Inverse hyperbolic problem by a finite time
of observations with arbitrary initial values

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Abstract. We consider a solution \( u(p, g, a, b) \) to an initial value-boundary value problem for a hyperbolic equation:

\[
\partial_t^2 u(x, t) = \Delta u(x, t) + p(x)u(x, t), \quad x \in \Omega, \quad 0 < t < T
\]

\[
u(x, 0) = a(x), \quad \partial_t u(x, 0) = b(x), \quad x \in \Omega,
\]

\[
u(x, t) = g(x, t), \quad x \in \partial \Omega, \quad 0 < t < T.
\]

and we discuss an inverse problem of determining a coefficient \( p(x) \) and \( a, b \) by observations of \( u(p, g, a, b) \) on a neighborhood \( \omega \) of \( \partial \Omega \) over a time interval \( (0, T) \) and \( u(p, g, a, b)(x, T_0), \partial_t u(p, g, a, b)(x, T_0), \) \( x \in \Omega \) with \( T_0 < T \). We prove that if \( T - T_0 \) and \( T_0 \) are larger than the diameter of \( \Omega \), then we can choose a finite number of Dirichlet boundary inputs \( g_1, ..., g_N \) by the Hilbert Uniqueness Method, so that the mapping

\[
\{u(p, g_j, a_j, b_j)\}_{\omega \times (0, T)}, u(p, g_j, a_j, b_j)(\cdot, T_0), \partial_t u(p, g_j, a_j, b_j)(\cdot, T_0)\}_{1 \leq j \leq N} \rightarrow \{p, a_j, b_j\}_{1 \leq j \leq N}
\]

is uniformly Lipschitz continuous with suitable Sobolev norms provided that \( \{p, a_j, b_j\}_{1 \leq j \leq N} \) remains some bounded set in a suitable Sobolev space. In our inverse problem, initial values are also unknown, and we do not assume any positivity of initial values at all. Our key is a Carleman estimate and the exact controllability in a Sobolev space of higher order.

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1. Introduction.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial \Omega$ and let us consider

$$\partial_t^2 u(x,t) = \Delta u(x,t) + p(x)u(x,t), \quad x \in \Omega, \ t > 0, \quad (1.1)$$

$$u(x,0) = a(x), \quad \partial_t u(x,0) = b(x), \quad x \in \Omega \quad (1.2)$$

and

$$u(x,t) = g(x,t), \quad x \in \partial \Omega, \ t > 0 \quad (1.3)$$

Let $g \in L^2(\partial \Omega \times (0,T))$ be given and smooth suitably. Then by $u = u(p,g,a,b)(x,t)$ we denote the solution to (1.1) - (1.3) within a suitable class which is described later. Let $\nu = \nu(x)$ be the unit outward normal vector to $\partial \Omega$ at $x$, $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$, and let $\omega \subset \Omega$ be a subdomain.

In this paper, we discuss an inverse problem of determining $p = p(x)$, $x \in \Omega$ by some observations. That is,

**Inverse Problems.** Let $0 < T_0 < T_1$ be given. Given $g_1, \ldots, g_N \in L^2(\partial \Omega \times (0,T))$, determine $p(x), a_j(x), b_j(x) \in \Omega, 1 \leq j \leq N$, by

$$u(p,g_j,a_j,b_j)|_{\omega \times (0,T)}, \quad u(p,g_j,a_j,b_j)(x,T_0), \quad x \in \Omega, \quad j = 1, \ldots, N.$$

This is an inverse problem by a finite time of interior observations. As long as inverse problems of determining coefficients in multidimensions by a finite number of observations without smallness are concerned, one main methodology is by a Carleman estimate, which was initiated by Bukhgeim and Klibanov [10]. See also Baudouin and Puel [1], Bellassoued [3], [4], Bellassoued and Yamamoto [6], [7], Bellassoued, Imanuvilov and Yamamoto [8], Imanuvilov, Isakov and Yamamoto [15], Imanuvilov and Yamamoto [16-19], Isakov [20], [21], Klibanov [23], [24], Klibanov
For example, by a method in Imanuvilov and Yamamoto [16], [17], we can prove

**Theorem 0.** Let a constant $M > 0$ and a smooth function $\eta$ on $\overline{\Omega}$ be arbitrarily given. Let $0 < t_1 < T_0$ and $\omega \subset \Omega$ be a subdomain such that $\partial \omega \supset \partial \Omega$. We set

$$U_0 = \{ p \in W^{1,\infty}(\Omega) ; \| p \|_{W^{1,\infty}(\Omega)} \leq M, p = \eta \text{ in } \omega \}.$$ 

Let $u = u(p, g, a, b) \in \cap_{j=0}^{2} C^j([T_0 - t_1, T_0 + t_1]; H^{3-j}(\Omega))$ satisfy (1.1),

$$u(x, T_0) = a(x), \quad \partial_t u(x, T_0) = b(x), \quad x \in \Omega$$

and

$$u|_{\partial \Omega \times (T_0 - t_1, T_0 + t_1)} = g$$

with suitable functions $a, b, g$. Moreover let

$$\| u(p, g, a, b) \|_{C^j([T_0 - t_1, T_0 + t_1]; H^{3-j}(\Omega))},$$

$$\| u(\tilde{p}, \tilde{g}, \tilde{a}, \tilde{b}) \|_{C^j([T_0 - t_1, T_0 + t_1]; H^{3-j}(\Omega))} \leq M, \quad j = 0, 1, 2.$$ 

We assume that there exists a constant $\delta_0 > 0$ such that

$$|a(x)|, \quad |\tilde{a}(x)| \geq \delta_0 \quad \text{on } \overline{\Omega \setminus \omega} \quad (1.4)$$

and that

$$t_1 > \inf_{x' \in \mathbb{R}^n \setminus (\Omega \setminus \omega)} \sup_{x \in \Omega} |x - x'|.$$ 

Then there exists a constant $C = C(M, \delta_0, \omega, T_0, t_1) > 0$ such that

$$\| p - \tilde{p} \|_{L^2(\Omega)} \leq C \left( \| u(p, g, a, b) - u(\tilde{p}, \tilde{g}, \tilde{a}, \tilde{b}) \|_{C^2([T_0 - t_1, T_0 + t_1]; L^2(\omega))} 
+ \| u(p, g, a, b) - u(\tilde{p}, \tilde{g}, \tilde{a}, \tilde{b}) \|_{C^1([T_0 - t_1, T_0 + t_1]; H^2(\omega))} 
+ \| a - \tilde{a} \|_{H^2(\Omega)} + \| b - \tilde{b} \|_{H^2(\Omega)} \right)$$
for all $p, \tilde{p} \in \mathcal{U}_0$.

As for the Lipschitz stability without assumption that $p, \tilde{p}$ are given in $\omega$, see for example Imanuvilov and Yamamoto [16], [17]. For completeness, we will prove Theorem 0 in Appendix I.

Here we assume that coefficients $p$ and $\tilde{p}$ are given in a boundary layer $\omega$ and we determine them in $\Omega \setminus \omega$ by observations of the solutions in $\omega \times (T_0 - t_1, T_0 + t_1)$. For the Lipschitz stability in our inverse problem, we have to assume the strict positivity (1.4) on $\overline{\Omega \setminus \omega}$ of the given displacement at $t = T_0$. The Lipschitz stability is mathematically satisfactory, but such a positivity condition is quite restrictive, because (1.4) requires that we have to control the spatial distribution of the state over a domain where coefficients are unknown. There have been many trials for relaxing (1.4) (e.g., $a \not\equiv 0$ in $\Omega$), but it remains a serious open problem. We note that with impulsive force terms represented by a Dirac delta function, the Lipschitz stability results are known with $a \equiv 0$ with some smallness assumption and see Glushkova [13], Li [32], Romanov [37], Romanov and Yamamoto [38].

On the other hand, by the Dirichlet-to-Neumann map requiring infinitely many observations, we know the sharp uniqueness results (e.g., Belishev [2], Kurylev and Lassas [30]) and stability (e.g., Bellassoued, Jellali and Yamamoto [5], Cipolatti and F.Lopez [11], Sun [39]).

This paper gives a partial answer to the longstanding open problem: the Lipschitz stability by a finite number of observations without any positivity assumptions. The characters of our inverse problem are:

1. We take observations $N$ times by suitably choosing boundary inputs $g_1, \ldots, g_N$.
2. Initial values at $t = 0$ are unknown, while we have to observe displacements
and velocities at a fixed intermediate time $T_0$ corresponding to $g_1, \ldots, g_N$.

We emphasize that we need not assume any positivity of functions in $\Omega$, and we assume that $g_1, \ldots, g_N$ are at our disposal.

For the statement of our main result, we introduce notations. Let $\partial_j = \frac{\partial}{\partial x_j}$, $1 \leq j \leq n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N} \cup \{0\}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\partial^\alpha_x = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$.

Let $C^{\ell,1}(\overline{\Omega})$, $\ell \in \mathbb{N}$, be the space of all the functions whose derivatives of orders $\leq \ell$ are all Lipschitz continuous on $\overline{\Omega}$ and we set $\|p\|_{C^{\ell,1}(\overline{\Omega})} = \|p\|_{C^{1}(\overline{\Omega})} + \sup_{|\alpha| = \ell, x, x' \in \Omega, x \neq x'} \frac{\partial^\alpha_x p(x) - \partial^\alpha_x p(x')}{|x - x'|}$. Let $M > 0$ be arbitrarily fixed and let $\omega \subset \Omega$ be a subdomain such that $\partial \omega \supset \partial \Omega$, $\eta$ be an arbitrarily given smooth function. Let $m \in \mathbb{N}$ satisfy

$$m > \frac{n}{4}.$$ 

Let

$$\mathcal{U} = \{p \in C^{2m}(\overline{\Omega}); \|p\|_{C^{2m}(\overline{\Omega})} \leq M, p = \eta \text{ in } \omega, p \leq 0 \text{ in } \Omega\}$$

and

$$\mathcal{V} = \{(a, b) \in H_0^{2m}(\Omega) \times H_0^{2m-1}(\Omega); \|a\|_{H^{2m}(\Omega)}, \|b\|_{H^{2m-1}(\Omega)} \leq M\}.$$ 

We define a Hilbert space

$$V_{2m} = \{g \in H^{2m}(0, T; L^2(\partial \Omega)); \partial^j_t g(\cdot, T_0) = 0, j = 0, 1, 2, \ldots, 2m - 1\}$$

with the scalar product $(g, h)_{V_{2m}} = (\partial^{2m}_t g, \partial^{2m}_t h)_{L^2(0, T_0; L^2(\partial \Omega))}$.

Now we are ready to state our main result.

**Theorem.** Let

$$\begin{cases} T - T_0 > \inf_{x' \in \mathbb{R}^n \setminus (\Omega \setminus \omega)} \sup_{x \in \Omega} |x - x'|, \\ T_0 > \inf_{x' \in \mathbb{R}^n \setminus (\Omega \setminus \omega)} \sup_{x \in \Omega} |x - x'|. \end{cases}$$

(1.5)
For $M > 0$, we can choose $N \in \mathbb{N}$ and $g_1, \ldots, g_N \in V_{2m}$ satisfying:

$$
\| p - \tilde{p} \|_{L^2(\Omega)} + \sum_{j=1}^{N} \| a_j - \tilde{a}_j \|_{H^1(\Omega)} + \sum_{j=1}^{N} \| b_j - \tilde{b}_j \|_{L^2(\Omega)}
\leq C \left( \sum_{j=1}^{N} \| u(p, g_j, a_j, b_j) - u(\tilde{p}, g_j, \tilde{a}_j, \tilde{b}_j) \|_{C^3([0,T]; L^2(\omega))} 
+ \sum_{j=1}^{N} \| (u(p, g_j, a_j, b_j) - u(\tilde{p}, g_j, \tilde{a}_j, \tilde{b}_j)) (\cdot, T_0) \|_{H^2(\Omega)} 
+ \sum_{j=1}^{N} \| (\partial_t u(p, g_j, a_j, b_j) - \partial_t u(\tilde{p}, g_j, \tilde{a}_j, \tilde{b}_j)) (\cdot, T_0) \|_{H^2(\Omega)} \right)
$$

for any $(a_j, b_j), (\tilde{a}_j, \tilde{b}_j) \in \mathcal{V}$ and $p, \tilde{p} \in \mathcal{U}$.

The theorem asserts that we can choose $N$ boundary inputs yielding the Lipschitz stability in estimating $p - \tilde{p}$ when both $p$ and $\tilde{p}$ vary in the admissible set. The essence of our main result is the uniform choice in $p$ of the boundary inputs guaranteeing the Lipschitz stability. By the classical exact controllability for the hyperbolic equation (e.g., Komornik [28], Lions [33]), we can directly see: if we will estimate $p - \tilde{p}$ with a fixed $\tilde{p} \in \mathcal{U}$, then the Lipschitz stability holds with a single suitable input $g$. Moreover precisely, we assume (1.5). For any given $\tilde{p} \in \mathcal{U}$, there exists $g \in V_{2m}$ such that

$$
\| p - \tilde{p} \|_{L^2(\Omega)} + \| a - \tilde{a} \|_{H^1(\Omega)} + \| b - \tilde{b} \|_{L^2(\Omega)}
\leq C \left( \| u(p, g, a, b) - u(\tilde{p}, g, \tilde{a}, \tilde{b}) \|_{C^3([0,T]; L^2(\omega))} + \| u(p, g, a, b) - u(\tilde{p}, g, \tilde{a}, \tilde{b}) \|_{C^1([0,T]; H^2(\omega))} 
+ \| (u(p, g, a, b) - u(\tilde{p}, g, \tilde{a}, \tilde{b})) (\cdot, T_0) \|_{H^2(\Omega)} 
+ \| (\partial_t u(p, g, a, b) - \partial_t u(\tilde{p}, g, \tilde{a}, \tilde{b})) (\cdot, T_0) \|_{H^2(\Omega)} \right)
$$

for any $(a, b), (\tilde{a}, \tilde{b}) \in \mathcal{V}$ and $p \in \mathcal{U}$. 

As is also seen from the proof;

(1) The number $N$ of observations increases as the a priori bound $M > 0$ is larger.

(2) The effective boundary inputs $g_1, ..., g_N$ can be constructed by the Hilbert Uniqueness Method (e.g., Komornik [28], Lions [33]). See the proof in Section 3 as for the construction. Moreover choices of such inputs $g_1, ..., g_N$ are rather generous (Remark in Section 3).

The paper is composed of three sections. In Section 2, we show a necessary exact controllability in a Sobolev spaces of higher orders, which may be an independent interest. In Section 3, we prove Theorem.

§2. Exact controllability in $H^{2m}(\Omega)$.

In the state space $L^2(\Omega) \times H^{-1}(\Omega)$ for a hyperbolic equation, the exact controllability has been studied extensively. Here we only refer to a few works: Komornik [28], Lasiecka and Triggiani [31], Lions [33] and the readers can consult them for comprehensive references. For the proof of our result, we need the exact controllability of displacement in Sobolev spaces of higher orders. The exact controllability in $H^1_0(\Omega) \times L^2(\Omega)$ has been discussed in Section 6 of Chapitre 1 in Komornik and Yamamoto [29], Lions [33] for $\partial_t^2 - \Delta$.

Let $\omega \subset \Omega$ be a subdomain such that $\partial \omega \supset \partial \Omega$ and let

$$T_0 \geq \inf_{x' \in \mathbb{R}^n \setminus (\Omega \setminus \omega)} \sup_{x \in \Omega} |x - x'|.$$  

Henceforth let $x_0 \in \mathbb{R}^n \setminus (\overline{\Omega \setminus \omega})$ satisfy

$$\inf_{x' \in \mathbb{R}^n \setminus (\Omega \setminus \omega)} \sup_{x \in \Omega} |x - x'| = \sup_{x \in \Omega} |x - x_0| < T_0.$$

Here $(\cdot, \cdot)$ and $\| \cdot \|$ denote the scalar product and the norm in $L^2(\Omega)$. Let $p \in \mathcal{U}_1 = \{p; \|p\|_{C^{2m-2,1}(\overline{\Omega})} \leq M, p \leq 0 \mbox{ in } \Omega\}$ be fixed and let us define an operator
A by $Au(x) = -\Delta u(x) - p(x)u(x)$ for $x \in \Omega$ and $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$. Then $A^{-1}$ exists and $\sigma(A)$ is entirely composed of eigenvalues with finite multiplicities. Let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ be the sequence of all the eigenvalues of $A$ where $\lambda_j$ appears repeatedly $k_j$-times, where $k_j$ is the multiplicity of $\lambda_j$. Let $\varphi_j$ be a unit eigenvector of $A$ for $\lambda_j$. Then $\{\varphi_j\}_{j \in \mathbb{N}}$ is an orthonormal basis in $L^2(\Omega)$ and

$$A^{-m}a = \sum_{j=1}^{\infty} \frac{(a, \varphi_j)}{\lambda_j^m} \varphi_j$$

in $L^2(\Omega)$.

Since $p \in C^{2m-2,1}(\Omega)$, elliptic regularity results (e.g., Theorem 8.13 (p.187) in Gilbarg and Trudinger [12]) yield

$$C_1^{-1} \|Au\| \leq \|u\|_{H^2(\Omega)} \leq \|Au\|, \quad u \in D(A).$$

Next, using $\|A^{k-1}u\| \leq C_1'|A^k u|$ for $u \in D(A^k)$ and $k \in \mathbb{N}$, we repeatedly apply Theorem 8.13 in [12] and we see

$$C_1^{-1} \|A^m u\| \leq \|u\|_{H^{2m}(\Omega)} \leq C_1 \|A^m u\|, \quad u \in D(A^m). \quad (2.2)$$

Here we note that the constant $C_1 > 0$ can be taken uniformly in $p \in U_1$. In particular, $D(A^m) \subset H^{2m}(\Omega)$, $m \in \mathbb{N}$. We regard $D(A^m)$ as a Banach space with the norm

$$\|a\|_{D(A^m)} = \|A^m a\| = \left( \sum_{j=1}^{\infty} \lambda_j^{2m} (a, \varphi_j)^2 \right)^{\frac{1}{2}}.$$ 

Moreover, for $\gamma \geq 0$, we can see that

$$A^\gamma a = \sum_{j=1}^{\infty} \lambda_j^\gamma (a, \varphi_j) \varphi_j,$$

$$D(A^\gamma) = \{a \in L^2(\Omega); \sum_{j=1}^{\infty} \lambda_j^{2\gamma} (a, \varphi_j)^2 < \infty\}. \quad (2.3)$$
Identifying a Banach space $D(A^0) = L^2(\Omega)$ with its dual $(L^2(\Omega))^*$, we have the dense and continuous embedding

$$D(A^m) \subset L^2(\Omega) \subset D(A^m)^*.$$ 

Henceforth $B(X,Y)$ denotes the Banach space of all bounded linear operators from a Banach space $X$ to another Banach space $Y$, $L^*$ denotes the dual operator of $L \in B(X,Y)$. We regard $A^m$ as a bounded linear operator from $D(A^m)$ to $L^2(\Omega)$:

$$A^m \in B(D(A^m), L^2(\Omega)).$$

Then $(A^m)^* \in B(L^2(\Omega), (D(A^m))^*)$. We denote the duality pairing between $D(A^m)^*$ and $D(A^m)$ by $(D(A^m))^* \cdot, \cdot >_{D(A^m)}$. We note that $(D(A^m))^* \cdot, \cdot >_{D(A^m)} = (\cdot, \cdot)_{L^2(\Omega)}$ if $\cdot, \cdot \in L^2(\Omega)$ (e.g., Brezis [9]).

Moreover we introduce a Banach space

$$V_{2m} = \{ g \in H^{2m}(0,T_0; L^2(\partial \Omega)); \partial^j_t g(\cdot, T_0) = 0, \ j = 0, 1, 2, \ldots, 2m - 1 \}$$

with the norm $\|g\|_{V_{2m}} = \|\partial^2_t g\|_{L^2(0,T_0; L^2(\partial \Omega))}$. Identifying a Banach space $L^2(0,T_0; L^2(\partial \Omega))$ with its dual, we have the dense and continuous embedding

$$V_{2m} \subset L^2(0,T_0; L^2(\partial \Omega)) \subset V_{2m}^*.$$ 

We denote the duality pairing between $g \in V_{2m}^*$ and $h \in V_{2m}$ by $\langle g, h \rangle$ and we note that $\langle g, h \rangle = (g, h)_{L^2(0,T_0; L^2(\partial \Omega))}$ for $g, h \in L^2(0,T_0; L^2(\partial \Omega))$.

Then

**Lemma 2.1.**

$$(A^m)^* a = A^m a, \quad a \in C_0^\infty(\Omega).$$

**Proof.** By the definition of the dual operator $(A^m)^*$, we have

$$\langle D(A^m) \cdot, (A^m)^* a, f \rangle_{D(A^m)} = (a, A^m f), \quad f \in D(A^m).$$
Since \( a \in C_0^\infty(\Omega) \subset \mathcal{D}(A^m) \), we see for example by using (2.3), that

\[
(\mathcal{D}(A^m))^* < (A^m)^* a, f >_{\mathcal{D}(A^m)} = (A^m a, f).
\]

By \( f, A^m a \in L^2(\Omega) \), we see that \( (A^m a, f) = (\mathcal{D}(A^m))^* < (A^m)^* a, f >_{\mathcal{D}(A^m)} \), so that \( (\mathcal{D}(A^m))^* < (A^m)^* a, f >_{\mathcal{D}(A^m)} = (A^m a, f) \) for all \( f \in \mathcal{D}(A^m) \). The proof of Lemma 2.1 is completed.

Next

**Lemma 2.2.**

\[
\|a\|_{\mathcal{D}(A^m)^*} = \left( \sum_{j=1}^{\infty} \left| \frac{(a, \varphi_j)}{\lambda_j^m} \right|^2 \right)^{\frac{1}{2}}, \quad a \in L^2(\Omega).
\]

**Proof.** Since \( A^m : \mathcal{D}(A^m) \rightarrow L^2(\Omega) \) is injective and surjective, and \((A^m)^{-1} \in B(L^2(\Omega), \mathcal{D}(A^m))\), we see that \(((A^m)^*)^{-1} \) is injective and \((A^m)^* \in B(\mathcal{D}(A^m)^*, L^2(\Omega))\) (e.g., Yosida [41]). Hence for any \( a \in \mathcal{D}(A^m)^* \), we have

\[
(\mathcal{D}(A^m))^* < a, f >_{\mathcal{D}(A^m)} = (A^m)^* ((A^m)^*)^{-1} a, f >_{\mathcal{D}(A^m)} = (a, A^m f)
\]

for \( f \in \mathcal{D}(A^m) \). Hence for \( a \in \mathcal{D}(A^m)^* \), we have

\[
\|a\|_{\mathcal{D}(A^m)^*} = \sup_{\|A^m f\| = 1} |(\mathcal{D}(A^m))^* < a, f >_{\mathcal{D}(A^m)}| = \sup_{\|A^m f\| = 1} ||((A^m)^*)^{-1} a, A^m f|| = \|((A^m)^*)^{-1} a\| = ||(A^m)^{-1} a\|.
\]

Henceforth let \( a \in L^2(\Omega) \). We have \((a, (A^m)^{-1} f) = (\mathcal{D}(A^m))^* < a, (A^m)^{-1} f >_{\mathcal{D}(A^m)} = ((A^m)^{-1})^* a, f \). On the other hand,

\[
(a, A^{-m} f) = a, \sum_{j=1}^{\infty} \frac{(f, \varphi_j)}{\lambda_j^m} \varphi_j = \left( \sum_{j=1}^{\infty} \frac{(a, \varphi_j)}{\lambda_j^m} \varphi_j, f \right).
\]

Hence

\[
(((A^m)^{-1})^* a, f) = \left( \sum_{j=1}^{\infty} \frac{(a, \varphi_j)}{\lambda_j^m} \varphi_j, f \right), \quad f \in L^2(\Omega),
\]
that is,

\[ ((A^m)^{-1})^* a = \sum_{j=1}^{\infty} \frac{(a, \varphi_j)}{\lambda_j^m} \varphi_j \]

in \( L^2(\Omega) \) for any \( a \in L^2(\Omega) \). Hence

\[ \|(A^m)^{-1})^* a\| = \left( \sum_{j=1}^{\infty} \left| \frac{(a, \varphi_j)}{\lambda_j^m} \right|^2 \right)^{\frac{1}{2}}. \]

In view of (2.4), the proof of Lemma 2.2 is completed.

We set

\[ H = \{ A^m a; a \in C_0^\infty(\Omega) \}^{D(A^m)^*}. \] (2.5)

Then

**Lemma 2.3.** \( H = D(A^m)^* \).

**Proof.** By Lemma 2.1, we have

\[ H = \{ (A^m)^* a; a \in C_0^\infty(\Omega) \}^{D(A^m)^*}. \]

Since \( A^m : D(A^m) \to L^2(\Omega) \) is injective and \( (A^m)^{-1} \in B(L^2(\Omega), D(A^m)) \), we see that \( ((A^m)^*)^{-1} \in B(D(A^m)^*, L^2(\Omega)) \) and \( D(A^m)^* = (A^m)^* L^2(\Omega) \). Hence it suffices to prove

\[ \{ (A^m)^* a; a \in C_0^\infty(\Omega) \}^{D(A^m)^*} = (A^m)^* L^2(\Omega). \] (2.6)

Since \( C_0^\infty(\Omega) \) is dense in \( L^2(\Omega) \), for any \( a \in L^2(\Omega) \), we can choose a sequence \( a_m \in C_0^\infty(\Omega) \), \( m \in \mathbb{N} \) such that \( a_m \to a \) in \( L^2(\Omega) \). By \( (A^m)^* \in B(L^2(\Omega), D(A^m)^*) \), we see that \( (A^m)^* a_m \to (A^m)^* a \) in \( D(A^m)^* \). Hence \( (A^m)^* a \in H \). Therefore (2.6) follows, and the proof of Lemma 2.3 is completed.

We consider

\[
\begin{cases}
\partial_t^2 v(x, t) = -Av(x, t), & x \in \Omega, \ 0 < t < T_0, \\
v(x, t) = 0, & x \in \partial\Omega, \ 0 < t < T_0, \\
v(\cdot, 0) = 0, & \partial_t v(\cdot, 0) = v_1 \in D(A^m)^*.
\end{cases}
\] (2.7)
By the transposition method (e.g., Komornik [28], Lions and Magenes [34]), there exists a unique solution \( v \in C^1([0, T_0]; \mathcal{D}(A^m)^*) \) to (2.7) and by \( v(x, t) = v(v_1)(x, t) \) we denote the solution to (2.7). Then we have

**Theorem 2.1.** Let us assume (2.1). Then there exist constants \( C_2, C_3 > 0 \) such that

\[
C_2\|v_1\|_{\mathcal{D}(A^m)^*} \leq \left\| \frac{\partial v(v_1)}{\partial \nu} \right\|_{V^*_{2m}} \leq C_3\|v_1\|_{\mathcal{D}(A^m)^*}
\]

for all \( v_1 \in \mathcal{D}(A^m)^* \).

**Proof.** By the limit passage and Lemmata 2.1 and 2.3, it is sufficient to prove the conclusion for \( v_1 = (A^m)^*a = A^ma \) with \( a \in C_0^\infty(\Omega) \). Then we have

\[
\frac{\partial v(v_1)}{\partial \nu}(x, t) = \sum_{j=1}^\infty (v_1, \varphi_j) \frac{\sin \sqrt{\lambda_j} t}{\sqrt{\lambda_j}} \frac{\partial \varphi_j}{\partial \nu}(x), \quad x \in \partial \Omega, \; 0 < t < T_0
\]

(e.g., [28]). We note that the right hand side is convergent in \( L^2(0, T_0; L^2(\partial \Omega)) \).

By the definition of the norm in \( V^*_{2m} \), we have

\[
\left\| \frac{\partial v(v_1)}{\partial \nu} \right\|_{V^*_{2m}} = \sup_{\|g\|_{V^*_{2m}} = 1} \left| \left( g, \frac{\partial v(v_1)}{\partial \nu} \right) \right| = \sup_{\|g\|_{V^*_{2m}} = 1} \left| \left( g, \frac{\partial v(v_1)}{\partial \nu} \right) \right|_{L^2(0, T_0; L^2(\partial \Omega))}.
\]

Since \( \partial_t^j g(\cdot, T_0) = 0, \; j = 0, 1, \cdots, 2m - 1 \), we integrate by parts to obtain

\[
\left( g, \frac{\partial v(v_1)}{\partial \nu} \right)_{L^2(0, T_0; L^2(\partial \Omega))} = \left( \partial_t^{2m} g, \int_0^t \frac{(t - \xi)^{2m-1}}{(2m - 1)!} \frac{\partial v(v_1)}{\partial \nu}(x, \xi)d\xi \right)_{L^2(0, T_0; L^2(\partial \Omega))}.
\]

Hence

\[
\left\| \frac{\partial v(v_1)}{\partial \nu} \right\|_{V^*_{2m}} = \left\| \int_0^t \frac{(t - \xi)^{2m-1}}{(2m - 1)!} \frac{\partial v(v_1)}{\partial \nu}(x, \xi)d\xi \right\|_{L^2(0, T_0; L^2(\partial \Omega))}.
\]
On the other hand, by integration by parts, we have

\[
\int_0^t \frac{(t - \xi)^{2m-1}}{(2m-1)!} \frac{\partial v(v_1)}{\partial \nu} (x, \xi) d\xi
\]

\[
= \frac{t^{2m-1}}{(2m-1)!} \sum_{j=1}^{\infty} \frac{(v_1, \varphi_j)}{\lambda_j} \frac{\partial \varphi_j}{\partial \nu}(x) - \frac{t^m}{(2m-2)!} \sum_{j=1}^{\infty} \frac{(v_1, \varphi_j) \cos \sqrt{\lambda_j} \xi}{\lambda_j} \frac{\partial \varphi_j}{\partial \nu}(x) d\xi
\]

\[
= \frac{t^{2m-1}}{(2m-1)!} \sum_{j=1}^{\infty} \frac{(v_1, \varphi_j)}{\lambda_j} \frac{\partial \varphi_j}{\partial \nu}(x) + \frac{t^{2m-3}}{(2m-2)!} \sum_{j=1}^{\infty} \frac{(v_1, \varphi_j)}{\lambda_j \sqrt{\lambda_j}} \sin \sqrt{\lambda_j} \xi \frac{\partial \varphi_j}{\partial \nu}(x) d\xi
\]

\[
= \cdots
\]

\[
= \sum_{k=1}^{m} (-1)^{k+1} \frac{t^{2m-(2k-1)}}{(2m-(2k-1))!} \sum_{j=1}^{\infty} \frac{(v_1, \varphi_j)}{\lambda_j^k} \frac{\partial \varphi_j}{\partial \nu}(x)
\]

\[
+ (-1)^m \sum_{j=1}^{\infty} \frac{(v_1, \varphi_j)}{\lambda_j^{m+\frac{3}{2}}} \sin \sqrt{\lambda_j} x \frac{\partial \varphi_j}{\partial \nu}(x).
\]

Moreover, since \( v_1 = A^m a \), for \( 1 \leq k \leq m \) we have \( (v_1, \lambda_j^{-k} \varphi_j) = (v_1, A^{-k} \varphi_j) = (A^{m-k} a, \varphi_j) \) and

\[
\sum_{j=1}^{\infty} \frac{(v_1, \varphi_j)}{\lambda_j^k} \varphi_j(x) = \sum_{j=1}^{\infty} (A^{m-k} a, \varphi_j) \varphi_j(x) = A^{m-k} a.
\]

Hence for \( 1 \leq k \leq m \), we have

\[
\sum_{j=1}^{\infty} \frac{(v_1, \varphi_j)}{\lambda_j^k} \frac{\partial \varphi_j}{\partial \nu}(x) = \frac{\partial}{\partial \nu} (A^{m-k} a) = 0
\]

because \( A^{m-k} a = 0 \) in \( \Omega \setminus \text{supp } a \). Hence

\[
\int_0^t \frac{(t - \xi)^{2m-1}}{(2m-1)!} \frac{\partial v(v_1)}{\partial \nu} (x, \xi) d\xi = (-1)^m \sum_{j=1}^{\infty} \frac{(v_1, \varphi_j)}{\lambda_j^{m+\frac{3}{2}}} \sin \sqrt{\lambda_j} x \frac{\partial \varphi_j}{\partial \nu}(x).
\]

Moreover

\[
C' \sum_{j=1}^{\infty} \beta_j^2 \leq \left\| \sum_{j=1}^{\infty} \frac{\beta_j \sin \sqrt{\lambda_j} t \partial \varphi_j}{\sqrt{\lambda_j}} \frac{\partial \varphi_j}{\partial \nu} \right\|_{L^2(0,T_0;L^2(\partial\Omega))}^2 \leq C' \sum_{j=1}^{\infty} \beta_j^2
\]
for all $\beta_j \in \mathbb{R}$, $j \in \mathbb{N}$, under assumption (2.1), and the constants $C'_2, C'_3$ can be chosen uniformly in $p \in U_1$ (e.g., Kazemi and Klibanov [22], Klibanov and Malinsky [25], Komornik [28]). Consequently
\[
C'_2 \sum_{j=1}^{\infty} \left| \frac{(v_1, \varphi_j)}{\lambda_j^m} \right|^2 \leq \left\| \sum_{j=1}^{\infty} \left( \frac{(v_1, \varphi_j)}{\lambda_j^m} \sin \sqrt{\lambda_j} t \frac{\partial \varphi_j}{\partial \nu} \right) \right\|_{L^2(0,T_0;L^2(\partial \Omega))}^2 \leq C'_3 \sum_{j=1}^{\infty} \left| \frac{(v_1, \varphi_j)}{\lambda_j^m} \right|^2.
\]

By $v_1 = A^m a \in L^2(\Omega)$, we can apply Lemma 2.2, so that the conclusion holds for $v_1 = (A^m)^* a$ with any $a \in C_0^\infty(\Omega)$.

We define an operator $K$ from $\mathcal{D}(A^m)^*$ to $V_{2m}^*$ by
\[
Kv_1 = \frac{\partial v(v_1)}{\partial \nu}.
\]

We set $\Lambda = K^*$. Then Theorem 2.1 implies that
\[
K^*: V_{2m} \longrightarrow \mathcal{D}(A^m)
\]
is surjective and bounded (e.g., Brezis [9]). By the open mapping theorem, $\Lambda \equiv (K^*)^{-1}: \mathcal{D}(A^m) \longrightarrow V_{2m}$ is bounded.

On the other hand, for $g \in V_{2m}$, there exists a unique solution $w = w(g) \in C^{2m}([0,T];L^2(\Omega))$ to
\[
\begin{align*}
\partial_t^2 w(x,t) &= \Delta w(x,t) + p(x)w(x,t), \quad x \in \Omega, \ 0 < t < T_0, \\
w(x,t) &= g(x,t), \quad x \in \partial \Omega, \ 0 < t < T_0, \\
w(\cdot,0) &= \partial_t w(x,0) = 0, \quad x \in \Omega.
\end{align*}
\]
(2.8)
The unique existence of $w(g)$ is proved by taking $t$-derivatives of $w$ and a usual a priori estimate (e.g., [28], [33]), and we can further prove more regularity by the transposition method (e.g., Lions and Magenes [34]), but we will omit the details.

In terms of the Hilbert Uniqueness Method (e.g., [28], [33]), noting (2.2), we have
Theorem 2.2. Let us assume (2.1). Then, for any \( a \in \mathcal{D}(A^m) \), we have

\[
    w(\Lambda(a))(x,T_0) = a(x), \quad x \in \Omega, \quad \|\Lambda(a)\|_{V_{2m}} \leq C_4 \|a\|_{H^{2m}(\Omega)}. \tag{2.9}
\]

The constant \( C_4 > 0 \) is taken uniformly in \( p \in \mathcal{U}_1 \).

This is the exact controllability for the displacement in a state space \( \mathcal{D}(A^m) \subset H^{2m}(\Omega) \). We conclude the section with Lemmata 2.4 and 2.5 which assert estimates for solutions to hyperbolic equations.

Lemma 2.4. Let \( u = u(p,g,a,b) \) satisfy

\[
    \begin{aligned}
    \partial_t^2 u(x,t) &= \Delta u(x,t) + p(x)u(x,t), \quad x \in \Omega, \; 0 < t < T, \\
    u(x,0) &= a(x), \quad \partial_t u(x,0) = b(x), \quad x \in \Omega, \\
    u(x,t) &= g(x,t), \quad x \in \partial\Omega, \; 0 < t < T.
    \end{aligned}
\]

Let \( \|p\|_{C^{2m-2,2}(\Omega)} \leq M \), and let \( a \in H^{2m}(\Omega) \), \( b \in H^{2m-1}(\Omega) \). Then for any subdomains \( \Omega' \) and \( \Omega'' \) such that \( \overline{\Omega'} \subset \Omega' \subset \overline{\Omega''} \subset \Omega \), there exist constants \( C_5 = C_5(M,T,\Omega'') > 0 \) and \( C_6 = C_6(M,T,\Omega',\Omega'') > 0 \) such that

\[
    \|\partial_t^j u\|_{C([0,T];H^1(\Omega'))} \leq C_5(\|a\|_{H^{2m}(\Omega)} + \|b\|_{H^{2m-1}(\Omega)} + \|g\|_{H^{2m}(0,T;L^2(\partial\Omega))}),
\]

\[
    j = 0,1,\cdots,2m-2 \tag{2.10}
\]

and

\[
    \|u\|_{C([0,T];H^{2m}(\Omega''))} \leq C_6(\|a\|_{H^{2m}(\Omega)} + \|b\|_{H^{2m-1}(\Omega)} + \|g\|_{H^{2m}(0,T;L^2(\partial\Omega))}). \tag{2.11}
\]

We can prove the lemma by taking \( t \)-derivatives of \( u \), and will give the proof in Appendix II for completeness.

Lemma 2.5. Let

\[
    T > 2 \inf_{x_0 \in \mathbb{R}^n \setminus (\Omega \setminus \omega)} \sup_{x \in \Omega} |x - x_0|
\]
and let
\[
\begin{align*}
\partial_t^2 w(x,t) &= \Delta w(x,t) + p(x)w(x,t) + f(x,t), \quad x \in \Omega, \ 0 < t < T, \\
w(x,t) &= 0, \quad x \in \partial \Omega, \ 0 < t < T,
\end{align*}
\]

where \(\|p\|_{L^\infty(\Omega)} \leq M\). Then there exists a constant \(C_7 = C_7(M, \Omega, T) > 0\) such that
\[
\|w(\cdot,0)\|_{H^1(\Omega)} + \|\partial_t w(\cdot,0)\|_{L^2(\Omega)} \leq C_7(\|f\|_{L^2(0,T;L^2(\Omega))} + \|w\|_{H^1(0,T;L^2(\omega))}).
\]

This is an observability inequality (e.g., Kazemi and Klibanov [22], Klibanov and Malinsky [25], Klibanov and Timonov [26], Komornik [28], Lions [33]) and for completeness we will prove it in Appendix III.

§3. Proof of Theorem.

First Step.

Since \(\mathcal{U}\) is relatively compact in \(C^{2m-1}(\overline{\Omega})\), for any \(m \in \mathbb{N}\) we can choose \(N = N(m) \in \mathbb{N}\) and \(p_j^m \in \mathcal{U}, 1 \leq j \leq N\) such that for any \(p \in \mathcal{U}\), there exists \(j_0 \in \{1,2,...,N\}\) satisfying
\[
\|p - p_j^m\|_{C^{2m-1}(\overline{\Omega})} \leq \frac{1}{m}. \tag{3.1}
\]

We arbitrarily fix
\[
a_0 \in H_0^{2m}(\Omega), \quad |a_0(x)| \geq 1, \quad x \in \Omega \setminus \omega. \tag{3.2}
\]

We set \(a_\ell(x) = \ell Ma_0(x)\) for \(\ell \in \mathbb{N}\). Then we have
\[
\|a_\ell\|_{H^{2m}(\Omega)} \leq C_0(\ell, M). \tag{3.3}
\]

Here \(\ell \in \mathbb{N}\) is a constant which we will choose later.

By Theorem 2.2, we can choose \(g_1,...,g_N \in V_{2m}\) such that
\[
u(p_j^m,g_j,0,0)(x,T_0) = a_\ell(x), \quad x \in \overline{\Omega}, \ 1 \leq j \leq N. \tag{3.4}
\]
Remark. Boundary controls $g_1, ..., g_N$ are suitable inputs guaranteeing the Lipschitz stability in our inverse problem. Let $\Lambda(p) : D(A^m) \rightarrow V_{2m}$ be a mapping defined by Theorem 2.2 with the coefficient $p$. Then we can represent

$$(g_1, ..., g_N) = (\Lambda(p^m_1)(a_\ell), ..., \Lambda(p^m_N)(a_\ell)).$$

Here, for simplicity, we choose controls $g_j$ which steer the system with the zero initial condition to $a_\ell(x) = u(x, T_0)$. We need not be restricted to the zero initial condition. By the proof, we can see that for the Lipschitz stability in our inverse problem, we can choose $h_j, 1 \leq j \leq N$, such that

$$u(p^m_j, h_j, \alpha_j, \beta_j)(x, T_0) = a_\ell(x), \quad x \in \Omega$$

for any $(\alpha_j, \beta_j) \in \mathcal{V}$, $1 \leq j \leq N$. Such boundary controls $h_j$ can exist in terms of Theorem 2.1 (also see [28], [33]). Thus such choices of boundary inputs are quite generous.

Here and henceforth $C, C_j$ denote generic positive constants which are dependent on $\Omega, \mathcal{U}, \mathcal{V}$ but independent of $\ell, m, M$ and choices of $p \in \mathcal{U}$, while $C(M, \ell), C_j(M, \ell)$ denote generic positive constants which are dependent on $M, \ell, \Omega, \mathcal{U}, \mathcal{V}$ but independent of $m$ and choices of $p \in \mathcal{U}$. We can understand that $C(M), C_j(M)$ denote a constant which are independent of $\ell$ but depend on $M, \Omega, \mathcal{U}, \mathcal{V}$.

Then by Theorem 2.2 and the second assumption in (1.5), we have

$$\|g_j\|_{V_{2m}} \leq C_1(M)\|a_\ell\|_{H^2m(\Omega)} \leq C_2(M, \ell). \quad (3.5)$$

Let $p \in \mathcal{U}$ be given arbitrarily. Then

$$\min_{1 \leq j \leq N} \|u(p, g_j, a, b) - a_\ell\|_{C([0,T_0];H^{2m}(\Omega))} \leq C_3(M) + \frac{C_4(M, \ell)}{m} \quad \text{for all } (a, b) \in \mathcal{V}. \quad (3.6)$$
In fact, by (3.1), there exists \( j_0 \in \{1, \ldots, N \} \) such that \( \|p - p_{j_0}^m\|_{C^{2m-1}(\Omega)} \leq \frac{1}{m} \).

Setting \( v = u(p, g_{j_0}, a, b) - u(p_{j_0}^m, g_{j_0}, 0, 0) \), we have

\[
\begin{aligned}
\frac{\partial^2 v(x, t)}{\partial t^2} &= \Delta v(x, t) + p(x)v(x, t) + (p(x) - p_{j_0}^m(x))u(p_{j_0}^m, g_{j_0}, 0, 0)(x, t), \quad x \in \Omega, \quad 0 < t < T, \\
v(x, 0) &= a(x), \quad \partial_t v(x, 0) = b(x), \quad x \in \Omega, \\
v(x, t) &= 0, \quad x \in \partial \Omega, \quad 0 < t < T.
\end{aligned}
\]

Then, by Lions and Magenes [34], we have

\[
\|v\|_{C([0, T]; H^{2m}(\Omega))} \\
\leq C_5(M)(\|a\|_{H^{2m}(\Omega)} + \|b\|_{H^{2m-1}(\Omega)} + \|(p - p_{j_0}^m)u(p_{j_0}^m, g_{j_0}, 0, 0)\|_{C([0, T]; H^{2m-1}(\Omega))}) \\
\leq C_5'(M)(\|a\|_{H^{2m}(\Omega)} + \|b\|_{H^{2m-1}(\Omega)}) + C_5(M)\|p - p_{j_0}^m\|_{C^{2m-1}(\Omega)}\|u(p_{j_0}^m, g_{j_0}, 0, 0)\|_{C([0, T]; H^{2m-1}(\Omega))} \\
\leq C_6(M) + \frac{C_7(M, \ell)}{m}.
\]

Here we used Lemma 2.4, \( \|a\|_{H^{2m}(\Omega)}, \|b\|_{H^{2m-1}(\Omega)} \leq M \) and (3.5). Since \( u(p_{j_0}^m, g_{j_0}, 0, 0)(x, T_0) = a_\ell(x), \ x \in \Omega \) by (3.4), the proof of (3.6) is completed.

By \( m > \frac{\ell}{3} \) and the Sobolev embedding, we have \( H^{2m}(\Omega) \subset C(\overline{\Omega}) \). In terms of (3.2) and (3.6), we see that for any \( p \in \mathcal{U} \), there exists \( j_0 \in \{1, \ldots, N \} \) such that

\[
|u(p, g_{j_0}, a, b)(x, T_0)| \geq \ell M - C_8(M) - \frac{C_9(M, \ell)}{m}, \quad x \in \overline{\Omega} \setminus \omega, \ \ (a, b) \in \mathcal{V}.
\]

We choose \( \ell \in \mathbb{N} \) sufficiently large so that \( \frac{(\ell - 1)M}{2} \geq C_8(M) \). For this \( \ell \), we choose \( m \in \mathbb{N} \) large such that \( \frac{(\ell - 1)M}{2} \geq \frac{C_9(M, \ell)}{m} \).

Thus: there exist \( g_1, \ldots, g_N \in V_{2m} \) such that for any \( p \in \mathcal{U} \), we can choose \( j_0 \in \{1, \ldots, N \} \) satisfying

\[
|u(p, g_{j_0}, a, b)(x, T_0)| \geq \left( \frac{(\ell - 1)M}{2} - C_8(M) \right) + \left( \frac{(\ell - 1)M}{2} - \frac{C_9(M, \ell)}{m} \right) + M \\
\geq M, \quad x \in \overline{\Omega} \setminus \omega, \ (a, b) \in \mathcal{V}.
\] (3.7)
Second Step.

We will complete the proof of Theorem.

We choose a subdomain \( \Omega' \subset \Omega \) such that \( \Omega \setminus \omega \subset \Omega' \) and \( \overline{\Omega'} \subset \Omega \). Then, by Lemma 2.4 and hyperbolic equation (1.1), we have

\[
\|u(p, g_j, a_j, b_j)\|_{C^2([0,T];H^2m(\Omega')) \cap C^2([0,T];H^{2m-2}(\Omega'))} \leq C_{10}(M),
\]

\[
\|u(\tilde{p}, g_j, \tilde{a}_j, \tilde{b}_j)\|_{C^2([0,T];H^2m(\Omega')) \cap C^2([0,T];H^{2m-2}(\Omega'))} \leq C_{10}(M)
\]

for \((p, a_j, b_j), (\tilde{p}, \tilde{a}_j, \tilde{b}_j) \in \mathcal{U} \times \mathcal{V} \). Setting \( t_1 = \min\{T - T_0, T_0\} \), we see by (1.5) that \( t_1 > \min_{x' \in \mathbb{R}^n \setminus (\Omega \setminus \omega)} \sup_{x \in \Omega} |x - x'| \). Therefore we can apply Theorem 0 in a domain \( \Omega' \) by replacing \( \omega \) by \( \omega \cap \Omega' \). Then, in terms of (3.7), assumption (1.4) is satisfied. Hence Theorem 0 yields that we can choose \( j \in \{1, ..., N\} \) such that

\[
\|p - \tilde{p}\|_{L^2(\Omega, \Sigma)} \leq C_{11} \left( \|u(p, g_j, a_j, b_j) - u(\tilde{p}, g_j, \tilde{a}_j, \tilde{b}_j)\|_{C^3([0,T];L^2(\omega))} + \|u(p, g_j, a_j, b_j) - u(\tilde{p}, g_j, \tilde{a}_j, \tilde{b}_j)\|_{C^1([0,T];H^2(\omega))} + \|u(p, g_j, a_j, b_j) - u(\tilde{p}, g_j, \tilde{a}_j, \tilde{b}_j)\|_{H^2(\Omega)} + \|\partial_t u(p, g_j, a_j, b_j) - \partial_t u(\tilde{p}, g_j, \tilde{a}_j, \tilde{b}_j)\|_{H^2(\Omega)} \right) = C_{11}^D_j.
\]

(3.8)

Since \( g_j \) depends on \( p, \tilde{p} \in \mathcal{U} \), we take the sum \( \sum_{j=1}^N D_j \), so that \( \|p - \tilde{p}\|_{L^2(\Omega)} \leq C_{11} \sum_{j=1}^N D_j \).

Next, setting \( v_j = u(p, g_j, a_j, b_j) - u(\tilde{p}, g_j, \tilde{a}_j, \tilde{b}_j) \), we have

\[
\begin{aligned}
\partial^2_t v_j(x, t) &= \Delta v_j(x, t) + p(x)v_j(x, t) + (p(x) - \tilde{p}(x))u(\tilde{p}, g_j, \tilde{a}_j, \tilde{b}_j)(x, t), \quad x \in \Omega, \ 0 < t < T, \\
v_j(x, 0) &= a_j(x) - \tilde{a}_j(x), \quad \partial_t v_j(x, 0) = b_j(x) - \tilde{b}_j(x), \quad x \in \Omega, \\
v_j(x, t) &= 0, \quad x \in \partial \Omega, \ 0 < t < T.
\end{aligned}
\]

In terms of (1.5), applying Lemma 2.5, we obtain

\[
\|a_j - \tilde{a}_j\|_{H^1(\Omega)} + \|b_j - \tilde{b}_j\|_{L^2(\Omega)} \leq C_{12}(\|(p - \tilde{p})u(\tilde{p}, g_j, \tilde{a}_j, \tilde{b}_j)\|_{L^2(0, T; L^2(\Omega))} + \|v_j\|_{H^1(0, T; L^2(\omega))}).
\]
Since \( p - \tilde{p} = 0 \) in \( \omega \), in terms of \( m > \frac{q}{4} \) and the Sobolev embedding, we have

\[
\| (p - \tilde{p}) u(\tilde{p}, g_j, \tilde{a}_j, \tilde{b}_j) \|_{L^2(0,T;L^2(\Omega'))} \leq \| (p - \tilde{p}) u(\tilde{p}, g_j, \tilde{a}_j, \tilde{b}_j) \|_{L^2(0,T;L^2(\Omega'))}
\]

\[
\leq \| p - \tilde{p} \|_{L^2(\Omega')} \| u(\tilde{p}, g_j, \tilde{a}_j, \tilde{b}_j) \|_{L^2(0,T;L^2(\Omega'))}
\]

\[
\leq C_{13} \| p - \tilde{p} \|_{L^2(\Omega')} \| u(\tilde{p}, g_j, \tilde{a}_j, \tilde{b}_j) \|_{L^2(0,T;H^{2m}(\Omega'))}.
\]

Hence (3.8) and Lemma 2.4 yield

\[
\| (p - \tilde{p}) u(\tilde{p}, g_j, \tilde{a}_j, \tilde{b}_j) \|_{L^2(0,T;L^2(\Omega))} \leq C_{14} \sum_{j=1}^{N} D_j.
\]

Therefore

\[
\| a_j - \tilde{a}_j \|_{H^1(\Omega)} + \| b_j - \tilde{b}_j \|_{L^2(\Omega)} \leq C_{14} D_j.
\]

Thus, with (3.8), the proof of Theorem is completed.

**Appendix I. Proof of Theorem 0.**

We prove Theorem 0 by modifying the argument in Imanuvilov and Yamamoto [16], [17]. We show a key Carleman estimate (Imanuvilov [14]). We set

\[
P v = \partial_t^2 v - \Delta v - p(x)v, \quad x \in \Omega, \ t > 0.
\]

Let \( x_0 \in \mathbb{R}^n \setminus (\Omega \setminus \omega) \) satisfy \( \inf_{x' \in \mathbb{R}^n \setminus (\Omega \setminus \omega)} \sup_{x \in \Omega} |x - x'| = \sup_{x \in \Omega} |x - x_0| \).

For this \( x_0 \) and \( \beta \in (0, 1) \), we define functions \( \psi = \psi(x,t) \) and \( \varphi = \varphi(x,t) \) by

\[
\psi(x,t) = |x - x_0|^2 - \beta |t - T_0|^2, \quad \varphi(x,t) = e^{\lambda \psi(x,t)}
\]

with a parameter \( \lambda > 0 \). Let \( 0 < t_1 < T_0 \).

**Lemma I.1.** Let \( \| p \|_{L^\infty(\Omega)} \leq M \) and let us assume that \( t_1 > \sup_{x \in \Omega} |x - x_0| \).

Then there exists \( \lambda_0 > 0 \) such that for all \( \lambda > \lambda_0 \) there exist \( s_0 = s_0(\lambda) > 0 \) and a constant \( C = C(s_0, \lambda_0, M, \Omega, T_0, t_1, x_0, \omega) > 0 \) such that

\[
\int_{T_0-t_1}^{T_0+t_1} \int_{\Omega} (s|\nabla_x y|^2 + s^3 y^2)e^{2s\varphi} dx dt \\
\leq C \int_{T_0-t_1}^{T_0+t_1} \int_{\Omega} |P y|^2 e^{2s\varphi} dx dt + C \int_{T_0-t_1}^{T_0+t_1} \int_{\omega} (s|\partial_\omega y|^2 + s^3 y^2)e^{2s\varphi} dx dt
\]
for all \( s > s_0(\lambda) \), provided that

\[
\begin{aligned}
& P y \in L^2(\Omega \times (T_0 - t_1, T_0 + t_1)), \\
& y \in H^1(\Omega \times (T_0 - t_1, T_0 + t_1)), \\
& y = 0 \quad \text{on } \partial \Omega \times (T_0 - t_1, T_0 + t_1), \\
& y(\cdot, T_0 - t_1) = \partial_t y(\cdot, T_0 - t_1) = y(\cdot, T_0 + t_1) = \partial_t y(\cdot, T_0 + t_1) = 0 \quad \text{in } \Omega.
\end{aligned}
\]  

(I.1)

Case 1.

\[ u(p, g, a, b) = u(\tilde{p}, \tilde{g}, \tilde{a}, \tilde{b}) \quad \text{on } \partial \Omega \times (T_0 - t_1, T_0 + t_1). \]

We set \( Q = \Omega \times (T_0 - t_1, T_0 + t_1) \). Moreover setting \( \tilde{y} = u(p, g, a, b) - u(\tilde{p}, \tilde{g}, \tilde{a}, \tilde{b}), R(x, t) = u(\tilde{p}, \tilde{g}, \tilde{a}, \tilde{b}) \) and \( f = p - \tilde{p} \), we have

\[
\begin{aligned}
& \partial^2_t \tilde{y}(x, t) = \Delta \tilde{y} + p(x)\tilde{y}(x, t) + f(x)R(x, t), \quad (x, t) \in Q, \\
& \tilde{y}(x, t) = 0, \quad x \in \partial \Omega, \quad T_0 - t_1 < t < T_0 + t_1.
\end{aligned}
\]

Here \( \tilde{y}, R \in \bigcap_{j=0}^2 C^j([T_0 - t_1, T_0 + t_1]; H^{3-j}(\Omega)) \). We set

\[ y = \tilde{y} - (a - \tilde{a}) - (b - \tilde{b})(t - T_0) \quad \text{in } Q. \]

Then

\[
\begin{aligned}
& \partial^2_t y(x, t) = \Delta y + p(x)y(x, t) + f(x)R(x, t) + (\Delta + p(x))((a - \tilde{a}) + (b - \tilde{b})(t - T_0)) \quad \text{in } Q, \\
& y(x, t) = 0, \quad x \in \partial \Omega, \quad T_0 - t_1 < t < T_0 + t_1, \\
& y(x, T_0) = \partial_t y(x, T_0) = 0, \quad x \in \Omega.
\end{aligned}
\]  

(I.2)

By an a priori estimate (e.g., Lions and Magenes [34]) and \( \|R\|_{\bigcap_{j=0}^2 C^j([T_0 - t_1, T_0 + t_1]; H^{3-j}(\Omega))} \leq C_1 M \), by using

\[ \Delta y = \partial^2_t y - py - fR - (\Delta + p(x))((a - \tilde{a}) + (b - \tilde{b})(t - T_0)) \quad \text{in } Q, \]

there exists a constant \( C_2 > 0 \) such that

\[ \|y\|_{H^2(Q)} \leq C_2(\|f\|_{L^2(\Omega)} + \|a - \tilde{a}\|_{H^2(\Omega)} + \|b - \tilde{b}\|_{H^2(\Omega)}). \]  

(I.3)
By the assumption on $t_1$, there exists $\beta \in (0, 1)$ such that
\[
\beta > \frac{\sup_{x \in \Omega} |x - x_0|^2}{t_1^2}.
\]

Therefore, by the definitions of $\psi$ and $\varphi$, we have
\[
\varphi(x, T_0) \geq 1, \quad x \in \overline{\Omega},
\]
\[
\varphi(x, T_0 - t_1) = \varphi(x, T_0 + t_1) < 1, \quad x \in \overline{\Omega}.
\] (I.4)

Therefore for given $\varepsilon > 0$, we can choose a sufficiently small $\delta = \delta(\varepsilon) > 0$ such that
\[
\varphi(x, t) \geq 1 - \varepsilon, \quad (x, t) \in \overline{\Omega} \times [T_0 - \delta, T_0 + \delta]
\] (I.5)

and
\[
\varphi(x, t) \leq 1 - 2\varepsilon, \quad (x, t) \in \overline{\Omega} \times ([T_0 - t_1, T_0 - t_1 + 2\delta] \cup [T_0 + t_1 - 2\delta, T_0 + t_1]).
\] (I.6)

In order to apply Lemma I.1, we have to introduce a cut-off function $\chi \in C_0^\infty(\mathbb{R})$ satisfying $0 \leq \chi \leq 1$ and
\[
\chi(t) = \begin{cases} 0 & t \in [T_0 - t_1, T_0 - t_1 + \delta] \cup [T_0 + t_1 - \delta, T_0 + t_1] \\ 1 & t \in [T_0 - t_1 + 2\delta, T_0 + t_1 - 2\delta]. \end{cases}
\] (I.7)

Henceforth $C > 0$ denotes generic constants which are dependent on $s_0, \lambda, M, \Omega, T, x_0, \omega, \beta, \chi$ and $\delta, \varepsilon$, but independent of $s > s_0$.

We set
\[
z = (\partial_t y)e^{s\varphi}\chi \in C([T_0 - t_1, T_0 + t_1]; H^1(\Omega)) \cap C^1([T_0 - t_1, T_0 + t_1]; L^2(\Omega)).
\] (I.8)

By (I.2), the function $z$ satisfies the equation
\[
Pz = (f\partial_t R)e^{s\varphi}\chi + e^{s\varphi}\chi(\Delta + p)(b - \tilde{b}) + s(-2\nabla_x \varphi \cdot \nabla_x z + 2(\partial_t \varphi)\partial_t z + (\square \varphi)z) - s^2(\partial_t \varphi)^2 \cdot (\nabla_x \varphi)^2)z + 2e^{s\varphi}(\partial_t^2 y)\partial_t \chi + (\partial_t y)e^{s\varphi} \partial_t^2 \chi \quad \text{in } Q.
\] (I.9)
In fact, we have
\[
\partial_t z = (\partial_t \partial_t y)e^{s\varphi} \chi + (\partial_t y)s(\partial_t \varphi)e^{s\varphi} \chi = (\partial_t \partial_t y)e^{s\varphi} \chi + s(\partial_t \varphi)z. \tag{I.10}
\]

Hence
\[
(\partial_t \partial_t y)e^{s\varphi} \chi = \partial_t z - s(\partial_t \varphi)z. \tag{I.11}
\]

Moreover, the differentiation of (I.10) yields
\[
\partial_t^2 z = (\partial_t^2 \partial_t y)e^{s\varphi} \chi + s(\partial_t \varphi)e^{s\varphi}(\partial_t \partial_t y) \chi + (sz\partial_t^2 \varphi + s(\partial_t \varphi)\partial_t z) = (\partial_t y)e^{s\varphi} \chi + s\partial_t \varphi(\partial_t z - sz\partial_t \varphi) + (s(\partial_t^2 \varphi)z + s(\partial_t \varphi)\partial_t z).
\]

At the second equality, we used also (I.11). Hence
\[
\Delta z = (\Delta \partial_t y)e^{s\varphi} \chi + 2s\nabla \varphi \cdot \nabla z + (s\Delta \varphi - s^2|\nabla \varphi|^2)z.
\]

We have
\[
\partial_t z = (\partial_t^2 y)e^{s\varphi} \chi + (\partial_t y)e^{s\varphi} \partial_t \chi + s(\partial_t \varphi)z.
\]

Similarly we have
\[
\partial_t^2 z = (\partial_t^2 y)e^{s\varphi} \chi + \{(\partial_t^2 y)e^{s\varphi} \chi\} s\partial_t \varphi + 2e^{s\varphi}(\partial_t^2 y)\partial_t \chi
\]
\[
+s(\partial_t y)(\partial_t \varphi)e^{s\varphi} \partial_t \chi + (\partial_t y)e^{s\varphi} \partial_t^2 \chi + s(\partial_t^2 \varphi)z + s(\partial_t \varphi)(\partial_t z)
\]
\[
=(\partial_t^3 y)e^{s\varphi} \chi + \{\partial_t z - (\partial_t y)e^{s\varphi} \partial_t \chi - s(\partial_t \varphi)z\} s\partial_t \varphi + 2e^{s\varphi}(\partial_t^2 y)\partial_t \chi
\]
\[
+s(\partial_t y)(\partial_t \varphi)e^{s\varphi} \partial_t \chi + (\partial_t y)e^{s\varphi} \partial_t^2 \chi + s(\partial_t^2 \varphi)z + s(\partial_t \varphi)(\partial_t z)
\]
\[
=(\partial_t^3 y)e^{s\varphi} \chi + 2s(\partial_t \varphi)\partial_t z + (s\partial_t^2 \varphi - s^2|\partial_t \varphi|^2)z
\]
\[
+2e^{s\varphi}(\partial_t^2 y)\partial_t \chi + (\partial_t y)e^{s\varphi} \partial_t^2 \chi.
\]

Therefore, by (I.2), we obtain (I.9).
In particular, setting \( w = \chi(\partial_t y) \) and \( s = 0 \) in (I.9), we have

\[
Pw = \chi f \partial_t R + \chi(\Delta + p)(b - \tilde{b}) + 2(\partial_t^2 y)\partial_t \chi + (\partial_t^2 \chi)\partial_t y \quad \text{in } Q. \tag{I.12}
\]

Now we will apply Lemma I.1 to equation (I.12). By (I.2), (I.3), (I.7) and (I.12), we see that \( Pw \in L^2(Q), w \in H^1(Q), \) and \( w = 0 \) on \( \partial \Omega \times (T_0 - t_1, T_0 + t_1) \). Hence by Lemma I.1, we obtain

\[
\int_Q (s^2 w^2 + s|\nabla_{x,t} w|^2)e^{2s\varphi}dxdt \leq C_3 \int_Q \chi^2 |f\partial_t R|^2 e^{2s\varphi}dxdt
\]

\[
+C_3 \int_Q \chi^2 |(\Delta + p)(b - \tilde{b})|^2 e^{2s\varphi}dxdt + C_3 \int_Q \{(\partial_t \chi)\partial_t^2 y + (\partial_t^2 \chi)\partial_t y\}^2 e^{2s\varphi}dxdt + C_3 D(y). \tag{I.13}
\]

Here and henceforth we set

\[
D(y) \equiv \int_{T_0-t_1}^{T_0+t_1} \int_\omega \frac{s^2 w^2 + s|\partial_t w|^2}{2}e^{2s\varphi}dxdt
\]

\[
\leq C_3 s^2 \int_{T_0-t_1}^{T_0+t_1} \int_\omega (|\partial_t y|^2 + |\partial_t^2 y|^2) e^{2s\varphi}dxdt.
\]

By (I.3), (I.6) and (I.7), we have

\[
\left| \int_Q \{(\partial_t \chi)\partial_t^2 y + (\partial_t^2 \chi)\partial_t y\}^2 e^{2s\varphi}dxdt \right|
\]

\[
= \left| \left( \int_{T_0-t_1+2\delta}^{T_0-t_1+\delta} + \int_{T_0+t_1-\delta}^{T_0+t_1-2\delta} \right) \int_\Omega \{(\partial_t \chi)\partial_t^2 y + (\partial_t^2 \chi)\partial_t y\}^2 e^{2s\varphi}dxdt \right|
\]

\[
\leq C_4 e^{2s(1-2\varepsilon)}\|y\|^2_{H^2(Q)}
\]

\[
\leq C_4 e^{2s(1-2\varepsilon)}(\|f\|^2_{L^2(\Omega)} + \|a - \tilde{a}\|^2_{H^2(\Omega)} + \|b - \tilde{b}\|^2_{H^2(\Omega)}).
\tag{I.14}
\]

Noting that \( z = we^{s\varphi} \), we have

\[
s^3 z^2 = s^3 w^2 e^{2s\varphi},
\]

\[
s|\nabla_{x,t} z|^2 = s|\nabla_{x,t} w + s(\nabla_{x,t} \varphi)w|^2 e^{2s\varphi} \leq C_4 (s|\nabla_{x,t} w|^2 e^{2s\varphi} + s^3 w^2 e^{2s\varphi}).
\]
Therefore by (I.13) and (I.14), we obtain
\[
\int_Q (s^3 z^2 + s |\nabla x, t z|^2) dx dt \leq C_5 \int_Q \chi^2 |f|^2 e^{2s_\varphi} dx dt \\
+C_5 e^{C_5 s} (\|a - \tilde{a}\|_{H^2(\Omega)} + \|b - \tilde{b}\|_{H^2(\Omega)}^2) + C_5 e^{2s(1-2\varepsilon)} \|f\|_{L^2(\Omega)}^2 + C_5 D(y). \\
(I.15)
\]

We multiply (I.9) by \( \partial_t z \) and integrate it over \( \Omega \times (T_0 - t_1, T_0) \):
\[
\int_{T_0 - t_1}^{T_0} \int_{\Omega} (P z) \partial_t z dx dt = \int_{T_0 - t_1}^{T_0} \int_{\Omega} f(\partial_t R)e^{s_\varphi} \chi \partial_t z dx dt \\
+ \int_{T_0 - t_1}^{T_0} \int_{\Omega} ((\Delta + p) (b - \tilde{b})) e^{s_\varphi} \chi \partial_t z dx dt \\
+ \int_{T_0 - t_1}^{T_0} \int_{\Omega} \{s (-2 \nabla_x \varphi \cdot (\partial_t z) \nabla x z + 2(\partial_t \varphi) |\partial_t z|^2 + (\Box \varphi) z \partial_t z) \\
- s^2 (|\partial_t \varphi|^2 - |\nabla_x \varphi|^2) z \partial_t z \} dx dt + \int_{T_0 - t_1}^{T_0} \int_{\Omega} 2 e^{s_\varphi}(\partial_t^2 y) (\partial_t \chi) \partial_t z dx dt \\
+ \int_{T_0 - t_1}^{T_0} \int_{\Omega} (\partial_t y) e^{s_\varphi}(\partial_t^2 \chi) \partial_t z dx dt. \\
(I.16)
\]

We denote the left and the right hand sides of (I.16) respectively by \( I_1 \) and \( I_2 \).

By the boundary conditions and the conditions at \( t = T_0 \) in (I.2), noting that
\( z(\cdot, T_0 - t_1) = \partial_t z(\cdot, T_0 - t_1) = 0 \), we integrate by parts, so that we have
\[
I_1 = \int_{T_0 - t_1}^{T_0} \int_{\Omega} \frac{1}{2} \left( \frac{\partial |\partial_t z|^2}{\partial t} + \frac{\partial |\nabla z|^2}{\partial t} - p \frac{\partial |z|^2}{\partial t} \right) dx dt \\
- \int_{\partial \Omega} \int_{T_0 - t_1}^{T_0} \frac{\partial}{\partial \nu} ((\partial_t y) e^{s_\varphi} \chi) \partial_t ((\partial_t y) e^{s_\varphi} \chi) dtd\sigma \\
\geq \frac{1}{2} \int_{\Omega} |\partial_t z(x, T_0)|^2 dx.
\]

On the other hand, by (I.2) and (I.8), we see that
\[
\partial_t z(x, T_0) = ((\partial_t^2 y) e^{s_\varphi} \chi)(x, T_0) + (s(\partial_t \varphi)e^{s_\varphi}(\partial_t y) \chi)(x, T_0) + ((\partial_t y) e^{s_\varphi} \partial_t \chi)(x, T_0) \\
= (\Delta y + py + fR + (\Delta + p) (a - \tilde{a}))(x, T_0) e^{s_\varphi(x, T_0)} \chi(T_0) \\
= (\tilde{a} f + (\Delta + p)(a - \tilde{a}))(x) e^{s_\varphi(x, T_0)}, \quad x \in \Omega.
\]

Therefore
\[
I_1 \geq \frac{1}{2} \int_{\Omega} |\tilde{a}(x)|^2 |f(x)|^2 e^{2s_\varphi(x, T_0)} dx - C_6 e^{2s_\varphi} \|a - \tilde{a}\|_{H^2(\Omega)}^2. \\
(I.17)
\]
Furthermore by the Cauchy-Schwarz inequality, we have
\[
\int_{T_0-t_1}^{T_0} \int_{\Omega} 2e^{s\varphi}(\partial_t^2 y)(\partial_t \chi) \partial_t z \, dx \, dt \\
\leq \int_{T_0-t_1}^{T_0} \int_{\Omega} |\partial_t z|^2 \, dx \, dt + \int_{T_0-t_1}^{T_0} \int_{\Omega} e^{2s\varphi}|(\partial_t^2 y)\partial_t \chi|^2 \, dx \, dt,
\]
and again the Cauchy-Schwarz inequality yields
\[
I_2 \leq C_6 \int_{T_0-t_1}^{T_0} \int_{\Omega} |f|^2 e^{2s\varphi} \chi^2 \, dx \, dt + C_6 e^{C_6s} \|b - \tilde{b}\|^2_{H^2(\Omega)} + C_6 \int_{T_0-t_1}^{T_0} \int_{\Omega} (s|\nabla_{x,t} z|^2 + s^3 |z|^2) \, dx \, dt \\
+C_6 \int_{T_0-t_1}^{T_0} \int_{\Omega} |\partial_t y|^2 e^{2s\varphi} |\partial_t^2 \chi|^2 \, dx \, dt + \int_{T_0-t_1}^{T_0} \int_{\Omega} e^{2s\varphi}|(\partial_t^2 y)\partial_t \chi|^2 \, dx \, dt.
\]
By noting (I.3), (I.7) and the fact that \(\partial_t y \in L^2(\Omega \times (T_0 - t_1, T_0))\), application of (I.15) yields
\[
I_2 \leq C_7 \int_Q |f|^2 \chi^2 e^{2s\varphi} \, dx \, dt + C_7 e^{C_7s} \|b - \tilde{b}\|^2_{H^2(\Omega)} + \|a - \tilde{a}\|^2_{H^2(\Omega)} \\
+C_7 e^{2s(1-2\varepsilon)} \|f\|^2_{L^2(\Omega)} + C_7 D(y) \\
+C_7 \left( \int_{T_0-t_1+2\delta}^{T_0-t_1+\delta} + \int_{T_0-t_1-\delta}^{T_0-t_1-2\delta} \right) \int_{\Omega} |\partial_t y|^2 + |\nabla_{x,t} \partial_t y|^2 e^{2s\varphi} \, dx \, dt \\
\leq C_8 \int_Q |f|^2 e^{2s\varphi} \, dx \, dt + C_8 e^{C_8s} \|b - \tilde{b}\|^2_{H^2(\Omega)} + \|a - \tilde{a}\|^2_{H^2(\Omega)} \\
+C_8 e^{2s(1-2\varepsilon)} \|f\|^2_{L^2(\Omega)} + C_8 D(y).
\]
Consequently (I.4), (I.17) and (I.18) imply
\[
\int_{\Omega} |f(x)|^2 e^{2s\varphi(x,T_0)} \, dx \\
\leq C_9 \int_Q |f|^2 e^{2s\varphi} \, dx \, dt + C_9 e^{C_9s} \|b - \tilde{b}\|^2_{H^2(\Omega)} + \|a - \tilde{a}\|^2_{H^2(\Omega)} \\
+C_9 e^{2s(1-2\varepsilon)} \|f\|^2_{L^2(\Omega)} + C_9 D(y).
\]
On the other hand,
\[
\int_Q |f(x)|^2 e^{2s\varphi} \, dx \, dt = \int_{\Omega} \left( \int_{T_0-t_1}^{T_0+t_1} |f(x)|^2 \exp \left( 2se^{\lambda \psi(x,t)} \right) \, dt \right) \, dx \\
\leq \int_{\Omega} \left( \int_{T_0-t_1}^{T_0+t_1} \exp \left( 2se^{\lambda |x-x_0|^2} (e^{-\lambda \beta t^2} - 1) \right) \, dt \right) |f(x)|^2 e^{2s\varphi(x,T_0)} \, dx \\
\leq \int_{\Omega} \left( \int_{T_0-t_1}^{T_0+t_1} \exp \left( 2s(e^{-\lambda \beta t^2} - 1) \right) \, dt \right) |f(x)|^2 e^{2s\varphi(x,T_0)} \, dx.
\]
The Lebesgue theorem implies
\[
\int_{T_0-t_1}^{T_0+t_1} \exp \left( 2s(e^{-\lambda \beta t^2} - 1) \right) dt = o(1)
\]
as \( s \to \infty \), so that
\[
\int_Q |f(x)|^2 e^{2s\varphi} dt dx = o(1) \int_\Omega |f(x)|^2 e^{2s\varphi(x,T_0)} dx.
\]
Therefore
\[
\int_\Omega |f(x)|^2 e^{2s\varphi(x,T_0)} dx \\
\leq o(1) \int_\Omega |f(x)|^2 e^{2s\varphi(x,T_0)} dx + C_9 e^{C_9 s}(\|b - \tilde{b}\|^2_{H^2(\Omega)} + \|a - \tilde{a}\|^2_{H^2(\Omega)}) \\
+ C_9 e^{2s(1-2\epsilon)} \|f\|^2_{L^2(\Omega)} + C_9 D(y)
\]
as \( s \to \infty \). Hence
\[
(1 - o(1)) e^{2s} \int_\Omega |f(x)|^2 dx \\
\leq C_{10} e^{2s(1-2\epsilon)} \|f\|^2_{L^2(\Omega)} + C_{10} e^{C_{10} s}(\|b - \tilde{b}\|^2_{H^2(\Omega)} + \|a - \tilde{a}\|^2_{H^2(\Omega)}) + C_{10} D(y).
\]
By choosing \( s > 0 \) sufficiently large and fixing, the proof of Theorem 0 is completed in Case 1.

Case 2.

\[
u(p, g, a, b) \neq u(\tilde{p}, \tilde{g}, \tilde{a}, \tilde{b}) \quad \text{on} \quad \partial \Omega \times (T_0 - t_1, T_0 + t_1).
\]
By the Sobolev extension theorem, we can choose \( h \in C^3([T_0-t_1, T_0+t_1]; L^2(\Omega)) \cap C^1([T_0-t_1, T_0+t_1]; H^2(\Omega)) \) such that
\[
\|h\|_{C^3([T_0-t_1, T_0+t_1]; L^2(\Omega))} + \|h\|_{C^1([T_0-t_1, T_0+t_1]; H^2(\Omega))} \\
\leq C_{11}(\|u(p, g, a, b) - u(\tilde{p}, \tilde{g}, \tilde{a}, \tilde{b})\|_{C^3([T_0-t_1, T_0+t_1]; L^2(\omega))} \\
+ \|u(p, g, a, b) - u(\tilde{p}, \tilde{g}, \tilde{a}, \tilde{b})\|_{C^1([T_0-t_1, T_0+t_1]; H^2(\omega))})
\]
(I.19)
and
\[ h = u(p, g, a, b) - u(\tilde{p}, \tilde{g}, \tilde{a}, \tilde{b}) \quad \text{in} \ \omega \times (T_0 - t_1, T_0 + t_1). \]

We set
\[ y_1 = u(p, g, a, b) - u(\tilde{p}, \tilde{g}, \tilde{a}, \tilde{b}) - h \quad \text{in} \ Q. \]

Since \( \partial \omega \supset \partial \Omega \), we have
\[
\begin{cases}
\partial_t^2 y_1(x, t) = \Delta y_1 + p(x)y(x, t) + f(x)R(x, t) + (\partial_t^2 - \Delta - p)h \quad \text{in} \ Q, \\
y_1(x, t) = 0, \quad x \in \partial \Omega, \ T_0 - t_1 < t < T_0 + t_1, \\
y_1(x, T_0) = a - \tilde{a} - h(\cdot, T_0), \\
\partial_t y_1(x, T_0) = b - \tilde{b} - (\partial_t h)(\cdot, T_0).
\end{cases}
\]

In terms of (I.19), we repeat the argument in Case 1, so that the proof of Theorem 0 is completed.

**Appendix II. Proof of Lemma 2.4.**

We set \( u_j = \partial^j_t u \). By \( 2m \)-times taking \( t \)-derivatives, we have
\[
\begin{cases}
\partial_t^2 u_1(x, t) = \Delta u_1(x, t) + p(x)u_1(x, t), \quad x \in \Omega, \ 0 < t < T, \\
u_1(x, 0) = b(x), \quad \partial_t u_1(x, 0) = (\Delta + p)a(x), \quad x \in \Omega, \\
u_1(x, t) = \partial_t g(x, t), \quad x \in \partial \Omega, \ 0 < t < T, \\
\partial_t^2 u_2(x, t) = \Delta u_2(x, t) + p(x)u_2(x, t), \quad x \in \Omega, \ 0 < t < T, \\
u_2(x, 0) = (\Delta + p)a(x), \quad \partial_t u_2(x, 0) = (\Delta + p)b(x), \quad x \in \Omega, \\
u_2(x, t) = \partial^2_t g(x, t), \quad x \in \partial \Omega, \ 0 < t < T, \\
\vdots \\
\partial_t^2 u_{2m-1}(x, t) = \Delta u_{2m-1}(x, t) + p(x)u_{2m-1}(x, t), \quad x \in \Omega, \ 0 < t < T, \\
u_{2m-1}(x, 0) = (\Delta + p)^{m-1}b(x), \quad \partial_t u_{2m-1}(x, 0) = (\Delta + p)^ma(x), \quad x \in \Omega, \\
u_{2m-1}(x, t) = \partial^{2m-1}_t g(x, t), \quad x \in \partial \Omega, \ 0 < t < T,
\end{cases}
\]
and

\[
\begin{cases}
\partial_t^2 u_{2m}(x, t) = \Delta u_{2m}(x, t) + p(x)u_{2m}(x, t), \quad x \in \Omega, \ 0 < t < T, \\
u_{2m}(x, 0) = (\Delta + p)^m a(x), \quad \partial_t u_{2m}(x, 0) = (\Delta + p)^m b(x), \quad x \in \Omega, \\
u_{2m}(x, t) = \partial_t^{2m} g(x, t), \quad x \in \partial\Omega, \ 0 < t < T.
\end{cases}
\]

By \(a \in H^{2m}(\Omega), \ b \in H^{2m-1}(\Omega)\) and \(g \in H^{2m}(0, T; L^2(\partial\Omega))\), the transposition method (e.g., [28], [34]) yields that \(\partial_t^j u \in C([0, T]; L^2(\Omega))\) and

\[
\|\partial_t^j u\|_{C([0, T]; L^2(\Omega))} \leq C_1 \left(\|a\|_{H^{2m}(\Omega)} + \|b\|_{H^{2m-1}(\Omega)} + \|g\|_{H^{2m}(0, T; L^2(\partial\Omega))}\right), \quad j = 0, 1, \cdots, 2m.
\]

(II.1)

On the other hand, we have

\[
\Delta u_j(x, t) + p(x)u_j(x, t) = \partial_t^2 u_j(x, t), \quad x \in \Omega, \ j = 0, 1, \cdots, 2m - 2 \quad (\text{II.2})
\]

for any \(t \in [0, T]\). Similarly to p.414 in [33], we can choose \(\mu \in W^{1, \infty}_0(\Omega)\) such that

\[
0 \leq \mu \leq 1 \text{ in } \Omega, \mu = 1 \text{ in } \Omega' \text{ and }
\]

\[
\frac{|\nabla \mu(x)|^2}{\mu(x)} \leq C_2(\Omega'), \quad x \in \Omega.
\]

(II.3)

Multiplying (II.2) with \(\mu u_j\), integrating in \(x\) and using the Green formula, we have

\[
\int_{\Omega} \mu |\nabla u_j(x, t)|^2 dx
= - \int_{\Omega} u_j(x, t) \nabla u_j(x, t) \cdot \nabla \mu(x) dx + \int_{\Omega} \frac{p u_j^2}{\mu} \mu dx - \int_{\Omega} \mu u_j \partial_t^2 u_j dx.
\]

Therefore the Cauchy-Schwarz inequality yields

\[
\int_{\Omega} \mu |\nabla u_j(x, t)|^2 dx \leq \int_{\Omega} \sqrt{\mu} |\nabla u_j| \frac{|\nabla \mu|}{\sqrt{\mu}} |u_j| dx + C_3 \int_{\Omega} |u_j|^2 dx + \int_{\Omega} |u_j| |\partial_t^2 u_j| dx
\]

\[
\leq \frac{1}{2} \int_{\Omega} \mu |\nabla u_j|^2 dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla \mu|^2}{\mu} |u_j|^2 dx
\]

\[
+ C_3 \int_{\Omega} |u_j|^2 dx + \frac{1}{2} \int_{\Omega} (|u_j|^2 + |\partial_t^2 u_j|^2) dx.
\]
By (II.3), we have
\[
\frac{1}{2} \int_{\Omega'} |\nabla \partial_t^j u(x, t)|^2 dx \leq \frac{1}{2} \int_{\Omega} |\nabla \partial_t^j u(x, t)|^2 dx
\]
\[
\leq C_4 \int_{\Omega} |\partial_t^j u|^2 dx + C_4 \int_{\Omega} |\partial_t^{j+2} u|^2 dx, \quad j = 0, ..., 2m - 2.
\]
Thus the proof of (2.10) is completed.

We choose subdomains $\Omega_1, ..., \Omega_{m-1}$ such that $\overline{\Omega'} \subset \Omega_{m-1} \subset \overline{\Omega_{m-2}} \subset \Omega_{m-2} \subset ... \subset \Omega_1 \subset \overline{\Omega_1} \subset \Omega'$. First $(\Delta + p) \partial_t^{2m-2} u(\cdot, t) = \partial_t^{2m} u(\cdot, t)$ in $\Omega$. By (2.10) and (II.1), applying the interior regularity (e.g., Theorem 8.8 (pp.183-184) in [12]), we have
\[
\|
partial_t^{2m-2} u(\cdot, t)\|_{H^2(\Omega_1)} \leq C_5(\|a\|_{H^{2m}(\Omega)} + \|b\|_{H^{2m-1}(\Omega)} + \|g\|_{H^{2m}(0,T;L^2(\partial\Omega))}). \quad (II.4)
\]
Next $(\Delta + p) \partial_t^{2m-4} u(\cdot, t) = \partial_t^{2m-2} u(\cdot, t)$ in $\Omega$. In terms of (II.4) and (2.10), we apply again the interior regularity (e.g., Theorem 8.10 (p.186) in [12]), so that
\[
\|
partial_t^{2m-4} u(\cdot, t)\|_{H^4(\Omega_2)} \leq C_6(\|a\|_{H^{2m}(\Omega)} + \|b\|_{H^{2m-1}(\Omega)} + \|g\|_{H^{2m}(0,T;L^2(\partial\Omega))}).
\]
Continuing the argument, we complete the proof of (2.11).

Appendix III. Proof of Lemma 2.5.

First, by a usual energy estimate, we can prove that there exists a constant $C_1 > 0$ such that
\[
E(t) \leq C_1 E(t') + C_1 \|f\|^2_{L^2(0,T;L^2(\Omega))} \quad \text{for any } t, t' \in [0,T]. \quad (III.1)
\]
Henceforth we set $Q_1 = \Omega \times (0,T)$, $T = 2t_0$ and
\[
E(t) = \int_{\Omega} (|\nabla w(x, t)|^2 + |\partial_t w(x, t)|^2) dx,
\]
and $C_j > 0$ denote generic constants which are dependent on $M, \Omega, T$. In terms of (III.1), it is sufficient to prove that
\[
\|w(\cdot, t_0)\|_{H^1(\Omega)} + \|\partial_t w(\cdot, t_0)\|_{L^2(\Omega)} \leq C(\|f\|_{L^2(0,T;L^2(\Omega))} + \|w\|_{H^1(0,T;L^2(\omega)))}.
\]
We choose $x_0 \in \mathbb{R}^n \setminus (\Omega \setminus \omega)$ such that
\[
\sup_{x \in \Omega} |x - x_0| = \inf_{x' \in \mathbb{R}^n \setminus (\Omega \setminus \omega)} \sup_{x \in \Omega} |x - x'|.
\]

By $t_0 > \sup_{x \in \Omega} |x - x_0|$, we can choose $\beta \in (0, 1)$ such that
\[
\beta > \frac{\sup_{x \in \Omega} |x - x_0|^2}{t_0^2}.
\]

Let
\[
\psi(x, t) = |x - x_0|^2 - \beta|t - t_0|^2, \quad \varphi(x, t) = e^{\lambda \psi(x, t)}.
\]

Then $\varphi(x, t_0) \geq 1$ and $\varphi(x, 0) = \varphi(x, 2t_0) < 1$ for $x \in \overline{\Omega}$. Therefore for given $\varepsilon > 0$, we can choose a sufficiently small $\delta = \delta(\varepsilon) > 0$ such that
\[
\begin{cases}
\varphi(x, t) \geq 1 - \varepsilon, & (x, t) \in \overline{\Omega} \times [t_0 - \delta, t_0 + \delta], \\
\varphi(x, t) \leq 1 - 2\varepsilon, & (x, t) \in \overline{\Omega} \times ([0, 2\delta] \cup [2t_0 - 2\delta, 2t_0]).
\end{cases} \tag{III.2}
\]

We take $\chi_1 \in C_0^\infty(\mathbb{R})$ satisfying $0 \leq \chi_1 \leq 1$ and
\[
\chi_1(t) = \begin{cases} 0 & t \in [0, \delta] \cup [2t_0 - \delta, 2t_0] \\
1 & t \in [2\delta, 2t_0 - 2\delta]. \end{cases} \tag{III.3}
\]

We set $w_1 = \chi_1 w$. Then
\[
\begin{cases}
\partial_t^2 w_1(x, t) = \Delta w_1 + p(x)w_1(x, t) + \chi_1 f(x, t) + 2(\partial_t \chi_1)\partial_t w + w \partial_t^2 \chi_1 & \text{in } Q_1, \\
w_1(x, t) = 0, & x \in \partial \Omega, 0 < t < 2t_0, \\
w_1(\cdot, 0) = w_1(\cdot, 2t_0) = \partial_t w_1(\cdot, 0) = \partial_t w_1(\cdot, 2t_0) = 0 & \text{in } \Omega.
\end{cases}
\]

Here, setting $T = t_0$, we apply Lemma I.1 in Appendix I to $w_1$, so that
\[
\int_{Q_1} (s|\nabla_{x,t} w_1|^2 + s^3|w_1|^2)e^{2s\varphi}dxdt \leq C_2 \int_{Q_1} |\chi_1 f|^2 e^{2s\varphi}dxdt
\]
\[
+ C_2 \int_{Q_1} (|\partial_t \chi_1| \partial_t |w|^2 + |w_1 \partial_t^2 \chi_1|^2)e^{2s\varphi}dxdt
\]
\[
+ C_2 s^3 \int_0^{2t_0} \int_{\omega} (|\partial_t w_1|^2 + |w_1|^2)e^{2s\varphi}dxdt
\]
for all large $s > 0$. The second term on the right hand side does not vanish only if
\[ \partial_t \chi_1, \partial_t^2 \chi_1 \neq 0, \]
so that in terms of (III.2) and (III.3), it is bounded by
\[ C_2 \int_0^{2t_0} E(t) dt e^{2s(1-2\varepsilon)}. \]

Hence, by (III.1), we have
\[ \int_\Omega \int_{t_0-\delta}^{t_0+\delta} (|\nabla_{x,t} w|^2 + |w|^2) e^{2s\varepsilon} dx dt \leq C_3 e^{C_3 s} \| f \|^2_{L^2(Q_1)} \]
\[ + C_3 e^{2s(1-2\varepsilon)} (E(t_0) + \| f \|^2_{L^2(Q_1)}) + C_3 e^{C_3 s} \| w \|^2_{H^1(0,T;L^2(\omega))}. \]

By (III.2) we see that
\[ e^{2s(1-\varepsilon)} \int_{t_0-\delta}^{t_0+\delta} E(t) dt \leq C_4 e^{C_4 s} \| f \|^2_{L^2(Q_1)} \]
\[ + C_4 e^{2s(1-2\varepsilon)} (E(t_0) + \| f \|^2_{L^2(Q_1)}) + C_4 e^{C_4 s} \| w \|^2_{H^1(0,T;L^2(\omega))}. \]

By (III.1) the left hand side is greater than or equal to
\[ e^{2s(1-\varepsilon)} (E(t_0) - \| f \|^2_{L^2(Q_1)})2\delta. \]

Noting the Poincaré inequality and choosing $s > 0$ sufficiently large and fixing, we complete the proof of Lemma 2.5.

**References**


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