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Analysis and Estimation of Error Constants for P_0 and P_1 Interpolations over Triangular Finite Elements

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Abstract

We give some fundamental results on the error constants for the piecewise constant interpolation function and the piecewise linear one over triangles. For the piecewise linear one, we mainly analyze the conforming case, but the present results also appear to be available for the non-conforming case. We obtain explicit relations for the upper bounds of the constants, and analyze dependence of such constants on the geometric parameters of triangles. In particular, we explicitly determine some special constants including the Babuška-Aziz constant, which plays an essential role in the interpolation error estimation of the linear triangular finite element. The obtained results are expected to be widely used for a priori and a posteriori error estimations in adaptive computation and numerical verification of numerical solutions based on the triangular finite elements. We also give some numerical results for the error constants and for a posteriori estimates of some eigenvalues related to the error constants.

Keywords : FEM, error estimates, triangular finite elements, interpolation error constants.
Mathematical Subject Classification 2000 : 65N15, 65N30

1 Introduction

The finite element method (FEM) is now recognized as a powerful numerical method for wide classes of partial differential equations. Furthermore, it also has sound mathematical bases such as highly refined a priori and a posteriori error estimations. In the classical a priori error analysis of FEM, the interpolation error analysis is essential to derive final error estimates in various norms and/or semi-norms [10, 11, 21]. In this process, there appear a number of positive constants besides the standard discretization (or mesh) parameter h and norms (or seminorms), but it has been very difficult to evaluate such constants explicitly. For quantitative purposes, however, it is indispensable to evaluate or bound them as accurately as possible, because sharper estimates enable more efficient finite element computations. Thus such evaluation has become progressively more important and has been attempted especially for adaptive finite element calculations based on a posteriori error estimation as well as for numerical verification by FEM [3, 6, 8, 10, 25]. In this paper, we will give some fundamental results on various interpolation error constants of the most popular triangular finite elements.

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More specifically, we derive some fundamental estimates for the interpolation error constants appearing in the popular P_0 (piecewise constant) and P_1 (piecewise linear) triangular finite elements. Inspired by the monumental paper of Babuška-Aziz [5], we analyze the dependence of several constants on the geometric parameters such as the maximum interior angle and the minimum edge length of a triangle more quantitatively than works precedent to ours. Among them, the optimal constant (C_4 in the present paper) appearing in the H^1 error estimate of the P_1 interpolation of H^2 functions over the unit isosceles right triangle is essential and frequently used, and it was explicitly evaluated firstly by Natterer [27]. On the other hand, this constant was shown to be closely related to the one (C_1 in this paper) presented and effectively used by Babuška and Aziz in conjunction with the maximum angle condition [5]. More precisely, C_1 gives an upper bound quite close to the optimal constant C_4, and the relation between C_4 and C_1 was further discussed in [25, 30]. Thus a precise estimation of these two constants is very important, and a number of researchers have given bounds for these using various approximation methods including numerical verification, see e. g. [4, 7, 22, 25, 26, 27, 30]. Furthermore, these constants can be also used to evaluate the interpolation error constants for the non-conforming P_1 triangle, as was mentioned in [19].

For the above Babuška-Aziz constant, we already succeeded in obtaining a value which is in a sense optimal [18]. That is, by analytically solving an eigenvalue problem for the 2D Laplacian over the above triangular domain, we showed that the constant can be easily determined from a solution of the simple transcendental equation $\mu^{-1} + \tan(\mu^{-1}) = 0$. In this process, we used the reflection (or symmetry) method [28]. In this paper, we will also give some additional results for exact values or bounds of various error constants. Moreover, we will present some explicit relations for the dependence of such constants on the geometry of triangles. In particular, emphasis is put on the maximum angle condition presented in [5]. We also give some analytical results based on asymptotic analysis with regards to the behaviors when a right triangle becomes very thin or slender. Such behaviors can be important for example in anisotropic triangulations, cf. [8].

Thus our results can be effectively used in the quantitative a priori and a posteriori error estimations of the finite element solutions by the P_1 triangular element and also those based on the P_0 triangle. The former is of course the most classical and fundamental one, but still in frequent use, while the latter appears in some mixed finite element methods and implicitly on various occasions. Moreover, we also give some concrete a posteriori error estimates to eigenvalues related to several error constants. Numerical results are also obtained for the error constants and a posteriori estimates of some eigenvalues.

The plan of this paper is as follows. Section 1 is the present one on some historical remarks and overview of our analysis. Section 2 gives necessary notations and concepts, and also introduces various error constants to be analyzed. Section 3 deals with estimation of various interpolation error constants, and Section 4 analyzes asymptotic behaviors of the error constants when the triangle is a thin right one. Section 5 gives application of our results to a posteriori estimation of some error constants by using the P_1 FEM. Section 6 is the one for numerical results, and Section 7 gives some concluding remarks and acknowledgements. Appendix is also attached to give some additional theoretical and numerical results related to Section 4.

2 Preliminaries: error constants

Let h, α and θ be positive constants such that

$$h > 0\ ,\ \ 0 < \alpha \leq 1\ ,\ \ (\frac{\pi}{3} \leq) \cos^{-1}\frac{\alpha}{2} \leq \theta < \pi\ . \tag{1}$$

Then we can define the triangle $T_{\alpha,\theta,h}$ by $\triangle OAB$ with three vertices $O(0,0)$, $A(h,0)$ and $B(\alpha h\cos\theta, \alpha h\sin\theta)$. From (1), AB turns out to be the edge of maximum length, i. e. $\overline{AB} \geq h \geq \alpha h$ with $h = \overline{OA}$ and $\alpha h = \overline{OB}$ being the medium and the minimum edge lengths, respectively. It is to be noted here that the notation h is mostly used as the largest edge length in standard textbooks such as [11], but our usage of h as the medium one may be convenient for the present purposes. A point on the closure of $T_{\alpha,\theta,h}$ is denoted by $x = \{x_1, x_2\}$, and the three edges e_1, e_2 and e_3 of $T_{\alpha,\theta,h}$ are defined as

$$e_1 = OA\,,\ \ e_2 = OB\,,\ \ e_3 = AB\,. \tag{2}$$

We can configure any triangle as $T_{\alpha,\theta,h}$ with some α, θ and h, by using an appropriate congruent transformation in $\mathbf{R}^2$. As the usage in [5], we will use abbreviated notations $T_{\alpha,\theta} = T_{\alpha,\theta,1}$, $T_\alpha = T_{\alpha,\pi/2}$ and $T = T_1$ (Fig. 1).

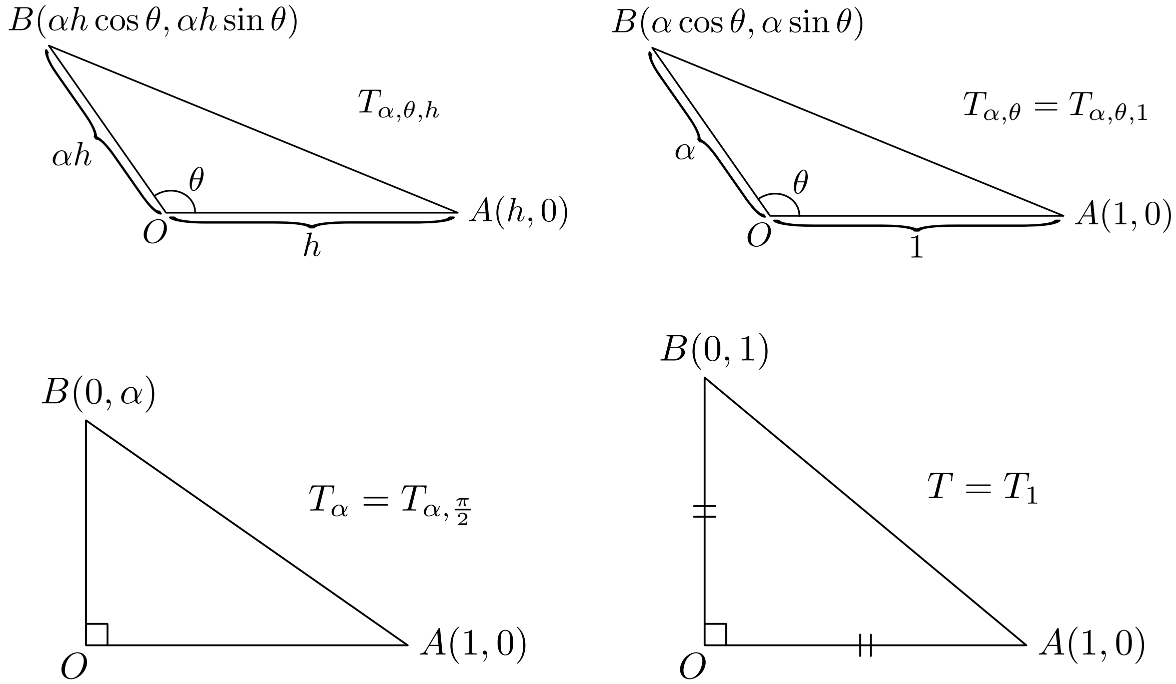


Figure 1: Notations for triangles : $T_{\alpha,\theta} = T_{\alpha,\theta,1}$, $T_\alpha = T_{\alpha,\pi/2}$, $T = T_1$

We will use the popular Hilbert space $L_2(T_{\alpha,\theta,h})$, and denote its norm by $\|\cdot\|_{T_{\alpha,\theta,h}}$, where the subscript $T_{\alpha,\theta,h}$ is often omitted if there is no fear of confusion. When we need to use the L_2 space and its norm for other domains such as Ω, we will use notations like $L_2(\Omega)$ and $\|\cdot\|_\Omega$.

Let us define the following closed linear spaces for functions over $T_{\alpha,\theta,h}$:

$$V^0_{\alpha,\theta,h} = \{v \in H^1(T_{\alpha,\theta,h}) \mid \int_{T_{\alpha,\theta,h}} v(x)\,dx = 0\}, \tag{3}$$

$$V^i_{\alpha,\theta,h} = \{v \in H^1(T_{\alpha,\theta,h}) \mid \int_{e_i} v\,ds = 0\} \quad (i = 1, 2, 3), \tag{4}$$

$$V^4_{\alpha,\theta,h} = \{v \in H^2(T_{\alpha,\theta,h}) \mid v(O) = v(A) = v(B) = 0\}, \tag{5}$$

where $H^1(T_{\alpha,\theta,h})$ and $H^2(T_{\alpha,\theta,h})$ are respectively the first- and second-order Sobolev spaces for real square integrable functions over $T_{\alpha,\theta,h}$ [2], and ds is the line element. For other domains like Ω, we will also use spaces such as $H^1(\Omega)$ and $H^2(\Omega)$ later. For the above spaces, we will again use abbreviated notations $V^i_{\alpha,\theta} = V^i_{\alpha,\theta,1}$, $V^i_\alpha = V^i_{\alpha,\pi/2}$ and $V^i = V^i_1$ $(0 \le i \le 4)$.

Let us consider the usual P_0 interpolation operator $\Pi^0_{\alpha,\theta,h}$ and P_1 one $\Pi^1_{\alpha,\theta,h}$ for functions on $T_{\alpha,\theta,h}$ [10, 11, 21]: $\Pi^0_{\alpha,\theta,h}v$ for $\forall v \in H^1(T_{\alpha,\theta,h})$ is a constant function well-defined by

$$(\Pi^0_{\alpha,\theta,h}v)(x) = \int_{T_{\alpha,\theta,h}} v(y)\,dy \Big/ \int_{T_{\alpha,\theta,h}} dy \quad (\forall x \in T_{\alpha,\theta,h}) \;, \tag{6}$$

while $\Pi^1_{\alpha,\theta,h}v$ for $\forall v \in H^2(T_{\alpha,\theta,h})$ is an at most linear polynomial function such that

$$(\Pi^1_{\alpha,\theta,h}v)(x) = v(x) \;\text{ for }\; x = O,\, A,\, B\,. \tag{7}$$

To give error estimates for these interpolation operators, it is natural to evaluate the positive constants defined by

$$C_i(\alpha,\theta,h) = \sup_{v \in V^i_{\alpha,\theta,h}\setminus\{0\}} \frac{\|v\|}{|v|_1} \quad (i = 0, 1, 2, 3)\,, \tag{8}$$

$$C_4(\alpha,\theta,h) = \sup_{v \in V^4_{\alpha,\theta,h}\setminus\{0\}} \frac{|v|_1}{|v|_2}\,, \tag{9}$$

$$C_5(\alpha,\theta,h) = \sup_{v \in V^4_{\alpha,\theta,h}\setminus\{0\}} \frac{\|v\|}{|v|_2}\,, \tag{10}$$

where $|v|_1 = (\sum_{i=1}^2 \|\partial v/\partial x_i\|^2)^{1/2}$, and $|v|_2 = (\sum_{i,j=1}^2 \|\partial^2 v/\partial x_i \partial x_j\|^2)^{1/2}$. When we need to specify a domain like Ω for the above semi-norms $|\cdot|_1$ and $|\cdot|_2$, we will use $|\cdot|_{1,\Omega}$ and $|\cdot|_{2,\Omega}$, respectively. The existence of these positive constants follows from the Rellich compactness theorem. Due to the properties to become clear soon, such constants together with some related ones are often called *interpolation error constants*. We will again use abbreviated notations $C_i(\alpha,\theta) = C_i(\alpha,\theta,1)$, $C_i(\alpha) = C_i(\alpha,\pi/2)$ and $C_i = C_i(1)$ for $0 \le i \le 5$.

By a simple scale change, we find that $C_i(\alpha,\theta,h) = hC_i(\alpha,\theta)$ $(i = 0,1,2,3,4)$ and $C_5(\alpha,\theta,h) = h^2 C_5(\alpha,\theta)$. These relations and constants are used to derive popular interpolation error estimates for $\Pi^i_{\alpha,\theta,h}$ $(i = 0, 1)$ applied to functions on $T_{\alpha,\theta,h}$ [10, 11, 21]:

$$\|v - \Pi^0_{\alpha,\theta,h}v\| \le C_0(\alpha,\theta)h|v|_1 \qquad ; \forall v \in H^1(T_{\alpha,\theta,h}), \tag{11}$$

$$|v - \Pi^1_{\alpha,\theta,h}v|_1 \le C_4(\alpha,\theta)h|v|_2 \qquad ; \forall v \in H^2(T_{\alpha,\theta,h}), \tag{12}$$

$$\|v - \Pi^1_{\alpha,\theta,h}v\| \le C_5(\alpha,\theta)h^2|v|_2 \qquad ; \forall v \in H^2(T_{\alpha,\theta,h}), \tag{13}$$

where we have used the facts that $v - \Pi^0_{\alpha,\theta,h}v \in V^0_{\alpha,\theta,h}$ for $v \in H^1(T_{\alpha,\theta,h})$ and $v - \Pi^1_{\alpha,\theta,h}v \in V^4_{\alpha,\theta,h}$ for $v \in H^2(T_{\alpha,\theta,h})$.

Moreover, for the partial derivative $\partial v/\partial x_1$ of $v \in H^2(T_{\alpha,\theta,h})$, we have

$$\left\|\frac{\partial(v - \Pi^1_{\alpha,\theta,h}v)}{\partial x_1}\right\| \le C_1(\alpha,\theta)h\left|\frac{\partial v}{\partial x_1}\right|_1 , \tag{14}$$

since $\partial(v - \Pi^1_{\alpha,\pi/2,h}v)/\partial x_1 \in V^1_{\alpha,\theta,h}$. On the other hand, to obtain an estimate in terms of $C_2(\alpha,\theta)$, we introduce rotation of the x_1-x_2 plane around the origin O by angle $\theta - \pi/2$ so that the edge OB becomes the ordinate. Then the coordinate transformation $\hat{x} = \Phi_\theta(x)$ between the original variable $x = \{x_1, x_2\}$ and the new one $\hat{x} = \{\hat{x}_1, \hat{x}_2\}$ is given by, together with the associated transformation $\hat{v} = v \circ \Phi_\theta^{-1}$ for $v \in H^2(T_{\alpha,\theta,h})$,

$$\hat{x}_1 = x_1\sin\theta - x_2\cos\theta\,,\quad \hat{x}_2 = x_1\cos\theta + x_2\sin\theta\,, \tag{15}$$

$$\hat{v}(\hat{x}) = v(x) = v(\hat{x}_1\sin\theta + \hat{x}_2\cos\theta, -\hat{x}_1\cos\theta + \hat{x}_2\sin\theta)\,. \tag{16}$$

Based on essentially the same arguments as for $\partial v/\partial x_1$, we can show for $\partial\hat{v}/\partial\hat{x}_2$ that

$$\left\|\frac{\partial(\hat{v} - \hat{\Pi}^1_{\alpha,\theta,h}\hat{v})}{\partial \hat{x}_2}\right\| \le C_2(\alpha,\theta)h\left|\frac{\partial \hat{v}}{\partial \hat{x}_2}\right|_1 , \tag{17}$$

where $\hat{\Pi}^1_{\alpha,\theta,h}$ is $\Pi^1_{\alpha,\theta,h}$ for the rotated $T_{\alpha,\theta,h}$. The above two estimates (14) and (17) are in a sense sharper than (12) as noted in [21]. Similar relation also holds for $C_3(\alpha,\theta)$.

Remark 1. *Refining the above arguments, we can obtain various* $anisotropic$ *error estimates* [8] *such as*

$$|v - \Pi^1_{\alpha,\theta,h}v|_1 \le h\sqrt{\sum_{i,j=1}^{2} c_{ij}\left\|\frac{\partial^2 v}{\partial x_i x_j}\right\|^2},$$

where c_{ij}*'s* $(1 \le i, j \le 2)$ *are constants similar to* $C_k(\alpha,\theta)$*'s* $(0 \le k \le 5)$ *and can take unequal values. We do not make such a refinement here, though it can be an interesting subject.*

Thus we can give quantitative interpolation estimates, provided that we succeed in evaluating or bounding the constants $C_i(\alpha,\theta)$'s explicitly. So we will try to bound these constants by fairly simple functions of α and θ. Notice here that each of such constants can be characterized by minimization of a kind of Rayleigh quotient. Then it is equivalent to finding the minimum eigenvalue of a certain eigenvalue problem expressed by a weak formulation, which is further expressed by a partial differential equation with some auxiliary conditions.

More specifically, we can characterize the constants $C_i(\alpha,\theta)$'s by minimization of Rayleigh's quotients $R^{(i)}_{\alpha,\theta}$'s :

$$C_i^{-2}(\alpha,\theta) = \inf_{v\in V^i_{\alpha,\theta}\backslash\{0\}} R^{(i)}_{\alpha,\theta}(v)\ ;\ \ R^{(i)}_{\alpha,\theta}(v) = \frac{|v|_1^2}{\|v\|^2} \qquad (i=0,1,2,3)\,, \tag{18}$$

$$C_4^{-2}(\alpha,\theta) = \inf_{v\in V^4_{\alpha,\theta}\backslash\{0\}} R^{(4)}_{\alpha,\theta}(v)\ ;\ \ R^{(4)}_{\alpha,\theta}(v) = \frac{|v|_2^2}{|v|_1^2}\,, \tag{19}$$

$$C_5^{-2}(\alpha,\theta) = \inf_{v\in V^4_{\alpha,\theta}\backslash\{0\}} R^{(5)}_{\alpha,\theta}(v)\ ;\ \ R^{(5)}_{\alpha,\theta}(v) = \frac{|v|_2^2}{\|v\|^2}\,, \tag{20}$$

where all notations and functions are for $T_{\alpha,\theta}$.

By the standard compactness arguments, each infimum above is actually a minimum, and is the smallest eigenvalue of a certain eigenvalue problem. For example, the eigenvalue problem associated with $C_0(\alpha,\theta)$ is to *find* $\lambda\in\mathbf{R}$ *and* $u\in V^0_{\alpha,\theta}\backslash\{0\}$ *that satisfy*

$$(\nabla u,\nabla v)_{T_{\alpha,\theta}} = \lambda(u,v)_{T_{\alpha,\theta}} \qquad (\forall v\in V^0_{\alpha,\theta})\,. \tag{21}$$

Here, $(\cdot,\cdot)_{T_{\alpha,\theta}}$ denotes the inner products of both $L_2(T_{\alpha,\theta})$ and $L_2(T_{\alpha,\theta})^2$, and ∇ is the gradient operator. When we consider the corresponding inner products for domains like Ω, we will use notations such as $(\cdot,\cdot)_\Omega$. The present eigenvalue problem is also expressed by a partial differential equation, a linear constraint for $V^0_{\alpha,\theta}$ and a boundary condition [25, 26]:

$$-\Delta u = \lambda u \ \text{ in } \ T_{\alpha,\theta}\,, \quad \int_{T_{\alpha,\theta}} u(x)\,dx = 0\,, \quad \frac{\partial u}{\partial n} = 0 \ \text{ on } \ \partial T_{\alpha,\theta}\,, \tag{22}$$

where $\frac{\partial}{\partial n}$ denotes the outward normal derivative on edges, and $\partial T_{\alpha,\theta}$ does the boundary of $T_{\alpha,\theta}$. The above boundary condition is the homogeneous Neumann one, and the desired minimum eigenvalue is the second (and positive) one for the same problem without the linear constraint.

For $C_1(\alpha,\theta)$, it is characterized in essentially the same fashion as (21), if the associated space $V^0_{\alpha,\theta}$ is replaced with $V^1_{\alpha,\theta}$. On the other hand, the equations corresponding to (22) become more complicated [25, 26]:

$$-\Delta u = \lambda u \ \text{ in } \ T_{\alpha,\theta}\,, \quad \int_0^1 u(x_1,0)\,dx_1 = 0\,, \quad \frac{\partial u}{\partial n} = \begin{cases} 0 & \text{on edges } OB \text{ and } AB\,, \\ c & \text{on edge } OA\,, \end{cases} \tag{23}$$

where c denotes an unknown constant to be decided simultaneously with u and λ. See also Section 5.3 of this paper.

The other constants are characterized similarly. For example, the eigenvalue problem associated to $C_4(\alpha,\theta)$ is to *find* $\lambda\in\mathbf{R}$ *and* $u\in V^4_{\alpha,\theta}\backslash\{0\}$ *that satisfy*

$$\sum_{i,j=1}^{2}(\partial^2 u/\partial x_i\partial x_j, \partial^2 v/\partial x_i\partial x_j)_{T_{\alpha,\theta}} = \lambda(\nabla u,\nabla v)_{T_{\alpha,\theta}} \qquad (\forall v\in V^4_{\alpha,\theta})\,. \tag{24}$$

But the partial differential equation related to the above and also that to $C_5(\alpha,\theta)$ are of fourth order ones with special linear constraints and boundary conditions, and are more difficult to deal with than the second order equations as in (22) and (23), cf. [4, 7]. Since $T_{\alpha,\theta}$ is a triangle, it is difficult to solve such eigenvalue problems explicitly even in the case of second-order equations, except in some rare cases to be shown later.

3 Estimation of interpolation error constants

It is in general difficult to obtain exact values of the error constants $C_i(\alpha,\theta)$'s except for very rare cases. In this section, we will first give some formulas to bound them in terms of their special values such as $C_i = C_i(1,\frac{\pi}{2},1)$'s. Such formula can be useful for various purposes, provided that the selected special values are evaluated with sufficient accuracy. So, we will also perform exact evaluation of some special constants.

3.1 Reconsideration of Natterer's results

Natterer [27] derived an upper bound formula for $C_4(\alpha,\theta)$ in terms of $C_4 = C_4(1,\frac{\pi}{2},1)$, α and θ. He also gave an upper bound for C_4, so that his formula has been effectively used in quantitative error estimates of finite element solutions including numerical verifications of various differential equations [25, 26]. Here we begin by applying his techniques to bound the error constants introduced in Section 2.

To this end, let us introduce the following simple affine transformation $\xi = \Psi_{\alpha,\theta}(x)$ between $x = \{x_1, x_2\} \in T_{\alpha,\theta}$ and $\xi = \{\xi_1,\xi_2\} \in T = T_{1,\frac{\pi}{2},1}$:

$$\xi_1 = x_1 - \frac{x_2\cos\theta}{\sin\theta}\,,\ \xi_2 = \frac{x_2}{\alpha\sin\theta}\quad;\quad x_1 = \xi_1 + \alpha\xi_2\cos\theta\,,\ x_2 = \alpha\xi_2\sin\theta\,. \tag{25}$$

By eigenvalue analysis of matrices resulting from the above transformation in the Rayleigh quotients from (18) through (20), we obtain the following results.

Theorem 1. *For $\alpha \in]0,+\infty[$ and $\theta \in]0,\pi[$, $C_i(\alpha,\theta)$'s are bounded as*

$$\psi_i(\alpha,\theta)C_i \le C_i(\alpha,\theta) \le \phi_i(\alpha,\theta)C_i \quad (0\le i\le 5)\,, \tag{26}$$

where $C_i = C_i(1,\frac{\pi}{2})$ $(0 \le i \le 5)$,

$$\psi_i(\alpha,\theta) = \sqrt{\frac{\nu_-(\alpha,\theta)}{2}}\quad (0\le i\le 3)\,,\ \psi_4(\alpha,\theta) = \frac{\nu_-(\alpha,\theta)}{\sqrt{2\nu_+(\alpha,\theta)}}\,,\ \psi_5(\alpha,\theta) = \frac{\nu_-(\alpha,\theta)}{2}\,, \tag{27}$$

$$\phi_i(\alpha,\theta) = \sqrt{\frac{\nu_+(\alpha,\theta)}{2}}\quad (0\le i\le 3)\,,\ \phi_4(\alpha,\theta) = \frac{\nu_+(\alpha,\theta)}{\sqrt{2\nu_-(\alpha,\theta)}}\,,\ \phi_5(\alpha,\theta) = \frac{\nu_+(\alpha,\theta)}{2} \tag{28}$$

with

$$\nu_-(\alpha,\theta) = 1+\alpha^2 - \sqrt{1+2\alpha^2\cos 2\theta+\alpha^4}\,,\ \nu_+(\alpha,\theta) = 1+\alpha^2 + \sqrt{1+2\alpha^2\cos 2\theta+\alpha^4}\,. \tag{29}$$

Remark 2. *In general, the upper bounds are more important than the lower ones, but the latter may be meaningful in evaluating the accuracy or efficiency of the boundings. The above estimates should be used essentially in the range $0<\alpha\le 1$ and $\frac{\pi}{3} \le \cos^{-1}\frac{\alpha}{2} \le \theta < \pi$, although they may be meaningful for general triangles without restrictions in* (1)*, and the upper bounds except for $i=4$ are uniformly bounded there. This fact means that these constants are robust to deformation of the triangle $T_{\alpha,\theta}$. On the other hand, the upper bound for $C_4(\alpha,\theta)$ is not so, and hence, to assure such uniform boundedness, we need the so-called minimum* *angle condition* [11] *: the minimum angle of $T_{\alpha,\theta}$ is bounded from below by a certain positive constant. This may be seen by using the identity $\nu_-(\alpha,\theta)\nu_+(\alpha,\theta) = 4\alpha^2\sin^2\theta$ and rewriting the upper bound inequality as*

$$C_4(\alpha,\theta) \le \frac{C_4}{\alpha\sin\theta}\left(\frac{\nu_+(\alpha,\theta)}{2}\right)^{\frac{3}{2}} \ . \tag{30}$$

Namely, we can see that the right-hand side diverges to $+\infty$ as $\alpha \to +0$ for each fixed $\theta \in]0,\pi[$. On the other hand, the minimum edge length α of $T_{\alpha,\theta}$ cannot approach to 0 under the minimum angle condition. It is also to be noted that the above inequality for $C_4(\alpha,\theta)$ is exactly the same as obtained by Natterer [27]*, although our notations are different from his. Actually, $C_4(\alpha,\theta)$ is uniformly bounded under the maximum angle condition of Babuška-Aziz* [5]*, which requires the maximum interior angle of $T_{\alpha,\theta}$ to be away from π by a positive constant and hence is weaker than the usual minimum angle one* [11]*. It is also known that this weaker condition is essential and cannot be relaxed any more* [5].

Proof. We will use the coordinate transformation (25) between $T_{\alpha,\theta}$ and T. By simple calculations, we have for $\tilde{v}(\xi_1,\xi_2) = v(x_1,x_2)$, i. e., $\tilde{v} = v\circ\Psi_{\alpha,\theta}^{-1}$ under the present transformation:

$$\sum_{i=1}^{2}\left(\frac{\partial v}{\partial x_i}\right)^2 = \frac{1}{\sin^2\theta}\left[\left(\frac{\partial\tilde{v}}{\partial\xi_1}\right)^2 - \frac{2\cos\theta}{\alpha}\frac{\partial\tilde{v}}{\partial\xi_1}\frac{\partial\tilde{v}}{\partial\xi_2} + \frac{1}{\alpha^2}\left(\frac{\partial\tilde{v}}{\partial\xi_2}\right)^2\right] \ ,$$

where v and $\tilde{v}$ are assumed to be sufficiently smooth. The two eigenvalues associated to the quadratic form for $\partial\tilde{v}/\partial\xi_i$ $(i=1,2)$ in $[\ \cdot\ ]$ above are two solutions of the characteristic equation $\mu^2 - \alpha^{-2}(1+\alpha^2)\mu + \alpha^{-2}\sin^2\theta = 0$, and are given by $\nu_-(\alpha,\theta)/(2\alpha^2)$ and $\nu_+(\alpha,\theta)/(2\alpha^2)$. Thus we can easily derive

$$\frac{\nu_-(\alpha,\theta)}{2\alpha^2\sin^2\theta}\sum_{i=1}^{2}\left(\frac{\partial\tilde{v}}{\partial\xi_i}\right)^2 \le \sum_{i=1}^{2}\left(\frac{\partial v}{\partial x_i}\right)^2 \le \frac{\nu_+(\alpha,\theta)}{2\alpha^2\sin^2\theta}\sum_{i=1}^{2}\left(\frac{\partial\tilde{v}}{\partial\xi_i}\right)^2 \ .$$

Moreover, the Jacobian of the present transformation is evaluated as $\partial(x_1,x_2)/\partial(\xi_1,\xi_2) = \alpha\sin\theta$. From these estimates and the identity $\nu_-(\alpha,\theta)\nu_+(\alpha,\theta) = 4\alpha^2\sin^2\theta$, we have

$$\|v\|_{T_{\alpha,\theta}}^2 = \alpha\sin\theta\|\tilde{v}\|_T^2 \ , \quad \frac{2\alpha\sin\theta}{\nu_+(\alpha,\theta)}|\tilde{v}|_{1,T}^2 \le |v|_{1,T_{\alpha,\theta}}^2 \le \frac{2\alpha\sin\theta}{\nu_-(\alpha,\theta)}|\tilde{v}|_{1,T}^2 \ , \tag{a.1}$$

where $|\cdot|_{1,T_{\alpha,\theta}}$, for example, denotes $|\cdot|_1$ for $T_{\alpha,\theta}$. The results for $i = 0,1,2,3$ are now easy to obtain by using the above and the definitions of the constants $C_i(\alpha,\theta)$'s, since the present transformation yields a bijection between $V_{\alpha,\theta}^i$ and V^i.

Similarly, we obtain

$$\sum_{i,j=1}^{2}\left(\frac{\partial^2 v}{\partial x_i\partial x_j}\right)^2 = \frac{1}{\sin^4\theta}\left[\left(\frac{\partial^2\tilde{v}}{\partial\xi_1^2}\right)^2 + \frac{1}{\alpha^4}\left(\frac{\partial^2\tilde{v}}{\partial\xi_2^2}\right)^2 + \frac{2(1+\cos^2\theta)}{\alpha^2}\left(\frac{\partial^2\tilde{v}}{\partial\xi_1\partial\xi_2}\right)^2 \right.$$
$$\left. + \frac{2\cos^2\theta}{\alpha^2}\frac{\partial^2\tilde{v}}{\partial\xi_1^2}\frac{\partial^2\tilde{v}}{\partial\xi_2^2} - \frac{4\cos\theta}{\alpha}\frac{\partial^2\tilde{v}}{\partial\xi_1^2}\frac{\partial^2\tilde{v}}{\partial\xi_1\partial\xi_2} - \frac{4\cos\theta}{\alpha^3}\frac{\partial^2\tilde{v}}{\partial\xi_2^2}\frac{\partial^2\tilde{v}}{\partial\xi_1\partial\xi_2}\right] \ .$$

Let us consider the following real symmetric matrix related to the quadratic form for $\partial^2\hat{v}/\partial\xi_1^2$, $\partial^2\hat{v}/\partial\xi_2^2$ and $\sqrt{2}\,\partial^2\hat{v}/\partial\xi_1\partial\xi_2$ in $[\,\cdot\,]$ of the right-hand side above :

$$\begin{pmatrix} 1 & \dfrac{\cos^2\theta}{\alpha^2} & -\dfrac{\sqrt{2}\cos\theta}{\alpha} \\ \dfrac{\cos^2\theta}{\alpha^2} & \dfrac{1}{\alpha^4} & -\dfrac{\sqrt{2}\cos\theta}{\alpha^3} \\ -\dfrac{\sqrt{2}\cos\theta}{\alpha} & -\dfrac{\sqrt{2}\cos\theta}{\alpha^3} & \dfrac{1+\cos^2\theta}{\alpha^2} \end{pmatrix} .$$

The associated characteristic equation is

$$\mu^3 - \alpha^{-4}\{1+(1+\cos^2\theta)\alpha^2+\alpha^4\}\mu^2 + \alpha^{-6}\sin^2\theta\{1+(1+\cos^2\theta)\alpha^2+\alpha^4\}\mu - \alpha^{-6}\sin^6\theta$$
$$= (\mu-\alpha^{-2}\sin^2\theta)\{\mu^2-\alpha^{-4}(1+2\alpha^2\cos^2\theta+\alpha^4)\mu+\alpha^{-4}\sin^4\theta\} = 0\,,$$

which has three eigenvalues $\nu_-^2(\alpha,\theta)/(4\alpha^4) \le \alpha^{-2}\sin^2\theta \le \nu_+^2(\alpha,\theta)/(4\alpha^4)$ with $\nu_-(\alpha,\theta)$ and $\nu_+(\alpha,\theta)$ defined by (29). Now we have the estimates

$$\frac{\nu_-^2(\alpha,\theta)}{4\alpha^4\sin^4\theta}\sum_{i,j=1}^{2}\left(\frac{\partial^2\tilde{v}}{\partial\xi_i\partial\xi_j}\right)^2 \le \sum_{i,j=1}^{2}\left(\frac{\partial^2 v}{\partial x_i\partial x_j}\right)^2 \le \frac{\nu_+^2(\alpha,\theta)}{4\alpha^4\sin^4\theta}\sum_{i,j=1}^{2}\left(\frac{\partial^2\tilde{v}}{\partial\xi_i\partial\xi_j}\right)^2 ,$$

which gives, as $(a.1)$,

$$\frac{4\alpha\sin\theta}{\nu_+^2(\alpha,\theta)}|\tilde{v}|_{2,T}^2 \le |v|_{2,T_{\alpha,\theta}}^2 \le \frac{4\alpha\sin\theta}{\nu_-^2(\alpha,\theta)}|\tilde{v}|_{2,T}^2\,. \qquad (a.2)$$

From $(a.1)$ and $(a.2)$, we obtain the results for $i=4,5$. □

As a corollary of the preceding theorem, we can bound each $C_i(\alpha,\theta)$ in terms of $C_i(\alpha)$ and θ. Such estimates can be effective when the dependence of the considered $C_i(\alpha)$ on α is known as we will see later. The bounding can be achieved by introducing the following affine transformation between $x=\{x_1,x_2\}\in T_{\alpha,\theta}$ and $\xi=\{\xi_1,\xi_2\}\in T_\alpha$:

$$\xi_1 = x_1 - x_2\cos\theta/\sin\theta\,,\ \ \xi_2 = x_2/\sin\theta\,, \qquad (31)$$

which is a bit different from that used by Babuška and Aziz [5]. But essentially the same can be attained by simply applying the results of Theorem 1 for $\alpha=1$, as may be seen by comparing (31) with (25).

Corollary 1. *For $\alpha\in]0,+\infty[$ and $\theta\in]0,\pi[$, $C_i(\alpha,\theta)$'s are bounded as*

$$\psi_i(\theta)C_i(\alpha) \le C_i(\alpha,\theta) \le \phi_i(\theta)C_i(\alpha) \quad (0\le i\le 5)\,, \qquad (32)$$

where $\psi(\theta)=\psi(1,\theta)$ and $\phi(\theta)=\phi(1,\theta)$ for $0\le i\le 5$. More specifically,

$$\psi_i(\theta)=\sqrt{1-|\cos\theta|} \quad (0\le i\le 3)\,,\ \ \psi_4(\theta)=\frac{1-|\cos\theta|}{\sqrt{1+|\cos\theta|}}\,,\ \ \psi_5(\theta)=1-|\cos\theta|\,, \qquad (33)$$

$$\phi_i(\theta)=\sqrt{1+|\cos\theta|} \quad (0\le i\le 3)\,,\ \ \phi_4(\theta)=\frac{1+|\cos\theta|}{\sqrt{1-|\cos\theta|}}\,,\ \ \phi_5(\theta)=1+|\cos\theta|\,. \qquad (34)$$

Remark 3. *The function form of $\phi_4(\theta)$ associated to $C_4(\alpha,\theta)$ is consistent with the maximum angle condition in* [5], *since $\phi_4(\theta)$ is bounded on $[\frac{\pi}{3}, \pi-\delta]$ for each sufficiently small fixed $\delta > 0$. Thus, $C_4(\alpha,\theta)$ is uniformly bounded for $0 < \alpha \le 1$ and $\frac{\pi}{3} \le \theta < \pi - \delta$, if we can show that $C_4(\alpha)$ is uniformly bounded for such α. Notice here that $C_4(\alpha) \le C_1 = C_2$ for $\alpha \le 1$, as will be seen in the subsequent section.*

3.2 Estimation of $C_4(\alpha,\theta)$ by $C_1(\alpha,\theta)$ and $C_2(\alpha,\theta)$

We can also give an upper bound for $C_4(\alpha,\theta)$ in terms of $C_1(\alpha,\theta)$ and $C_2(\alpha,\theta)$.

Theorem 2. *For $\forall\, \alpha \in]0,+\infty[$ and $\forall\, \theta \in]0,\pi[$, $C_4(\alpha,\theta)$ is bounded as*

$$C_4(\alpha,\theta) \le \frac{1}{\sqrt{2}\,\sin\theta}\nu(\alpha,\theta) \le C_1 \frac{1+|\cos\theta|}{\sin\theta}\sqrt{\frac{\nu_+(\alpha,\theta)}{2}}\,, \tag{35}$$

where $\nu_+(\alpha,\theta)$ is defined by (29), *and $\nu(\alpha,\theta)$ by*

$$\begin{aligned}\nu(\alpha,\theta) = \Big[&C_1^2(\alpha,\theta) + C_2^2(\alpha,\theta) + 2\,C_1(\alpha,\theta)C_2(\alpha,\theta)\cos^2\theta \\ &+ (C_1(\alpha,\theta)+C_2(\alpha,\theta))\sqrt{C_1^2(\alpha,\theta)+C_2^2(\alpha,\theta)+2\,C_1(\alpha,\theta)C_2(\alpha,\theta)\cos 2\theta}\,\Big]^{1/2}.\end{aligned} \tag{36}$$

In particular, the maximum angle condition applies to the present estimate (35), *cf.* [5, 21].

Remark 4. *It is also possible to bound $C_4(\alpha,\theta)$ by $C_1(\alpha,\theta)$ and $C_3(\alpha,\theta)$ in a similar manner, although we omit the explicit expressions. It may be meaningful to compare two estimates* (30) *and* (35) *for $C_4(\alpha,\theta)$*:

$$C_4(\alpha,\theta) \le \frac{C_4}{\alpha\sin\theta}\left(\frac{\nu_+(\alpha,\theta)}{2}\right)^{\frac{3}{2}} =: \gamma_1(\alpha,\theta)\,,$$
$$C_4(\alpha,\theta) \le C_1\frac{1+|\cos\theta|}{\sin\theta}\sqrt{\frac{\nu_+(\alpha,\theta)}{2}} =: \gamma_2(\alpha,\theta)\,.$$

Noting the relations $2\alpha|\cos\theta| \le \sqrt{1+2\alpha^2\cos 2\theta+\alpha^4} \le 1+\alpha^2$ and $2\alpha \le 1+\alpha^2$, we find

$$\frac{\alpha(1+|\cos\theta|)}{1+\alpha^2}\frac{C_1}{C_4} \le \frac{\gamma_2(\alpha,\theta)}{\gamma_1(\alpha,\theta)} = \frac{2\alpha(1+|\cos\theta|)}{1+\alpha^2+\sqrt{1+2\alpha^2\cos 2\theta+\alpha^4}}\frac{C_1}{C_4} \le \frac{C_1}{C_4}\,.$$

It is known that $C_4 \approx 0.489$ by numerical computations without verification [4, 22, 30]. *On the other hand, it is theoretically shown that C_1 is an upper bound of C_4, but is quite close to C_4 as shown by numerically verified bounding $0.492 < C_1 < 0.493$* [25, 26, 18, 19], *cf. also Theorems 3 and 4 later. Thus the above estimate shows that* (35) *is practically better than* (30) *for almost all values of α and θ. As a practical upper bound for C_4, Siganevich* [30] *recommended 0.5.*

Proof. From the definition, we have

$$C_4^2(\alpha,\theta) = \sup_{v\in V^4_{\alpha,\theta}\setminus\{0\}} \frac{|v|_1^2}{|v|_2^2} = \sup_{v\in V^4_{\alpha,\theta}\setminus\{0\}} \frac{\|\partial v/\partial x_1\|^2 + \|\partial v/\partial x_2\|^2}{|\partial v/\partial x_1|_1^2 + |\partial v/\partial x_2|_1^2}\,. \qquad (b.1)$$

Recall here the transformation rules (15) and (16). Then, for the present $v \in V^4_{\alpha,\theta} \setminus \{0\}$ and the associated $\hat{v} = v \circ \Phi_\theta^{-1}$, we can show as (14) and (17) that

$$\|\partial v/\partial x_1\| \le C_1(\alpha,\theta)|\partial v/\partial x_1|_1\,, \quad \|\partial \hat{v}/\partial \hat{x}_2\| \le C_2(\alpha,\theta)|\partial \hat{v}/\partial \hat{x}_2|_1\,, \qquad (b.2)$$

where $\partial \hat{v}/\partial \hat{x}_2 = \cos\theta\,\partial v/\partial x_1 + \sin\theta\,\partial v/\partial x_2$ at $x = \{x_1,x_2\} \in T_{\alpha,\theta}$ and $\hat{x} = \{\hat{x}_1,\hat{x}_2\} = \Phi_\theta(x)$. Then $\partial v/\partial x_2 = (\partial \hat{v}/\partial \hat{x}_2 - \cos\theta\,\partial v/\partial x_1)/\sin\theta$ can be evaluated as

$$\|\partial v/\partial x_2\|^2 \le \frac{1}{\sin^2\theta}\left[\|\partial \hat{v}/\partial \hat{x}_2\|^2 + 2|\cos\theta|\cdot\|\partial \hat{v}/\partial \hat{x}_2\|\cdot\|\partial v/\partial x_1\| + \cos^2\theta\,\|\partial v/\partial x_1\|^2\right]\,.$$

By $(b.2)$ and the present inequality, we can bound $\|\partial v/\partial x_1\|^2 + \|\partial v/\partial x_2\|^2$ from above as

$$\begin{aligned}
&\|\partial v/\partial x_1\|^2 + \|\partial v/\partial x_2\|^2 \\
&\le \frac{1}{\sin^2\theta}\left[\|\partial v/\partial x_1\|^2 + 2|\cos\theta|\cdot\|\partial v/\partial x_1\|\cdot\|\partial \hat{v}/\partial \hat{x}_2\| + \|\partial \hat{v}/\partial \hat{x_2}\|^2\right] \\
&\le \frac{1}{\sin^2\theta}\left[C_1^2(\alpha,\theta)|\partial v/\partial x_1|_1^2 + 2\,C_1(\alpha,\theta)C_2(\alpha,\theta)|\cos\theta|\cdot|\partial v/\partial x_1|_1\cdot|\partial \hat{v}/\partial \hat{x}_2|_1\right. \\
&\qquad\qquad \left. + \,C_2^2(\alpha,\theta)|\partial \hat{v}/\partial \hat{x}_2|_1^2\right]\,. \qquad (b.3)
\end{aligned}$$

To evaluate $|\partial \hat{v}/\partial \hat{x}_2|_1$ above, we again use the relation $\partial \hat{v}/\partial \hat{x}_2 = \cos\theta\,\partial v/\partial x_1 + \sin\theta\,\partial v/\partial x_2$. Then we find

$$|\partial \hat{v}/\partial \hat{x}_2|_1 \le |\cos\theta|\cdot|\partial v/\partial x_1|_1 + \sin\theta\,|\partial v/\partial x_2|_1\,.$$

Substituting the above into the right-hand side of $(b.3)$, we obtain

$$\begin{aligned}
&\|\partial v/\partial x_1\|^2 + \|\partial v/\partial x_2\|^2 \\
&\le \frac{1}{\sin^2\theta}\left[\left\{C_1^2(\alpha,\theta) + 2\,C_1(\alpha,\theta)C_2(\alpha,\theta)\cos^2\theta + C_2^2(\alpha,\theta)\cos^2\theta\right\}\,|\partial v/\partial x_1|_1^2\right. \\
&\quad + 2\,C_2(\alpha,\theta)\left\{C_1(\alpha,\theta) + C_2(\alpha,\theta)\right\}\sin\theta|\cos\theta|\cdot|\partial v/\partial x_1|_1\cdot|\partial v/\partial x_2|_1 \\
&\quad \left. + \,C_2^2(\alpha,\theta)\sin^2\theta\,|\partial v/\partial x_2|_1^2\right]\,.
\end{aligned}$$

By eigenvalue analysis of the quadratic form above for $|\partial v/\partial x_1|_1$ and $|\partial v/\partial x_2|_1$, we have, by using $\nu(\alpha,\theta)$ in (36),

$$\|\partial v/\partial x_1\|^2 + \|\partial v/\partial x_2\|^2 \le \frac{\nu^2(\alpha,\theta)}{2\,\sin^2\theta}\left[|\partial v/\partial x_1|_1^2 + |\partial v/\partial x_2|_1^2\right]\,,$$

which gives the former part of (35) by $(b.1)$.

To derive the latter part of (35), we should use $C_i(\alpha,\theta) \le \phi_i(\alpha,\theta)C_i$ $(i = 1,2)$ in (26) and the identities $\phi_1(\alpha,\theta) = \phi_2(\alpha,\theta) = \sqrt{\dfrac{\nu_+(\alpha,\theta)}{2}}$ and $C_1 = C_2$. □

3.3 Determination of some constants

Theorem 1 tells us that we can obtain upper bounds of the constants $C_i(\alpha,\theta)$ $(0 \le i \le 5)$, if correct values of $C_i = C_i(1, \frac{\pi}{2}, 1)$ are known. The upper bounds thus evaluated may be rough but anyway correct, so that they can be used for various theoretical purposes. According to some preceding works [18, 19, 25, 26], such exact evaluation is possible at least for C_0 and $C_1 = C_2$. We will quote the results below, together with an additional result for C_3.

Theorem 3. *It holds for* $C_i = C_i(1, \frac{\pi}{2}, 1)$ $(0 \le i \le 3)$ *that*

1) $i = 0: C_0 = \dfrac{1}{\pi}$,

2) $i = 1, 2: C_1 = C_2$, *and is given as the maximum positive solution of the transcendental equation for* μ:

$$\frac{1}{\mu} + \tan\frac{1}{\mu} = 0\,. \tag{37}$$

The concrete value of C_1 *can be obtained numerically with verification. For example, we have the estimation*

$$0.49282 < C_1 < 0.49293\,. \tag{38}$$

3) $i = 3:\ \ C_3 = \dfrac{C_1}{\sqrt{2}}\,,\quad 0.34847 < C_3 < 0.34856\,.$

Remark 5. i) *Numerical computation without verification gives* $C_1 = 0.49291245\cdots$ *and* $C_3 = 0.34854173\cdots$. *The present transcendental equation can be commonly seen in vibration analysis of strings with special boundary conditions* [28]. *The constant* C_1 *plays an important role in various situations and is called the Babuška-Aziz constant in* [18, 19].
ii) *At present, exact values of* C_4 *and* C_5 *are not known to the best of the authors' knowledge. Fortunately,* C_1 $(= C_2)$ *is a nice upper bound of* C_4 *as we will see in Sections 4.2 and 6.2. Numerically,* $C_4 \approx 0.489$ *as was reported in* [4, 22, 30]. *As for* C_5, *estimate* $C_5 < 0.361$ *is a correct but probably rough one given in* [14], *while an exact lower bound estimation is* $C_5 \ge [(15+\sqrt{193})/1440]^{1/2} = 0.1416...$, *which is derived by the Ritz-Galerkin method using* $x_1+x_2-x_1^2-x_2^2$ *and* x_1x_2 *as the basis of the trial space employed in* [25]. *Our own numerical computations suggest that* $C_5 \lessapprox 0.168$.

Proof. Refer [19, 25, 26] for 1) and [18, 19] for 2), respectively. For 3), we can prove by using the results for C_1 and a kind of symmetry method. We will proceed in three steps.

1] Similarly to (21), the eigenvalue problem associated to C_3 is given by: *Find* $\{\lambda, u\} \in \mathbf{R} \times V^3\backslash\{0\}$ *such that*

$$(\nabla u, \nabla v)_T = \lambda(u, v)_T \quad (\forall v \in V^3)\,. \tag{c.1}$$

Here, T is the unit right isosceles triangle $T_{1,\frac{\pi}{2},1}$, $V^3 = V^3_{1,\frac{\pi}{2},1}$ is defined by (4), and the inner products are those for T. Notice that we are interested only in the minimum eigenvalue and the associated eigenfunctions.

Let us divide T into two congruent parts using the line $x_2 = x_1$, which is also the line of symmetry for T. Moreover, one of the congruent parts is denoted by $\tilde{T}$:

$$\tilde{T} = \{x = \{x_1, x_2\} \in T\,;\ x_1 > x_2\}\,.$$

The eigenfunction $u \neq 0$ can be uniquely decomposed into the symmetric part u_s and the antisymmetric one u_a:

$$u = u_s + u_a\,,$$

where the symmetry and antisymmetry are those with respect to $x_2 = x_1$. Due to the orthogonalities of u_s and u_a for the bilinear forms $(\cdot,\cdot)_T$ and $(\nabla\cdot,\nabla\cdot)_T$, u_s and u_a can be dealt with separately: u_s and u_a both belong to V^3 and satisfy (c.1) for the minimum eigenvalue λ.

2] We first consider the case where $u_s \neq 0$. We can see that the restriction $\tilde{u}$ of u_s to $\tilde{T}$ is not zero and satisfies the following eigenvalue problem related to $\tilde{T}$:

$$\tilde{u} \in \tilde{V}^3\backslash\{0\}\,;\;\; (\nabla\tilde{u},\nabla\tilde{v})_{\tilde{T}} = \lambda(\tilde{u},\tilde{v})_{\tilde{T}} \quad (\forall\tilde{v} \in \tilde{V}^3)\,, \qquad (c.2)$$

where λ is identical to the former one, the inner products are the L_2 ones for $\tilde{T}$, and $\tilde{V}^3$ is defined by

$$\tilde{V}^3 = \{\tilde{v} \in H^1(\tilde{T})\,;\; \int_0^{\frac{1}{2}} \tilde{v}(1-s,s)\,ds = 0\}\,.$$

Now we can see that this is essentially the same problem as the eigenvalue problem for $C_1(1,\frac{\pi}{2},\frac{1}{\sqrt{2}})$, since $\tilde{T}$ is congruent to $T_{1,\frac{\pi}{2},\frac{1}{\sqrt{2}}}$. It is also fairly easy to see that the eigenpair for the minimum eigenvalue of (c.2) satisfies (c.1), if the eigenfunction is extended to whole T symmetrically with respect to $x_2 = x_1$. Thus $\tilde{u}$ is an eigenfunction for the minimum eigenvalue of (c.2) in the present case. Then we find that $C_3 = C_1/\sqrt{2}$, since $C_1(\alpha,\theta,1/\sqrt{2}) = C_1(\alpha,\theta)/\sqrt{2}$ as we have seen in Section 2. Of course, this conclusion is derived under the assumption that $u_s \neq 0$.

3] Secondly, we consider the case where $u_a \neq 0$. Due to the antisymmetry, the trace of u_a to the line of symmetry $x_2 = x_1$ inside T is shown to be 0. Moreover, any antisymmetric function in $H^1(T)$ automatically satisfies the line integration condition imposed on V^3. Thus the restriction $u^\dagger$ of u_a to $\tilde{T}$ is not zero and is an eigenfunction of the eigenvalue problem:

$$u^\dagger \in V^\dagger\backslash\{0\}\,;\;\; (\nabla u^\dagger,\nabla v^\dagger)_{\tilde{T}} = \lambda(u^\dagger,v^\dagger)_{\tilde{T}} \quad (\forall v^\dagger \in V^\dagger)\,, \qquad (c.3)$$

where λ is identical to the former one, and $V^\dagger$ is defined by

$$V^\dagger = \{v^\dagger \in H^1(\tilde{T})\,;\; v^\dagger(s,s) = 0\;\; (0 < s < \frac{1}{2})\}\,.$$

If we consider the reflection with respect to the line $x_1 = 1/2$, $(c.3)$ becomes the problem of the same form with $V^\dagger$ replaced by

$$V^* = \{v^* \in H^1(\tilde{T})\,;\; v^*(1-s,s) = 0\;\; (0 < s < \frac{1}{2})\}\,.$$

Clearly, the eigenvalues remain the same under such a transformation. Since $V^* \subset \tilde{V}^3$, the minimum eigenvalue of $(c.3)$ cannot be smaller than that of $(c.2)$, as can be seen by considering the characterization of the minimum eigenvalue by the Rayleigh quotient. Thus it is sufficient to consider the case where $u_s \neq 0$ only, and the proof is complete. □

3.4 Application to interpolation and a priori error estimates

In this subsection, we show how to apply the obtained results to interpolation error estimates and some a priori error estimates for FEM.

From the preceding considerations, especially equations (11) through (13) and Theorems 1 and 2, we have for example the following P_0 and P_1 interpolation error estimates:

$$\|v - \Pi^0_{\alpha,\theta,h} v\| \le C_0 \phi_0(\alpha,\theta) h |v|_1 \quad ; \forall v \in H^1(T_{\alpha,\theta,h}), \tag{39}$$

$$|v - \Pi^1_{\alpha,\theta,h} v|_1 \le C_1 \frac{1 + |\cos\theta|}{\sin\theta} \left[\frac{\nu_+(\alpha,\theta)}{2}\right]^{\frac{1}{2}} h |v|_2 \quad ; \forall v \in H^2(T_{\alpha,\theta,h}), \tag{40}$$

$$\|v - \Pi^1_{\alpha,\theta,h} v\| \le C_5 \phi_5(\alpha,\theta) h^2 |v|_2 \quad ; \forall v \in H^2(T_{\alpha,\theta,h}). \tag{41}$$

These may be rough but are still correct quantitative upper bounds, provided that the values of C_0, C_1 and C_5 or at least their upper bounds are known. For C_0 and C_1, we have obtained exact values in Theorem 3, while, presumably, C_5 has been evaluated only approximately as was noted in Remark 5.

As was already noted, such error bounds are available for triangles of general configuration by applying appropriate congruent transformations [5, 10, 11, 21]. Then such interpolation error estimates can be directly used in a priori error estimates of finite element solutions. In what follows, we will briefly explain an example of such process. See e. g. [11] for the details.

As a model problem, let us consider the Dirichlet problem of the Poisson equation over an bounded polygonal domain $\Omega \subset \mathbf{R}^2$: *given* $f \in L_2(\Omega)$, *find* $u \in H^1_0(\Omega)$ *such that* $-\Delta u = f$ *in* Ω. Here, $H^1_0(\Omega)$ is the popular subspace of $H^1(\Omega)$ with the homogeneous Dirichlet condition imposed. In the standard weak formulation, the condition for $u \in H^1_0(\Omega)$ is stated as

$$(\nabla u, \nabla v)_\Omega = (f, v)_\Omega \quad \left(\forall v \in H^1_0(\Omega)\right), \tag{42}$$

where the L_2-type inner products are those for Ω. As is well known, this is a well-posed problem so that we can define an operator G by $G : f \in L_2(\Omega) \mapsto u \in H^1_0(\Omega)$.

To apply the FEM based on the P_1 triangle to this problem, we consider a regular family of triangulations $\{\mathcal{T}^\eta\}_{\eta>0}$ of Ω, and then construct a P_1 finite element space $V^\eta \subset H^1_0(\Omega)$ for each $\mathcal{T}^\eta$, cf. [11] for the terminology *regular*. Here we have used η instead of the standard notation h for the discretization parameter, which has been already used in a different meaning. The finite element approximation $u_\eta \in V^\eta$ of the above u is now uniquely determined by imitating (42) in V^η:

$$(\nabla u_\eta, \nabla v_\eta)_\Omega = (f, v_\eta)_\Omega \quad (\forall v_\eta \in V^\eta). \tag{43}$$

An important fact in the error analysis of the Ritz-Galerkin FEM is the following best approximation property [11]:

$$|u - u_\eta|_{1,\Omega} = \min_{v_\eta \in V^\eta} |u - v_\eta|_{1,\Omega}, \tag{44}$$

where $|\cdot|_{1,\Omega}$ is $|\cdot|_1$ for Ω (similar usages will frequently appear hereafter). Another important one is the L_2-error estimate based on the Aubin-Nitsche trick [11]:

$$\|u - u_\eta\|_\Omega \le |u - u_\eta|_{1,\Omega} \inf_{v_\eta \in V^\eta} \sup_{g \in L_2(\Omega)\setminus\{0\}} \frac{|Gg - v_\eta|_{1,\Omega}}{\|g\|_\Omega}. \tag{45}$$

From (44), an error estimation based on the interpolation function $\Pi^{\eta,1}u \in V^\eta$ using the vertex values of u is given by

$$|u - u_\eta|_{1,\Omega} \le |u - \Pi^{\eta,1}u|_{1,\Omega}\,. \tag{46}$$

Clearly, the global interpolation operator $\Pi^{\eta,1}$ is closely related to the local one $\Pi^1_{\alpha,\theta,h}$. That is, for each triangle $K \in \mathcal{T}^\eta$, we can find a $T_{\alpha,\theta,h}$ congruent to K under a congruent transformation $\Phi_K : K \to T_{\alpha,\theta,h}$, and it then holds that $(\Pi^{\eta,1}u)|_K = [\Pi^1_{\alpha,\theta,h}\{(u|K) \circ \Phi_K^{-1}\}] \circ \Phi_K$. If $u \in H^2(\Omega)$, we have, using notations $\{\alpha_K, \theta_K, h_K\}$ for $\{\alpha, \theta, h\}$ of $T_{\alpha,\theta,h}$ associated to K,

$$|u - \Pi^{\eta,1}u|^2_{1,\Omega} = \sum_{K \in \mathcal{T}^\eta} |u - \Pi^{\eta,1}u|^2_{1,K} \le \sum_{K \in \mathcal{T}^\eta} h_K^2 C_4^2(\alpha_K, \theta_K)|u|^2_{2,K}\,. \tag{47}$$

Thus we obtain from (46) an a priori error estimate

$$|u - u_\eta|_{1,\Omega} \le |u - \Pi^{\eta,1}u|_{1,\Omega} \le C_{4,\eta}\eta|u|_{2,\Omega}\,, \tag{48}$$

where $C_{4,\eta}$ and η are defined by

$$C_{4,\eta} = \max_{K \in \mathcal{T}^\eta} C_4(\alpha_K, \theta_K)\,, \quad \eta = \max_{K \in \mathcal{T}^\eta} h_K\,. \tag{49}$$

To evaluate $C_{4,\eta}$ from above, we can utilize various upper bounds already derived for $C_4(\alpha,\theta)$, an example of which can be also found in (40). In problems more general than (42), we may also need upper bounds for $C_5(\alpha,\theta)$ to obtain global L_2 error bounds, although we can avoid the use of such bounds to a certain extent by adopting the Aubin-Nitsche trick [11]. The constants $C_i(\alpha,\theta)$ for $0 \le i \le 3$ may appear to be subsidiary here, but they actually play essential roles in the analysis of the non-conforming P_1 FEM as is noted in [19].

In order to apply the above to verification of various differential equations by FEM, it is often required to evaluate norms or semi-norms of the solutions by various data. A typical example is to give upper bounds of $|u|_{2,\Omega}$ in (48) by a norm of f. In the present case, we can use the well-known relation $|u|_{2,\Omega} \le \|f\|_\Omega$, provided that Ω is convex in addition to the assumptions already stated [15]. Then we have

$$|u - u_\eta|_{1,\Omega} \le C_{4,\eta}\eta\|f\|_\Omega\,, \tag{50}$$

and moreover, by applying (45) with v_η taken as $\Pi^{\eta,1}(Gg)$,

$$\|u - u_\eta\|_\Omega \le C_{4,\eta}\eta|u - u_\eta|_{1,\Omega} \le C^2_{4,\eta}\eta^2\|f\|_\Omega\,, \tag{51}$$

where we have used the estimate $|Gg - \Pi^{\eta,1}(Gg)|_{1,\Omega} \le C_{4,\eta}\eta|Gg|_{2,\Omega} \le C_{4,\eta}\eta\|g\|_\Omega$. The present estimation can be compared with the L_2 interpolation estimate

$$\|u - \Pi^{\eta,1}u\|_\Omega \le C_{5,\eta}\eta^2|u|_{2,\Omega} \le C_{5,\eta}\eta^2\|f\|_\Omega \quad \text{with } C_{5,\eta} = \max_{K \in \mathcal{T}^\eta} C_5(\alpha_K, \theta_K)\,. \tag{52}$$

Such evaluations become much more difficult for general problems, but have been gradually realized in various cases.

3.5 Application to a posteriori error estimates

A posteriori error estimation is also feasible and effective in various situations by using the interpolation error constants considered in the preceding subsections. So, before closing the present section, we also show how to apply the obtained results to a posteriori error estimates for FEM. Here we only explain a special and rather classical approach [12, 20, 24] briefly, but we can find a number of literatures on this subject.

Let q be an element of $H(\mathrm{div};\Omega) := \{q \in L_2(\Omega)^2 \mid \ \mathrm{div}\, q \in L_2(\Omega)\}$ [12, 20]. Such q can be also chosen from the narrower space $H^1(\Omega)^2$. Then, some simple calculations give, with the same notations as those in Section 3.4,

$$\begin{aligned} |u-u_\eta|_{1,\Omega}^2 &= (\nabla(u-u_\eta), \nabla(u-u_\eta))_\Omega = (u-u_\eta, -\Delta u)_\Omega - ((\nabla(u-u_\eta), \nabla u_\eta)_\Omega \\ &= (u-u_\eta, f)_\Omega + ((\nabla(u-u_\eta), q - \nabla u_\eta - q)_\Omega \\ &= (u-u_\eta, f + \mathrm{div}\, q)_\Omega + ((\nabla(u-u_\eta), q - \nabla u_\eta)_\Omega \\ &\leq \|u-u_\eta\|_\Omega \cdot \|f + \mathrm{div}\, q\|_\Omega + |u-u_\eta|_{1,\Omega} \cdot \|q - \nabla u_\eta\|_\Omega \, . \end{aligned}$$

Applying (51) to the above, we have $|u-u_\eta|_{1,\Omega}^2 \leq (C_{4,\eta}\eta \, \|f + \mathrm{div}\, q\|_\Omega + \|q - \nabla u_\eta\|_\Omega)|u-u_\eta|_{1,\Omega}$, and hence

$$|u-u_\eta|_{1,\Omega} \leq C_{4,\eta}\eta \, \|f + \mathrm{div}\, q\|_\Omega + \|q - \nabla u_\eta\|_\Omega \, . \tag{53}$$

Here, the constant $C_{4,\eta}$ appears again, and this estimate becomes an a posteriori one, provided that q is specified somehow. The most elegant but quite a restrictive choice is based on the hypercircle method [12, 20], where q is chosen so that $f + \mathrm{div}\, q = 0$ and hence the use of $C_{4,\eta}$ becomes unnecessary. More common and practical approach is to obtain q by post-processing of u_η, for example, by averaging or smoothing ∇u_η so as to belong to $H(\mathrm{div};\Omega)$. For this approach to be really effective, it is at least necessary that $\|q - \nabla u_\eta\|_\Omega = O(\eta)$, and preferably $\|f + \mathrm{div}\, q\|_\Omega = o(\eta)$. Combining (53) with (45) as in (51), we can also obtain a kind of a posteriori L_2-error estimate as

$$\|u-u_\eta\|_\Omega \leq C_{4,\eta}^2 \eta^2 \, \|f + \mathrm{div}\, q\|_\Omega + C_{4,\eta}\eta \|q - \nabla u_\eta\|_\Omega \, . \tag{54}$$

In summary, we can effectively utilize error constants such as $C_{4,\eta}$ also in a posteriori error estimation of finite element solutions.

4 Dependence of $\mathrm{C}_i(\alpha)$ on α

Up to now, we have given some basic results for dependence of error constants on h, α and θ. In this section, we will consider the dependence of such constants on $\alpha > 0$ in the special case when $\theta = \pi/2$ and $h = 1$. Actually, we need their behaviors in the range $0 < \alpha \leq 1$, and, in view of (32), we want to find their maxima or nice upper bounds there. Furthermore, the limiting case $\alpha \to +0$ is of some practical interests in the so-called anisotropic mesh refinements [1, 13].

4.1 Definitions and notations

Since each $C_i(\alpha) = C_i(\alpha, \pi/2, 1)$ is defined through minimization of a Rayleigh quotient in terms of norms and/or seminorms over T_α (see (18) through (20)), it is natural to introduce the following transformation $\xi = \Psi_\alpha(x)$ between $x = \{x_1, x_2\} \in T_\alpha$ and $\xi = \{\xi_1, \xi_2\} \in T$:

$$\xi_1 = x_1 \,, \;\; \xi_2 = x_2/\alpha \,, \tag{55}$$

together with the associated transformation $\tilde{v} = v \circ \Psi_\alpha^{-1}$ between functions v over T_α and $\tilde{v}$ over T: $\tilde{v}(\xi) = v(x) = v(\xi_1, \alpha\xi_2)$. Notice that $\Psi_\alpha = \Psi_{\alpha,\pi/2}$ for $\Psi_{\alpha,\theta}$ in (25).

Then we have the following expressions to (semi-)norms for T_α in terms of those for T:

$$\|v\|^2_{T_\alpha} = \alpha \|\tilde{v}\|^2_T \,, \tag{56}$$

$$|v|^2_{1,T_\alpha} = \alpha a^{(1)}_\alpha(\tilde{v}) \,; \;\; a^{(1)}_\alpha(\tilde{v}) := \left\| \frac{\partial \tilde{v}}{\partial \xi_1} \right\|^2_T + \alpha^{-2} \left\| \frac{\partial \tilde{v}}{\partial \xi_2} \right\|^2_T \,, \tag{57}$$

$$|v|^2_{2,T_\alpha} = \alpha a^{(2)}_\alpha(\tilde{v}) \,; \;\; a^{(2)}_\alpha(\tilde{v}) := \left\| \frac{\partial^2 \tilde{v}}{\partial \xi_1^2} \right\|^2_T + 2\alpha^{-2} \left\| \frac{\partial^2 \tilde{v}}{\partial \xi_1 \partial \xi_2} \right\|^2_T + \alpha^{-4} \left\| \frac{\partial^2 \tilde{v}}{\partial \xi_2^2} \right\|^2_T \,, \tag{58}$$

where, for example in (56), $v \in L_2(T_\alpha)$ and $\tilde{v} \in L_2(T)$ with $v = \tilde{v} \circ \Psi_\alpha$. By using these α-dependent quadratic forms, the Rayleigh quotients $R^{(i)}_\alpha(v) = R^{(i)}_{\alpha,\pi/2}(v)$ $(0 \leq i \leq 5)$ for $R^{(i)}_{\alpha,\theta}$'s in (18) through (20) are expressed as

$$R^{(i)}_\alpha(v) = \tilde{R}^{(i)}_\alpha(\tilde{v}) := \frac{a^{(1)}_\alpha(\tilde{v})}{\|\tilde{v}\|^2_T} \,; \;\; v \in V^i_\alpha \setminus \{0\} \,, \; \tilde{v} = v \circ \Psi_\alpha^{-1} \in V^i \setminus \{0\} \quad (0 \leq i \leq 3) \,, \tag{59}$$

$$R^{(4)}_\alpha(v) = \tilde{R}^{(4)}_\alpha(\tilde{v}) := \frac{a^{(2)}_\alpha(\tilde{v})}{a^{(1)}_\alpha(\tilde{v})} \,; \;\; v \in V^4_\alpha \setminus \{0\} \,, \; \tilde{v} = v \circ \Psi_\alpha^{-1} \in V^4 \setminus \{0\} \,, \tag{60}$$

$$R^{(5)}_\alpha(v) = \tilde{R}^{(5)}_\alpha(\tilde{v}) := \frac{a^{(2)}_\alpha(\tilde{v})}{\|\tilde{v}\|^2_T} \,; \;\; v \in V^4_\alpha \setminus \{0\} \,, \; \tilde{v} = v \circ \Psi_\alpha^{-1} \in V^4 \setminus \{0\} \,. \tag{61}$$

We can now analyze the constants $C_i(\alpha)$'s over the common triangle T, at the expense of explicit appearance of the parameter α in the Rayleigh quotients.

We also present the bilinear forms associated to the quadratic forms $a_\alpha^{(i)}(\cdot)$'s for $i=1,2$:

$$a_\alpha^{(1)}(u,v) := \left(\frac{\partial u}{\partial x_1},\frac{\partial v}{\partial x_1}\right)_T + \alpha^{-2}\left(\frac{\partial u}{\partial x_2},\frac{\partial v}{\partial x_2}\right)_T \qquad ;\ \forall u,\ v\in H^1(T)\,, \tag{62}$$

$$a_\alpha^{(2)}(u,v) := \left(\frac{\partial^2 u}{\partial x_1^2},\frac{\partial^2 v}{\partial x_1^2}\right)_T + 2\alpha^{-2}\left(\frac{\partial^2 u}{\partial x_1\partial x_2},\frac{\partial^2 v}{\partial x_1\partial x_2}\right)_T + \alpha^{-4}\left(\frac{\partial^2 u}{\partial x_2^2},\frac{\partial^2 v}{\partial x_2^2}\right)_T \qquad ;\ \forall u,\ v\in H^2(T)\,. \tag{63}$$

Here, for simplicity, we use u and v instead of $\tilde{u}$ and $\tilde{v}$, and the variable is denoted by $x=\{x_1,x_2\}$ instead of $\xi=\{\xi_1,\xi_2\}$.

The following function spaces will play important roles later:

$$H^{k,Z}(T)=\{v\in H^k(T)\,;\ \partial v/\partial x_2=0\}\quad (k=1,2)\,, \tag{64}$$

$$V^{i,Z}=\{v\in V^i\,;\ \partial v/\partial x_2=0\}\qquad (0\le i\le 4)\,, \tag{65}$$

which are actually identified with the spaces of functions dependent only on the variable x_1 as we will see later. By considering bilinear forms $a^{(i)}(\cdot,\cdot)$ for $i=1,\,2$ over the above type of function spaces, we are naturally led to the following bilinear forms:

$$a_Z^{(1)}(u,v) := \left(\frac{\partial u}{\partial x_1},\frac{\partial v}{\partial x_1}\right)_T \qquad ;\ \forall u,\ v\in H^1(T)\,, \tag{66}$$

$$a_Z^{(2)}(u,v) := \left(\frac{\partial^2 u}{\partial x_1^2},\frac{\partial^2 v}{\partial x_1^2}\right)_T \qquad ;\ \forall u,\ v\in H^2(T)\,. \tag{67}$$

Although these are defined over the whole H^1 and H^2 spaces for convenience, the partial derivatives above can be actually replaced with the ordinary ones when they are considered over the respective $H^{1,Z}$ and $H^{2,Z}$ spaces.

As a characterization of the above $H^{1,Z}(T)$, let us state a fundamental lemma to be used for our analysis. Its proof is omitted here since it can be performed by slightly modifying that for Theorem 3.1.4′ of [16]. Of course, essentially the same conclusions are drawn for other spaces in (64) and (65).

Lemma 1. *Any $v\in H^{1,Z}(T)$ can be identified with a function v^* of single variable x_1 :*

$$v(x_1,x_2)=v^*(x_1)\ \text{ for a.e. }\ x=\{x_1,x_2\}\in T\,. \tag{68}$$

Remark 6. *The present lemma does not necessarily hold for general domains. It holds for a domain $\Omega\subset\mathbf{R}^2$ which is "connected in x_2 direction" in the sense: For any two points x and x^* in Ω with a common x_1 component, the segment connecting these points is contained in Ω.*

4.2 Monotonicity and upper bounds of $\mathbf{C}_i(\alpha)$

We first derive some fundamental results for $C_i(\alpha)$'s for $0<\alpha\le 1$, especially for their upper bounds. With this regard, we owe much the following results to the analysis by Babuška and Aziz [5]. In particular, the estimation $C_4(\alpha)\le C_1$ below is an important consequence derived in [5] and also in [25, 30], and so we here call C_1 the Babuška-Aziz constant.

Theorem 4. *$C_i(\alpha) = C_i(\alpha, \pi/2, 1)$ $(0 \le i \le 5)$ are continuous positive-valued functions of $\alpha \in\ ]0, +\infty[$ (here we consider also for $\alpha > 1$). In addition, except for $i = 4$, they are monotonically increasing in α. Thus, in particular,*

$$C_i(\alpha) \le C_i\ (= C_i(1))\ \ ;\ \forall\alpha \in]0,1] \quad (i = 0, 1, 2, 3, 5)\,. \tag{69}$$

On the other hand, it holds for $i = 4$ that

$$C_4(\alpha) \le \max\{C_1(\alpha), C_2(\alpha)\} \le C_1(= C_2)\,;\ \forall\alpha \in]0,1]\,. \tag{70}$$

Remark 7. *It is also possible to show the continuity of constants $C_i(\alpha, \theta)$'s with respect to $\{\alpha, \theta\} \in]0, +\infty[\times]0, \pi[$ by slightly generalizing the arguments below. For $C_4(\alpha)$, bounding (70) assures that it is bounded from above by a monotonically increasing function of α. Moreover, numerical results suggest that it is also monotonically increasing as will be seen later, although we do not have fully theoretical results at present. Existence of $C_i(+0)$ for $i \ne 4$ is clear from the monotonicity stated above, although we will make more detailed analysis in the subsequent subsection including the case $i = 4$.*

Proof. We just give sketches since the arguments employed here are rather standard. As was mentioned in Section 4.1 and also used in [5], we consider the Rayleigh quotients $\tilde{R}^{(i)}_\alpha$'s for functions over the common domain T.

For the continuity, we first note that each Rayleigh quotient for a fixed $\tilde{v} \ne 0$ is a continuous positive function of α, so that its infimum over all $\tilde{v}$ is uniformly bounded over any compact interval for α of the form $[\alpha_1, \alpha_2];\ 0 < \alpha_1 < \alpha_2 < +\infty$. It is also clear that the infimum for each $\alpha > 0$ is actually the minimum and cannot be zero (i. e., it is positive), as is shown by the usual arguments based on the Rellich compactness theorem and the reduction to absurdity. Then we can assure the existence of both $\overline{\lim}_{\beta\to\alpha} C_i^{-2}(\beta)\ (\le C_i^{-2}(\alpha))$ and $\underline{\lim}_{\beta\to\alpha} C_i^{-2}(\beta)$ for each $\alpha > 0$ and $i\,;\ 0 \le i \le 5$. Choosing an appropriate bounded sequence in V^i associated to the above lower limit, we can prove $C_i^{-2}(\alpha) \le \underline{\lim}_{\beta\to\alpha} C_i^{-2}(\beta)$, i. e., the continuity at α, by adopting the weakly lower semi-continuity of the numerator and the continuity of the denominator appearing in the definition of $\tilde{R}^{(i)}_\alpha$ with respect to the metric of V^i. Here, the Rellich type compactness theorem is again needed, and arguments similar to those in the subsequent subsection are used as well.

For the monotonicity and (70), we omit the proof since they can be concluded in completely the same fashion as in [5]. □

4.3 Asymptotic behaviors of constants as $\alpha \to +0$

We will now analyze the asymptotic behaviors of the constants $C_i(\alpha)$'s $(0 \le i \le 5)$ as $\alpha \to +0$ by adopting various techniques developed e. g. in [23]. In particular, the right limit values $C_i(+0)$'s are given by zeros of certain transcendental equations (derived from eigenvalue problems of ordinary differential equations, ODE's) in terms of the hypergeometric functions [32]. For example, $C_2(+0)^{-1}$ is equal to the first positive zero of the Bessel function $J_0(\cdot)$. Moreover, these right limits give lower bounds for respective $C_i(\alpha)$'s, including the non-trivial case $i = 4$. Such results can be of use for understanding and analyzing the so-called “anisotropic triangulations” discussed e. g. in [1, 8, 13].

4.3.1 Main results

We first present the main results as a theorem below.

Theorem 5. *For each* i $(0 \le i \le 5)$, $C_i(+0) = \lim_{\alpha \to +0} C_i(\alpha)$ *exists and is positive. Moreover, they are the lower limits of the respective constants, i. e.,* $C_i(+0) = \inf_{\alpha > 0} C_i(\alpha)$ *for* $0 \le i \le 5$. *They are characterized by the relations* $C_i(+0) = 1/\sqrt{\lambda^{(i)}}$ *for* $0 \le i \le 5$, *where* $\lambda^{(i)}$*'s are the minimum eigenvalues of the following eigenvalue problems:*

$0 \le i \le 3$: *Find* $\lambda (= \lambda^{(i)}) \in \mathbf{R}$ *and* $u \in V^{i,Z} \setminus \{0\}$ *such that*

$$a_Z^{(1)}(u, v) = \lambda (u, v)_T \ ; \ \forall v \in V^{i,Z} \, , \tag{71}$$

$i = 4$: *Find* $\lambda (= \lambda^{(4)}) \in \mathbf{R}$ *and* $u \in V^{4,Z} \setminus \{0\}$ *such that*

$$a_Z^{(2)}(u, v) = \lambda a_Z^{(1)}(u, v) \ ; \ \forall v \in V^{4,Z} \, , \tag{72}$$

$i = 5$: *Find* $\lambda (= \lambda^{(5)}) \in \mathbf{R}$ *and* $u \in V^{4,Z} \setminus \{0\}$ *such that*

$$a_Z^{(2)}(u, v) = \lambda (u, v)_T \ ; \ \forall v \in V^{4,Z} \, . \tag{73}$$

These eigenvalue problems are also expressed by those for the following 2nd- or 4th-order ordinary differential equations for $u = u(s)$ *over the interval* $[0, 1]$.

$$i = 0: \ -[(1-s)u'(s)]' = \lambda^{(0)}(1-s)u(s) \ \ (0 < s < 1), \ \int_0^1 (1-s)u(s)\, ds = u'(0) = 0, \tag{74}$$

$$i = 1: \ -[(1-s)u'(s)]' = \lambda^{(1)}(1-s)u(s) + C \ \ (0 < s < 1), \ \int_0^1 u(s)\, ds = u'(0) = 0, \tag{75}$$

$$i = 2: \qquad -[(1-s)u'(s)]' = \lambda^{(2)}(1-s)u(s) \quad (0 < s < 1), \ \ u(0) = 0 \, , \tag{76}$$

$i = 3$: *essentially the same as for* $i = 1$;

$$-[(1-s)u'(s)]' = \lambda^{(3)}(1-s)u(s) + C \quad (0 < s < 1), \ \int_0^1 u(s)\, ds = u'(0) = 0 \, , \tag{77}$$

$i = 4$: *actually reduces to the case* $i = 1$;

$$[(1-s)u''(s)]'' = -\lambda^{(4)}[(1-s)u'(s)]' \ \ (0 < s < 1), \ \ u(0) = u(1) = u''(0) = 0 \, , \tag{78}$$

$$i = 5: \ [(1-s)u''(s)]'' = \lambda^{(5)}(1-s)u(s) \ \ (0 < s < 1), \ \ u(0) = u(1) = u''(0) = 0 \, . \tag{79}$$

Here, C *is an unknown constant to be determined simultaneously with* u *and* $\lambda^{(i)}$ $(i = 1, 3)$.

Remark 8. *In* (74), *the two conditions* $\int_0^1 (1-s)u(s)\, ds = 0$ *and* $u'(0) = 0$ *are actually identical for smooth* u *as may be seen by integrating the differential equation in* (74) *from* $s = 0$ *to* $s = 1$. *In the above, the numbers of boundary conditions are smaller than the orders of differential equations. This is mainly because the ordinary differential equations above have singularities in their coefficients at* $s = 1$, *so that the usual full numbers of boundary conditions are excessive to decide eigenfunctions in respective spaces* V^j*'s* $(0 \le j \le 4)$. *The eigenfunctions and eigenvalues are determined by using the hypergeometric functions* [32], *and the results are summarized in Appendix.*

4.3.2 Proof of main results

Let us now prove Theorem 5. The statements for us to show range from $i = 0$ to $i = 5$, but the methods and techniques to be employed are more or less alike. Among them, the analysis for $i = 4$ appears to be the most complicated, so that we will present the proof almost exclusively in this particular case. We will proceed in several steps.

1] To analyze asymptotic behaviors of $C_4(\alpha)$, let us define $\lambda_4(\alpha)$ for $\alpha > 0$ by $\lambda_4(\alpha) := C_4^{-2}(\alpha) > 0$, that is,

$$\lambda_4(\alpha) = \inf_{v \in V^4 \setminus \{0\}} \tilde{R}_\alpha^{(4)}(v)\,; \quad \tilde{R}_\alpha^{(4)}(v) = \frac{a_\alpha^{(2)}(v)}{a_\alpha^{(1)}(v)}\,, \tag{80}$$

where, for simplicity, v is used instead of $\tilde{v}$ unlike in (60). By the standard arguments, the infimum is shown to be actually the minimum, and is attained by a certain $u \in V^4 \setminus \{0\}$.

Moreover, $\{\lambda_4(\alpha), u\}$ is an eigenpair of the following eigenvalue problem:

$$a_\alpha^{(2)}(u, v) = \lambda_4(\alpha) a_\alpha^{(1)}(u, v)\,; \quad \forall v \in V^4\,, \tag{81}$$

where $a_\alpha^{(i)}(u, v)$ for $i = 1, 2$ are the bilinear forms associated to $a_\alpha^{(i)}(\cdot)$'s, see (62) and (63). Of course, the present $\lambda_4(\alpha) > 0$ is the minimum eigenvalue of (81).

Since $\tilde{R}_\alpha^{(4)}(v)$ is a homogeneous form of order 0, we can normalize the eigenfunction u as

$$a_\alpha^{(1)}(u) = 1\,. \tag{82}$$

2] Let us show that $\lambda_4(+0) = \lim_{\alpha \to +0} \lambda_4(\alpha)$ exists and is positive. Taking $v \in V^4 \setminus \{0\}$ in (80) as $v(x_1, x_2) = x_1(1 - x_1)$, we can see that $\lambda_4(\alpha)$ is uniformly bounded for $\alpha \in]0, \infty[$, and hence $\alpha = 0$ is an accumulation point. In particular, both $\lambda_4^* := \underline{\lim}_{\alpha \to +0} \lambda_4(\alpha) \geq 0$ and $\lambda_4^\dagger := \overline{\lim}_{\alpha \to +0} \lambda_4(\alpha)$ exist. Then we can find a sequence $\{\alpha_n\}_{n=1}^\infty$ in $]0, 1]$ such that

$$\lim_{n \to \infty} \alpha_n = 0\,, \quad \lim_{n \to \infty} \lambda_4(\alpha_n) = \lambda_4^*\,. \tag{83}$$

We must show that λ_4^* coincides with $\lambda_4^\dagger$ to conclude the existence of the right limit $\lambda_4(+0)$.

Associated to the above sequence $\{\alpha_n\}$, there exists a sequence $\{u_n\}$ in $V^4 \setminus \{0\}$ such that each member satisfies (81) and (82), i. e., $a_{\alpha_n}^{(1)}(u_n) = 1$, $a_{\alpha_n}^{(2)}(u_n) = \lambda_4(\alpha_n)$, and

$$a_{\alpha_n}^{(2)}(u_n, v) = \lambda_4(\alpha_n) a_{\alpha_n}^{(1)}(u_n, v)\,; \quad \forall v \in V^4\,. \tag{84}$$

Since $|u|_{1,T}^2 = \sum_{i=1}^2 \|\partial u/\partial x_i\|_T^2 \leq a_\alpha^{(1)}(u)$ and $|u|_{2,T}^2 = \sum_{i,j=1}^2 \|\partial^2 u/\partial x_i \partial x_j\|_T^2 \leq a_\alpha^{(2)}(u)$ for $\alpha > 0$, we have for $\{u_n\}$ that

$$|u_n|_{1,T}^2 + |u_n|_{2,T}^2 \leq 1 + \lambda_4(\alpha_n) \;\; (n = 1, 2, \ldots)\,. \tag{85}$$

That is, $\{u_n\}$ is bounded with respect to the semi-norms of $H^1(T)$ and $H^2(T)$ appearing above. Moreover, we can show that $\{\|u_n\|_T\}$ is also bounded by noting that $\{u_n\}$ is a sequence in V^4 and utilizing the Rellich theorem.

Thus $\{u_n\}$ is a bounded sequence in $H^2(T)$, so that there exists a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$ for convenience, and $u_0 \in V^4$ such that, for $n \to \infty$,

$$u_n \to u_0 \;\text{ weakly in } V^4 \subset H^2(T)\,, \text{ and strongly in } H^1(T)\,, \tag{86}$$

where the strong convergence is concluded by the Rellich theorem. Substituting $v_Z \in V^{4,Z}$ as $v \in V^4$ in (84) and then taking the limit for $n \to \infty$, we find

$$a_Z^{(2)}(u_0, v_Z) = \lambda_4^* a_Z^{(1)}(u_0, v_Z)\,;\ \forall v_Z \in V^{4,Z}\,. \tag{87}$$

Furthermore, since $\|\partial u_n/\partial x_2\|_T^2 = \alpha_n^2 \left[a_{\alpha_n}^{(1)}(u_n) - \|\partial u_n/\partial x_1\|_T^2\right]$ from (62), we can show that

$$\partial u_0/\partial x_2 = 0\,,\ \text{i. e.,}\ \ u_0 \in V^{4,Z}\,. \tag{88}$$

Thus we have obtained (72), provided that $u_0 \neq 0$. For the moment, we cannot exclude the possibility that $u_0 = 0$, so that we will now consider the two cases below.

3] (Case$\,:u_0 \neq 0$) In this case, $\{\lambda_4^*, u_0\} \in \mathbf{R} \times V^{4,Z}$ is an eigenpair of (87), and is also associated with the following minimization problem:

$$\lambda = \inf_{v \in V^{4,Z}\setminus\{0\}} \frac{a_Z^{(2)}(v)}{a_Z^{(1)}(v)}\,. \tag{89}$$

It is not difficult to show that this minimization problem has a minimum $\mu_4 > 0$, which is at the same time the minimum eigenvalue of (87) and whose arbitrary minimizer $v^Z \in V^{4,Z}\setminus\{0\}$ is an associated eigenfunction. Noting that λ_4^* is an eigenvalue of (87) and

$$\mu_4 = \inf_{v \in V^{4,Z}\setminus\{0\}} \frac{a_Z^{(2)}(v)}{a_Z^{(1)}(v)} = \frac{a_Z^{(2)}(v^Z)}{a_Z^{(1)}(v^Z)} = \frac{a_\alpha^{(2)}(v^Z)}{a_\alpha^{(1)}(v^Z)} = \tilde{R}_\alpha^{\{4\}}(v^Z) \geq \lambda_4(\alpha) \quad \text{for } \forall \alpha \in]0,\infty[\,, \tag{90}$$

we have $\mu_4 \leq \lambda_4^* = \underline{\lim}_{\alpha\to+0}\,\lambda_4(\alpha) \leq \lambda_4^\dagger = \overline{\lim}_{\alpha\to+0}\,\lambda_4(\alpha) \leq \mu_4$, that is, λ_4^* coincides with $\lambda_4^\dagger$ and also with μ_4, so that it is the minimum eigenvalue of (87) and u_0 is an associated eigenfunction. Thus, if $u_0 \neq 0$ for all possible subsequences, λ_4^* is uniquely determined independently of the sequences like original $\{u_n\}$, so that the present λ_4^* is the true right limit $\lambda_4(+0)$. Furthermore, from the above consideration, λ_4^* is also the upper limit of $\lambda_4(\alpha)$ for $\alpha \in]0,\infty[$, that is, $1/\sqrt{\lambda_4^*}$ is the lower limit of $C_4(\alpha)$. Of course, such conclusions are justified provided that u_0 cannot be 0. By using $v(x_1,x_2) = \sin \pi x_1 \in V^{4,Z}\setminus\{0\}$ in the Rayleigh quotient appearing in (89), we can also show that

$$0 < \mu_4 = \lambda_4^* \leq \pi^2 < 10\,. \tag{91}$$

4] (Case$\,:u_0 = 0$) Let us define w_n by $w_n = \alpha_n^{-1}\partial u_n/\partial x_2\ \ (n = 1, 2, ...)$. Then we can see that $w_n \in V^2 \subset H^1(T)$. Since $u_n \to u_0 = 0$ strongly in $H^1(T)$ and $a_{\alpha_n}^{(1)}(u_n) = 1$, it holds that $\|w_n\|_T^2 = 1 - \|\partial u_n/\partial x_1\|_T^2 \to 1$. Moreover, $a_{\alpha_n}^{(2)}(u_n) = \lambda_4(\alpha_n)$, i. e.,

$$\|\partial^2 u_n/\partial x_1^2\|_T^2 + 2\|\partial w_n/\partial x_1\|_T^2 + \alpha_n^{-2}\|\partial w_n/\partial x_2\|_T^2 = \lambda_4(\alpha_n) \quad (n = 1, 2, ...)\,, \tag{92}$$

is uniformly bounded, so that $\{w_n\}$ is bounded in $H^1(T)$ and $\|\partial w_n/\partial x_2\|_T \to 0$ $(n \to \infty)$. Thus, further choosing a subsequence of $\{w_n\}$ and denoting it by the same notation for simplicity, we can show the existence of $w_0 \in V^{2,Z} \setminus \{0\}$ with $\|w_0\|_T = 1$ such that, for $n \to \infty$,

$$w_n \to w_0 \ \text{ weakly in } V^2 \subset H^1(T)\,,\ \text{and strongly in } L^2(T)\,. \tag{93}$$

Let v^* be an arbitrary function of x_1 such that $v^* \in C^2([0,1])$ with $v^*(0) = 0$, and take $v \in V^4$ in (84) as $v(x_1, x_2) = v^*(x_1)x_2$. For simplicity, we will identify v^* with $v^* \otimes 1_{x_2}$, where 1_{x_2} is the unit constant function of x_2. Then (84) becomes

$$\alpha_n \left(\frac{\partial^2 u_n}{\partial x_1^2}, \frac{\partial^2 v}{\partial x_1^2}\right)_T + 2\left(\frac{\partial w_n}{\partial x_1}, \frac{\partial v^*}{\partial x_1}\right)_T = \lambda_4(\alpha_n)[\alpha_n\left(\frac{\partial u_n}{\partial x_1}, \frac{\partial v}{\partial x_1}\right)_T + (w_n, v^*)_T]\,. \tag{94}$$

Letting $n \to \infty$ above, we find that $w_0 \in V^{2,Z} \setminus \{0\}$ satisfies

$$2\left(\frac{\partial w_0}{\partial x_1}, \frac{\partial v^*}{\partial x_1}\right)_T = \lambda_4^*(w_0, v^*)_T\,,\ \ \text{i.}\,e.,\ \ a_Z^{(1)}(w_0, v^*) = \frac{1}{2}\lambda_4^*(w_0, v^*)_T\,. \tag{95}$$

Moreover, the above holds even for $\forall v^*$ taken from $V^{2,Z}$, since any functions in $V^{2,Z}$ can be approximated by C^2 functions of x_1 vanishing at $x_1 = 0$. Thus the present relation can be viewed as an eigenvalue problem which has $\{\lambda_4^*/2, w_0\}$ as an eigenpair. As usual, we can show that all the eigenvalues are positive, so that $\lambda_4^* > 0$.

By Lemma 1, w_0 can be identified with a function w^* of a single variable x_1, so that (95) can be expressed by

$$\int_0^1 (1-x_1)\frac{dw^*}{dx_1}(x_1)\frac{dv^*}{dx_1}(x_1)\,dx_1 = \frac{1}{2}\lambda_4^*\int_0^1 (1-x_1)w^*(x_1)v^*(x_1)\,dx_1\,. \tag{96}$$

Taking v^* from $C_0^\infty(]0,1[)$, we have, in the distributional sense (and actually in the classical sense as well) on the interval $]0,1[$,

$$-\frac{d}{dx_1}\left[(1-x_1)\frac{dw^*}{dx_1}(x_1)\right] = \frac{1}{2}\lambda_4^*(1-x_1)w^*(x_1)\,. \tag{97}$$

Moreover, it follows from the condition $w_0 \in V^{2,Z}$ that $w^*(0) = 0$. Since $\lambda_4^* > 0$, the general solution of the above is of the form, for arbitrary constants c_1 and c_2,

$$w^*(x_1) = c_1 J_0\left(\sqrt{\frac{\lambda_4^*}{2}}(1-x_1)\right) + c_2 Y_0\left(\sqrt{\frac{\lambda_4^*}{2}}(1-x_1)\right)\,, \tag{98}$$

where $J_0(\cdot)$ and $Y_0(\cdot)$ are the 0-th order Bessel functions of the first and second kinds, respectively. As is well known, $J_0(\cdot)$ is sufficiently smooth, while $Y_0(\cdot)$ is of the form $Y_0(s) = c_3 \log s + r(s)$ for $s > 0$, where $c_3 \neq 0$ is a constant and $r(s)$ is a sufficiently smooth remainder term [32]. Consequently, c_2 must be zero for w_0 to belong to $V^{2,Z} \subset H^1(T)$. Then by considering the conditions $w^*(0) = 0$ and $c_1 \neq 0$, $J_0(\sqrt{\lambda_4^*/2})$ must be zero, that is, $\sqrt{\lambda_4^*/2}$ is equal to a positive zero of $J_0(\cdot)$. In fact $J_0(\cdot)$ has countably many positive zeros without any accumulation points except $+\infty$ [32]. Denoting the smallest positive zero by $\gamma_0 > 0$, we have

$$\lambda_4^* \geq 2\gamma_0^2\,. \tag{99}$$

We can show that $\gamma_0 > 2.25 = 9/4$, so that $\lambda_4^* > 10$. Comparing this with (91), i. e., $10 > \mu_4 \geq \sup_{\alpha>0}\lambda_4(\alpha) \geq \lambda_4^*$, we have a contradiction, and can exclude the possibility that $u_0 = 0$.

Remark 9. *Although it is well known that $\gamma_0 = 2.4048...$ numerically, we must verify that $\gamma_0 > 2.25$ for strict analysis. This can be done for example by using the well-known power series expansion $J_0(s) = \sum_{m=0}^{\infty} \frac{1}{(m!)^2}(-\frac{s^2}{4})^m$ and numerical verification techniques, cf.* [18, 33].

5] We have now proved that λ_4^* and $u_0 \neq 0$ are actually the minimum eigenvalue and the associated eigenfunction of (72), respectively, and that $C_4(+0) = 1/\sqrt{\lambda_4^*} = \inf_{\alpha>0} C_4(\alpha)$.

It is not difficult to prove (78). That is, the differential equation can be obtained just as we derived (97) from (96), while $u(0) = u(1) = 0$ follow from the condition $u_0 \in V^{4,Z}$. Finally, $u''(0) = 0$ is obtained as a natural boundary condition associated to (72).

Let us also show that (78) reduces to (75). Denoting u' by v and then integrating the differential equation in (78) with respect to the variable s, we have

$$-[(1-s)v'(s)]' = \lambda^{(4)}[(1-s)v(s)] + C\,, \tag{100}$$

which coincides with the differential equation in (75) after rewriting v as u. The boundary condition $v'(0) = 0$ follows from $u''(0) = 0$, and the condition $\int_0^1 v(s)\,ds = 0$ is derived from the relation $\int_0^1 u'(s)\,ds = u(1) - u(0) = 0$. Once v is determined, u can be reconstructed by integration: $u(s) = \int_0^s v(t)\,dt$. Consequently, the present case $i = 4$ reduces to the case $i = 1$.

6] In the cases other than $i = 4$, the analyses are a bit easier since the denominators of $\tilde{R}_\alpha^{(i)}$'s do not depend on α. For example, (71) and (73) can be derived easily. We just show, in the case of $i = 1$, how to derive (75) from (71).

For $i = 1$, $u = u(x_1, x_2)$ and $v = v(x_1, x_2)$ in $V^{1,Z}$ can be identified with functions $u^* = u^*(x_1)$ and $v^* = v^*(x_1)$, respectively, so that (71) for $i = 1$ can be expressed by, as (96),

$$\int_0^1 (1-x_1)\frac{du^*}{dx_1}(x_1)\frac{dv^*}{dx_1}(x_1)\,dx_1 = \lambda \int_0^1 (1-x_1)u^*(x_1)v^*(x_1)\,dx_1\,. \tag{101}$$

Let us consider dw^*/dx_1 for $\forall w^* \in C_0^\infty(]0,1[)$. Then it can be identified with a function in $V^{1,Z}$, so that, by substituting it into (101) as v^*, we have in the sense of distribution that

$$\frac{d^2}{dx_1^2}\left[(1-x_1)\frac{du^*}{dx_1}(x_1)\right] = -\lambda\frac{d}{dx_1}[(1-x_1)u^*(x_1)]\,, \tag{102}$$

from which we obtain the differential equation in (75). The condition $\int_0^1 u(s)\,ds = 0$ follows from $u \in V^{1,Z}$, while $u'(0) = 0$ is a natural boundary condition associated to (71) for $i = 1$.

Remark 10. *Table* 1 *shows numerical results for $C_i(+0)$ $(0 \le i \le 5)$ by Mathematica®, cf. Appendix.*

Table 1: Numerical values of $C_i(+0)$ $(0 \le i \le 5)$

i	0	1, 3, 4	2	5
$C_i(+0)$	0.26098	0.32454	0.41583	0.10790

5 A posteriori estimation of some constants

It is in general very difficult to determine exact values of various constants defined in Section 2 for $T_{\alpha,\theta}$ of general shape. Numerically, we can adopt the FEM to obtain approximate values to such constants as may be found e. g. in [4, 7, 22, 30], but their quantitative error estimates are often unavailable. In this section, as an application of our results, let us give a kind of a posteriori estimation of $C_i(\alpha,\theta)$'s $(0 \le i \le 3)$ by adopting the P_1 (piecewise linear) FEM. At present, our approach gives only approximate or numerical boundings of constants, but they can be turned into mathematically correct boundings provided that appropriate numerical verification methods are introduced.

Our approach is based on the classical a priori error estimates for the finite element approximations to the smallest non-zero eigenvalue of the (minus) Laplacian with the Neumann or Dirichlet boundary conditions, cf. e. g. Schultz [29].

5.1 Preliminaries

First, let us make some preparations. Let Ω be a bounded convex polygonal domain. In the present applications, it is often the triangular domain $T_{\alpha,\theta}$. Let us also consider a closed linear subspace $H^1_s(\Omega)$ of $H^1(\Omega)$, which can be infinite-dimensional and satisfies

$$H^1_s(\Omega) \neq \{0\}\ , \quad 1 \notin H^1_s(\Omega)\ , \tag{103}$$

where 1 is the constant function of unit value in Ω. A typical example of such $H^1_s(\Omega)$ is $H^1_0(\Omega)$ considered in Section 3.4.

As a generalization of (42), we can consider the problem of finding $u \in H^1_s(\Omega)$, for a given $f \in L_2(\Omega)$, such that

$$(\nabla u, \nabla v)_\Omega = (f, v)_\Omega \quad (\forall v \in H^1_s(\Omega))\ . \tag{104}$$

The uniqueness and existence of u in $H^1_s(\Omega)$ are almost trivial, so that we can define an operator G_s by

$$G_s : f \in L_2(\Omega) \mapsto u \in H^1_s(\Omega) \text{ determined by } (104)\ . \tag{105}$$

As a generalization of the problem related to (18), let us also consider a minimization problem for the Rayleigh quotient

$$R^s(v) := \frac{|v|^2_{1,\Omega}}{\|v\|^2_\Omega}\ ;\ \ v \in H^1_s(\Omega) \setminus \{0\}\ . \tag{106}$$

The minimum actually exists and is positive under (103) as may be shown by the compactness arguments. Moreover, denoting the minimum and an associated minimizer by $\lambda > 0$ and $u \in H^1_s(\Omega) \setminus \{0\}$, respectively, they satisfy

$$(\nabla u, \nabla v)_\Omega = \lambda (u, v)_\Omega \quad (\forall v \in H^1_s(\Omega))\ . \tag{107}$$

By using G_s in (104), the present $u \in H^1_s(\Omega)$ is shown to satisfy $u = \lambda G_s u$.

To apply the P_1 FEM to the above two problems, we first introduce a regular family of triangulations $\{\mathcal{T}^\eta\}_{\eta>0}$ of Ω as was mentioned in Section 3.4, and then construct the piecewise linear finite element space $S^\eta \subset H^1(\Omega)$ for each $\mathcal{T}^\eta$ as

$$S^\eta := \{v_\eta \in C(\overline{\Omega}) \mid\ v_\eta|K \text{ is a linear function for each } K \in \mathcal{T}^\eta\}\ . \tag{108}$$

For $u \in H^2(\Omega) \subset C(\overline{\Omega})$, we can define the piecewise linear interpolant $\Pi^{\eta,1}u \in S^\eta$ by

$$(\Pi^{\eta,1}u)(x^*) = u(x^*) \text{ for any vertex } x^* \text{ of } \mathcal{T}^\eta \, . \tag{109}$$

We will also use the parameters $\eta = \max_{K \in \mathcal{T}^\eta} h_K$, $C_{4,\eta} = \max_{K \in \mathcal{T}^\eta} C_4(\alpha_K, \theta_K)$ and $C_{5,\eta} = \max_{K \in \mathcal{T}^\eta} C_5(\alpha_K, \theta_K)$ defined in Section 3.4. Then we have the following interpolation estimates for the above u as was discussed in Section 3.4 :

$$|u - \Pi^{\eta,1}u|_{1,\Omega} \le C_{4,\eta}\eta |u|_{2,\Omega} \ , \quad \|u - \Pi^{\eta,1}u\|_\Omega \le C_{5,\eta}\eta^2 |u|_{2,\Omega} \, . \tag{110}$$

To construct approximate problems to (104) and the minimization of (106), let us consider the subspace $S^{\eta,s}$ of S^η defined by

$$S^{\eta,s} := S^\eta \cap H^1_s(\Omega) \ , \tag{111}$$

which we assume to be different from $\{0\}$. Of course, various other finite-dimensional subspaces of $H^1_s(\Omega)$ are available in place of $S^{\eta,s}$, but the above one is theoretically simple and also practically favorable in many cases.

Then an approximation to (104) is to find $u_\eta \in S^{\eta,s}$, for a given $f \in L_2(\Omega)$, such that

$$(\nabla u_\eta, \nabla v_\eta)_\Omega = (f, v_\eta)_\Omega \quad (\forall v_\eta \in S^{\eta,s}) \, . \tag{112}$$

The uniqueness and existence of u_η in $S^{\eta,s}$ are trivial, so that we can define an operator G^η_s approximating G_s by

$$G^\eta_s : f \in L_2(\Omega) \mapsto u_\eta \in S^{\eta,s} \text{ determined by } (112) \, . \tag{113}$$

As generalizations of (44) and (45), we have

$$|G_s f - G^\eta_s f|_{1,\Omega} = \min_{v_\eta \in S^{\eta,s}} |G_s f - v_\eta|_{1,\Omega} \ , \tag{114}$$

$$\|G_s f - G^\eta_s f\|_\Omega \le |G_s f - G^\eta_s f|_{1,\Omega} \inf_{v_\eta \in S^{\eta,s}} \sup_{g \in L_2(\Omega)\setminus\{0\}} \frac{|G_s g - v_\eta|_{1,\Omega}}{\|g\|_\Omega} \, . \tag{115}$$

On the other hand, an approximation problem related to $R^s(\cdot)$ is to find its minimum in $S^{\eta,s} \setminus \{0\}$. In this case, the existence of the minimum is again trivial, and the minimum λ^η and an associated minimizer $u_\eta \in S^{\eta,s} \setminus \{0\}$ satisfies the relation analogous to (107) :

$$(\nabla u_\eta, \nabla v_\eta)_\Omega = \lambda^\eta (u_\eta, v_\eta)_\Omega \quad (\forall v_\eta \in S^{\eta,s}) \, . \tag{116}$$

The following results are easy to derive but will play an essential role in our approach, cf. e. g. Theorem 8.3 of [29].

Lemma 2. *Let λ and λ^η be respectively defined by $\lambda = \min_{v \in H^1_s(\Omega)\setminus\{0\}} R^s(v)$ and $\lambda^\eta = \min_{v_\eta \in S^{\eta,s}\setminus\{0\}} R^s(v_\eta)$, and $u \in H^1_s(\Omega)$ be an minimizer associated to λ such that $\|u\|_\Omega = 1$. Then it holds that, for $\forall v_\eta \in S^{\eta,s} \setminus \{0\}$ with $\|u - v_\eta\|_\Omega < 1$,*

$$\lambda \le \lambda^\eta \le \lambda + \frac{|u - v_\eta|^2_{1,\Omega}}{(1 - \|u - v_\eta\|_\Omega)^2} \, . \tag{117}$$

The following results are also well known and will be used later, cf. [15].

Lemma 3. *Let us consider the problem: for the present Ω and a given $f \in L_2(\Omega)$, find $u \in H^1(\Omega)$ such that*

$$(\nabla u, \nabla v)_\Omega = (f, v)_\Omega \quad (\forall v \in H^1(\Omega)) \,. \tag{118}$$

Then such u exists if and only if

$$(f, 1)_\Omega = 0 \,, \tag{119}$$

and is unique up to an additive arbitrary constant function. Moreover, $u \in H^2(\Omega)$ with

$$|u|_{2,\Omega} \leq \|\triangle u\|_\Omega = \|f\|_\Omega \,. \tag{120}$$

Remark 11. *To assure the uniqueness to u, we can for example impose the condition $(u, 1)_\Omega = 0$ on u. The present problem corresponds to the one for the Poisson equation with the homogeneous Neumann boundary condition:*

$$-\triangle u = f \ \text{ in } \ \Omega \,, \quad \frac{\partial u}{\partial n} = 0 \ \text{ on } \ \partial\Omega \,. \tag{121}$$

5.2 A posteriori estimation of $C_0(\alpha, \theta)$

We first give a posteriori estimates to $C_0(\alpha, \theta)$. In this case, $\Omega = T_{\alpha,\theta}$ and $H^1_s(\Omega) = V^0_{\alpha,\theta}$. Let us define an orthogonal projection operator $P^0_{\alpha,\theta} : L_2(T_{\alpha,\theta}) \to L^0_2(T_{\alpha,\theta}) := \{g \in L_2(_{\alpha,\theta}) \mid (g, 1)_{T_{\alpha,\theta}} = 0\}$ by

$$P^0_{\alpha,\theta} g := g - \frac{\int_{T_{\alpha,\theta}} g(x)\, dx}{\int_{T_{\alpha,\theta}} dx} = g - \frac{(g, 1)_{T_{\alpha,\theta}}}{|T_{\alpha,\theta}|} \quad (\forall g \in L_2(T_{\alpha,\theta})) \,, \tag{122}$$

where $|T_{\alpha,\theta}|$ denotes the measure of $T_{\alpha,\theta}$. We can easily show that $P^0_{\alpha,\theta}|H^1(T_{\alpha,\theta})$ is an orthogonal projection operator from $H^1(T_{\alpha,\theta})$ to $V^0_{\alpha,\theta}$ with respect to the standard inner product of $H^1(T_{\alpha,\theta})$: $(u, v)_{1,T_{\alpha,\theta}} := (u, v)_{T_{\alpha,\theta}} + (\nabla u, \nabla v)_{T_{\alpha,\theta}}$ ($\forall u,\ v \in H^1(T_{\alpha,\theta})$). We also denote the present G_s, G^η_s, S^η and $S^{\eta,s}$ respectively by $G^0_{\alpha,\theta}$, $G^{\eta,0}_{\alpha,\theta}$, $S^\eta_{\alpha,\theta}$ and $S^{\eta,0}_{\alpha,\theta}$. Since $S^\eta_{\alpha,\theta}$ contains the constant functions, we find that

$$S^{\eta,0}_{\alpha,\theta} = P^0_{\alpha,\theta} S^\eta_{\alpha,\theta} \,. \tag{123}$$

From now on, we will omit the subscript $T_{\alpha,\theta}$ for the norms, semi-norms and inner products related to this domain. Noting that $\nabla P^0_{\alpha,\theta} v = \nabla v$ and $(f, P^0_{\alpha,\theta} v) = (P^0_{\alpha,\theta} f, v)$ for $\forall v \in H^1(T_{\alpha,\theta})$, eq. (104) for the present $u \in V^0_{\alpha,\theta}$ becomes

$$(\nabla u, \nabla v) = (P^0_{\alpha,\theta} f, v) \quad (\forall v \in H^1(T_{\alpha,\theta})) \,, \tag{124}$$

which reduces to (118) under (119). Likewise, eq. (107) for the present $\{\lambda, u\} \in \mathbf{R} \times V^0_{\alpha,\theta} \backslash \{0\}$ becomes

$$(\nabla u, \nabla v) = \lambda(u, v) \quad (\forall v \in H^1(T_{\alpha,\theta})) \,, \tag{125}$$

since $P^0_{\alpha,\theta} u = u$. By Lemma 3, the above u belongs to $H^2(T_{\alpha,\theta}) \cap V^0_{\alpha,\theta}$ with

$$|u|_2 \leq \lambda \|u\| \,. \tag{126}$$

Under the preceding preparations, let us apply Lemma 2 to estimate the minimum eigenvalue $\lambda^{\eta,0}_{\alpha,\theta}$ of (116) in terms of the one $\lambda^0_{\alpha,\theta}$ of (107) or (125). The minimizer associated to $\lambda^0_{\alpha,\theta}$ is denoted by $u^0_{\alpha,\theta}$ with the normalization condition $\|u^0_{\alpha,\theta}\| = 1$. As v_η in (117), we can take various candidates from $S^{\eta,0}_{\alpha,\theta}$. One possibility is to utilize the interpolant $\Pi^{\eta,1}u^0_{\alpha,\theta} \in S^\eta_{\alpha,\theta}$ of $u^0_{\alpha,\theta}$. Unfortunately, it may be outside $S^{\eta,0}_{\alpha,\theta}$, but its projection $P^0_{\alpha,\theta}\Pi^{\eta,1}u^0_{\alpha,\theta}$ can be used thanks to (123). By taking advantage of properties of the orthogonal projection (122), we find that

$$|u^0_{\alpha,\theta} - P^0_{\alpha,\theta}\Pi^{\eta,1}u^0_{\alpha,\theta}|_1 = |u^0_{\alpha,\theta} - \Pi^{\eta,1}u^0_{\alpha,\theta}|_1 \ , \tag{127}$$

$$\|u^0_{\alpha,\theta} - P^0_{\alpha,\theta}\Pi^{\eta,1}u^0_{\alpha,\theta}\| = \|P^0_{\alpha,\theta}(u^0_{\alpha,\theta} - \Pi^{\eta,1}u^0_{\alpha,\theta})\| \leq \|u^0_{\alpha,\theta} - \Pi^{\eta,1}u^0_{\alpha,\theta}\| \ . \tag{128}$$

Using (110) and (126), we can evaluate the above in terms of η, $\lambda^0_{\alpha,\theta}$, $C_{4,\eta}$ and $C_{5,\eta}$. Unfortunately, we have not necessarily obtained accurate theoretical upper bounds for $C_{5,\eta}$ as was noted in Section 3.3. So we should try to avoid the use of such a constant.

Another possibility is to use $\tilde{u}^0_{\eta,\alpha,\theta} := \lambda^0_{\alpha,\theta}G^{\eta,0}_{\alpha,\theta}u^0_{\alpha,\theta}$, which is surely in $S^{\eta,0}_{\alpha,\theta}$ and is suggested by the identity $u^0_{\alpha,\theta} = \lambda^0_{\alpha,\theta}G^0_{\alpha,\theta}u^0_{\alpha,\theta}$. For this choice, we have

$$|u^0_{\alpha,\theta} - \tilde{u}^0_{\eta,\alpha,\theta}|_1 \leq |u^0_{\alpha,\theta} - P^0_{\alpha,\theta}\Pi^{\eta,1}u^0_{\alpha,\theta}|_1 = |u^0_{\alpha,\theta} - \Pi^{\eta,1}u^0_{\alpha,\theta}|_1 \ , \tag{129}$$

$$\|u^0_{\alpha,\theta} - \tilde{u}^0_{\eta,\alpha,\theta}\| \leq |u^0_{\alpha,\theta} - \tilde{u}^0_{\eta,\alpha,\theta}|_1 \inf_{v_\eta \in S^{\eta,0}_{\alpha,\theta}} \sup_{g \in L_2(T_{\alpha,\theta})\backslash\{0\}} \frac{|G^0_{\alpha,\theta}g - v_\eta|_1}{\|g\|} \ . \tag{130}$$

In this case, we only need former part of (110), that is, the values of η, $\lambda^0_{\alpha,\theta}$ and $C_{4,\eta}$, and can actually avoid the use of $C_{5,\eta}$.

Based on the above considerations, we have now the following two a priori error estimates.

Lemma 4 (A priori estimates for $\lambda^{\eta,0}_{\alpha,\theta}$). *Let $\lambda^0_{\alpha,\theta}$ and $\lambda^{\eta,0}_{\alpha,\theta}$ be defined as above. Then, if $C_{5,\eta}\eta^2\lambda^0_{\alpha,\theta} < 1$,*

$$\lambda^0_{\alpha,\theta} \leq \lambda^{\eta,0}_{\alpha,\theta} \leq \lambda^0_{\alpha,\theta} + \frac{(C_{4,\eta}\eta\lambda^0_{\alpha,\theta})^2}{(1 - C_{5,\eta}\eta^2\lambda^0_{\alpha,\theta})^2} \ . \tag{131}$$

Similarly, if $C^2_{4,\eta}\eta^2\lambda^0_{\alpha,\theta} < 1$, then

$$\lambda^0_{\alpha,\theta} \leq \lambda^{\eta,0}_{\alpha,\theta} \leq \lambda^0_{\alpha,\theta} + \frac{(C_{4,\eta}\eta\lambda^0_{\alpha,\theta})^2}{(1 - C^2_{4,\eta}\eta^2\lambda^0_{\alpha,\theta})^2} \ . \tag{132}$$

Remark 12. *In actual application of the above estimates, where the exact value of $C_{4,\eta}$ ($C_{5,\eta}$, resp.) may not be available, we can use an appropriate upper bound $\tilde{C}_{4,\eta}$ ($\tilde{C}_{5,\eta}$, resp.). From the considerations in Section 3.3 for concrete values of these constants, (131) would give a better bounding than (132), if an accurate upper bound $\tilde{C}_{5,\eta}$ of $C_{5,\eta}$ becomes available.*

Let us define two functions related to (131) and (132):

$$\varphi_{0,1}(t) := t + \frac{(C_{4,\eta}\eta t)^2}{(1 - C_{5,\eta}\eta^2 t)^2} \qquad (0 < t < \frac{1}{C_{5,\eta}\eta^2}) \ , \tag{133}$$

$$\varphi_{0,2}(t) := t + \frac{(C_{4,\eta}\eta t)^2}{(1 - C^2_{4,\eta}\eta^2 t)^2} \qquad (0 < t < \frac{1}{(C_{4,\eta}\eta)^2}) \ , \tag{134}$$

where t is the variable, while other quantities are considered just parameters. Since these two functions are continuous and monotonically increasing on their domains of definition, they have their inverse functions, which are defined in $]0, +\infty[$ and will be denoted by the popular notations $\varphi_{0,1}^{-1}$ and $\varphi_{0,2}^{-1}$. Of course, these inverse functions are also continuous and monotonically increasing. Then we can easily obtain the following a posteriori estimates or boundings of $\lambda_{\alpha,\theta}^{0}$ by numerically obtained $\lambda_{\alpha,\theta}^{\eta,0}$.

Theorem 6 (A posteriori estimates for $\lambda_{\alpha,\theta}^{0}$). *Let $\lambda_{\alpha,\theta}^{0}$, $\lambda_{\alpha,\theta}^{\eta,0}$, $\varphi_{0,1}^{-1}$ and $\varphi_{0,2}^{-1}$ be defined as above. Then it holds that*

$$\varphi_{0,1}^{-1}(\lambda_{\alpha,\theta}^{\eta,0}) \le \lambda_{\alpha,\theta}^{0} \le \lambda_{\alpha,\theta}^{\eta,0} \quad \text{if} \ \ \lambda_{\alpha,\theta}^{\eta,0} < \frac{1}{C_{5,\eta}\eta^2} \, , \tag{135}$$

$$\varphi_{0,2}^{-1}(\lambda_{\alpha,\theta}^{\eta,0}) \le \lambda_{\alpha,\theta}^{0} \le \lambda_{\alpha,\theta}^{\eta,0} \quad \text{if} \ \ \lambda_{\alpha,\theta}^{\eta,0} < \frac{1}{(C_{4,\eta}\eta)^2} \, . \tag{136}$$

Proof. From the preceding theorem, we have, for example, $(0 <)\lambda_{\alpha,\theta}^{0} \le \varphi_{0,1}(\lambda_{\alpha,\theta}^{0}) \le \varphi_{0,1}(\lambda_{\alpha,\theta}^{\eta,0})$ if $\lambda_{\alpha,\theta}^{\eta,0} < 1/(C_{5,\eta}\eta^2)$. Then (135) follows immediately by operating $\varphi_{0,1}^{-1}$ to this inequality, while (136) can be obtained similarly. □

It is now straightforward to obtain boundings to the constant $C_0(\alpha, \theta)$. For example, we have from (135) that

$$1/\sqrt{\lambda_{\alpha,\theta}^{\eta,0}} \le C_0(\alpha,\theta) \le 1/\sqrt{\varphi_{0,1}^{-1}(\lambda_{\alpha,\theta}^{\eta,0})} \quad \text{if} \ \ \lambda_{\alpha,\theta}^{\eta,0} < \frac{1}{C_{5,\eta}\eta^2} \, . \tag{137}$$

The results (135) and (136) can be also viewed as a posteriori error estimates for $\lambda_{\alpha,\theta}^{\eta,0}$, since (135), for example, can be rewritten as $0 \le \lambda_{\alpha,\theta}^{\eta,0} - \lambda_{\alpha,\theta}^{0} \le \lambda_{\alpha,\theta}^{\eta,0} - \varphi_{0,1}^{-1}(\lambda_{\alpha,\theta}^{\eta,0})$.

Remark 13. *The results in Lemma 4 and Theorem 6, i. e., estimates (131), (132), (135) and (136), also hold for λ and λ^η of Lemma 2 in the case where $H_s^1(\Omega) = H_0^1(\Omega)$ and $S^{\eta,s} = S^\eta \cap H_0^1(\Omega)$. In such a case, Lemma 3 cannot be used, but the corresponding G_s has the property $G_s : L_2(\Omega) \to H_0^1(\Omega) \cap H^2(\Omega)$ with $|G_s f|_2 \le \|f\|$ $(\forall f \in L_2(\Omega))$, since Ω is a bounded convex polygonal domain* [15]. *Moreover, we cannot utilize projection operators like $P_{\alpha,\theta}^{0}$ above, but, instead, we can take full advantage of the property $\Pi^{\eta,1} G_s f \in S^{\eta,s}$ $(\forall f \in L_2(\Omega))$ for the present $\Pi^{\eta,1}$, G_s and $S^{\eta,s}$. The present case is related to the approximation of the Poincaré constant* [2], *which is essentially the numerical evaluation of the smallest eigenvalue of $-\Delta$ with the homogeneous Dirichlet boundary condition. Similarly, $C_0(\alpha, \theta)$ is associated with the second eigenvalue of $-\Delta$ over $\Omega = T_{\alpha,\theta}$ with the homogeneous Neumann boundary condition, as was noted in Section 2.*

5.3 A posteriori estimation of $C_i(\alpha, \theta)$'s $(i = 1, 2, 3)$

Secondly, we give a posteriori estimates to $C_i(\alpha, \theta)$'s $(1 \le i \le 3)$. In these cases, let us choose or use the notations $\Omega = T_{\alpha,\theta}$, $H_s^1(\Omega) = V_{\alpha,\theta}^{i}$, $G_s = G_{\alpha,\theta}^{i}$, $G_s^\eta = G_{\alpha,\theta}^{\eta,i}$, $S^\eta = S_{\alpha,\theta}^{\eta}$ and $S^{\eta,s} = S_{\alpha,\theta}^{\eta,i}$ for each $i \in \{1, 2, 3\}$. Let us define an operator $P_{\alpha,\theta}^{i} : H^1(T_{\alpha,\theta}) \to V_{\alpha,\theta}^{i}$ $(i \in \{1, 2, 3\})$ by

$$P_{\alpha,\theta}^{i} v := v - \frac{1}{|e_i|} \int_{e_i} v \, ds \qquad (\forall v \in V_{\alpha,\theta}^{i}) \, , \tag{138}$$

where $|e_i|$ denotes the length of edge e_i. Unlike $P^0_{\alpha,\theta}$, the above operators are not well-defined over $L_2(T_{\alpha,\theta})$, but the following relations similar to (123) still hold :

$$S^{\eta,i}_{\alpha,\theta} = P^i_{\alpha,\theta} S^{\eta}_{\alpha,\theta} \ \ (1 \le i \le 3) \ . \tag{139}$$

Suggested by [26], let us introduce quadratic functions f_i's $(1 \le i \le 3)$ of $x = \{x_1, x_2\}$ by

$$f_i(x_1, x_2) := \frac{|e_i|}{4|T_{\alpha,\theta}|}[(x_1 - x^i_1)^2 + (x_2 - x^i_2)^2] \ , \tag{140}$$

where

$$x^1 = \{x^1_1, x^1_2\} = B(\alpha \cos\theta, \alpha \sin\theta), \ x^2 = \{x^2_1, x^2_2\} = A(1, 0), \ x^3 = \{x^3_1, x^3_2\} = O(0, 0). \tag{141}$$

These functions are sufficiently smooth and satisfy

$$\frac{\partial f_i}{\partial n} = \delta_{ij} \ \text{ on } \ e_j \ \ \text{ for } \ \forall i, \ \forall j \in \{1, 2, 3\} \ . \tag{142}$$

Then, for $\forall v \in H^1(T_{\alpha,\theta})$, we find that

$$\int_{e_i} v \ ds = (\nabla f_i, \nabla v) + (\Delta f_i, v) \ , \tag{143}$$

so that (138) can be rewritten by

$$P^i_{\alpha,\theta} v := v - \frac{1}{|e_i|}[(\nabla f_i, \nabla v) + (\Delta f_i, v)] \qquad (\forall v \in H^1(T_{\alpha,\theta})) \ . \tag{144}$$

Similarly to (124), eq. (104) for the present $u \in V^i_{\alpha,\theta}$ becomes

$$(\nabla u, \nabla v) = (f, P^i_{\alpha,\theta} v) \qquad (\forall v \in H^1(T_{\alpha,\theta})) \ , \tag{145}$$

which can be rewritten by

$$\left(\nabla(u + \frac{(f,1)}{|e_i|} f_i), \nabla v\right) = (f - \frac{(f,1)}{|e_i|}\Delta f_i, v) \qquad (\forall v \in H^1(T_{\alpha,\theta})) \ . \tag{146}$$

By Lemma 3, we find that $u + \frac{(f,1)}{|e_i|} f_i \in H^2(T_{\alpha,\theta})$ with

$$\left| u + \frac{(f,1)}{|e_i|} f_i \right|_2 \le \| f - \frac{(f,1)}{|e_i|}\Delta f_i \| \ , \tag{147}$$

and hence, by using the triangle and Schwarz inequalities,

$$|u|_2 \le \|f\| + \frac{|(f,1)|}{|e_i|}(|f_i|_2 + \|\Delta f_i\|) \le \|f\| \left[1 + \frac{\sqrt{|T_{\alpha,\theta}|}}{|e_i|}(|f_i|_2 + \|\Delta f_i\|) \right] \ . \tag{148}$$

Clearly, it holds that

$$|T_{\alpha,\theta}| = \frac{\alpha}{2}\sin\theta\ ,\ \ |e_1| = 1\ ,\ \ |e_2| = \alpha\ ,\ \ |e_3| = \sqrt{1+\alpha^2-2\alpha\cos\theta}\ ,$$
$$|f_i|_2 = \frac{\sqrt{2}}{2}\|\Delta f_i\|\ ,\ \ \Delta f_i(x_1,x_2) = \frac{|e_i|}{|T_{\alpha,\theta}|}\ , \tag{149}$$

so that we have, for $\forall i \in \{1,2,3\}$,

$$|u|_2 \le (2+\frac{\sqrt{2}}{2})\|f\|\ . \tag{150}$$

Similarly to (145), eq.(107) for the present $\{\lambda, u\} \in \mathbf{R} \times V^i_{\alpha,\theta}$ $(1 \le i \le 3)$ becomes

$$(\nabla u, \nabla v) = \lambda(u, P^i_{\alpha,\theta} v) \qquad (\forall v \in H^1(T_{\alpha,\theta}))\ . \tag{151}$$

Thus, we can utilize the results for (145) by taking f in (145) as λu in (151). The approximation problems corresponding to (112) and (116) are also given by using $S^{\eta,i}_{\alpha,\theta}$'s $(1 \le i \le 3)$. Then, just like Lemma 4 and Theorem 6 for $C_0(\alpha,\theta)$, we have the following results for $C_i(\alpha,\theta)$'s $(1 \le i \le 3)$.

Theorem 7 (A priori and a posteriori estimates for $\lambda^{\eta,i}_{\alpha,\theta}$'s $(1 \le i \le 3)$). *For each $i \in \{1,2,3\}$, let $\lambda^i_{\alpha,\theta}$ and $\lambda^{\eta,i}_{\alpha,\theta}$ be respectively the smallest eigenvalues of* (107) *and* (116) *in the present case where $H^1_s(\Omega) = V^i_{\alpha,\theta}$ and $S^{\eta,s} = S^{\eta,i}_{\alpha,\theta}$. Then, if $(MC_{4,\eta}\eta)^2\lambda^i_{\alpha,\theta} < 1$ with $M := 2+\sqrt{2}/2$, it holds that*

$$\lambda^i_{\alpha,\theta} \le \lambda^{\eta,i}_{\alpha,\theta} \le \lambda^i_{\alpha,\theta} + \frac{(MC_{4,\eta}\eta\lambda^i_{\alpha,\theta})^2}{(1-M^2C^2_{4,\eta}\eta^2\lambda^i_{\alpha,\theta})^2}\ . \tag{152}$$

and, if $\lambda^{\eta,i}_{\alpha,\theta} < \dfrac{1}{(MC_{4,\eta}\eta)^2}$,

$$\varphi_i^{-1}(\lambda^{\eta,i}_{\alpha,\theta}) \le \lambda^i_{\alpha,\theta} \le \lambda^{\eta,i}_{\alpha,\theta}\ , \tag{153}$$

where

$$\varphi_i(t) := t + \frac{(MC_{4,\eta}\eta t)^2}{(1-M^2C^2_{4,\eta}\eta^2 t)^2} \qquad \left(0 < t < \frac{1}{(MC_{4,\eta}\eta)^2}\ ;\ \ 1 \le i \le 3\right)\ , \tag{154}$$

which is continuous and monotonically increasing.

Remark 14. *Because of the factor $M \approx 2.7071...$, efficiency of* (152) *is worse than that of* (132). *In the present case, estimates corresponding to* (131) *and using $C_{5,\eta}$ do not appear to be fully effective unlike in the preceding subsection. This is attributed to the fact that we cannot at present obtain desirable estimates for $\|u - P^i_{\alpha,\theta}\Pi^{\eta,1}u\|$ $(\forall u \in V^i_{\alpha,\theta} \cap H^2(T_{\alpha,\theta});\ 1 \le i \le 3)$, since $P^i_{\alpha,\theta}$ is not definable over $L_2(T_{\alpha,\theta})$ and hence we cannot take advantage of the best approximation property with respect to the L_2 norm.*

6 Numerical results

We performed numerical computations to see the actual dependence of various constants on α and θ. Furthermore, we also utilized the obtained exact values or upper bounds of such constants to give quantitative a posteriori error estimates for some eigenvalue problems.

6.1 Computational methods

To obtain approximate values of error constants, we can utilize the FEM quite effectively. In particular, we used the most popular P_1 triangular finite element for numerical computations of $C_i(\alpha, \theta)$'s for $0 \leq i \leq 3$ by preparing appropriate triangulations of $T_{\alpha,\theta}$. For $C_4(\alpha, \theta)$ and $C_5(\alpha, \theta)$, it is natural to use various triangular finite elements for Kirchhoff plate bending problems, since the associated partial differential equations are of 4th order as is noted in Section 2. In our actual computations, we used the discrete Kirchhoff triangular element presented in [17]. On the other hand, we can also use the Siganevich approach for computation of $C_4(\alpha, \theta)$, which adopts the P_1 triangle and a kind of penalty method for a system of 2nd order partial differential equations similar to the incompressible Stokes system [30]. This method also works well if the penalty parameter is carefully chosen.

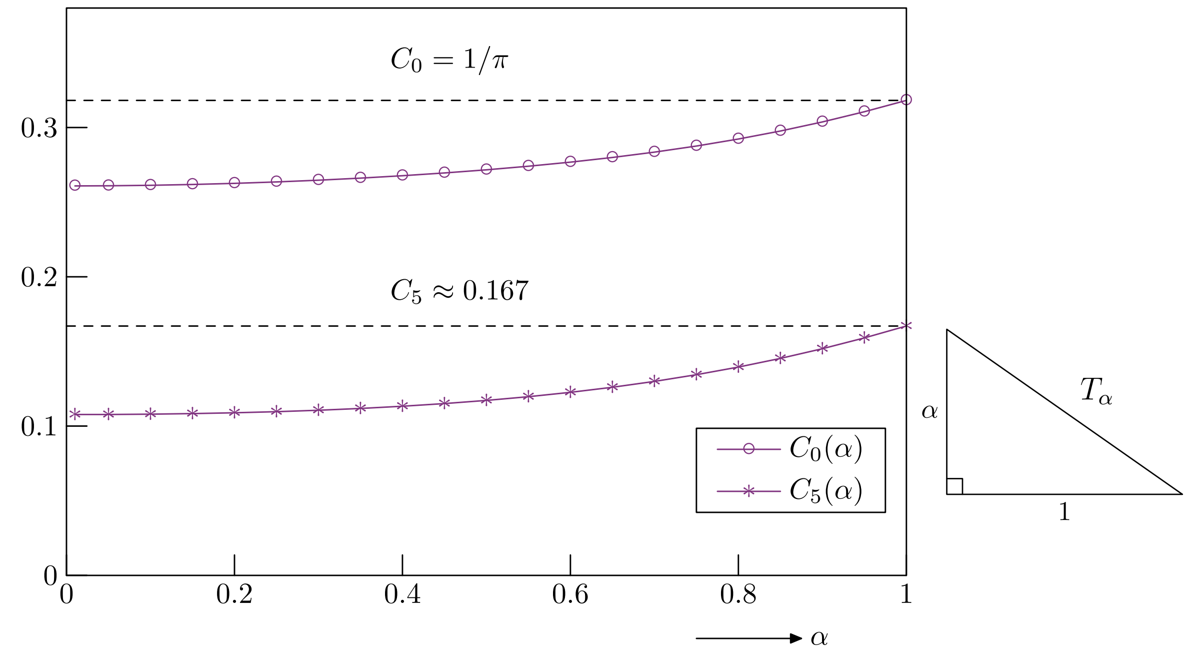
In every case, we have a matrix eigenvalue problem as the discretization of the original eigenvalue problem described by a weak form. More specifically, it is a generalized matrix eigenvalue problem with respect to unknown eigenvectors of nodal values of approximate eigenfunctions, and it can be solved for example by the inverse iteration method and the subspace iteration method [9]. A difficulty in deriving such matrix eigenvalue problems is how to deal with linear constraint conditions imposed on the spaces $V^i_{\alpha,\theta}$ for $i = 0, 1, 2, 3$. Similar constraint conditions are also necessary to deal with, if we compute $C_4(\alpha, \theta)$ by the Siganevich method. On the other hand, we do not have such a difficulty in computing $C_4(\alpha, \theta)$ and $C_5(\alpha, \theta)$ by Kirchhoff type triangular elements, where the linear constraints $v(O) = v(A) = v(B) = 0$ for $V^4_{\alpha,\theta}$ can be handled as homogeneous "point" conditions.

One possible method is to eliminate some unknown nodal values by using the linear constraints, but then we have non-sparse coefficient matrices in general. Another method is to use the Lagrange multiplier method, which does not essentially destroy the global sparseness of the matrices. We tested both approaches with reasonable results. Various iteration methods may be also available for the same purposes.

The numerical results below are obtained by FEM in the double or quadruple precision arithmetics, without evaluating the errors strictly by the interval analysis. But their accuracy appears to be reasonable at least in graphical level, since finer mesh computations give essentially the same graphs. We hope that effective verification methods will be established in near future, so that numerical results can be of strictly mathematical significance.

6.2 Numerical results for error constants

Here, we first show some results for $C_i(\alpha)$'s $(0 \leq i \leq 5)$ by the P_1 finite element and the Kirchhoff triangular element in [17] with the uniform triangulation of the domain T_α. In such calculations, T_α is subdivided into a number of small triangles congruent to $T_{\alpha,\pi/2,h}$ with e. g. $h = 1/20$. The penalty method in [30] is also tested to calculate $C_4(\alpha)$ approximately.



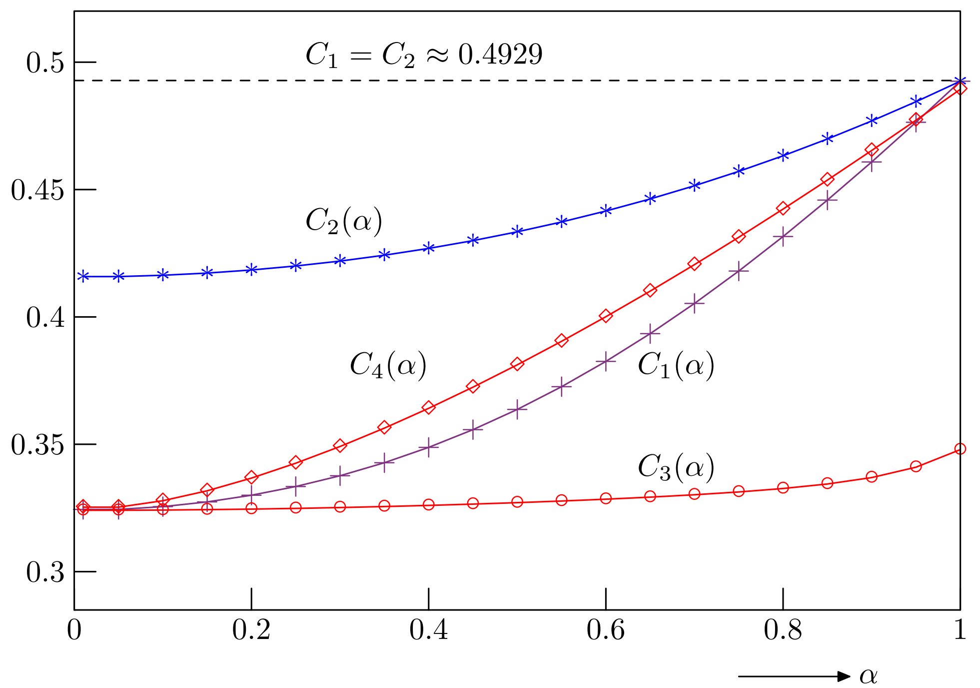


Figure 2: Numerically obtained graphs for $C_i(\alpha)$'s $\quad (0 \le i \le 5;\ 0 < \alpha \le 1)$

Figure 2 consists of two parts and illustrates the graphs of approximate $C_i(\alpha)$'s $(0 \le i \le 5)$ versus $\alpha \in]0,1]$. Exact values of C_0 and $C_1 = C_2$ together with an approximate value of C_5 are also included as horizontal lines in graphs. At $\alpha = 1$, the approximate values coincide well with the available exact ones in Theorem 3, and we can numerically see that $C_1 (= C_2)$ is a nice upper bound of C_4. For general α, the monotonically increasing behaviors theoretically predicted for $C_i(\alpha)$'s $(i = 0, 1, 2, 3, 5)$ as well as the relation $C_4(\alpha) \le \min\{C_1(\alpha), C_2(\alpha)\}$ are also well observable in the graphs. The present numerical results suggest that $C_4(\alpha)$ is also monotonically increasing, but we have not succeeded in proving such a conjecture. Moreover, when $\alpha \approx 0$, the numerical results agree well with the exact right limits given in Table 1 based on the asymptotic analysis.

For $C_4(\alpha)$, we tested two methods, that is, the P_1 triangle with the penalty method and the Kirchhoff triangle. These two methods turned out to give almost the same results if the meshes are relatively fine and the penalty parameter is appropriately chosen. The graph for $C_4(\alpha)$ in Fig. 2 is actually obtained by the Kirchhoff element, but is indistinguishable in graphical level from the one by the penalty method.

We also performed numerical computations to see the validity and effectiveness of the upper bounds for $C_4(\alpha, \theta)$ given in Corollary 1 and Theorem 2. We here show just one example with $0 < \alpha \le 1$ and $\theta = 2\pi/3$. That is, we numerically compare $C_4(\alpha, \theta)$, $C_4^{(1)}(\alpha, \theta) := C_4(\alpha)\phi_4(\theta)$ and $C_4^{(2)}(\alpha, \theta) := \nu(\alpha, \theta)/(\sqrt{2}\sin\theta)$ for $\theta = 2\pi/3$, where the latter two functions come from (32) and (35). In the computations, $C_4(\alpha)$, $C_1(\alpha, 2\pi/3)$ and $C_2(\alpha, 2\pi/3)$ were also obtained numerically for use in the above two upper bound formulas, and the uniform meshes were again employed. The results are shown in Fig. 3, and we can see that both $C_4^{(1)}(\alpha, 2\pi/3)$ and $C_4^{(2)}(\alpha, 2\pi/3)$ give upper bounds to $C_4^{(1)}(\alpha, 2\pi/3)$ numerically. Moreover, at least in the present case, $C_4^{(2)}(\alpha, 2\pi/3)$ is superior to $C_4^{(1)}(\alpha, 2\pi/3)$ as an upper bound.

Figures 4 and 5 illustrate numerically obtained contour lines for $C_i(\alpha, \theta)$'s in the $\alpha-\theta$ polar coordinates, where the abscissa denotes $\alpha\cos\theta$, and the ordinate does $\alpha\sin\theta$. The unit circle $\alpha = 1$ is also shown by a dotted curve. The minimum required range for α and θ is specified by (1), but the contour lines are shown for wider ranges, so that we can easily see global behaviors of error constants. These results can be also useful for practical adaptive computations to specify constants in error indicators approximately. Of course, for strict mathematical analysis like numerical verification, we need correct upper bounds to error constants. The contour lines are sometimes cut off in the portions where the expected accuracy may be insufficient. For example, when $\alpha \approx 0$ or $|\theta - \pi/2| \approx \pi/2$, it requires extraordinarily fine meshes to retain sufficient accuracy. The behavior of $C_4(\alpha, \theta)$ appears to be the most complicated among all the constants, and the necessity of the maximum angle condition can be visually recognized. The other constants seem to be uniformly bounded over the unit disk $\alpha \le 1$.

6.3 A posteriori estimates of eigenvalues

To apply the results in Section 5, let us consider a posteriori estimates or boundings for $C_0 = C_0(1, \pi/2)$ and $C_1 = C_1(1, \pi/2)$ based on the P_1 FEM. We denote the associated eigenvalues by $\lambda_0 = C_0^{-2}$ and $\lambda_1 = C_1^{-2}$, and the results will be shown for these eigenvalues. We can also give a posteriori estimates to general $C_i(\alpha, \theta)$'s for $i = 0, 1, 2, 3$ by the same approach.

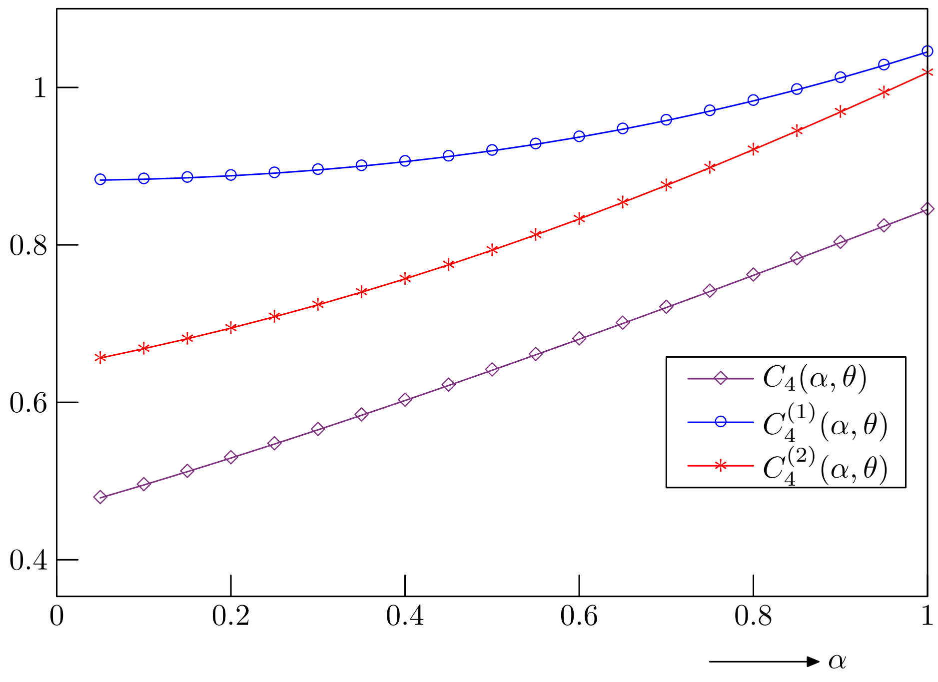


Figure 3: Two upper bounds of $C_4(\alpha,\theta)$ for $\theta = 2\pi/3$

Table 2 gives the boundings for λ_0 based on (135) and (136) of Theorem 6 and those for λ_1 based on (153) of Theorem 7. We tested several meshes, which are uniform ones composed of small triangles similar to the entire domain T as is shown in the same table. The values of parameters $\tilde{C}_{4,\eta}$, $\tilde{C}_{5,\eta}$ and η that are necessary to use the formulas (135), (136) and (153) are specified here as

$$\tilde{C}_{4,\eta} = 0.5\,, \quad \tilde{C}_{5,\eta} = 0.17\,, \quad \eta = 1/N\,, \tag{155}$$

where N is the number of elements along each edge of T ($N = 4$ in the figure of Table 2). Notice here that $\tilde{C}_{4,\eta} = 0.5$ is a theoretical upper bound of $C_{4,\eta}$ (cf. Remark 12), but the above $\tilde{C}_{5,\eta} = 0.17$ is only a numerically obtained approximate upper bound of $C_{5,\eta}$ at present. We tested (135) only to see its effectiveness experimentally.

We can observe that the present simple methods can actually bound C_0 and C_1 from both above and below. As is expected, (135) gives better lower bounds than (136) for coarser meshes. Table 2 also shows that the lower bounds obtained for C_1 are in general rougher than those for C_0. This is probably attributed to the existence of the factor $M = 2 + \sqrt{2}/2$. Even in this case, we can obtain reasonable results by mesh refinement.

As another application of our method, let us consider the bounding of the first eigenvalue for $-\Delta$ subjected to the Dirichlet boundary condition for the right n-polygonal domain Ω_n ($n \geq 3$), circumscribing the unit disk Ω_∞ centered at the origin. In this case, the formulas in Lemma 4 and Theorem 6 can be used without modifications as is noted in Remark 13, since each Ω_n is convex. It is well known that the first eigenvalue for Ω_n is monotonically increasing in n and is bounded from above by that for Ω_∞. The eigenvalues for $n = 4$ and $n = \infty$ are known as $\pi^2/2$ and the square of the first zero of the Bessel function J_0, respectively, but it is difficult to determine the exact values for general n. So we will numerically evaluate

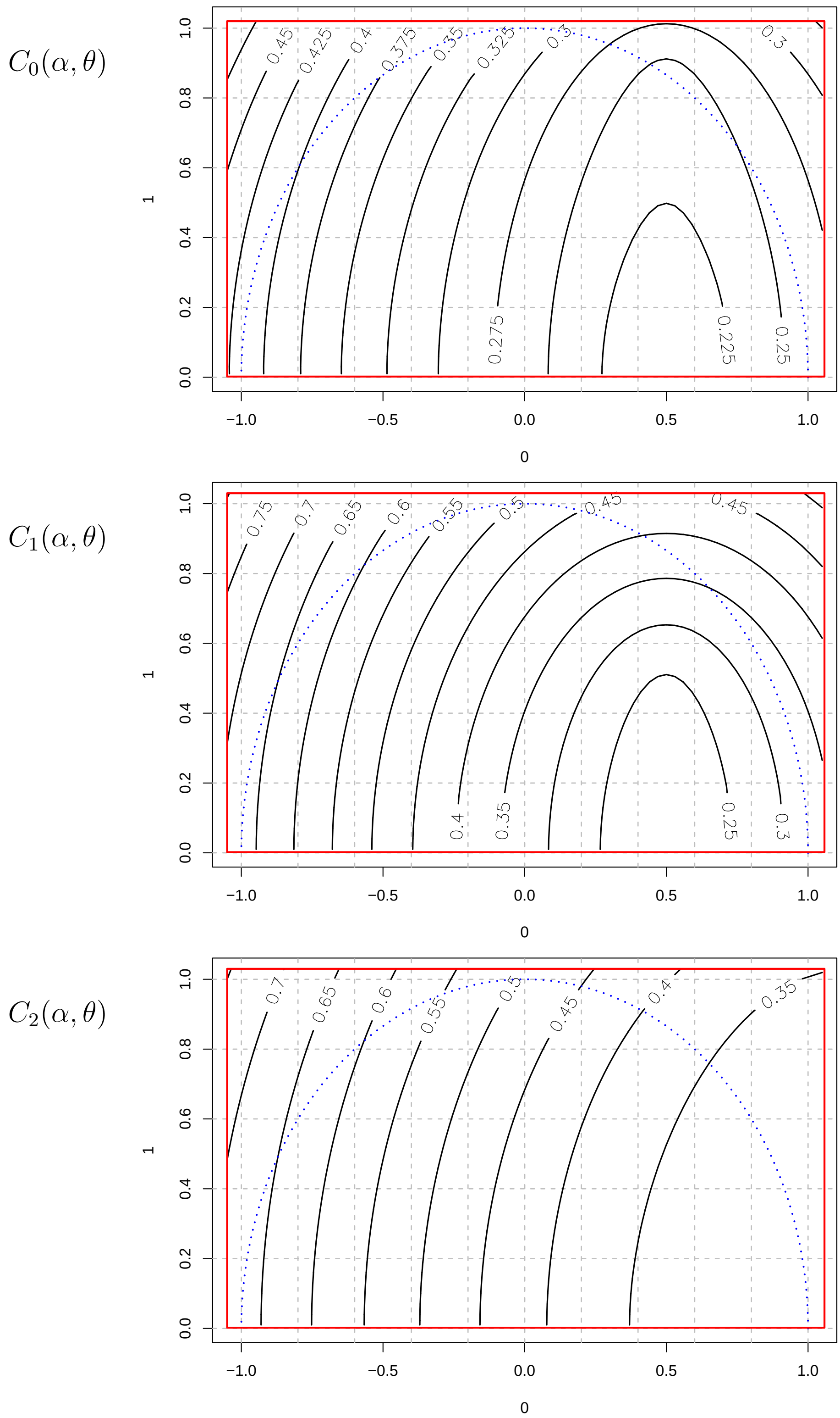


Figure 4: Contour lines of $C_i(\alpha, \theta)$ for $i = 0, 1, 2$

$C_3(\alpha, \theta)$

$C_4(\alpha, \theta)$

$C_5(\alpha, \theta)$

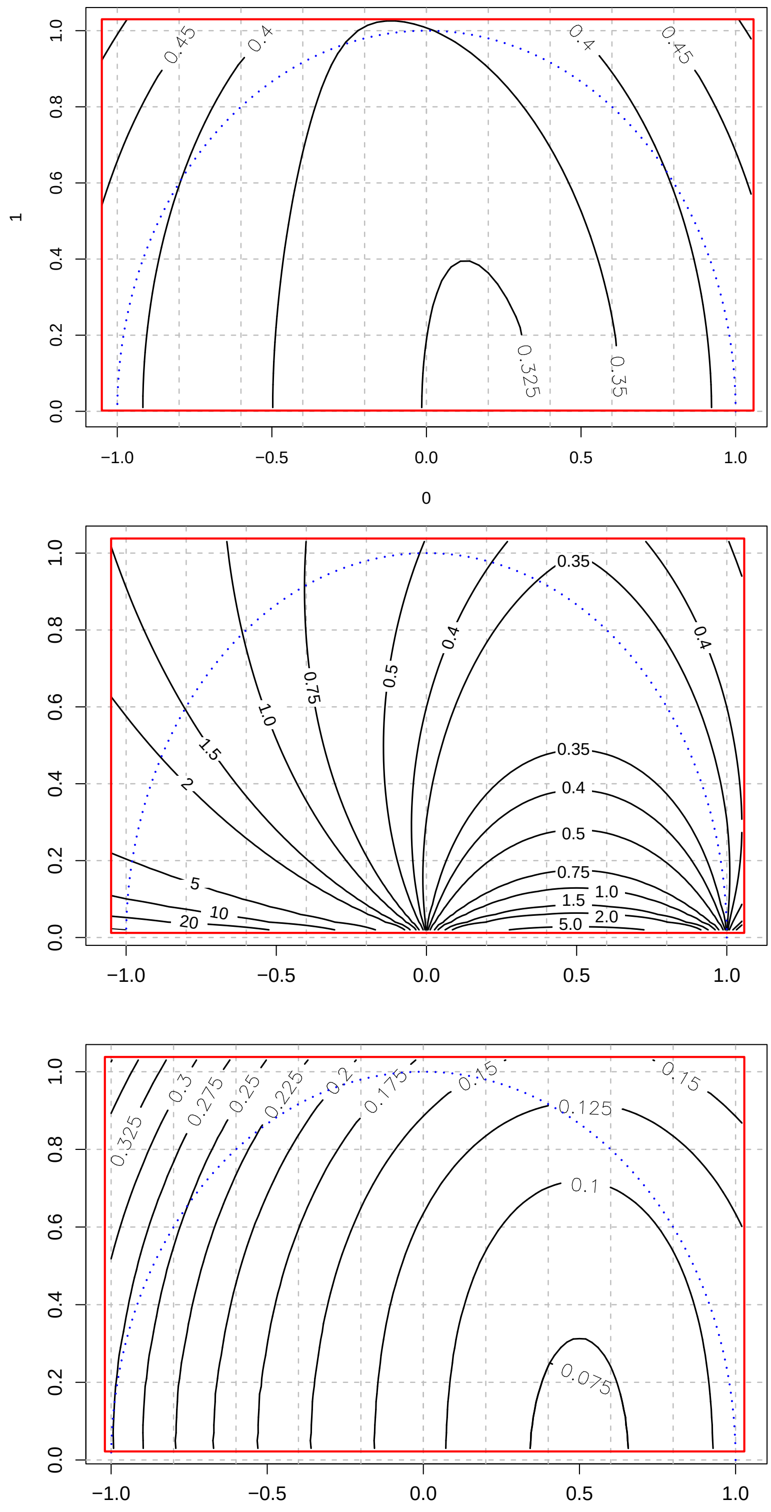


Figure 5: Contour lines of $C_i(\alpha, \theta)$ for $i = 3, 4, 5$

such eigenvalues for several n with a posteriori estimates. At present, such estimates are just numerical, but they will be strictly mathematical estimates when appropriate verification methods become available.

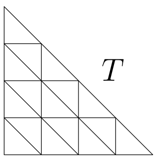
As meshes, we first triangulate the right triangle $\triangle OAB$ with $OA = 1$, $AB = \tan(\pi/n)$ and $\angle OAB = \pi/2$ just as we did for T and T_α in the preceding problems by dividing each edges uniformly into N segments. Then by a reflection and rotations, we can obtain whole meshes for Ω_n, see Fig. 6. Then we can use (136) with

$$\tilde{C}_{4,\eta} = 0.5, \quad \eta = \begin{cases} \sqrt{3}/N & \text{if } \ n = 3 \\ 1/N & \text{if } \ n \geq 4 \end{cases}, \tag{156}$$

where $\alpha \leq 1$ in all the cases.

The obtained results are summarized in Table 3, from which we can experimentally see the effectiveness of our bounding method.

Table 2: A posteriori estimates for λ_0 and λ_1



$\eta = 1/N, \quad N = 4$ in the left figure

† Approximate eigenvalue is outside the domain of definition for φ_1^{-1}.

N	bounds for λ_0 by $\varphi_{0,1}^{-1}$	bounds for λ_0 by $\varphi_{0,2}^{-1}$	bounds for λ_1 by φ_1^{-1}
2	$5.9890 < \lambda_0 < 11.7155$	$6.5550 < \lambda_0 < 11.7155$	$\lambda_1 < 4.3071^\dagger$
3	$7.8874 < \lambda_0 < 10.7213$	$8.1463 < \lambda_0 < 10.7213$	$1.9780 < \lambda_1 < 4.2102$
4	$8.7512 < \lambda_0 < 10.3570$	$8.8616 < \lambda_0 < 10.3570$	$2.6006 < \lambda_1 < 4.1713$
8	$9.6055 < \lambda_0 < 9.9946$	$9.6143 < \lambda_0 < 9.9946$	$3.6537 < \lambda_1 < 4.1304$
16	$9.8054 < \lambda_0 < 9.9012$	$9.8060 < \lambda_0 < 9.9012$	$3.9982 < \lambda_1 < 4.1196$
32	$9.8537 < \lambda_0 < 9.8776$	$9.8537 < \lambda_0 < 9.8776$	$4.0864 < \lambda_1 < 4.1168$
64	$9.8656 < \lambda_0 < 9.8716$	$9.8656 < \lambda_0 < 9.8716$	$4.1085 < \lambda_1 < 4.1161$
(∞)	$\lambda_0 = \pi^2 = 9.869604...$		$\lambda_1 \approx 4.115858$

7 Concluding remarks

We have obtained some explicit relations for the dependence of several interpolation error constants on geometric parameters of triangular finite elements. In particular, we have succeeded in determining some special constants including the Babuška-Aziz constant from very simple

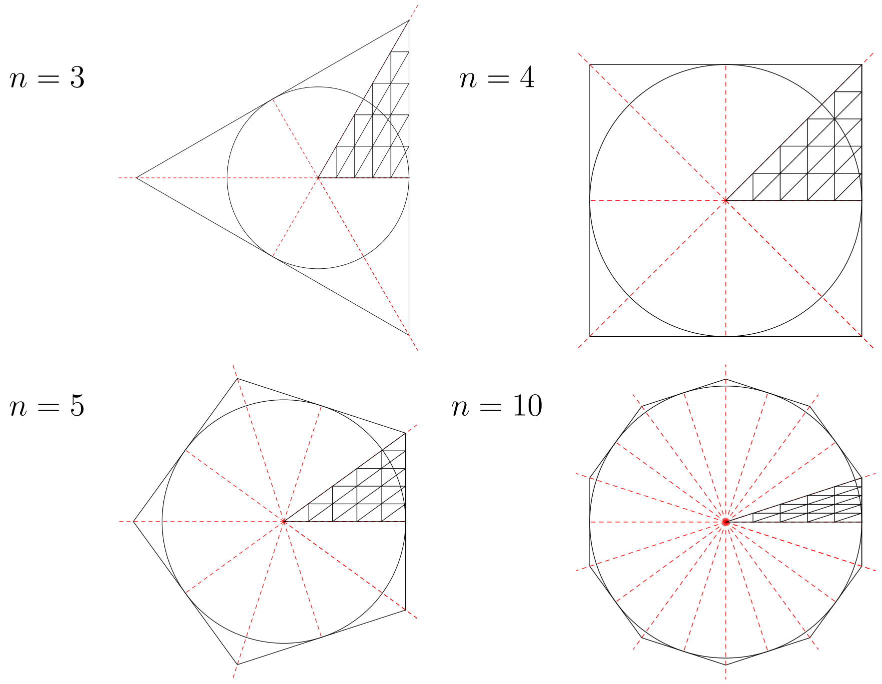


Figure 6: Meshes for n-polygonal domains Ω_n with $N = 5 : n = 3,\ 4,\ 5, 10$

equations. We can effectively utilize these results to give upper bounds of various a priori and a posteriori error estimates of finite element solutions based on the P_1 and/or P_0 approximate functions. Some numerical results were also given to see the effectiveness of our analysis and the actual behaviors of the error constants. To obtain more clear picture for the dependence of the interpolation error constants, we should also perform various analyses including numerical analysis with verifications, asymptotic analysis etc.

We have mainly considered the conforming P_1 triangle, which can naturally construct subspaces of H^1 space over the entire domain. But there also exists a non-conforming counterpart, which is also based on the piecewise linear polynomials but uses as nodes the midpoints of edges or edges themselves [11, 31]. Analysis of such an element is more complicated, since we must additionally evaluate the errors induced by the interelement discontinuity of the approximate functions. Still we can obtain some results for the interpolation errors as suggested in [19] by using the constants for the P_0 and the conforming P_1 triangles. We will report more refined results to the non-conforming P_1 triangle in subsequent papers.

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Table 3: A posteriori estimates for the first eigenvalue λ associated to Ω_n

n	N	bounds for λ	N	bounds for λ	N	bounds for λ
3	5	$3.9082 < \lambda < 4.4963$	10	$4.2688 < \lambda < 4.4147$	100	$4.3853 < \lambda < 4.3868$
4	5	$4.7700 < \lambda < 5.0211$	10	$4.8954 < \lambda < 4.9569$	100	$4.9344 < \lambda < 4.9351$
5	5	$5.0049 < \lambda < 5.2826$	10	$5.1590 < \lambda < 5.2273$	100	$5.2075 < \lambda < 5.2082$
6	5	$5.1387 < \lambda < 5.4323$	10	$5.3114 < \lambda < 5.3839$	100	$5.3659 < \lambda < 5.3667$
7	5	$5.2220 < \lambda < 5.5257$	10	$5.4078 < \lambda < 5.4831$	100	$5.4666 < \lambda < 5.4674$
8	5	$5.2774 < \lambda < 5.5879$	10	$5.4727 < \lambda < 5.5498$	100	$5.5346 < \lambda < 5.5354$
9	5	$5.3160 < \lambda < 5.6313$	10	$5.5185 < \lambda < 5.5969$	100	$5.5827 < \lambda < 5.5836$
10	5	$5.3440 < \lambda < 5.6628$	10	$5.5520 < \lambda < 5.6313$	100	$5.6181 < \lambda < 5.6190$

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A Determination of $\lambda^{(i)} = C_i^{-2}(+0) \quad (0 \le i \le 5)$

Recall Theorem 5 for the determination relations of $C_i(+0) := \lim_{\alpha \to +0} C_i(\alpha)$ or $\lambda^{(i)} = C_i^{-2}(+0)$ $(i = 0, 1, 2, 3, 4, 5)$. Fortunately, all the ordinary differential equations (ODE) appearing there can be solved by means of the hypergeometric functions including the Bessel functions [32], so that we can obtain the determination equations as transcendental ones in terms of such functions. All the numerical results below are obtained by using Mathematica®.

A.1 $\lambda^{(0)}$

From Theorem 5, the ODE and the boundary condition in this case are given by

$$-((1-s)u'(s))' = \lambda^{(0)}(1-s)u(s) \ \text{ for } \ s \in]0,1[, \ \ u'(0) = 0\,. \tag{157}$$

The general solution of the above ODE that can be identified with an element of $H^1(T) \supset V^{0,Z}$ is expressed by

$$u(s) = c^{(1)} J_0(\sqrt{\lambda^{(0)}}(1-s))\,, \tag{158}$$

where $c^{(1)}$ is an arbitrary constant and J_0 is the 0-th order Bessel function of the first kind. Actually $Y_0(\sqrt{\lambda^{(0)}}(1-s))$ (Y_0= the 0-th order Bessel function of the second kind) also satisfies the ODE but cannot be identified with an element of $H^1(T)$. Thus applying the boundary condition above and the relation $J_1 = -J_0'$, we have the following equation for $\lambda^{(0)}$:

$$J_1(\sqrt{\lambda^{(0)}}) = 0\,, \tag{159}$$

which means that $\sqrt{\lambda^{(0)}}$ is the smallest positive zero of J_1. Thus we can obtain approximate values of $\lambda^{(0)}$ and $C_0(0+)$ as

$$\lambda^{(0)} \approx 3.83171^2\,, \quad C_0(+0) \approx 0.260980\,. \tag{160}$$

A.2 $\lambda^{(1)} = \lambda^{(3)} = \lambda^{(4)}$

Similarly, the ODE, the linear constraint and the boundary condition in this case are given by (75) as

$$-((1-s)u'(s))' = \lambda^{(1)}(1-s)u(s) + C \ \text{ for } \ s \in]0,1[, \ \int_0^1 u(s)ds = 0, \ \ u'(0) = 0\,, \tag{161}$$

where C is an arbitrary constant. Then the general solution of ODE that can be identified with an element of $H^1(T) \supset V^{1,Z}$ is expressed by

$$u(s) = c^{(1)} J_0(\sqrt{\lambda^{(1)}}(1-s)) - C\,(1-s)\,{}_1F_2(1; 3/2, 3/2; -\lambda^{(1)}(1-s)^2/4)\,, \tag{162}$$

where $c^{(1)}$ is an arbitrary constant, and ${}_1F_2(\cdot;\cdot,\cdot,\cdot;\cdot)$ is a kind of hypergeometric function. Using the linear constraint and the boundary condition to the above, we have the following determination equation for $\lambda = \lambda^{(1)}$:

$$\frac{\lambda}{4}\,{}_0F_1(;2;-\frac{\lambda}{4})\,{}_2F_3(1,1;\frac{3}{2},\frac{3}{2},2;-\frac{\lambda}{4}) + {}_1F_2(\frac{1}{2};1,\frac{3}{2};-\frac{\lambda}{4})\,{}_1F_2(1;\frac{1}{2},\frac{3}{2};-\frac{\lambda}{4}) = 0\,, \tag{163}$$

where ${}_0F_1$ and ${}_2F_3$ are also hypergeometric functions. Solving this equation numerically, we have the following approximate values:

$$\lambda^{(1)} \approx 3.08126^2, \quad C_1(+0) \approx 0.324542\,. \tag{164}$$

A.3 $\lambda^{(2)}$

By Theorem 5, the ODE and the boundary condition associated with $\lambda^{(2)}$ are given as

$$-((1-s)u'(s))' = \lambda^{(2)}(1-s)u(s) \ \text{ for } \ s \in]0,1[\,, \ \ u(0) = 0\,. \tag{165}$$

Then the general solution of the above ODE belonging to $H^1(T) \supset V^{2,Z}$ is the same as (158):

$$u(x) = c^{(1)} J_0(\sqrt{\lambda^{(2)}}(1-s)) \tag{166}$$

so that the determination equation for $\lambda^{(2)}$ is obtained as

$$J_0(\sqrt{\lambda^{(2)}}) = 0\,. \tag{167}$$

Thus $\sqrt{\lambda^{(2)}}$ is the minimum positive zero of J_0. Approximately, we have

$$\lambda^{(2)} \approx 2.40483^2, \quad C_2(+0) \approx 0.415831\,. \tag{168}$$

A.4 $\lambda^{(5)}$

By Theorem 5, the ODE and the boundary conditions associated to $\lambda^{(5)}$ are given as

$$((1-s)u''(s))'' = \lambda^{(5)}(1-s)u(s) \ \text{ for } \ s \in]0,1[\,, \ \ u(0) = u(1) = u''(0) = 0\,. \tag{169}$$

Then the general solution of the ODE belonging to $H^2(T) \supset V^{4,Z}$ is

$$\begin{aligned} u(s) =& c^{(1)}\,{}_0F_3(;\frac{1}{2},\frac{3}{4},\frac{3}{4};\frac{\lambda^{(5)}(1-s)^4}{256}) + c^{(2)}(1-s)\,{}_0F_3(;\frac{3}{4},1,\frac{5}{4};\frac{\lambda^{(5)}(1-s)^4}{256}) \\ &+ c^{(3)}(1-s)^2\,{}_0F_3(;\frac{5}{4},\frac{5}{4},\frac{3}{2};\frac{\lambda^{(5)}(1-s)^4}{256})\,, \end{aligned} \tag{170}$$

where $c^{(1)}$, $c^{(2)}$ and $c^{(3)}$ are arbitrary constants, and ${}_0F_3$ is a hypergeometric function. Then, introducing two functions $f(\lambda,t) = t\,{}_0F_3(;\frac{3}{4},1,\frac{5}{4};\frac{\lambda}{256}t^4)$ and $g(\lambda,t) = t^2\,{}_0F_3(;\frac{5}{4},\frac{5}{4},\frac{3}{2};\frac{\lambda}{256}t^4)$, the determination equation for $\lambda = \lambda^{(5)}$ is given by

$$f''(\lambda,1)g(\lambda,1) - g''(\lambda,1)f(\lambda,1) = 0\,, \tag{171}$$

where $'' = \partial^2/\partial t^2$. Approximately, we find that

$$\lambda^{(5)} \approx 9.26775^2, \quad C_5(+0) \approx 0.107901\,. \tag{172}$$

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