpha xy\beta \succcurlyeq_f \gamma \implies \alpha yx\beta \succcurlyeq_f \gamma. \end{array}\right\} \qquad \text{(exchange law)}$$

Proof Omitted.

Remark 3.1 The repetition law is related to but different from

$$\alpha \preccurlyeq_f \alpha. \qquad \text{(reflexivity law)}$$

The $\preccurlyeq_f$ is reflexive iff $\#B = 1$.

Theorem 3.2 The f-validity relation $\preccurlyeq_f$ satisfies the following law:

$$\left.\begin{array}{l} \alpha \preccurlyeq_f x,\ x\beta \preccurlyeq_f \delta \implies \alpha\beta \preccurlyeq_f \delta, \\ \alpha \succcurlyeq_f x,\ x\beta \succcurlyeq_f \delta \implies \alpha\beta \succcurlyeq_f \delta. \end{array}\right\} \qquad \text{(cut law)}$$

If B is a distributive lattice, then $\preccurlyeq_f$ satisfies the following law:

$$\left.\begin{array}{l} \alpha \preccurlyeq_f x\gamma, \\ x\beta \preccurlyeq_f \delta \end{array}\right\} \implies \alpha\beta \preccurlyeq_f \delta\gamma. \qquad \text{(strong cut law)}$$

Proof In order to prove the strong cut law, let $a = \inf f\alpha$, $b = \inf f\beta$, $c = \sup f\gamma$, $d = \sup f\delta$, and $e = fx$. Then the premise implies that $a \leq e \vee c$ and $e \wedge b \leq d$ holds. Therefore, if B is distributive, then

$$a \wedge b \leq (e \vee c) \wedge b \leq (e \wedge b) \vee c \leq d \vee c.$$

Therefore $\inf f(\alpha\beta) = a \wedge b \leq d \vee c = \sup f(\delta\gamma)$, hence $\alpha\beta \preccurlyeq_f \delta\gamma$. If $\gamma = \varepsilon$ or $\beta = \varepsilon$, then $c = 0$ or $b = 1$, and so the above displayed reasoning works without the distributive law. Thus the cut law holds with no additional conditions.

Definition 3.1 If a relation on A^* satisfies the repetition law, weakening law, contraction law, exchange law, and *cut law*, we say that it is a **latticed relation**. Also, if a relation on A^* satisfies the repetition law, weakening law, contraction law, exchange law, and *strong cut law*, we say that it is a **strongly latticed relation**.

Thus, the f-validity relation $\preccurlyeq_f$ is a latticed relation, and if B is distributive, then $\preccurlyeq_f$ is a strongly latticed relation.

Theorem 3.3 The image fA of f contains $0, 1$ iff the f-validity relation $\preccurlyeq_f$ satisfies the following **end laws**:

$$\text{there exists an element } x \in A \text{ such that } x \preccurlyeq_f \varepsilon, \qquad \text{(lower end law)}$$
$$\text{there exists an element } x \in A \text{ such that } x \succcurlyeq_f \varepsilon. \qquad \text{(upper end law)}$$

Proof This is because an element $x \in A$ satisfies $fx = 0$ iff $x \preccurlyeq_f \varepsilon$, while x satisfies $fx = 1$ iff $x \succcurlyeq_f \varepsilon$.

Theorem 3.4 According as $0 = \inf fA$ or $1 = \sup fA$, the f-validity relation $\preccurlyeq_f$ satisfies the following inf-emptiness law or the sup-emptiness law (the union of these laws will be called the **emptiness laws**):

$$\alpha \preccurlyeq_f \varepsilon \iff \alpha \preccurlyeq_f y \text{ for every element } y \in A, \qquad \text{(inf-emptiness law)}$$
$$\alpha \succcurlyeq_f \varepsilon \iff \alpha \succcurlyeq_f y \text{ for every element } y \in A. \qquad \text{(sup-emptiness law)}$$

Proof Assume $0 = \inf fA$. If an element $\alpha \in A^*$ satisfies $\alpha \preccurlyeq_f y$ for every element $y \in A$, then $\inf f\alpha \leq fy$ for every element $y \in A$, and so $\inf f\alpha \leq \inf fA = 0$, hence $\alpha \preccurlyeq_f \varepsilon$. Conversely if $\alpha \preccurlyeq_f \varepsilon$, then $\alpha \preccurlyeq_f y$ for every element $y \in A$ by the weakening law. The rest of the proof is omitted.

Theorem 3.5 Let $(x, y) \mapsto x \wedge y$ and $(x, y) \mapsto x \vee y$ be binary global operations on A. Then f is a $\{\wedge, \vee\}$-homomorphism iff the f-validity relation $\preccurlyeq_f$ satisfies the following two **junction laws**:

$$x \wedge y \preccurlyeq_f x, \qquad x \wedge y \preccurlyeq_f y, \qquad xy \preccurlyeq_f x \wedge y, \qquad \text{(conjunction law)}$$
$$x \vee y \succcurlyeq_f x, \qquad x \vee y \succcurlyeq_f y, \qquad xy \succcurlyeq_f x \vee y. \qquad \text{(disjunction law)}$$

Proof Assume that f is a $\{\wedge\}$-homomorphism. Then, for each $(x, y) \in A \times A$, we have $f(x \wedge y) = fx \wedge fy$, hence $f(x \wedge y) \leq fx$, $f(x \wedge y) \leq fy$, and $fx \wedge fy \leq f(x \wedge y)$. These inequalities show that $\preccurlyeq_f$ satisfies the conjunction law. Similarly, if f is a $\{\vee\}$-homomorphism, then $\preccurlyeq_f$ satisfies the disjunction law.

Conversely, assume that $\preccurlyeq_f$ satisfies the conjunction law. Then for each $(x, y) \in A \times A$, we have $f(x \wedge y) \leq fx$, $f(x \wedge y) \leq fy$, and $fx \wedge fy \leq f(x \wedge y)$, hence $f(x \wedge y) = fx \wedge fy$. Thus f is a $\{\wedge\}$-homomorphism. Similarly, if $\preccurlyeq_f$ satisfies the disjunction law, then f is a $\{\vee\}$-homomorphism.

A subset of B is called a **sublattice** of B if it is closed under the operations $\wedge$ and $\vee$.

Theorem 3.6 The image fA of f is a sublattice of B iff the f-validity relation $\preccurlyeq_f$ satisfies the following **quasi-junction laws**:

for each $(x, y) \in A \times A$, there exists an element $z \in A$ such that $z \preccurlyeq_f x$, $z \preccurlyeq_f y$, and $xy \preccurlyeq_f z$. (quasi-conjunction law) for each $(x, y) \in A \times A$, there exists an element $z \in A$ such that $z \succcurlyeq_f x,\ z \succcurlyeq_f y$, and $xy \succcurlyeq_f z$. (quasi-disjunction law).

Proof Let $(x, y) \in A \times A$. If fA is closed under the operation $\wedge$, then there exists an element $z \in A$ such that $fx \wedge fy = fz$, hence $fz \le fx,\ fz \le fy$, and $fx \wedge fy \le fz$, which imply $z \preccurlyeq_f x,\ z \preccurlyeq_f y$, and $xy \preccurlyeq_f z$ respectively. Conversely, if there exists an element $z \in A$ which satisfies $z \preccurlyeq_f x,\ z \preccurlyeq_f y$, and $xy \preccurlyeq_f z$, then $fz \le fx,\ fz \le fy$, and $fx \wedge fy \le fz$, hence $fx \wedge fy = fz \in fA$. The rest of the proof is omitted.

Theorem 3.7 Assume that B is a Boolean lattice, and let $\lozenge$ denote the complement on B. Furthermore let $x \mapsto x^{\lozenge}$ be an unary global operation on A. Then f is a $\{\lozenge\}$-homomorphism iff the f-validity relation $\preccurlyeq_f$ satisfies the following **negation laws**:

$$xx^{\lozenge} \preccurlyeq_f \varepsilon, \qquad \text{(lower negation law)}$$
$$xx^{\lozenge} \succcurlyeq_f \varepsilon. \qquad \text{(upper negation law)}$$

Proof If f is a $\{\lozenge\}$-homomorphism, then $fx \wedge f(x^{\lozenge}) = fx \wedge (fx)^{\lozenge} = 0$ and $fx \vee f(x^{\lozenge}) = fx \vee (fx)^{\lozenge} = 1$ for every element $x \in A$, and so $\preccurlyeq_f$ satisfies the negation laws. Conversely, assume that $\preccurlyeq_f$ satisfies the negation laws. Then for every element $x \in A$, we have $fx \wedge f(x^{\lozenge}) = 0$ and $fx \vee f(x^{\lozenge}) = 1$, hence $f(x^{\lozenge}) = (fx)^{\lozenge}$ by the uniqueness of the complement. Thus f is a $\{\lozenge\}$-homomorphism.

Remark 3.2 As in [2], in dealing with several lattices simultaneously, we wish to use different symbols for meets and joins in different lattices, for instance, $\cap$ and $\cup$ for a lattice A, $\wedge$ and $\vee$ for a lattice B, $\sqcap$ and $\sqcup$ for a lattice C, and so on. Then, how about complements in Boolean lattices? The best way is to use symbols made of those for meets and joins. For instance, if the lattices $A, B, C, \ldots$ above are Boolean, then use $\circ$ for A, $\lozenge$ for B, $\square$ for C, and so on. This is the reason why an unusual symbol $\lozenge$ is used for complements or negations in this paper and [2]. In [2], the usual symbol $\neg$ for negations is used for another kind of negation.

Theorem 3.8 Assume that B is a Boolean lattice, and let $\lozenge$ and $\Rightarrow$ be the complement and implication on B. Furthermore let $x \mapsto x^{\lozenge}$ and $(x, y) \mapsto x \Rightarrow y$ be unary and binary global operations on A. Then f is a $\{\lozenge, \Rightarrow\}$-isomorphism iff the f-validity relation $\preccurlyeq_f$ satisfies the following **implication laws** in addition to the negation laws:

$$x^{\lozenge} \preccurlyeq_f x \Rightarrow y, \qquad \text{(first implication law)}$$
$$y \preccurlyeq_f x \Rightarrow y, \qquad \text{(second implication law)}$$
$$x \Rightarrow y \preccurlyeq_f x^{\lozenge} y. \qquad \text{(third implication law)}$$

Proof Assume that f is a $\{\Diamond,\Rightarrow\}$-homomorphism. Then $\preccurlyeq_{\mathsf{f}}$ satisfies the negation laws by Theorem 3.7. Furthermore we have

$$\mathsf{f}(x \Rightarrow y) = \mathsf{f}x \Rightarrow \mathsf{f}y = (\mathsf{f}x)^{\Diamond} \vee \mathsf{f}y = \mathsf{f}(x^{\Diamond}) \vee \mathsf{f}y.$$

Hence we have three inequalities

$$\mathsf{f}(x^{\Diamond}) \leq \mathsf{f}(x \Rightarrow y), \qquad \mathsf{f}y \leq \mathsf{f}(x \Rightarrow y), \qquad \mathsf{f}(x \Rightarrow y) \leq \mathsf{f}(x^{\Diamond}) \vee \mathsf{f}y,$$

which imply that the implication laws hold.

Conversely, assume that $\preccurlyeq_{\mathsf{f}}$ satisfies the negation laws and implication laws. Then f is a $\{\Diamond\}$-homomorphism by Theorem 3.7. Furthermore, $(\mathsf{f}x)^{\Diamond} = \mathsf{f}(x^{\Diamond}) \leq \mathsf{f}(x \Rightarrow y)$ and $\mathsf{f}y \leq \mathsf{f}(x \Rightarrow y)$ by the first and second implication laws. Hence $\mathsf{f}x \Rightarrow \mathsf{f}y = (\mathsf{f}x)^{\Diamond} \vee \mathsf{f}y \leq \mathsf{f}(x \Rightarrow y)$. On the other hand, $\mathsf{f}(x \Rightarrow y) \leq \mathsf{f}(x^{\Diamond}) \vee \mathsf{f}y \leq (\mathsf{f}x)^{\Diamond} \vee \mathsf{f}y = \mathsf{f}x \Rightarrow \mathsf{f}y$ by the third implication law. Therefore $\mathsf{f}(x \Rightarrow y) = \mathsf{f}x \Rightarrow \mathsf{f}y$. Thus f is an $\{\Rightarrow\}$-homomorphism.

Definition 3.2 Assume that B is a Boolean lattice, and let $\Diamond$ and $\Rightarrow$ denote the complement and implication on B. Also, assume that $x \wedge y,\ x \vee y,\ x^{\Diamond},\ x \Rightarrow y$ are global operations on A, and $\mathsf{f} \in \mathsf{A} \to \mathsf{B}$ is a $\{\wedge, \vee, \Diamond, \Rightarrow\}$-homomorphism. Then we say that f is a **Boolean representation** of A on B with respect to the operations $\wedge, \vee, \Diamond, \Rightarrow$.

Definition 3.3 Let $x \wedge y,\ x \vee y,\ x^{\Diamond},\ x \Rightarrow y$ be global operations on A. Then a relation $\preccurlyeq$ on A^* is called a **Boolean relation** if it is a strongly latticed relation and satisfies the junction laws, negation laws, and implication laws with respect to the operations $\wedge, \vee, \Diamond, \Rightarrow$.

Also, a relation $\preccurlyeq$ on A^* is said to be **weakly Boolean**, if it satisfies the repetition law, weakening law, contraction law, exchange law, and the following four laws with respect to the operations $\wedge, \vee, \Diamond, \Rightarrow$:

$$\left.\begin{array}{l} xy\alpha \preccurlyeq \beta \implies x \wedge y, \alpha \preccurlyeq \beta, \\ \alpha \preccurlyeq x\beta,\ \alpha \preccurlyeq y\beta \implies \alpha \preccurlyeq x \wedge y, \beta, \end{array}\right\} \qquad \text{(strong conjunction law)}$$

$$\left.\begin{array}{l} x\alpha \preccurlyeq \beta,\ y\alpha \preccurlyeq \beta \implies x \vee y, \alpha \preccurlyeq \beta, \\ \alpha \preccurlyeq xy\beta \implies \alpha \preccurlyeq x \vee y, \beta, \end{array}\right\} \qquad \text{(strong disjunction law)}$$

$$\left.\begin{array}{l} \alpha \preccurlyeq x\beta \implies x^{\Diamond}\alpha \preccurlyeq \beta, \\ x\alpha \preccurlyeq \beta \implies \alpha \preccurlyeq x^{\Diamond}\beta, \end{array}\right\} \qquad \text{(strong negation law)}$$

$$\left.\begin{array}{l} \alpha \preccurlyeq x\beta,\ y\alpha \preccurlyeq \beta \implies x \Rightarrow y, \alpha \preccurlyeq \beta, \\ x\alpha \preccurlyeq y\beta \implies \alpha \preccurlyeq x \Rightarrow y, \beta. \end{array}\right\} \qquad \text{(strong implication law)}$$

The strong conjunction law and strong disjunction law put together will be called the **strong junction laws**. Obviously, the largest relation on A^* is Boolean and weakly Boolean. We call it the **trivial** relation.

Note that weakly Boolean relations need not satisfy the cut law. The relationship between the Boolean relations and the weakly Boolean relations will be clarified in the next subsection.

Theorem 3.9 Assume that B is a Boolean lattice and f is a Boolean representation with respect to the global operations $x \wedge y$, $x \vee y$, $x^{\Diamond}$, $x \Rightarrow y$ on A. Then the f-validity relation $\preccurlyeq_f$ is a Boolean relation with respect to the operations.

Proof This is a synthesis of Theorems 3.1, 3.2, 3.5, and 3.8.

Theorem 3.10 Assume that $fA = B$ and that the f-validity relation $\preccurlyeq_f$ satisfies either the strong conjunction law with respect to a global operation $x \wedge y$ on A or the strong disjunction law with respect to a global operation $x \vee y$ on A. Then B is a distributive lattice.

Proof Consider the case where $\preccurlyeq_f$ satisfies the strong conjunction law. Then, since $xy \preccurlyeq_f x$ and $xy \preccurlyeq_f y$ hold by the repetition law, weakening law, and exchange law, it follows that $\preccurlyeq_f$ satisfies the conjunction law. Therefore, f is a $\{\wedge\}$-homomorphism, as shown in the proof of Theorem 3.5. Let $a, b, c, d \in B$ and assume $d \leq a \vee c$, $d \leq b \vee c$. Then there exist elements $x, y, z, w \in A$ such that $a = fx$, $b = fy$, $c = fz$, $d = fw$, which necessarily satisfy $w \preccurlyeq_f xz$, $w \preccurlyeq_f yz$. Therefore $w \preccurlyeq_f x \wedge y, z$ by the strong conjunction law, hence $fw \leq f(x \wedge y) \vee fz$. Since $f(x \wedge y) = fx \wedge fy$, we conclude that $d \leq (a \wedge b) \vee c$ holds. Therefore B is distributive.

Theorem 3.11 Assume that the latticed representation (A, B, f) satisfies the following condition for all elements $\alpha \in A^*$:

$$\left.\begin{array}{l} a = \inf f\alpha \implies a = \sup\bigl(fA \cap (\leftarrow a]\bigr), \\ a = \sup f\alpha \implies a = \inf\bigl(fA \cap [a \rightarrow)\bigr), \end{array}\right\} \tag{3.2}$$

where $(\leftarrow a] = \{b \in B \mid b \leq a\}$ and $[a \rightarrow) = \{b \in B \mid a \leq b\}$. Define the **pullback** $\leq_f$ of $\leq$ by f to be the relation on A defined by

$$x \leq_f y \iff fx \leq fy.$$

Then an element $(x_1 \cdots x_m, y_1 \cdots y_n) \in A^* \times A^*$ satisfies $x_1 \cdots x_m \preccurlyeq_f y_1 \cdots y_n$ iff the following holds for all elements $(x, y) \in A^2$:

$$x \leq_f x_i \ (i = 1, \ldots, m), \ y_j \leq_f y \ (j = 1, \ldots, n) \implies x \leq_f y \tag{3.3}$$

Proof Let $a = \inf\{fx_1, \ldots, fx_m\}$ and $b = \sup\{fy_1, \ldots, fy_n\}$. Then the following holds:

$$\begin{aligned} x \leq_f x_i \ (i = 1, \ldots, m) &\iff fx \leq a, \\ y_j \leq_f y \ (j = 1, \ldots, n) &\iff b \leq fy. \end{aligned}$$

Therefore, we may argue as follows to complete the proof:

$$\begin{aligned}
&(3.3) \text{ holds for all elements } (x, y) \in A^2 \\
&\iff \text{if elements } x, y \in A \text{ satisfy } fx \le a \text{ and } b \le fy, \text{ then } fx \le fy \\
&\iff \text{if elements } c, d \in fA \text{ satisfy } c \le a \text{ and } b \le d, \text{ then } c \le d \\
&\iff \sup\bigl(fA \cap (\leftarrow a]\bigr) \le \inf\bigl(fA \cap [b \rightarrow)\bigr) \\
&\iff a \le b \\
&\iff x_1 \cdots x_m \preccurlyeq_f y_1 \cdots y_n.
\end{aligned}$$

3.2 Analysis of latticed relations

Throughout this subsection, we let A be a set and $\preccurlyeq$ be a relation on A^* which satisfies the repetition law, weakening law, contraction law, and exchange law. Here we search for the laws which are equivalent to the strong cut law, junction laws, negation laws, and implication laws on $\preccurlyeq$.

Theorem 3.12 Let $x \wedge y$ be a binary global operation on A, and assume that $\preccurlyeq$ satisfies the cut law. Then the following three laws are equivalent:

($\wedge$1) $x \wedge y \preccurlyeq x,\ x \wedge y \preccurlyeq y$.

($\wedge$2) $xy\alpha \preccurlyeq \beta \implies x \wedge y, \alpha \preccurlyeq \beta$.

($\wedge$3) $\alpha \preccurlyeq x \wedge y, \beta \implies \alpha \preccurlyeq x\beta,\ \alpha \preccurlyeq y\beta$.

Also, the following law ($\wedge$4) and ($\wedge$5) are equivalent and ($\wedge$6) imply them. If $\preccurlyeq$ satisfies the strong cut law, then ($\wedge$6) is equivalent to ($\wedge$4) and to ($\wedge$5):

($\wedge$4) $xy \preccurlyeq x \wedge y$.

($\wedge$5) $x \wedge y, \alpha \preccurlyeq \beta \implies xy\alpha \preccurlyeq \beta$.

($\wedge$6) $\alpha \preccurlyeq x\beta,\ \alpha \preccurlyeq y\beta \implies \alpha \preccurlyeq x \wedge y, \beta$.

Let $x \vee y$ be a binary global operation on A, and assume that $\preccurlyeq$ satisfies the cut law. Then the following three laws are equivalent:

($\vee$1) $x \preccurlyeq x \vee y,\ y \preccurlyeq x \vee y$.

($\vee$2) $\alpha \preccurlyeq xy\beta \implies \alpha \preccurlyeq x \vee y, \beta$.

($\vee$3) $x \vee y, \alpha \preccurlyeq \beta \implies x\alpha \preccurlyeq \beta,\ y\alpha \preccurlyeq \beta$.

Also, the following law ($\vee$4) and ($\vee$5) are equivalent and ($\vee$6) imply them. If $\preccurlyeq$ satisfies the strong cut law, then ($\vee$6) is equivalent to ($\vee$4) and to ($\vee$5):

($\vee$4) $x \vee y \preccurlyeq xy$.

($\vee$5) $\alpha \preccurlyeq x \vee y, \beta \implies \alpha \preccurlyeq xy\beta$.

($\vee$6) $x\alpha \preccurlyeq \beta,\ y\alpha \preccurlyeq \beta \implies x \vee y, \alpha \preccurlyeq \beta.$

Proof The law ($\wedge$2) is derived from ($\wedge$1) by the cut law, exchange law, and contraction law. We have $xy \preccurlyeq x$ and $xy \preccurlyeq y$ by the repetition law, weakening law, and exchange law, and so ($\wedge$2) with $\alpha = \varepsilon$ and $\beta = x$ or y implies ($\wedge$1). The law ($\wedge$3) is derived from ($\wedge$1) by the cut law. We have $x \wedge y \preccurlyeq x \wedge y$ by the repetition law, and so ($\wedge$3) with $\alpha = x \wedge y$ and $\beta = \varepsilon$ implies ($\wedge$1).

The law ($\wedge$5) is derived from ($\wedge$4) by the cut law. The law ($\wedge$5) with $\alpha = \varepsilon$ and $\beta = x \wedge y$ implies ($\wedge$4). The law ($\wedge$6) is derived from ($\wedge$4) by the strong cut law, exchange law, and contraction law. The law ($\wedge$6) with $\alpha = xy$ and $\beta = \varepsilon$ implies ($\wedge$4).

Since the dual $\succcurlyeq$ of $\preccurlyeq$ is also a latticed relation and dual of the strong cut law is the strong cut law itself, the rest holds by the duality.

Theorem 3.13 Let $x^\lozenge$ be an unary global operation on A, and assume that $\preccurlyeq$ satisfies the strong cut law. Then the following four laws are equivalent:

($\lozenge$1) $xx^\lozenge \preccurlyeq \varepsilon$ (the lower negation law).

($\lozenge$2) $\alpha \preccurlyeq x\beta,\ \alpha \preccurlyeq x^\lozenge\beta \implies \alpha \preccurlyeq \beta.$

($\lozenge$3) $\alpha \preccurlyeq x\beta \implies x^\lozenge\alpha \preccurlyeq \beta.$

($\lozenge$4) $\alpha \preccurlyeq x^\lozenge\beta \implies x\alpha \preccurlyeq \beta.$

Also, the following four laws are equivalent:

($\lozenge$5) $\varepsilon \preccurlyeq xx^\lozenge$ (the upper negation law).

($\lozenge$6) $x\alpha \preccurlyeq \beta,\ x^\lozenge\alpha \preccurlyeq \beta \implies \alpha \preccurlyeq \beta.$

($\lozenge$7) $x\alpha \preccurlyeq \beta \implies \alpha \preccurlyeq x^\lozenge\beta.$

($\lozenge$8) $x^\lozenge\alpha \preccurlyeq \beta \implies \alpha \preccurlyeq x\beta.$

Proof The laws ($\lozenge$2), ($\lozenge$3), and ($\lozenge$4) are all derived from ($\lozenge$1) by the strong cut law, exchange law, and contraction law. We have $xx^\lozenge \preccurlyeq x$ and $xx^\lozenge \preccurlyeq x^\lozenge$ by the repetition law, weakening law, and exchange law, and so the law ($\lozenge$2) with $\alpha = xx^\lozenge$ and $\beta = \varepsilon$ implies ($\lozenge$1). Similarly, the law ($\lozenge$3) with $\alpha = x$ and $\beta = \varepsilon$ together with the exchange law implies ($\lozenge$1), and the law ($\lozenge$4) with $\alpha = x^\lozenge$ and $\beta = \varepsilon$ implies ($\lozenge$1). The rest holds by duality.

Theorem 3.14 Let $x \Rightarrow y$ be a binary global operation on A, and assume that $\preccurlyeq$ satisfies the strong cut law and the negation laws with respect to an unary operation $x^\lozenge$ on A. Then the following three conditions are equivalent:

($\Rightarrow$1) $x^\lozenge \preccurlyeq x \Rightarrow y,\ y \preccurlyeq x \Rightarrow y$ (the first and second implication laws united).

($\Rightarrow$2) $x\alpha \preccurlyeq y\beta \implies \alpha \preccurlyeq x \Rightarrow y, \beta.$

(⇒3) $x \Rightarrow y, \alpha \preccurlyeq \beta \implies \alpha \preccurlyeq x\beta,\ y\alpha \preccurlyeq \beta$.

Also, the following four laws are equivalent:

(⇒4) $x \Rightarrow y \preccurlyeq x^{\Diamond}y$ (the third implication law).

(⇒5) $\alpha \preccurlyeq x \Rightarrow y, \beta \implies x\alpha \preccurlyeq y\beta$.

(⇒6) $\alpha \preccurlyeq x\beta,\ y\alpha \preccurlyeq \beta \implies x \Rightarrow y, \alpha \preccurlyeq \beta$.

(⇒7) $x, x \Rightarrow y \preccurlyeq y$ (cut-implication law).

Proof Assume (⇒1) and $x\alpha \preccurlyeq y\beta$. Then applying the cut law to $x\alpha \preccurlyeq y\beta$ and $y \preccurlyeq x \Rightarrow y$, we have $x\alpha \preccurlyeq x \Rightarrow y, \beta$. Also, applying the weakening law and exchange law to $x^{\Diamond} \preccurlyeq x \Rightarrow y$, we have $x^{\Diamond}\alpha \preccurlyeq x \Rightarrow y, \beta$. Hence $\alpha \preccurlyeq x \Rightarrow y, \beta$ by the law (◊6). Thus (⇒1) implies (⇒2).

Assume (⇒1) and $x \Rightarrow y, \alpha \preccurlyeq \beta$. Then applying the cut law to $x \Rightarrow y, \alpha \preccurlyeq \beta$ and $x^{\Diamond} \preccurlyeq x \Rightarrow y$, we have $x^{\Diamond}\alpha \preccurlyeq \beta$, hence $\alpha \preccurlyeq x\beta$ by the law (◊8). Also, applying the cut law to $x \Rightarrow y, \alpha \preccurlyeq \beta$ and $y \preccurlyeq x \Rightarrow y$, we have $y\alpha \preccurlyeq \beta$. Thus (⇒1) implies (⇒3).

We have $xx^{\Diamond} \preccurlyeq y$ by the lower negation law and weakening law, and so (⇒2) with $\alpha = x^{\Diamond}$ and $\beta = \varepsilon$ implies $x^{\Diamond} \preccurlyeq x \Rightarrow y$. We have $xy \preccurlyeq y$ by the repetition law and weakening law, and so (⇒2) with $\alpha = y$ and $\beta = \varepsilon$ implies $y \preccurlyeq x \Rightarrow y$. Thus (⇒2) implies (⇒1).

We have $x \Rightarrow y \preccurlyeq x \Rightarrow y$ by the repetition law, and so (⇒3) with $\alpha = \varepsilon$ and $\beta = x \Rightarrow y$ implies $\varepsilon \preccurlyeq x, x \Rightarrow y$ and $y \preccurlyeq x \Rightarrow y$. Applying the law (◊3) to $\varepsilon \preccurlyeq x, x \Rightarrow y$, we have $x^{\Diamond} \preccurlyeq x \Rightarrow y$. Thus (⇒3) implies (⇒1).

The laws (◊4) and (◊7) show that (⇒4) and (⇒7) are equivalent.

Assume (⇒7) and $\alpha \preccurlyeq x \Rightarrow y, \beta$. Then applying the exchange law and strong cut law, we have $x\alpha \preccurlyeq y\beta$. Thus (⇒7) implies (⇒5).

Assume (⇒7) and $\alpha \preccurlyeq x\beta,\ y\alpha \preccurlyeq \beta$. Then applying the strong cut law, exchange law, and contraction law, we have $x \Rightarrow y, \alpha \preccurlyeq \beta$. Thus (⇒7) implies (⇒6).

We have $x \Rightarrow y \preccurlyeq x \Rightarrow y$ by the repetition law, and so (⇒5) with $\alpha = x \Rightarrow y$ and $\beta = \varepsilon$ implies (⇒7).

We have $x \preccurlyeq xy$ and $yx \preccurlyeq y$ by the repetition law, weakening law, and exchange law, and so (⇒6) with $\alpha = x$ and $\beta = y$ yields $x \Rightarrow y, x \preccurlyeq y$, hence $x, x \Rightarrow y \preccurlyeq y$ by the exchange law. Thus (⇒6) implies (⇒7).

Theorem 3.15 Assume that $\preccurlyeq$ satisfies the cut law, the strong negation law with respect to an unary global operation ◊ on A, and either of the strong junction laws with respect to binary operations $\wedge, \vee$ on A. Then $\preccurlyeq$ satisfies the strong cut law.

Proof Consider the case where $\preccurlyeq$ satisfies the strong conjunction law. Assume $\alpha \preccurlyeq x\gamma$ and $x\beta \preccurlyeq \delta$. Then $\alpha\beta \preccurlyeq x\delta\gamma$ and $x\alpha\beta \preccurlyeq \delta\gamma$ by the weakening law and exchange law, and applying the strong negation law to $x\alpha\beta \preccurlyeq \delta\gamma$, we have $\alpha\beta \preccurlyeq x^{\Diamond}\delta\gamma$, so $\alpha\beta \preccurlyeq x^{\Diamond} \wedge x, \delta\gamma$ by the strong conjunction law. Also,

$x^\Diamond \wedge x \preccurlyeq \varepsilon$ by the repetition law, strong negation law, and strong conjunction law. Therefore $\alpha\beta \preccurlyeq \delta\gamma$ by the cut law. Thus $\preccurlyeq$ satisfies the strong cut law.

Theorem 3.16 Assume that $\preccurlyeq$ satisfies the cut law and the strong implication law with respect to a binary operation $\Rightarrow$ on A. Then $\preccurlyeq$ satisfies the strong cut law.

Proof Assume $\alpha \preccurlyeq x\gamma$ and $x\beta \preccurlyeq \delta$. Then $\alpha\beta \preccurlyeq x\delta\gamma$ and $x\alpha\beta \preccurlyeq \delta\gamma$ by the weakening law and exchange law, and so $x \Rightarrow x, \alpha\beta \preccurlyeq \delta\gamma$ by the strong implication law. Also, $\varepsilon \preccurlyeq x \Rightarrow x$ by the repetition law and strong implication law. Therefore $\alpha\beta \preccurlyeq \delta\gamma$ by the cut law. Thus $\preccurlyeq$ satisfies the strong cut law.

Theorem 3.17 Let $x \wedge y$, $x \vee y$, $x^\Diamond$, $x \Rightarrow y$ be global operations on A. Then the following conditions are equivalent.

(1) The $\preccurlyeq$ is Boolean with respect to $\wedge, \vee, \Rightarrow, \Diamond$.

(2) The $\preccurlyeq$ is weakly Boolean with respect to $\wedge, \vee, \Rightarrow, \Diamond$ and satisfies the cut law.

Proof This is a consequence of Theorems 3.12, 3.13, 3.14, and 3.15 or 3.16.

Theorem 3.18 Assume that $\preccurlyeq$ satisfies the cut law. Then according as $\preccurlyeq$ satisfies the quasi-conjunction law or the quasi-disjunction law, $\preccurlyeq$ satisfies the following n-tuple quasi-conjunction law or the n-tuple quasi-disjunction law for each $n \geq 1$:

for each element $(x_1, \dots, x_n) \in A^n$, there exists an element $y \in A$ such that $y \preccurlyeq x_i$ $(i = 1, \dots, n)$, and $x_1 \cdots x_n \preccurlyeq y$. ($n$-tuple quasi-conjunction law)

for each element $(x_1, \dots, x_n) \in A^n$, there exists an element $y \in A$ such that $y \succcurlyeq x_i$ $(i = 1, \dots, n)$, and $x_1 \cdots x_n \succcurlyeq y$. ($n$-tuple quasi-disjunction law)

Proof Assume that the quasi-conjunction law holds. We will prove that the n-tuple quasi-conjunction law holds for $n = 1, 2, \dots$ by induction on n. If $n = 1$, the n-tuple quasi-conjunction law holds by the repetition law. Therefore assume $n \geq 2$ and $(x_1, \dots, x_n) \in A^n$. By the induction hypothesis, there exists an element $z \in A$ such that $z \preccurlyeq x_i$ $(i = 1, \dots, n-1)$ and $x_1 \cdots x_{n-1} \preccurlyeq z$. Also, by the quasi-conjunction law, there exists an element $y \in A$ such that $y \preccurlyeq z$, $y \preccurlyeq x_n$, and $zx_n \preccurlyeq y$. Applying the cut law to $y \preccurlyeq z$ and $z \preccurlyeq x_i$, we have $y \preccurlyeq x_i$ $(i = 1, \dots, n-1)$. Also, applying the cut law to $x_1 \cdots x_{n-1} \preccurlyeq z$ and $zx_n \preccurlyeq y$, we have $x_1 \cdots x_n \preccurlyeq y$. Thus the n-tuple quasi-conjunction law holds for $n = 1, 2, \dots$. The rest of the proof is omitted.

3.3 Restrictions of and extensions to latticed relations

Theorem 3.19 Let A be a set and $\preccurlyeq$ be a latticed relation on A^*. Let $\models$ be the restriction of $\preccurlyeq$ to $A^* \times A$. Then $\models$ is a **partially latticed relation** in the sense that it satisfies the following five laws:

$$x \models x \qquad \text{(repetition law)}$$
$$\alpha \models y \implies x\alpha \models y \qquad \text{(partial weakening law)}$$
$$xx\alpha \models y \implies x\alpha \models y \qquad \text{(partial contraction law)}$$
$$\alpha xy\beta \models z \implies \alpha yx\beta \models z \qquad \text{(partial exchange law)}$$
$$\alpha \models x,\ x\beta \models y \implies \alpha\beta \models y \qquad \text{(partial cut law)}$$

Furthermore, let $\sqsubseteq$ be the restriction of $\preccurlyeq$ to $A \times A$. Then $\sqsubseteq$ is a **preorder** in the sense that it satisfies the following laws:

$$x \sqsubseteq x, \qquad \text{(reflexivity law)}$$
$$x \sqsubseteq y,\ y \sqsubseteq z \implies x \sqsubseteq z. \qquad \text{(transitivity law)}$$

Proof Omitted.

Theorem 3.20 Let A be a set and $\models$ be a partially latticed relation between A^* and A. Define the relation $\preccurlyeq$ on A^* by

$$\alpha \preccurlyeq y_1 \cdots y_n$$
$$\iff \alpha \models z \text{ for every element } z \in A \text{ such that } y_i \models z\ (i = 1, \ldots, n).$$

Then $\preccurlyeq$ satisfies the inf-emptiness law and is the largest of the latticed relations on A^* which extend $\models$ (for this reason, we call $\preccurlyeq$ the **largest latticed extension** of $\models$).

Proof There is a sophisticated proof in good perspective. The proof given here is the contrary, but requires no preliminaries.

If $\alpha \models y$, then for any $z \in A$ such that $y \models z$, we have $\alpha \models z$ by the partial cut law, hence $\alpha \preccurlyeq y$. Conversely if $\alpha \preccurlyeq y$, then since $y \models y$ by the repetition law, we have $\alpha \models y$. Therefore, $\preccurlyeq$ is an extension of $\models$, and in particular satisfies the repetition law.

The definition of $\preccurlyeq$ implies that $\alpha \preccurlyeq \varepsilon$ iff $\alpha \models z$ for every element $z \in A$. Since $\preccurlyeq$ is an extension of $\models$, it follows that $\preccurlyeq$ satisfies the inf-emptiness law.

In order to prove the weakening law, assume $\alpha \preccurlyeq y_1 \cdots y_n$ and $x \in A$. Then for any $z \in A$ such that $y_i \models z$ $(i = 1, \ldots, n)$, we have $\alpha \models z$, so $x\alpha \models z$ by the partial weakening law. Therefore $x\alpha \preccurlyeq y_1 \cdots y_n$. Also, for any $z \in A$ such that $x \models z$ and $y_i \models z$ $(i = 1, \ldots, n)$, since $y_i \models z$ $(i = 1, \ldots, n)$, we have $\alpha \models z$. Therefore $\alpha \preccurlyeq xy_1 \cdots y_n$.

In order to prove the contraction law, assume $xx\alpha \preccurlyeq y_1 \cdots y_n$. Then for any $z \in A$ such that $y_i \models z$ $(i = 1, \ldots, n)$, we have $xx\alpha \models z$, so $x\alpha \models z$ by the partial contraction law. Therefore $x\alpha \preccurlyeq y_1 \cdots y_n$. If $\alpha \preccurlyeq xxy_1 \cdots y_n$, then since $\{x, x, y_1, \ldots, y_n\} = \{x, y_1, \ldots, y_n\}$, we have $\alpha \preccurlyeq xy_1 \cdots y_n$.

The exchange law for $\preccurlyeq$ is derived from the partial exchange law for $\vDash$ and the fact that the definition of $\preccurlyeq$ does not depend on the numbering of $y_1, \ldots, y_n$.

In order to prove the cut law, first assume $\alpha \preccurlyeq x$ and $x\beta \preccurlyeq y_1 \cdots y_n$. Then for any $z \in A$ such that $y_i \vDash z$ $(i = 1, \ldots, n)$, we have $x\beta \vDash z$, and $\alpha \vDash x$ because $\preccurlyeq$ extends $\vDash$, so $\alpha\beta \vDash z$ by the partial cut law. Therefore $\alpha\beta \preccurlyeq y_1 \cdots y_n$. Next assume $x \preccurlyeq x_1 \cdots x_m$ and $\delta \preccurlyeq xy_1 \cdots y_n$. Then, for any $z \in A$ such that $x_i \vDash z$ $(i = 1, \ldots, m)$, we have $x \vDash z$, so if furthermore $y_j \vDash z$ $(j = 1, \ldots, n)$, then $\delta \vDash z$. Therefore $\delta \preccurlyeq x_1 \cdots x_m y_1 \cdots y_n$.

We have proved that $\preccurlyeq$ is a latticed relation on A^* which extends $\vDash$.

Let R be an arbitrary latticed relation on A^* which extends $\vDash$. We have to show $R \subseteq \preccurlyeq$. If $\alpha \mathrel{R} \varepsilon$, then for any element $z \in A$, we have $\alpha \mathrel{R} z$ by the weakening law for R, and then $\alpha \vDash z$ because R is an extension of $\vDash$, and so $\alpha \preccurlyeq \varepsilon$ by the definition of $\preccurlyeq$. Therefore assume $\alpha \mathrel{R} y_1 \cdots y_n$ with $n \geq 1$. Then for any $z \in A$ such that $y_i \vDash z$ $(i = 1, \ldots, n)$, we have $y_i \mathrel{R} z$ $(i = 1, \ldots, n)$ because R is an extension of $\vDash$, hence $\alpha \mathrel{R} z$ by the cut law, exchange law, and contraction law for R, and so $\alpha \vDash z$ because R is an extension of $\vDash$. Therefore $\alpha \preccurlyeq y_1 \cdots y_n$. Thus we have proved $R \subseteq \preccurlyeq$. Therefore, $\preccurlyeq$ is the largest of the latticed relations on A^* which extend $\vDash$.

Theorem 3.21 Let A be a set and $\sqsubseteq$ be a preorder on A. For each element $(x_1 \cdots x_m, y_1 \cdots y_n) \in A^* \times A^*$, let us write $x_1 \cdots x_m \preccurlyeq y_1 \cdots y_n$ if it satisfies the following condition for all elements $(x, y) \in A^2$:

$$x \sqsubseteq x_i \ (i = 1, \ldots, m), \ y_j \sqsubseteq y \ (j = 1, \ldots, n) \implies x \sqsubseteq y.$$

Then the relation $\preccurlyeq$ on A^* thus defined satisfies the emptiness laws and is the largest of the latticed relations on A^* which extend $\sqsubseteq$ (for this reason, we call $\preccurlyeq$ the **largest latticed extension** of $\sqsubseteq$).

Proof Omitted, because it is similar to the proof of Theorem 3.20. There is also a sophisticated proof in good perspective, but it requires certain preliminaries.

Remark 3.3 The pullback $\leq_f$ of $\leq$ by f in Theorem 3.11 is equal to the restriction of the f-validity relation $\preccurlyeq_f$ to $A \times A$, and so is a preorder by Theorem 3.19. Thus, Theorem 3.11 shows that $\preccurlyeq_f$ is equal to the largest latticed extension of $\leq_f$.

Theorem 3.22 Let A be a set and $\preccurlyeq$ be a latticed relation on A^*. Assume that $\preccurlyeq$ satisfies the quasi-disjunction law and inf-emptiness law. Then $\preccurlyeq$ is equal to the largest latticed extension of the partially latticed relation $\vDash$ obtained by restricting $\preccurlyeq$ to $A^* \times A$.

Proof Let $\vDash^*$ denote the largest latticed extension of $\vDash$. Since $\preccurlyeq \subseteq \vDash^*$ and both $\preccurlyeq$ and $\vDash^*$ satisfy the inf-emptiness law, we only need to show that if $\alpha \vDash^* y_1 \cdots y_n$ with $n \geq 1$, then $\alpha \preccurlyeq y_1 \cdots y_n$. Since $\preccurlyeq$ satisfies the quasi-disjunction law, there exists an element $z \in A$ such that $y_j \preccurlyeq z$ $(i = 1, \ldots, n)$ and $z \preccurlyeq y_1 \cdots y_n$. Since $\preccurlyeq$ is an extension of $\vDash$, we have $y_j \vDash z$ $(i = 1, \ldots, n)$, and so $\alpha \vDash z$ by the definition of $\vDash^*$, hence $\alpha \preccurlyeq z$. Applying the cut law to $\alpha \preccurlyeq z$ and $z \preccurlyeq y_1 \cdots y_n$, we conclude that $\alpha \preccurlyeq y_1 \cdots y_n$ holds as desired.

Theorem 3.23 Let A be a set and $\preccurlyeq$ be a latticed relation on A^*. Assume that $\preccurlyeq$ satisfies the quasi-junction laws and emptiness laws. Then $\preccurlyeq$ is equal to the largest latticed extension of the preorder $\sqsubseteq$ on A obtained by restricting $\preccurlyeq$ to $A \times A$.

Proof Let $\sqsubseteq^*$ denote the largest latticed extension of $\sqsubseteq$. Since $\preccurlyeq \subseteq \sqsubseteq^*$ and both $\preccurlyeq$ and $\sqsubseteq^*$ satisfy the emptiness laws, we only need to show that if $x_1 \cdots x_m \sqsubseteq^* y_1 \cdots y_n$ with $m \geq 1$ and $n \geq 1$, then $x_1 \cdots x_m \preccurlyeq y_1 \cdots y_n$. Since $\preccurlyeq$ satisfies the quasi-junction laws, there exists an element $x \in A$ such that $x \preccurlyeq x_i$ $(i = 1, \ldots, m)$ and $x_1 \cdots x_m \preccurlyeq x$, and also there exists an element $y \in A$ such that $y_j \preccurlyeq y$ $(j = 1, \ldots, n)$ and $y \preccurlyeq y_1 \cdots y_n$. Since $\preccurlyeq$ is an extension of $\sqsubseteq$, we have $x \sqsubseteq x_i$ $(i = 1, \ldots, m)$ and $y_j \sqsubseteq y$ $(j = 1, \ldots, n)$, and so $x \sqsubseteq y$ by the definition of $\sqsubseteq^*$, hence $x \preccurlyeq y$. By repeated applications of the cut law to $x_1 \cdots x_m \preccurlyeq x$, $x \preccurlyeq y$, and $y \preccurlyeq y_1 \cdots y_n$, we conclude that $x_1 \cdots x_m \preccurlyeq y_1 \cdots y_n$ holds as desired.

Theorem 3.24 Let A be a set and $\preccurlyeq$ be a latticed relation on A^*. Assume that $\preccurlyeq$ is equal to the largest latticed extension of the partially latticed relation $\vDash$ obtained by restricting $\preccurlyeq$ to $A^* \times A$ and that $\preccurlyeq$ satisfies the following conditions with respect to binary global operations $x \vee y$ and $x \Rightarrow y$ on A:

$$\left.\begin{array}{l} x \preccurlyeq x \vee y, \\ y \preccurlyeq x \vee y, \end{array}\right\} \tag{3.4}$$

$$x \vee y, x \Rightarrow z, y \Rightarrow z \preccurlyeq z, \tag{3.5}$$

$$x\alpha \preccurlyeq y \implies \alpha \preccurlyeq x \Rightarrow y. \tag{deduction law}$$

Then $\preccurlyeq$ satisfies the strong disjunction law with respect to the operation $\vee$.

Proof Assume $x\alpha \preccurlyeq z$ and $y\alpha \preccurlyeq z$. Then, we have $\alpha \preccurlyeq x \Rightarrow z$ and $\alpha \preccurlyeq y \Rightarrow z$ by the deduction law. By repeated applications of the cut law, exchange law, and contraction law to these two relations and (3.5), we have $x \vee y, \alpha \preccurlyeq z$. Thus $\preccurlyeq$ satisfies the following law:

$$x\alpha \preccurlyeq z,\ y\alpha \preccurlyeq z \implies x \vee y, \alpha \preccurlyeq z. \tag{3.6}$$

This law with $\alpha = \varepsilon$ becomes

$$x \vDash z,\ y \vDash z \implies x \vee y \vDash z,$$

because $\vDash$ is the restriction of $\preccurlyeq$ to $A^* \times A$. Therefore $\preccurlyeq$ satisfies the law $x \vee y \preccurlyeq xy$, because $\preccurlyeq$ is the largest latticed extension of $\vDash$. This law and (3.4) constitute the disjunction law. Therefore, $\preccurlyeq$ satisfies the union

$$\alpha \preccurlyeq xy\beta \iff \alpha \preccurlyeq x \vee y, \beta. \tag{3.7}$$

of the laws ($\vee$2) and ($\vee$5) of Theorem 3.12. This contains one of the two laws which constitute the strong disjunction law. It remains to prove another:

$$x\alpha \preccurlyeq \beta,\ y\alpha \preccurlyeq \beta \implies x \vee y, \alpha \preccurlyeq \beta. \tag{3.8}$$

In proving this, we may assume $\beta \neq \varepsilon$, because $\preccurlyeq$ satisfies the inf-emptiness law by Theorem 3.20. Also, (3.7) implies that, when $\beta \neq \varepsilon$, (3.8) is equivalent to (3.6). Therefore $\preccurlyeq$ satisfies (3.8).

3.4 Latticed representations made of latticed relations

Theorem 3.25 Let A be a set and $\preccurlyeq$ be a latticed relation on A^*. Assume that $\preccurlyeq$ satisfies the quasi-junction laws and end laws. Then there exists a latticed representation f of A onto a lattice B such that $\preccurlyeq$ is equal to the f-validity relation $\preccurlyeq_f$.

Remark 3.4 Theorems 3.3 and 3.6 show that the converse of Theorem 3.25 is also true.

Proof Let $\sqsubseteq$ be the restriction of $\preccurlyeq$ to $A \times A$. Then $\sqsubseteq$ is a preorder by Theorem 3.19. Let $\equiv$ be the relation on A defined by

$$x \equiv y \iff x \sqsubseteq y \text{ and } x \sqsupseteq y.$$

Then $\equiv$ is an equivalence relation. Let f be the projection of A onto the quotient set $B = A/{\equiv}$ of A by $\equiv$. Then we can define the order $\leq$ on B by

$$fx \leq fy \iff x \sqsubseteq y.$$

Thus, $\sqsubseteq$ is equal to the pullback $\leq_f$ of $\leq$ by f, and the following holds:

$$fx \leq fy \iff x \preccurlyeq y.$$

Since $\preccurlyeq$ satisfies the end laws, there exist elements $\underline{x}, \overline{x} \in A$ such that $\underline{x} \preccurlyeq \varepsilon$ and $\varepsilon \preccurlyeq \overline{x}$. By the weakening law, every element $x \in A$ satisfies $\underline{x} \preccurlyeq x \preccurlyeq \overline{x}$, hence $f\underline{x} \leq fx \leq f\overline{x}$. Thus $f\underline{x}$ and $f\overline{x}$ are the smallest and the largest elements of B respectively.

Let x, y be arbitrary elements of A. Then by the quasi-conjunction law, there exists an element $z \in A$ such that $z \preccurlyeq x$, $z \preccurlyeq y$, and $xy \preccurlyeq z$, which satisfies $fz \leq fx$ and $fz \leq fy$. Conversely, if an element $z' \in A$ satisfies $fz' \leq fx$ and $fz' \leq fy$, then $z' \preccurlyeq x$ and $z' \preccurlyeq y$, and applying the cut law, exchange law, and contraction law to $z' \preccurlyeq x$, $z' \preccurlyeq y$, and $xy \preccurlyeq z$, we have $z' \preccurlyeq z$, hence

$fz' \leq fz$. This implies $fz = \inf\{fx, fy\}$. Similarly by the quasi-disjunction law, there exists an element $w \in A$ such that $w \not\succcurlyeq x$, $w \not\succcurlyeq y$, and $xy \not\succcurlyeq w$, which satisfies $fw = \sup\{fx, fy\}$.

Thus B is a lattice which has the smallest element and the largest element, and f is a latticed representation of A onto B, and so we can apply results in §3.1 to (A, B, f).

Since $\preccurlyeq$ satisfies the weakening law, end laws, and cut law, it follows that $\preccurlyeq$ satisfies the emptiness laws. Therefore by Theorem 3.23, $\preccurlyeq$ is equal to the largest latticed extension of $\sqsubseteq$. Also, since $fA = B$, (3.2) is obviously satisfied for all elements $\alpha \in A^*$, and so by Theorem 3.11, the f-validity relation $\preccurlyeq_f$ is equal to the largest latticed extension of $\leq_f$ (cf. Remark 3.3). Since $\sqsubseteq$ is equal to $\leq_f$, we conclude that $\preccurlyeq$ is equal to $\preccurlyeq_f$.

Theorem 3.26 Let A be a set and $\preccurlyeq$ be a latticed relation on A^*. Assume that $\preccurlyeq$ satisfies the end laws. Also, let $x \wedge y$ and $x \vee y$ be binary global operations on A, and assume that $\preccurlyeq$ satisfies the strong conjunction law and disjunction law or satisfies the conjunction law and strong disjunction law with respect to the operations $\wedge, \vee$. Then $\preccurlyeq$ satisfies the strong cut law.

Proof Theorem 3.12 shows that $\preccurlyeq$ satisfies the junction laws with respect to $\wedge, \vee$, and so satisfies the quasi-junction laws. Also, Theorem 3.25 shows that there exists a latticed representation f of A onto a lattice B such that $\preccurlyeq$ is equal to the f-validity relation $\preccurlyeq_f$. Since $fA = B$, Theorem 3.10 shows that B is distributive. Therefore, Theorem 3.2 shows that $\preccurlyeq$ satisfies the strong cut law.

3.5 Specific latticed representations

Theorem 3.27 Let A be a set and B be a subset of A. Define the relation $\preccurlyeq_B$ on A^* by

$$\alpha \preccurlyeq_B \beta \iff \alpha \nsubseteq B \text{ or } \beta \cap B \neq \emptyset. \tag{3.9}$$

Then $\preccurlyeq_B$ is a strongly latticed relation.

Proof Let 1_B be the characteristic mapping of B. Then 1_B is a latticed representation of A on the lattice $\mathbb{T} = \{0, 1\}$. The following holds:

$$\begin{aligned}
\alpha \preccurlyeq_B \beta &\iff \alpha \nsubseteq B \text{ or } \beta \cap B \neq \emptyset \\
&\iff \begin{cases} \text{there exists an element } x \in \alpha \text{ such that } 1_B x = 0 \\ \text{or there exists an element } y \in \beta \text{ such that } 1_B y = 1 \end{cases} \\
&\iff \inf 1_B \alpha = 0 \text{ or } \sup 1_B \beta = 1 \\
&\iff \inf 1_B \alpha \leq \sup 1_B \beta.
\end{aligned}$$

Therefore, $\preccurlyeq_B$ is equal to the 1_B-validity relation $\preccurlyeq_{1_B}$. Since $\mathbb{T}$ is distributive, $\preccurlyeq_B$ is a strongly latticed relation by Theorems 3.1 and 3.2.

Theorem 3.28 Let A be a set, φ be a closure operator on $(\mathcal{P}A, \subseteq)$, and define the relation $\preccurlyeq_\varphi$ on A^* by

$$\alpha \preccurlyeq_\varphi \beta \iff \varphi\alpha \supseteq \bigcap_{y \in \beta} \varphi\{y\},$$

where if $\beta = \varepsilon$ then $\bigcap_{y\in\beta} \varphi\{y\} = A$. Then $\preccurlyeq_\varphi$ is a latticed relation.

Proof Let $\mathcal{B}$ be the fixture domain of φ, and define the order $\leq$ on $\mathcal{B}$ and the mapping $f \in A \to \mathcal{B}$ by

$$X \leq Y \iff X \supseteq Y, \qquad\qquad fx = \varphi\{x\}.$$

Then by Theorem 2.10, $\mathcal{B}$ is a lattice with $\min \mathcal{B} = A$, $\max \mathcal{B} = \varphi\emptyset$, and f is a latticed representation of A on $\mathcal{B}$. Let $\alpha, \beta \in A^*$ and define

$$\mathcal{X} = \big\{\varphi\{x\} \mid x \in \alpha\big\}, \qquad \mathcal{Y} = \big\{\{x\} \mid x \in \alpha\big\}, \qquad \mathcal{Z} = \big\{\varphi\{y\} \mid y \in \beta\big\}.$$

Then $\mathcal{X}, \mathcal{Z} \subseteq \mathcal{B}$, $\mathcal{Y} \subseteq \mathcal{P}A$, and the following holds:

$$\begin{aligned} f\alpha &= \{fx \mid x \in \alpha\} = \big\{\varphi\{x\} \mid x \in \alpha\big\} = \mathcal{X} = \{\varphi\eta \mid \eta \in \mathcal{Y}\} = \varphi\mathcal{Y}, \\ f\beta &= \{fy \mid y \in \beta\} = \big\{\varphi\{y\} \mid y \in \beta\big\} = \mathcal{Z}. \end{aligned}$$

Therefore the following holds by Theorem 2.10:

$$\begin{aligned} &\inf_{\mathcal{B}} f\alpha = \varphi\left(\sup_{\mathcal{P}A} \mathcal{X}\right) = \varphi\left(\sup_{\mathcal{P}A} \varphi\mathcal{Y}\right) = \varphi\left(\sup_{\mathcal{P}A} \mathcal{Y}\right) = \varphi\left(\bigcup_{x\in\alpha}\{x\}\right) = \varphi\alpha, \\ &\sup_{\mathcal{B}} f\beta = \inf_{\mathcal{P}A} \mathcal{Z} = \bigcap_{y\in\beta} \varphi\{y\}. \end{aligned}$$

Therefore, $\preccurlyeq_\varphi$ is equal to the f-validity relation $\preccurlyeq_f$ and so is a latticed relation by Theorems 3.1 and 3.2.

Theorem 3.29 Let A be a set and B be a subset of A. Define the relation $\vDash_B$ between A^* and A by

$$\alpha \vDash_B y \iff \alpha \nsubseteq B \text{ or } y \in B. \tag{3.10}$$

Then $\vDash_B$ is a partially latticed relation.

Proof This is a consequence of Theorems 3.27 and 3.19, because $\vDash_B$ is the restriction of the relation $\preccurlyeq_B$ defined in Theorem 3.27.

Theorem 3.30 Le A be a set, φ be a closure operator on $(\mathcal{P}A, \subseteq)$, and define the relation $\vDash_\varphi$ between A^* and A by

$$\alpha \vDash_\varphi y \iff \varphi\alpha \ni y.$$

Then $\vDash_\varphi$ is a partially latticed relation.

Proof Since φ is a closure operator, it follows that $\varphi\alpha \ni y$ iff $\varphi\alpha \supseteq \varphi\{y\}$. Therefore, $\vDash_\varphi$ is the restriction of the relation $\preccurlyeq_\varphi$ defined in Theorem 3.28, and the result follows from Theorems 3.28 and 3.19.

Theorem 3.31 Let A be a set and φ be a closure operator on $(\mathcal{P}A, \subseteq)$. Then the relation $\preccurlyeq_\varphi$ defined in Theorem 3.28 is the largest latticed extension of the relation $\vDash_\varphi$ defined in Theorem 3.30.

Proof As follows:

$$\begin{aligned} &\alpha \preccurlyeq_\varphi y_1 \cdots y_n \\ &\iff \varphi\alpha \supseteq \varphi\{y_1\} \cap \cdots \cap \varphi\{y_n\} \\ &\iff \varphi\alpha \ni z \text{ for every element } z \in A \text{ such that } \varphi\{y_i\} \ni z \ (i = 1, \ldots, n) \\ &\iff \alpha \vDash_\varphi z \text{ for every element } z \in A \text{ such that } y_i \vDash_\varphi z \ (i = 1, \ldots, n). \end{aligned}$$

3.6 Closure of the laws on relations

Definition 3.4 Let S and T be sets, and let L be a law on the relations between S and T. Then L is said to be **$\cap$-closed** if the set of the relations between S and T which satisfy L is $\cap$-closed, when regarded as a subset of $\mathcal{P}(S \times T)$.

Theorem 3.32 All the laws which define the latticed relations, partially latticed relations, Boolean relations, and weakly Boolean relations are $\cap$-closed.

The proof is easy and omitted (cf. §5.2).

4 Theories for logics

Let A be a set. Then a **logic** or a **relational logic** on A is a relation R between A^* and A. For the logic R, if a subset B of A satisfies the condition

$$\alpha \subseteq B,\ y \in A,\ \alpha\, R\, y \implies y \in B, \tag{4.1}$$

which is an abbreviation by sequence convention for the condition

$$\{x_1, \ldots, x_n\} \subseteq B,\ y \in A,\ x_1 \cdots x_n\, R\, y \implies y \in B,$$

then we call B an **R-theory** or say that B is **closed** under R or that R **closes** B. Obviously, A itself is an R-theory. Notice that (4.1) implies the following:

$$y \in A,\ \varepsilon\, R\, y \implies y \in B.$$

Therefore, defining the **R-core** A_R in A by

$$A_R = \{y \in A \mid \varepsilon\, R\, y\},$$

we have that every R-theory contains the R-core. Consequently, $\emptyset$ is an R-theory, iff $A_R = \emptyset$.

Example 4.1 Let $(A, (\alpha_\lambda)_{\lambda\in\Lambda})$ be an algebra. Define a logic R on A by

$$x_1 \cdots x_n \, R \, y \iff y = \alpha_\lambda(x_1, \ldots, x_n) \text{ for some } \lambda \in \Lambda.$$

Then $A_R = \emptyset$, and R-theories in A are nothing but support subalgebras of A. It is known that conversely, if R is a logic on a set A and satisfies $A_R = \emptyset$, then there exists a family $(\alpha_\lambda)_{\lambda\in\Lambda}$ of operations on A from which R is obtained as above.

Theorem 4.1 Let R be a logic on a set A. Then the set of all R-theories in A is $\cap$-closed in $\mathcal{P}A$.

Proof Let $(B_i)_{i\in I}$ be a family of R-theories and define $B = \bigcap_{i\in I} B_i$. If $I = \emptyset$, then $B = A$, which is an R-theory. Assume $I \neq \emptyset$. If $\alpha \subseteq B$, $y \in A$, and $\alpha \, R \, y$, then for all $i \in I$, since $\alpha \subseteq B_i$ and B_i is an R-theory, we have $y \in B_i$, hence $y \in B$. Therefore, B is an R-theory.

Definition 4.1 Let X be a subset of a set A and R be a logic on A. Then by Theorem 4.1, there exists the smallest of the R-theories in A which contain X, which we denote by $[X]_R$ and call the **R-closure** of X.

Theorem 4.2 Let Q, R be logics on a set A, and assume $R \subseteq Q$. Then every Q-theory in A is an R-theory. Consequently, $[X]_R \subseteq [X]_Q$ for every subset X of A.

Proof Let B be a Q-theory. If $\alpha \subseteq B$, $y \in A$, and $\alpha \, R \, y$, then $\alpha \, Q \, y$, and so $y \in B$. Therefore B is an R-theory. Since $[X]_Q$ is an R-theory and $X \subseteq [X]_Q$, Definition 4.1 implies that $[X]_R \subseteq [X]_Q$ holds.

Theorem 4.3 Let R be a logic on a set A and D be a subset of A. Then the mapping $X \mapsto [X \cup D]_R$ is a finitary closure operator on $(\mathcal{P}A, \subseteq)$, and its fixture domain is equal to the set of the R-theories which contain D. Consequently, the set of the R-theories containing D is $\cap$-closed in $\mathcal{P}A$ and quasi-finitary.

Proof The latter assertion is derived from the former by Theorems 2.10 (cf. Theorem 4.1) and 2.3. As for the former, we may assume $D = \emptyset$ by virtue of Theorem 2.11. By Definition 4.1 and Theorem 2.4, the mapping $X \mapsto [X]_R$ is a closure operator and its fixture domain is equal to the set of the R-theories. Thus, defining $(X)_R = \bigcup_{Y\in\mathcal{P}'X} [Y]_R$, we only need to show $[X]_R \subseteq (X)_R$. First $X \subseteq (X)_R$, because $\{x\} \subseteq \big[\{x\}\big]_R \subseteq (X)_R$ for each $x \in X$. Next suppose $x_1, \ldots, x_n \in (X)_R$, $y \in A$, and $x_1 \cdots x_n \, R \, y$. Then there exist sets $Y_1, \ldots, Y_n \in \mathcal{P}'X$ such that $x_i \in [Y_i]_R$ $(i = 1, \ldots, n)$. Define $Y = \bigcup_{i=1}^n Y_i$. Then $Y \in \mathcal{P}'X$ and $x_i \in [Y_i]_R \subseteq [Y]_R$ $(i = 1, \ldots, n)$, so $y \in [Y]_R \subseteq (X)_R$. This argument works even if $n = 0$. Thus $(X)_R$ is closed under R, and therefore $[X]_R \subseteq (X)_R$ as desired.

Theorem 4.4 Let A be a set and R be a partially latticed logic on A. Then the following holds for each subset X of A and for each element $\alpha \in A^*$:

$$[X]_R = \{y \in A \mid \text{there exists an element } \alpha \in A^* \text{ such that } \alpha \subseteq X \text{ and } \alpha\, R\, y\},$$
$$[\alpha]_R = \{y \in A \mid \alpha\, R\, y\}.$$

Consequently $A_R = [\emptyset]_R$.

Proof Let Y be the right-hand side of the first equation. Then $X \subseteq Y$ by the repetition law, and $Y \subseteq [X]_R$ because $X \subseteq [X]_R$ and $[X]_R$ is closed under R. Thus we only need to show that Y is closed under R. Assume $y_1, \ldots, y_n \in Y$, $z \in A$, and

$$y_1 \cdots y_n\, R\, z.$$

If $n = 0$, then $z \in Y$ by the definition of Y. Assume $n \geq 1$. Then, for each $i \in \{1, \ldots, n\}$, there exists an element $\alpha_i \in A^*$ such that $\alpha_i \subseteq X$ and

$$\alpha_i\, R\, y_i.$$

By repeated applications of the partial cut law and partial exchange law to the above $n + 1$ displayed R-relations, we get $\alpha_1 \cdots \alpha_n\, R\, z$, hence $z \in Y$ as desired. We have proved the first equation. The second one is derived from the first by the partial weakening law, partial contraction law, and partial exchange law.

Theorem 4.5 Let A be a set and Q, R be logics on A. Assume that Q is a partially latticed relation. Then the following three conditions are equivalent:

(1) $R \subseteq Q$.

(2) $[X]_R \subseteq [X]_Q$ for every subset X of A.

(3) Every Q-theory in A is an R-theory.

Proof Theorem 4.2 says that (1) implies (3). If (3) holds, then since $[X]_Q$ is an R-theory containing X, we have $[X]_R \subseteq [X]_Q$, and so (2) holds. Suppose (2) holds. If $(\alpha, y) \in A^* \times A$ satisfies $\alpha\, R\, y$, then $y \in [\alpha]_R$, so $y \in [\alpha]_Q$ by (2), hence $\alpha\, Q\, y$ by Theorem 4.4. Therefore (1) holds.

Remark 4.1 Since both the mapping $X \mapsto [X]_R$ and $X \mapsto [X]_Q$ are finitary by Theorems 4.3, it follows from Theorem 2.3 that the conditions (1) - (3) of Theorem 4.5 is equivalent to the condition obtained from (2) by replacing "subsets" by "finite subsets." Similar remarks apply to Theorems 5.6, 5.7, and 7.1.

Theorem 4.6 Let R be a logic on a set A and X be a subset of A. Then $[X]_R$ is the union of the **R-descendants** X_n ($n = 0, 1, \ldots$) of X, where $X_0 = X$ and X_n ($n \geq 1$) is the inductively defined set of the elements $y \in A$ such that $x_1 \cdots x_m\, R\, y$ for some elements $x_i \in X_{l_i}$ ($i = 1, \ldots, m$) with $n = 1 + \sum_{i=1}^{m} l_i$.

Remark 4.2 If $m = 0$ in the definition of X_n ($n \geq 1$), then $x_1 \cdots x_m \, R \, y$ means $\varepsilon \, R \, y$ and $n = 1 + \sum_{i=1}^{m} l_i$ means $n = 1$. Therefore, the R-core A_R is contained in the first R-descendant X_1.

Proof Let $B = \bigcup_{n \geq 0} X_n$. First, we will show that every element $y \in X_n$ ($n = 0, 1, \ldots$) belongs to $[X]_R$ by induction on n. This holds when $n = 0$ because $X_0 = X \subseteq [X]_R$. Assume $n \geq 1$. Since $A_R \subseteq [X]_R$, we may assume $y \notin A_R$. Then $x_1 \cdots x_m \, R \, y$ ($m \geq 1$) for some elements $x_i \in X_{l_i}$ ($i = 1, \ldots, m$) with $n = 1 + \sum_{i=1}^{m} l_i$, and so $x_1, \ldots, x_m \in [X]_R$ by the induction hypothesis. Therefore $y \in [X]_R$. We have shown $B \subseteq [X]_R$. Suppose $x_1, \ldots, x_m \in B$, $y \in A$, and $x_1 \cdots x_m \, R \, y$. If $m = 0$, then $y \in A_R$, and so $y \in X_1$ by Remark 4.2. If $m \geq 1$, then for $i = 1, \ldots, m$, there exists a non-negative integer l_i such that $x_i \in X_{l_i}$, and defining $n = 1 + \sum_{i=1}^{m} l_i$, we have $y \in X_n$. Therefore $y \in B$ in either case. Thus B is closed under R. Since $X \subseteq B \subseteq [X]_R$, we conclude that $[X]_R = B$ holds.

Remark 4.3 In addition to Theorem 4.6, it is known that an element $y \in A$ belongs to $[X]_R$ iff there exists elements $x_1, \ldots, x_n \in A$ such that $x_n = y$ and, for each $i \in \{1, \ldots, n\}$, either $x_i \in X$ or there exist numbers $j_1, \ldots, j_k \in \{1, \ldots, i-1\}$ satisfying $x_{j_1} \cdots x_{j_k} \, R \, x_i$.

5 Deduction pairs

Let A be a set. Then a **deduction pair** on A is a pair (R, D) of a logic R on A and a subset D of A. We call R and D the **rule** and **basis** of the deduction pair.

5.1 Deduction relations

Let A be a set and (R, D) be a deduction pair on A. Then we define the logic R^D on A by

$$\alpha \, R^D \, y \iff [\alpha \cup D]_R \ni y \tag{5.1}$$

for each $(\alpha, y) \in A^* \times A$. We call R^D the **D-closure** of R. We will denote R^D also by $\vDash_{R,D}$ and call it the **partial deduction relation** determined by (R, D) in order to relate R^D to the **deduction relation** $\preccurlyeq_{R,D}$ on A^* defined by

$$\alpha \preccurlyeq_{R,D} \beta \iff [\alpha \cup D]_R \supseteq \bigcap_{y \in \beta} \big[\{y\} \cup D\big]_R \tag{5.2}$$

for each $(\alpha, \beta) \in A^* \times A^*$. Notice that $\vDash_{R,D}$ is the restriction of $\preccurlyeq_{R,D}$ to $A^* \times A$ and that $\alpha \preccurlyeq \varepsilon$ holds iff $[\alpha \cup D]_R = A$.

Theorem 5.1 Let (R, D) be a deduction pair on a set A. Then the following holds for the D-closure R^D of R and the R^D-core A_{R^D}.

(1) $R \subseteq R^D$.

(2) $A_{R^D} = [D]_R$, hence in particular $D \subseteq A_{R^D}$.

(3) If a deduction pair (Q, C) on A satisfies $R \subseteq Q$ and $D \subseteq C$, then $R^D \subseteq Q^C$.

Proof (1) If $(\alpha, y) \in A^* \times A$ satisfies $\alpha \, R \, y$, then $y \in [\alpha]_R$, and since $[\alpha]_R \subseteq [\alpha \cup D]_R$ by Theorems 4.3, we have $\alpha \, R^D \, y$. Thus (1) holds.

(2) is a consequence of (5.1) with $\alpha = \varepsilon$.

(3) If $(\alpha, y) \in A^* \times A$ satisfies $\alpha \, R^D \, y$, then $y \in [\alpha \cup D]_R$, and since $[\alpha \cup D]_R \subseteq [\alpha \cup C]_Q$ by Theorems 4.2 and 4.3, we have $\alpha \, Q^C \, y$. Thus (3) holds.

Theorem 5.2 Let (R, D) be a deduction pair on a set A. Then R^D is a partially latticed relation, and $\preccurlyeq_{R,D}$ is the largest latticed extension of R^D.

Proof Define $\varphi \in \mathcal{P}A \to \mathcal{P}A$ by $\varphi X = [X \cup D]_R$. Then φ is a closure operator by Theorems 4.3, and R^D is equal to the relation $\vDash_\varphi$ defined in Theorem 3.30, and $\preccurlyeq_{R,D}$ is equal to the relation $\preccurlyeq_\varphi$ defined in Theorem 3.28. Therefore the theorem is a consequence of Theorems 3.30 and 3.31.

Theorem 5.3 Let (R, D) be a deduction pair on a set A. Then the following holds.

(1) $[X]_{R^D} = [X \cup D]_R$ for every subset X of A.

(2) The set of the R^D-theories in A is equal to that of the R-theories which contain D.

Consequently, if R is a logic on A, then $[X]_{R^\emptyset} = [X]_R$ for every subset X of A, and the set of the $R^\emptyset$-theories in A is equal to that of the R-theories.

Proof Since R^D is a partially latticed relation by Theorem 5.2, if follows from Theorem 4.4 that an element $y \in A$ belongs to $[X]_{R^D}$ iff there exists an element $Y \in \mathcal{P}'X$ such that $y \in [Y \cup D]_R$. Since the mapping $X \mapsto [X \cup D]_R$ is finitary by Theorem 4.3, we conclude that $y \in [X]_{R^D}$ iff $y \in [X \cup D]_R$. Thus (1) holds. Theorem 4.3 also shows that the fixture domains of the mappings $X \mapsto [X]_{R^D}$ and $X \mapsto [X \cup D]_R$ are equal to the set of the R^D-theories and that of the R-theories which contain D. Therefore (2) is a consequence of (1).

Theorem 5.4 Let (R, D) be a deduction pair on a set A. Then in order that $R^D = R$ holds, either of the following conditions is necessary and sufficient.

(1) R is a partially latticed relation and every R-theory in A contains D.

(2) R is a partially latticed relation and $D \subseteq A_R$.

Consequently, a logic R on A satisfies $R^\emptyset = R$ iff R is a partially latticed relation.

Proof Since R^{D} is a partially latticed relation by Theorem 5.2, $\mathsf{R}^{\mathsf{D}} = \mathsf{R}$ holds only if R is a partially latticed relation. Therefore assume that R is a partially latticed relation. Then by Theorems 4.5 and 5.3, $\mathsf{R}^{\mathsf{D}} = \mathsf{R}$ holds iff every R-theory contains D. Since $[\emptyset]_{\mathsf{R}} = \mathsf{A}_{\mathsf{R}}$ by Theorem 4.4, every R-theory contains D iff $\mathsf{D} \subseteq \mathsf{A}_{\mathsf{R}}$.

Corollary 5.4.1 Let A be a set and D be its subset. Regard logics on A as subsets of $\mathsf{A}^* \times \mathsf{A}$. Then the mapping $\mathsf{R} \mapsto \mathsf{R}^{\mathsf{D}}$ which maps each logic R on A to its D-closure R^{D} is a closure operator on $(\mathcal{P}(\mathsf{A}^* \times \mathsf{A}), \subseteq)$.

Proof By virtue of Theorem 5.1, we only need to show $(\mathsf{R}^{\mathsf{D}})^{\mathsf{D}} = \mathsf{R}^{\mathsf{D}}$. Let $\mathsf{Q} = \mathsf{R}^{\mathsf{D}}$. Then Q is a partially latticed relation by Theorem 5.2 and satisfies $\mathsf{D} \subseteq \mathsf{A}_{\mathsf{Q}}$ by Theorem 5.1, so $\mathsf{Q}^{\mathsf{D}} = \mathsf{Q}$ by Theorem 5.4, hence $(\mathsf{R}^{\mathsf{D}})^{\mathsf{D}} = \mathsf{R}^{\mathsf{D}}$.

Incidentally, it is known that more generally $(\mathsf{R}^{\mathsf{D}})^{\mathsf{C}} = \mathsf{R}^{\mathsf{D} \cup \mathsf{C}}$ holds.

Theorem 5.5 Let (R, D) be a deduction pair on a set A. Then R^{D} is the smallest of the logics Q on A which satisfy the following conditions.

(1) Q is a partially latticed relation.

(2) $\mathsf{R} \subseteq \mathsf{Q}$ and $\mathsf{D} \subseteq \mathsf{A}_{\mathsf{Q}}$.

Proof If $\mathsf{Q} = \mathsf{R}^{\mathsf{D}}$, then Theorems 5.1 and 5.2 shows that Q satisfies the above conditions. Conversely if Q satisfies the above conditions, then $\mathsf{R}^{\mathsf{D}} \subseteq \mathsf{Q}^{\mathsf{D}} = \mathsf{Q}$ by Theorems 5.1 and 5.4 (cf. Corollary 5.4.1).

Theorem 5.6 Let (Q, C) and (R, D) be deduction pairs on a set A. Then the following four conditions are equivalent.

(1) $\mathsf{R}^{\mathsf{D}} \subseteq \mathsf{Q}^{\mathsf{C}}$.

(2) $[\mathsf{X} \cup \mathsf{D}]_{\mathsf{R}} \subseteq [\mathsf{X} \cup \mathsf{C}]_{\mathsf{Q}}$ for every subset X of A.

(3) Every Q-theory in A containing C is an R-theory containing D.

(4) $\mathsf{R} \subseteq \mathsf{Q}^{\mathsf{C}}$ and $\mathsf{D} \subseteq \mathsf{A}_{\mathsf{Q}^{\mathsf{C}}}$.

Proof Since Q^{C} is a partially latticed relation by Theorem 5.2, Theorems 4.5 and 5.3 show that (1) - (3) are equivalent, and Theorem 5.5 shows that (4) implies (1). Conversely under (1), we have $\mathsf{R} \subseteq \mathsf{R}^{\mathsf{D}} \subseteq \mathsf{Q}^{\mathsf{C}}$ and $\mathsf{D} \subseteq \mathsf{A}_{\mathsf{R}^{\mathsf{D}}} \subseteq \mathsf{A}_{\mathsf{Q}^{\mathsf{C}}}$ by Theorem 5.1, and thus (4) holds.

Theorem 5.7 Let (R, D) be a deduction pair on A and Q be a partially latticed logic on A. Then the following four conditions are equivalent.

(1) $\mathsf{R}^{\mathsf{D}} \subseteq \mathsf{Q}$.

(2) $[\mathsf{X} \cup \mathsf{D}]_{\mathsf{R}} \subseteq [\mathsf{X}]_{\mathsf{Q}}$ for every subset X of A.

(3) Every Q-theory in A is an R-theory containing D.

(4) $R \subseteq Q$ and $D \subseteq A_Q$.

Also, the following three conditions are equivalent.

(5) $Q \subseteq R^D$.

(6) $[X]_Q \subseteq [X \cup D]_R$ for every subset X of A.

(7) Every R-theory in A containing D is a Q-theory.

Therefore, the following four conditions are equivalent.

(8) $Q = R^D$.

(9) $[X]_Q = [X \cup D]_R$ for every subset X of A.

(10) The set of the Q-theories in A is equal to the set of the R-theories in A containing D.

(11) $R \subseteq Q$, $D \subseteq A_Q$, and $Q \subseteq R^D$.

Proof Theorem 5.4 shows $Q^\emptyset = Q$. Therefore, the conditions (1) - (4) of Theorem 5.6 with $C = \emptyset$ are equivalent to the conditions (1) - (4) of Theorem 5.7. Also, the conditions (1) - (3) of Theorem 5.6 with (Q, C) and (R, D) interchanged and then with C replaced by $\emptyset$ are equivalent to the conditions (5) - (7) of Theorem 5.7.

5.2 Generational laws for relations

Let A, B be sets and (R', D') be a deduction pair on the direct product $A' = A \times B$. Then (R', D') is also called a **generational law** on the relations between A and B, and if a relation R between A and B regarded as a subset of A' is closed under R' and contains D', we say that R satisfy the generational law (R', D').

As an immediate consequence of this definition, we have that the R'-closure $[D']_{R'}$ of D' regarded as a relation between A and B is the smallest of the relations between A and B which satisfy the generational law (R', D'). Also, generational laws for relations are $\cap$-closed in the sense of Definition 3.4 (cf. Theorem 3.32).

Example 5.1 The equivalence law may be regarded as a generational law. Let A be a set and define $A' = A \times A$. Let R' be the union of the logics S' and T' on A', each defined by the fractional expression:

$$S' = \frac{(x, y)}{(y, x)} \qquad (x, y \in A),$$

$$T' = \frac{(x, y)\ (y, z)}{(x, z)} \qquad (x, y, z \in A).$$

Define the subset D' of A' by

$$D' = \{(x, x) \mid x \in A\}.$$

Let R be a relation on A and regard it as a subset of A'. Then R is reflexive iff R contains D', R is symmetric iff R is closed under S', and R is transitive iff R is closed under T'. Therefore, R is an equivalence relation iff R contains D' and is closed under R'. Thus, a relation on A is an equivalence relation iff it satisfies the generational law (R', D').

Example 5.2 The Boolean law is also regarded as a generational law. Let A be a set and define

$$\vec{A} = A^* \times A^*.$$

Denote the elements $(\alpha, \beta) \in \vec{A}$ by $\alpha \to \beta$ or $\beta \leftarrow \alpha$, and call them **sequents**. Let $\vec{R}$ be the union of the following sixteen logics on $\vec{A}$, each defined by the fractional expression:

$$\frac{\alpha \to \beta}{x\alpha \to \beta}, \qquad \frac{\alpha \leftarrow \beta}{x\alpha \leftarrow \beta}, \qquad \text{(weakening)}$$

$$\frac{xx\alpha \to \beta}{x\alpha \to \beta}, \qquad \frac{xx\alpha \leftarrow \beta}{x\alpha \leftarrow \beta}, \qquad \text{(contraction)}$$

$$\frac{\alpha xy\beta \to \gamma}{\alpha yx\beta \to \gamma}, \qquad \frac{\alpha xy\beta \leftarrow \gamma}{\alpha yx\beta \leftarrow \gamma}, \qquad \text{(exchange)}$$

$$\frac{\alpha \to x \quad x\beta \to \gamma}{\alpha\beta \to \gamma}, \qquad \frac{\alpha \leftarrow x \quad x\beta \leftarrow \gamma}{\alpha\beta \leftarrow \gamma}. \qquad \text{(cut)}$$

Let $\vec{D}$ be the set of the sequents on the following list, where $\wedge, \vee, \Diamond, \Rightarrow$ are global operations on A:

$$
\begin{array}{llll}
x \to x, & & & \text{(repetition seq.)} \\
x \wedge y \to x, & x \wedge y \to y, & xy \to x \wedge y, & \text{(conjunction seq.)} \\
x \vee y \leftarrow x, & x \vee y \leftarrow y, & xy \leftarrow x \vee y, & \text{(disjunction seq.)} \\
xx^{\Diamond} \to \varepsilon, & xx^{\Diamond} \leftarrow \varepsilon, & & \text{(negation seq.)} \\
x^{\Diamond} \to x \Rightarrow y, & y \to x \Rightarrow y, & x \Rightarrow y \to x^{\Diamond}y. & \text{(implication seq.)}
\end{array}
$$

Let R be a relation on A^* and regard R as a subset of $\vec{A}$. Then R contains $\vec{D}$ and is closed under $\vec{R}$ iff R satisfies the repetition law, weakening law, contraction law, exchange law, cut law, junction laws, negation laws, and implication laws with respect to the operations $\wedge, \vee, \Diamond, \Rightarrow$. Therefore, R satisfies the generational law $(\vec{R}, \vec{D})$ iff R is a Boolean relation with respect to $\wedge, \vee, \Diamond, \Rightarrow$.

6 Logical spaces

A **logical space** is a pair $(\mathsf{A}, \mathcal{B})$ of a non-empty set A and a subset $\mathcal{B}$ of $\mathcal{P}\mathsf{A}$, which we call the set of the **given theories** of the logical space. A logic R on A is said to be $\mathcal{B}$**-sound** or called a $\mathcal{B}$**-logic**, if every element of $\mathcal{B}$ is closed under R. A subset X of A is called a $\mathcal{B}$**-theory**, if X is closed under every $\mathcal{B}$-logic on A. We call $\bigcap_{\mathsf{B}\in\mathcal{B}} \mathsf{B}$ the $\mathcal{B}$**-core**. Elements and subsets of A are said to be $\mathcal{B}$**-sound** if they are contained in the $\mathcal{B}$-core. A $\mathcal{B}$-sound element is also called a $\mathcal{B}$**-tautology**.

As immediate consequences of the above definitions, we first have that all given theories of a logical space $(\mathsf{A}, \mathcal{B})$ are $\mathcal{B}$-theories and hence that $\mathcal{B}$-logics are the only logics on A that close every $\mathcal{B}$-theory. Notice that there is a Galois correspondence in the background.

Theorem 6.1 Let $(\mathsf{A}, \mathcal{B})$ be a logical space. Then the following holds.

(1) There exists the largest $\mathcal{B}$-logic on A, which we denote by Q for the time being.

(2) A logic R on A is a $\mathcal{B}$-logic iff R is contained in Q.

Let X be a subset of A. Then the following holds.

(3) The X is a $\mathcal{B}$-theory iff X is a Q-theory.

(4) The Q-closure $[\mathsf{X}]_{\mathsf{Q}}$ of X is the smallest of the $\mathcal{B}$-theories which contain X.

Proof Let $\mathcal{R}$ be the set of the $\mathcal{B}$-logics on A, regard $\mathcal{R}$ as a subset of $\mathcal{P}(\mathsf{A}^* \times \mathsf{A})$, and define $\mathsf{Q} = \bigcup_{\mathsf{R}\in\mathcal{R}} \mathsf{R}$. If $\alpha \subseteq \mathsf{B} \in \mathcal{B}$, $y \in \mathsf{A}$, and $\alpha\,\mathsf{Q}\,y$, then $\alpha\,\mathsf{R}\,y$ for some $\mathsf{R} \in \mathcal{R}$, and so $y \in \mathsf{B}$. Thus Q is also a $\mathcal{B}$-logic on A and so is the largest one. Suppose $\mathsf{R} \subseteq \mathsf{Q}$. Then every element $\mathsf{B} \in \mathcal{B}$ is a Q-theory and so is an R-theory by Theorem 4.2. Therefore, R is also a $\mathcal{B}$-logic. Since Q is a $\mathcal{B}$-logic, $\mathcal{B}$-theories are Q-theories. Conversely if X is a Q-theory, then X is an R-theory for each $\mathsf{R} \in \mathcal{R}$ again by Theorem 4.2, and so is a $\mathcal{B}$-theory. Thus we have proved (1) - (3). Finally, (4) is a direct consequence of (3) and Definition 4.1.

Theorem 6.2 Let $(\mathsf{A}, \mathcal{B})$ be a logical space. Then the following holds for the largest $\mathcal{B}$-logic Q on A and each $(\alpha, y) \in \mathsf{A}^* \times \mathsf{A}$:

$$\alpha\,\mathsf{Q}\,y \iff y \in \bigcap_{\alpha\subseteq\mathsf{B}\in\mathcal{B}} \mathsf{B}$$

Proof Define the logic P on A by

$$\alpha\,\mathsf{P}\,y \iff y \in \bigcap_{\alpha\subseteq\mathsf{B}\in\mathcal{B}} \mathsf{B}.$$

If $\alpha \subseteq \mathsf{B} \in \mathcal{B}$, $y \in \mathsf{A}$, and $\alpha\,\mathsf{P}\,y$, then $y \in \mathsf{B}$ by the definition of P, which shows that P is a $\mathcal{B}$-logic, hence $\mathsf{P} \subseteq \mathsf{Q}$. If $\alpha\,\mathsf{Q}\,y$ and $\alpha \subseteq \mathsf{B} \in \mathcal{B}$, then since Q is a $\mathcal{B}$-logic, we have $y \in \mathsf{B}$. This shows $\mathsf{Q} \subseteq \mathsf{P}$. Therefore $\mathsf{Q} = \mathsf{P}$.

Theorem 6.3 Let $(A, \mathcal{B})$ be a logical space and Q be the largest $\mathcal{B}$-logic on A. Then Q is a partially latticed relation, and the following holds for each subset X of A and for each element $\alpha \in A^*$:

$$[X]_Q = \{y \in A \mid \text{there exists an element } \alpha \in A^* \text{ such that } \alpha \subseteq X \text{ and } \alpha\, Q\, y\},$$
$$[\alpha]_Q = \{y \in A \mid \alpha\, Q\, y\}.$$

Proof For each set $B \in \mathcal{B}$, define the logic $\models_B$ on A by (3.10). Then $Q = \bigcap_{B \in \mathcal{B}} \models_B$ by Theorem 6.2. Therefore the former assertion is a consequence of Theorems 3.29 and 3.32. The latter assertion is a consequence of the former and Theorem 4.4.

Alternate proof It follows from Theorem 5.3 and Theorem 5.1 that $Q^\emptyset$ is a $\mathcal{B}$-logic and $Q \subseteq Q^\emptyset$. Therefore $Q = Q^\emptyset$, and so Q is a partially latticed relation by Theorem 5.2, and the second equation holds. Since the mapping $X \mapsto [X]_Q$ is finitary by Theorem 4.3, the first equation follows from the second.

Theorem 6.4 Let $(A, \mathcal{B})$ be a logical space and Q be the largest $\mathcal{B}$-logic on A. Then the following holds for the $\mathcal{B}$-core C of A:

$$C = A_Q = [\emptyset]_Q.$$

Also, C is the smallest $\mathcal{B}$-theory in A.

Proof This is a consequence of Theorems 6.2, 6.3, and 6.1 (cf. Theorem 4.4).

Theorem 6.5 Let $(A, \mathcal{B}_i)$ be a logical space for $i = 1, 2$. Then the following three conditions are equivalent.

(1) The set of the $\mathcal{B}_1$-logics is equal to that of the $\mathcal{B}_2$-logics.

(2) The set of the $\mathcal{B}_1$-theories is equal to that of the $\mathcal{B}_2$-theories.

(3) The largest $\mathcal{B}_1$-logic is equal to the largest $\mathcal{B}_2$-logic.

Under these equivalent conditions, the $\mathcal{B}_1$-core is equal to the $\mathcal{B}_2$-core.

Proof As for the former assertion, obviously (1) implies (3). Also, Theorem 6.1 and the remark before it shows that (3) implies (2) and (2) implies (1). The latter assertion is a consequence of Theorem 6.4.

Definition 6.1 If logical spaces $(A, \mathcal{B}_1)$ and $(A, \mathcal{B}_2)$ satisfy the three equivalent conditions of Theorem 6.5, we say that $(A, \mathcal{B}_1)$ and $(A, \mathcal{B}_2)$ are **equivalent**.

Lemma 6.1 Let $(A, \mathcal{B})$ and $(A, \mathcal{B}')$ be logical spaces and assume $\mathcal{B} \subseteq \mathcal{B}' \subseteq \overline{\mathcal{B}^\cap}$, where $\overline{\mathcal{B}^\cap}$ is the quasi-finitary $\cap$-closure of $\mathcal{B}$ in $\mathcal{P}A$. Then $(A, \mathcal{B})$ is equivalent to $(A, \mathcal{B}')$.

Proof Since $\mathcal{B} \subseteq \mathcal{B}'$, every $\mathcal{B}'$-logic is a $\mathcal{B}$-logic. Let R be a $\mathcal{B}$-logic and $\mathcal{T}$ be the set of the R-theories. Then $\mathcal{B} \subseteq \mathcal{T}$, so $\mathcal{B}' \subseteq \overline{\mathcal{B}^\cap} \subseteq \overline{\mathcal{T}^\cap} = \mathcal{T}$ by Theorems 2.7 and 4.3. Thus R is a $\mathcal{B}'$-logic. Therefore the lemma holds by Definition 6.1.

The following theorem is implicit in a paper by Matsuda [6]

Theorem 6.6 Let $(\mathsf{A}, \mathcal{B})$ be a logical space. Then the set of the $\mathcal{B}$-theories is equal to the quasi-finitary $\cap$-closure $\overline{\mathcal{B}^\cap}$ of $\mathcal{B}$ in $\mathcal{P}\mathsf{A}$.

Proof Lemma 6.1 and Theorem 6.5 show that the set of the $\mathcal{B}$-theories is equal to the set of the $\overline{\mathcal{B}^\cap}$-theories. Therefore, we may assume $\mathcal{B} = \overline{\mathcal{B}^\cap}$, and need to show that every $\mathcal{B}$-theory X is super-covered by $\mathcal{B}$. Therefore, let $\mathsf{Y} \in \mathcal{P}'\mathsf{X}$. Define $\mathsf{B}' = \bigcap_{\mathsf{Y} \subseteq \mathsf{B} \in \mathcal{B}} \mathsf{B}$. Then since $\mathcal{B}$ is $\cap$-closed in $\mathcal{P}\mathsf{A}$, we have $\mathsf{Y} \subseteq \mathsf{B}' \in \mathcal{B}$. Let $\mathsf{Y} = \{y_1, \ldots, y_n\}$ and define $\alpha = y_1 \cdots y_n \in \mathsf{A}^*$. Then $\mathsf{B}' = \bigcap_{\alpha \subseteq \mathsf{B} \in \mathcal{B}} \mathsf{B}$, so by Theorem 6.2, every element $y \in \mathsf{B}'$ satisfies $\alpha \, \mathsf{Q} \, y$ for the largest $\mathcal{B}$-logic Q. Since $\alpha \subseteq \mathsf{X}$ and X is closed under Q, we have $\mathsf{B}' \subseteq \mathsf{X}$, hence $\mathsf{Y} \subseteq \mathsf{B}' \subseteq \mathsf{X}$ and $\mathsf{B}' \in \mathcal{B}$. Thus X is super-covered by $\mathcal{B}$, as desired.

Corollary 6.6.1 Two logical spaces $(\mathsf{A}, \mathcal{B})$ and $(\mathsf{A}, \mathcal{B}')$ are equivalent iff $\overline{\mathcal{B}^\cap} = \overline{\mathcal{B}'^\cap}$.

Corollary 6.6.2 Let $(\mathsf{A}, \mathcal{B})$ be a logical space and X be a $\mathcal{B}$-theory different from A. Then $\mathsf{X} \subseteq \bigcup_{\mathsf{B} \in \mathcal{B} - \{\mathsf{A}\}} \mathsf{B}$.

Proof This is because $\bigcup_{\mathsf{X} \in \overline{\mathcal{B}^\cap} - \{\mathsf{A}\}} \mathsf{X} = \bigcup_{\mathsf{B} \in \mathcal{B} - \{\mathsf{A}\}} \mathsf{B}$ by Theorems 2.5 and 2.6.

Definition 6.2 We put logical spaces $(\mathsf{A}, \mathcal{B})$ into the following three **classes** in view of Theorem 6.6:

Class 1: $\overline{\mathcal{B}^\cap} = \mathcal{B}$, that is, $\mathcal{B}$ is $\cap$-closed in $\mathcal{P}\mathsf{A}$ and quasi-finitary.

Class 2: $\overline{\mathcal{B}^\cap} = \mathcal{B}^\cap \neq \mathcal{B}$, that is, $\mathcal{B}$ is not $\cap$-closed in $\mathcal{P}\mathsf{A}$ and the $\cap$-closure $\mathcal{B}^\cap$ of $\mathcal{B}$ is quasi-finitary.

Class 3: $\overline{\mathcal{B}^\cap} \neq \mathcal{B}^\cap$, that is, the $\cap$-closure $\mathcal{B}^\cap$ of $\mathcal{B}$ in $\mathcal{P}\mathsf{A}$ is not quasi-finitary.

Since $\overline{\mathcal{B}^\cap} \supseteq \mathcal{B}^\cap \supseteq \mathcal{B}$, we have $\overline{\mathcal{B}^\cap} = \mathcal{B}$ iff $\overline{\mathcal{B}^\cap} = \mathcal{B}^\cap = \mathcal{B}$. Therefore, every logical space belongs to one and only one of the above classes.

Theorem 6.7 The following conditions on a logical space $(\mathsf{A}, \mathcal{B})$ are equivalent.

(1) $(\mathsf{A}, \mathcal{B})$ belongs to the 1st class.

(2) $\mathcal{B}$ is the set of the $\mathcal{B}$-theories in A.

(3) $\mathcal{B}$ is the set of the $\mathcal{B}'$-theories for a logical space $(\mathsf{A}, \mathcal{B}')$.

(4) $\mathcal{B}$ is the set of the R-theories for a partially latticed logic R on A.

(5) $\mathcal{B}$ is the set of the R-theories for a logic R on A.

(6) For some deduction pair (R, D) on A, $\mathcal{B}$ is the set of the R-theories which contain D.

Proof Theorem 6.6 shows that (1) implies (2). Obviously (2) implies (3). Theorems 6.1 and 6.3 show that (3) implies (4). Obviously, (4) implies (5), and (5) implies (6) with $\mathsf{D} = \emptyset$. Theorem 4.3 shows that (6) implies (1).

It is known that the sentence logical space in the propositional logic belongs to the 2nd class. Takaoka [7] has shown that the sentence logical space in monophasic case logic [2] belongs to the 3rd class under certain conditions. Notice that every logical space $(\mathsf{A}, \mathcal{B})$ is equivalent to the logical space $(\mathsf{A}, \overline{\mathcal{B}^{\cap}})$ which is in the 1st class.

Incidentally, Example 4.1 and Theorem 6.7 show that if A is an algebra and $\mathcal{B}$ is the set of the support subalgebras of A, then the logical space $(\mathsf{A}, \mathcal{B})$ belongs to the 1st class. Also, if A is a non-discrete topological space which satisfies the Fréchet's separation axiom and $\mathcal{B}$ is the set of the open sets of A, then the logical space $(\mathsf{A}, \mathcal{B})$ belongs to the 2nd class, while if $\mathcal{B}$ is the set of the closed sets of A, then the logical space $(\mathsf{A}, \mathcal{B})$ belongs to the 3rd class.

7 Completeness of deduction pairs

Throughout this section, $(\mathsf{A}, \mathcal{B})$ is a logical space, Q is the largest $\mathcal{B}$-logic on A, and C is the $\mathcal{B}$-core.

Definition 7.1 Let (R, D) be a deduction pair on A.

- (R, D) is said to be $\mathcal{B}$-**sound** if $\mathsf{R}^{\mathsf{D}} \subseteq \mathsf{Q}$.
- (R, D) is said to be $\mathcal{B}$-**sufficient** if $\mathsf{Q} \subseteq \mathsf{R}^{\mathsf{D}}$.
- (R, D) is said to be $\mathcal{B}$-**complete** if $\mathsf{Q} = \mathsf{R}^{\mathsf{D}}$.
- (R, D) is said to be $\mathcal{B}$-**core-sound** if $[\mathsf{D}]_{\mathsf{R}} \subseteq \mathsf{C}$.
- (R, D) is said to be $\mathcal{B}$-**core-sufficient** if $\mathsf{C} \subseteq [\mathsf{D}]_{\mathsf{R}}$.
- (R, D) is said to be $\mathcal{B}$-**core-complete** if $\mathsf{C} = [\mathsf{D}]_{\mathsf{R}}$.
- (R, D) is said to be $\mathcal{B}$-**extra-complete**, if (R, D) is $\mathcal{B}$-sound and every R-theory containing D belongs to the $\cap$-closure $\mathcal{B}^{\cap}$ of $\mathcal{B}$ in $\mathcal{P}\mathsf{A}$.
- (R, D) is said to be $\mathcal{B}$-**super-complete**, if the set of the R-theories containing D is equal to $\mathcal{B}$.

Theorem 7.1 Let (R, D) be a deduction pair on A. Then the following four conditions are equivalent.

(1) (R, D) is $\mathcal{B}$-sound.

(2) $[X \cup D]_R \subseteq [X]_Q$ for every subset X of A.

(3) Every $\mathcal{B}$-theory in A is an R-theory containing D.

(4) $R \subseteq Q$ and $D \subseteq C$, that is, both R and D are $\mathcal{B}$-sound.

Also, the following three conditions are equivalent.

(5) (R, D) is $\mathcal{B}$-sufficient.

(6) $[X]_Q \subseteq [X \cup D]_R$ for every subset X of A.

(7) Every R-theory in A containing D is a $\mathcal{B}$-theory.

Therefore, the following four conditions are equivalent.

(8) (R, D) is $\mathcal{B}$-complete.

(9) $[X]_Q = [X \cup D]_R$ for every subset X of A.

(10) The set of the $\mathcal{B}$-theories in A is equal to the set of the R-theories in A containing D.

(11) $R \subseteq Q$, $D \subseteq C$, and $Q \subseteq R^D$.

Proof The Q is a partially latticed relation by Theorem 6.3, the $\mathcal{B}$-theories are identical with the Q-theories by Theorem 6.1, and $C = A_Q$ by Theorem 6.4. Therefore, Theorem 7.1 is a consequence of Theorem 5.7.

Theorem 7.2 Let (R, D) be a deduction pair on A. If (R, D) is $\mathcal{B}$-sound, then (R, D) is $\mathcal{B}$-core-sound. If (R, D) is $\mathcal{B}$-sufficient, then (R, D) is $\mathcal{B}$-core-sufficient. Therefore if (R, D) is $\mathcal{B}$-complete, then (R, D) is $\mathcal{B}$-core-complete.

Proof If (R, D) is $\mathcal{B}$-sound, then $[X \cup D]_R \subseteq [X]_Q$ for every subset X of A by Theorem 7.1, hence in particular $[D]_R \subseteq [\emptyset]_Q$, and since $C = [\emptyset]_Q$ by Theorem 6.4, we conclude that $[D]_R \subseteq C$ holds. The rest of the proof is similar.

Theorem 7.3 Let (R, D) be a deduction pair on A. If (R, D) is $\mathcal{B}$-core-sufficient and there exists a mapping $\phi \in A^* \times A \to A$ which satisfies

$$\alpha \, Q \, y \implies \varepsilon \, Q \, \phi(\alpha, y), \qquad \varepsilon \, R^D \, \phi(\alpha, y) \implies \alpha \, R^D \, y$$

for each $(\alpha, y) \in A^* \times A$, then (R, D) is $\mathcal{B}$-sufficient.

Similarly, if (R, D) is $\mathcal{B}$-core-sound and there exists a mapping $\phi \in A^* \times A \to A$ which satisfies

$$\alpha \, Q \, y \impliedby \varepsilon \, Q \, \phi(\alpha, y), \qquad \varepsilon \, R^D \, \phi(\alpha, y) \impliedby \alpha \, R^D \, y$$

for each $(\alpha, y) \in A^* \times A$, then (R, D) is $\mathcal{B}$-sound.

Proof As for the sufficiency, by Theorem 6.4 and the $\mathcal{B}$-core-sufficiency of (R, D), we have

$$\varepsilon\, \mathsf{Q}\, y \iff y \in \mathsf{C} \implies y \in [\mathsf{D}]_{\mathsf{R}} \iff \varepsilon\, \mathsf{R}^{\mathsf{D}}\, y$$

for all $y \in \mathsf{A}$. Therefore,

$$\alpha\, \mathsf{Q}\, y \implies \varepsilon\, \mathsf{Q}\, \phi(\alpha, y) \implies \varepsilon\, \mathsf{R}^{\mathsf{D}}\, \phi(\alpha, y) \implies \alpha\, \mathsf{R}^{\mathsf{D}}\, y$$

for all $(\alpha, y) \in \mathsf{A}^* \times \mathsf{A}$, hence $\mathsf{Q} \subseteq \mathsf{R}^{\mathsf{D}}$. The rest of the proof is similar.

Theorem 7.4 Let (R, D) be a deduction pair on A. Then the following conditions are equivalent.

(1) (R, D) is $\mathcal{B}$-extra-complete.

(2) (R, D) is $\mathcal{B}$-complete and $(\mathsf{A}, \mathcal{B})$ belongs to the 1st or the 2nd class.

Proof Let $\mathcal{D}$ be the set of the R-theories in A which contain D.

Assume (1). Then $\mathcal{D} \subseteq \mathcal{B}^{\cap} \subseteq \overline{\mathcal{B}^{\cap}}$. In particular $\mathcal{D} \subseteq \overline{\mathcal{B}^{\cap}}$, so (R, D) is $\mathcal{B}$-sufficient by Theorems 6.6 and 7.1. Therefore (R, D) is $\mathcal{B}$-complete, and so $\overline{\mathcal{B}^{\cap}} = \mathcal{D}$ by the same theorems. Therefore $\overline{\mathcal{B}^{\cap}} = \mathcal{B}^{\cap}$, and thus $(\mathsf{A}, \mathcal{B})$ belongs to the 1st or the 2nd class.

Assume (2).Then (R, D) is $\mathcal{B}$-sound, and $\mathcal{D} = \overline{\mathcal{B}^{\cap}} = \mathcal{B}^{\cap}$ by Theorem 7.1. Thus, (R, D) is $\mathcal{B}$-extra-complete.

Theorem 7.5 Let (R, D) be a deduction pair on A. Then the following conditions are equivalent.

(1) (R, D) is $\mathcal{B}$-super-complete.

(2) (R, D) is $\mathcal{B}$-extra-complete and $\mathcal{B}^{\cap} = \mathcal{B}$.

(3) (R, D) is $\mathcal{B}$-complete and $(\mathsf{A}, \mathcal{B})$ belongs to the 1st class.

Proof By Theorem 7.4, (2) holds iff (R, D) is $\mathcal{B}$-complete and $\overline{\mathcal{B}^{\cap}} = \mathcal{B}^{\cap} = \mathcal{B}$. This equation means that $(\mathsf{A}, \mathcal{B})$ belongs to the 1st class. Thus (2) is equivalent to (3).

Let $\mathcal{D}$ be the set of the R-theories in A which contain D, and assume (1). Then, since $\mathcal{D} = \mathcal{B}$, we have the following. First, R is $\mathcal{B}$-sound because it closes every element of $\mathcal{B}$. Secondly, D is $\mathcal{B}$-sound because it is contained in every element of $\mathcal{B}$ and hence in C. Thirdly, every member of $\mathcal{D}$ is a $\mathcal{B}$-theory. Therefore, (R, D) is $\mathcal{B}$-complete by Theorem 7.1. Furthermore, $\overline{\mathcal{B}^{\cap}} = \mathcal{D} = \mathcal{B}$ by Theorems 6.6 and 7.1, so $(\mathsf{A}, \mathcal{B})$ belongs to the 1st class.

Assume (3). Then $\mathcal{D} = \overline{\mathcal{B}^{\cap}} = \mathcal{B}$ by Theorems 7.1 and 6.6, and so (R, D) is $\mathcal{B}$-super-complete.

Theorem 7.6 The following holds for each logical space $(\mathsf{A}, \mathcal{B})$.

(1) If a deduction pair on A is $\mathcal{B}$-super-complete, then $(\mathsf{A}, \mathcal{B})$ belongs to the 1st class. Conversely, if $(\mathsf{A}, \mathcal{B})$ belongs to the 1st class, then every $\mathcal{B}$-complete deduction pair on A is $\mathcal{B}$-super-complete.

(2) If a deduction pair on A is $\mathcal{B}$-extra-complete but not $\mathcal{B}$-super-complete, then $(\mathsf{A}, \mathcal{B})$ belongs to the 2nd class. Conversely, if $(\mathsf{A}, \mathcal{B})$ belongs to the 2nd class, then every $\mathcal{B}$-complete deduction pair on A is $\mathcal{B}$-extra-complete but not $\mathcal{B}$-super-complete.

(3) If a deduction pair on A is $\mathcal{B}$-complete but not $\mathcal{B}$-extra-complete, then $(\mathsf{A}, \mathcal{B})$ belongs to the 3rd class. Conversely, if $(\mathsf{A}, \mathcal{B})$ belongs to the 3rd class, then no deduction pair on A is $\mathcal{B}$-extra-complete.

Proof (1) and (3) are restatements of part of Theorems 7.5 and 7.4 respectively. If a deduction pair (R, D) on A is $\mathcal{B}$-extra-complete but not $\mathcal{B}$-super-complete, then (R, D) is $\mathcal{B}$-complete and $(\mathsf{A}, \mathcal{B})$ belongs to the 1st or the 2nd class by Theorem 7.4, and $(\mathsf{A}, \mathcal{B})$ belongs to the 2nd class by Theorem 7.5. Conversely if $(\mathsf{A}, \mathcal{B})$ belongs to the 2nd class, then a $\mathcal{B}$-complete deduction pair on A is $\mathcal{B}$-extra-complete by Theorem 7.4 but not $\mathcal{B}$-super-complete by Theorem 7.5.

Theorem 7.7 If logical spaces $(\mathsf{A}, \mathcal{B})$ and $(\mathsf{A}, \mathcal{B}')$ are equivalent to each other, then the $\mathcal{B}$-completeness and the $\mathcal{B}$-core-completeness are respectively identical with the $\mathcal{B}'$-completeness and the $\mathcal{B}'$-core-completeness, and similarly for the soundness and the sufficiency.

Proof This is because the largest $\mathcal{B}$-logic is equal to the largest $\mathcal{B}'$-logic and the $\mathcal{B}$-core is equal to the $\mathcal{B}'$-core by Theorem 6.5.

Corollary 7.7.1 Suppose logical spaces $(\mathsf{A}, \mathcal{B})$ and $(\mathsf{A}, \mathcal{B}')$ satisfy $\mathcal{B}^\cap = \mathcal{B}'^\cap$. Then the $\mathcal{B}$-extra-completeness is identical with the $\mathcal{B}'$-extra-completeness.

Proof Since $\overline{\mathcal{B}^\cap} = \overline{\mathcal{B}'^\cap}$, $(\mathsf{A}, \mathcal{B})$ and $(\mathsf{A}, \mathcal{B}')$ are equivalent by Corollary 6.6.1, and so the $\mathcal{B}$-soundness is identical with the $\mathcal{B}'$-soundness. Hence the above result.

8 Consistency and classes

In §7, we have observed the interrelations between the completeness and the classification in Definition 6.2 of the logical spaces. Here is given an account of the interrelations between the classification and the consistency.

Definition 8.1 Let $(\mathsf{A}, \mathcal{B})$ be a logical space and Q be the largest $\mathcal{B}$-logic on A. Then a subset X of A is said to be $\mathcal{B}$-**consistent** if $[\mathsf{X}]_\mathsf{Q} \neq \mathsf{A}$. If a singleton $\{\mathsf{x}\}$ is $\mathcal{B}$-consistent, we say that x is $\mathcal{B}$-consistent.

Theorem 8.1 Let $(A, \mathcal{B})$ be a logical space and Q be the largest $\mathcal{B}$-logic on A. Then a finite subset $\{x_1, \ldots, x_n\}$ of A is $\mathcal{B}$-inconsistent iff every element $y \in A$ satisfies $x_1 \cdots x_n \, Q \, y$.

Proof This is a direct consequence of Theorem 6.3.

Theorem 8.2 Let $(A, \mathcal{B})$ be a logical space, Q be the largest $\mathcal{B}$-logic on A, and assume that there exists a $\mathcal{B}$-inconsistent finite subset $\{x_1, \ldots, x_n\}$ of A. Then the following conditions on a subset X of A are equivalent.

(1) X is $\mathcal{B}$-inconsistent.

(2) $x_1, \ldots, x_n \in [X]_Q$.

(3) There exists an element $\alpha \in A^*$ such that $\alpha \subseteq X$ and $\alpha \, Q \, x_i$ for every number $i \in \{1, \ldots, n\}$.

(4) There exists an element $\alpha \in A^*$ such that $\alpha \subseteq X$ and $\alpha \, Q \, y$ for every element $y \in A$.

(5) There exists a $\mathcal{B}$-inconsistent finite subset of X.

Proof If (1) holds, then since $[X]_Q = A$, (2) holds. Assume that (2) holds. Then Theorem 6.3 shows that, for each $i = 1, \ldots, n$, there exists an element $\alpha_i \in A^*$ such that $\alpha_i \subseteq X$ and $\alpha_i \, Q \, x_i$. Since Q satisfies the partial weakening law and partial exchange law by Theorem 6.3, $\alpha = \alpha_1 \cdots \alpha_n$ satisfies $\alpha \subseteq X$ and $\alpha \, Q \, x_i$ for every number $i \in \{1, \ldots, n\}$. Thus (3) holds. Assume that (3) holds and let $y \in A$. Then

$$\begin{aligned} \alpha \, Q \, x_i \quad & (i = 1, \ldots, n), \\ x_1 \cdots x_n \, Q \, y \end{aligned}$$

by the assumption and Theorem 8.1. By repeated applications of the partial cut law, partial exchange law, and partial contraction law to the above $n + 1$ displayed Q-relations, we have $\alpha \, Q \, y$. Thus (4) holds. If (4) holds, then α is $\mathcal{B}$-inconsistent by Theorem 8.1, and so (5) holds. If (5) holds, then since every subset Y of X satisfies $[Y]_Q \subseteq [X]_Q$ by Theorems 4.3, (1) holds.

Theorem 8.3 Let $(A, \mathcal{B})$ be a logical space and Q be the largest $\mathcal{B}$-logic on A. Then the following holds.

(1) $[Y]_Q = \bigcap_{Y \subseteq B \in \mathcal{B}} B$ for each $Y \in \mathcal{P}'A$.

(2) $(A, \mathcal{B})$ belongs to the 1st or the 2nd class iff $[X]_Q = \bigcap_{X \subseteq B \in \mathcal{B}} B$ for each $X \in \mathcal{P}A$.

Proof (1) Let $Y = \{y_1, \dots, y_n\}$ and define $\alpha = y_1 \cdots y_n \in A^*$. Then

$$[Y]_Q = [\alpha]_Q = \{y \in A \mid \alpha\, Q\, y\} = \bigcap_{\alpha \subseteq B \in \mathcal{B}} B = \bigcap_{Y \subseteq B \in \mathcal{B}} B$$

by Theorems 6.3 and 6.2. An alternate proof is as follows:

$$[Y]_Q = \bigcap_{Y \subseteq X \in \overline{\mathcal{B}^\cap}} X = \bigcap_{Y \subseteq X \in \mathcal{B}^\cap} X = \bigcap_{Y \subseteq B \in \mathcal{B}} B$$

by Theorems 6.1, 6.6, 2.6, and 2.5.

(2) If $(A, \mathcal{B})$ belongs to the 1st or the 2nd class, then $\overline{\mathcal{B}^\cap} = \mathcal{B}^\cap$, so

$$[X]_Q = \bigcap_{X \subseteq Y \in \overline{\mathcal{B}^\cap}} Y = \bigcap_{X \subseteq Y \in \mathcal{B}^\cap} Y = \bigcap_{X \subseteq B \in \mathcal{B}} B$$

by Theorems 6.1, 6.6, and 2.5. Conversely, if $[X]_Q = \bigcap_{X \subseteq B \in \mathcal{B}} B$ for each $X \in \mathcal{P}A$, then in particular for each $X \in \overline{\mathcal{B}^\cap}$, we have $X = [X]_Q = \bigcap_{X \subseteq B \in \mathcal{B}} B \in \mathcal{B}^\cap$ by Theorem 6.6, and therefore $\overline{\mathcal{B}^\cap} = \mathcal{B}^\cap$.

Definition 8.2 Let $(A, \mathcal{B})$ be a logical space and X be a subset of A. Then a **$\mathcal{B}$-model** for X is a set $B \in \mathcal{B} - \{A\}$ such that $X \subseteq B$.

Therefore, the subset X has a $\mathcal{B}$-model iff X belongs to the downward closure $\overleftarrow{\mathcal{B} - \{A\}}$ of $\mathcal{B} - \{A\}$, and iff $\bigcap_{X \subseteq B \in \mathcal{B}} B \neq A$.

In the rest of this section, we let $(A, \mathcal{B})$ be a logical space, Q be the largest $\mathcal{B}$-logic on A, $\mathcal{C}$ be the set of the $\mathcal{B}$-consistent subsets of A, and $\mathcal{D}$ be the set of the maximal elements of the ordered set $(\mathcal{C}, \subseteq)$.

Theorem 8.4 The following holds.

(1) $\overleftarrow{\mathcal{B} - \{A\}} \subseteq \mathcal{C} = \overleftarrow{\overline{\mathcal{B}^\cap} - \{A\}} \subseteq \overline{\overleftarrow{\mathcal{B} - \{A\}}}$.

(2) Every finite set in $\mathcal{C}$ belongs to $\overleftarrow{\mathcal{B} - \{A\}}$.

(3) If $(A, \mathcal{B})$ belongs to the 1st or the 2nd class, then $\mathcal{C} = \overleftarrow{\mathcal{B} - \{A\}}$.

(4) There exists $\mathcal{B}$-inconsistent finite subset of A iff $\mathcal{P}'A \not\subseteq \overleftarrow{\mathcal{B} - \{A\}}$.

(5) If there exists a $\mathcal{B}$-inconsistent finite subset of A, then $\mathcal{C} = \overline{\overleftarrow{\mathcal{B} - \{A\}}} = \overleftarrow{\mathcal{D}}$.

(6) $\mathcal{D}$ is equal to the set of the maximal elements of the ordered set $(\overline{\mathcal{B}^\cap} - \{A\}, \subseteq)$.

(7) If $X \in \mathcal{D}$, then $X = [X]_Q$.

Proof (1) If $X \in \overleftarrow{\overline{\mathcal{B}^\cap} - \{A\}}$, then there exists a set $Y \in \overline{\mathcal{B}^\cap} - \{A\}$ such that $X \subseteq Y$, and since $[X]_Q \subseteq Y \neq A$ by Theorem 6.6, we have $X \in \mathcal{C}$. Thus $\overleftarrow{\mathcal{B} - \{A\}} \subseteq \overleftarrow{\overline{\mathcal{B}^\cap} - \{A\}} \subseteq \mathcal{C}$. Conversely if $X \in \mathcal{C}$, then $X \subseteq [X]_Q \in \overline{\mathcal{B}^\cap} - \{A\}$ by Theorems 6.1 and 6.6, hence $X \in \overleftarrow{\overline{\mathcal{B}^\cap} - \{A\}}$. Thus $\mathcal{C} = \overleftarrow{\overline{\mathcal{B}^\cap} - \{A\}}$. Consequently, $\mathcal{C}$ is downward. Suppose $X \in \mathcal{C}$ and $Y \in \mathcal{P}'X$. Then $\bigcap_{Y \subseteq B \in \mathcal{B}} B = [Y]_Q \neq A$ by Theorem 8.3, and so $Y \in \overleftarrow{\mathcal{B} - \{A\}}$. Thus $\mathcal{C} \subseteq \overleftarrow{\overleftarrow{\mathcal{B} - \{A\}}}$ by Theorem 2.9.

(2) This is a consequence of (1) and Theorem 2.6, although already proved in the above proof of $\mathcal{C} \subseteq \overleftarrow{\overleftarrow{\mathcal{B} - \{A\}}}$ with $X = Y$.

(3) Suppose $(A, \mathcal{B})$ belongs to the 1st or the 2nd class. Then if $X \in \mathcal{C}$, $A \neq [X]_Q = \bigcap_{X \subseteq B \in \mathcal{B}} B$ by Theorem 8.3, and so $X \in \overleftarrow{\mathcal{B} - \{A\}}$. This together with (1) implies $\mathcal{C} = \overleftarrow{\mathcal{B} - \{A\}}$.

(4) This is a consequence of (1) and (2).

(5) Assume that there exists a $\mathcal{B}$-inconsistent finite subset of A. Then Theorems 8.2 and 2.1 imply that $\mathcal{C}$ is finitary. Therefore, $\mathcal{C} = \overleftarrow{\overleftarrow{\mathcal{B} - \{A\}}}$ by (1) and Theorem 2.9. Also, the ordered set $(\mathcal{C}, \subseteq)$ is inductive by Theorem 2.2 and a remark before Theorem 2.1, and so $\mathcal{C} \subseteq \overleftarrow{\mathcal{D}}$ by Zorn's lemma. Since $\mathcal{C}$ is downward by (1), conversely $\overleftarrow{\mathcal{D}} \subseteq \mathcal{C}$.

(6) This is a consequence of (1).

(7) This is because $X \subseteq [X]_Q$ and $\big[[X]_Q\big]_Q = [X]_Q \neq A$.

Theorem 8.5 Assume that, for each element $x \in A$, there exists an element $x^\Diamond \in A$ which satisfies

$$x^\Diamond \in B \iff x \notin B$$

for all $B \in \mathcal{B} - \{A\}$ (we call $x^\Diamond$ a **$\mathcal{B}$-complement** of x). Then the following holds.

(1) Q satisfies the law "$x\alpha\, Q\, y,\ x^\Diamond \alpha\, Q\, y \implies \alpha\, Q\, y$."

(2) For each $x \in A$, $\{x, x^\Diamond\}$ is $\mathcal{B}$-inconsistent.

Proof For each $B \in \mathcal{B} - \{A\}$, define the relation $\preccurlyeq_B$ on A^* by (3.9). Then the characteristic mapping 1_B of B is a latticed representation of A on the lattice $\mathbb{T} = \{0, 1\}$, and $\preccurlyeq_B$ is equal to the 1_B-validity relation $\preccurlyeq_{1_B}$. Since $\mathbb{T}$ is Boolean, $\preccurlyeq_B$ is a strongly latticed relation by Theorems 3.1 and 3.2. Furthermore, since $x^\Diamond \in B$ iff $x \notin B$, we have $1_B(x^\Diamond) = (1_B x)^\Diamond$, where $\Diamond$ on the right-hand side is the complement in $\mathbb{T}$. Therefore by Theorem 3.7, $\preccurlyeq_B$ satisfies the negation laws. Let $\preccurlyeq\, = \bigcap_{B \in \mathcal{B} - \{A\}} \preccurlyeq_B$. Then by Theorem 3.32, $\preccurlyeq$ is also a strongly latticed relation satisfying the negation laws. Therefore by Theorem 3.13, $\preccurlyeq$ satisfies the law ($\Diamond$6), and $xx^\Diamond \preccurlyeq y$ holds for any elements $x, y \in A$. Since Q is the restriction of $\preccurlyeq$ to $A^* \times A$ by Theorem 6.2, we conclude that (1) and (2) hold.

Theorem 8.6 Assume that each element $x \in A$ has its $\mathcal{B}$-complement $x^\Diamond$. Then the following conditions are equivalent.

(1) $(A, \mathcal{B})$ belongs to the 1st or the 2nd class.

(2) $\mathcal{D} \subseteq \mathcal{B} - \{A\}$.

(3) $\mathcal{C} \subseteq \overleftarrow{\mathcal{B} - \{A\}}$.

Proof Assume (1). Then $\overline{\mathcal{B}^{\cap}} = \mathcal{B}^{\cap}$, an so by Theorem 8.4, $\mathcal{D}$ is equal to the set of the maximal elements of $\mathcal{B}^{\cap} - \{A\}$. Since maximal elements of $\mathcal{B}^{\cap} - \{A\}$ belong to $\mathcal{B} - \{A\}$, we conclude that (2) holds.

Assume (2). Then $\overleftarrow{\mathcal{D}} \subseteq \overleftarrow{\mathcal{B} - \{A\}}$. Also, since there exists a $\mathcal{B}$-inconsistent finite subset $\{x, x^{\Diamond}\}$ by Theorem 8.5, we have $\mathcal{C} = \overleftarrow{\mathcal{D}}$ by Theorem 8.4. Therefore (3) holds.

Finally, we assume (3) and show that (1) holds. By virtue of Theorem 8.3, we only need to show that every element $X \in \mathcal{P}A$ satisfies $[X]_Q = \bigcap_{X \subseteq B \in \mathcal{B}} B$. Obviously $[X]_Q \subseteq \bigcap_{X \subseteq B \in \mathcal{B}} B = \bigcap_{X \subseteq B \in \mathcal{B} - \{A\}} B$, so we will show that every element $y \in \bigcap_{X \subseteq B \in \mathcal{B} - \{A\}} B$ belongs to $[X]_Q$. If $X \subseteq B \in \mathcal{B} - \{A\}$, then $y \in B$, and so $y^{\Diamond} \notin B$. Therefore, there does not exist a set B such that $\{y^{\Diamond}\} \cup X \subseteq B \in \mathcal{B} - \{A\}$, and so (3) yields $[\{y^{\Diamond}\} \cup X]_Q = A$, hence $y \in [\{y^{\Diamond}\} \cup X]_Q$. Therefore by Theorem 6.3, there exists an element $\alpha \in A^*$ such that $\alpha \subseteq X$ and $y^{\Diamond} \alpha \, Q \, y$. Since also $y \alpha \, Q \, y$ by Theorem 6.3, we have $\alpha \, Q \, y$ by Theorem 8.5. Thus $y \in [X]_Q$ as desired. The argument used here is due to [7].

Remark 8.1 Under the assumption of Theorem 8.6, we have $\mathcal{B} - \{A\} \subseteq \mathcal{D}$ and $\overleftarrow{\mathcal{B} - \{A\}} \subseteq \mathcal{C} = \overline{\overleftarrow{\mathcal{B} - \{A\}}}$ by Theorem 8.4 and 8.5. Therefore, the three equivalent conditions in Theorem 8.6 are furthermore equivalent to the following conditions.

(4) $\mathcal{D} = \mathcal{B} - \{A\}$.

(5) $\mathcal{C} = \overleftarrow{\mathcal{B} - \{A\}}$.

(6) $\overline{\overleftarrow{\mathcal{B} - \{A\}}} = \overleftarrow{\mathcal{B} - \{A\}}$.

Since $\overleftarrow{\mathcal{B} - \{A\}}$ is downward, Theorem 2.1 shows that the above conditions are furthermore equivalent to the following condition.

(7) $\overleftarrow{\mathcal{B} - \{A\}}$ is finitary.

9 Generalizations to functional logical spaces

In §6, we have defined a logical space to be a pair $(A, \mathcal{B})$ of a non-empty set A and a subset $\mathcal{B}$ of $\mathcal{P}A$. Here we generalize the notion utilizing the fact that $\mathcal{P}A$ is identified with $A \to \mathbb{T}$.

9.1 Functional logical spaces

A **functional logical space** is a pair $(A, \mathcal{F})$ of a non-empty set A and a subset $\mathcal{F}$ of $A \to \mathbb{B}$, where $\mathbb{B}$ is a lattice which has the smallest element and the largest element and **non-trivial** in the sense that $\#\mathbb{B} \geq 2$.

If $(A, \mathcal{F})$ is a $\mathbb{B}$-valued functional logical space, then each element $f \in \mathcal{F}$ is a latticed representation of A on $\mathbb{B}$ in the sense of §3. Conversely, if $(f_i)_{i \in I}$ is a family of latticed representations of a non-empty set A on a non-trivial lattice $\mathbb{B}$, then $(A, \{f_i \mid i \in I\})$ is a $\mathbb{B}$-valued functional logical space. In fact, every functional logical space $(A, \mathcal{F})$ is "equivalent" to a functional logical space $(A, \{f\})$ made of a single latticed representation f of A on some non-trivial lattice. In a different context, every functional logical space is "equivalent" to a $\mathbb{T}$-valued functional logical space.

Henceforth in this section, we assume that $(A, \mathcal{F})$ is a $\mathbb{B}$-valued functional logical space and denote the order, smallest element, and largest element of $\mathbb{B}$ by $\leq, 0$, and 1. Then, for each $f \in \mathcal{F}$ and $a \in \mathbb{B}$, we define

$$A_{f,a} = \{x \in A \mid a \leq fx\}.$$

We will denote $A_{f,1}$ also by A_f:

$$A_f = \{x \in A \mid fx = 1\} = f^{-1}1.$$

Furthermore, we define

$$\mathcal{B} = \{A_{f,a} \mid f \in \mathcal{F},\ 0 \neq a \in \mathbb{B}\}.$$

Then $(A, \mathcal{B})$ is a logical space in the sense of §6, and we have defined various notions for it, such as the $\mathcal{B}$-logics, $\mathcal{B}$-theories, $\mathcal{B}$-core, $\mathcal{B}$-completeness, and so on. We call them the **$\mathcal{F}$-logics, $\mathcal{F}$-theories, $\mathcal{F}$-core, $\mathcal{F}$-completeness**, and so on. Furthermore we define, for each $f \in \mathcal{F}$, the relation $\vDash_f$ between A^* and A by

$$\alpha \vDash_f y \iff \inf f\alpha \leq fy. \tag{9.1}$$

We call $\vDash_f$ the **partial f-validity relation**, because it is equal to the restriction to $A^* \times A$ of the f-validity relation $\preccurlyeq_f$ defined by (3.1):

$$\alpha \preccurlyeq_f \beta \iff \inf f\alpha \leq \sup f\beta.$$

Furthermore, we define the relation $\vDash$ between A^* and A by

$$\alpha \vDash y \iff \alpha \vDash_f y \text{ for every } f \in \mathcal{F}, \tag{9.2}$$

which we call the **partial $\mathcal{F}$-validity relation**. It is equal to the restriction to $A^* \times A$ of the **$\mathcal{F}$-validity relation** $\preccurlyeq$ defined by

$$\alpha \preccurlyeq \beta \iff \alpha \preccurlyeq_f \beta \text{ for every } f \in \mathcal{F}. \tag{9.3}$$

Theorem 9.1 The partial $\mathcal{F}$-validity relation $\vDash$ is equal to the largest $\mathcal{F}$-logic on A. Also, elements $x_1, \ldots, x_n, y \in A$ satisfy $x_1 \cdots x_n \vDash y$ iff they satisfy the condition

$$fx_1 \geq a, \ \ldots, \ fx_n \geq a \implies fy \geq a \tag{9.4}$$

for every $f \in \mathcal{F}$ and every $a \in \mathbb{B}$.

Proof Let $(A, \mathcal{B})$ be the logical space associated with $(A, \mathcal{F})$ and Q be the largest $\mathcal{B}$-logic on A. Then by Theorems 6.2, elements $x_1, \ldots, x_n, y \in A$ satisfy $x_1 \cdots x_n \, Q \, y$ iff they satisfy the condition

$$\{x_1, \ldots, x_n\} \subseteq A_{f,a} \implies y \in A_{f,a}$$

for every element $(f, a) \in \mathcal{F} \times \mathbb{B}$ (notice that this condition for $a = 0$ is always satisfied because $A_{f,0} = A$). This condition is equivalent to (9.4). Furthermore, $x_1, \ldots, x_n, y \in A$ and $f \in \mathcal{F}$ satisfy (9.4) for all $a \in \mathbb{B}$ iff they satisfy $\inf\{fx_1, \ldots, fx_n\} \leq fy$, that is, $x_1 \cdots x_n \vDash_f y$. Thus $x_1, \ldots, x_n, y$ satisfy $x_1 \cdots x_n \, Q \, y$ iff they satisfy $x_1 \cdots x_n \vDash y$.

The following result justifies and amplifies the definitions in §6.

Theorem 9.2 The following holds for the partial $\mathcal{F}$-validity relation $\vDash$.

(1) A logic R on A is $\mathcal{F}$-sound iff it satisfies the condition

$$\alpha \, R \, y \implies \alpha \vDash y$$

for every element $(\alpha, y) \in A^* \times A$.

(2) A subset X of A is an $\mathcal{F}$-theory iff it satisfies the condition

$$\alpha \subseteq X, \ \alpha \vDash y \implies y \in X$$

for every element $(\alpha, y) \in A^* \times A$.

(3) An element $x \in A$ is an $\mathcal{F}$-tautology iff $\varepsilon \vDash x$.

Proof Since $\vDash$ is the largest $\mathcal{F}$-logic by Theorem 9.1, the above results are simple restatements of part of Theorems 6.1 and 6.4, although (3) is also a direct consequence of the definition of $\vDash$.

Theorem 9.3 The partial $\mathcal{F}$-validity relation $\vDash$ is a partially latticed relation.

Proof This is a direct consequence of Theorems 9.1 and 6.3 or results in §3.

The following result justifies and amplifies Definition 7.1. Recall that, if (R, D) is a deduction pair on A, then $\vDash_{R,D}$ is an alternate expression of the D-closure R^D of R defined by (5.1) and is called the partial deduction relation.

Theorem 9.4 Let (R, D) be a deduction pair on A. Then the following holds.

(1) (R, D) is $\mathcal{F}$-sound iff it satisfies the condition

$$\alpha \vDash_{R,D} y \implies \alpha \vDash y$$

for every element $(\alpha, y) \in A^* \times A$.

(2) (R, D) is $\mathcal{F}$-sufficient iff it satisfies the condition

$$\alpha \vDash y \implies \alpha \vDash_{R,D} y$$

for every element $(\alpha, y) \in A^* \times A$.

(3) (R, D) is $\mathcal{F}$-complete iff it satisfies the condition

$$\alpha \vDash_{R,D} y \iff \alpha \vDash y$$

for every element $(\alpha, y) \in A^* \times A$.

Proof Since $\vDash$ is equal to the largest $\mathcal{F}$-logic on A by Theorem 9.1 and $\vDash_{R,D}$ is the D-closure R^D of R, the above is a restatement of part of Definition 7.1.

Theorem 9.5 Let (R, D) be a deduction pair on A. If (R, D) is $\mathcal{F}$-core-sufficient and there exists a mapping $\phi \in A^* \times A \to A$ which satisfies

$$\alpha \vDash y \implies \varepsilon \vDash \phi(\alpha, y), \qquad \varepsilon \vDash_{R,D} \phi(\alpha, y) \implies \alpha \vDash_{R,D} y$$

for each $(\alpha, y) \in A^* \times A$, then (R, D) is $\mathcal{F}$-sufficient.

Similarly, if (R, D) is $\mathcal{F}$-core-sound and there exists a mapping $\phi \in A^* \times A \to A$ which satisfies

$$\alpha \vDash y \impliedby \varepsilon \vDash \phi(\alpha, y), \qquad \varepsilon \vDash_{R,D} \phi(\alpha, y) \impliedby \alpha \vDash_{R,D} y$$

for each $(\alpha, y) \in A^* \times A$, then (R, D) is $\mathcal{F}$-sound.

Proof Since $\vDash$ is equal to the largest $\mathcal{F}$-logic on A by Theorem 9.1 and $\vDash_{R,D}$ is the D-closure R^D of R, the above is a consequence of Theorem 7.3.

9.2 Fundamental theorem of completeness

We continue consideration of the $\mathbb{B}$-valued functional logical space $(A, \mathcal{F})$. Here we prove a theorem of vital importance on the $\mathcal{F}$-completeness of deduction pairs on A. Contrary to the previous subsection, the $\mathcal{F}$-validity relation $\preccurlyeq$ plays the principal role, and the f-validity relations $\preccurlyeq_f$ $(f \in \mathcal{F})$ play supporting roles.

In addition to the notation and terminology so far used, we define

$$\vec{A} = A^* \times A^*,$$

denote the elements $(\alpha, \beta) \in \vec{A}$ by $\alpha \to \beta$ as in Example 5.2, and define

$$\vec{C} = \{\alpha \to \beta \in \vec{A} \mid \alpha \preccurlyeq \beta\},$$
$$\vec{A}_f = \{\alpha \to \beta \in \vec{A} \mid \alpha \preccurlyeq_f \beta\}$$

for each $f \in \mathcal{F}$, and

$$\vec{\mathcal{F}} = \{\vec{A}_f \mid f \in \mathcal{F}\}.$$

Then $(\vec{A}, \vec{\mathcal{F}})$ is a logical space, and $\vec{C}$ is equal to the $\vec{\mathcal{F}}$-core of $\vec{A}$, because

$$\vec{C} = \bigcap_{f \in \mathcal{F}} \vec{A}_f = \bigcap_{F \in \vec{\mathcal{F}}} F.$$

Theorem 9.6 Let (R, D) be a deduction pair on A. Then the following holds on the deduction relation $\preccurlyeq_{R,D}$ defined by (5.2).

(1) If $\preccurlyeq_{R,D}$ is contained in $\preccurlyeq$, then (R, D) is $\mathcal{F}$-sound.

(2) If $\preccurlyeq_{R,D}$ contains $\preccurlyeq$, then (R, D) is $\mathcal{F}$-sufficient.

(3) If $\preccurlyeq_{R,D}$ is equal to $\preccurlyeq$, then (R, D) is $\mathcal{F}$-complete.

Proof Since the restrictions of $\preccurlyeq_{R,D}$ and $\preccurlyeq$ to $A^* \times A$ are equal to $\vDash_{R,D}$ and $\vDash$ respectively, the above holds by Theorem 9.4.

Recall here from §5.2 that a deduction pair on $\vec{A}$ is also called a generational law on the relations on A^*.

Theorem 9.7 (Fundamental theorem of completeness) Let (R, D) be an $\mathcal{F}$-sound deduction pair on A, and let $(\vec{R}, \vec{D})$ be a deduction pair on $\vec{A}$. Assume that the following two conditions are satisfied.

(1) $\vec{C} \subseteq [\vec{D}]_{\vec{R}}$.

(2) The deduction relation $\preccurlyeq_{R,D}$ satisfies the generational law $(\vec{R}, \vec{D})$.

Then (R, D) is $\mathcal{F}$-complete.

Proof Define the subset $\vec{A}_{R,D}$ of $\vec{A}$ by

$$\vec{A}_{R,D} = \{\alpha \to \beta \in \vec{A} \mid \alpha \preccurlyeq_{R,D} \beta\}.$$

Then (2) is equivalent to the condition

(3) $\vec{A}_{R,D}$ is closed under $\vec{R}$ and $\vec{D} \subseteq \vec{A}_{R,D}$.

Hence $[\vec{D}]_{\vec{R}} \subseteq \vec{A}_{R,D}$. Therefore $\vec{C} \subseteq \vec{A}_{R,D}$ by (1), which means that $\preccurlyeq$ is contained in $\preccurlyeq_{R,D}$. Therefore (R, D) is $\mathcal{F}$-sufficient by Theorem 9.6. Since we are assuming (R, D) is $\mathcal{F}$-sound, we conclude that (R, D) is $\mathcal{F}$-complete.

Remark 9.1 In [2], the content of §5.2 is not included and so the condition (2) of Theorem 9.7 is replaced by its equivalent (3).

The following five theorems supplement Theorem 9.7.

Theorem 9.8 The $\mathcal{F}$-validity relation $\preccurlyeq$ is a latticed relation.

Proof Since the f-validity relation $\preccurlyeq_{\mathsf{f}}$ is a latticed relation for each $\mathsf{f} \in \mathcal{F}$ by Theorems 3.1 and 3.2, so is $\preccurlyeq$ by Theorem 3.32.

Theorem 9.9 If the $\mathcal{F}$-validity relation $\preccurlyeq$ satisfies the quasi-disjunction law and inf-emptiness law, then $\preccurlyeq$ is equal to the largest latticed extension of the partial $\mathcal{F}$-validity relation $\vDash$.

Proof This is a consequence of Theorem 9.8 and Theorem 3.22.

Theorem 9.10 Assume that the $\mathcal{F}$-validity relation $\preccurlyeq$ is equal to the largest latticed extension of the partial $\mathcal{F}$-validity relation $\vDash$, and let (R, D) be a deduction pair on A. Then, in order that (R, D) is $\mathcal{F}$-complete, it is necessary and sufficient that the deduction relation $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ is equal to $\preccurlyeq$.

Proof Sufficiency has been shown in Theorem 9.6. Assume that(R, D) is $\mathcal{F}$-complete. Then $\vDash_{\mathsf{R},\mathsf{D}}$ is equal to $\vDash$ by Theorem 9.4. Since $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ is the largest latticed extension of $\vDash_{\mathsf{R},\mathsf{D}}$ by Theorem 5.2, while $\preccurlyeq$ is the largest latticed extension of $\vDash$ by our assumption, we conclude that $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ is equal to $\preccurlyeq$.

Theorem 9.11 Assume that the $\mathcal{F}$-validity relation $\preccurlyeq$ is equal to the largest latticed extension of the partial $\mathcal{F}$-validity relation $\vDash$. Let (R, D) be an $\mathcal{F}$-sound deduction pair on A and $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ be a deduction pair on $\vec{\mathsf{A}}$. Then the following conditions are equivalent.

(1) (R, D) and $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ satisfy the conditions (1) (2) of Theorem 9.7

(2) (R, D) is $\mathcal{F}$-complete and $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ is $\vec{\mathcal{F}}$-core-complete.

(3) $\vec{\mathsf{C}} = [\vec{\mathsf{D}}]_{\vec{\mathsf{R}}} = \vec{\mathsf{A}}_{\mathsf{R},\mathsf{D}}$.

Proof Assume (2). Then $\preccurlyeq$ is equal to $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ by Theorem 9.10, hence $\vec{\mathsf{C}} = \vec{\mathsf{A}}_{\mathsf{R},\mathsf{D}}$. Also, $\vec{\mathsf{C}} = [\vec{\mathsf{D}}]_{\vec{\mathsf{R}}}$ by the definition of $\vec{\mathcal{F}}$-core-completeness. Therefore (3) holds.

Assume (3). Then the conditions $\vec{\mathsf{C}} = [\vec{\mathsf{D}}]_{\vec{\mathsf{R}}}$ and $\vec{\mathsf{A}}_{\mathsf{R},\mathsf{D}} = [\vec{\mathsf{D}}]_{\vec{\mathsf{R}}}$ imply the conditions (1) and (2) of Theorem 9.7 respectively. Therefore (1) holds.

Assume (1). Then (R, D) is $\mathcal{F}$-complete by Theorem 9.7, and so $\vec{\mathsf{C}} = \vec{\mathsf{A}}_{\mathsf{R},\mathsf{D}}$ as shown in the first paragraph. Also $\vec{\mathsf{C}} \subseteq [\vec{\mathsf{D}}]_{\vec{\mathsf{R}}} \subseteq \vec{\mathsf{A}}_{\mathsf{R},\mathsf{D}}$ as shown in the proof of Theorem 9.7, hence $\vec{\mathsf{C}} = [\vec{\mathsf{D}}]_{\vec{\mathsf{R}}}$. Thus (2) holds.

Theorem 9.12 Assume that the $\mathcal{F}$-validity relation $\preccurlyeq$ is equal to the largest latticed extension of the partial $\mathcal{F}$-validity relation $\vDash$. Let (R, D) be an $\mathcal{F}$-sound deduction pair on A. Then (R, D) is $\mathcal{F}$-complete iff there exists a generational law $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ on the relations on A^* such that both $\preccurlyeq$ and $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ are the smallest of the relations on A^* which satisfy $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$.

Proof There exists some $\vec{\mathcal{F}}$-core-complete deduction pair $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ on $\vec{\mathsf{A}}$, and if (R, D) is $\mathcal{F}$-complete, then $\vec{\mathsf{C}} = [\vec{\mathsf{D}}]_{\vec{\mathsf{R}}} = \vec{\mathsf{A}}_{\mathsf{R},\mathsf{D}}$ by Theorem 9.11. Conversely if there exists a deduction pair $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ on $\vec{\mathsf{A}}$ such that $\vec{\mathsf{C}} = [\vec{\mathsf{D}}]_{\vec{\mathsf{R}}} = \vec{\mathsf{A}}_{\mathsf{R},\mathsf{D}}$, then (R, D) is $\mathcal{F}$-complete by Theorem 9.11. As noticed in §5.2, if $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ is a deduction pair on $\vec{\mathsf{A}}$, then $[\vec{\mathsf{D}}]_{\vec{\mathsf{R}}}$ regarded as a relation on A^* is the smallest of the relations on A^* which satisfy the generational law $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$. Therefore this theorem holds.

9.3 Boolean logical spaces

We continue consideration of the $\mathbb{B}$-valued functional logical space $(\mathsf{A}, \mathcal{F})$. Here we assume in addition that $(\mathsf{A}, \mathcal{F})$ is a **Boolean logical space** in the following sense:

(1) $\mathbb{B}$ is a Boolean lattice with respect to the meet $\wedge$, join $\vee$, complement $\Diamond$, and implication $\Rightarrow$.

(2) $\mathsf{x} \wedge \mathsf{y}$, $\mathsf{x} \vee \mathsf{y}$, $\mathsf{x}^{\Diamond}$, $\mathsf{x} \Rightarrow \mathsf{y}$ are global operations on A.

(3) every element of $\mathcal{F}$ is a Boolean representation of A on $\mathbb{B}$ with respect to the operations $\wedge, \vee, \Diamond, \Rightarrow$.

In particular, if $\mathbb{B} = \mathbb{T}$, then we call $(\mathsf{A}, \mathcal{F})$ a **binary logical space**. It is known that every Boolean logical space is "equivalent" to a binary logical space.

Theorem 9.13 The $\mathcal{F}$-validity relation $\preccurlyeq$ is a Boolean relation with respect to the operations $\wedge, \vee, \Diamond, \Rightarrow$, and is the largest latticed extension of the partial $\mathcal{F}$-validity relation $\vDash$. If $\mathcal{F} \neq \emptyset$, then $\preccurlyeq$ is non-trivial.

Proof Since each f-validity relation $\preccurlyeq_{\mathsf{f}}$ is Boolean by Theorem 3.9, so is $\preccurlyeq$ by Theorem 3.32. In particular, $\preccurlyeq$ satisfies the quasi-disjunction law. It also satisfies the lower negation law and conjunction law, and so satisfies the lower end law by Theorems 3.12. By using the cut law and weakening law, we conclude that $\preccurlyeq$ satisfies the inf-emptiness law. Therefore, $\preccurlyeq$ is the largest latticed extension of $\vDash$ by Theorem 9.9. If $\mathcal{F} \neq \emptyset$ and $\preccurlyeq$ is the trivial relation, then so is each f-validity relation $\preccurlyeq_{\mathsf{f}}$, and so $\varepsilon \preccurlyeq_{\mathsf{f}} \varepsilon$ holds, which implies that $1 = 0$, hence $\#\mathbb{B} = 1$ contrary to our assumption.

Because of Theorem 9.13, Theorems 9.10 - 9.12 apply to $(\mathsf{A}, \mathcal{F})$. In particular, we have the following.

Theorem 9.14 Let (R, D) be a deduction pair on A. Then (R, D) is $\mathcal{F}$-complete iff the deduction relation $\preccurlyeq_{R,D}$ is equal to the $\mathcal{F}$-validity relation $\preccurlyeq$. Therefore, (R, D) is $\mathcal{F}$-complete only if $\preccurlyeq_{R,D}$ is a Boolean relation with respect to the operations $\wedge, \vee, \lozenge, \Rightarrow$.

Proof Since $\preccurlyeq$ is the largest latticed extension of the partial $\mathcal{F}$-validity relation $\vDash$ by Theorem 9.13, the former assertion holds by Theorem 9.10. The latter assertion is a consequence of the former and Theorem 9.13.

Theorem 9.15 Let (R, D) be an $\mathcal{F}$-core-sufficient deduction pair on A and assume that R contains the modus ponens $\dfrac{x \quad x \Rightarrow y}{y}$. Then (R, D) is $\mathcal{F}$-sufficient.

Proof Define the mapping $\phi \in A^* \times A \to A$ by

$$\phi(x_1 \cdots x_n, y) = x_n \Rightarrow (\cdots \Rightarrow (x_2 \Rightarrow (x_1 \Rightarrow y)) \cdots),$$

where the right-hand side with $n = 0$ means y. Then since the $\mathcal{F}$-validity relation $\preccurlyeq$ is Boolean by Theorem 9.13 and the partial $\mathcal{F}$-validity relation $\vDash$ is the restriction of $\preccurlyeq$ to $A^* \times A$, Theorem 3.14 shows that

$$x_1 \cdots x_n \vDash y \iff \varepsilon \vDash \phi(x_1 \cdots x_n, y)$$

holds. In particular, ϕ satisfies

$$\alpha \vDash y \implies \varepsilon \vDash \phi(\alpha, y)$$

for every element $(\alpha, y) \in A^* \times A$. Assume $\varepsilon \vDash_{R,D} \phi(x_1 \cdots x_n, y)$. Then

$$x_n \Rightarrow (\cdots \Rightarrow (x_2 \Rightarrow (x_1 \Rightarrow y)) \cdots) \in [D]_R \subseteq [\{x_1, \ldots, x_n\} \cup D]_R.$$

Since R contains the modus ponens, it inductively follows that

$$x_{n-i} \Rightarrow (\cdots \Rightarrow (x_2 \Rightarrow (x_1 \Rightarrow y)) \cdots) \in [\{x_1, \ldots, x_n\} \cup D]_R$$

for $i = 1, \ldots, n$. Hence $y \in [\{x_1, \ldots, x_n\} \cup D]_R$, that is, $x_1 \cdots x_n \vDash_{R,D} y$. Thus ϕ satisfies

$$\varepsilon \vDash_{R,D} \phi(\alpha, y) \implies \alpha \vDash_{R,D} y.$$

for every element $(\alpha, y) \in A^* \times A$. The result now follows from Theorem 9.5.

Corollary 9.15.1 The deduction pair $(\wp, C)$ on A consisting of the modus ponens $\wp = \dfrac{x \quad x \Rightarrow y}{y}$ and the $\mathcal{F}$-core C is $\mathcal{F}$-complete.

Proof This is because $(\wp, C)$ is $\mathcal{F}$-sufficient by Theorem 9.15 and $\mathcal{F}$-sound by Theorems 7.1, 9.2, 9.13, and 3.14.

10 Deduction pairs and Boolean relations

Throughout this section, we let A be a non-empty set, and $x \wedge y$, $x \vee y$, $x^{\Diamond}$, $x \Rightarrow y$ be global operations on A, and (R, D) be a deduction pair on A.

In view of Theorem 9.14, here we consider the condition which (R, D) has to satisfy in order for the deduction relation $\preccurlyeq_{R,D}$ to be a Boolean relation or a weakly Boolean relation with respect to the operations $\wedge, \vee, \Diamond, \Rightarrow$.

Since $\preccurlyeq_{R,D}$ is a latticed relation by Theorem 5.2, we may apply the results in §3.2 to $\preccurlyeq_{R,D}$. In particular, Theorem 3.17 shows that $\preccurlyeq_{R,D}$ is Boolean iff it is weakly Boolean . Therefore, we only need to consider the condition which (R, D) has to satisfy in order for $\preccurlyeq_{R,D}$ to satisfy the strong cut law, junction laws, negation laws, and implication laws.

We will abbreviate $\big[\{x_1, \ldots, x_m\} \cup D\big]_R$ to $[x_1, \ldots, x_m, D]_R$. Thus

$$\begin{aligned} x_1 \cdots x_m \preccurlyeq_{R,D} y_1 \cdots y_n &\iff [x_1, \ldots, x_m, D]_R \supseteq [y_1, D]_R \cap \cdots \cap [y_n, D]_R, \\ x_1 \cdots x_m \preccurlyeq_{R,D} y &\iff [x_1, \ldots, x_m, D]_R \ni y, \\ \varepsilon \preccurlyeq_{R,D} y &\iff [D]_R \ni y, \end{aligned}$$

and Theorem 5.1 shows that the following holds:

$$x_1 \cdots x_m \, R \, y \implies x_1 \cdots x_m \preccurlyeq_{R,D} y.$$

The following logics $\wp$ and $\&$ on A will play important roles:

$$\wp = \frac{x \quad x \Rightarrow y}{y}, \qquad\qquad \& = \frac{x \quad y}{x \wedge y}.$$

The logic $\wp$ is the modus ponens.

If $(A, \mathcal{F})$ is a Boolean logical space, then the $\mathcal{F}$-validity relation $\preccurlyeq$ is a Boolean relation by Theorem 9.13, and so both $\wp$ and $\&$ are contained in $\preccurlyeq$ by Theorem 3.14 and the definition of the Boolean relations. Therefore by Theorem 9.2, both $\wp$ and $\&$ are $\mathcal{F}$-sound.

Lemma 10.1 If R contains $\wp$, then $\preccurlyeq_{R,D}$ satisfies the following laws:

$$x, x \Rightarrow y \preccurlyeq_{R,D} y, \qquad \text{(cut-implication law)}$$

$$\alpha \preccurlyeq_{R,D} x \Rightarrow y \implies x\alpha \preccurlyeq_{R,D} y. \qquad \text{(reverse deduction law)}$$

Proof Since $x, x \Rightarrow y \, \wp \, y$ and $\wp \subseteq R$, we have $x, x \Rightarrow y \, R \, y$, hence the law (1) holds. The law (2) may be derived from (1) by the cut law and exchange law.

Theorem 10.1 Assume that R contains $\wp$. Then $\preccurlyeq_{R,D}$ satisfies

$$x\alpha \preccurlyeq_{R,D} y \implies \alpha \preccurlyeq_{R,D} x \Rightarrow y \qquad \text{(deduction law)}$$

iff $\preccurlyeq_{R,D}$ satisfies the following laws:

(1) $\varepsilon \preccurlyeq_{R,D} x \Rightarrow x$.

(2) $y \preccurlyeq_{R,D} x \Rightarrow y$ (the second implication law in Theorem 3.8).

(3) If $z_1 \cdots z_k \, R \, y$ $(k \geq 0)$, then $x \Rightarrow z_1, \cdots, x \Rightarrow z_k \preccurlyeq_{R,D} x \Rightarrow y$.

Proof Assume that $\preccurlyeq_{R,D}$ satisfies the deduction law. Then since $\preccurlyeq_{R,D}$ satisfies $x \preccurlyeq_{R,D} x$ and $xy \preccurlyeq_{R,D} y$ by the repetition law and weakening law, (1) and (2) hold. In order to prove (3), assume $z_1 \cdots z_k \, R \, y$. Then we have

$$z_1 \cdots z_k \preccurlyeq_{R,D} y.$$

If $k = 0$, this becomes $\varepsilon \preccurlyeq_{R,D} y$, hence $\varepsilon \preccurlyeq_{R,D} x \Rightarrow y$ as desired by the weakening law and deduction law. Therefore assume $k \geq 1$. Then, since $x, x \Rightarrow z_i \preccurlyeq_{R,D} z_i$ by Lemma 10.1, we have

$$x, x \Rightarrow z_1, \cdots, x \Rightarrow z_k \preccurlyeq_{R,D} z_i \qquad (i = 1, \ldots, k)$$

by the weakening law and exchange law. By repeated applications of the cut law, exchange law, and contraction law to the above $k+1$ displayed relations, we have $x, x \Rightarrow z_1, \cdots, x \Rightarrow z_k \preccurlyeq_{R,D} y$, hence $x \Rightarrow z_1, \cdots, x \Rightarrow z_k \preccurlyeq_{R,D} x \Rightarrow y$ as desired.

Conversely, assume that $\preccurlyeq_{R,D}$ satisfies (1) - (3), and in order to prove that $\preccurlyeq_{R,D}$ satisfies the deduction law, assume $x\alpha \preccurlyeq_{R,D} y$. Furthermore, let $\alpha = x_1 \cdots x_m$ and define $X = \{x, x_1, \ldots, x_m\} \cup D$. Then, since $x\alpha \preccurlyeq_{R,D} y$, we have $[X]_R \ni y$, and so by Theorem 4.6, y belongs to the R-descendant X_n of X for some non-negative integer n. We will show $\alpha \preccurlyeq_{R,D} x \Rightarrow y$ by induction on n.

Assume $n = 0$. Then $y \in X$. If $y = x$, then (1) and the weakening law certainly imply $\alpha \preccurlyeq_{R,D} x \Rightarrow y$. Therefore assume $y \in \{x_1, \ldots, x_m\} \cup D$. Then $\alpha \preccurlyeq_{R,D} y$, and since $y \preccurlyeq_{R,D} x \Rightarrow y$ by (2), we have $\alpha \preccurlyeq_{R,D} x \Rightarrow y$ as desired by the cut law.

Assume $n \geq 1$. Then, since $y \in X_n$, we have $z_1 \cdots z_k \, R \, y$ for some elements $z_i \in X_{l_i}$ $(i = 1, \ldots, k)$ with $n = 1 + \sum_{i=1}^{k} l_i$. Therefore

$$x \Rightarrow z_1, \cdots, x \Rightarrow z_k \preccurlyeq_{R,D} x \Rightarrow y$$

by (3), and

$$\alpha \preccurlyeq_{R,D} x \Rightarrow z_i \qquad (i = 1, \ldots, k)$$

by the induction hypothesis. By repeated applications of the cut law, exchange law, and contraction law to the above $k+1$ displayed relations, we have $\alpha \preccurlyeq_{R,D} x \Rightarrow y$ as desired.

Lemma 10.2 If $\preccurlyeq_{R,D}$ satisfies the following laws, then $\preccurlyeq_{R,D}$ satisfies the laws (1) and (2) of Theorem 10.1:

(1) $\varepsilon \preccurlyeq_{R,D} x^{\Diamond} \vee x$.

(2) $y \preccurlyeq_{R,D} x \vee y$.

(3) $x^{\Diamond} \vee y \preccurlyeq_{R,D} x \Rightarrow y$.

Proof Applying the cut law to (1) and (3) with $x = y$, we have $\varepsilon \preccurlyeq_{R,D} x \Rightarrow x$. Applying the cut law to (2) with x replaced by $x^{\Diamond}$ and (3), we have $y \preccurlyeq_{R,D} x \Rightarrow y$.

Lemma 10.3 If R contains &, then $\preccurlyeq_{R,D}$ satisfies the following laws:

(1) $xy \preccurlyeq_{R,D} x \wedge y$.

(2) $x \wedge y, \beta \preccurlyeq_{R,D} \alpha \implies xy\beta \preccurlyeq_{R,D} \alpha$.

Proof Since $xy \mathbin{\&} x \wedge y$ and $\& \subseteq R$, we have $xy \mathrel{R} x \wedge y$, hence (1) holds. Applying the cut law to (1) and the premise of (2), we have the conclusion of (2).

Theorem 10.2 Assume that R contains $\wp \cup \&$ and $\preccurlyeq_{R,D}$ satisfies the following laws. Then $\preccurlyeq_{R,D}$ is a Boolean relation with respect to the operations $\wedge, \vee, \Diamond, \Rightarrow$:

(1) $\varepsilon \preccurlyeq_{R,D} x^{\Diamond} \vee x$,

(2) $x \wedge y \preccurlyeq_{R,D} x$,

(3) $x \wedge y \preccurlyeq_{R,D} y$,

(4) $x \preccurlyeq_{R,D} x \vee y$,

(5) $y \preccurlyeq_{R,D} x \vee y$,

(6) $x^{\Diamond} \vee y \preccurlyeq_{R,D} x \Rightarrow y$,

(7) $x \vee y, x \Rightarrow z, y \Rightarrow z \preccurlyeq_{R,D} z$,

(8) If $z_1 \cdots z_k \mathrel{R} y$ $(k \geq 0)$, then $x \Rightarrow z_1, \cdots, x \Rightarrow z_k \preccurlyeq_{R,D} x \Rightarrow y$.

Proof The $\preccurlyeq_{R,D}$ is a latticed relation by Theorem 5.2 and satisfies the second implication law by Lemma 10.2. Therefore, we need to show that $\preccurlyeq_{R,D}$ satisfies the strong cut law, junction laws, negation laws, first implication law, and third implication law.

We note that $\preccurlyeq_{R,D}$ is the largest latticed extension of $\vDash_{R,D}$ by Theorem 5.2 and so $\preccurlyeq_{R,D}$ satisfies the inf-emptiness law by Theorem 3.20. Also, $\preccurlyeq_{R,D}$ satisfies the cut-implication law, reverse deduction law, and deduction law by Lemmas 10.1 and 10.2 and Theorem 10.1.

The laws (2), (3) and Lemma 10.3 show that $\preccurlyeq_{R,D}$ satisfies the conjunction law.

Since $\preccurlyeq_{R,D}$ satisfies the strong disjunction law by Theorem 3.24, $\preccurlyeq_{R,D}$ satisfies the disjunction law by Theorem 3.12.

We have $x^{\Diamond} \preccurlyeq_{R,D} x^{\Diamond} \vee y$ and $x^{\Diamond} \vee y \preccurlyeq_{R,D} x \Rightarrow y$ by (4) and (6), hence $x^{\Diamond} \preccurlyeq_{R,D} x \Rightarrow y$ by the cut law. Thus the first implication law holds.

Applying the reverse deduction law to the first implication law, we have $xx^{\Diamond} \preccurlyeq_{R,D} y$. Since y is arbitrary and $\preccurlyeq_{R,D}$ satisfies the inf-emptiness law, it follows that $xx^{\Diamond} \preccurlyeq_{R,D} \varepsilon$. Thus the lower negation law holds.

Since $\varepsilon \preccurlyeq_{R,D} x^{\Diamond} \vee x$ by (1) and $\preccurlyeq_{R,D}$ satisfies the disjunction law, we have $\varepsilon \preccurlyeq_{R,D} x^{\Diamond}x$ by Theorem 3.12, hence $\varepsilon \preccurlyeq_{R,D} xx^{\Diamond}$ by the exchange law. Thus the upper negation law holds.

Since $\preccurlyeq_{R,D}$ satisfies the junction laws and negation laws, it follows from Theorem 3.12 that $\preccurlyeq_{R,D}$ satisfies the end laws. As shown above, $\preccurlyeq_{R,D}$ satisfies the conjunction law and strong disjunction law. Therefore by Theorem 3.26, $\preccurlyeq_{R,D}$ satisfies the strong cut law.

Since $\preccurlyeq_{R,D}$ satisfies the strong cut law, negation laws, and cut-implication law by Lemma 10.1, Theorem 3.14 shows that $\preccurlyeq_{R,D}$ satisfies the third implication law.

Corollary 10.2.1 Assume that R contains $\wp \cup \&$ and that $[D]_R$ contains all of the elements of A in the following shape. Then $\preccurlyeq_{R,D}$ is a Boolean relation with respect to the operations $\wedge, \vee, \Diamond, \Rightarrow$:

(1) $x^{\Diamond} \vee x$.

(2) $(x \wedge y) \Rightarrow x$.

(3) $(x \wedge y) \Rightarrow y$.

(4) $x \Rightarrow (x \vee y)$.

(5) $y \Rightarrow (x \vee y)$.

(6) $(x^{\Diamond} \vee y) \Rightarrow (x \Rightarrow y)$.

(7) $\big((x \Rightarrow z) \wedge (y \Rightarrow z)\big) \Rightarrow \big((x \vee y) \Rightarrow z\big)$.

(8) $\big((x \Rightarrow z_1) \wedge \cdots \wedge (x \Rightarrow z_k)\big) \Rightarrow (x \Rightarrow y)$ where $z_1 \cdots z_k \, R \, y$.

Proof Since R contains $\wp \cup \&$, we may use the reverse deduction law of Lemma 10.1 and the laws of Lemma 10.3. Also, since

$$\big((x \Rightarrow z) \wedge (y \Rightarrow z)\big) \Rightarrow \big((x \vee y) \Rightarrow z\big) \in [D]_R,$$

we have

$$\varepsilon \preccurlyeq_{R,D} \big((x \Rightarrow z) \wedge (y \Rightarrow z)\big) \Rightarrow \big((x \vee y) \Rightarrow z\big).$$

Therefore, we conclude that $\preccurlyeq_{R,D}$ satisfies the law $x \vee y, x \Rightarrow z, y \Rightarrow z \preccurlyeq_{R,D} z$. Similarly, we see that $\preccurlyeq_{R,D}$ satisfies all the laws listed in Theorem 10.2. Therefore $\preccurlyeq_{R,D}$ is a Boolean relation.

Corollary 10.2.2 Assume that $R = \wp \cup \&$ and $[D]_R$ contains all of the elements of A in the following shape. Then $\preccurlyeq_{R,D}$ is a Boolean relation with respect to the operations $\wedge, \vee, \Diamond, \Rightarrow$:

(1) $x^{\Diamond} \vee x$.

(2) $(x \wedge y) \Rightarrow x$.

(3) $(x \wedge y) \Rightarrow y$.

(4) $x \Rightarrow (x \vee y)$.

(5) $y \Rightarrow (x \vee y)$.

(6) $(x^{\Diamond} \vee y) \Rightarrow (x \Rightarrow y)$.

(7) $\big((x \Rightarrow z) \wedge (y \Rightarrow z)\big) \Rightarrow \big((x \vee y) \Rightarrow z\big)$.

(8) $\big((z \Rightarrow x) \wedge (z \Rightarrow y)\big) \Rightarrow \big(z \Rightarrow (x \wedge y)\big)$.

(9) $\big((z \Rightarrow x) \wedge \big(z \Rightarrow (x \Rightarrow y)\big)\big) \Rightarrow (z \Rightarrow y)$.

Proof Since $R = \wp \cup \&$, we have $z_1 \cdots z_k \, R \, y$ iff $z_1 \cdots z_k \, \wp \, y$ or $z_1 \cdots z_k \, \& \, y$. Therefore, the elements $\big((x \Rightarrow z_1) \wedge \cdots \wedge (x \Rightarrow z_k)\big) \Rightarrow (x \Rightarrow y)$ on the list of Corollary 10.2.1 are the elements

$$\big((x \Rightarrow y) \wedge \big(x \Rightarrow (y \Rightarrow z)\big)\big) \Rightarrow (x \Rightarrow z),$$
$$\big((x \Rightarrow y) \wedge (x \Rightarrow z)\big) \Rightarrow \big(x \Rightarrow (y \wedge z)\big).$$

This completes the proof.

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