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by

Rolci Cipolatti and Masahiro Yamamoto

UNIVERSITY OF TOKYO
GRADUATE SCHOOL OF MATHEMATICAL SCIENCES
KOMABA, TOKYO, JAPAN

INVERSE HYPERBOLIC PROBLEM BY A FINITE TIME OF OBSERVATIONS WITH ARBITRARY INITIAL VALUES

Rolci Cipolatti [1] and Masahiro Yamamoto[2]

[1] Departamento de Métodos Matemáticos
Instituto de Matemática - Universidade Federal do Rio de Janeiro
Ilha do Fundão - Caixa Postal 68530 - CEP 21941-972
Rio de Janeiro, Brasil
e-mail: cipolatti@im.ufrj.br
[2] Department of Mathematical Sciences, The University of Tokyo
Komaba Meguro Tokyo 153-8914 Japan
e-mail:myama@ms.u-tokyo.ac.jp

Abstract. We consider a solution $u(p,g,a,b)$ to an initial value-boundary value problem for a hyperbolic equation:

$$\partial_t^2 u(x,t) = \Delta u(x,t) + p(x)u(x,t), \quad x \in \Omega,\, 0 < t < T$$

$$u(x,0) = a(x), \quad \partial_t u(x,0) = b(x), \qquad x \in \Omega,$$

$$u(x,t) = g(x,t), \qquad x \in \partial\Omega,\, 0 < t < T.$$

and we discuss an inverse problem of determining a coefficient $p(x)$ and a,b by observations of $u(p,g,a,b)(x,t)$ in a neighbourhood ω of $\partial\Omega$ over a time interval $(0,T)$ and $u(p,g,a,b)(x,T_0)$, $\partial_t u(p,g,a,b)(x,T_0)$, $x \in \Omega$ with $T_0 < T$. We prove that if $T - T_0$ and T_0 are larger than the diameter of Ω, then we can choose a finite number of Dirichlet boundary inputs $g_1, ..., g_N$ by the Hilbert Uniqueness Method, so that the mapping

$$\{u(p,g_j,a_j,b_j)|_{\omega\times(0,T)}, u(p,g_j,a_j,b_j)(\cdot,T_0), \partial_t u(p,g_j,a_j,b_j)(\cdot,T_0)\}_{1\le j\le N} \longrightarrow \{p,a_j,b_j\}_{1\le j\le N}$$

is uniformly Lipschitz continuous with suitable Sobolev norms provided that $\{p,a_j,b_j\}_{1\le j\le N}$ remains some bounded set in a suitable Sobolev space. In our inverse problem, initial values are also unknown, and we do not assume any positivity of initial values at all. Our key is a Carleman estimate and the exact controllability in a Sobolev space of higher order.

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§1. Introduction.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ and let us consider

$$\partial_t^2 u(x,t) = \Delta u(x,t) + p(x)u(x,t), \qquad x \in \Omega,\, t > 0, \tag{1.1}$$

$$u(x,0) = a(x), \quad \partial_t u(x,0) = b(x), \qquad x \in \Omega \tag{1.2}$$

and

$$u(x,t) = g(x,t), \qquad x \in \partial\Omega,\, t > 0 \tag{1.3}$$

Let $g \in L^2(\partial\Omega \times (0,T))$ be given and smooth suitably. Then by $u = u(p,g,a,b)(x,t)$ we denote the solution to (1.1) - (1.3) within a suitable class which is described later. Let $\nu = \nu(x)$ be the unit outward normal vector to $\partial\Omega$ at x, $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$, and let $\omega \subset \Omega$ be a subdomain.

In this paper, we discuss an inverse problem of determining $p = p(x)$, $x \in \Omega$ by some observations. That is,

Inverse Problems. Let $0 < T_0 < T_1$ be given. Given $g_1, ..., g_N \in L^2(\partial\Omega \times (0,T))$, determine $p(x)$, $a_j(x)$, $b_j(x) \in \Omega$, $1 \le j \le N$, by

$$u(p,g_j,a_j,b_j)|_{\omega\times(0,T))}, \qquad u(p,g_j,a_j,b_j)(x,T_0), \quad x \in \Omega, \qquad j = 1, ..., N.$$

This is an inverse problem by a finite time of interior observations. As long as inverse problems of determining coefficients in multidimensions by a finite number of observations without smallness are concerned, one main methodology is by a Carleman estimate, which was initiated by Bukhgeim and Klibanov [10]. See also Baudouin and Puel [1], Bellassoued [3], [4], Bellassoued and Yamamoto [6], [7], Bellassoued, Imanuvilov and Yamamoto [8], Imanuvilov, Isakov and Yamamoto [15], Imanuvilov and Yamamoto [16-19], Isakov [20], [21], Klibanov [23], [24], Klibanov and Timonov [26], Klibanov and Yamamoto [27], Puel and Yamamoto [35], [36], Yamamoto [40].

For example, by a method in Imanuvilov and Yamamoto [16], [17], we can prove

Theorem 0. *Let a constant $M > 0$ and a smooth function η on $\overline{\Omega}$ be arbitrarily given. Let $0 < t_1 < T_0$ and $\omega \subset \Omega$ be a subdomain such that $\partial\omega \supset \partial\Omega$. We set*

$$\mathcal{U}_0 = \{p \in W^{1,\infty}(\Omega);\ \|p\|_{W^{1,\infty}(\Omega)} \le M,\ p = \eta \ in\ \omega\}.$$

Let $u = u(p,g,a,b) \in \cap_{j=0}^2 C^j([T_0 - t_1, T_0 + t_1]; H^{3-j}(\Omega))$ satisfy (1.1),

$$u(x,T_0) = a(x), \quad \partial_t u(x,T_0) = b(x), \qquad x \in \Omega$$

and

$$u|_{\partial\Omega\times(T_0-t_1,T_0+t_1)} = g$$

with suitable functions a, b, g. Moreover let

$$\|u(p,g,a,b)\|_{C^j([T_0-t_1,T_0+t_1];H^{3-j}(\Omega))},$$

$$\|u(\widetilde{p},\widetilde{g},\widetilde{a},\widetilde{b})\|_{C^j([T_0-t_1,T_0+t_1];H^{3-j}(\Omega))} \le M, \quad j = 0,1,2.$$

We assume that there exists a constant $\delta_0 > 0$ such that

$$|a(x)|, \quad |\widetilde{a}(x)| \ge \delta_0 \qquad on\ \overline{\Omega \setminus \omega} \tag{1.4}$$

and that

$$t_1 > \inf_{x' \in \mathbb{R}^n \setminus (\Omega\setminus\omega)} \sup_{x\in\Omega} |x - x'|.$$

Then there exists a constant $C = C(M, \delta_0, \omega, T_0, t_1) > 0$ such that

$$\|p - \widetilde{p}\|_{L^2(\Omega)} \le C\Bigg(\|u(p,g,a,b) - u(\widetilde{p},\widetilde{g},\widetilde{a},\widetilde{b})\|_{C^3([T_0-t_1,T_0+t_1];L^2(\omega))}$$
$$+ \|u(p,g,a,b) - u(\widetilde{p},\widetilde{g},\widetilde{a},\widetilde{b})\|_{C^1([T_0-t_1,T_0+t_1];H^2(\omega))}$$
$$+ \|a - \widetilde{a}\|_{H^2(\Omega)} + \|b - \widetilde{b}\|_{H^2(\Omega)} \Bigg)$$

for all $p, \widetilde{p} \in \mathcal{U}_0$.

As for the Lipschitz stability without assumption that $p, \widetilde{p}$ are given in ω, see for example Imanuvilov and Yamamoto [16], [17]. For completeness, we will prove Theorem 0 in Appendix I.

Here we assume that coefficients p and $\widetilde{p}$ are given in a boundary layer ω and we determine them in $\Omega \setminus \omega$ by observations of the solutions in $\omega \times (T_0 - t_1, T_0 + t_1)$. For the Lipschitz stability in our inverse problem, we have to assume the strict positivity (1.4) on $\overline{\Omega \setminus \omega}$ of the given displacement at $t = T_0$. The Lipschitz stability is mathematically satisfactory, but such a positivity condition is quite restrictive, because (1.4) requires that we have to control the spatial distribution of the state over a domain where coefficients are uknown. There have been many trials for relaxing (1.4) (e.g., $a \not\equiv 0$ in Ω), but it remains a serious open problem. We note that with impulsive force terms represented by a Dirac delta function, the Lipschitz stability results are known with $a \equiv 0$ with some smallness assumption and see Glushkova [13], Li [32], Romanov [37], Romanov and Yamamoto [38].

On the other hand, by the Dirichlet-to-Neumann map requiring infinitely many observations, we know the sharp uniqueness results (e.g., Belishev [2], Kurylev and Lassas [30]) and stability (e.g., Bellassoued, Jellali and Yamamoto [5], Cipolatti and F.Lopez [11], Sun [39]).

This paper gives a partial answer to the longstanding open problem: the Lipschitz stability by a finite number of observations without any positivity assumptions. The characters of our inverse problem are:

(1) We take observations N times by suitably choosing boundary inputs $g_1, ..., g_N$.

(2) Initial values at $t = 0$ are unknown, while we have to observe displacements

and velocities at a fixed intermediate time T_0 corresponding to $g_1, ..., g_N$.

We emphasize that we need not assume any positivity of functions in Ω, and we assume that $g_1, ..., g_N$ are at our disposal.

For the statement of our main result, we intoduce notations. Let $\partial_j = \frac{\partial}{\partial x_j}$, $1 \le j \le n$, $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N} \cup \{0\}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\partial_x^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. Let $C^{\ell,1}(\overline{\Omega})$, $\ell \in \mathbb{N}$, be the space of all the functions whose derivatives of orders $\le \ell$ are all Lipschitz continuous on $\overline{\Omega}$ and we set $\|p\|_{C^{\ell,1}(\overline{\Omega})} = \|p\|_{C^1(\overline{\Omega})} + \sup_{|\alpha|=\ell, x, x' \in \Omega, x \ne x'} \frac{|\partial_x^\alpha p(x) - \partial_x^\alpha p(x')|}{|x-x'|}$. Let $M > 0$ be arbitrarily fixed and let $\omega \subset \Omega$ be a subdomain such that $\partial\omega \supset \partial\Omega$, η be an arbitrarily given smooth function. Let $m \in \mathbb{N}$ satisfy

$$m > \frac{n}{4}.$$

Let

$$\mathcal{U} = \{p \in C^{2m}(\overline{\Omega});\ \|p\|_{C^{2m}(\overline{\Omega})} \le M,\ p = \eta \quad \text{in } \omega,\ p \le 0 \,\text{in}\, \Omega\}$$

and

$$\mathcal{V} = \{(a, b) \in H_0^{2m}(\Omega) \times H_0^{2m-1}(\Omega);\ \|a\|_{H^{2m}(\Omega)}, \|b\|_{H^{2m-1}(\Omega)} \le M\}.$$

We define a Hilbert space

$$V_{2m} = \{g \in H^{2m}(0, T; L^2(\partial\Omega));\ \partial_t^j g(\cdot, T_0) = 0,\ j = 0, 1, 2, \cdots, 2m-1\}$$

with the scalar product $(g, h)_{V_{2m}} = (\partial_t^{2m} g, \partial_t^{2m} h)_{L^2(0,T_0;L^2(\partial\Omega))}$.

Now we are ready to state our main result.

Theorem. *Let*

$$\begin{cases} T - T_0 > \inf_{x' \in \mathbb{R}^n \setminus (\Omega \setminus \omega)} \sup_{x \in \Omega} |x - x'|, \\ T_0 > \inf_{x' \in \mathbb{R}^n \setminus (\Omega \setminus \omega)} \sup_{x \in \Omega} |x - x'|. \end{cases} \tag{1.5}$$

For $M>0$, we can choose $N\in\mathbb{N}$ and $g_1,...,g_N\in V_{2m}$ satisfying:

$$
\begin{aligned}
&\|p-\widetilde{p}\|_{L^2(\Omega)}+\sum_{j=1}^N\|a_j-\widetilde{a}_j\|_{H^1(\Omega)}+\sum_{j=1}^N\|b_j-\widetilde{b}_j\|_{L^2(\Omega)}\\
\le& C\left(\sum_{j=1}^N\|u(p,g_j,a_j,b_j)-u(\widetilde{p},g_j,\widetilde{a}_j,\widetilde{b}_j)\|_{C^3([0,T];L^2(\omega))}\right.\\
&+\sum_{j=1}^N\|u(p,g_j,a_j,b_j)-u(\widetilde{p},g_j,\widetilde{a}_j,\widetilde{b}_j)\|_{C^1([0,T];H^2(\omega))}\\
&+\sum_{j=1}^N\|(u(p,g_j,a_j,b_j)-u(\widetilde{p},g_j,\widetilde{a}_j,\widetilde{b}_j))(\cdot,T_0)\|_{H^2(\Omega)}\\
&\left.+\sum_{j=1}^N\|(\partial_t u(p,g_j,a_j,b_j)-\partial_t u(\widetilde{p},g_j,\widetilde{a}_j,\widetilde{b}_j))(\cdot,T_0)\|_{H^2(\Omega)}\right)
\end{aligned}
$$

for any $(a_j,b_j),(\widetilde{a}_j,\widetilde{b}_j)\in\mathcal{V}$ and $p,\widetilde{p}\in\mathcal{U}$.

The theorem asserts that we can choose N boundary inputs yielding the Lipschitz stability in estimating $p-\widetilde{p}$ when both p and $\widetilde{p}$ vary in the admissible set. The essence of our main result is the uniform choice in p of the boundary inputs guaranteeing the Lipschitz stability. By the classical exact controllability for the hyperbolic equation (e.g., Komornik [28], Lions [33]), we can directly see: if we will estimate $p-\widetilde{p}$ with a fixed $\widetilde{p}\in\mathcal{U}$, then the Lipschitz stability holds with a single suitable input g. Moreover precisely, we assume (1.5). For any given $\widetilde{p}\in\mathcal{U}$, there exists $g\in V_{2m}$ such that

$$
\begin{aligned}
&\|p-\widetilde{p}\|_{L^2(\Omega)}+\|a-\widetilde{a}\|_{H^1(\Omega)}+\|b-\widetilde{b}\|_{L^2(\Omega)}\\
\le& C\Big(\|u(p,g,a,b)-u(\widetilde{p},g,\widetilde{a},\widetilde{b})\|_{C^3([0,T];L^2(\omega))}+\|u(p,g,a,b)-u(\widetilde{p},g,\widetilde{a},\widetilde{b})\|_{C^1([0,T];H^2(\omega))}\\
&+\|(u(p,g,a,b)-u(\widetilde{p},g,\widetilde{a},\widetilde{b}))(\cdot,T_0)\|_{H^2(\Omega)}\\
&+\|(\partial_t u(p,g,a,b)-\partial_t u(\widetilde{p},g,\widetilde{a},\widetilde{b}))(\cdot,T_0)\|_{H^2(\Omega)}\Big)
\end{aligned}
$$

for any $(a,b),(\widetilde{a},\widetilde{b})\in\mathcal{V}$ and $p\in\mathcal{U}$.

As is also seen from the proof;

(1) The number N of observations increases as the a priori bound $M > 0$ is larger.

(2) The effective boundary inputs $g_1, ..., g_N$ can be constructed by the Hilbert Uniqueness Method (e.g., Komornik [28], Lions [33]). See the proof in Section 3 as for the construction. Moreover choices of such inputs $g_1, ..., g_N$ are rather generous (Remark in Section 3).

The paper is composed of three sections. In Section 2, we show a necessary exact controllability in a Sobolev spaces of higher orders, which may be an independent interest. In Section 3, we prove Theorem.

§2. Exact controllability in $H^{2m}(\Omega)$.

In the state space $L^2(\Omega) \times H^{-1}(\Omega)$ for a hyperbolic equation, the exact controllabililty has been studied extensively. Here we only refer to a ver few works: Komornik [28], Lasiecka and Triggiani [31], Lions [33] and the readers can consult them for comprehensive references. For the proof of our result, we need the exact controllability of displacement in Sobolev spaces of higher orders. The exact controllability in $H_0^1(\Omega) \times L^2(\Omega)$ has been discussed in Section 6 of Chapitre 1 in Komornik and Yamamoto [29], Lions [33] for $\partial_t^2 - \Delta$.

Let $\omega \subset \Omega$ be a subdomain such that $\partial\omega \supset \partial\Omega$ and let

$$T_0 > \inf_{x' \in \mathbb{R}^n \setminus (\Omega\setminus\omega)} \sup_{x\in\Omega} |x - x'|. \tag{2.1}$$

Henceforth let $x_0 \in \mathbb{R}^n \setminus \overline{(\Omega \setminus \omega)}$ satisfy

$$\inf_{x' \in \mathbb{R}^n \setminus (\Omega\setminus\omega)} \sup_{x\in\Omega} |x - x'| = \sup_{x\in\Omega} |x - x_0| < T_0.$$

Here $(\cdot, \cdot)$ and $\| \cdot \|$ denote the scalar product and the norm in $L^2(\Omega)$. Let $p \in \mathcal{U}_1 = \{p;\ \|p\|_{C^{2m-2,1}(\overline{\Omega})} \le M,\ p \le 0 \quad \text{in } \Omega\}$ be fixed and let us define an operator

A by $Au(x) = -\Delta u(x) - p(x)u(x)$ for $x \in \Omega$ and $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Then A^{-1} exists and $\sigma(A)$ is entirely composed of eigenvalues with finite multiplicities. Let $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$ be the sequence of all the eigenvalues of A where λ_j appears repeatedly k_j-times, where k_j is the multiplicity of λ_j. Let φ_j be a unit eigenvector of A for λ_j. Then $\{\varphi_j\}_{j\in\mathbb{N}}$ is an orthonormal basis in $L^2(\Omega)$ and

$$A^{-m}a = \sum_{j=1}^{\infty} \frac{(a,\varphi_j)}{\lambda_j^m}\varphi_j$$

in $L^2(\Omega)$.

Since $p \in C^{2m-2,1}(\overline{\Omega})$, elliptic regularity results (e.g., Theorem 8.13 (p.187) in Gilbarg and Trudinger [12]) yield

$$C_1^{-1}\|Au\| \le \|u\|_{H^2(\Omega)} \le \|Au\|, \quad u \in \mathcal{D}(A).$$

Next, using $\|A^{k-1}u\| \le C_1'\|A^k u\|$ for $u \in \mathcal{D}(A^k)$ and $k \in \mathbb{N}$, we repeatedly apply Theorem 8.13 in [12] and we see

$$C_1^{-1}\|A^m u\| \le \|u\|_{H^{2m}(\Omega)} \le C_1\|A^m u\|, \quad u \in \mathcal{D}(A^m). \tag{2.2}$$

Here we note that the constant $C_1 > 0$ can be taken uniformly in $p \in \mathcal{U}_1$. In particular, $D(A^m) \subset H^{2m}(\Omega)$, $m \in \mathbb{N}$. We regard $\mathcal{D}(A^m)$ as a Banach space with the norm

$$\|a\|_{\mathcal{D}(A^m)} = \|A^m a\| = \left(\sum_{j=1}^{\infty} \lambda_j^{2m}(a,\varphi_j)^2\right)^{\frac{1}{2}}.$$

Moreover, for $\gamma \ge 0$, we can see that

$$\begin{aligned} &A^\gamma a = \sum_{j=1}^{\infty} \lambda_j^\gamma (a,\varphi_j)\varphi_j, \\ &\mathcal{D}(A^\gamma) = \{a \in L^2(\Omega);\ \sum_{j=1}^{\infty} \lambda_j^{2\gamma}(a,\varphi_j)^2 < \infty\}. \end{aligned} \tag{2.3}$$

Identifying a Banach space $D(A^0) = L^2(\Omega)$ with its dual $(L^2(\Omega))^*$, we have the dense and continuous embedding

$$\mathcal{D}(A^m) \subset L^2(\Omega) \subset \mathcal{D}(A^m)^*.$$

Henceforth $B(X, Y)$ denotes the Banach space of all bounded linear operators from a Banach space X to another Banach space Y, L^* denotes the dual operator of $L \in B(X, Y)$. We regard A^m as a bounded linear operator from $\mathcal{D}(A^m)$ to $L^2(\Omega)$: $A^m \in B(\mathcal{D}(A^m), L^2(\Omega))$. Then $(A^m)^* \in B(L^2(\Omega), (\mathcal{D}(A^m))^*)$. We denote the duality pairing between $\mathcal{D}(A^m)^*$ and $\mathcal{D}(A^m)$ by ${}_{(\mathcal{D}(A^m))^*} < \cdot, \cdot >_{\mathcal{D}(A^m)}$. We note that ${}_{(\mathcal{D}(A^m))^*} < \varphi, \psi >_{\mathcal{D}(A^m)} = (\varphi, \psi)$ if $\varphi, \psi \in L^2(\Omega)$ (e.g., Brezis [9]).

Moreover we introduce a Banach space

$$V_{2m} = \{g \in H^{2m}(0, T_0; L^2(\partial\Omega));\ \partial_t^j g(\cdot, T_0) = 0,\ j = 0, 1, 2, \cdots, 2m-1\}$$

with the norm $\|g\|_{V_{2m}} = \|\partial_t^{2m} g\|_{L^2(0,T_0;L^2(\partial\Omega))}$. Identifying a Banach space $L^2(0, T_0; L^2(\partial\Omega))$ with its dual, we have the dense and continuous embedding

$$V_{2m} \subset L^2(0, T_0; L^2(\partial\Omega)) \subset V_{2m}^*.$$

We denote the duality pairing between $g \in V_{2m}^*$ and $h \in V_{2m}$ by $< g, h >$ and we note that $< g, h > = (g, h)_{L^2(0,T_0;L^2(\partial\Omega))}$ for $g, h \in L^2(0, T_0; L^2(\partial\Omega))$.

Then

Lemma 2.1.

$$(A^m)^* a = A^m a, \qquad a \in C_0^\infty(\Omega).$$

Proof. By the definition of the dual operator $(A^m)^*$, we have

$${}_{(\mathcal{D}(A^m))^*} < (A^m)^* a, f >_{\mathcal{D}(A^m)} = (a, A^m f), \qquad f \in \mathcal{D}(A^m).$$

Since $a \in C_0^\infty(\Omega) \subset \mathcal{D}(A^m)$, we see for example by using (2.3), that

$$_{(\mathcal{D}(A^m))^*} < (A^m)^* a, f >_{\mathcal{D}(A^m)} = (A^m a, f).$$

By $f, A^m a \in L^2(\Omega)$, we see that $(A^m a, f) = _{(\mathcal{D}(A^m))^*}< A^m a, f >_{\mathcal{D}(A^m)}$, so that $_{(\mathcal{D}(A^m))^*} < (A^m)^* a, f >_{\mathcal{D}(A^m)} = _{(\mathcal{D}(A^m))^*}< A^m a, f >_{\mathcal{D}(A^m)}$ for all $f \in \mathcal{D}(A^m)$. The proof of Lemma 2.1 is completed.

Next

Lemma 2.2.

$$\|a\|_{\mathcal{D}(A^m)^*} = \left(\sum_{j=1}^{\infty} \left| \frac{(a, \varphi_j)}{\lambda_j^m} \right|^2 \right)^{\frac{1}{2}}, \qquad a \in L^2(\Omega).$$

Proof. Since $A^m : \mathcal{D}(A^m) \longrightarrow L^2(\Omega)$ is injective and surjective, and $(A^m)^{-1} \in B(L^2(\Omega), \mathcal{D}(A^m))$, we see that $((A^m)^*)^{-1}$ is injective and $((A^m)^*)^{-1} \in B(\mathcal{D}(A^m)^*, L^2(\Omega))$ (e.g., Yosida [41]). Hence for any $a \in \mathcal{D}(A^m)^*$, we have

$$_{(\mathcal{D}(A^m))^*} < a, f >_{\mathcal{D}(A^m)} = _{(\mathcal{D}(A^m))^*}< (A^m)^*((A^m)^*)^{-1} a, f >_{\mathcal{D}(A^m)} = (((A^m)^*)^{-1} a, A^m f)$$

for $f \in \mathcal{D}(A^m)$. Hence for $a \in \mathcal{D}(A^m)^*$, we have

$$\begin{aligned} \|a\|_{\mathcal{D}(A^m)^*} &= \sup_{\|A^m f\|=1} |_{(\mathcal{D}(A^m))^*} < a, f >_{\mathcal{D}(A^m)} | = \sup_{\|A^m f\|=1} |(((A^m)^*)^{-1} a, A^m f)| \\ &= \|((A^m)^*)^{-1} a\| = \|((A^m)^{-1})^* a\|. \end{aligned} \tag{2.4}$$

Henceforth let $a \in L^2(\Omega)$. We have $(a, (A^m)^{-1} f) = _{(\mathcal{D}(A^m))^*}< a, (A^m)^{-1} f) >_{\mathcal{D}(A^m)} = (((A^m)^{-1})^* a, f)$. On the other hand,

$$(a, A^{-m} f) = \left(a, \sum_{j=1}^{\infty} \frac{(f, \varphi_j)}{\lambda_j^m} \varphi_j \right) = \left(\sum_{j=1}^{\infty} \frac{(a, \varphi_j)}{\lambda_j^m} \varphi_j, f \right).$$

Hence

$$(((A^m)^{-1})^* a, f) = \left(\sum_{j=1}^{\infty} \frac{(a, \varphi_j)}{\lambda_j^m} \varphi_j, f \right), \quad f \in L^2(\Omega),$$

that is,

$$((A^m)^{-1})^* a = \sum_{j=1}^{\infty} \frac{(a, \varphi_j)}{\lambda_j^m} \varphi_j$$

in $L^2(\Omega)$ for any $a \in L^2(\Omega)$. Hence

$$\|((A^m)^{-1})^* a\| = \left(\sum_{j=1}^{\infty} \left| \frac{(a, \varphi_j)}{\lambda_j^m} \right|^2 \right)^{\frac{1}{2}}.$$

In view of (2.4), the proof of Lemma 2.2 is completed.

We set

$$H = \overline{\{A^m a;\, a \in C_0^\infty(\Omega)\}}^{\mathcal{D}(A^m)^*}. \tag{2.5}$$

Then

Lemma 2.3. $H = \mathcal{D}(A^m)^*$.

Proof. By Lemma 2.1, we have

$$H = \overline{\{(A^m)^* a;\, a \in C_0^\infty(\Omega)\}}^{\mathcal{D}(A^m)^*}.$$

Since $A^m : \mathcal{D}(A^m) \longrightarrow L^2(\Omega)$ is injective and $(A^m)^{-1} \in B(L^2(\Omega), \mathcal{D}(A^m))$, we see that $((A^m)^*)^{-1} \in B(\mathcal{D}(A^m)^*, L^2(\Omega))$ and $\mathcal{D}(A^m)^* = (A^m)^* L^2(\Omega)$. Hence it suffices to prove

$$\overline{\{(A^m)^* a;\, a \in C_0^\infty(\Omega)\}}^{\mathcal{D}(A^m)^*} = (A^m)^* L^2(\Omega). \tag{2.6}$$

Since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, for any $a \in L^2(\Omega)$, we can choose a sequence $a_m \in C_0^\infty(\Omega)$, $m \in \mathbb{N}$ such that $a_m \longrightarrow a$ in $L^2(\Omega)$. By $(A^m)^* \in B(L^2(\Omega), \mathcal{D}(A^m)^*)$, we see that $(A^m)^* a_m \longrightarrow (A^m)^* a$ in $\mathcal{D}(A^m)^*$. Hence $(A^m)^* a \in H$. Therefore (2.6) follows, and the proof of Lemma 2.3 is completed.

We consider

$$\begin{cases} \partial_t^2 v(x,t) = -Av(x,t), & x \in \Omega,\ 0 < t < T_0, \\ v(x,t) = 0, & x \in \partial\Omega,\ 0 < t < T_0, \\ v(\cdot, 0) = 0, \quad \partial_t v(\cdot, 0) = v_1 \in \mathcal{D}(A^m)^*. \end{cases} \tag{2.7}$$

By the transposition method (e.g., Komornik [28], Lions and Magenes [34]), there exists a unique solution $v \in C^1([0,T_0];\mathcal{D}(A^m)^*)$ to (2.7) and by $v(x,t) = v(v_1)(x,t)$ we denote the solution to (2.7). Then we have

Theorem 2.1. *Let us assume (2.1). Then there exist constants $C_2, C_3 > 0$ such that*

$$C_2\|v_1\|_{\mathcal{D}(A^m)^*} \le \left\|\frac{\partial v(v_1)}{\partial\nu}\right\|_{V_{2m}^*} \le C_3\|v_1\|_{\mathcal{D}(A^m)^*}$$

for all $v_1 \in \mathcal{D}(A^m)^*$.

Proof. By the limit passage and Lemmata 2.1 and 2.3, it is sufficient to prove the conclusion for $v_1 = (A^m)^*a = A^m a$ with $a \in C_0^\infty(\Omega)$. Then we have

$$\frac{\partial v(v_1)}{\partial\nu}(x,t) = \sum_{j=1}^{\infty}(v_1,\varphi_j)\frac{\sin\sqrt{\lambda_j}t}{\sqrt{\lambda_j}}\frac{\partial\varphi_j}{\partial\nu}(x), \quad x \in \partial\Omega,\ 0<t<T_0$$

(e.g., [28]). We note that the right hand side is convergent in $L^2(0,T_0;L^2(\partial\Omega))$. By the definition of the norm in V_{2m}^*, we have

$$\left\|\frac{\partial v(v_1)}{\partial\nu}\right\|_{V_{2m}^*} = \sup_{\|g\|_{V_{2m}}=1}\left|\left\langle g, \frac{\partial v(v_1)}{\partial\nu}\right\rangle\right| = \sup_{\|g\|_{V_{2m}}=1}\left|\left(g, \frac{\partial v(v_1)}{\partial\nu}\right)_{L^2(0,T_0;L^2(\partial\Omega))}\right|.$$

Since $\partial_t^j g(\cdot,T_0) = 0$, $j = 0,1,\cdots,2m-1$, we integrate by parts to obtain

$$\left(g, \frac{\partial v(v_1)}{\partial\nu}\right)_{L^2(0,T_0;L^2(\partial\Omega))} = \left(\partial_t^{2m}g, \int_0^t\frac{(t-\xi)^{2m-1}}{(2m-1)!}\frac{\partial v(v_1)}{\partial\nu}(x,\xi)d\xi\right)_{L^2(0,T_0;L^2(\partial\Omega))}.$$

Hence

$$\left\|\frac{\partial v(v_1)}{\partial\nu}\right\|_{V_{2m}^*} = \left\|\int_0^t\frac{(t-\xi)^{2m-1}}{(2m-1)!}\frac{\partial v(v_1)}{\partial\nu}(x,\xi)d\xi\right\|_{L^2(0,T_0;L^2(\partial\Omega))}.$$

On the other hand, by integration by parts, we have

$$
\begin{aligned}
&\int_0^t \frac{(t-\xi)^{2m-1}}{(2m-1)!}\frac{\partial v(v_1)}{\partial\nu}(x,\xi)d\xi\\
=&\frac{t^{2m-1}}{(2m-1)!}\sum_{j=1}^{\infty}\frac{(v_1,\varphi_j)}{\lambda_j}\frac{\partial\varphi_j}{\partial\nu}(x)-\int_0^t\frac{(\xi-t)^{2m-2}}{(2m-1)!}\sum_{j=1}^{\infty}\frac{(v_1,\varphi_j)}{\lambda_j}\cos\sqrt{\lambda_j}\xi\frac{\partial\varphi_j}{\partial\nu}(x)d\xi\\
=&\frac{t^{2m-1}}{(2m-1)!}\sum_{j=1}^{\infty}\frac{(v_1,\varphi_j)}{\lambda_j}\frac{\partial\varphi_j}{\partial\nu}(x)+\int_0^t\frac{(\xi-t)^{2m-2}}{(2m-2)!}\sum_{j=1}^{\infty}\frac{(v_1,\varphi_j)}{\lambda_j\sqrt{\lambda_j}}\sin\sqrt{\lambda_j}\xi\frac{\partial\varphi_j}{\partial\nu}(x)d\xi\\
=&\frac{t^{2m-1}}{(2m-1)!}\sum_{j=1}^{\infty}\frac{(v_1,\varphi_j)}{\lambda_j}\frac{\partial\varphi_j}{\partial\nu}(x)-\frac{t^{2m-3}}{(2m-3)!}\sum_{j=1}^{\infty}\frac{(v_1,\varphi_j)}{\lambda_j^2}\frac{\partial\varphi_j}{\partial\nu}(x)\\
+&\int_0^t\frac{(\xi-t)^{2m-4}}{(2m-4)!}\sum_{j=1}^{\infty}\frac{(v_1,\varphi_j)}{\lambda_j^2}\cos\sqrt{\lambda_j}\xi\frac{\partial\varphi_j}{\partial\nu}(x)d\xi\\
=&\cdots\\
=&\sum_{k=1}^{m}(-1)^{k+1}\frac{t^{2m-(2k-1)}}{(2m-(2k-1))!}\sum_{j=1}^{\infty}\frac{(v_1,\varphi_j)}{\lambda_j^k}\frac{\partial\varphi_j}{\partial\nu}(x)\\
+&(-1)^m\sum_{j=1}^{\infty}\frac{(v_1,\varphi_j)}{\lambda_j^{m+\frac{1}{2}}}\sin\sqrt{\lambda_j}x\frac{\partial\varphi_j}{\partial\nu}(x).
\end{aligned}
$$

Moreover, since $v_1 = A^m a$, for $1 \le k \le m$ we have $(v_1, \lambda_j^{-k}\varphi_j) = (v_1, A^{-k}\varphi_j) = (A^{m-k}a, \varphi_j)$ and

$$
\sum_{j=1}^{\infty}\frac{(v_1,\varphi_j)}{\lambda_j^k}\varphi_j(x)=\sum_{j=1}^{\infty}(A^{m-k}a,\varphi_j)\varphi_j(x)=A^{m-k}a.
$$

Hence for $1 \le k \le m$, we have

$$
\sum_{j=1}^{\infty}\frac{(v_1,\varphi_j)}{\lambda_j^k}\frac{\partial\varphi_j}{\partial\nu}(x)=\frac{\partial}{\partial\nu}(A^{m-k}a)=0
$$

because $A^{m-k}a = 0$ in $\Omega\setminus\operatorname{supp}a$. Hence

$$
\int_0^t\frac{(t-\xi)^{2m-1}}{(2m-1)!}\frac{\partial v(v_1)}{\partial\nu}(x,\xi)d\xi=(-1)^m\sum_{j=1}^{\infty}\frac{(v_1,\varphi_j)}{\lambda_j^{m+\frac{1}{2}}}\sin\sqrt{\lambda_j}x\frac{\partial\varphi_j}{\partial\nu}(x).
$$

Moreover

$$
C_2'\sum_{j=1}^{\infty}\beta_j^2\le\left\|\sum_{j=1}^{\infty}\beta_j\frac{\sin\sqrt{\lambda_j}t}{\sqrt{\lambda_j}}\frac{\partial\varphi_j}{\partial\nu}\right\|^2_{L^2(0,T_0;L^2(\partial\Omega))}\le C_3'\sum_{j=1}^{\infty}\beta_j^2
$$

for all $\beta_j \in \mathbb{R}$, $j \in \mathbb{N}$, under assumption (2.1), and the constants C_2', C_3' can be chosen uniformly in $p \in \mathcal{U}_1$ (e.g., Kazemi and Klibanov [22], Klibanov and Malinsky [25], Komornik [28]). Consequently

$$\begin{aligned} &C_2' \sum_{j=1}^{\infty} \left| \frac{(v_1, \varphi_j)}{\lambda_j^m} \right|^2 \le \left\| \sum_{j=1}^{\infty} \frac{(v_1, \varphi_j)}{\lambda_j^m} \frac{\sin \sqrt{\lambda_j} t}{\sqrt{\lambda_j}} \frac{\partial \varphi_j}{\partial \nu} \right\|^2_{L^2(0,T_0;L^2(\partial\Omega))} \\ \le &C_3' \sum_{j=1}^{\infty} \left| \frac{(v_1, \varphi_j)}{\lambda_j^m} \right|^2 . \end{aligned}$$

By $v_1 = A^m a \in L^2(\Omega)$, we can apply Lemma 2.2, so that the conclusion holds for $v_1 = (A^m)^* a$ with any $a \in C_0^\infty(\Omega)$.

We define an operator K from $\mathcal{D}(A^m)^*$ to V_{2m}^* by

$$K v_1 = \frac{\partial v(v_1)}{\partial \nu}.$$

We set $\Lambda = K^*$. Then Theorem 2.1 implies that

$$K^* : V_{2m} \longrightarrow \mathcal{D}(A^m)$$

is surjective and bounded (e.g., Brezis [9]). By the open mapping theorem, $\Lambda \equiv (K^*)^{-1} : \mathcal{D}(A^m) \longrightarrow V_{2m}$ is bounded.

On the other hand, for $g \in V_{2m}$, there exists a unique solution $w = w(g) \in C^{2m}([0,T]; L^2(\Omega))$ to

$$\begin{cases} \partial_t^2 w(x,t) = \Delta w(x,t) + p(x) w(x,t), & x \in \Omega,\ 0 < t < T_0, \\ w(x,t) = g(x,t), \qquad x \in \partial\Omega,\ 0 < t < T_0, \\ w(\cdot, 0) = \partial_t w(x,0) = 0, \quad x \in \Omega. \end{cases} \tag{2.8}$$

The unique existence of $w(g)$ is proved by taking t-derivatives of w and a usual a priori estimate (e.g., [28], [33]), and we can further prove more regularity by the transposition method (e.g., Lions and Magenes [34]), but we will omit the details. In terms of the Hilbert Uniqueness Method (e.g., [28], [33]), noting (2.2), we have

Theorem 2.2. *Let us assume (2.1). Then, for any $a \in \mathcal{D}(A^m)$, we have*

$$w(\Lambda(a))(x, T_0) = a(x), \quad x \in \Omega, \qquad \|\Lambda(a)\|_{V_{2m}} \le C_4 \|a\|_{H^{2m}(\Omega)}. \tag{2.9}$$

The constant $C_4 > 0$ is taken uniformly in $p \in \mathcal{U}_1$.

This is the exact controllability for the displacement in a state space $\mathcal{D}(A^m) \subset H^{2m}(\Omega)$. We conclude the section with Lemmata 2.4 and 2.5 which assert estimates for solutions to hyperbolic equations.

Lemma 2.4. *Let $u = u(p, g, a, b)$ satisfy*

$$\begin{cases} \partial_t^2 u(x,t) = \Delta u(x,t) + p(x)u(x,t), & x \in \Omega,\ 0 < t < T, \\ u(x,0) = a(x), \quad \partial_t u(x,0) = b(x), & x \in \Omega, \\ u(x,t) = g(x,t), \qquad x \in \partial\Omega,\ 0 < t < T. \end{cases}$$

Let $\|p\|_{C^{2m-2,1}(\overline{\Omega})} \le M$, and let $a \in H^{2m}(\Omega)$, $b \in H^{2m-1}(\Omega)$. Then for any subdomains Ω' and Ω'' such that $\overline{\Omega''} \subset \Omega' \subset \overline{\Omega'} \subset \Omega$, there exist constants $C_5 = C_5(M, T, \Omega'') > 0$ and $C_6 = C_6(M, T, \Omega', \Omega'') > 0$ such that

$$\|\partial_t^j u\|_{C([0,T];H^1(\Omega'))} \le C_5(\|a\|_{H^{2m}(\Omega)} + \|b\|_{H^{2m-1}(\Omega)} + \|g\|_{H^{2m}(0,T;L^2(\partial\Omega))}),$$
$$j = 0, 1, \cdots 2m-2 \tag{2.10}$$

and

$$\|u\|_{C([0,T];H^{2m}(\Omega''))} \le C_6(\|a\|_{H^{2m}(\Omega)} + \|b\|_{H^{2m-1}(\Omega)} + \|g\|_{H^{2m}(0,T;L^2(\partial\Omega))}). \tag{2.11}$$

We can prove the lemma by taking t-derivatives of u, and will give the proof in Appendix II for completeness.

Lemma 2.5. *Let*

$$T > 2 \inf_{x_0 \in \mathbb{R}^n \setminus (\Omega \setminus \omega)} \sup_{x \in \Omega} |x - x_0|$$

and let

$$\begin{cases} \partial_t^2 w(x,t) = \Delta w(x,t) + p(x)w(x,t) + f(x,t), & x \in \Omega,\ 0 < t < T, \\ \\ w(x,t) = 0, \qquad x \in \partial\Omega,\ 0 < t < T, \end{cases}$$

where $\|p\|_{L^\infty(\Omega)} \le M$. *Then there exists a constant* $C_7 = C_7(M, \Omega, T) > 0$ *such that*

$$\|w(\cdot,0)\|_{H^1(\Omega)} + \|\partial_t w(\cdot,0)\|_{L^2(\Omega)} \le C_7(\|f\|_{L^2(0,T;L^2(\Omega))} + \|w\|_{H^1(0,T;L^2(\omega))}).$$

This is an observability inequality (e.g., , Kazemi and Klibanov [22], Klibanov and Malinsky [25], Klibanov and Timonov [26], Komornik [28], Lions [33]) and for completeness we will prove it in Appendix III.

§3. Proof of Theorem.

First Step.

Since $\mathcal{U}$ is relatively compact in $C^{2m-1}(\overline{\Omega})$, for any $m \in \mathbb{N}$ we can chooose $N = N(m) \in \mathbb{N}$ and $p_j^m \in \mathcal{U}$, $1 \le j \le N$ such that for any $p \in \mathcal{U}$, there exists $j_0 \in \{1, 2, ..., N\}$ satisfying

$$\|p - p_{j_0}^m\|_{C^{2m-1}(\overline{\Omega})} \le \frac{1}{m}. \tag{3.1}$$

We arbitrarily fix

$$a_0 \in H_0^{2m}(\Omega), \quad |a_0(x)| \ge 1, \quad x \in \overline{\Omega \setminus \omega}. \tag{3.2}$$

We set $a_\ell(x) = \ell M a_0(x)$ for $\ell \in \mathbb{N}$. Then we have

$$\|a_\ell\|_{H^{2m}(\Omega)} \le C_0(\ell, M). \tag{3.3}$$

Here $\ell \in \mathbb{N}$ is a constant which we will choose later.

By Theorem 2.2, we can choose $g_1, ..., g_N \in V_{2m}$ such that

$$u(p_j^m, g_j, 0, 0)(x, T_0) = a_\ell(x), \qquad x \in \overline{\Omega},\ 1 \le j \le N. \tag{3.4}$$

Remark. Boundary controls $g_1, ..., g_N$ are suitable inputs guaranteeing the Lipschitz stability in our inverse problem. Let $\Lambda(p) : \mathcal{D}(A^m) \longrightarrow V_{2m}$ be a mapping defined by Theorem 2.2 with the coefficient p. Then we can represent

$$(g_1, ..., g_N) = (\Lambda(p_1^m)(a_\ell), ..., \Lambda(p_N^m)(a_\ell)).$$

Here, for simplicity, we choose controls g_j which steer the system with the zero initial condition to $a_\ell(x) = u(x, T_0)$. We need not be restricted to the zero initial condition. By the proof, we can see that for the Lipschitz stability in our inverse problem, we can choose h_j, $1 \le j \le N$, such that

$$u(p_j^m, h_j, \alpha_j, \beta_j)(x, T_0) = a_\ell(x), \qquad x \in \Omega$$

for any $(\alpha_j, \beta_j) \in \mathcal{V}$, $1 \le j \le N$. Such boundary controls h_j can exists in terms of Theorem 2.1 (also see [28], [33]). Thus such choices of boundary inputs are quite generous.

Here and henceforth C, C_j denote generic positive constants which are dependent on $\Omega, \mathcal{U}, \mathcal{V}$ but independent of ℓ, m, M and choices of $p \in \mathcal{U}$, while $C(M, \ell), C_j(M, \ell)$ denote generic positive constants which are dependent on $M, \ell, \Omega, \mathcal{U}, \mathcal{V}$ but independent of m and choices of $p \in \mathcal{U}$. We can understand that $C(M)$, $C_j(M)$ denote a constant which are independent of ℓ but depend on $M, \Omega, \mathcal{U}, \mathcal{V}$.

Then by Theorem 2.2 and the second assumption in (1.5), we have

$$\|g_j\|_{V_{2m}} \le C_1(M)\|a_\ell\|_{H^{2m}(\Omega)} \le C_2(M, \ell). \tag{3.5}$$

Let $p \in \mathcal{U}$ be given arbitrarily. Then

$$\begin{aligned} &\min_{1 \le j \le N} \|u(p, g_j, a, b) - a_\ell\|_{C([0,T_0];H^{2m}(\Omega))} \\ \le &C_3(M) + \frac{C_4(M, \ell)}{m} \quad \text{for all } (a, b) \in \mathcal{V}. \end{aligned} \tag{3.6}$$

In fact, by (3.1), there exists $j_0 \in \{1, ..., N\}$ such that $\|p - p^m_{j_0}\|_{C^{2m-1}(\overline{\Omega})} \le \frac{1}{m}$.

Setting $v = u(p, g_{j_0}, a, b) - u(p^m_{j_0}, g_{j_0}, 0, 0)$, we have

$$\begin{cases} \partial_t^2 v(x,t) = \Delta v(x,t) + p(x)v(x,t) + (p(x) - p^m_{j_0}(x))u(p^m_{j_0}, g_{j_0}, 0, 0)(x,t), \quad x \in \Omega,\ 0 < t < T, \\ v(x,0) = a(x), \quad \partial_t v(x,0) = b(x), \qquad x \in \Omega, \\ v(x,t) = 0, \qquad x \in \partial\Omega,\ 0 < t < T. \end{cases}$$

Then, by Lions and Magenes [34], we have

$$\begin{aligned} &\|v\|_{C([0,T];H^{2m}(\Omega))} \\ \le& C_5(M)(\|a\|_{H^{2m}(\Omega)} + \|b\|_{H^{2m-1}(\Omega)} + \|(p - p^m_{j_0})u(p^m_{j_0}, g_{j_0}, 0, 0)\|_{C([0,T];H^{2m-1}(\Omega))}) \\ \le& C_5'(M)(\|a\|_{H^{2m}(\Omega)} + \|b\|_{H^{2m-1}(\Omega)}) + C_5'(M)\|p - p^m_{j_0}\|_{C^{2m-1}(\overline{\Omega})}\|u(p^m_{j_0}, g_{j_0}, 0, 0)\|_{C([0,T];H^{2m-1}(\Omega))} \\ \le& C_6(M) + \frac{C_7(M,\ell)}{m}. \end{aligned}$$

Here we used Lemma 2.4, $\|a\|_{H^{2m}(\Omega)}, \|b\|_{H^{2m-1}(\Omega)} \le M$ and (3.5). Since $u(p^m_{j_0}, g_{j_0}, 0, 0)(x, T_0) = a_\ell(x)$, $x \in \Omega$ by (3.4), the proof of (3.6) is completed.

By $m > \frac{n}{4}$ and the Sobolev embedding, we have $H^{2m}(\Omega) \subset C(\overline{\Omega})$. In terms of (3.2) and (3.6), we see that for any $p \in \mathcal{U}$, there exists $j_0 \in \{1, ..., N\}$ such that

$$|u(p, g_{j_0}, a, b)(x, T_0)| \ge \ell M - C_8(M) - \frac{C_9(M,\ell)}{m}, \quad x \in \overline{\Omega \setminus \omega},\ (a,b) \in \mathcal{V}.$$

We choose $\ell \in \mathbb{N}$ suffciently large so that $\frac{(\ell-1)M}{2} \ge C_8(M)$. For this ℓ, we choose $m \in \mathbb{N}$ large such that $\frac{(\ell-1)M}{2} \ge \frac{C_9(M,\ell)}{m}$.

Thus: there exist $g_1, ..., g_N \in V_{2m}$ such that for any $p \in \mathcal{U}$, we can choose $j_0 \in \{1, ..., N\}$ satisfying

$$\begin{aligned} &|u(p, g_{j_0}, a, b)(x, T_0)| \ge \left(\frac{(\ell-1)M}{2} - C_8(M)\right) + \left(\frac{(\ell-1)M}{2} - \frac{C_9(M,\ell)}{m}\right) + M \\ \ge& M, \qquad x \in \overline{\Omega \setminus \omega},\ (a,b) \in \mathcal{V}. \end{aligned} \tag{3.7}$$

Second Step.

We will complete the proof of Theorem.

We choose a subdomain $\Omega' \subset \Omega$ such that $\overline{\Omega \setminus \omega} \subset \Omega'$ and $\overline{\Omega'} \subset \Omega$. Then, by Lemma 2.4 and hyperbolic equation (1.1), we have

$$\|u(p, g_j, a_j, b_j)\|_{C([0,T];H^{2m}(\Omega')) \cap C^2([0,T];H^{2m-2}(\Omega'))} \le C_{10}(M),$$

$$\|u(\widetilde{p}, g_j, \widetilde{a}_j, \widetilde{b}_j)\|_{C([0,T];H^{2m}(\Omega')) \cap C^2([0,T];H^{2m-2}(\Omega'))} \le C_{10}(M)$$

for $(p, a_j, b_j), (\widetilde{p}, \widetilde{a}_j, \widetilde{b}_j) \in \mathcal{U} \times \mathcal{V}$. Setting $t_1 = \min\{T - T_0, T_0\}$, we see by (1.5) that $t_1 > \inf_{x' \in \mathbb{R}^n \setminus (\Omega \setminus \omega)} \sup_{x \in \Omega} |x - x'|$. Therefore we can Theorem 0 in a domain Ω' by replacing ω by $\omega \cap \Omega'$. Then, in terms of (3.7), assumption (1.4) is satisfied. Hence Theorem 0 yields that we can choose $j \in \{1, ..., N\}$ such that

$$\begin{aligned}
\|p - \widetilde{p}\|_{L^2(\Omega \setminus \overline{\omega})} \le C_{11} \Bigg(& \|u(p, g_j, a_j, b_j) - u(\widetilde{p}, g_j, \widetilde{a}_j, \widetilde{b}_j)\|_{C^3([0,T];L^2(\omega))} \\
& + \|u(p, g_j, a_j, b_j) - u(\widetilde{p}, g_j, \widetilde{a}_j, \widetilde{b}_j)\|_{C^1([0,T];H^2(\omega))} \\
& + \|(u(p, g_j, a_j, b_j) - u(\widetilde{p}, g_j, \widetilde{a}_j, \widetilde{b}_j))(\cdot, T_0)\|_{H^2(\Omega)} \\
& + \|(\partial_t u(p, g_j, a_j, b_j) - \partial_t u(\widetilde{p}, g_j, \widetilde{a}_j, \widetilde{b}_j))(\cdot, T_0)\|_{H^2(\Omega)} \Bigg) \\
\equiv & C_{11} D_j. \qquad (3.8)
\end{aligned}$$

Since g_j depends on $p, \widetilde{p} \in \mathcal{U}$, we take the sum $\sum_{j=1}^N D_j$, so that $\|p - \widetilde{p}\|_{L^2(\Omega)} \le C_{11} \sum_{j=1}^N D_j$.

Next, setting $v_j = u(p, g_j, a_j, b_j) - u(\widetilde{p}, g_j, \widetilde{a}_j, \widetilde{b}_j)$, we have

$$\begin{cases}
\partial_t^2 v_j(x,t) = \Delta v_j(x,t) + p(x) v_j(x,t) + (p(x) - \widetilde{p}(x)) u(\widetilde{p}, g_j, \widetilde{a}_j, \widetilde{b}_j)(x,t), & x \in \Omega,\ 0 < t < T, \\
v_j(x,0) = a_j(x) - \widetilde{a}_j(x), \quad \partial_t v_j(x,0) = b_j(x) - \widetilde{b}_j(x), & x \in \Omega, \\
v_j(x,t) = 0, & x \in \partial\Omega,\ 0 < t < T.
\end{cases}$$

In terms of (1.5), applying Lemma 2.5, we obtain

$$\begin{aligned}
& \|a_j - \widetilde{a}_j\|_{H^1(\Omega)} + \|b_j - \widetilde{b}_j\|_{L^2(\Omega)} \\
\le & C_{12} (\|(p - \widetilde{p}) u(\widetilde{p}, g_j, \widetilde{a}_j, \widetilde{b}_j)\|_{L^2(0,T;L^2(\Omega))} + \|v_j\|_{H^1(0,T;L^2(\omega))}).
\end{aligned}$$

Since $p-\widetilde{p}=0$ in ω, in terms of $m>\frac{n}{4}$ and the Sobolev embedding, we have

$$\begin{aligned}
&\|(p-\widetilde{p})u(\widetilde{p},g_j,\widetilde{a}_j,\widetilde{b}_j)\|_{L^2(0,T;L^2(\Omega))} \le \|(p-\widetilde{p})u(\widetilde{p},g_j,\widetilde{a}_j,\widetilde{b}_j)\|_{L^2(0,T;L^2(\Omega'))}\\
\le&\|p-\widetilde{p}\|_{L^2(\Omega')}\|u(\widetilde{p},g_j,\widetilde{a}_j,\widetilde{b}_j)\|_{L^2(0,T;L^\infty(\Omega'))}\\
\le&C_{13}\|p-\widetilde{p}\|_{L^2(\Omega')}\|u(\widetilde{p},g_j,\widetilde{a}_j,\widetilde{b}_j)\|_{L^2(0,T;H^{2m}(\Omega'))}.
\end{aligned}$$

Hence (3.8) and Lemma 2.4 yield

$$\|(p-\widetilde{p})u(\widetilde{p},g_j,\widetilde{a}_j,\widetilde{b}_j)\|_{L^2(0,T;L^2(\Omega))} \le C_{14}\sum_{j=1}^N D_j.$$

Therefore

$$\|a_j-\widetilde{a}_j\|_{H^1(\Omega)}+\|b_j-\widetilde{b}_j\|_{L^2(\Omega)} \le C_{14}D_j.$$

Thus, with (3.8), the proof of Theorem is completed.

Appendix I. Proof of Theorem 0.

We prove Theorem 0 by modifying the argument in Imanuvilov and Yamamoto [16], [17]. We show a key Carleman estimate (Imanuvilov [14]). We set

$$Pv=\partial_t^2 v-\Delta v-p(x)v, \qquad x\in\Omega,\ t>0.$$

Let $x_0\in\mathbb{R}^n\setminus\overline{(\Omega\setminus\omega)}$ satisfy $\inf_{x'\in\mathbb{R}^n\setminus(\Omega\setminus\omega)}\sup_{x\in\Omega}|x-x'|=\sup_{x\in\Omega}|x-x_0|$. For this x_0 and $\beta\in(0,1)$, we define functions $\psi=\psi(x,t)$ and $\varphi=\varphi(x,t)$ by

$$\psi(x,t)=|x-x_0|^2-\beta|t-T_0|^2, \qquad \varphi(x,t)=e^{\lambda\psi(x,t)}$$

with a parameter $\lambda>0$. Let $0<t_1<T_0$.

Lemma I.1. *Let $\|p\|_{L^\infty(\Omega)}\le M$ and let us assume that $t_1>\sup_{x\in\Omega}|x-x_0|$. Then there exists $\lambda_0>0$ such that for all $\lambda>\lambda_0$ there exist $s_0=s_0(\lambda)>0$ and a constant $C=C(s_0,\lambda_0,M,\Omega,T_0,t_1,x_0,\omega)>0$ such that*

$$\begin{aligned}
&\int_{T_0-t_1}^{T_0+t_1}\int_\Omega (s|\nabla_{x,t}y|^2+s^3y^2)e^{2s\varphi}dxdt\\
\le&C\int_{T_0-t_1}^{T_0+t_1}\int_\Omega |Py|^2e^{2s\varphi}dxdt+C\int_{T_0-t_1}^{T_0+t_1}\int_\omega (s|\partial_t y|^2+s^3y^2)e^{2s\varphi}dxdt
\end{aligned}$$

for all $s > s_0(\lambda)$, provided that

$$\begin{cases} Py \in L^2(\Omega \times (T_0 - t_1, T_0 + t_1)), \quad y \in H^1(\Omega \times (T_0 - t_1, T_0 + t_1)), \\ y = 0 \quad \text{on } \partial\Omega \times (T_0 - t_1, T_0 + t_1), \\ y(\cdot, T_0 - t_1) = \partial_t y(\cdot, T_0 - t_1) = y(\cdot, T_0 + t_1) = \partial_t y(\cdot, T_0 + t_1) = 0 \quad \text{in } \Omega. \end{cases} \tag{I.1}$$

Case 1.

$$u(p, g, a, b) = u(\widetilde{p}, \widetilde{g}, \widetilde{a}, \widetilde{b}) \quad \text{on } \partial\Omega \times (T_0 - t_1, T_0 + t_1).$$

We set $Q = \Omega \times (T_0 - t_1, T_0 + t_1)$. Moreover setting $\widetilde{y} = u(p, g, a, b) - u(\widetilde{p}, g, \widetilde{a}, \widetilde{b})$, $R(x,t) = u(\widetilde{p}, g, \widetilde{a}, \widetilde{b})$ and $f = p - \widetilde{p}$, we have

$$\begin{cases} \partial_t^2 \widetilde{y}(x,t) = \Delta \widetilde{y} + p(x)\widetilde{y}(x,t) + f(x)R(x,t), \qquad (x,t) \in Q, \\ \widetilde{y}(x,t) = 0, \qquad x \in \partial\Omega,\ T_0 - t_1 < t < T_0 + t_1. \end{cases}$$

Here $\widetilde{y}, R \in \bigcap_{j=0}^2 C^j([T_0 - t_1, T_0 + t_1]; H^{3-j}(\Omega))$. We set

$$y = \widetilde{y} - (a - \widetilde{a}) - (b - \widetilde{b})(t - T_0) \quad \text{in } Q.$$

Then

$$\begin{cases} \partial_t^2 y(x,t) = \Delta y + p(x)y(x,t) + f(x)R(x,t) + (\Delta + p(x))((a - \widetilde{a}) + (b - \widetilde{b})(t - T_0)) \quad \text{in } Q, \\ y(x,t) = 0, \qquad x \in \partial\Omega,\ T_0 - t_1 < t < T_0 + t_1, \\ y(x, T_0) = \partial_t y(x, T_0) = 0, \qquad x \in \Omega. \end{cases} \tag{I.2}$$

By an a priori estimate (e.g., Lions and Magenes [34]) and $\|R\|_{\cap_{j=0}^2 C^j([T_0 - t_1, T_0 + t_1]; H^{3-j}(\Omega))} \le C_1 M$, by using

$$\Delta y = \partial_t^2 y - py - fR - (\Delta + p(x))((a - \widetilde{a}) + (b - \widetilde{b})(t - T_0)) \quad \text{in } Q,$$

there exists a constant $C_2 > 0$ such that

$$\|y\|_{H^2(Q)} \le C_2(\|f\|_{L^2(\Omega)} + \|a - \widetilde{a}\|_{H^2(\Omega)} + \|b - \widetilde{b}\|_{H^2(\Omega)}). \tag{I.3}$$

By the assumption on t_1, there exists $\beta \in (0,1)$ such that

$$\beta > \frac{\sup_{x\in\Omega} |x - x_0|^2}{t_1^2}.$$

Therefore, by the definitions of ψ and φ, we have

$$\begin{aligned} &\varphi(x, T_0) \ge 1, \qquad x \in \overline{\Omega}, \\ &\varphi(x, T_0 - t_1) = \varphi(x, T_0 + t_1) < 1, \qquad x \in \overline{\Omega}. \end{aligned} \tag{I.4}$$

Therefore for given $\varepsilon > 0$, we can choose a sufficiently small $\delta = \delta(\varepsilon) > 0$ such that

$$\varphi(x,t) \ge 1 - \varepsilon, \quad (x,t) \in \overline{\Omega} \times [T_0 - \delta, T_0 + \delta] \tag{I.5}$$

and

$$\varphi(x,t) \le 1 - 2\varepsilon, \quad (x,t) \in \overline{\Omega} \times ([T_0 - t_1, T_0 - t_1 + 2\delta] \cup [T_0 + t_1 - 2\delta, T_0 + t_1]). \tag{I.6}$$

In order to apply Lemma I.1, we have to introduce a cut-off function $\chi \in C_0^\infty(\mathbb{R})$ satisfying $0 \le \chi \le 1$ and

$$\chi(t) = \begin{cases} 0 & t \in [T_0 - t_1, T_0 - t_1 + \delta] \cup [T_0 + t_1 - \delta, T_0 + t_1] \\ 1 & t \in [T_0 - t_1 + 2\delta, T_0 + t_1 - 2\delta]. \end{cases} \tag{I.7}$$

Henceforth $C > 0$ denotes generic constants which are dependent on s_0, λ, M, Ω, T, x_0, ω, β, χ and δ, ε, but independent of $s > s_0$.

We set

$$z = (\partial_t y)e^{s\varphi}\chi \in C([T_0 - t_1, T_0 + t_1]; H^1(\Omega)) \cap C^1([T_0 - t_1, T_0 + t_1]; L^2(\Omega)). \tag{I.8}$$

By (I.2), the function z satisfies the equation

$$\begin{aligned} Pz = (f\partial_t R)e^{s\varphi}\chi + e^{s\varphi}\chi(\Delta + p)(b - \widetilde{b}) + s(-2\nabla_x\varphi \cdot \nabla_x z + 2(\partial_t\varphi)\partial_t z + (\Box\varphi)z) \\ -s^2(|\partial_t\varphi|^2 - |\nabla_x\varphi|^2)z + 2e^{s\varphi}(\partial_t^2 y)\partial_t\chi + (\partial_t y)e^{s\varphi}\partial_t^2\chi \quad \text{in } Q. \end{aligned} \tag{I.9}$$

In fact, we have

$$\partial_i z = (\partial_i\partial_t y)e^{s\varphi}\chi + (\partial_t y)s(\partial_i\varphi)e^{s\varphi}\chi = (\partial_i\partial_t y)e^{s\varphi}\chi + s(\partial_i\varphi)z. \tag{I.10}$$

Hence

$$(\partial_i\partial_t y)e^{s\varphi}\chi = \partial_i z - s(\partial_i\varphi)z. \tag{I.11}$$

Moreover, the differentiation of (I.10) yields

$$\partial_i^2 z = (\partial_i^2\partial_t y)e^{s\varphi}\chi + s(\partial_i\varphi)e^{s\varphi}(\partial_i\partial_t y)\chi + (sz\partial_i^2\varphi + s(\partial_i\varphi)\partial_i z)$$

$$=(\partial_i^2\partial_t y)e^{s\varphi}\chi + s\partial_i\varphi(\partial_i z - sz\partial_i\varphi) + (s(\partial_i^2\varphi)z + s(\partial_i\varphi)\partial_i z).$$

At the second equality, we used also (I.11). Hence

$$\Delta z = (\Delta\partial_t y)e^{s\varphi}\chi + 2s\nabla\varphi\cdot\nabla z + (s\Delta\varphi - s^2|\nabla\varphi|^2)z.$$

We have

$$\partial_t z = (\partial_t^2 y)e^{s\varphi}\chi + (\partial_t y)e^{s\varphi}\partial_t\chi + s(\partial_t\varphi)z.$$

Similarly we have

$$\partial_t^2 z = (\partial_t^3 y)e^{s\varphi}\chi + \{(\partial_t^2 y)e^{s\varphi}\chi\}s\partial_t\varphi + 2e^{s\varphi}(\partial_t^2 y)\partial_t\chi$$

$$+s(\partial_t y)(\partial_t\varphi)e^{s\varphi}\partial_t\chi + (\partial_t y)e^{s\varphi}\partial_t^2\chi + s(\partial_t^2\varphi)z + s(\partial_t\varphi)(\partial_t z)$$

$$=(\partial_t^3 y)e^{s\varphi}\chi + \{\partial_t z - (\partial_t y)e^{s\varphi}\partial_t\chi - s(\partial_t\varphi)z\}s\partial_t\varphi + 2e^{s\varphi}(\partial_t^2 y)\partial_t\chi$$

$$+s(\partial_t y)(\partial_t\varphi)e^{s\varphi}\partial_t\chi + (\partial_t y)e^{s\varphi}\partial_t^2\chi + s(\partial_t^2\varphi)z + s(\partial_t\varphi)(\partial_t z)$$

$$=(\partial_t^3 y)e^{s\varphi}\chi + 2s(\partial_t\varphi)\partial_t z + (s\partial_t^2\varphi - s^2|\partial_t\varphi|^2)z$$

$$+2e^{s\varphi}(\partial_t^2 y)\partial_t\chi + (\partial_t y)e^{s\varphi}\partial_t^2\chi.$$

Therefore, by (I.2), we obtain (I.9).

In particular, setting $w = \chi(\partial_t y)$ and $s = 0$ in (I.9), we have

$$Pw = \chi f \partial_t R + \chi(\Delta + p)(b - \widetilde{b}) + 2(\partial_t^2 y)\partial_t \chi + (\partial_t^2 \chi)\partial_t y \quad \text{in } Q. \tag{I.12}$$

Now we will apply Lemma I.1 to equation (I.12). By (I.2), (I.3), (I.7) and (I.12), we see that $Pw \in L^2(Q)$, $w \in H^1(Q)$, and $w = 0$ on $\partial\Omega \times (T_0 - t_1, T_0 + t_1)$. Hence by Lemma I.1, we obtain

$$\begin{aligned}&\int_Q (s^3 w^2 + s|\nabla_{x,t} w|^2) e^{2s\varphi} dxdt \le C_3 \int_Q \chi^2 |f \partial_t R|^2 e^{2s\varphi} dxdt \\ &+C_3 \int_Q \chi^2 |(\Delta + p)(b - \widetilde{b})|^2 e^{2s\varphi} dxdt + C_3 \int_Q \{(\partial_t \chi)\partial_t^2 y + (\partial_t^2 \chi)\partial_t y\}^2 e^{2s\varphi} dxdt + C_3 D(y).\end{aligned} \tag{I.13}$$

Here and henceforth we set

$$\begin{aligned}D(y) \equiv &\int_{T_0 - t_1}^{T_0 + t_1} \int_\omega (s^3 w^2 + s|\partial_t w|^2) e^{2s\varphi} dxdt \\ \le &C_3 s^3 \int_{T_0 - t_1}^{T_0 + t_1} \int_\omega (|\partial_t y|^2 + |\partial_t^2 y|^2) e^{2s\varphi} dxdt.\end{aligned}$$

By (I.3), (I.6) and (I.7), we have

$$\begin{aligned}&\left| \int_Q \{(\partial_t \chi)\partial_t^2 y + (\partial_t^2 \chi)\partial_t y\}^2 e^{2s\varphi} dxdt \right| \\ =&\left| \left(\int_{T_0 - t_1 + \delta}^{T_0 - t_1 + 2\delta} + \int_{T_0 + t_1 - 2\delta}^{T_0 + t_1 - \delta} \right) \int_\Omega \{(\partial_t \chi)\partial_t^2 y + (\partial_t^2 \chi)\partial_t y\}^2 e^{2s\varphi} dxdt \right| \\ \le &C_4 e^{2s(1-2\varepsilon)} \|y\|_{H^2(Q)}^2 \\ \le &C_4 e^{2s(1-2\varepsilon)} (\|f\|_{L^2(\Omega)}^2 + \|a - \widetilde{a}\|_{H^2(\Omega)}^2 + \|b - \widetilde{b}\|_{H^2(\Omega)}^2).\end{aligned} \tag{I.14}$$

Noting that $z = we^{s\varphi}$, we have

$$s^3 z^2 = s^3 w^2 e^{2s\varphi},$$

$$s|\nabla_{x,t} z|^2 = s|\nabla_{x,t} w + s(\nabla_{x,t}\varphi) w|^2 e^{2s\varphi} \le C_4 (s|\nabla_{x,t} w|^2 e^{2s\varphi} + s^3 w^2 e^{2s\varphi}).$$

Therefore by (I.13) and (I.14), we obtain

$$\int_Q (s^3z^2 + s|\nabla_{x,t}z|^2)dxdt \le C_5 \int_Q \chi^2|f|^2 e^{2s\varphi}dxdt$$
$$+C_5e^{C_5 s}(\|a-\widetilde{a}\|^2_{H^2(\Omega)} + \|b-\widetilde{b}\|^2_{H^2(\Omega)}) + C_5e^{2s(1-2\varepsilon)}\|f\|^2_{L^2(\Omega)} + C_5D(y). \tag{I.15}$$

We multiply (I.9) by $\partial_t z$ and integrate it over $\Omega \times (T_0 - t_1, T_0)$:

$$\int_{T_0-t_1}^{T_0}\int_\Omega (Pz)\partial_t z dxdt = \int_{T_0-t_1}^{T_0}\int_\Omega f(\partial_t R)e^{s\varphi}\chi\partial_t z dxdt$$
$$+\int_{T_0-t_1}^{T_0}\int_\Omega ((\Delta+p)(b-\widetilde{b}))e^{s\varphi}\chi\partial_t z dxdt$$
$$+\int_{T_0-t_1}^{T_0}\int_\Omega \{s(-2\nabla_x\varphi\cdot(\partial_t z)\nabla_x z + 2(\partial_t\varphi)|\partial_t z|^2 + (\Box\varphi)z\partial_t z)$$
$$-s^2(|\partial_t\varphi|^2 - |\nabla_x\varphi|^2)z\partial_t z\}dxdt + \int_{T_0-t_1}^{T_0}\int_\Omega 2e^{s\varphi}(\partial_t^2 y)(\partial_t\chi)\partial_t z dxdt$$
$$+\int_{T_0-t_1}^{T_0}\int_\Omega (\partial_t y)e^{s\varphi}(\partial_t^2\chi)\partial_t z dxdt. \tag{I.16}$$

We denote the left and the right hand sides of (I.16) respectively by I_1 and I_2. By the boundary conditions and the conditions at $t = T_0$ in (I.2), noting that $z(\cdot, T_0 - t_1) = \partial_t z(\cdot, T_0 - t_1) = 0$, we integrate by parts, so that we have

$$I_1 = \int_{T_0-t_1}^{T_0}\int_\Omega \frac{1}{2}\left(\frac{\partial|\partial_t z|^2}{\partial t} + \frac{\partial|\nabla z|^2}{\partial t} - p\frac{\partial|z|^2}{\partial t}\right)dxdt$$
$$-\int_{\partial\Omega}\int_{T_0-t_1}^{T_0} \frac{\partial}{\partial\nu}((\partial_t y)e^{s\varphi}\chi)\partial_t((\partial_t y)e^{s\varphi}\chi)dtd\sigma$$
$$\ge \frac{1}{2}\int_\Omega |\partial_t z(x,T_0)|^2 dx.$$

On the other hand, by (I.2) and (I.8), we see that

$$\partial_t z(x,T_0) = ((\partial_t^2 y)e^{s\varphi}\chi)(x,T_0) + (s(\partial_t\varphi)e^{s\varphi}(\partial_t y)\chi)(x,T_0) + ((\partial_t y)e^{s\varphi}\partial_t\chi)(x,T_0)$$
$$=(\Delta y + py + fR + (\Delta+p)(a-\widetilde{a}))(x,T_0)e^{s\varphi(x,T_0)}\chi(T_0)$$
$$=(\widetilde{a}f + (\Delta+p)(a-\widetilde{a}))(x)e^{s\varphi(x,T_0)}, \quad x \in \Omega.$$

Therefore

$$I_1 \ge \frac{1}{2}\int_\Omega |\widetilde{a}(x)|^2|f(x)|^2e^{2s\varphi(x,T_0)}dx - C_6e^{2sC_6}\|a-\widetilde{a}\|^2_{H^2(\Omega)}. \tag{I.17}$$

Furthermore by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\int_{T_0-t_1}^{T_0}\int_\Omega 2e^{s\varphi}(\partial_t^2 y)(\partial_t\chi)\partial_t z dxdt\\ \le&\int_{T_0-t_1}^{T_0}\int_\Omega |\partial_t z|^2 dxdt+\int_{T_0-t_1}^{T_0}\int_\Omega e^{2s\varphi}|(\partial_t^2 y)\partial_t\chi|^2 dxdt,\end{aligned}$$

and again the Cauchy-Schwarz inequality yields

$$\begin{aligned}I_2\le\ & C_6\int_{T_0-t_1}^{T_0}\int_\Omega |f|^2 e^{2s\varphi}\chi^2 dxdt+C_6e^{C_6 s}\|b-\widetilde{b}\|^2_{H^2(\Omega)}+C_6\int_{T_0-t_1}^{T_0}\int_\Omega (s|\nabla_{x,t}z|^2+s^3|z|^2)dxdt\\ &+C_6\int_{T_0-t_1}^{T_0}\int_\Omega |\partial_t y|^2e^{2s\varphi}|\partial_t^2\chi|^2 dxdt+\int_{T_0-t_1}^{T_0}\int_\Omega e^{2s\varphi}|(\partial_t^2 y)\partial_t\chi|^2dxdt.\end{aligned}$$

By noting (I.3), (I.7) and the fact that $\partial_t y\in L^2(\Omega\times(T_0-t_1,T_0))$, application of (I.15) yields

$$\begin{aligned}&I_2\le C_7\int_Q |f|^2\chi^2e^{2s\varphi}dxdt+C_7e^{C_7 s}(\|b-\widetilde{b}\|^2_{H^2(\Omega)}+\|a-\widetilde{a}\|^2_{H^2(\Omega)})\\ &+C_7e^{2s(1-2\varepsilon)}\|f\|^2_{L^2(\Omega)}+C_7D(y)\\ &+C_7\left(\int_{T_0-t_1+\delta}^{T_0-t_1+2\delta}+\int_{T_0+t_1-2\delta}^{T_0+t_1-\delta}\right)\int_\Omega(|\partial_t y|^2+|\nabla_{x,t}\partial_t y|^2)e^{2s\varphi}dxdt\\ &\le C_8\int_Q |f|^2e^{2s\varphi}dtdx+C_8e^{C_8 s}(\|b-\widetilde{b}\|^2_{H^2(\Omega)}+\|a-\widetilde{a}\|^2_{H^2(\Omega)})\\ &+C_8e^{2s(1-2\varepsilon)}\|f\|^2_{L^2(\Omega)}+C_8D(y).\end{aligned}\tag{I.18}$$

Consequently (1.4), (I.17) and (I.18) imply

$$\begin{aligned}&\int_\Omega |f(x)|^2e^{2s\varphi(x,T_0)}dx\\ \le& C_9\int_Q |f|^2e^{2s\varphi}dtdx+C_9e^{C_9 s}(\|b-\widetilde{b}\|^2_{H^2(\Omega)}+\|a-\widetilde{a}\|^2_{H^2(\Omega)})\\ &+C_9e^{2s(1-2\varepsilon)}\|f\|^2_{L^2(\Omega)}+C_9D(y).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_Q |f(x)|^2e^{2s\varphi}dtdx=\int_\Omega\left(\int_{T_0-t_1}^{T_0+t_1}|f(x)|^2\exp\left(2se^{\lambda\psi(x,t)}\right)dt\right)dx\\ \le&\int_\Omega\left(\int_{T_0-t_1}^{T_0+t_1}\exp\left(2se^{\lambda|x-x_0|^2}(e^{-\lambda\beta t^2}-1)\right)dt\right)|f(x)|^2e^{2s\varphi(x,T_0)}dx\\ \le&\int_\Omega\left(\int_{T_0-t_1}^{T_0+t_1}\exp\left(2s(e^{-\lambda\beta t^2}-1)\right)dt\right)|f(x)|^2e^{2s\varphi(x,T_0)}dx.\end{aligned}$$

The Lebesgue theorem implies

$$\int_{T_0-t_1}^{T_0+t_1} \exp\left(2s(e^{-\lambda\beta t^2}-1)\right) dt = o(1)$$

as $s \longrightarrow \infty$, so that

$$\int_Q |f(x)|^2 e^{2s\varphi} dtdx = o(1) \int_\Omega |f(x)|^2 e^{2s\varphi(x,T_0)} dx.$$

Therefore

$$\begin{aligned}
&\int_\Omega |f(x)|^2 e^{2s\varphi(x,T_0)} dx \\
\le& o(1) \int_\Omega |f(x)|^2 e^{2s\varphi(x,T_0)} dx + C_9 e^{C_9 s}(\|b-\widetilde{b}\|^2_{H^2(\Omega)} + \|a-\widetilde{a}\|^2_{H^2(\Omega)}) \\
+& C_9 e^{2s(1-2\varepsilon)} \|f\|^2_{L^2(\Omega)} + C_9 D(y)
\end{aligned}$$

as $s \longrightarrow \infty$. Hence

$$\begin{aligned}
&(1-o(1))\, e^{2s} \int_\Omega |f(x)|^2 dx \\
\le& C_{10} e^{2s(1-2\varepsilon)} \|f\|^2_{L^2(\Omega)} + C_{10} e^{C_{10} s}(\|b-\widetilde{b}\|^2_{H^2(\Omega)} + \|a-\widetilde{a}\|^2_{H^2(\Omega)}) + C_{10} D(y).
\end{aligned}$$

By choosing $s > 0$ sufficiently large and fixing, the proof of Theorem 0 is completed in Case 1.

Case 2.

$$u(p,g,a,b) \ne u(\widetilde{p},\widetilde{g},\widetilde{a},\widetilde{b}) \quad \text{on } \partial\Omega \times (T_0-t_1, T_0+t_1).$$

By the Sobolev extension theorem, we can choose $h \in C^3([T_0-t_1, T_0+t_1]; L^2(\Omega)) \cap C^1([T_0-t_1, T_0+t_1]; H^2(\Omega))$ such that

$$\begin{aligned}
&\|h\|_{C^3([T_0-t_1,T_0+t_1];L^2(\Omega))} + \|h\|_{C^1([T_0-t_1,T_0+t_1];H^2(\Omega))} \\
\le& C_{11}(\|u(p,g,a,b) - u(\widetilde{p},\widetilde{g},\widetilde{a},\widetilde{b})\|_{C^3([T_0-t_1,T_0+t_1];L^2(\omega))} \\
+& \|u(p,g,a,b) - u(\widetilde{p},\widetilde{g},\widetilde{a},\widetilde{b})\|_{C^1([T_0-t_1,T_0+t_1];H^2(\omega))} \qquad \text{(I.19)}
\end{aligned}$$

and

$$h = u(p,g,a,b) - u(\widetilde{p},\widetilde{g},\widetilde{a},\widetilde{b}) \quad \text{in } \omega\times(T_0-t_1,T_0+t_1).$$

We set

$$y_1 = u(p,g,a,b) - u(\widetilde{p},\widetilde{g},\widetilde{a},\widetilde{b}) - h \quad \text{in } Q.$$

Since $\partial\omega \supset \partial\Omega$, we have

$$\begin{cases} \partial_t^2 y_1(x,t) = \Delta y_1 + p(x)y(x,t) + f(x)R(x,t) + (\partial_t^2 - \Delta - p)h \quad \text{in } Q, \\ y_1(x,t) = 0, \qquad x\in\partial\Omega,\ T_0-t_1<t<T_0+t_1, \\ y_1(x,T_0) = a - \widetilde{a} - h(\cdot,T_0), \\ \partial_t y_1(x,T_0) = b - \widetilde{b} - (\partial_t h)(\cdot,T_0). \end{cases}$$

In terms of (I.19), we repeat the argument in Case 1, so that the proof of Theorem 0 is completed.

Appendix II. Proof of Lemma 2.4.

We set $u_j = \partial_t^j u$. By $2m$-times taking t-derivatives, we have

$$\begin{cases} \partial_t^2 u_1(x,t) = \Delta u_1(x,t) + p(x)u_1(x,t), \quad x\in\Omega,\ 0<t<T, \\ u_1(x,0) = b(x), \quad \partial_t u_1(x,0) = (\Delta+p)a(x), \qquad x\in\Omega, \\ u_1(x,t) = \partial_t g(x,t), \qquad x\in\partial\Omega,\ 0<t<T, \end{cases}$$

$$\begin{cases} \partial_t^2 u_2(x,t) = \Delta u_2(x,t) + p(x)u_2(x,t), \quad x\in\Omega,\ 0<t<T, \\ u_2(x,0) = (\Delta+p)a(x), \quad \partial_t u_2(x,0) = (\Delta+p)b(x), \quad x\in\Omega, \\ u_2(x,t) = \partial_t^2 g(x,t), \qquad x\in\partial\Omega,\ 0<t<T, \end{cases}$$

$$\cdots$$

$$\begin{cases} \partial_t^2 u_{2m-1}(x,t) = \Delta u_{2m-1}(x,t) + p(x)u_{2m-1}(x,t), \quad x\in\Omega,\ 0<t<T, \\ u_{2m-1}(x,0) = (\Delta+p)^{m-1}b(x), \quad \partial_t u_{2m-1}(x,0) = (\Delta+p)^m a(x), \quad x\in\Omega, \\ u_{2m-1}(x,t) = \partial_t^{2m-1} g(x,t), \qquad x\in\partial\Omega,\ 0<t<T, \end{cases}$$

and

$$\begin{cases} \partial_t^2 u_{2m}(x,t) = \Delta u_{2m}(x,t) + p(x)u_{2m}(x,t), \quad x \in \Omega,\, 0<t<T, \\ u_{2m}(x,0) = (\Delta + p)^m a(x), \quad \partial_t u_{2m}(x,0) = (\Delta+p)^m b(x), \quad x \in \Omega, \\ u_{2m}(x,t) = \partial_t^{2m} g(x,t), \qquad x \in \partial\Omega,\, 0<t<T. \end{cases}$$

By $a \in H^{2m}(\Omega)$, $b \in H^{2m-1}(\Omega)$ and $g \in H^{2m}(0,T;L^2(\partial\Omega))$, the transposition method (e.g., [28], [34]) yields that $\partial_t^j u \in C([0,T];L^2(\Omega))$ and

$$\|\partial_t^j u\|_{C([0,T];L^2(\Omega))} \le C_1(\|a\|_{H^{2m}(\Omega)} + \|b\|_{H^{2m-1}(\Omega)} + \|g\|_{H^{2m}(0,T;L^2(\partial\Omega))}), \quad j = 0,1,\cdots,2m. \tag{II.1}$$

On the other hand, we have

$$\Delta u_j(x,t) + p(x)u_j(x,t) = \partial_t^2 u_j(x,t), \quad x \in \Omega,\, j = 0,1,\cdots,2m-2 \tag{II.2}$$

for any $t \in [0,T]$. Similarly to p.414 in [33], we can choose $\mu \in W_0^{1,\infty}(\Omega)$ such that $0 \le \mu \le 1$ in Ω, $\mu = 1$ in Ω' and

$$\frac{|\nabla \mu(x)|^2}{\mu(x)} \le C_2(\Omega'), \quad x \in \Omega. \tag{II.3}$$

Multiplying (II.2) with μu_j, integrating in x and using the Green formula, we have

$$\begin{aligned} &\int_\Omega \mu |\nabla u_j(x,t)|^2 dx \\ =& -\int_\Omega u_j(x,t)\nabla u_j(x,t)\cdot \nabla\mu(x) dx + \int_\Omega p u_j^2 \mu dx - \int_\Omega \mu u_j \partial_t^2 u_j dx. \end{aligned}$$

Therefore the Cauchy-Schwarz inequality yields

$$\begin{aligned} &\int_\Omega \mu|\nabla u_j(x,t)|^2 dx \le \int_\Omega \sqrt{\mu}|\nabla u_j|\frac{|\nabla\mu|}{\sqrt{\mu}}|u_j| dx + C_3\int_\Omega |u_j|^2 dx + \int_\Omega |u_j||\partial_t^2 u_j| dx \\ \le& \frac{1}{2}\int_\Omega \mu|\nabla u_j|^2 dx + \frac{1}{2}\int_\Omega \frac{|\nabla\mu|^2}{\mu}|u_j|^2 dx \\ &+ C_3\int_\Omega |u_j|^2 dx + \frac{1}{2}\int_\Omega (|u_j|^2 + |\partial_t^2 u_j|^2) dx. \end{aligned}$$

By (II.3), we have

$$\frac{1}{2}\int_{\Omega'} |\nabla \partial_t^j u(x,t)|^2 dx \le \frac{1}{2}\int_{\Omega} |\nabla \partial_t^j u(x,t)|^2 dx$$
$$\le C_4 \int_{\Omega} |\partial_t^j u|^2 dx + C_4 \int_{\Omega} |\partial_t^{j+2} u|^2 dx, \quad j = 0, ..., 2m-2.$$

Thus the proof of (2.10) is completed.

We choose subdomains $\Omega_1, ..., \Omega_{m-1}$ such that $\overline{\Omega''} \subset \Omega_{m-1} \subset \overline{\Omega_{m-1}} \subset \Omega_{m-2} \subset \overline{\Omega_{m-2}} \subset .... \subset \Omega_1 \subset \overline{\Omega_1} \subset \Omega'$. First $(\Delta + p)\partial_t^{2m-2} u(\cdot,t) = \partial_t^{2m} u(\cdot,t)$ in Ω. By (2.10) and (II.1), applying the interior regularity (e.g., Theorem 8.8 (pp.183-184) in [12]), we have

$$\|\partial_t^{2m-2} u(\cdot,t)\|_{H^2(\Omega_1)} \le C_5(\|a\|_{H^{2m}(\Omega)} + \|b\|_{H^{2m-1}(\Omega)} + \|g\|_{H^{2m}(0,T;L^2(\partial\Omega))}). \quad \text{(II.4)}$$

Next $(\Delta + p)\partial_t^{2m-4} u(\cdot,t) = \partial_t^{2m-2} u(\cdot,t)$ in Ω. In terms of (II.4) and (2.10), we apply again the interior regularity (e.g., Theorem 8.10 (p.186) in [12]), so that

$$\|\partial_t^{2m-4} u(\cdot,t)\|_{H^4(\Omega_2)} \le C_6(\|a\|_{H^{2m}(\Omega)} + \|b\|_{H^{2m-1}(\Omega)} + \|g\|_{H^{2m}(0,T;L^2(\partial\Omega))}).$$

Continuing the argument, we complete the proof of (2.11).

Appendix III. Proof of Lemma 2.5.

First, by a usual energy estimate, we can prove that there exists a constant $C_1 > 0$ such that

$$E(t) \le C_1 E(t') + C_1 \|f\|^2_{L^2(0,T;L^2(\Omega))} \quad \text{for any } t, t' \in [0,T]. \quad \text{(III.1)}$$

Henceforth we set $Q_1 = \Omega \times (0,T)$, $T = 2t_0$ and

$$E(t) = \int_{\Omega} (|\nabla w(x,t)|^2 + |\partial_t w(x,t)|^2) dx,$$

and $C_j > 0$ denote generic constants which are dependent on M, Ω, T. In terms of (III.1), it is sufficient to prove that

$$\|w(\cdot,t_0)\|_{H^1(\Omega)} + \|\partial_t w(\cdot,t_0)\|_{L^2(\Omega)} \le C(\|f\|_{L^2(0,T;L^2(\Omega))} + \|w\|_{H^1(0,T;L^2(\omega))}).$$

We choose $x_0 \in \overline{\mathbb{R}^n \setminus (\Omega \setminus \omega)}$ such that

$$\sup_{x\in\Omega} |x - x_0| = \inf_{x'\in\mathbb{R}^n\setminus(\Omega\setminus\omega)} \sup_{x\in\Omega} |x - x'|.$$

By $t_0 > \sup_{x\in\Omega} |x - x_0|$, we can choose $\beta \in (0, 1)$ such that

$$\beta > \frac{\sup_{x\in\Omega} |x - x_0|^2}{t_0^2}.$$

Let

$$\psi(x,t) = |x - x_0|^2 - \beta|t - t_0|^2, \qquad \varphi(x,t) = e^{\lambda\psi(x,t)}.$$

Then $\varphi(x, t_0) \ge 1$ and $\varphi(x, 0) = \varphi(x, 2t_0) < 1$ for $x \in \overline{\Omega}$. Therefore for given $\varepsilon > 0$, we can choose a sufficiently small $\delta = \delta(\varepsilon) > 0$ such that

$$\begin{cases} \varphi(x,t) \ge 1 - \varepsilon, & (x,t) \in \overline{\Omega} \times [t_0 - \delta, t_0 + \delta], \\ \varphi(x,t) \le 1 - 2\varepsilon, & (x,t) \in \overline{\Omega} \times ([0, 2\delta] \cup [2t_0 - 2\delta, 2t_0]). \end{cases} \tag{III.2}$$

We take $\chi_1 \in C_0^\infty(\mathbb{R})$ satisfying $0 \le \chi_1 \le 1$ and

$$\chi_1(t) = \begin{cases} 0 & t \in [0, \delta] \cup [2t_0 - \delta, 2t_0] \\ 1 & t \in [2\delta, 2t_0 - 2\delta]. \end{cases} \tag{III.3}$$

We set $w_1 = \chi_1 w$. Then

$$\begin{cases} \partial_t^2 w_1(x,t) = \Delta w_1 + p(x)w_1(x,t) + \chi_1 f(x,t) + 2(\partial_t\chi_1)\partial_t w + w\partial_t^2\chi_1 \quad \text{in } Q_1, \\ w_1(x,t) = 0, \qquad x \in \partial\Omega,\ 0 < t < 2t_0, \\ w_1(\cdot, 0) = w_1(\cdot, 2t_0) = \partial_t w_1(\cdot, 0) = \partial_t w_1(\cdot, 2t_0) = 0 \quad \text{in } \Omega. \end{cases}$$

Here, setting $T = t_0$, we apply Lemma I.1 in Appendix I to w_1, so that

$$\begin{aligned} &\int_{Q_1} (s|\nabla_{x,t} w_1|^2 + s^3|w_1|^2) e^{2s\varphi} dxdt \le C_2 \int_{Q_1} |\chi_1 f|^2 e^{2s\varphi} dxdt \\ &+ C_2 \int_{Q_1} (|(\partial_t\chi_1)\partial_t w|^2 + |w_1 \partial_t^2 \chi_1|^2) e^{2s\varphi} dxdt \\ &+ C_2 s^3 \int_0^{2t_0} \int_\omega (|\partial_t w_1|^2 + |w_1|^2) e^{2s\varphi} dxdt \end{aligned}$$

for all large $s > 0$. The second term on the right hand side does not vanish only if $\partial_t \chi_1, \partial_t^2 \chi_1 \neq 0$, so that in terms of (III.2) and (III.3), it is bounded by

$$C_2 \int_0^{2t_0} E(t) dt e^{2s(1-2\varepsilon)}.$$

Hence, by (III.1), we have

$$\int_\Omega \int_{t_0-\delta}^{t_0+\delta} (|\nabla_{x,t} w|^2 + |w|^2) e^{2s\varphi} dx dt \leq C_3 e^{C_3 s} \|f\|^2_{L^2(Q_1)}$$
$$+ C_3 e^{2s(1-2\varepsilon)} (E(t_0) + \|f\|^2_{L^2(Q_1)}) + C_3 e^{C_3 s} \|w\|^2_{H^1(0,T;L^2(\omega))}.$$

By (III.2) we see that

$$e^{2s(1-\varepsilon)} \int_{t_0-\delta}^{t_0+\delta} E(t) dt \leq C_4 e^{C_4 s} \|f\|^2_{L^2(Q_1)}$$
$$+ C_4 e^{2s(1-2\varepsilon)} (E(t_0) + \|f\|^2_{L^2(Q_1)}) + C_4 e^{C_4 s} \|w\|^2_{H^1(0,T_0;L^2(\omega))}.$$

By (III.1) the left hand side is greater than or equal to

$$e^{2s(1-\varepsilon)} (E(t_0) - \|f\|^2_{L^2(Q_1)}) 2\delta.$$

Noting the Poincaré inequality and choosing $s > 0$ sufficiently large and fixing, we complete the proof of Lemma 2.5.

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ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012