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The $(\mathfrak{g}, K)$-module structures of the principal series representations of $SL(4, R)$

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THE $(\mathfrak{g}, K)$-MODULE STRUCTURES OF THE PRINCIPAL SERIES REPRESENTATIONS OF $SL(4, \mathrm{R})$

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Abstract

We describe explicitly the $(\mathfrak{g}, K)$-module structures of the principal series representations of $SL(4, \mathbf{R})$.

1 Introduction

When we study a representation of a reductive Lie group, we often investigate its associated $(\mathfrak{g}, K)$-module first, because it has an advantage of being purely algebraic. By using this, we sometimes have "algorithmic" or effective computable results for the representation in question.

In this paper, we describe explicitly the $(\mathfrak{g}, K)$-module structures of the principal series representations of $SL(4, \mathbf{R})$. The method of investigating $(\mathfrak{g}, K)$-module structures for groups of higher rank used in this paper was originally found by Takayuki Oda in the case of $Sp(2, \mathbf{R})$ ([6]). Furthermore Tadashi Miyazaki [3, 4] applied the same method for the cases $Sp(3, \mathbf{R})$ and $SL(3, \mathbf{R})$. There is another method different from ours, see, e.g., Howe [2] for $GL(3, \mathbf{R})$.

Let us formulate the problem more precisely for a general real semisimple Lie group G with a maximal compact subgroup K. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively. Moreover let (π, H_π) be a standard representation of G with the K-finite part $H_{\pi,K}$. Since (π, H_π) is realized as a subspace of $L^2(K)$, we can see the K-module structure of π by the Peter-Weyl theorem. Thus, to accomplish the investigation of the $(\mathfrak{g}, K)$-module structure of π, it is sufficient to investigate the action of $\mathfrak{p}$ or $\mathfrak{p}_{\mathbf{C}}$ because of the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. To investigate the action of $\mathfrak{p}_{\mathbf{C}}$, we define Γ_i^τ as follows. For a K-type (τ, V_τ) of π and a K-homomorphism $\eta : V_\tau \to H_{\pi,K}$, we define a K-homomorphism $\widetilde{\eta} : \mathfrak{p}_{\mathbf{C}} \otimes V_\tau \to H_{\pi,K}$ by $X \otimes v \mapsto \pi(X)\eta(v)$, where we regard $\mathfrak{p}_{\mathbf{C}}$ as a K-module by the adjoint action of K and denote the differential of π by the same symbol π. Let $\mathfrak{p}_{\mathbf{C}} \otimes V_\tau \simeq \bigoplus_i V_{\tau_i}$ be the decomposition into irreducible K-modules, and fix an injection I_i^τ from V_{τ_i} into $\mathfrak{p}_{\mathbf{C}} \otimes V_\tau$ for each i. We then define a linear map $\Gamma_i^\tau \colon \operatorname{Hom}_K(V_\tau, H_{\pi,K}) \to \operatorname{Hom}_K(V_{\tau_i}, H_{\pi,K})$ by $\eta \mapsto \widetilde{\eta} \circ I_i^\tau$. This Γ_i^τ characterizes the action of $\mathfrak{p}_{\mathbf{C}}$. Now the problem is to describe explicitly Γ_i^τ.

The key to describing Γ_i^τ is to find a "good" basis in the space $\operatorname{Hom}_K(V_\tau, H_{\pi,K})$ of intertwining operators (which is not one-dimensional in general) with respect to which we can easily describe Γ_i^τ. We take as the good basis a basis induced from *elementary functions* in H_π.

As a result of this paper, we obtain some relations between vectors in $H_\pi[\tau]$ and $H_\pi[\tau_i]$. Here $H_\pi[\tau]$ means the τ-isotypic component of H_π. We can use these relations to obtain explicit formulae of principal series Whittaker functions of $SL(4, \mathbf{R})$ with a certain K-type from those with another K-type. Tatsuo Hina, Taku Ishii and Takayuki Oda [1] give the explicit formulae of principal series Whittaker functions of $SL(4, \mathbf{R})$ with the minimal K-type. So there is a possibility that we derive explicit formulae of principal series Whittaker functions of $SL(4, \mathbf{R})$ with an arbitrary K-type using the relations obtained in this paper. Passing to the various realizations or models of the principal series representation, including Whittaker models, one can have an infinite number of difference-differential operators. It seems interesting problem to have explicit formulae of these realizations (see Miyazaki [5] for $SL(3, \mathbf{R})$).

Here are the contents of this paper. In section 2, we recall the structure of $G = SL(4, \mathbf{R})$ and define the principal series representation of G. In section 3, we study representations of $K = SO(4)$. In section 4, we investigate the K-module structure of the principal series representation by the Peter-Weyl theorem, where we also define the elementary functions. In section 5, which is the main part of this paper, we define Γ_i^τ for $G = SL(4, \mathbf{R})$ and also $\overline{\Gamma}_i^\tau$ which is essentially the same as Γ_i^τ. Then we describe $\overline{\Gamma}_i^\tau$ explicitly in Theorem 5.2. We also give the action of $\mathfrak{p}_{\mathbf{C}}$ explicitly by using Γ_i^τ in Theorem 5.5.

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2 Preliminaries

2.1 The structure of $SL(4, \mathbf{R})$

Let G be the special linear group $SL(4, \mathbf{R})$ of degree four. We take $K = SO(4)$ as a maximal compact subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively:

$$\mathfrak{g} = \{X \in \mathfrak{gl}(4, \mathbf{R}) \mid \operatorname{tr} X = 0\}, \quad \mathfrak{k} = \{X \in \mathfrak{g} \mid X + {}^tX = 0\}.$$

Then $\mathfrak{g}$ has the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{p} = \{X \in \mathfrak{g} \mid X = {}^tX\}$.

We define subgroups N, A and M of G by

$$N = \left\{ \begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \in G \right\}, \qquad A = \left\{ \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_3 & \\ & & & a_4 \end{pmatrix} \middle| \begin{array}{c} a_k \in \mathbf{R}_{>0} \\ a_1a_2a_3a_4 = 1 \end{array} \right\},$$

$$M = \left\{ \begin{pmatrix} \varepsilon_1 & & & \\ & \varepsilon_2 & & \\ & & \varepsilon_3 & \\ & & & \varepsilon_4 \end{pmatrix} \middle| \begin{array}{c} \varepsilon_k \in \{\pm 1\} \\ \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = 1 \end{array} \right\}.$$

Let $\mathfrak{a}$ and $\mathfrak{n}$ be the Lie algebras of A and N respectively. If we denote by $E_{kl} \in M(4, \mathbf{R})$ the matrix unit with its (k, l)-th component 1 and remaining components 0, then $\mathfrak{n} = \bigoplus_{1 \le k < l \le 4} \mathbf{R}E_{kl}$. We take a basis $\{H_1, H_2, H_3\}$ of $\mathfrak{a}$ defined by

$$H_1 = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & -1 \end{pmatrix}, \qquad H_3 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

For $1 \le k \le 4$, we define a linear form e_k of $\mathfrak{a}$ by $\mathfrak{a} \ni \operatorname{diag}(t_1, t_2, t_3, t_4) \mapsto t_k \in \mathbf{C}$. Then the set of the restricted roots for $(\mathfrak{a}, \mathfrak{g})$ is given by $\Sigma = \{e_k - e_l \mid 1 \le k, l \le 4,\ k \ne l\}$, and the subset $\Sigma^+ = \{e_k - e_l \mid 1 \le k < l \le 4\}$ forms a positive root system. The half sum ρ of the positive roots is given by

$$\rho = \frac{1}{2} \sum_{e \in \Sigma^+} e = \frac{1}{2}(3e_1 + e_2 - e_3 - 3e_4).$$

2.2 Definition of the principal series representations

To define the principal series representation, we fix $\nu \in \operatorname{Hom}_{\mathbf{R}}(\mathfrak{a}, \mathbf{C})$ and a character σ of M. We identify ν with $(\nu_1, \nu_2, \nu_3) \in \mathbf{C}^3$ defined by $\nu_k = \nu(H_k)$ $(k = 1, 2, 3)$. Note that the half sum ρ of the positive roots has the coordinate $(\rho_1, \rho_2, \rho_3) = (3, 2, 1)$ through this identification. We also identify σ with $(\sigma_1, \sigma_2, \sigma_3) \in \{0, 1\}^{\oplus 3}$ defined by

$$\sigma(\varepsilon) = \varepsilon_1^{\sigma_1}\varepsilon_2^{\sigma_2}\varepsilon_3^{\sigma_3} \quad \text{for } \varepsilon = \operatorname{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in M.$$

We define a quasicharacter $e^{\nu+\rho} : A \to \mathbf{C}^\times$ by

$$e^{\nu+\rho}(a) = a_1^{\nu_1+\rho_1} a_2^{\nu_2+\rho_2} a_3^{\nu_3+\rho_3} \quad \text{for } a = \operatorname{diag}(a_1, a_2, a_3, a_4) \in A.$$

Then the principal series representation is defined as follows.

Definition 2.1. *Let $H_{(\nu,\sigma)}$ be the space of the locally integrable functions on G satisfying the equation*

$$f(namg) = e^{\nu+\rho}(a)\sigma(m)f(g)$$

for $(n, a, m, g) \in N \times A \times M \times G$, and $f|_K \in L^2(K)$. Then the principal series representation $\pi_{(\nu,\sigma)}$ is the right representation of G on $H_{(\nu,\sigma)}$.

3 Representations of K

We study representations of K. We realize representations of K as $SU(2) \times SU(2)$-modules by using a covering map $\varphi : SU(2) \times SU(2) \to K$, which is defined in subsection 3.1. In subsection 3.3, the adjoint action of K on $\mathfrak{p}_{\mathbf{C}}$ is studied. In subsection 3.4, we give $I_i^\tau : V_{\tau_i} \to \mathfrak{p}_{\mathbf{C}} \otimes V_\tau$, which is explained in introduction.

3.1 The covering group of K

Let

$$\mathbf{H} = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \middle| \, a, b \in \mathbf{C} \right\}.$$

We regard $\mathbf{H}$ as a four-dimensional real Euclidean space with an inner product $(x, y) = \operatorname{tr} xy^*$ for $x, y \in \mathbf{H}$. Here $y^* = {}^t\bar{y}$. We define for $(g_1, g_2) \in SU(2) \times SU(2)$, an automorphism $\varphi(g_1, g_2)$ of $\mathbf{H}$ by

$$\varphi(g_1, g_2)(x) = g_1 x g_2^{-1} \quad \text{for } x \in \mathbf{H}.$$

Then $\varphi(g_1, g_2)$ preserves the inner product and the orientation, hence we have a homomorphism

$$\varphi : SU(2) \times SU(2) \longrightarrow SO(\mathbf{H}) \simeq SO(4) = K.$$

We observe that φ is surjective and the kernel of φ is $\{\pm(1_2, 1_2)\}$. If we take

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{-1} & \\ & -\sqrt{-1} \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} & \sqrt{-1} \\ \sqrt{-1} & \end{pmatrix}$$

as an orthonormal basis in $\mathbf{H}$, then $\varphi : SU(2) \times SU(2) \to K$ is given by

$$\varphi(g_1, g_2) = \begin{pmatrix} \operatorname{Re} a\overline{a'} + \operatorname{Re} b\overline{b'} & -\operatorname{Im} a\overline{a'} + \operatorname{Im} b\overline{b'} & \operatorname{Re} a\overline{b'} - \operatorname{Re} b\overline{a'} & -\operatorname{Im} a\overline{b'} - \operatorname{Im} b\overline{a'} \\ \operatorname{Im} a\overline{a'} + \operatorname{Im} b\overline{b'} & \operatorname{Re} a\overline{a'} - \operatorname{Re} b\overline{b'} & \operatorname{Im} a\overline{b'} - \operatorname{Im} b\overline{a'} & \operatorname{Re} a\overline{b'} + \operatorname{Re} b\overline{a'} \\ -\operatorname{Re} ab' + \operatorname{Re} ba' & \operatorname{Im} ab' + \operatorname{Im} ba' & \operatorname{Re} aa' + \operatorname{Re} bb' & -\operatorname{Im} aa' + \operatorname{Im} bb' \\ -\operatorname{Im} ab' + \operatorname{Im} ba' & -\operatorname{Re} ab' - \operatorname{Re} ba' & \operatorname{Im} aa' + \operatorname{Im} bb' & \operatorname{Re} aa' - \operatorname{Re} bb' \end{pmatrix}$$

for

$$(g_1, g_2) = \left(\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \begin{pmatrix} a' & b' \\ -\overline{b'} & \overline{a'} \end{pmatrix} \right) \in SU(2) \times SU(2).$$

Here $\operatorname{Re} z$ and $\operatorname{Im} z$ mean the real part and the imaginary part of $z \in \mathbf{C}$, respectively.

For later use, we give the explicit description of the differential $d\varphi : \mathfrak{su}(2)_{\mathbf{C}} \oplus \mathfrak{su}(2)_{\mathbf{C}} \to \mathfrak{k}_{\mathbf{C}}$. We take a basis $\{H, E, F\}$ of $\mathfrak{su}(2)_{\mathbf{C}}$ defined by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

If we define $Y_k \in \mathfrak{k}_{\mathbf{C}}$ $(1 \le k \le 6)$ by

$$Y_1 = \left(\begin{array}{cc|cc} & \sqrt{-1} & & \\ -\sqrt{-1} & & & \\ \hline & & & \sqrt{-1} \\ & & -\sqrt{-1} & \end{array} \right), \quad Y_2 = \frac{1}{2}\left(\begin{array}{c|c} & C_{11} \\ \hline -C_{11} & \end{array} \right), \quad Y_3 = \frac{1}{2}\left(\begin{array}{c|c} & C_{22} \\ \hline -C_{22} & \end{array} \right),$$

$$Y_4 = \left(\begin{array}{cc|cc} & -\sqrt{-1} & & \\ \sqrt{-1} & & & \\ \hline & & & \sqrt{-1} \\ & & -\sqrt{-1} & \end{array} \right), \quad Y_5 = \frac{1}{2}\left(\begin{array}{c|c} & C_{12} \\ \hline -C_{21} & \end{array} \right), \quad Y_6 = \frac{1}{2}\left(\begin{array}{c|c} & -C_{21} \\ \hline C_{12} & \end{array} \right),$$

where

$$C_{11} = \begin{pmatrix} -1 & \sqrt{-1} \\ \sqrt{-1} & 1 \end{pmatrix}, \quad C_{12} = \begin{pmatrix} 1 & -\sqrt{-1} \\ \sqrt{-1} & 1 \end{pmatrix}, \quad C_{21} = \begin{pmatrix} 1 & \sqrt{-1} \\ -\sqrt{-1} & 1 \end{pmatrix}, \quad C_{22} = \begin{pmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & -1 \end{pmatrix},$$

then the correspondence between $\mathfrak{su}(2)_{\mathbf{C}} \oplus \mathfrak{su}(2)_{\mathbf{C}}$ and $\mathfrak{k}_{\mathbf{C}}$ through $d\varphi$ is given as follows:

$$\begin{array}{lll} (H, 0) \longmapsto Y_1, & (E, 0) \longmapsto Y_2, & (F, 0) \longmapsto Y_3, \\ (0, H) \longmapsto Y_4, & (0, E) \longmapsto Y_5, & (0, F) \longmapsto Y_6. \end{array}$$

3.2 Representations of K

We first study representations of $SU(2)$. For an integer $m \geq 0$, let V_m be the subspace of degree m homogeneous polynomials of two variables z_1, z_2 in the polynomial ring $\mathbf{C}[z_1, z_2]$. For $g \in SU(2)$ with $g^{-1} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ and $f \in V_m$, we set

$$\tau_m(g)f(z_1, z_2) = f(az_1 + bz_2, -\bar{b}z_1 + \bar{a}z_2).$$

It is known that irreducible representations of $SU(2)$ are exhausted by $\{(\tau_m, V_m) \mid m \in \mathbf{Z}_{\geq 0}\}$ up to equivalence. We take $\{v_k^{(m)} = z_1^k z_2^{m-k} \mid 0 \leq k \leq m\}$ as a basis of V_m. If we denote the differential of τ_m by the same symbol τ_m, then the action on this basis of $\{H, E, F\} \subset \mathfrak{su}(2)_{\mathbf{C}}$ is given by

$$\tau_m(H)v_k^{(m)} = (m-2k)v_k^{(m)}, \qquad \tau_m(E)v_k^{(m)} = -kv_{k-1}^{(m)}, \qquad \tau_m(F)v_k^{(m)} = (k-m)v_{k+1}^{(m)}.$$

We write for $\mathbf{m} = (m, n) \in \mathbf{Z}_{\geq 0}^2$, $\tau_{\mathbf{m}} = \tau_m \boxtimes \tau_n$ and $V_{\mathbf{m}} = V_m \boxtimes V_n$ which is a representation of $SU(2) \times SU(2)$. We take a basis $\{v_{(k,l)}^{\mathbf{m}} = v_k^{(m)} \otimes v_l^{(n)} \mid 0 \leq k \leq m,\ 0 \leq l \leq n\}$ of $V_{\mathbf{m}}$, which is called the *standard basis*.

The representation $(\tau_{\mathbf{m}}, V_{\mathbf{m}})$ of $SU(2) \times SU(2)$ induces a representation of $K = SO(4)$ by $\tau_{\mathbf{m}}(k) = \tau_{\mathbf{m}}(\varphi^{-1}(k))$ for $k \in K$. This definition is well-defined if $\tau_{\mathbf{m}}(\mathrm{Ker}\,\varphi) = 1$, i.e., $m + n \equiv 0 \pmod 2$. Thus we find that $\widehat{K} = \{(\tau_{\mathbf{m}}, V_{\mathbf{m}}) \mid \mathbf{m} \in L\}$, where we put $L = \{\mathbf{m} = (m, n) \in \mathbf{Z}_{\geq 0}^2 \mid m + n \equiv 0 \pmod 2\}$. We say that an element $\mathbf{m} = (m, n)$ of L is *even* if both m and n are even, and *odd* if both m and n are odd.

For later use, we here study the dual representation of $(\tau_{\mathbf{m}}, V_{\mathbf{m}})$. We first note that the dual representation (τ_m^*, V_m^*) of the representation (τ_m, V_m) of $SU(2)$ is isomorphic to (τ_m, V_m) again, since irreducible $(m+1)$-dimensional representation of $SU(2)$ is unique up to isomorphism. Hence we also note that the dual representation $(\tau_{\mathbf{m}}^*, V_{\mathbf{m}}^*)$ of the representation $(\tau_{\mathbf{m}}, V_{\mathbf{m}})$ of K is isomorphic to $(\tau_{\mathbf{m}}, V_{\mathbf{m}})$ again. In the next lemma, we give an isomorphism between $(\tau_{\mathbf{m}}, V_{\mathbf{m}})$ and $(\tau_{\mathbf{m}}^*, V_{\mathbf{m}}^*)$.

Lemma 3.1. *For $\mathbf{m} = (m, n) \in L$, let $\{v_{(k,l)}^{\mathbf{m}*} \mid 0 \leq k \leq m,\ 0 \leq l \leq n\}$ be the dual basis of the standard basis $\{v_{(k,l)}^{\mathbf{m}} \mid 0 \leq k \leq m,\ 0 \leq l \leq n\}$. Then $(\tau_{\mathbf{m}}^*, V_{\mathbf{m}}^*)$ is isomorphic to $(\tau_{\mathbf{m}}, V_{\mathbf{m}})$ as a K-module via*

$$v_{(k,l)}^{\mathbf{m}*} \longmapsto (-1)^{k+l} \binom{m}{k} \binom{n}{l} v_{\mathbf{m}-(k,l)}^{\mathbf{m}}$$

for $0 \leq k \leq m$, $0 \leq l \leq n$.

Proof. From Miyazaki [4, Lemma 3.4], V_m^* is isomorphic to V_m as an $SU(2)$-module via

$$v_k^{(m)*} \longmapsto (-1)^k \binom{m}{k} v_{m-k}^{(m)}.$$

Thus we obtain the lemma. □

3.3 The adjoint action of K on $\mathfrak{p}_{\mathbf{C}}$

The next proposition states the K-module structure of $\mathfrak{p}_{\mathbf{C}}$. Recall that we think of $\mathfrak{p}_{\mathbf{C}}$ as a K-module by the adjoint action of K.

Proposition 3.2. *As a K-module, $\mathfrak{p}_{\mathbf{C}}$ is isomorphic to $V_{(2,2)}$ via $X_{(x,y)} \mapsto v_{(x,y)}^{(2,2)}$ $(x, y = 0, 1, 2)$, where $X_{(x,y)} \in \mathfrak{p}_{\mathbf{C}}$ are defined as follows:*

$$X_{(0,0)} = 2\left(\begin{array}{c|c} & \\ \hline & C_{11} \end{array}\right), \qquad X_{(0,1)} = \left(\begin{array}{c|c} & C_{11} \\ \hline C_{11} & \end{array}\right), \qquad X_{(0,2)} = 2\left(\begin{array}{c|c} C_{11} & \\ \hline & \end{array}\right),$$

$$X_{(1,0)} = \left(\begin{array}{c|c} & C_{12} \\ \hline C_{21} & \end{array}\right), \qquad X_{(1,1)} = \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline & & -1 & \\ & & & -1 \end{array}\right), \qquad X_{(1,2)} = -\left(\begin{array}{c|c} & C_{21} \\ \hline C_{12} & \end{array}\right),$$

$$X_{(2,0)} = -2\left(\begin{array}{c|c} C_{22} & \\ \hline & \end{array}\right), \qquad X_{(2,1)} = \left(\begin{array}{c|c} & C_{22} \\ \hline C_{22} & \end{array}\right), \qquad X_{(2,2)} = -2\left(\begin{array}{c|c} & \\ \hline & C_{22} \end{array}\right).$$

Proof. Because $X_{(0,0)}$ satisfies

$$[Y_1, X_{(0,0)}] = [Y_4, X_{(0,0)}] = 2X_{(0,0)}, \quad [Y_2, X_{(0,0)}] = [Y_5, X_{(0,0)}] = 0,$$

we find that $X_{(0,0)}$ is a highest weight vector with weight $(2,2)$. Thus $\mathfrak{p}_{\mathbf{C}}$ contains a subrepresentation isomorphic to $V_{(2,2)}$. But, because $\dim \mathfrak{p}_{\mathbf{C}} = \dim V_{(2,2)} = 9$, we see that $\mathfrak{p}_{\mathbf{C}}$ is actually isomorphic to $V_{(2,2)}$. We can take an isomorphism from $\mathfrak{p}_{\mathbf{C}}$ to $V_{(2,2)}$ which maps $X_{(0,0)}$ to $v^{(2,2)}_{(0,0)}$. The remaining correspondence between $X_{(x,y)}$ and $v^{(2,2)}_{(x,y)}$ can be easily checked. For example, since $\tau_{(2,2)}(F,0)v^{(2,2)}_{(0,0)} = -2v^{(2,2)}_{(1,0)}$, we see that $v^{(2,2)}_{(1,0)}$ corresponds to

$$-\frac{1}{2}[Y_3, X_{(0,0)}] = X_{(1,0)}.$$

□

3.4 Clebsch-Gordan coefficients for $\mathfrak{p}_{\mathbf{C}} \otimes V_{\mathbf{m}}$

We study the irreducible decomposition of a tensor product representation $\mathfrak{p}_{\mathbf{C}} \otimes V_{\mathbf{m}}$ for $\mathbf{m} = (m,n) \in L$ as a K-module. Since $\mathfrak{p}_{\mathbf{C}}$ is isomorphic to $V_{(2,2)}$, we first investigate the $\mathfrak{su}(2)_{\mathbf{C}}$-module $V_2 \otimes V_m$.

It follows from the Clebsch-Gordan theorem that $V_2 \otimes V_m$ generically decomposes three irreducible subrepresentations of $\mathfrak{su}(2)_{\mathbf{C}}$:

$$V_2 \otimes V_m \simeq \bigoplus_{i=-1,0,1} V_{m+2i}.$$

Here some components may vanish. An injection from V_{m+2i} into $V_2 \otimes V_m$ for $i = -1, 0, 1$ is given in the following lemma.

Lemma 3.3. *When V_{m+2i}-component of $V_2 \otimes V_m$ does not vanish, we define a linear map $I^m_i : V_{m+2i} \to V_2 \otimes V_m$ by*

$$I^m_i(v^{(m+2i)}_k) = \sum_{x=0}^{2} A^{[m,i]}_{[k,x]} v^{(2)}_x \otimes v^{(m)}_{k+1-i-x}.$$

Here we put $v^{(m)}_k = 0$ unless $0 \le k \le m$ and the coefficients $A^{[m,i]}_{[k,x]}$ are defined as follows:

$$\begin{aligned}
A^{[m,1]}_{[k,0]} &= \frac{(m-k+2)(m-k+1)}{(m+2)(m+1)}, & A^{[m,1]}_{[k,1]} &= \frac{2(m-k+2)k}{(m+2)(m+1)}, & A^{[m,1]}_{[k,2]} &= \frac{k(k-1)}{(m+2)(m+1)},\\
A^{[m,0]}_{[k,0]} &= \frac{m-k}{m}, & A^{[m,0]}_{[k,1]} &= \frac{2k-m}{m}, & A^{[m,0]}_{[k,2]} &= -\frac{k}{m},\\
A^{[m,-1]}_{[k,0]} &= 1, & A^{[m,-1]}_{[k,1]} &= -2, & A^{[m,-1]}_{[k,2]} &= 1.
\end{aligned}$$

Then I^m_i is a generator of $\mathrm{Hom}_{\mathfrak{su}(2)_{\mathbf{C}}}(V_{m+2i}, V_2 \otimes V_m)$.

Proof. We first note that $I^m_i(v^{(m+2i)}_0) \in V_2 \otimes V_m$ is a highest weight vector with weight $m+2i$. Indeed, we have

$$\begin{aligned}
\tau_2 \otimes \tau_m(H)\left(I^m_i(v^{(m+2i)}_0)\right) &= \sum_{x=0}^{2} A^{[m,i]}_{[0,x]}\, \tau_2 \otimes \tau_m(H)\left(v^{(2)}_x \otimes v^{(m)}_{1-i-x}\right)\\
&= \sum_{x=0}^{2} A^{[m,i]}_{[0,x]} \left(\tau_2(H)(v^{(2)}_x) \otimes v^{(m)}_{1-i-x} + v^{(2)}_x \otimes \tau_m(H)(v^{(m)}_{1-i-x})\right)\\
&= \sum_{x=0}^{2} A^{[m,i]}_{[0,x]} \left((2-2x)v^{(2)}_x \otimes v^{(m)}_{1-i-x} + (m-2(1-i-x))v^{(2)}_x \otimes v^{(m)}_{1-i-x}\right)\\
&= (m+2i) I^m_i(v^{(m+2i)}_0),
\end{aligned}$$

and moreover,

$$\begin{aligned}\tau_2 \otimes \tau_m(E)\left(I_i^m(v_0^{(m+2i)})\right) &= \sum_{x=1}^{2} A_{[0,x]}^{[m,i]}\left(-x\cdot v_{x-1}^{(2)} \otimes v_{1-i-x}^{(m)}\right) + \sum_{x=0}^{2} A_{[0,x]}^{[m,i]}\left(-(1-i-x)v_x^{(2)} \otimes v_{-i-x}^{(m)}\right)\\ &= -\sum_{x=0}^{2}\left((x+1)A_{[0,x+1]}^{[m,i]} + (1-i-x)A_{[0,x]}^{[m,i]}\right) v_x^{(2)} \otimes v_{-i-x}^{(m)}\\ &= 0,\end{aligned}$$

since $(x+1)A_{[0,x+1]}^{[m,i]} + (1-i-x)A_{[0,x]}^{[m,i]} = 0$ for $i = -1, 0, 1$ and $x = 0, 1, 2$. Here we put $A_{[0,x]}^{[m,i]} = 0$ unless $0 \leq x \leq 2$.

Thus, to complete the proof, it suffices to show that

$$\tau_2 \otimes \tau_m(F)\left(I_i^m(v_k^{(m+2i)})\right) = I_i^m\left(\tau_{m+2i}(F)(v_k^{(m+2i)})\right).$$

This equation is implied by

$$(x-3)A_{[k,x-1]}^{[m,i]} + (k+1-i-x-m)A_{[k,x]}^{[m,i]} = (k-m-2i)A_{[k+1,x]}^{[m,i]},$$

which can be checked by direct computation. □

We next investigate $\mathfrak{p}_{\mathbf{C}} \otimes V_{\mathbf{m}}$. From above arguments, $\mathfrak{p}_{\mathbf{C}} \otimes V_{\mathbf{m}}$ generically decomposes nine irreducible subrepresentations:

$$\mathfrak{p}_{\mathbf{C}} \otimes V_{\mathbf{m}} \simeq \bigoplus_{i,j=-1,0,1} V_{\mathbf{m}+2(i,j)}.$$

Here some components may vanish.

Proposition 3.4. *When $V_{\mathbf{m}+2(i,j)}$-component of $\mathfrak{p}_{\mathbf{C}} \otimes V_{\mathbf{m}}$ does not vanish, we define a linear map $I_{(i,j)}^{\mathbf{m}} : V_{\mathbf{m}+2(i,j)} \to \mathfrak{p}_{\mathbf{C}} \otimes V_{\mathbf{m}}$ by*

$$I_{(i,j)}^{\mathbf{m}}(v_{(k,l)}^{\mathbf{m}+2(i,j)}) = \sum_{x=0}^{2}\sum_{y=0}^{2} A_{[(k,l),(x,y)]}^{[\mathbf{m},(i,j)]}\, X_{(x,y)} \otimes v_{(k,l)+(1,1)-(i,j)-(x,y)}^{\mathbf{m}}.$$

Here we put $v_{(k,l)}^{\mathbf{m}} = 0$ unless $0 \leq k \leq m$ and $0 \leq l \leq n$, and the coefficients $A_{[(k,l),(x,y)]}^{[\mathbf{m},(i,j)]}$ are defined by

$$A_{[(k,l),(x,y)]}^{[\mathbf{m},(i,j)]} = A_{[k,x]}^{[m,i]} \cdot A_{[l,y]}^{[n,j]}.$$

Then $I_{(i,j)}^{\mathbf{m}}$ is a generator of $\mathrm{Hom}_{\mathfrak{k}_{\mathbf{C}}}(V_{\mathbf{m}+2(i,j)}, \mathfrak{p}_{\mathbf{C}} \otimes V_{\mathbf{m}})$.

Proof. This follows immediately from Lemma 3.3. □

Lemma 3.5. *The coefficients $A_{[(k,l),(x,y)]}^{[\mathbf{m},(i,j)]}$ satisfy following relations.*

- $A_{[\mathbf{m}+2(i,j)-(k,l),(x,y)]}^{[\mathbf{m},(i,j)]} = (-1)^{i+j} A_{[(k,l),(2,2)-(x,y)]}^{[\mathbf{m},(i,j)]}$
- $2\left\{(k-i+1)A_{[(k,l),(0,y)]}^{[\mathbf{m},(i,j)]} + (m-k+i+1)A_{[(k,l),(2,y)]}^{[\mathbf{m},(i,j)]}\right\} = (im+i^2+i-2)A_{[(k,l),(1,y)]}^{[\mathbf{m},(i,j)]}$
- $2\left\{(l-j+1)A_{[(k,l),(x,0)]}^{[\mathbf{m},(i,j)]} + (n-l+j+1)A_{[(k,l),(x,2)]}^{[\mathbf{m},(i,j)]}\right\} = (jn+j^2+j-2)A_{[(k,l),(x,1)]}^{[\mathbf{m},(i,j)]}$

Proof. From the definition of $A_{[k,x]}^{[m,i]}$ in Lemma 3.3, it is easy to see that the coefficients $A_{[k,x]}^{[m,i]}$ satisfy

$$A_{[m+2i-k,x]}^{[m,i]} = (-1)^{i+1} A_{[k,2-x]}^{[m,i]},$$
$$2\left\{(k-i+1)A_{[k,0]}^{[m,i]} + (m-k+i+1)A_{[k,2]}^{[m,i]}\right\} = (im+i^2+i-2)A_{[k,1]}^{[m,i]}.$$

Because $A_{[(k,l),(x,y)]}^{[\mathbf{m},(i,j)]} = A_{[k,x]}^{[m,i]} \cdot A_{[l,y]}^{[n,j]}$, the lemma follows from these equations. □

4 The structure of the principal series representation as a K-module

We first recall the Peter-Weyl theorem for the compact group K. Next we investigate the K-module structure of the principal series representation by embedding it into $L^2(K)$.

$L^2(K)$ has a $K \times K$-bimodule structure by the two sided regular action:

$$((k_1, k_2)f)(x) = f(k_1^{-1} x k_2), \quad f \in L^2(K),\ x \in K,\ (k_1, k_2) \in K \times K.$$

We define a homomorphism $\Phi_{\mathbf{m}} : V_{\mathbf{m}}^* \boxtimes V_{\mathbf{m}} \to L^2(K)$ of $K \times K$-bimodules by

$$w \otimes v \longmapsto (k \mapsto \langle w, \tau_{\mathbf{m}}(k) v \rangle),$$

where $\langle\ ,\ \rangle$ is the canonical pairing on $V_{\mathbf{m}}^* \boxtimes V_{\mathbf{m}}$. Then the Peter-Weyl theorem tells that

$$\Phi = \widehat{\bigoplus_{\mathbf{m} \in L}} \Phi_{\mathbf{m}} : \widehat{\bigoplus_{\mathbf{m} \in L}} V_{\mathbf{m}}^* \boxtimes V_{\mathbf{m}} \longrightarrow L^2(K)$$

is an isomorphism of $K \times K$-bimodules. Here $\widehat{\bigoplus}$ means a Hilbert space direct sum.

We next consider the restriction map $r_K : H_{(\nu,\sigma)} \to L^2(K)$. It is injective because $G = NAK$, and if we regard $L^2(K)$ as a K-module by the regular right action only, then r_K is a homomorphism of K-modules. The image of r_K is

$$L^2_{(M,\sigma)}(K) = \{ f \in L^2(K) \mid f(mk) = \sigma(m) f(k) \ \text{ for a.e. } \ m \in M,\ k \in K \}.$$

$L^2_{(M,\sigma)}(K)$ has an irreducible decomposition $L^2_{(M,\sigma)}(K) \simeq \widehat{\bigoplus}_{\mathbf{m} \in L} V_{\mathbf{m}}^*[\sigma] \otimes V_{\mathbf{m}}$ as a K-module. Here $V_{\mathbf{m}}^*[\sigma]$ means the σ-isotypic component in $(\tau_{\mathbf{m}}^*|_M, V_{\mathbf{m}}^*)$, that is

$$V_{\mathbf{m}}^*[\sigma] = \{ w \in V_{\mathbf{m}}^* \mid \tau_{\mathbf{m}}^*(m) w = \sigma(m) w,\ m \in M \}.$$

Hence we obtain an isomorphism of K-modules

$$r_K^{-1} \circ \Phi : \widehat{\bigoplus_{\mathbf{m} \in L}} V_{\mathbf{m}}^*[\sigma] \otimes V_{\mathbf{m}} \longrightarrow H_{(\nu,\sigma)}.$$

In order to accomplish the investigation of the K-module structure of $H_{(\nu,\sigma)}$, it remains for us to decide $V_{\mathbf{m}}^*[\sigma]$. We first consider $V_{\mathbf{m}}[\sigma]$. Let $\mathbf{m} = (m, n)$. Since M is generated by three elements

$$m_1 = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \quad m_2 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \quad m_3 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix},$$

we see that $v \in V_{\mathbf{m}}$ is in $V_{\mathbf{m}}[\sigma]$ if and only if $\tau_{\mathbf{m}}(m_i) v = (-1)^{\sigma_i} v$ for $i = 1, 2, 3$. On the other hand, from the definition of $(\tau_{\mathbf{m}}, V_{\mathbf{m}})$ and because the inverse images of m_1, m_2, m_3 under $\varphi : SU(2) \times SU(2) \to K$ are given by

$$\varphi^{-1}(m_1) = \left\{ \pm \left(\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix} \right) \right\}, \quad \varphi^{-1}(m_2) = \left\{ \pm \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \right\},$$

$$\varphi^{-1}(m_3) = \left\{ \pm \left(\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} \right) \right\},$$

we have

$$\begin{aligned} \tau_{\mathbf{m}}(m_1) v^{\mathbf{m}}_{(k,l)} &= (-1)^{m + \frac{m+n}{2}} v^{\mathbf{m}}_{\mathbf{m}-(k,l)}, \\ \tau_{\mathbf{m}}(m_2) v^{\mathbf{m}}_{(k,l)} &= (-1)^{k+l} v^{\mathbf{m}}_{\mathbf{m}-(k,l)}, \\ \tau_{\mathbf{m}}(m_3) v^{\mathbf{m}}_{(k,l)} &= (-1)^{k+l+\frac{m+n}{2}} v^{\mathbf{m}}_{(k,l)}. \end{aligned}$$

Thus if $v^{\mathbf{m}}_{\mathbf{m}-(k,l)} + (-1)^{\varepsilon} v^{\mathbf{m}}_{(k,l)} \in V_{\mathbf{m}}$ with $\varepsilon \in \{0,1\}$ lies in $V_{\mathbf{m}}[\sigma]$, then we have

$$\begin{aligned} \sigma_1 &\equiv \varepsilon + m + \frac{m+n}{2} \pmod 2, \\ \sigma_2 &\equiv \varepsilon + k + l \pmod 2, \\ \sigma_3 &\equiv k + l + \frac{m+n}{2} \pmod 2, \end{aligned}$$

or equivalently

$$\begin{aligned} m &\equiv \sigma_1 + \sigma_2 + \sigma_3 \pmod 2, \\ \varepsilon &\equiv \sigma_1 + m + \frac{m+n}{2} \pmod 2, \\ k + l &\equiv \sigma_3 + \frac{m+n}{2} \pmod 2. \end{aligned}$$

Therefore, we see that if $m \not\equiv \sigma_1 + \sigma_2 + \sigma_3 \pmod 2$ then $V_{\mathbf{m}}[\sigma] = 0$, and that if $m \equiv \sigma_1 + \sigma_2 + \sigma_3 \pmod 2$ then

$$V_{\mathbf{m}}[\sigma] = \bigoplus_{(k,l)\in Z(\sigma;\mathbf{m})} \mathbf{C}(v^{\mathbf{m}}_{\mathbf{m}-(k,l)} + (-1)^{\varepsilon(\sigma;\mathbf{m})} v^{\mathbf{m}}_{(k,l)}).$$

Here $\varepsilon(\sigma;\mathbf{m}) \in \{0,1\}$ is defined by

$$\varepsilon(\sigma;\mathbf{m}) \equiv \sigma_1 + m + \frac{m+n}{2} \pmod 2,$$

and $Z(\sigma;\mathbf{m})$ is defined as follows:

- If $\mathbf{m}$ is even and $\varepsilon(\sigma;\mathbf{m}) = 1$, then

$$Z(\sigma;\mathbf{m}) = \left\{ (k,l) \in \mathbf{Z}^2 \,\middle|\, \begin{array}{c} 0 \le k \le m \text{ and } 0 \le l \le n/2 - 1, \text{ or } 0 \le k \le m/2 - 1 \text{ and } l = n/2 \\ k + l \equiv \sigma_3 + (m+n)/2 \pmod 2 \end{array} \right\}.$$

- If $\mathbf{m}$ is even and $\varepsilon(\sigma;\mathbf{m}) = 0$, then

$$Z(\sigma;\mathbf{m}) = \left\{ (k,l) \in \mathbf{Z}^2 \,\middle|\, \begin{array}{c} 0 \le k \le m \text{ and } 0 \le l \le n/2 - 1, \text{ or } 0 \le k \le m/2 \text{ and } l = n/2 \\ k + l \equiv \sigma_3 + (m+n)/2 \pmod 2 \end{array} \right\}.$$

- If $\mathbf{m}$ is odd, then

$$Z(\sigma;\mathbf{m}) = \left\{ (k,l) \in \mathbf{Z}^2 \,\middle|\, 0 \le k \le m \text{ and } 0 \le l \le \frac{n-1}{2},\ k + l \equiv \sigma_3 + \frac{m+n}{2} \pmod 2 \right\}.$$

By the correspondence between $V_{\mathbf{m}}$ and $V^*_{\mathbf{m}}$ in Lemma 3.1, we note that if $m \not\equiv \sigma_1 + \sigma_2 + \sigma_3 \pmod 2$ then $V^*_{\mathbf{m}}[\sigma] = 0$, and if $m \equiv \sigma_1 + \sigma_2 + \sigma_3 \pmod 2$ then $\{v^{\mathbf{m}*}_{\mathbf{m}-(k,l)} + (-1)^{\varepsilon(\sigma;\mathbf{m})} v^{\mathbf{m}*}_{(k,l)} \mid (k,l) \in Z(\sigma;\mathbf{m})\}$ is a basis of $V^*_{\mathbf{m}}[\sigma]$.

From above arguments, we obtain the following.

Proposition 4.1. *As a K-module, the principal series representation $H_{(\nu,\sigma)}$ has an irreducible decomposition*

$$H_{(\nu,\sigma)} \simeq \widehat{\bigoplus_{\mathbf{m}\in L}} V^*_{\mathbf{m}}[\sigma] \otimes V_{\mathbf{m}}.$$

Here if $m \not\equiv \sigma_1 + \sigma_2 + \sigma_3 \pmod 2$ then $V^*_{\mathbf{m}}[\sigma] = 0$, and if $m \equiv \sigma_1 + \sigma_2 + \sigma_3 \pmod 2$ then

$$V^*_{\mathbf{m}}[\sigma] = \bigoplus_{(k,l)\in Z(\sigma;\mathbf{m})} \mathbf{C}(v^{\mathbf{m}*}_{\mathbf{m}-(k,l)} + (-1)^{\varepsilon(\sigma;\mathbf{m})} v^{\mathbf{m}*}_{(k,l)}).$$

Corollary 4.2. *Let $d(\sigma;\mathbf{m})$ be the dimension of the space $\mathrm{Hom}_K(V_{\mathbf{m}}, H_{(\nu,\sigma)})$ of intertwining operators.*

1. *When* $\sigma_1+\sigma_2+\sigma_3 \equiv 0 \pmod 2$*:*

   - *If* $\mathbf{m}=(m,n)$ *is even, then*

$$d(\sigma;\mathbf{m}) = \begin{cases} (mn+m+n)/4+1 & \textit{for } (\sigma_1,\sigma_2,\sigma_3)=(0,0,0) \textit{ and } m\equiv n \pmod 4, \\ (mn+m+n+2)/4-1 & \textit{for } (\sigma_1,\sigma_2,\sigma_3)=(0,0,0) \textit{ and } m\not\equiv n \pmod 4, \\ (mn+m+n)/4 & \textit{for } (\sigma_1,\sigma_2,\sigma_3)\neq(0,0,0) \textit{ and } m\equiv n \pmod 4, \\ (mn+m+n+2)/4 & \textit{for } (\sigma_1,\sigma_2,\sigma_3)\neq(0,0,0) \textit{ and } m\not\equiv n \pmod 4. \end{cases}$$

   - *If* $\mathbf{m}=(m,n)$ *is odd, then* $d(\sigma;\mathbf{m})=0$.

2. *When* $\sigma_1+\sigma_2+\sigma_3 \equiv 1 \pmod 2$*:*

   - *If* $\mathbf{m}=(m,n)$ *is even, then* $d(\sigma;\mathbf{m})=0$.
   - *If* $\mathbf{m}=(m,n)$ *is odd, then*

$$d(\sigma;\mathbf{m}) = \frac{1}{4}(mn+m+n+1).$$

We define an element $s(\mathbf{m};(p_1,p_2),(q_1,q_2))$ of $H_{(\nu,\sigma)}$ for $\mathbf{m}=(m,n)\in L$, $(p_1,p_2)\in Z(\sigma;\mathbf{m})$, $0\le q_1\le m$ and $0\le q_2\le n$ by

$$s(\mathbf{m};(p_1,p_2),(q_1,q_2)) = r_K^{-1}\circ\Phi((v^{\mathbf{m}*}_{\mathbf{m}-(p_1,p_2)}+(-1)^{\varepsilon(\sigma;\mathbf{m})}v^{\mathbf{m}*}_{(p_1,p_2)})\otimes v^{\mathbf{m}}_{(q_1,q_2)}).$$

We call it the *elementary function.* Moreover we define a K-homomorphism $\eta^{\mathbf{m}}_{(p_1,p_2)}\in \mathrm{Hom}_K(V_{\mathbf{m}},H_{(\nu,\sigma)})$ for $(p_1,p_2)\in Z(\sigma;\mathbf{m})$ by

$$\eta^{\mathbf{m}}_{(p_1,p_2)}(v^{\mathbf{m}}_{(q_1,q_2)}) = s(\mathbf{m};(p_1,p_2),(q_1,q_2)).$$

Then $\{\eta^{\mathbf{m}}_{(p_1,p_2)} \mid (p_1,p_2)\in Z(\sigma;\mathbf{m})\}$ is a basis of $\mathrm{Hom}_K(V_{\mathbf{m}},H_{(\nu,\sigma)})$, and we call it the *induced basis from the elementary functions.*

5 The structure of the principal series representation as a $\mathfrak{g}$-module

Let $H_{(\nu,\sigma),K}$ be the K-finite part of $H_{(\nu,\sigma)}$. To investigate the $\mathfrak{g}$-module structure of $H_{(\nu,\sigma),K}$, it is sufficient to investigate the action of $\mathfrak{p}_{\mathbf{C}}$ because of the Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$.

In this section, we define a linear map $\Gamma^{\mathbf{m}}_{(i,j)}$, and also $\overline{\Gamma}^{\mathbf{m}}_{(i,j)}$ by modifying $\Gamma^{\mathbf{m}}_{(i,j)}$ a little, which is easier to treat than $\Gamma^{\mathbf{m}}_{(i,j)}$. We then describe $\overline{\Gamma}^{\mathbf{m}}_{(i,j)}$ explicitly in Theorem 5.2. This $\Gamma^{\mathbf{m}}_{(i,j)}$ or $\overline{\Gamma}^{\mathbf{m}}_{(i,j)}$ characterizes the action of $\mathfrak{p}_{\mathbf{C}}$. We give an explicit description of the action of $\mathfrak{p}_{\mathbf{C}}$ by using $\Gamma^{\mathbf{m}}_{(i,j)}$ in Theorem 5.5.

Notation . *For* $\mathbf{m}=(m,n)\in\mathbf{Z}^2_{\ge 0}$ *and a vector space* W*, we denote by* $M(\mathbf{m},W)$ *the set of* $(m+1)\times(n+1)$ *matrices whose components are elements of* W*. For convenience, the indices of rows and columns of matrices in* $M(\mathbf{m},W)$ *start from* 0*. For example, the element of* $M(\mathbf{m},W)$ *whose* (k,l)*-th component is* $w_{kl}\in W$*, which is denoted by* $(w_{kl})_{kl}$*, is*

$$\begin{array}{c c} & \begin{array}{cccccc} 0 & 1 & \dots & l & \dots & n \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ k \\ \vdots \\ m \end{array} & \begin{pmatrix} w_{00} & w_{01} & & \dots & & w_{0n} \\ w_{10} & w_{11} & & \dots & & w_{1n} \\ & & & & & \\ \vdots & \vdots & & w_{kl} & & \vdots \\ & & & & & \\ w_{m0} & w_{m1} & & \dots & & w_{mn} \end{pmatrix} \end{array}.$$

5.1 Definition of $\Gamma^{\mathbf{m}}_{(i,j)}$ and $\overline{\Gamma}^{\mathbf{m}}_{(i,j)}$

For a K-type $(\tau_{\mathbf{m}}, V_{\mathbf{m}})$ of $\pi_{(\nu,\sigma)}$ and $\eta \in \mathrm{Hom}_K(V_{\mathbf{m}}, H_{(\nu,\sigma),K})$, we define a K-homomorphism $\tilde{\eta} : \mathfrak{p}_{\mathbf{C}} \otimes V_{\mathbf{m}} \to H_{(\nu,\sigma),K}$ by $X \otimes v \mapsto \pi_{(\nu,\sigma)}(X)\eta(v)$. Here we denote the differential of $\pi_{(\nu,\sigma)}$ by the same symbol $\pi_{(\nu,\sigma)}$. Then, for $-1 \le i \le 1$ and $-1 \le j \le 1$, we define a linear map

$$\Gamma^{\mathbf{m}}_{(i,j)} : \mathrm{Hom}_K(V_{\mathbf{m}}, H_{(\nu,\sigma),K}) \longrightarrow \mathrm{Hom}_K(V_{\mathbf{m}+2(i,j)}, H_{(\nu,\sigma),K}).$$

by $\eta \mapsto \tilde{\eta} \circ I^{\mathbf{m}}_{(i,j)}$.

Let $J_{\mathbf{m}}$ be an injective linear map from $\mathrm{Hom}_K(V_{\mathbf{m}}, H_{(\nu,\sigma),K})$ into $M(\mathbf{m}, H_{(\nu,\sigma),K})$ defined by

$$J_{\mathbf{m}}(\eta) = (\eta(v^{\mathbf{m}}_{(q_1,q_2)}))_{q_1 q_2} \quad \text{for } \eta \in \mathrm{Hom}_K(V_{\mathbf{m}}, H_{(\nu,\sigma),K}).$$

Put $S(\mathbf{m}; (p_1,p_2)) = J_{\mathbf{m}}(\eta^{\mathbf{m}}_{(p_1,p_2)})$. Then $\{S(\mathbf{m}; (p_1,p_2)) \mid (p_1,p_2) \in Z(\sigma; \mathbf{m})\}$ is a basis of the image of $J_{\mathbf{m}}$ in $M(\mathbf{m}, H_{(\nu,\sigma),K})$. We also put $\overline{\Gamma}^{\mathbf{m}}_{(i,j)} = J_{\mathbf{m}+2(i,j)} \circ \Gamma^{\mathbf{m}}_{(i,j)} \circ J_{\mathbf{m}}^{-1}$ whose domain is the image of $J_{\mathbf{m}}$ in $M(\mathbf{m}, H_{(\nu,\sigma),K})$:

$$\begin{array}{ccc}
\mathrm{Hom}_K(V_{\mathbf{m}}, H_{(\nu,\sigma),K}) & \xrightarrow{\Gamma^{\mathbf{m}}_{(i,j)}} & \mathrm{Hom}_K(V_{\mathbf{m}+2(i,j)}, H_{(\nu,\sigma),K}) \\
{\scriptstyle J_{\mathbf{m}}}\big\downarrow & & \big\downarrow{\scriptstyle J_{\mathbf{m}+2(i,j)}} \\
\mathrm{Image}(J_{\mathbf{m}}) & \xrightarrow[\overline{\Gamma}^{\mathbf{m}}_{(i,j)}]{} & \mathrm{Image}(J_{\mathbf{m}+2(i,j)}) \\
\cap & & \cap \\
M(\mathbf{m}, H_{(\nu,\sigma),K}) & & M(\mathbf{m}+2(i,j), H_{(\nu,\sigma),K})
\end{array}$$

Note that describing $\Gamma^{\mathbf{m}}_{(i,j)}$ with respect to the induced basis $\{\eta^{\mathbf{m}}_{(p_1,p_2)} \mid (p_1,p_2) \in Z(\sigma;\mathbf{m})\}$ is equivalent to describing $\overline{\Gamma}^{\mathbf{m}}_{(i,j)}$ with respect to $\{S(\mathbf{m}; (p_1,p_2)) \mid (p_1,p_2) \in Z(\sigma;\mathbf{m})\}$. Thus we also call $\{S(\mathbf{m}; (p_1,p_2)) \mid (p_1,p_2) \in Z(\sigma;\mathbf{m})\}$ the induced basis.

5.2 An explicit description of $\overline{\Gamma}^{\mathbf{m}}_{(i,j)}$

We give the Iwasawa decomposition of $X_{(x,y)} \in \mathfrak{p}_{\mathbf{C}}$, which is needed in the proof of the next theorem.

Lemma 5.1. *The basis $\{X_{(x,y)} \mid 0 \le x, y \le 2\}$ of $\mathfrak{p}_{\mathbf{C}}$ given in Proposition 3.2 have the following expressions according to the Iwasawa decomposition $\mathfrak{g}_{\mathbf{C}} = \mathfrak{n}_{\mathbf{C}} \oplus \mathfrak{a}_{\mathbf{C}} \oplus \mathfrak{k}_{\mathbf{C}}$:*

$$\begin{aligned}
X_{(0,0)} &= 4\sqrt{-1}E_{34} - 2H_3 - Y_1 - Y_4, \\
X_{(0,1)} &= 2(-E_{13} + \sqrt{-1}E_{14} + \sqrt{-1}E_{23} + E_{24}) - 2Y_2, \\
X_{(0,2)} &= 4\sqrt{-1}E_{12} - 2(H_1 - H_2) - Y_1 + Y_4, \\
X_{(1,0)} &= 2(E_{13} - \sqrt{-1}E_{14} + \sqrt{-1}E_{23} + E_{24}) - 2Y_5, \\
X_{(1,1)} &= H_1 + H_2 - H_3, \\
X_{(1,2)} &= -2(E_{13} + \sqrt{-1}E_{14} - \sqrt{-1}E_{23} + E_{24}) - 2Y_6, \\
X_{(2,0)} &= -4\sqrt{-1}E_{12} - 2(H_1 - H_2) + Y_1 - Y_4, \\
X_{(2,1)} &= 2(E_{13} + \sqrt{-1}E_{14} + \sqrt{-1}E_{23} - E_{24}) - 2Y_3, \\
X_{(2,2)} &= -4\sqrt{-1}E_{34} - 2H_3 + Y_1 + Y_4.
\end{aligned}$$

Here $E_{kl} \in \mathfrak{n}_{\mathbf{C}}$ and $H_k \in \mathfrak{a}_{\mathbf{C}}$ are defined in subsection 2.1, and $Y_k \in \mathfrak{k}_{\mathbf{C}}$ are defined in subsection 3.1.

Proof. This may be checked by direct computation. □

Now we state an explicit description of $\overline{\Gamma}^{\mathbf{m}}_{(i,j)}$ with respect to the induced bases.

Theorem 5.2. *For $\mathbf{m} = (m,n) \in L$ and $-1 \le i, j \le 1$ such that $d(\sigma;\mathbf{m}) \ne 0$ and $d(\sigma; \mathbf{m}+2(i,j)) \ne 0$, we have*

$$\overline{\Gamma}^{\mathbf{m}}_{(i,j)}(S(\mathbf{m}; (p_1,p_2))) = \sum_{(k,l)=(0,0),\pm(1,1),\pm(1,-1)} \gamma^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(k,l)]}\, S(\mathbf{m}+2(i,j); (p_1,p_2)+(i,j)+(k,l)). \quad (*)$$

Here $\gamma^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(k,l)]} \in \mathbf{C}$ *are given by*

$$\begin{aligned}
\gamma^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(1,1)]} &= (-2\nu_3 - 2\rho_3 + m - 2p_1 + n - 2p_2)A^{[\mathbf{m},(i,j)]}_{[\mathbf{m}+2(i,j)-((p_1,p_2)+(i,j)+(1,1)),(0,0)]},\\
\gamma^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),-(1,1)]} &= (-2\nu_3 - 2\rho_3 - m + 2p_1 - n + 2p_2)A^{[\mathbf{m},(i,j)]}_{[\mathbf{m}+2(i,j)-((p_1,p_2)+(i,j)-(1,1)),(2,2)]},\\
\gamma^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(1,-1)]} &= (-2\nu_1 - 2\rho_1 + 2\nu_2 + 2\rho_2 + m - 2p_1 - n + 2p_2)A^{[\mathbf{m},(i,j)]}_{[\mathbf{m}+2(i,j)-((p_1,p_2)+(i,j)+(1,-1)),(0,2)]},\\
\gamma^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(-1,1)]} &= (-2\nu_1 - 2\rho_1 + 2\nu_2 + 2\rho_2 - m + 2p_1 + n - 2p_2)A^{[\mathbf{m},(i,j)]}_{[\mathbf{m}+2(i,j)-((p_1,p_2)+(i,j)+(-1,1)),(2,0)]},\\
\gamma^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(0,0)]} &= (\nu_1 + \rho_1 + \nu_2 + \rho_2 - \nu_3 - \rho_3 + im + jn + i^2 + j^2 + i + j - 4)A^{[\mathbf{m},(i,j)]}_{[\mathbf{m}+2(i,j)-((p_1,p_2)+(i,j)),(1,1)]},
\end{aligned}$$

where we put $A^{[\mathbf{m},(i,j)]}_{[(p_1',p_2'),(x,y)]} = 0$ *unless* $0 \le p_1' \le m + 2i$ *and* $0 \le p_2' \le n + 2j$*. In the right hand side of* $(*)$*, we put*

$$S(\mathbf{m} + 2(i,j); (p_1', p_2')) = 0$$

if $p_1' < 0$ *or* $p_2' < 0$*, and*

$$S(\mathbf{m} + 2(i,j); (p_1', p_2')) = (-1)^{\varepsilon(\sigma;\mathbf{m}+2(i,j))} S(\mathbf{m} + 2(i,j); \mathbf{m} + 2(i,j) - (p_1', p_2'))$$

if $(p_1', p_2') \notin Z(\sigma; \mathbf{m} + 2(i,j))$*,* $p_1' \ge 0$ *and* $p_2' \ge 0$*.*

Proof. Let $\mathbf{E}_{(k,l)} \in M(\mathbf{m} + 2(i,j), \mathbf{C})$ be the matrix unit with its (k,l)-th component 1 and remaining components 0. Let us consider a linear map $\Psi : M(\mathbf{m} + 2(i,j), H_{(\nu,\sigma)}) \to M(\mathbf{m} + 2(i,j), \mathbf{C})$ defined by $(f_{q_1'q_2'})_{q_1'q_2'} \mapsto (f_{q_1'q_2'}(1_4))_{q_1'q_2'}$. Since

$$\begin{aligned}
s(\mathbf{m} + 2(i,j); (p_1', p_2'), (q_1', q_2'))(1_4) &= \langle v^{\mathbf{m}+2(i,j)*}_{\mathbf{m}+2(i,j)-(p_1',p_2')} + (-1)^{\varepsilon(\sigma;\mathbf{m}+2(i,j))} v^{\mathbf{m}+2(i,j)*}_{(p_1',p_2')}, v^{\mathbf{m}+2(i,j)}_{(q_1',q_2')}\rangle\\
&= \delta_{\mathbf{m}+2(i,j)-(p_1',p_2'),(q_1',q_2')} + (-1)^{\varepsilon(\sigma;\mathbf{m}+2(i,j))}\delta_{(p_1',p_2'),(q_1',q_2')},
\end{aligned}$$

we have

$$\Psi(S(\mathbf{m} + 2(i,j); (p_1', p_2'))) = S(\mathbf{m} + 2(i,j); (p_1', p_2'))(1_4) = \mathbf{E}_{\mathbf{m}+2(i,j)-(p_1',p_2')} + (-1)^{\varepsilon(\sigma;\mathbf{m}+2(i,j))}\,\mathbf{E}_{(p_1',p_2')}.$$

Here $\delta_{(p_1',p_2'),(q_1',q_2')} = \delta_{p_1'q_1'}\delta_{p_2'q_2'}$ and $\delta_{p'q'}$ is the Kronecker delta. Hence $\{\Psi(S(\mathbf{m} + 2(i,j); (p_1', p_2'))) \mid (p_1', p_2') \in Z(\sigma; \mathbf{m} + 2(i,j))\}$ are linearly independent over $\mathbf{C}$. Thus, in order to prove the theorem, it is sufficient to compare the values of the both side of $(*)$ at $1_4 \in G$.

Since the value of the right hand side of the equation $(*)$ at $1_4 \in G$ becomes

$$\begin{aligned}
&\overline{\Gamma}^{\mathbf{m}}_{(i,j)}(S(\mathbf{m}; (p_1, p_2)))(1_4)\\
&= J_{\mathbf{m}+2(i,j)} \circ \Gamma^{\mathbf{m}}_{(i,j)} \circ J^{-1}_{\mathbf{m}}(S(\mathbf{m}; (p_1, p_2)))(1_4)\\
&= J_{\mathbf{m}+2(i,j)} \circ \Gamma^{\mathbf{m}}_{(i,j)}(\eta^{\mathbf{m}}_{(p_1,p_2)})(1_4)\\
&= J_{\mathbf{m}+2(i,j)}(\tilde{\eta}^{\mathbf{m}}_{(p_1,p_2)} \circ I^{\mathbf{m}}_{(i,j)})(1_4)\\
&= \left(\tilde{\eta}^{\mathbf{m}}_{(p_1,p_2)} \circ I^{\mathbf{m}}_{(i,j)}(v^{\mathbf{m}+2(i,j)}_{(q_1',q_2')})(1_4)\right)_{q_1'q_2'}\\
&= \sum_{x=0}^{2}\sum_{y=0}^{2}\left(A^{[\mathbf{m},(i,j)]}_{[(q_1',q_2'),(x,y)]}\{\tilde{\eta}^{\mathbf{m}}_{(p_1,p_2)}(X_{(x,y)} \otimes v^{\mathbf{m}}_{(q_1',q_2')+(1,1)-(i,j)-(x,y)})\}(1_4)\right)_{q_1'q_2'}\\
&= \sum_{x=0}^{2}\sum_{y=0}^{2}\left(A^{[\mathbf{m},(i,j)]}_{[(q_1',q_2'),(x,y)]}\{\pi_{(\nu,\sigma)}(X_{(x,y)})s(\mathbf{m}; (p_1, p_2), (q_1', q_2') + (1,1) - (i,j) - (x,y))\}(1_4)\right)_{q_1'q_2'},
\end{aligned}$$

we first compute $\{\pi_{(\nu,\sigma)}(X_{(x,y)})s(\mathbf{m}; (p_1, p_2), (q_1, q_2))\}(1_4)$. By the definition of principal series representation, we have

$$\{\pi_{(\nu,\sigma)}(E_{kl})s(\mathbf{m}; (p_1, p_2), (q_1, q_2))\}(1_4) = 0$$

for $1 \le k < l \le 4$, and

$$\begin{aligned}
\{\pi_{(\nu,\sigma)}(H_k)s(\mathbf{m}; (p_1, p_2), (q_1, q_2))\}(1_4) &= (\nu_k + \rho_k)s(\mathbf{m}; (p_1, p_2), (q_1, q_2))(1_4)\\
&= (\nu_k + \rho_k)(\delta_{\mathbf{m}-(p_1,p_2),(q_1,q_2)} + (-1)^{\varepsilon(\sigma;\mathbf{m})}\delta_{(p_1,p_2),(q_1,q_2)})
\end{aligned}$$

for $k=1,2,3$. In addition, from the definition of the elementary function $s(\mathbf{m};(p_1,p_2),(q_1,q_2))$, we have

$$\begin{aligned}
\{\pi_{(\nu,\sigma)}(Y_1)s(\mathbf{m};(p_1,p_2),(q_1,q_2))\}(1_4) &= (m-2q_1)s(\mathbf{m};(p_1,p_2),(q_1,q_2))(1_4)\\
&= (m-2q_1)(\delta_{\mathbf{m}-(p_1,p_2),(q_1,q_2)}+(-1)^{\varepsilon(\sigma;\mathbf{m})}\delta_{(p_1,p_2),(q_1,q_2)}),\\
\{\pi_{(\nu,\sigma)}(Y_2)s(\mathbf{m};(p_1,p_2),(q_1,q_2))\}(1_4) &= -q_1s(\mathbf{m};(p_1,p_2),(q_1-1,q_2))(1_4)\\
&= -q_1(\delta_{\mathbf{m}-(p_1,p_2),(q_1-1,q_2)}+(-1)^{\varepsilon(\sigma;\mathbf{m})}\delta_{(p_1,p_2),(q_1-1,q_2)}),\\
\{\pi_{(\nu,\sigma)}(Y_3)s(\mathbf{m};(p_1,p_2),(q_1,q_2))\}(1_4) &= (q_1-m)s(\mathbf{m};(p_1,p_2),(q_1+1,q_2))(1_4)\\
&= (q_1-m)(\delta_{\mathbf{m}-(p_1,p_2),(q_1+1,q_2)}+(-1)^{\varepsilon(\sigma;\mathbf{m})}\delta_{(p_1,p_2),(q_1+1,q_2)}),\\
\{\pi_{(\nu,\sigma)}(Y_4)s(\mathbf{m};(p_1,p_2),(q_1,q_2))\}(1_4) &= (n-2q_2)s(\mathbf{m};(p_1,p_2),(q_1,q_2))(1_4)\\
&= (n-2q_2)(\delta_{\mathbf{m}-(p_1,p_2),(q_1,q_2)}+(-1)^{\varepsilon(\sigma;\mathbf{m})}\delta_{(p_1,p_2),(q_1,q_2)}),\\
\{\pi_{(\nu,\sigma)}(Y_5)s(\mathbf{m};(p_1,p_2),(q_1,q_2))\}(1_4) &= -q_2s(\mathbf{m};(p_1,p_2),(q_1,q_2-1))(1_4)\\
&= -q_2(\delta_{\mathbf{m}-(p_1,p_2),(q_1,q_2-1)}+(-1)^{\varepsilon(\sigma;\mathbf{m})}\delta_{(p_1,p_2),(q_1,q_2-1)}),\\
\{\pi_{(\nu,\sigma)}(Y_6)s(\mathbf{m};(p_1,p_2),(q_1,q_2))\}(1_4) &= (q_2-n)s(\mathbf{m};(p_1,p_2),(q_1,q_2+1))(1_4)\\
&= (q_2-n)(\delta_{\mathbf{m}-(p_1,p_2),(q_1,q_2+1)}+(-1)^{\varepsilon(\sigma;\mathbf{m})}\delta_{(p_1,p_2),(q_1,q_2+1)}).
\end{aligned}$$

Therefore, by using the Iwasawa decomposition in Lemma 5.1, we have

$$\begin{aligned}
&\{\pi_{(\nu,\sigma)}(X_{(0,0)})s(\mathbf{m};(p_1,p_2),(q_1,q_2))\}(1_4)\\
&\quad= (-2\nu_3-2\rho_3-m+2q_1-n+2q_2)(\delta_{\mathbf{m}-(p_1,p_2),(q_1,q_2)}+(-1)^{\varepsilon(\sigma;\mathbf{m})}\delta_{(p_1,p_2),(q_1,q_2)}),\\
&\{\pi_{(\nu,\sigma)}(X_{(0,1)})s(\mathbf{m};(p_1,p_2),(q_1,q_2))\}(1_4)\\
&\quad= 2q_1(\delta_{\mathbf{m}-(p_1,p_2),(q_1-1,q_2)}+(-1)^{\varepsilon(\sigma;\mathbf{m})}\delta_{(p_1,p_2),(q_1-1,q_2)}),\\
&\{\pi_{(\nu,\sigma)}(X_{(0,2)})s(\mathbf{m};(p_1,p_2),(q_1,q_2))\}(1_4)\\
&\quad= (-2\nu_1-2\rho_1+2\nu_2+2\rho_2-m+2q_1+n-2q_2)(\delta_{\mathbf{m}-(p_1,p_2),(q_1,q_2)}+(-1)^{\varepsilon(\sigma;\mathbf{m})}\delta_{(p_1,p_2),(q_1,q_2)}),\\
&\{\pi_{(\nu,\sigma)}(X_{(1,0)})s(\mathbf{m};(p_1,p_2),(q_1,q_2))\}(1_4)\\
&\quad= 2q_2(\delta_{\mathbf{m}-(p_1,p_2),(q_1,q_2-1)}+(-1)^{\varepsilon(\sigma;\mathbf{m})}\delta_{(p_1,p_2),(q_1,q_2-1)}),\\
&\{\pi_{(\nu,\sigma)}(X_{(1,1)})s(\mathbf{m};(p_1,p_2),(q_1,q_2))\}(1_4)\\
&\quad= (\nu_1+\rho_1+\nu_2+\rho_2-\nu_3-\rho_3)(\delta_{\mathbf{m}-(p_1,p_2),(q_1,q_2)}+(-1)^{\varepsilon(\sigma;\mathbf{m})}\delta_{(p_1,p_2),(q_1,q_2)}),\\
&\{\pi_{(\nu,\sigma)}(X_{(1,2)})s(\mathbf{m};(p_1,p_2),(q_1,q_2))\}(1_4)\\
&\quad= -2(q_2-n)(\delta_{\mathbf{m}-(p_1,p_2),(q_1,q_2+1)}+(-1)^{\varepsilon(\sigma;\mathbf{m})}\delta_{(p_1,p_2),(q_1,q_2+1)}),\\
&\{\pi_{(\nu,\sigma)}(X_{(2,0)})s(\mathbf{m};(p_1,p_2),(q_1,q_2))\}(1_4)\\
&\quad= (-2\nu_1-2\rho_1+2\nu_2+2\rho_2+m-2q_1-n+2q_2)(\delta_{\mathbf{m}-(p_1,p_2),(q_1,q_2)}+(-1)^{\varepsilon(\sigma;\mathbf{m})}\delta_{(p_1,p_2),(q_1,q_2)}),\\
&\{\pi_{(\nu,\sigma)}(X_{(2,1)})s(\mathbf{m};(p_1,p_2),(q_1,q_2))\}(1_4)\\
&\quad= -2(q_1-m)(\delta_{\mathbf{m}-(p_1,p_2),(q_1+1,q_2)}+(-1)^{\varepsilon(\sigma;\mathbf{m})}\delta_{(p_1,p_2),(q_1+1,q_2)}),\\
&\{\pi_{(\nu,\sigma)}(X_{(2,2)})s(\mathbf{m};(p_1,p_2),(q_1,q_2))\}(1_4)\\
&\quad= (-2\nu_3-2\rho_3+m-2q_1+n-2q_2)(\delta_{\mathbf{m}-(p_1,p_2),(q_1,q_2)}+(-1)^{\varepsilon(\sigma;\mathbf{m})}\delta_{(p_1,p_2),(q_1,q_2)}).
\end{aligned}$$

Now let us compute $\overline{\Gamma}^{\mathbf{m}}_{(i,j)}(S(\mathbf{m};(p_1,p_2)))(1_4)$. By above equations, we have

$$\begin{aligned}
&\overline{\Gamma}^{\mathbf{m}}_{(i,j)}(S(\mathbf{m};(p_1,p_2)))(1_4)\\
&= \Big(\{-2\nu_3-2\rho_3-m+2(q_1'+1-i)-n+2(q_2'+1-j)\}A^{[\mathbf{m},(i,j)]}_{[(q_1',q_2'),(0,0)]}\\
&\qquad\qquad \times(\delta_{\mathbf{m}-(p_1,p_2),(q_1',q_2')+(1,1)-(i,j)}+(-1)^{\varepsilon(\sigma;\mathbf{m})}\delta_{(p_1,p_2),(q_1',q_2')+(1,1)-(i,j)})\Big)_{q_1'q_2'}\\
&\quad+\Big(\{-2\nu_3-2\rho_3+m-2(q_1'-1-i)+n-2(q_2'-1-j)\}A^{[\mathbf{m},(i,j)]}_{[(q_1',q_2'),(2,2)]}\\
&\qquad\qquad \times(\delta_{\mathbf{m}-(p_1,p_2),(q_1',q_2')-(1,1)-(i,j)}+(-1)^{\varepsilon(\sigma;\mathbf{m})}\delta_{(p_1,p_2),(q_1',q_2')-(1,1)-(i,j)})\Big)_{q_1'q_2'}
\end{aligned}$$

$$
\begin{aligned}
&+ \Big(\{-2\nu_1 - 2\rho_1 + 2\nu_2 + 2\rho_2 - m + 2(q_1' + 1 - i) + n - 2(q_2' - 1 - j)\} A^{[\mathbf{m},(i,j)]}_{[(q_1',q_2'),(0,2)]} \\
&\qquad\qquad \times \big(\delta_{\mathbf{m}-(p_1,p_2),(q_1',q_2')+(1,-1)-(i,j)} + (-1)^{\varepsilon(\sigma;\mathbf{m})} \delta_{(p_1,p_2),(q_1',q_2')+(1,-1)-(i,j)}\big)\Big)_{q_1'q_2'} \\
&+ \Big(\{-2\nu_1 - 2\rho_1 + 2\nu_2 + 2\rho_2 + m - 2(q_1' - 1 - i) - n + 2(q_2' + 1 - j)\} A^{[\mathbf{m},(i,j)]}_{[(q_1',q_2'),(2,0)]} \\
&\qquad\qquad \times \big(\delta_{\mathbf{m}-(p_1,p_2),(q_1',q_2')+(-1,1)-(i,j)} + (-1)^{\varepsilon(\sigma;\mathbf{m})} \delta_{(p_1,p_2),(q_1',q_2')+(-1,1)-(i,j)}\big)\Big)_{q_1'q_2'} \\
&+ \Big((\nu_1 + \rho_1 + \nu_2 + \rho_2 - \nu_3 - \rho_3) A^{[\mathbf{m},(i,j)]}_{[(q_1',q_2'),(1,1)]} \big(\delta_{\mathbf{m}-(p_1,p_2),(q_1',q_2')-(i,j)} + (-1)^{\varepsilon(\sigma;\mathbf{m})} \delta_{(p_1,p_2),(q_1',q_2')-(i,j)}\big)\Big)_{q_1'q_2'} \\
&+ \Big(2(q_1' + 1 - i) A^{[\mathbf{m},(i,j)]}_{[(q_1',q_2'),(0,1)]} \big(\delta_{\mathbf{m}-(p_1,p_2),(q_1',q_2')-(i,j)} + (-1)^{\varepsilon(\sigma;\mathbf{m})} \delta_{(p_1,p_2),(q_1',q_2')-(i,j)}\big)\Big)_{q_1'q_2'} \\
&+ \Big(2(q_2' + 1 - j) A^{[\mathbf{m},(i,j)]}_{[(q_1',q_2'),(1,0)]} \big(\delta_{\mathbf{m}-(p_1,p_2),(q_1',q_2')-(i,j)} + (-1)^{\varepsilon(\sigma;\mathbf{m})} \delta_{(p_1,p_2),(q_1',q_2')-(i,j)}\big)\Big)_{q_1'q_2'} \\
&+ \Big(-2(q_2' - 1 - j - n) A^{[\mathbf{m},(i,j)]}_{[(q_1',q_2'),(1,2)]} \big(\delta_{\mathbf{m}-(p_1,p_2),(q_1',q_2')-(i,j)} + (-1)^{\varepsilon(\sigma;\mathbf{m})} \delta_{(p_1,p_2),(q_1',q_2')-(i,j)}\big)\Big)_{q_1'q_2'} \\
&+ \Big(-2(q_1' - 1 - i - m) A^{[\mathbf{m},(i,j)]}_{[(q_1',q_2'),(2,1)]} \big(\delta_{\mathbf{m}-(p_1,p_2),(q_1',q_2')-(i,j)} + (-1)^{\varepsilon(\sigma;\mathbf{m})} \delta_{(p_1,p_2),(q_1',q_2')-(i,j)}\big)\Big)_{q_1'q_2'} \\
&= \alpha^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(1,1)]} \, \mathbf{E}_{\mathbf{m}+2(i,j)-((p_1,p_2)+(i,j)+(1,1))} + (-1)^{\varepsilon(\sigma;\mathbf{m})} \beta^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),-(1,1)]} \, \mathbf{E}_{(p_1,p_2)+(i,j)-(1,1)} \\
&\quad + \alpha^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),-(1,1)]} \, \mathbf{E}_{\mathbf{m}+2(i,j)-((p_1,p_2)+(i,j)-(1,1))} + (-1)^{\varepsilon(\sigma;\mathbf{m})} \beta^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(1,1)]} \, \mathbf{E}_{(p_1,p_2)+(i,j)+(1,1)} \\
&\quad + \alpha^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(1,-1)]} \, \mathbf{E}_{\mathbf{m}+2(i,j)-((p_1,p_2)+(i,j)+(1,-1))} + (-1)^{\varepsilon(\sigma;\mathbf{m})} \beta^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(-1,1)]} \, \mathbf{E}_{(p_1,p_2)+(i,j)+(-1,1)} \\
&\quad + \alpha^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(-1,1)]} \, \mathbf{E}_{\mathbf{m}+2(i,j)-((p_1,p_2)+(i,j)+(-1,1))} + (-1)^{\varepsilon(\sigma;\mathbf{m})} \beta^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(1,-1)]} \, \mathbf{E}_{(p_1,p_2)+(i,j)+(1,-1)} \\
&\quad + \alpha^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(0,0)]} \, \mathbf{E}_{\mathbf{m}+2(i,j)-((p_1,p_2)+(i,j))} + (-1)^{\varepsilon(\sigma;\mathbf{m})} \beta^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(0,0)]} \, \mathbf{E}_{(p_1,p_2)+(i,j)} \,.
\end{aligned}
$$

Here

$$
\begin{aligned}
\alpha^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(1,1)]} &= (-2\nu_3 - 2\rho_3 + m - 2p_1 + n - 2p_2) A^{[\mathbf{m},(i,j)]}_{[\mathbf{m}+2(i,j)-((p_1,p_2)+(i,j)+(1,1)),(0,0)]}, \\
\alpha^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),-(1,1)]} &= (-2\nu_3 - 2\rho_3 - m + 2p_1 - n + 2p_2) A^{[\mathbf{m},(i,j)]}_{[\mathbf{m}+2(i,j)-((p_1,p_2)+(i,j)-(1,1)),(2,2)]}, \\
\alpha^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(1,-1)]} &= (-2\nu_1 - 2\rho_1 + 2\nu_2 + 2\rho_2 + m - 2p_1 - n + 2p_2) A^{[\mathbf{m},(i,j)]}_{[\mathbf{m}+2(i,j)-((p_1,p_2)+(i,j)+(1,-1)),(0,2)]}, \\
\alpha^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(-1,1)]} &= (-2\nu_1 - 2\rho_1 + 2\nu_2 + 2\rho_2 - m + 2p_1 + n - 2p_2) A^{[\mathbf{m},(i,j)]}_{[\mathbf{m}+2(i,j)-((p_1,p_2)+(i,j)+(-1,1)),(2,0)]}, \\
\alpha^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(0,0)]} &= (\nu_1 + \rho_1 + \nu_2 + \rho_2 - \nu_3 - \rho_3) A^{[\mathbf{m},(i,j)]}_{[\mathbf{m}+2(i,j)-((p_1,p_2)+(i,j)),(1,1)]} \\
&\quad + 2(m + 1 - p_1) A^{[\mathbf{m},(i,j)]}_{[\mathbf{m}+2(i,j)-((p_1,p_2)+(i,j)),(0,1)]} + 2(n + 1 - p_2) A^{[\mathbf{m},(i,j)]}_{[\mathbf{m}+2(i,j)-((p_1,p_2)+(i,j)),(1,0)]} \\
&\quad + 2(p_2 + 1) A^{[\mathbf{m},(i,j)]}_{[\mathbf{m}+2(i,j)-((p_1,p_2)+(i,j)),(1,2)]} + 2(p_1 + 1) A^{[\mathbf{m},(i,j)]}_{[\mathbf{m}+2(i,j)-((p_1,p_2)+(i,j)),(2,1)]}, \\
\beta^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(1,1)]} &= (-2\nu_3 - 2\rho_3 + m - 2p_1 + n - 2p_2) A^{[\mathbf{m},(i,j)]}_{[(p_1,p_2)+(i,j)+(1,1),(2,2)]}, \\
\beta^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),-(1,1)]} &= (-2\nu_3 - 2\rho_3 - m + 2p_1 - n + 2p_2) A^{[\mathbf{m},(i,j)]}_{[(p_1,p_2)+(i,j)-(1,1),(0,0)]}, \\
\beta^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(1,-1)]} &= (-2\nu_1 - 2\rho_1 + 2\nu_2 + 2\rho_2 + m - 2p_1 - n + 2p_2) A^{[\mathbf{m},(i,j)]}_{[(p_1,p_2)+(i,j)+(1,-1),(2,0)]}, \\
\beta^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(-1,1)]} &= (-2\nu_1 - 2\rho_1 + 2\nu_2 + 2\rho_2 - m + 2p_1 + n - 2p_2) A^{[\mathbf{m},(i,j)]}_{[(p_1,p_2)+(i,j)+(-1,1),(0,2)]}, \\
\beta^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(0,0)]} &= (\nu_1 + \rho_1 + \nu_2 + \rho_2 - \nu_3 - \rho_3) A^{[\mathbf{m},(i,j)]}_{[(p_1,p_2)+(i,j),(1,1)]} \\
&\quad + 2(p_1 + 1) A^{[\mathbf{m},(i,j)]}_{[(p_1,p_2)+(i,j),(0,1)]} + 2(p_2 + 1) A^{[\mathbf{m},(i,j)]}_{[(p_1,p_2)+(i,j),(1,0)]} \\
&\quad + 2(n + 1 - p_2) A^{[\mathbf{m},(i,j)]}_{[(p_1,p_2)+(i,j),(1,2)]} + 2(m + 1 - p_1) A^{[\mathbf{m},(i,j)]}_{[(p_1,p_2)+(i,j),(2,1)]}.
\end{aligned}
$$

By using the relations in Lemma 3.5, we find that

$$
\gamma^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(k,l)]} = \alpha^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(k,l)]} = (-1)^{i+j} \beta^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(k,l)]}.
$$

Moreover we see that

$$\varepsilon(\sigma;\mathbf{m}+2(i,j)) \equiv \sigma_1 + m + 2i + \frac{m+2i+n+2j}{2} \equiv i + j + \varepsilon(\sigma;\mathbf{m}) \pmod 2.$$

Hence $\overline{\Gamma}^{\mathbf{m}}_{(i,j)}(S(\mathbf{m};(p_1,p_2)))(1_4)$ becomes

$$\sum_{(k,l)=(0,0),\pm(1,1),\pm(1,-1)} \gamma^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(k,l)]} \left(\mathbf{E}_{\mathbf{m}+2(i,j)-((p_1,p_2)+(i,j)+(k,l))} + (-1)^{\varepsilon(\sigma;\mathbf{m}+2(i,j))}\,\mathbf{E}_{(p_1,p_2)+(i,j)+(k,l)}\right).$$

This equals the value of the left hand side of $(*)$ at $1_4 \in G$, thus we complete the proof. □

5.3 Projections for $\mathfrak{p}_{\mathbf{C}} \otimes V_{\mathbf{m}}$

To give the action of $\mathfrak{p}_{\mathbf{C}}$, we need a projection from $\mathfrak{p}_{\mathbf{C}} \otimes V_{\mathbf{m}}$ onto $V_{\mathbf{m}+2(i,j)}$. We first give a projection from $V_2 \otimes V_m$ onto V_{m+2i}.

Lemma 5.3. *When V_{m+2i}-component of $V_2 \otimes V_m$ does not vanish, we define a linear map $P^m_i : V_2 \otimes V_m \to V_{m+2i}$ by*

$$P^m_i(v^{(2)}_x \otimes v^{(m)}_k) = B^{[m,i]}_{[k,x]} v^{(m+2i)}_{k+i+x-1}.$$

Here we put $v^{(m+2i)}_k = 0$ unless $0 \le k \le m+2i$, and the coefficients $B^{[m,i]}_{[k,x]}$ are defined as follows:

$$\begin{aligned}
&B^{[m,1]}_{[k,0]} = 1, && B^{[m,1]}_{[k,1]} = 1, && B^{[m,1]}_{[k,2]} = 1,\\
&B^{[m,0]}_{[k,0]} = \frac{2k}{m+2}, && B^{[m,0]}_{[k,1]} = \frac{2k-m}{m+2}, && B^{[m,0]}_{[k,2]} = -\frac{2(m-k)}{m+2},\\
&B^{[m,-1]}_{[k,0]} = \frac{k(k-1)}{(m+1)m}, && B^{[m,-1]}_{[k,1]} = -\frac{(m-k)k}{(m+1)m}, && B^{[m,-1]}_{[k,2]} = \frac{(m-k)(m-k-1)}{(m+1)m}.
\end{aligned}$$

Then P^i_m is the generator of $\mathrm{Hom}_{\mathfrak{su}(2)_{\mathbf{C}}}(V_2 \otimes V_m, V_{m+2i})$ such that $P^m_i \circ I^m_i = \mathrm{id}_{V_{m+2i}}$.

Proof. We follow the proof of Miyazaki [4, Lemma 6.1]. The composite map

$$V_2 \otimes V_m \simeq V_2^* \otimes V_m^* \simeq (V_2 \otimes V_m)^* \ni f \mapsto f \circ I^m_i \in V^*_{m+2i} \simeq V_{m+2i}$$

is a surjective $\mathfrak{su}(2)_{\mathbf{C}}$-homomorphism from $V_2 \otimes V_m$ onto V_{m+2i}, where the isomorphism from V_m^* to V_m have been defined in the proof of Lemma 3.1. Therefore we get the assertion by multiplying this composite map by some scalar so that it satisfies $P^m_i \circ I^m_i = \mathrm{id}_{V_{m+2i}}$. □

Proposition 5.4. *When $V_{\mathbf{m}+2(i,j)}$-component of $\mathfrak{p}_{\mathbf{C}} \otimes V_{\mathbf{m}}$ does not vanish, we define a linear map $P^{\mathbf{m}}_{(i,j)} : \mathfrak{p}_{\mathbf{C}} \otimes V_{\mathbf{m}} \to V_{\mathbf{m}+2(i,j)}$ by*

$$P^{\mathbf{m}}_{(i,j)}(X_{(x,y)} \otimes v^{\mathbf{m}}_{(k,l)}) = B^{[\mathbf{m},(i,j)]}_{[(k,l),(x,y)]} v^{\mathbf{m}+2(i,j)}_{(k,l)+(i,j)+(x,y)-(1,1)}.$$

Here we put $v^{\mathbf{m}+2(i,j)}_{(k,l)} = 0$ unless $0 \le k \le m+2i$ and $0 \le l \le n+2j$, and the coefficients $B^{[\mathbf{m},(i,j)]}_{[(k,l),(x,y)]}$ are defined by

$$B^{[\mathbf{m},(i,j)]}_{[(k,l),(x,y)]} = B^{[m,i]}_{[k,x]} \cdot B^{[n,j]}_{[l,y]}.$$

Then $P^{\mathbf{m}}_{(i,j)}$ is the generator of $\mathrm{Hom}_{\mathfrak{k}_{\mathbf{C}}}(\mathfrak{p}_{\mathbf{C}} \otimes V_{\mathbf{m}}, V_{\mathbf{m}+2(i,j)})$ such that $P^{\mathbf{m}}_{(i,j)} \circ I^{\mathbf{m}}_{(i,j)} = \mathrm{id}_{V_{\mathbf{m}+2(i,j)}}$.

Proof. This follows immediately from Lemma 5.3. □

5.4 The action of $\mathfrak{p}_{\mathbf{C}}$ on the elementary functions

We give an explicit description of the action of $\mathfrak{p}_{\mathbf{C}}$ on the elementary functions, which compose a basis of $H_{(\nu,\sigma),K}$.

Theorem 5.5. *The action of $X_{(x,y)} \in \mathfrak{p}_{\mathbf{C}}$ on the elementary function $s(\mathbf{m};(p_1,p_2),(q_1,q_2))$ for $\mathbf{m} = (m,n) \in L$, $(p_1,p_2) \in Z(\sigma;\mathbf{m})$, $0 \le q_1 \le m$ and $0 \le q_2 \le n$ is given by*

$$\pi_{(\nu,\sigma)}(X_{(x,y)})s(\mathbf{m};(p_1,p_2),(q_1,q_2))$$
$$= \sum_{\substack{i,j=-1,0,1 \\ (k,l)=(0,0),\pm(1,1),\pm(1,-1)}} \gamma^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(k,l)]} B^{[\mathbf{m},(i,j)]}_{[(q_1,q_2),(x,y)]} s(\mathbf{m}+2(i,j);(p_1,p_2)+(i,j)+(k,l),(q_1,q_2)+(i,j)+(x,y)-(1,1)).$$

In the right hand side of the above equation, we put

$$s(\mathbf{m}+2(i,j);(p_1',p_2'),(q_1',q_2')) = 0$$

if $p_1' < 0$, $p_2' < 0$, $q_1' < 0$, $q_1' > m+2i$, $q_2' < 0$ or $q_2' > n+2i$, and

$$s(\mathbf{m}+2(i,j);(p_1',p_2'),(q_1',q_2')) = (-1)^{\varepsilon(\sigma;\mathbf{m}+2(i,j))} s(\mathbf{m}+2(i,j);\mathbf{m}+2(i,j)-(p_1',p_2'),(q_1',q_2'))$$

if $(p_1',p_2') \notin Z(\sigma;\mathbf{m}+2(i,j))$, $p_1' \ge 0$, $p_2' \ge 0$, $0 \le q_1' \le m+2i$ and $0 \le q_2' \le n+2j$.

Proof. We see that

$$\begin{aligned}
\pi_{(\nu,\sigma)}(X_{(x,y)})s(\mathbf{m};(p_1,p_2),(q_1,q_2)) &= \pi_{(\nu,\sigma)}(X_{(x,y)})\eta^{\mathbf{m}}_{(p_1,p_2)}(v^{\mathbf{m}}_{(q_1,q_2)}) \\
&= \tilde{\eta}^{\mathbf{m}}_{(p_1,p_2)}(X_{(x,y)} \otimes v^{\mathbf{m}}_{(q_1,q_2)}) \\
&= \tilde{\eta}^{\mathbf{m}}_{(p_1,p_2)}\left(\sum_{i,j=-1,0,1} I^{\mathbf{m}}_{(i,j)} \circ P^{\mathbf{m}}_{(i,j)}(X_{(x,y)} \otimes v^{\mathbf{m}}_{(q_1,q_2)}) \right) \\
&= \sum_{i,j=-1,0,1} \Gamma^{\mathbf{m}}_{(i,j)}(\eta^{\mathbf{m}}_{(p_1,p_2)}) \circ P^{\mathbf{m}}_{(i,j)}(X_{(x,y)} \otimes v^{\mathbf{m}}_{(q_1,q_2)}).
\end{aligned}$$

By Theorem 5.2 and Proposition 5.4, we have

$$\Gamma^{\mathbf{m}}_{(i,j)}(\eta^{\mathbf{m}}_{(p_1,p_2)}) = \sum_{(k,l)=(0,0),\pm(1,1),\pm(1,-1)} \gamma^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(k,l)]}\, \eta^{\mathbf{m}+2(i,j)}_{(p_1,p_2)+(i,j)+(k,l)}$$

and

$$P^{\mathbf{m}}_{(i,j)}(X_{(x,y)} \otimes v^{\mathbf{m}}_{(q_1,q_2)}) = B^{[\mathbf{m},(i,j)]}_{[(q_1,q_2),(x,y)]} v^{\mathbf{m}+2(i,j)}_{(q_1,q_2)+(i,j)+(x,y)-(1,1)},$$

hence

$$\pi_{(\nu,\sigma)}(X_{(x,y)})s(\mathbf{m};(p_1,p_2),(q_1,q_2))$$
$$= \sum_{\substack{i,j=-1,0,1 \\ (k,l)=(0,0),\pm(1,1),\pm(1,-1)}} \gamma^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(k,l)]} B^{[\mathbf{m},(i,j)]}_{[(q_1,q_2),(x,y)]} \eta^{\mathbf{m}+2(i,j)}_{(p_1,p_2)+(i,j)+(k,l)}(v^{\mathbf{m}+2(i,j)}_{(q_1,q_2)+(i,j)+(x,y)-(1,1)})$$
$$= \sum_{\substack{i,j=-1,0,1 \\ (k,l)=(0,0),\pm(1,1),\pm(1,-1)}} \gamma^{[\mathbf{m},(i,j)]}_{[(p_1,p_2),(k,l)]} B^{[\mathbf{m},(i,j)]}_{[(q_1,q_2),(x,y)]} s(\mathbf{m}+2(i,j);(p_1,p_2)+(i,j)+(k,l),(q_1,q_2)+(i,j)+(x,y)-(1,1)).$$

□

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