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by

G. Bayarmagnai

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

THE $(\mathfrak{g}, K)$-MODULE STRUCTURES OF PRINCIPAL SERIES OF SU(2,2)

G. BAYARMAGNAI

Abstract. We completely describe the $(\mathfrak{g}, K)$-module structures of the principal series representations of $SU(2,2)$.

Introduction. The purpose of this paper is to describe completely the $(\mathfrak{g}, K)$-module structure of the principal series representations of $SU(2,2)$, parabolically induced with respect to the minimal parabolic subgroup P_{min}.

This is motivated by the problem of the determination of the precise formulas for various spherical models of the standard representations of $SU(2,2)$. Among others we are interested in the Whittaker models (Bayarmagnai [1], Hayata [3], Ishii [4], Miyazaki-Oda [6]).

Our method of proof is similar to that of a recent paper of Oda [5], which describes the $(\mathfrak{g}, K)$-module structure of standard representations of $Sp(2,\mathbb{R})$. Namely we utilize the concept of simple K-modules with marking, to overcome the problem of multiplicities in K-types.

Our main results are Theorem 3.6 and Theorem 3.7 which are shortly explained below. The template of the formulas is the following:

$$\mathcal{C}_{[\pm,\pm;\pm]}\mathbf{S}^{(m)} = \mathbf{S}^{(m')}\Gamma_{[\pm,\pm;\pm]}.$$

Here $\mathbf{S}^{(m)}$ is the matrix consisting of elementary functions in the representation identified with a closed subspace of $L^2(K)$, $\mathcal{C}_{[\pm,\pm;\pm]}$ is a matrix with entries either in $\mathfrak{p}^+$ or in $\mathfrak{p}^-$, and $\Gamma_{[\pm,\pm;\pm]}$ is a constant matrix whose entries consists of linear forms in the parameters of the representation. The last is called a matrix of intertwining constants.

Let us recall the Casimir equation for the Casimir operator $\mathcal{C}$:

$$\mathcal{C}v = \gamma(\mathcal{C})v,$$

where γ is the infinitesimal character and v is a differential vector. Our formula is a "covariant" analogue of this. The details of each symbol is explained in the text.

In the section 1 we have collected the necessary facts of $SU(2,2)$, related subgroups and Lie algebras. The *marked basis* of each continuous simple K-submodule of the principal series representation of $SU(2,2)$ is introduced in terms of the elementary functions in the section 2. We begin section 3 by computing the Clebsch-Gordan coefficients of finite dimensional representations of K (Proposition 3.2 and Proposition 3.3). Then we introduce our main result concerning the $\mathfrak{g}_{\mathbb{C}}$-module (Theorem 3.6 and Theorem 3.7), and finally give some examples.

According to this way, the case of real symplectic group of rank 3 is also due to Miyazaki [7].

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1. Preliminaries

In this section, we recall some definitions and results which will be needed in the sequel. For more details, for instance, we refer to [3].

1.1. **Basic notions.** Let G be the special unitary group defined by

$$SU(2,2) = \left\{ g \in SL_4(\mathbb{C}) \mid {}^t\bar{g} I_{2,2} g = I_{2,2},\ I_{2,2} = \begin{pmatrix} 1_2 & \\ & -1_2 \end{pmatrix} \right\}$$

and K be a maximal compact subgroup of G given by the fixed part $K = G^\theta$ of the Cartan involution $\theta(g) = {}^t\bar{g}^{-1}, g \in G$:

$$K = S(U(2) \times U(2)) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} :\ a, b \in U(2), \det(ab) = 1 \right\}.$$

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ ($\mathfrak{p} = \mathfrak{g}^{-\theta}$) be the Cartan symmetric decomposition associated to the involution θ. For $x \in M_2(\mathbb{C})$ we set

$$p_+(x) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \text{ and } p_-(x) = \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}.$$

Let $H_i = p_+(e_{ii}) + p_-(e_{ii})$ $(i = 1, 2)$, where e_{ij} the matrix unit $M_2(\mathbb{R})$ with 1 in the (i, j)-entry and zero elsewhere. Then the space $\mathfrak{a}$ spanned by H_1, H_2 over $\mathbb{R}$ is a maximally abelian subalgebra of $\mathfrak{p}$. Let $\{\lambda_1, \lambda_2\}$ be a basis of the dual space $\mathfrak{a}^*$ such that $\lambda_i(H_j) = \delta_{ij}$. Then the restricted root system for $\mathbf{\Phi}(\mathfrak{g}, \mathfrak{a})$ is of type C_2, namely

$$\mathbf{\Phi}(\mathfrak{g}, \mathfrak{a}) = \{\pm\lambda_1, \pm\lambda_2 \pm 2\lambda_1, \pm 2\lambda_2\}.$$

Choose $\lambda_1 - \lambda_2$ and $2\lambda_2$ as simple roots of $\mathbf{\Phi}(\mathfrak{g}, \mathfrak{a})$. Denote by E_{ij} the matrix units in $M_4(\mathbb{C})$ for $0 \le i, j \le 4$. Then the corresponding root spaces of dimension two and one are given by

$$\mathfrak{g}_{\lambda_1 - \lambda_2} = \mathbb{R} \cdot E_1 \oplus \mathbb{R} \cdot E_2 \text{ and } \mathfrak{g}_{2\lambda_2} = \mathbb{R} \cdot E_0,$$

with $E_0 = \kappa^{-1} E_{24} \kappa, E_1 = \kappa^{-1}(E_{12} - E_{43})\kappa$ and $E_2 = \kappa^{-1}(iE_{12} + iE_{43})\kappa$. Here $\kappa = \frac{1}{\sqrt{2}} \begin{pmatrix} 1_2 & 1_2 \\ -i1_2 & i1_2 \end{pmatrix}$ with $i = \sqrt{-1}$.

We put $A = \exp(\mathfrak{a})$, $M = Z_A(K)$, and choose a minimal parabolic subgroup P_{min} with Langlands decomposition $P_{min} = MAN$ with the unipotent subgroup N :

$$N = \left\{ \kappa^{-1} \left(\begin{array}{cc|cc} 1 & n_0 & & \\ & 1 & & \\ \hline & & 1 & \\ & & -\bar{n}_0 & 1 \end{array} \right) \left(\begin{array}{cc|cc} 1 & & n_1 & n_2 \\ & 1 & \bar{n}_2 & n_3 \\ \hline & & 1 & \\ & & & 1 \end{array} \right) \kappa \;\middle|\; \begin{array}{c} n_1, n_3 \in \mathbb{R}, \\ n_0, n_2 \in \mathbb{C} \end{array} \right\}.$$

1.2. **The K-modules.** The group $\tilde{K} = SU(2) \times SU(2) \times U(1)$ is a twofold covering of K with a projection given by

$$pr(g_1, g_2; u) = \mathrm{diag}(ug_1, u^{-1}g_2),$$

where $g_1, g_2 \in SU(2)$ and $u \in U(1)$. The kernel of this homomorphism is

$$\mathrm{Ker}(pr) = \{\pm(1_2, 1_2; 1)\}.$$

Let (τ_m, V_m) be the m-th symmetric tensor representation of the group $SU(2)$. Then the unitary dual of K can be parameterized by the set

$$\hat{K} = \{(\tau_{[m_1,m_2;\ l]}, V_{m_1 m_2}) \mid m_1, m_2 \in \mathbb{N} \cup 0,\ l \in \mathbb{Z},\ m_1 + m_2 + l \in 2\mathbb{Z}\}.$$

Here $V_{m_1 m_2}$ is the outer tensor product of the spaces V_{m_1} and V_{m_2}, and if $g_1, g_2 \in SU(2)$ and $u \in U(1)$, then the action is

$$\tau_{[m_1,m_2;\ l]}(g_1, g_2; u) = \mathrm{sym}^{m_1}(g_1) \otimes \mathrm{sym}^{m_2}(g_2) \otimes u^l.$$

We fix now a basis for $\mathfrak{k}_{\mathbb{C}} = \mathrm{Lie}(K)_{\mathbb{C}}$:

$$I_{2,2} = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} h^1 = \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}, h^2 = \begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix},$$
$$e^1_{\pm} = \begin{pmatrix} e_{\pm} & 0 \\ 0 & 0 \end{pmatrix}, e^2_{\pm} = \begin{pmatrix} 0 & 0 \\ 0 & e_{\pm} \end{pmatrix},$$

where $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Using these basis, we write the action $\tau_{[m_1,m_2;l]}$ on $V_{m_1 m_2}$ explicitly .

Lemma 1.1. *Let $\{f_i\}_{0 \le i \le m_j}$ be a basis of V_{m_j} as $SU(2)$-module for $j = 0, 1$. For a given K-module $(\tau_{[m_1,m_2;l]}, V_{m_1 m_2})$ the set*

$$\{f_{pq} : f_{pq} = f_p \otimes f_q,\ 0 \le p \le m_1, 0 \le q \le m_2\}$$

forms a basis of $V_{m_1 m_2}$ as K-module and the infinitesimal actions of K on $V_{m_1 m_2}$ are expressed by

$$\begin{array}{ll} h^1(f_{pq}) = (2p - m_1) f_{pq}, & h^2(f_{pq}) = (2q - m_2) f_{pq}, \\ e^1_+(f_{pq}) = (m_1 - p) f_{p+1,q}, & e^2_+(f_{pq}) = (m_2 - q) f_{p,q+1}, \\ e^1_-(f_{pq}) = p f_{p-1,q}, & e^2_-(f_{pq}) = q f_{p,q-1}, \\ I_{2,2} f_{pq} = l f_{pq}. & \end{array}$$

Proof. It is a well known standard fact. □

For a simple K-module τ, we can normalize the one dimensional space of K-homomorphisms of τ onto itself, by the following definition.

Definition 1.1. *A simple K-module τ equipped with a canonical basis is called a* marked simple K-module *or a* simple K-module with marking.

1.3. **Iwasawa decomposition.** The set $\{E_{i,j+2}, E_{i+2,j} \mid i,j = 1,2\}$ forms a basis of the 8-dimensional vector space $\mathfrak{p}_{\mathbb{C}}$ and one has

$$E_{i+2,j} = p_+(e_{ij}) \qquad \text{and} \qquad E_{i,j+2} = p_-(e_{ij}),$$

where $i,j = 1,2$.

Lemma 1.2. *Put*

$$\begin{aligned} E_{2\lambda_1} &= \kappa^{-1}E_{13}\kappa,\ E^1_{\lambda_1+\lambda_2} = \kappa^{-1}E_{14}\kappa,\ E^1_{\lambda_1-\lambda_2} = \kappa^{-1}E_{43}\kappa, \\ E_{2\lambda_2} &= \kappa^{-1}E_{24}\kappa,\ E^2_{\lambda_1+\lambda_2} = \kappa^{-1}E_{23}\kappa,\ E^2_{\lambda_1-\lambda_2} = \kappa^{-1}E_{12}\kappa. \end{aligned}$$

Then we have

$$\begin{aligned} p_\pm(e_{ii}) &= \tfrac{1}{2}(\pm\sqrt{-1}E_{2\lambda_i} + H_i \pm \tfrac{1}{2}(I_{2,2} - \epsilon(i)(h^1 - h^2))), \\ p_\pm(e_{ij}) &= \tfrac{1}{2}(-\epsilon(i)E^j_{\lambda_1-\lambda_2} \mp \sqrt{-1}E^i_{\lambda_1+\lambda_2}) - \epsilon(j)\begin{cases} e^j_{\epsilon(j)}, & \text{if} \quad (+) \\ e^i_{-\epsilon(i)}, & \text{if} \quad (-) \end{cases} \end{aligned}$$

where $\epsilon(i) := \mathrm{sign}(-1)^i \quad (i \neq j,\ i,j \in \{1,2\})$.

Proof. We can show this by direct computation. □

1.4. **Principal series representations.** Let P_{min} be a minimal parabolic subgroup of G with Langlands decomposition $P_{min} = MAN$ with $M = Z_A(K)$. In particularly, the subgroup M of P_{min} is identified with

$$M = \{[e^{\sqrt{-1}\theta}]\gamma^j \mid \theta \in \mathbb{R}, j = \pm 1\}$$

where $\gamma = \mathrm{diag}(1,-1,1,-1) \in G$ and

$$[e^{\sqrt{-1}\theta}] = \mathrm{diag}(e^{\sqrt{-1}\theta}, e^{-\sqrt{-1}\theta}, e^{\sqrt{-1}\theta}, e^{-\sqrt{-1}\theta}).$$

For an integer s and a character e of the group μ_2, we define a unitary character of M by

$$\sigma_{s,e}([e^{\sqrt{-1}\theta}]\gamma^j) = e(-1)^j e^{\sqrt{-1}\theta s}.$$

Let ρ be the half sum of the positive roots and define a character $e^{\mu+\rho}$ of A:

$$e^{\mu+\rho}(a) = e^{(\mu+\rho)\log(a)} \qquad (\mu = (\mu_1,\mu_2) \in \mathrm{Lie}(A)).$$

We extend it to a character of AN so that the restriction to N is trivial. Define an admissible character of P_{min} by tensoring these characters. Then we get the induced representation (π, H_π) usually denoted by $\pi = \mathrm{Ind}^G_{P_{min}}(\sigma_{s,e} \otimes e^{\mu+\rho} \otimes 1_N)$ and called the *principal series representation* of G. By definition the representation space H_π of G can be realized on the Hilbert space

$$\begin{aligned} L^2_{(\sigma_{s,e})}(K) = \ &\{f \in L^2(K) \mid \\ &\qquad f(mk) = \sigma_{n,e}(m)f(k) \text{ for } m \in M, k \in K,\ a.e.\} \end{aligned}$$

with G-action defined by

$$(\pi(g)f)(k) = a(kg)^{\nu+\rho} f(k(kg)), \ k \in K, g \in G,$$

where $kg = n(kg)a(kg)m(kg)k(kg)$ is the Iwasawa decomposition of the element kg.

2. The structure of K-types of the principal series representation

In this section we express the K-isotypic components of H_π in terms of the elementary functions obtained from the tautological representation of $SU(2)$. Combining it with Lemma 1.2, the K-module structures on H_π^K is described explicitly.

2.1. **Elementary functions in $L^2(K)$.** Let us recall the parametrization of the unitary dual of $SU(2)$. Let $S(x)$ ($x \in SU(2)$) be a square matrix function associated to $SU(2)$ given by

$$S(x) = \begin{pmatrix} s_1(x) & s_2(x) \\ -\bar{s}_2(x) & \bar{s}_1(x) \end{pmatrix}, \quad \text{with } \det(S(x)) = 1.$$

Then we have $S(xy) = S(x)S(y)$ and $s_i(-x) = -s_i(x)$ for $i = 1, 2$. Consider $S(x)$ as a linear transformation from (X, Y) to (X', Y'), *i.e.*,

$$(X', Y') = (X, Y) \begin{pmatrix} s_1(x) & s_2(x) \\ -\bar{s}_2(x) & \bar{s}_1(x) \end{pmatrix},$$

where X, Y are independent variables. For each positive integer $n \geq 2$, there is a linear transformation

$$\mathrm{Sym}^{(n)}(S(x)) = \{s_{ij}^{(n)}(x)\}_{0 \leq i,j \leq n}$$

between the homogeneous forms of (X, Y) and (X', Y') of degree n via

$$((X')^n, (X')^{n-1}Y', ..., (Y')^n) = (X^n, X^{n-1}Y, ..., Y^n) \cdot \mathrm{Sym}^{(n)}(S(x)).$$

First recall the following well-known observation without proof.

Lemma 2.1. *The $n+1$ entries of each i-th row vector of $\mathrm{Sym}^{(n)}(S(x))$ make a canonical basis of the irreducible right $SU(2)$-representation of dimension $n+1$ in $L^2(SU(2))$. In particular, we have*

1. $\mathrm{Sym}^{(n)}(S(xy)) = \mathrm{Sym}^{(n)}(S(x))\mathrm{Sym}^{(n)}(S(y)), \ x, y \in SU(2),$

2. $\mathrm{Sym}^{(n)}(S(x)) = \mathbf{diag}_{0 \leq i \leq n}(e^{\sqrt{-1}t(n-2i)})$ if $x = \mathbf{diag}(e^{\sqrt{-1}t}, e^{-\sqrt{-1}t})$

with $t \in \mathbb{R}$.

2.2. **Elementary functions in $L^2(\tilde{K})$.** Fix positive integers m_1, m_2 and an integer l. Put $m = [m_1, m_2; l]$. For each quadruple $(i, j, p, q) \in \mathbb{Z}_+^4$ such that $i, p \leq m_1$ and $j, q \leq m_2$, we define a $\mathbb{C}$-valued function on $\tilde{K}$ by

$$S_{ij,pq}(g_1, g_2, u) = s_{ip}^{(m_1)}(g_1) s_{jq}^{(m_2)}(g_2) u^l,$$

where $g_1, g_2 \in SU(2)$ and $u \in U(1)$. For a fixed pair (i, j), a space $W_{ij}^{(m)}$ generated by

$$\{S_{ij,pq} \mid 0 \leq p \leq m_1, 0 \leq q \leq m_2\}$$

is a $\tilde{K}$-module with the action τ_m defined by

$$\tau_m(g_1, g_2; u) S_{ij,pq}(x, y; v) = S_{ij,pq}(xg_1, yg_2; vu)$$

for $g_1, g_2, x, y \in SU(2)$ and $u, v \in U(1)$. Note that for each pair (i, j), we have that $(\tau_m, W_{00}^{(m)}) \cong (\tau_m, W_{ij}^{(m)})$ and the τ_m-isotypic component in the right $\tilde{K}$-module $L^2(\tilde{K})$ is just the sum of all spaces $W_{ij}^{(m)}$, where $0 \leq i \leq m_1, 0 \leq j \leq m_2$.

2.3. **K-isotypic components of the principal series representations.** For $x \in SU(2)$, Lemma 2.1 implies that

$$\mathrm{Sym}^{(n)}(S(-x)) = (-1)^n \mathrm{Sym}^{(n)}(S(x)),$$

hence $S_{ij,pq}(k) = S_{ij,pq}(-(1_2, 1_2; 1)k)$ for $k \in \tilde{K}$ when $m_1 + m_2 + l \in 2\mathbb{Z}$. Therefore in this case the functions $S_{ij,pq}(k)$ are well defined on K *i.e.*, we may say that

$$\hat{K} = \{(\tau_m, W_{00}^{(m)}) \mid m = [m_1, m_2; l], m_1 + m_2 + l \in 2\mathbb{Z}\}.$$

Note also that Lemma 2.1 shows $S_{ij,pq}(k) = \delta_{ij,pq}$ at the point $k = 1_4$. This property will be used several times later.

Set $\sigma = \sigma_{s,e}$. Since $L^2_\sigma(K) \subset L^2(K)$, as a right unitary representation of K, it has an irreducible decomposition of $K \times K$-bimodules

$$L^2_\sigma(K) \cong \hat{\oplus}_{\tau \in \hat{K}} \{(\tau^* \mid_M)[\sigma^{-1}] \otimes \tau\}$$

by the Peter-Weyl theorem. Here $(\tau^* \mid M)[\sigma^{-1}]$ is the σ^{-1}-isotypic component in $\tau^* \mid_M$. Hence one can explicitly describe the K-isotypic components of the principal series representation π.

Lemma 2.2. (cf. [3,3.4]) *Assume $m_1 + m_2 \geq \mid s \mid$ and $l \equiv 2m_2 + s + 1 - e(-1)$ (mod 4). Then the τ_m-isotypic component $H_\pi(\tau_m)$ in the principal series representation π is isomorphic to*

$$\oplus_\gamma W_\gamma^{(m)} \text{ with } \gamma = (t, (m_1 + m_2 + s)/2 - t)),$$

where t runs over integers satisfying,

$$\begin{cases} 0 \leq t \leq \min(m_1, m_2), & \text{if } s < \min(m_1 - m_2, m_2 - m_1) \\ (m_1 - m_2 + s)/2 \leq t \leq m_1, & \text{if } s \geq \max(m_2 - m_1, m_2 - m_1) \end{cases}$$

and when $\min(m_1 - m_2, m_2 - m_1) \le s < \max(m_1 - m_2, m_2 - m_1)$

$$\begin{cases} 0 \le t \le \min(m_1, m_2), & \text{if } m_1 > m_2 \\ (m_1 - m_2 + s)/2 \le t \le (m_1 + m_2 + s)/2, & \text{if } m_1 < m_2. \end{cases}$$

Extending the notion given in Definition 1.1 slightly, we can define a set of markings for each isotypic companent of $L^2(K)$.

Definition 2.1. *For each possible pair (i, j), the marking on the simple K-module $(\tau_m, W_{ij}^{(m)})$ specified by the basis $\{S_{ij,pq} \mid 0 \le p \le m_1, 0 \le q \le m_2\}$ is called the* marking by elementary functions.

Conventions. Fix π and a marked simple K-module τ_m in $\pi\mid_K$ with $m = [m_1, m_2; l]$. Denote by $I(\pi, \tau_m)$ the set of all γ such that $\gamma = (t, (m_1 + m_2 + s)/2 - t))$ as in Lemma 2.2 and $W_\gamma^{(m)}$ occurs in $\pi\mid_K$. Then the multiplicity $\mathrm{m}(\pi, \tau_m)$ of τ_m in $\pi\mid_K$ is the cardinality of the finite set $I(\pi, \tau_m)$.

When $\gamma \in I(\pi, \tau_m)$, there is a K-isomorphism from V_m onto $W_\gamma^{(m)}$ by sending the set of marked basis onto the set of marked elementary functions and hence denote this K-isomorphism by $[\gamma]$.

3. $(\mathfrak{g}, K)$-MODULE STRUCTURES

In this section we investigate the action $\mathfrak{g} = \mathrm{Lie}(G)$ (or $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$) on the subspace $H_{\pi,K}$ of the K-finite vectors in the representation space H_π. Because of the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, it is suffices to investigate the action of $\mathfrak{p}$ or $\mathfrak{p}_{\mathbb{C}}$.

3.1. **Clebsch-Gordan coefficients.** The adjoint representation of K on $\mathfrak{p}_{\mathbb{C}}$ splits into two irreducible components, *i.e.*, the holomorphic part $\mathfrak{p}_+$ generated by the set of matrix units $\{E_{ij} \mid i = 1, 2, j = 3, 4\}$ over $\mathbb{C}$ and the antiholomorphic part $\mathfrak{p}_-$ generated by the set $\{E_{ij} \mid i = 3, 4, j = 1, 2\}$ over $\mathbb{C}$.

Lemma 3.1. (cf. [3,3.10]) *We have the K-isomorphisms*

$$(Ad, \mathfrak{p}_+) \cong \tau_{[1,1;2]} \quad \text{and} \quad (Ad, \mathfrak{p}_-) \cong \tau_{[1,1;-2]}$$

given by

$$(E_{23}, E_{13}, E_{24}, E_{14}) \to (f_{00}, f_{10}, -f_{01}, -f_{11}),$$
$$(E_{41}, E_{31}, E_{42}, E_{32}) \to (f_{00}, f_{01}, -f_{10}, -f_{11}).$$

Let (τ_m, V_m) $(m = [m_1, m_2; l])$ be an irreducible representation of K.

$\mathfrak{p}_+$-side. By the well known Clebsch-Gordan theorem and Lemma 3.1, the irreducible components in the K-module $\mathfrak{p}_+ \otimes_{\mathbb{C}} \tau_m$ are precisely the K-representations

$$\{\ \tau_{[m_1+e_1, m_2+e_2;\ l+2]} \mid e_1, e_2 \in \{\pm 1\}\},$$

and we will denote these by $\tau_{[\pm,\pm;+]}$ or $\tau_{[e_1,e_2;+]}$ respectively.

When $\tau_{[\pm,\pm;+]}$ is non zero, we now express the canonical basis vectors of $\tau_{[\pm,\pm;+]}$ in terms of the basis vectors of $\mathfrak{p}_+ \otimes_{\mathbb{C}} \tau_m$ induced from those of $\mathfrak{p}_+$ and τ. In this case, denote by $I_{[\pm,\pm;+]}$ a generator of the vector space $\mathrm{Hom}_K(\tau_{[e_1,e_2;+]}, \mathfrak{p}_+ \otimes_{\mathbb{C}} \tau_m)$, which is unique up to constant multiple.

Proposition 3.2. *The image of the (p,q)-th canonical basis vector f'_{pq} of $\tau_{[e_1,e_2;+]}$ under the K-homomorphism $I_{[e_1,e_2;+]}$ is given by*

i. If $(e_1, e_2) = (-1,-1)$ then

$$E_{23} \otimes f_{p+1q+1} - E_{13} \otimes f_{pq+1} + E_{24} \otimes f_{p+1q} - E_{14} \otimes f_{pq} \ ,$$

ii. If $(e_1, e_2) = (+1,-1)$ then

$$(1 - \mathbf{c}^1_p)(E_{23} \otimes f_{pq+1} + E_{24} \otimes f_{pq}) + \mathbf{c}^1_p(E_{13} \otimes f_{p-1q+1} + E_{14} \otimes f_{p-1q}),$$

iii. If $(e_1, e_2) = (-1,+1)$ then

$$(1 - \mathbf{c}^2_q)(E_{13} \otimes f_{pq} - E_{23} \otimes f_{p+1q}) + \mathbf{c}^2_q(E_{24} \otimes f_{p+1q-1} - E_{14} \otimes f_{pq-1}),$$

iv. If $(e_1, e_2) = (+1,+1)$ then

$$\begin{aligned} &-(1 - \mathbf{c}^2_q)((1 - \mathbf{c}^1_p)E_{23} \otimes f_{pq} + \mathbf{c}^1_p E_{13} \otimes f_{p-1q}) \\ &+\mathbf{c}^2_q((1 - \mathbf{c}^1_p)E_{24} \otimes f_{pq-1} + \mathbf{c}^1_p E_{14} \otimes f_{p-1q-1}) \end{aligned}$$

with the coefficients expressed as follows

$$\mathbf{c}^1_p = \frac{p}{m_1 + 1}, \ \mathbf{c}^2_q = \frac{q}{m_2 + 1}$$

where $0 \le p \le m_1 + e_1$ and $0 \le q \le m_2 + e_2$, respectively.

Proof. Denote by u_{pq} the element in $\mathfrak{p}_+ \otimes_{\mathbb{C}} \tau_m$ defined in our Proposition. To prove $I_{[e_1,e_2;+]}(f'_{pq}) = u_{pq}$, it is enough to show that the correspondence $f_{pq} \to u_{pq}$ is a K-module homomorphism by utilizing the infinitesimal representation of K. Note that the algebra generated by h^1, h^2 and $I_{2,2}$ form a Cartan subalgebra. We first claim that the weight of the vector $u_{m_1-1m_2-1}$ is identified with

$$E_{23} \otimes f_{m_1m_2} + E_{13} \otimes f_{m_1-1m_2} + E_{24} \otimes f_{m_1m_2-1} + E_{14} \otimes f_{m_1-1m_2-1}$$

is the same as the weight of f_{m_1-1,m_2-1} in $\tau_{[-,-;+]}$. It is obvious that $I_{2,2} \cdot u_{m_1-1m_2-1} = (l+2)u_{m_1-1m_2-1}$. By Lemma 1.2 and Lemma 3.1, it follows that

$$\begin{aligned} h^1 \cdot E_{14} \otimes f_{m_1-1m_2-1} &= (1 + 2(m_1 - 1) - m_1)E_{14} \otimes f_{m_1-1m_2-1}, \\ h^1 \cdot E_{13} \otimes f_{m_1-1m_2} &= (1 + 2(m_1 - 1) - m_1)E_{13} \otimes f_{m_1-1m_2}, \\ h^1 \cdot E_{24} \otimes f_{m_1m_2-1} &= (-1 + 2m_1 - m_1)E_{24} \otimes f_{m_1m_2-1}, \\ h^1 \cdot E_{23} \otimes f_{m_1m_2} &= (m_1 + 1 - 2)E_{23} \otimes f_{m_1m_2}. \end{aligned}$$

Hence the eigenvalue of u_{m_1-1,m_2-1} under h^1 is just $m_1 - 1$. Similarly, one can check that the eigenvalue via h^2 is equal to $m_2 - 1$. The next claim is

$$u_{p-1,q} = \frac{e^1_- \cdot u_{p,q}}{p}$$

for all possible values of (p, q). By using Lemma 1.2 and Lemma 3.1 again, we obtain that

$$\begin{aligned} e_-^1 \cdot E_{23} \otimes f_{p+1q+1} &= (p+1)E_{23} \otimes f_{pq+1}, \\ e_-^1 \cdot E_{13} \otimes f_{pq+1} &= E_{23} \otimes f_{pq+1} + pE_{13} \otimes f_{pq+1}, \\ e_-^1 \cdot E_{24} \otimes f_{p+1q} &= (p+1)E_{24} \otimes f_{pq}, \\ e_-^1 \cdot E_{14} \otimes f_{pq} &= E_{24} \otimes f_{pq} + p \cdot E_{14} \otimes f_{pq}. \end{aligned}$$

Hence the claim follows from the above. Similarly, for all possible indices (p, q), we can show that $u_{pq-1} = e_-^2 \cdot u_{pq}/q$. Therefore the natural correspondence $f_{pq} \to u_{pq}$ gives a non zero K-isomorphism. □

$\mathfrak{p}_-$-side. Since $(Ad, \mathfrak{p}_-) \cong \tau_{[1,1;-2]}$, the tensor product $\mathfrak{p}_- \otimes_{\mathbb{C}} \tau_m$ has four irreducible K-components:

$$\{ \tau_{[m_1+e_1, m_2+e_2; l-2]} \mid e_1, e_2 \in \{\pm 1\}\}$$

and we will denote these by $\tau_{[e_1,e_2;-]}$ respectively. Let $I_{[e_1,e_2;-]}$ be a generator of the vector space $\mathrm{Hom}_K(\tau_{[e_1,e_2;-]}, \mathfrak{p}_- \otimes_{\mathbb{C}} \tau_m)$ when $\tau_{[e_1,e_2;-]}$ is non zero. Now similarly as Proposition 3.2 we have the following:

Proposition 3.3. *The image of the (p, q)-th canonical basis vector f'_{pq} of $\tau_{[e_1,e_2;-]}$ under the K-homomorphism $I_{[e_1,e_2;-]}$ is given by*

i. If $(e_1, e_2) = (-1, -1)$ then

$$E_{41} \otimes f_{p+1q+1} + E_{42} \otimes f_{pq+1} - E_{31} \otimes f_{p+1q} - E_{32} \otimes f_{pq} \ ,$$

ii. If $(e_1, e_2) = (+1, -1)$ then

$$(1 - \mathbf{c}_p^1)(E_{31} \otimes f_{pq} - E_{41} \otimes f_{pq+1}) + \mathbf{c}_p^1(E_{42} \otimes f_{p-1q+1} - E_{32} \otimes f_{p-1q}),$$

iii. If $(e_1, e_2) = (-1, +1)$ then

$$(1 - \mathbf{c}_q^2)(E_{42} \otimes f_{pq} + E_{41} \otimes f_{p+1q}) + \mathbf{c}_q^2(E_{31} \otimes f_{p+1q-1} + E_{32} \otimes f_{pq-1}),$$

iv. If $(e_1, e_2) = (+1, +1)$ then

$$\begin{aligned} &-(1 - \mathbf{c}_q^2)((1 - \mathbf{c}_p^1)E_{41} \otimes f_{pq} - \mathbf{c}_p^1 E_{42} \otimes f_{p-1q}) \\ &-\mathbf{c}_q^2((1 - \mathbf{c}_p^1)E_{31} \otimes f_{pq-1} - \mathbf{c}_p^1 E_{32} \otimes f_{p-1q-1}), \end{aligned}$$

with the coefficients $\mathbf{c}_p^1$ and $\mathbf{c}_q^2$ described in Proposition 3.2.

Proof. The proof is quite similar to the proof of Proposition 3.2. □

3.2. **Matrix form of the Clebsch-Gordan decompositions.** For the further convenience, it is useful to describe the K-isomorphisms $I_{[e_1,e_2;\pm]}$ described in Proposition 3.2 and 3.3 in terms of the canonical basis of V_m.

To the set of all canonical basis $\{f_{pq} \mid 0 \le p \le m_1, 0 \le q \le m_2\}$ of the simple K-module V_m, we associate a row vector of size $(m_1+1)(m_2+1)$ with entries f_{pq} given by

$$\mathbf{F}_\tau = (f_{00}, f_{01} ..., f_{0m_2}, f_{10}, f_{11}, ..., f_{m_1,m_2-1}, f_{m_1 m_2}).$$

$\mathfrak{p}^+$-side. Define a matrix $\mathcal{C}_{[-,-;+]} = \{C_{ij}\}$ of size $(m_1 m_2) \times (m_1 + 1)(m_2 + 1)$ with entries consisting of elements in $\mathfrak{p}^+$ by

$$\begin{aligned} C_{m_2p+q+1,(m_2+1)p+q+1} &= -E_{14}, \\ C_{m_2p+q+1,(m_2+1)p+q+2} &= -E_{13}, \\ C_{m_2p+q+1,(m_2+1)(p+1)+q+1} &= E_{24}, \\ C_{m_2p+q+1,(m_2+1)(p+1)+q+2} &= E_{23}, \end{aligned}$$

for each $0 \leq p \leq m_1 - 1$ and $0 \leq q \leq m - 1$, but all other entries are 0.

Define a matrix $\mathcal{C}_{[+,-;+]} = \{C_{ij}\}$ of size $(m_1+2)m_2 \times (m_1+1)(m_2+1)$ with entries consisting of elements in $\mathfrak{p}^+$ by

$$\begin{aligned} C_{m_2p+q+1,(m_2+1)p+q+1} &= (1 - \mathbf{c}_p^1)E_{24}, \\ C_{m_2p+q+1,(m_2+1)p+q+2} &= (1 - \mathbf{c}_p^1)E_{23}, \\ C_{m_2p+q+1,(m_2+1)(p-1)+q+1} &= \mathbf{c}_p^1 E_{14}, \\ C_{m_2p+q+1,(m_2+1)(p-1)+q+2} &= \mathbf{c}_p^1 E_{13}, \end{aligned}$$

for $0 \leq p \leq m_1 + 1$ and $0 \leq q \leq m_2 - 1$, but all other entries are 0.

Define a matrix $\mathcal{C}_{[-,+;+]} = \{C_{ij}\}$ of size $m_1(m_2+2) \times (m_1+1)(m_2+1)$ with entries consisting of elements in $\mathfrak{p}^+$ by

$$\begin{aligned} C_{(m_2+2)p+q+1,(m_2+1)p+q+1} &= (1 - \mathbf{c}_q^2)E_{13}, \\ C_{(m_2+2)p+q+1,(m_2+1)p+q} &= -\mathbf{c}_q^2 E_{14}, \\ C_{(m_2+2)p+q+1,(m_2+1)(p+1)+q+1} &= -(1 - \mathbf{c}_q^2)E_{23}, \\ C_{(m_2+2)p+q+1,(m_2+1)(p+1)+q} &= \mathbf{c}_q^2 E_{24}, \end{aligned}$$

for $0 \leq p \leq m_1 + 1$ and $0 \leq q \leq m_2 - 1$, but all other entries are 0.

Define a matrix $\mathcal{C}_{[+,+;+]} = \{C_{ij}\}$ of size $(m_1 + 2)(m_2 + 2) \times (m_1 + 1)(m_2 + 1)$ with entries consisting of elements in $\mathfrak{p}^+$ by

$$\begin{aligned} C_{(m_2+2)p+q+1,(m_2+1)p+q+1} &= -(1 - \mathbf{c}_p^1)(1 - \mathbf{c}_q^2)E_{23}, \\ C_{(m_2+2)p+q+1,(m_2+1)p+q} &= (1 - \mathbf{c}_p^1)\mathbf{c}_q^2 E_{24}, \\ C_{(m_2+2)p+q+1,(m_2+1)(p-1)+q+1} &= -\mathbf{c}_p^1(1 - \mathbf{c}_q^2)E_{13}, \\ C_{(m_2+2)p+q+1,(m_2+1)(p-1)+q} &= \mathbf{c}_p^1 \mathbf{c}_q^2 E_{14}, \end{aligned}$$

for each $0 \leq p \leq m_1 + 1$ and $0 \leq q \leq m_2 + 1$, but all other entries are 0. Then Proposition 3.2 reads as the following proposition .

Proposition 3.4. *Let $\mathcal{C}_{[e_1,e_2;+]}, F_\tau$ be as above. Then for each pair e_1, e_2 the simple K-module $V_{[e_1,e_2;+]}$ is generated by the entries of the matrix ${\mathcal{C}_{[e_1,e_2;+]}}^t F_\tau$. Moreover, these entries make a set of canonical basis.*

Proof. Note that for the (i,j)-th entry of $\mathcal{C}_{[e_1,e_2;+]}$, the index i indicates the i-th coordinate in $\mathbf{F}_{[\mathbf{e_1},\mathbf{e_2};+]}$ and the index j indicates the j-th coordinate in $\mathbf{F}_\tau$. The i-th coordinate in $\mathbf{F}_{[\mathbf{e_1},\mathbf{e_2};+]}$ is uniquely expressed as

$$i = (m_2 + 1 + e_2)p + q + 1$$

for some pair (p,q) so that $0 \leq p \leq m_1 + e_1$ and $0 \leq q \leq m_2 + e_2$. Hence it is just the (p,q)-th canonical basis vector in $\tau_{[e_1,e_2;+]}$ by definition of

$\mathcal{C}_{[e_1,e_2;+]}$. Similarly, the j-th coordinate in $\mathbf{F}_\tau$ corresponds to the (p,q)-th basis vector in τ. Thus the proposition follows from Proposition 3.2. $\square$

$\mathfrak{p}_-$-side. Define a matrix $\mathcal{C}_{[-,-;-]} = \{C_{ij}\}$ of size $m_1 m_2 \times (m_1+1)(m_2+1)$ with entries consisting of elements in $\mathfrak{p}^-$ by

$$\begin{aligned}
C_{m_2p+q+1,(m_2+1)p+q+1} &= -E_{32},\\
C_{m_2p+q+1,(m_2+1)p+q+2} &= E_{42},\\
C_{m_2p+q+1,(m_2+1)(p+1)+q+1} &= -E_{31},\\
C_{m_2p+q+1,(m_2+1)(p+1)+q+2} &= E_{41},
\end{aligned}$$

for $0 \le i \le m_1 - 1$ and $0 \le q \le m_2 - 1$, but all other entries are 0.

Define a matrix $\mathcal{C}_{[+,-;-]} = \{C_{ij}\}$ of size $(m_1+2)m_2 \times (m_1+1)(m_2+1)$ with entries consisting of elements in $\mathfrak{p}^-$ by

$$\begin{aligned}
C_{m_2p+q+1,(m_2+1)p+q+1} &= (1-\mathbf{c}_p^1)E_{31},\\
C_{m_2p+q+1,(m_2+1)p+q+2} &= -(1-\mathbf{c}_p^1)E_{41},\\
C_{m_2p+q+1,(m_2+1)(p-1)+q+1} &= -\mathbf{c}_p^1 E_{32},\\
C_{m_2p+q+1,(m_2+1)(p-1)+q+2} &= \mathbf{c}_p^1 E_{42},
\end{aligned}$$

for $0 \le p \le m_1 + 1$ and $0 \le q \le m_2 - 1$, but all other entries are 0.

Define a matrix $\mathcal{C}_{[-,+;-]} = \{C_{ij}\}$ of size $m_1(m_2+2) \times (m_1+1)(m_2+1)$ with entries consisting of elements in $\mathfrak{p}^-$ by

$$\begin{aligned}
C_{(m_2+2)p+q+1,(m_2+1)p+q+1} &= (1-\mathbf{c}_q^2)E_{42},\\
C_{(m_2+2)p+q+1,(m_2+1)p+q} &= \mathbf{c}_q^2 E_{32},\\
C_{(m_2+2)p+q+1,(m_2+1)(p+1)+q+1} &= (1-\mathbf{c}_q^2)E_{41},\\
C_{(m_2+2)p+q+1,(m_2+1)(p+1)+q} &= \mathbf{c}_q^2 E_{31},
\end{aligned}$$

for $0 \le p \le m_1 - 1$ and $0 \le q \le m_2 + 1$, but all other entries are 0.

Define a matrix $\mathcal{C}_{[+,+;-]} = \{C_{ij}\}$ of size $(m_1+2)(m_2+2) \times (m_1+1)(m_2+1)$ with entries consisting of elements in $\mathfrak{p}^-$ by

$$\begin{aligned}
C_{(m_2+2)p+q+1,(m_2+1)p+q+1} &= -(1-\mathbf{c}_p^1)(1-\mathbf{c}_q^2)E_{41},\\
C_{(m_2+2)p+q+1,(m_2+1)p+q} &= -(1-\mathbf{c}_p^1)\mathbf{c}_q^2 E_{31},\\
C_{(m_2+2)p+q+1,(m_2+1)(p-1)+q+1} &= \mathbf{c}_p^1(1-\mathbf{c}_q^2)E_{42},\\
C_{(m_2+2)p+q+1,(m_2+1)(p-1)+q} &= \mathbf{c}_p^1\mathbf{c}_q^2 E_{32},
\end{aligned}$$

for each $0 \le p \le m_1 + 1$ and $0 \le q \le m_2 + 1$, but all other entries are 0. Then Proposition 3.3 reads as the following proposition .

Proposition 3.5. *Let $\mathcal{C}_{[e_1,e_2;-]}, F_\tau$ be as above. Then for each pair e_1, e_2 the simple K-module $V_{[e_1,e_2;-]}$ is generated by the entries of the matrix ${\mathcal{C}_{[e_1,e_2;-]}}^t F_\tau$. Moreover, these entries make a set of canonical basis.*

Proof. The proof is similar to the proof of Proposition 3.4. $\square$

3.3. **The Dirac-Schmid operators.** In this subsection we discuss the main result of this paper, that is, to compute the matrix forms of intertwining constants explicitly.

$\mathfrak{p}_+$-side. Note that the homomorphisms $[\gamma]$ with $\gamma \in I(\pi, \tau_m)$ defined in the section 2 form a basis of the vector space $\mathrm{Hom}_K(\tau_m, H_\pi(\tau_m))$ and hence we fix this basis for each τ_m in π. Take an element $i \in \mathrm{Hom}_K(\tau_m, H_\pi(\tau_m))$, then the $(\mathfrak{g}, K)$-module property of H_π^K gives us the canonical surjective K-homomorphism

$$\mathfrak{p}_+ \otimes_{\mathbb{C}} \tau_m \to \mathfrak{p}_+ \mathrm{Im}(\tau_m).$$

For the K-module $\tau_{[e_1,e_2;+]}$, by composing this K-homomorphism with the injection $\tau_{[e_1,e_2;+]} \subset \mathfrak{p}_+ \otimes_{\mathbb{C}} \tau_m$, we obtain a $\mathbb{C}$-linear map ϕ

$$\phi : \mathrm{Hom}_K(\tau_m, H_\pi(\tau_m)) \mapsto \mathrm{Hom}_K(\tau_{[e_1,e_2;+]}, H_\pi(\tau_{[e_1,e_2;+]})),$$

which is determining the action of $\mathfrak{p}_+$ on H_π^K.

Our goal is to determine the matrix representation $\Gamma_{[e_1,e_2;+]}$ of ϕ *i.e.*, to find a matrix $\Gamma_{[e_1,e_2;+]}$ such that

$$\phi\Big(\sum_{\gamma \in I(\pi,\tau_m)} [\gamma]\Big) = \Big(\sum_{\gamma' \in I(\pi,\tau_{m'})} [\gamma']\Big) \times \Gamma_{[e_1,e_2;+]},$$

where $m' = [e_1, e_2; +]$. Therefore we have to compute the image (under ϕ) of the K-isomorphism $[\gamma] : \tau_m \to W_\gamma^{(m)}$ for each $\gamma \in I(\pi, \tau_m)$, that is, to express the K-homomorphism ϕ_γ in the commutative diagram

$$\begin{tikzcd}
\tau_{[e_1,e_2;+]} \arrow[r] \arrow[dr, dashed, "\phi_\gamma"'] & \mathfrak{p}_+ \otimes_{\mathbb{C}} \tau_m \arrow[d, "{[\gamma]}"] & \\
& \mathfrak{p}_+ W_\gamma^{(m)} \arrow[r] & H_\pi(\tau_{[e_1,e_2;+]})
\end{tikzcd}$$

Diagram 1.

in terms of the fixed basis $[\gamma']$ with $\gamma' \in I(\pi, \tau_{[e_1,e_2;+]})$.

Set $\nu = (m_1 + m_2 + s)/2$. For each τ_m, we regard the vector space $\mathrm{Hom}_K(\tau_m, H_\pi(\tau_m))$ as a subspace of the $\nu + 1$-dimensional vector space $\mathrm{Hom}_K(\tau_m, \oplus_\gamma W_\gamma^{(m)})$ with γ running over all positive integer pairs (t_1, t_2) such that $t_1 + t_2 = \nu$ and hence define $\Gamma_{[e_1,e_2;+]}$ as a matrix of size $(\nu + 1 + (e_1 + e_2)/2) \times (\nu + 1)$.

Remark 3.1. *The size of the matrix $\Gamma_{[e_1,e_2;\pm]}$ is defined by the multiplicities of τ_m and $\tau_{[e_1,e_2;\pm]}$. The explicit formula of $m(\pi, \tau_{[e_1,e_2;\pm]})$ seems to be involved. Therefore here we define that multiplicity as the cardinality of the set $I(\pi, \tau_m)$.*

Fix a K-module τ_m with $m = [m_1, m_2; l]$. Set $r = (s + l)/2$ and $m' = [m_1+e_1, m_2+e_2; l+2]$. In the following list, we use the coefficients $\mathbf{c}_p^1$ and $\mathbf{c}_q^2$ defined in Proposition 3.2.

1. Define a matrix $\Gamma_{[-,-;+]} = \{a_{ij}\}_{0\le i\le \nu-1, 0\le j\le \nu}$ of size $\nu \times (\nu+1)$ so that its all non zero entries are given by

$$\begin{aligned} &a_{t-1,t} = a_t \quad && \text{if } (t, \nu-t) \in I(\pi, \tau_m),\ (t-1, \nu-t) \in I(\pi, \tau_{m'}),\\ &a_{t,t} = b_t && \text{if } (t, \nu-t) \in I(\pi, \tau_m),\ (t, \nu-t-1) \in I(\pi, \tau_{m'}). \end{aligned}$$

where

$$\begin{aligned} a_t &= \frac{1}{2}(\mu_2 + 1 + m_1 + r - 2t),\\ b_t &= -\frac{1}{2}(\mu_1 - 1 - m_2 + r - 2t), \end{aligned}$$

for $\gamma = (t, \nu - t) \in I(\pi, \tau)$.

2. Define a matrix $\Gamma_{[+,+;+]} = \{a_{ij}\}_{0\le i\le \nu+1, 0\le j\le \nu}$ of size $(\nu+2)\times(\nu+1)$ so that its all non zero entries are given by

$$\begin{aligned} &a_{t,t} = a_t \quad && \text{if } (t, \nu-t) \in I(\pi, \tau_m),\ (t, \nu-t+1) \in I(\pi, \tau_{m'}),\\ &a_{t+1,t} = b_t && \text{if } (t, \nu-t) \in I(\pi, \tau_m),\ (t+1, \nu-t) \in I(\pi, \tau_{m'}). \end{aligned}$$

where

$$\begin{aligned} a_t &= \frac{1}{2}(\mu_2 + 1 + m_1 + r - 2t)(1 - \mathbf{c}^1_t)\mathbf{c}^2_{\nu-t+1},\\ b_t &= -\frac{1}{2}(\mu_1 + 3 + 2m_1 + m_2 + r - 2t)\mathbf{c}^1_{t+1}(1 - \mathbf{c}^2_{\nu-t}), \end{aligned}$$

for $\gamma = (t, \nu - t) \in I(\pi, \tau)$.

3. Define a square matrix $\Gamma_{[-,+;+]} = \{a_{ij}\}_{0\le i\le \nu, 0\le j\le \nu}$ of size $(\nu+1)\times(\nu+1)$ so that its all non zero entries are given by

$$\begin{aligned} &a_{t-1,t} = a_t \quad && \text{if } (t, \nu-t) \in I(\pi, \tau_m),\ (t-1, \nu-t+1) \in I(\pi, \tau_{m'})\\ &a_{t,t} = b_t && \text{if } (t, \nu-t) \in I(\pi, \tau_m),\ (t, \nu-t) \in I(\pi, \tau_{m'}). \end{aligned}$$

where

$$\begin{aligned} a_t &= \frac{1}{2}(\mu_2 + 1 + m_1 + r - 2t)\mathbf{c}^2_{\nu-t+1},\\ b_t &= \frac{1}{2}(\mu_1 + 1 + m_2 + r - 2t)(1 - \mathbf{c}^2_{\nu-t}), \end{aligned}$$

for $\gamma = (t, \nu - t) \in I(\pi, \tau)$.

4. Define a square matrix $\Gamma_{[+,-;+]} = \{a_{ij}\}_{0\le i\le \nu, 0\le j\le \nu}$ of size $(\nu+1)\times(\nu+1)$ so that its all non zero entries are given by

$$\begin{aligned} &a_{t,t} = a_t \quad && \text{if } (t, \nu-t) \in I(\pi, \tau_m),\ (t, \nu-t) \in I(\pi, \tau_{m'}),\\ &a_{t+1,t} = b_t && \text{if } (t, \nu-t) \in I(\pi, \tau_m),\ (t+1, \nu-t-1) \in I(\pi, \tau_{m'}). \end{aligned}$$

where

$$\begin{aligned} a_t &= \frac{1}{2}(\mu_2 + 1 + m_1 + r - 2t)(1 - \mathbf{c}^1_t),\\ b_t &= \frac{1}{2}(\mu_1 + 1 + 2m_1 - m_2 + r - 2t)\mathbf{c}^1_{t+1}, \end{aligned}$$

for $\gamma = (t, \nu - t) \in I(\pi, \tau)$.

Our main result is these constructions of $\Gamma_{[e_1,e_2;+]}$. In the following, we show that these matrices are the desired ones.

Theorem 3.6. *Let (e_1, e_2) be a pair so that $e_1, e_2 \in \{\pm 1\}$. Then the matrix $\Gamma_{[e_1,e_2;+]}$ defined above is the $\mathbb{C}$-linear homomorphism between the vector spaces* $\mathrm{Hom}_K(\tau_m, H_\pi(\tau_m))$ *and* $\mathrm{Hom}_K(\tau_{[e_1,e_2;+]}, H_\pi(\tau_{[e_1,e_2;+]}))$.

Proof. We only consider the case $(e_1, e_2) = (-1, -1)$, because the remaining cases are proved similarly. Set $m' = [m_1 - 1, m_2 - 1; l + 2]$ and fix a basis vector $[\gamma]$. From the K-equivariant property of ϕ_γ induced from $[\gamma]$ in the Diagram 1, the image of a fixed basis element $f_{pq}^{(m')}$ in $V_{m'}$ can be expressed as

$$\phi_\gamma(f_{pq}^{(m')}) = \sum_{\gamma' \in I(\pi, m')} c_{\gamma'} S_{\gamma', pq}^{(m')}(x).$$

Note that we omit the index (m) of basis vectors for only τ_m *i.e.*, write f_{pq} instead of $f_{pq}^{(m)}$. Consider the above expression at $x = 1_4$, by using $S_{\gamma,pq}(1_4) = \delta_{\gamma,pq}$, we then get

$$\phi_\gamma(f_{pq}^{(m')})(1_4) = c_{\gamma'}, \text{ if } \gamma' = (p, q).$$

On the other hand, the commutativity of the Diagram 1 and Proposition 3.2 imply that $\phi_\gamma(f_{pq}^{(m')})$ is equal to

$$E_{23} S_{\gamma,(p+1q+1)}(k) - E_{13} S_{\gamma,(pq+1)}(k) + E_{24} S_{\gamma,(p+1q)}(k) - E_{14} S_{\gamma,(pq)}(k).$$

Note that $X S_{\gamma,pq}(k) = 0$ for any $X \in \mathfrak{n}$. By considering the Iwasawa decomposition of E_{ij} $(i = 1, 2, j = 3, 4)$ given Lemma 1.1, one can calculate that

$$\begin{aligned}
(E_{13} S_{\gamma,(pq)})(1_4) &= \frac{1}{2}\Big(H_1 + \frac{1}{2}(I_{2,2} + h^1 - h^2)\Big) S_{\gamma,(pq)}(k) \mid_{k=1_4} \\
&= \frac{1}{4}(2\mu_1 + 6 + l + (2p - m_1) - (2q - m_2)) S_{\gamma,(pq)}(1_4), \\
(E_{24} S_{\gamma,(pq)})(1_4) &= \frac{1}{2}\Big((H_2 + \frac{1}{2}(I_{2,2} - h^1 + h^2)\Big) S_{\gamma,pq}(k) \mid_{k=1_4} \\
&= \frac{1}{4}\Big(2\mu_2 + 2 + l - (2p - m_1) + (2q - m_2)\Big) S_{\gamma,(pq)}(1_4),
\end{aligned}$$

$$\begin{aligned}
(E_{14} S_{\gamma,(pq)})(1_4) &= -e_+^2 S_{\gamma,(pq)}(k) \mid_{k=1_4} = (q - m_2) S_{\gamma,(p,q+1)}(1_4), \\
(E_{23} S_{\gamma,(p,q)})(1_4) &= e_-^1 S_{\gamma,(p,q)}(k) \mid_{k=1_4} = p S_{\gamma,(p-1,q)}(1_4).
\end{aligned}$$

Combining these observations, we obtain that $\phi_\gamma(f_{pq}^{(m')})(1_4)$ is equal to

$$\begin{aligned}
&\frac{1}{2}\Big(\mu_2 + q - p + \frac{m_1 - m_2 + l}{2}\Big) S_{\gamma,(p+1,q)}(1_4) + S_{\gamma,(p,q+1)}(1_4) \times \\
&\Big(-\frac{1}{2}\Big(\mu_1 + 2 + p - q + \frac{m_2 - m_1 + l}{2}\Big) + p + 1 - (q - m_2)\Big)
\end{aligned}$$

Using $S_{\gamma,pq}(1_4) = \delta_{\gamma,pq}$ again, one has

$$\gamma' \text{ is equal to } \gamma - (1, 0) \text{ or } \gamma - (0, 1)$$

and hence the corresponding coefficients $c_{\gamma'}$ are just

$$c_{\gamma'} = \frac{1}{2}\Big[\mu_2 + 1 + m_1 + \frac{s+l}{2} - 2t\Big]$$

and

$$c_{\gamma'} = -\frac{1}{2}\Big[\mu_1 - 1 - m_2 + \frac{l+s}{2} - 2t\Big],$$

respectively when $\gamma = (t, \nu - t) \in I(\pi, \tau)$. It shows the coincidence of $\Gamma_{[-,-;+]}$ with ϕ. $\square$

$\mathfrak{p}_-$-side. By the same computation as the case $\mathfrak{p}_+$-side we obtain similar results for the matrix form of the $\mathbb{C}$-linear map

$$\Gamma_{[e_1,e_2;-]} : \mathrm{Hom}_K(\tau_m, H_\pi(\tau_m)) \to \mathrm{Hom}_K(\tau_{[e_1,e_2;-]}, H_\pi(\tau_{[e_1,e_2;-]})).$$

1. Define a matrix $\Gamma_{[-,-;-]} = \{a_{ij}\}_{0\le i\le \nu-1, 0\le j\le \nu}$ of size $\nu \times (\nu+1)$ so that its all non zero entries are given by

$$\begin{aligned} a_{t,t} &= a_t \quad \text{if } (t, \nu-t) \in I(\pi,\tau_m),\ (t, \nu-t-1) \in I(\pi, \tau_{m'}),\\ a_{t-1,t} &= b_t \quad \text{if } (t, \nu-t) \in I(\pi,\tau_m),\ (t-1, \nu-t) \in I(\pi, \tau_{m'}). \end{aligned}$$

where

$$\begin{aligned} a_t &= \frac{1}{2}(\mu_2 + 1 - m_1 - r + 2t),\\ b_t &= -\frac{1}{2}(\mu_1 - 1 - 2m_1 - m_2 - r + 2t), \end{aligned}$$

for $\gamma = (t, \nu - t) \in I(\pi, \tau)$.

2. Define a matrix $\Gamma_{[+,+;-]} = \{a_{ij}\}_{0\le i\le \nu+1, 0\le j\le \nu}$ of size $(\nu+2)\times(\nu+1)$ so that its all non zero entries are given by

$$\begin{aligned} a_{t+1,t} &= a_t \quad \text{if } (t, \nu-t) \in I(\pi,\tau_m),\ (t+1, \nu-t) \in I(\pi, \tau_{m'}),\\ a_{t,t} &= b_t \quad \text{if } (t, \nu-t) \in I(\pi,\tau_m),\ (t, \nu-t+1) \in I(\pi, \tau_{m'}). \end{aligned}$$

where

$$\begin{aligned} a_t &= \frac{1}{2}(\mu_2 + 1 - m_1 - r + 2t)\mathbf{c}^1_{t+1}(1 - \mathbf{c}^2_{\nu-t}),\\ b_t &= -\frac{1}{2}(\mu_1 + 3 + m_2 - r + 2t)(1 - \mathbf{c}^1_t)\mathbf{c}^2_{\nu-t+1}, \end{aligned}$$

for $\gamma = (t, \nu - t) \in I(\pi, \tau)$.

3. Define a square matrix $\Gamma_{[-,+;-]} = \{a_{ij}\}_{0\le i\le \nu, 0\le j\le \nu}$ of size $(\nu+1)\times(\nu+1)$ so that its all non zero entries are given by

$$\begin{aligned} a_{t,t} &= a_t \quad \text{if } (t, \nu-t) \in I(\pi,\tau_m),\ (t, \nu-t) \in I(\pi, \tau_{m'}),\\ a_{t-1,t} &= b_t \quad \text{if } (t, \nu-t) \in I(\pi,\tau_m),\ (t-1, \nu-t+1) \in I(\pi, \tau_{m'}). \end{aligned}$$

where

$$a_t = \frac{1}{2}(\mu_2 + 1 - m_1 - r + 2t)(1 - \mathbf{c}^2_{\nu-t}),$$
$$b_t = \frac{1}{2}(\mu_1 + 1 - 2m_1 + m_2 - r + 2t)\mathbf{c}^2_{\nu-t+1},$$

for $\gamma = (t, \nu - t) \in I(\pi, \tau)$.

4. Define a square matrix $\Gamma_{[+,-;-]} = \{a_{ij}\}_{0\le i\le\nu, 0\le j\le\nu}$ of size $(\nu+1)\times(\nu+1)$ so that its all non zero entries are given by

$$a_{t+1,t} = a_t \quad \text{if } (t, \nu-t) \in I(\pi, \tau_m),\ (t+1, \nu-t-1) \in I(\pi, \tau_{m'}),$$
$$a_{t,t} = b_t \quad \text{if } (t, \nu-t) \in I(\pi, \tau_m),\ (t, \nu-t) \in I(\pi, \tau_{m'}).$$

where

$$a_t = \frac{1}{2}(\mu_2 + 1 - m_1 - r + 2t)\mathbf{c}^1_{t+1},$$
$$b_t = \frac{1}{2}(\mu_1 + 1 - m_2 - r + 2t)(1 - \mathbf{c}^1_t),$$

for $\gamma = (t, \nu - t) \in I(\pi, \tau)$.

Thus we have the following results similar to that of $\mathfrak{p}_+$-side.

Theorem 3.7. *Let (e_1, e_2) be a pair so that $e_1, e_2 \in \{\pm 1\}$. Then the matrix $\Gamma_{[e_1,e_2;-]}$ defined above is the $\mathbb{C}$-linear homomorphism between the vector spaces $\mathrm{Hom}_K(\tau_m, H_\pi(\tau_m))$ and $\mathrm{Hom}_K(\tau_{[e_1,e_2;-]}, H_\pi(\tau_{[e_1,e_2;-]}))$.*

Proof. Set $m' = [m_1+e_1, m_2+e_2; l-2]$ and fix a basis vector $[\gamma]$. From the K-equivariant property of ϕ_γ induced from $[\gamma]$ in the Diagram 1 , the image of a fixed basis element $f^{(m')}_{pq}$ in $V_{m'}$ can be expressed as

$$\phi_\gamma(f^{(m')}_{pq}) = \sum_{\gamma' \in I(\pi, m')} c_{\gamma'} S^{(m')}_{\gamma', pq}(x).$$

The commutativity of the Diagram 1 and Proposition 3.3 imply that $\phi_\gamma(f^{(m')}_{pq})$ is equal to

$$E_{41}S_{\gamma,(p+1q+1)}(k) + E_{42}S_{\gamma,(pq+1)}(k) - E_{31}S_{\gamma,(p+1q)}(k) - E_{32}S_{\gamma,(pq)}(k).$$

Combining the fact $XS_{\gamma,pq}(k) = 0$ for any $X \in \mathfrak{n}$ and the Iwasawa decomposition of E_{ji} $(i = 1, 2, j = 3, 4)$ given Lemma 1.1, one can also calculate that

$$(E_{31}S_{\gamma,pq})(1_4) = \frac{1}{2}\Big(H_1 - \frac{1}{2}(I_{2,2} + h^1 - h^2)\Big)S_{\gamma,pq}(k)\ |_{k=1_4}$$
$$= \frac{1}{4}(2\mu_1 + 6 - l - (2p - m_1) + (2q - m_2))S_{\gamma,pq}(1_4),$$
$$(E_{42}S_{\gamma,pq})(1_4) = \frac{1}{2}\Big(H_2 - \frac{1}{2}(I_{2,2} - h^1 + h^2)\Big)S_{\gamma,pq}(k)\ |_{k=1_4}$$
$$= \frac{1}{4}\Big(2\mu_2 + 2 - l + (2p - m_1) - (2q - m_2)\Big)S_{\gamma,pq}(1_4),$$

$$(E_{32}S_{\gamma,pq})(1_4) = -e_+^1 S_{\gamma,pq}(k) \mid_{k=1_4} = (p-m_1)S_{\gamma,p+1q}(1_4),$$
$$(E_{41}S_{\gamma,pq})(1_4) = e_-^2 S_{\gamma,pq}(k) \mid_{k=1_4} = (q+a_2)S_{\gamma,pq-1}(1_4).$$

It follows that $\phi_\gamma(f_{pq}^{(m')})(1_4)$ is equal to

$$\Big(-\frac{1}{2}\Big(\mu_1 + q - p - \frac{m_2 - m_1 + l}{2}\Big) + q + m_1 - p\Big) S_{\gamma,p+1q}(1_4)$$
$$-\frac{1}{2}\Big(\mu_2 + p - q - \frac{m_1 - m_2 + l}{2}\Big) S_{\gamma,pq+1}(1_4).$$

As seen in the previous lemma

$$\gamma' \text{ is equal to } \gamma - (0,1) \text{ or } \gamma - (1,0)$$

and hence the corresponding coefficients $c_{\gamma'}$ are just

$$c_{\gamma'} = \frac{1}{2}\Big[\mu_2 + 1 - m_1 - r + 2t\Big]$$

and

$$c_{\gamma'} = -\frac{1}{2}\Big[\mu_1 - 1 - 2m_1 - m_2 - r + 2t\Big],$$

respectively when $\gamma = (t, \nu - t) \in I(\pi, \tau)$. It shows the coincidence of $\Gamma_{[e_1,e_2;-]}$ with ϕ. □

3.4. **Matrix representations.** We now describe the relations between the matrices $\mathcal{C}_{[e_1,e_2;\pm]}$ and $\Gamma_{[e_1,e_2;\pm]}$ in terms of the marked elementary basis functions in the K-isotypic component of π. Fix τ_m with $m = [m_1, m_2; l]$. For a pair (i,j) such that $i + j = \nu$ and $i, j \in \mathbb{Z}_+$, we define a row matrix $\mathbf{F}_{(i,j)}^{(m)}$ of size $1 \times (m_1+1)(m_2+1)$ with entries in the set of all marked elementary functions of $W_{ij}^{(m)}$ introduced in Definition 2.1 as follows

$$\mathbf{F}_\gamma^{(m)} = (S_{\gamma,00}, S_{\gamma,01} ..., S_{\gamma,0m_2}, S_{\gamma,10}, S_{\gamma,11}, ..., S_{\gamma,m_1(m_2-1)}, S_{\gamma,m_1m_2})$$

with $\gamma = (i,j)$. To the K-isotypic component of τ_m in π we associate a matrix $\mathbf{S}^{(m)}$ of size $(m_1+1)(m_2+1) \times (\nu+1)$ such that the non zero columns are those ${}^{\mathbf{t}}\mathbf{F}_\gamma^{(m)}$ with entries in the K-isotypic component $H_\pi(\tau_m)$, that is,

$$\mathbf{S}^{(m)} = [{}^{\mathbf{t}}\mathbf{F}_{(0,\nu)}^{(m)}, ..., {}^{\mathbf{t}}\mathbf{F}_{(\nu,0)}^{(m)}],$$

where the symbol ^{t} is the transpose and $\mathbf{F}_\gamma^{(m)} = \mathbf{0}$ when $\gamma \notin I(\pi, \tau_m)$.

Now we are in a position to state the main result which includes all results in this paper.

Theorem 3.8. *Let $\tau_{[e_1,e_2;\pm]}$ be a simple K-submodule of the K-module $\mathfrak{p}_\pm \otimes_{\mathbb{C}} \tau_m$ for a given simple K-module τ_m and the K-module* $(\mathrm{Ad}, \mathfrak{p}_\pm)$. *Then we have that*

$$\mathcal{C}_{[e_1,e_2;\pm]}\mathbf{S}^{(m)} = \mathbf{S}^{([e_1,e_2;\pm])}\Gamma_{[e_1,e_2;\pm]},$$

where the product of the entries of matrices of the left hand side is the differential operation.

3.5. **Examples of contiguous relations and their composites.** Here are some examples of contiguous relations along the multiplicity one K-types in a given principal series representation π. We refer the reader to [5] for further reference and contiguous relations.

Let $\tau = \tau_{[m_1,m_2;l]}$ be a K-submodule of $\pi = \mathrm{Ind}_P^G(\sigma_{s,e} \otimes e^{\mu+\rho} \otimes 1_N)$. Then Lemma 2.2 implies that $[\pi\mid_K : \tau] = 1$ if and only if

$$\mid s \mid = m_1 + m_2 \text{ and } l = 2m_2 + s + 1 - e(-1)\ (mod\ 4).$$

Hence, in this case, we may assume that the size of the matrices $\Gamma_{[+,-;\pm]}, \Gamma_{[+,-;\pm]}$ are just 1×1 $i, e.$, they are constants and $\Gamma_{[+,+;\pm]}$ is of size 2×1, because the other entries are zero. Although there is no $\Gamma_{[-,-;\pm]}$, since $\tau_{[-,-;\pm]}$ does not occur in π.

Note that $H_\pi(\tau) \cong W^{(m)}_{(m_1,m_2)}$ if $s \geq 0$ and $H_\pi(\tau) \cong W^{(m)}_{(0,0)}$ if $s \leq 0$. Put

$$\nu_1 = \frac{l + m_1 - m_2}{2} \quad \text{and} \quad \nu_2 = \frac{l + m_2 - m_1}{2}.$$

Formula 3.9. *Assume $s \geq 0$. Then we have*

$$\begin{aligned}
\mathcal{C}_{[+,-;+]}{}^{\mathbf{t}}\mathbf{F}^{\tau}_{(m_1,m_2)} &= \frac{1}{2}(\mu_1 + 1 + \nu_1)^{\mathbf{t}}\mathbf{F}^{\tau[+,-;+]}_{(+,-)},\\
\mathcal{C}_{[-,+;+]}{}^{\mathbf{t}}\mathbf{F}^{\tau}_{(m_1,m_2)} &= \frac{1}{2}(\mu_2 + 1 + \nu_2)^{\mathbf{t}}\mathbf{F}^{\tau[-,+;+]}_{(-,+)},\\
\mathcal{C}_{[+,-;-]}{}^{\mathbf{t}}\mathbf{F}^{\tau}_{(m_1,m_2)} &= \frac{1}{2}(\mu_2 + 1 - \nu_2)^{\mathbf{t}}\mathbf{F}^{\tau[+,-;-]}_{(+,-)},\\
\mathcal{C}_{[-,+;-]}{}^{\mathbf{t}}\mathbf{F}^{\tau}_{(m_1,m_2)} &= \frac{1}{2}(\mu_1 + 1 - \nu_1)^{\mathbf{t}}\mathbf{F}^{\tau[-,+;-]}_{(-,+)}.
\end{aligned}$$

Here the symbol $(\pm,\pm)$ means $(m_1 \pm 1, m_2 \pm 1)$, respectively.

Formula 3.10. *Assume $s \leq 0$ and set $n = (0,0)$. Then we have*

$$\begin{aligned}
\mathcal{C}_{[+,-;+]}{}^{\mathbf{t}}\mathbf{F}^{\tau}_{n} &= \frac{1}{2}(\mu_2 + 1 + \nu_1)^{\mathbf{t}}\mathbf{F}^{\tau[+,-;+]}_{n},\\
\mathcal{C}_{[-,+;+]}{}^{\mathbf{t}}\mathbf{F}^{\tau}_{n} &= \frac{1}{2}(\mu_1 + 1 + \nu_2)^{\mathbf{t}}\mathbf{F}^{\tau[-,+;+]}_{n},\\
\mathcal{C}_{[+,-;-]}{}^{\mathbf{t}}\mathbf{F}^{\tau}_{n} &= \frac{1}{2}(\mu_1 + 1 - \nu_2)^{\mathbf{t}}\mathbf{F}^{\tau[+,-;-]}_{n},\\
\mathcal{C}_{[-,+;-]}{}^{\mathbf{t}}\mathbf{F}^{\tau}_{n} &= \frac{1}{2}(\mu_2 + 1 - \nu_1)^{\mathbf{t}}\mathbf{F}^{\tau[-,+;-]}_{n}.
\end{aligned}$$

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Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo, 153-8914 Japan.
E-mail address : magnai1@ms.u-tokyo.ac.jp

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ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012