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Abstract

We consider an inverse problem of determining a coefficient matrix and an initial value for a first order hyperbolic system. Assuming that the boundary values over a time interval are known, we characterize coefficient matrices and initial values, and prove the uniqueness of some components of the matrix function. The proof is based on a transformation formula and the spectral properties of the corresponding nonsymmetric ordinary differential operator.

1 Introduction and the main result

We will consider the following initial value / boundary value problem:

$$\frac{\partial u}{\partial t}(t,x) = B_{2n}\frac{\partial u}{\partial x}(t,x) + P(x)u(t,x) \qquad -T < t < T, \quad 0 < x < 1 \tag{1.1}$$

with boundary conditions

$$u_{\ell+n}(t,0) = h_\ell u_\ell(t,0) \qquad \ell = 1,2,\cdots,n, \quad -T \le t \le T \tag{1.2}$$

$$u_{\ell+n}(t,1) = H_\ell u_\ell(t,1) \qquad \ell = 1,2,\cdots,n, \quad -T \le t \le T \tag{1.3}$$

and with initial conditions

$$u(0,x) = a(x) \qquad 0 \le x \le 1. \tag{1.4}$$

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Here, let $n \in \mathbf{N}, h_\ell, H_\ell \in \mathbf{R} \ \setminus \{-1, 1\}$, $\ell = 1, 2, \cdots, n$, and let

$$u(t,x) = \begin{pmatrix} u_1(t,x) \\ u_2(t,x) \\ \vdots \\ u_{2n}(t,x) \end{pmatrix}, \quad B_{2n} = \begin{pmatrix} 0 & E_n \\ E_n & 0 \end{pmatrix}, \quad E_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix},$$

$$P(x) = \begin{pmatrix} p_{1,1}(x) & p_{1,2}(x) & \dots & p_{1,2n}(x) \\ p_{2,1}(x) & p_{2,2}(x) & \dots & p_{2,2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ p_{2n,1}(x) & p_{2n,2}(x) & \dots & p_{2n,2n}(x) \end{pmatrix}, \quad a(x) = \begin{pmatrix} a_1(x) \\ a_2(x) \\ \vdots \\ a_{2n}(x) \end{pmatrix},$$

and u_ℓ, $p_{k,\ell}$, $1 \le k, \ell \le 2n$ be real-valued. Henceforth 0 denotes zero matrices whose sizes may change line by lne, and $(M)_{k,\ell}$ denotes the (k, ℓ) -component of a matrix M. Moreover we assume also the compatibility condition:

$$\begin{cases} a_{\ell+n}(0) = h_\ell a_\ell(0) \\ a_{\ell+n}(1) = H_\ell a_\ell(1) \end{cases} \qquad \ell = 1, 2, \cdots, n, \quad -T \le t \le T. \tag{1.5}$$

System (1.1) describes some vibrating system. For example, we consider a governing equation of an electric oscillation in parallel n transmission lines:

$$\begin{pmatrix} L(x) & 0 \\ 0 & C(x) \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} I \\ V \end{pmatrix} + \begin{pmatrix} 0 & E_n \\ E_n & 0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} I \\ V \end{pmatrix} + \begin{pmatrix} R(x) & 0 \\ 0 & G(x) \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix} = 0. \tag{1.6}$$

Here $I = I(t,x)$ and $V = V(t,x)$ are vector-valued functions whose j-th components are respectively the current and the voltage of the j-th transmission line. Moreover we assume that the electromagnetic properties of the n lines are not homogeneous in x and the coefficients $R,\ L,\ C,\ G$ depend on $x \in (0,1)$. The parameters $R,\ L,\ C,\ G$ are called a resistance matrix, an inductance matrix, a capacity matrix and a conductance matrix respectively. If there exists a scalar function $r(x) > 0$ such that

$$L(x)C(x) = r(x)E_n, \tag{1.7}$$

we can reduce system (1.6) to (1.1). In fact, in terms of (1.7), we can reduce (1.6) to a equation in the form

$$\frac{\partial}{\partial t} \begin{pmatrix} I \\ V \end{pmatrix} = -\frac{1}{r(x)} \begin{pmatrix} 0 & C(x) \\ L(x) & 0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} I \\ V \end{pmatrix} + \begin{pmatrix} \widetilde{R}(x) & 0 \\ 0 & \widetilde{G}(x) \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix}. \tag{1.8}$$

Changing variables as

$$z \equiv \int_0^x \sqrt{r(y)}dy, \qquad s \equiv t,$$

and

$$u(s,z) \equiv V(t, x(z))$$
$$v(s,z) \equiv -\frac{1}{\sqrt{r(x(z))}} L(x(z)) I(t, x(z)),$$

we obtain the following system:

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = B_{2n}\frac{\partial}{\partial z}\begin{pmatrix} u \\ v \end{pmatrix} + \widetilde{P}(z)\begin{pmatrix} u \\ v \end{pmatrix}.$$

We will investigate

Inverse Problem

Determine a coefficient matrix $P(x)$ and an initial value $a(x)$ from the boundary values $u(t,0), u(t,1), -T \leq t \leq T$.

For inverse problems for one-dimensional first-order system such as (1.1), the method of characteristics is applicable (e.g. Chapter 5 in Romanov [8]). However such a method cannot characterize coefficients and initial values yielding the same boundary values, although boundary data $u(t,0)$, $u(t,1)$, $-T \leq t \leq T$, can simultaneously identify a coefficient matrix and an initial value. For inverse problems for first-order systems, see also Blagoveshchenskii [1]. For the corresponding inverse spectral problems with $n = 1$, see Ning [6], Ning and Yamamoto [7], Trooshin and Yamamoto [11], Yamamoto [12].

In this paper, we will study the uniqueness in our inverse problem. Here, we will only consider the case of $n = 2$. The basic properties for $n = 2$ such as the asymptotic behaviour of eigenvalues, are very different from $n = 1$, and already the case $n = 2$ needs essentially different treatments.

In general, the uniqueness does not hold, as the following example shows.

Example

Let

$$P(x) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad a(x) = \begin{pmatrix} e^{-x} \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$Q(x) = \begin{pmatrix} 0 & 0 & 2x & 0 \\ 0 & 0 & 0 & 0 \\ 2x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad b(x) = \begin{pmatrix} e^{-x^2} \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

and $h_\ell = H_\ell = 0$, $\ell = 1, 2$. Then we can verify that the solution to

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = B_4 \frac{\partial u}{\partial x}(t,x) + P(x)u(t,x) & -T < t < T, \quad 0 < x < 1 \\ u_3(t,0) = u_4(t,0) = 0, \quad -T \le t \le T \\ u_3(t,1) = u_4(t,1) = 0, \quad -T \le t \le T \\ u(0,x) = a(x) \end{cases}$$

is

$$u(t,x) = \begin{pmatrix} e^{-x} \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

while the solution to

$$\begin{cases} \frac{\partial \widetilde{u}}{\partial t}(t,x) = B_4 \frac{\partial \widetilde{u}}{\partial x}(t,x) + Q(x)\widetilde{u}(t,x) & -T < t < T, \quad 0 < x < 1 \\ \widetilde{u}_3(t,0) = \widetilde{u}_4(t,0) = 0, \quad -T \le t \le T \\ \widetilde{u}_3(t,1) = \widetilde{u}_4(t,1) = 0, \quad -T \le t \le T \\ \widetilde{u}(0,x) = b(x) \end{cases}$$

is

$$\widetilde{u}(t,x) = \begin{pmatrix} e^{-x^2} \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore we obtain the same boundary value:

$$u(t,0) = \widetilde{u}(t,0) = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad u(t,1) = \widetilde{u}(t,1) = \begin{pmatrix} e^{-1} \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Consequently, the uniqueness does not hold, even though we restrict the coefficient matrices $P(x)$ in (1.1) to a form $\begin{pmatrix} 0 & P_1(x) \\ P_1(x) & 0 \end{pmatrix}$ with 2×2 matrix $P_1(x)$. $\square$

We will find a condition for the uniqueness to our inverse problem, and the condition should be sufficiently general. Here and henceforth, by $u = u_{P,a}(t,x)$ we denote the solution to

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = B_4 \frac{\partial u}{\partial x}(t,x) + P(x)u(t,x) & -T < t < T, \quad 0 < x < 1 \\ u_{\ell+2}(t,0) = h_\ell u_\ell(t,0) \quad \ell = 1,2 & -T \le t \le T \\ u_{\ell+2}(t,1) = H_\ell u_\ell(t,1) \quad \ell = 1,2 & -T \le t \le T \\ u(0,x) = a(x), \end{cases} \tag{1.9}$$

provided that $h_\ell, H_\ell \in \mathbf{R} \backslash \{-1, 1\}$ are fixed.

Throughout this paper, we assume that the solution $u_{P,a}(t,x)$ is sufficiently smooth. By using an energy estimate we can prove that there exists at most one solution. Moreover the existence of the solution can be proved, and the sufficient smoothness can be proved by compatibility conditions of a and P. We will omit details of the unique existence of $u_{P,a}$ in order to concentrate on the inverse problem.

Henceforth $L^2(0,1)$ and $H^1(0,1)$ are the usual Lebesgue space and Sobolev space of complex-valued functions.

We set

$$M_T(P,a) \equiv \left\{ (Q,b) \in \{C^1[0,1]\}^{20} \ ; \ u_{Q,b}(t,0) = u_{P,a}(t,0), \quad u_{Q,b}(t,1) = u_{P,a}(t,1) \quad -T < t < T \right\}$$

for arbitrarily fixed (P,a) guaranteeing the unique existence of smooth $u_{P,a}$. We can immediately see that $(P,a) \in M_T(P,a)$. If $M_T(P,a)$ has only one element (P,a), then uniqueness in our inverse problem would be true. Thus it is sufficient to characterize the set $M_T(P,a)$.

Definition 1.1 *We define an operator A_P acting from $\{L_2(0,1)\}^4$ to $\{L_2(0,1)\}^4$, by*

$$\begin{cases} (A_P u)(x) = B_4 \frac{du}{dx}(x) + P(x)u(x), \qquad 0 < x < 1 \\ D(A_P) = \{u \in \{H^1(0,1)\}^4; u_{\ell+2}(0) - h_\ell u_\ell(0) = 0, \quad u_{\ell+2}(1) - H_\ell u_\ell(1) = 0, \quad \ell = 1,2\}. \end{cases} \tag{1.10}$$

Definition 1.2 *For an eigenvalue λ of A_P, we call $\phi \neq 0$ a root vector of an operator A_P for λ if $(A_P - \lambda)^k \phi = 0$ for some $k \in \mathbf{N}$. We call $\dim\{\phi; (A_P - \lambda)^k \phi = 0 \text{ for some } k \in \mathbf{N}\}$ and $\dim\{\phi; (A_P - \lambda)\phi = 0\}$ the algebraic multiplicity and the geometric multiplicity of λ, respectively.*

In order to state the main result, we assume the following three conditions:

(I): For each root vector f^* of the adjoint operator A_P^* for A_P, the fixed initial value $a(x)$ satisfies

$$(a, f^*)_{\{L_2(0,1)\}^4} \neq 0. \tag{1.11}$$

(II): The following quadratic equation in α has two distinct roots:

$$\det\left\{\alpha E_2 - \begin{pmatrix} e^{-2\nu_1} & 0 \\ 0 & e^{-2\nu_2} \end{pmatrix} G(\widetilde{\theta}^P)(1) \begin{pmatrix} e^{-2\mu_1} & 0 \\ 0 & e^{-2\mu_2} \end{pmatrix} G(\theta^P)(1)^{-1}\right\} = 0 \tag{1.12}$$

where $h_\ell = \tanh \mu_\ell, \quad H_\ell = -\tanh \nu_\ell$, and

$$\theta^P(x) = (\theta^P_{k,\ell}(x))_{k,\ell=1,2} = \left(\frac{1}{2}(p_{k,\ell}(x) + p_{k,\ell+2}(x) + p_{k+2,\ell}(x) + p_{k+2,\ell+2}(x))\right)_{k,\ell=1,2}, \tag{1.13}$$

$$\widetilde{\theta}^P(x) = (\widetilde{\theta}^P_{k,\ell}(x))_{k,\ell=1,2} = \left(\frac{1}{2}(-p_{k,\ell}(x) + p_{k,\ell+2}(x) + p_{k+2,\ell}(x) - p_{k+2,\ell+2}(x))\right)_{k,\ell=1,2}, \tag{1.14}$$

and by $G(\Theta)(x)$ for a 2×2 -matrix $\Theta = \Theta(x)$, we denote the solution to

$$\frac{d}{dx}(G(\Theta)(x)) + \Theta(x)G(\Theta)(x) = 0, \quad 0 < x < 1 \tag{1.15}$$

with the condition $G(\Theta)(0) = E_2$.

(III): For an arbitrary eigenvalue λ of A_P, we assume that the geometric multiplicity of λ is 1.

Remark 1.3 *Since Condition* (II) *holds if the determinant of quadratic equation* (1.12) *in* α, *is not zero, we can assert that the condition holds generically. Condition* (III) *is always true for* $n = 1$. *By Theorem 2.1 stated in Section 2, if Condition* (II) *holds, then the geometric multiplicities of the eigenvalues is one except for a finite number of eigenvalues. Moreover we can assert that Condition* (III) *holds also generically. In fact, let* φ *and* ψ *be the solutions to* $(A_P - \lambda)u = 0$ *with conditions at* $x = 0$

$$\varphi(0) = \begin{pmatrix} 1 \\ 0 \\ h_1 \\ 0 \end{pmatrix}, \quad \psi(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ h_2 \end{pmatrix},$$

respectively. Let $u \neq 0$ *satisfy* $(A_P - \lambda)u = 0$. *Then* $u = \alpha\varphi + \beta\psi$ *with some* $\alpha, \beta \in \mathrm{C}$. *Since* $u \in D(A_P)$, *we have*

$$(\varphi_3(1) - H_1\varphi_1(1))\alpha + (\psi_3(1) - H_1\psi_1(1))\beta = 0$$

and

$$(\varphi_4(1) - H_2\varphi_2(1))\alpha + (\psi_4(1) - H_2\psi_2(1))\beta = 0.$$

If either of $\varphi_3(1) - H_1\varphi_1(1)$, $\psi_3(1) - H_1\psi_1(1)$, $\varphi_4(1) - H_2\varphi_2(1)$ *and* $\psi_4(1) - H_2\psi_2(1)$ *is not zero, then* $\alpha = \gamma\beta$ *or* $\beta = \gamma\alpha$ *where* γ *is independent of* α *and* β. *Hence* $u = \beta(\gamma\varphi + \psi)$ *or* $u = \alpha(\varphi + \gamma\psi)$. *That is,* $\{u; (A_P - \lambda)u = 0\}$ *is spanned by one vector, which means that the geometric multiplicity of* λ *is one. Therefore Condition* (III) *breaks only if* $\varphi_3(1) - H_1\varphi_1(1) = \psi_3(1) - H_1\psi_1(1) = \varphi_4(1) - H_2\varphi_2(1) = \psi_4(1) - H_2\psi_2(1) = 0$. *Thanks to the transformation formula* (2.10) (Theorem 2.5) *with* $P = 0$, *the condition* $\varphi(1)$ *can be described by*

$$\varphi(1) = R(1)\varphi_0(1, \lambda) + \int_0^1 K(y, 1)\varphi_0(y, \lambda)dy$$

where

$$\varphi_0(x, \lambda) = \begin{pmatrix} \frac{h_1+1}{2}e^{\lambda x} - \frac{h_1-1}{2}e^{-\lambda x} \\ 0 \\ \frac{h_1+1}{2}e^{\lambda x} + \frac{h_1-1}{2}e^{-\lambda x} \\ 0 \end{pmatrix}.$$

Thus $\varphi_3(1) - H_1\varphi_1(1) = \varphi_4(1) - H_2\varphi_2(1) = 0$ *are given by two equations involving* λ *and* K, R. *We note that* K *and* R *are determined by* h_1, h_2 *and* P. *From* $\psi_3(1) - H_1\psi_1(1) = \psi_4(1) - H_2\psi_2(1) = 0$, *we can obtain similar equations. Hence for given* h_1, h_2, H_1, H_2, *if* (λ, P) *does not satisfy those four equations, then Condition* (III) *holds true. In this sense, Condition* (III) *holds generically.*

Let $P(x)$ and $Q(x)$ be 4×4 -matrix functions. Here let 4×4-matrix function

$$R(x) = \begin{pmatrix} R^1(x) & R^2(x) \\ R^2(x) & R^1(x) \end{pmatrix} \tag{1.16}$$

with 2×2 -matrix functions $R^j(x)$, $j = 1, 2$, satisfy the following system of eight ordinary differential equations

$$\begin{cases} (B_4R'(x) + Q(x)R(x) - R(x)P(x))_{k,\ell} + (B_4R'(x) + Q(x)R(x) - R(x)P(x))_{k+2,\ell+2} = 0 \\ (B_4R'(x) + Q(x)R(x) - R(x)P(x))_{k,\ell+2} + (B_4R'(x) + Q(x)R(x) - R(x)P(x))_{k+2,\ell} = 0, \end{cases} \tag{1.17}$$

$$0 < x < 1, \quad k, \ell = 1, 2$$

and $R(0) = E_4$. Here and henceforth we set $R'(x) = \frac{dR}{dx}(x)$. By the theory of ordinary differential equations, we can prove that such an $R(x)$ exists uniquely.

Now we are ready to state our main result characterizing $M_T(P, a)$.

Theorem 1.4 *Let* (P, a) *satisfy Conditions* (I), (II) *and* (III) *and let* $a \in \{C^3[0, 1]\}^4 \cap D(A^2)$. *We assume that* $T \geq 2$. *Then*

$$(Q, b) \in M_T(P, a)$$

if and only if the following conditions hold:

$$R(1) = E_4 \tag{1.18}$$

$$(B_4R'(x) + Q(x)R(x) - R(x)P(x))_{k,\ell} = 0, \quad k, \ell = 1, 2 \quad 0 < x < 1 \tag{1.19}$$

$$(B_4R'(x) + Q(x)R(x) - R(x)P(x))_{k,\ell+2} = 0, \quad k, \ell = 1, 2 \quad 0 < x < 1 \tag{1.20}$$

$$b(x) = R(x)a(x). \tag{1.21}$$

The theorem gives the uniqueness for some components. For example, we can prove obtain the following result by verifying that (1.19) and (1.20) yield $p_{1,\ell} = q_{1,\ell}$, $1 \le \ell \le 4$ when $p_{k,\ell} = q_{k,\ell} = 0$ for $2 \le k \le 4$ and $1 \le \ell \le 4$.

Corollary 1.5 *If we restrict a class of coefficient matrices to the matrix with the form*

$$\left\{ \begin{pmatrix} a(x) & b(x) & c(x) & d(x) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad a, b, c, d \in C^1[0,1] \right\}$$

and the initial value is known, then the solution to the inverse problem is unique under Conditions (I) - (III).

In Section 2, we state spectral properties of the operator A_P and in Sections 3-4, we prove them. Section 5 is devoted to the proof of Theorem 1.4.

2 Spectral property of A_P and transformation formulae

In this section, we will first present the spectral property of A_P defined by (1.10), and such properties are necessary for the proof of Theorem 1.4. There are very few works concerning spectral properties for a nonsymmetric operator of ordinary differential equations and Theorems 2.1 and 2.3 may be independent interests. On the other hand, there are many results on the spectral properties for the classical Sturm-Liouville problem and readers can consult Levitan and Sargsjan [4], Naimark [5] as monographs. For $n = 1$, see Trooshin snd Yamamoto [10].

Let $\sigma(A_P)$ denote the spectrum of the operator A_P and let $i = \sqrt{-1}$.

We present the asymptotic behaviour of $\sigma(A_P)$.

Theorem 2.1 *There exist $N \in \mathbf{N}$ and $\Sigma_1, \Sigma_2 \subset \sigma(A_P)$ such that*

$$\sigma(A_P) = \Sigma_1 \cup \Sigma_2, \qquad \Sigma_1 \cap \Sigma_2 = \emptyset.$$

(1) *Let equation* (1.12) *possess distinct roots* α_1 *and* α_2. *Then the following* (a) *and* (b) *hold.*

(a)*:* Σ_1 *consists of* $2(2N-1)$ *eigenvalues by taking the algebraic multiplicities into consideration, and is included in*

$$\left\{\lambda\ ;\ |Im\lambda - \widetilde{\alpha}| < N\pi - \frac{\pi}{2}\right\}.$$

Here and henceforth we set

$$\widetilde{\alpha} = \frac{1}{4}\mathrm{Im}\log\alpha_1 + \frac{1}{4}\mathrm{Im}\log\alpha_2$$

and we take the principal value of the logarithm : $-\pi < \mathrm{Im}\log\alpha_j \le \pi,\ j = 1, 2.$

(b)*: All the elements of* Σ_2 *are eigenvalues whose algebraic multiplicities are one, and*

$$\Sigma_2 \subset \left\{\lambda \;\middle|\; |Im\lambda - \widetilde{\alpha}| > N\pi - \frac{\pi}{2}\right\}$$

and with suitable numbering $\{\lambda_{j,m}\}_{j=1,2,|m|\ge N, m\in\mathbf{Z}}$ *of* $\sigma(A)$*, the eigenvalues have an asymptotic behaviour*

$$\lambda_{j,m} = \frac{1}{2}\log\alpha_j + m\pi i + O\left(\frac{1}{|m|}\right) \tag{2.1}$$

as $|m| \to \infty$.

(2) *Let* (1.12) *possess the multiple root* $\alpha_1 = \alpha_2 \equiv \alpha$. *Then* Σ_1 *has the same property as in Case* (1) *and we can number all the eigenvalues of* Σ_2 *by* $\{\lambda_{j,m}\}_{j=1,2,|m|\ge N, m\in\mathbf{Z}}$ *such that* $\lambda_{1,m} = \lambda_{2,m}$ *may happen, but* $\lambda_{j,m} \ne \lambda_{j',m'}$ *for* $j, j' = 1, 2$ *if* $m \ne m'$*, and*

$$\lambda_{j,m} = \frac{1}{2}\log\alpha + m\pi i + O\left(\frac{1}{\sqrt{|m|}}\right) \tag{2.2}$$

as $|m| \to \infty$. *Moreover for sufficiently large* $|m|$*, the algebraic multiplicities of* $\lambda_{1,m}$ *and* $\lambda_{2,m}$ *are one if* $\lambda_{1,m} \ne \lambda_{2,m}$ *and are two if* $\lambda_{1,n} = \lambda_{2,n}$.

The asymptotic behaviour in the case of $\alpha_1 \ne \alpha_2$ has two branches whose real parts are close to $\frac{1}{2}\,\mathrm{Re}\log\alpha_1$ and $\frac{1}{2}\,\mathrm{Re}\log\alpha_2$, and is very different from the case of $n = 1$.

Next we discuss the completeness of eigenvectors.

Definition 2.2 *We call $\{b_m\}_{m\in\mathbf{Z}}$ a Riesz basis in $\{L_2(0,1)\}^4$ if each $u \in \{L_2(0,1)\}^4$ has a unique expansion*

$$u = \sum_{m\in\mathbf{Z}} c_m b_m, \qquad c_m \in \mathbf{C}$$

and there exists a positive number M, which is independent of the choice of u, such that

$$M^{-1}\sum_{m\in\mathbf{Z}} |c_m|^2 \le ||u||^2_{\{L_2(0,1)\}^4} \le M \sum_{m\in\mathbf{Z}} |c_m|^2.$$

We state the completeness of the root vectors.

Theorem 2.3 *Let* (1.12) *have two distinct roots. Then the set of all the root vectors of A_P is a Riesz basis in $\{L_2(0,1)\}^4$.*

In Theorem 2.3, we note that we need not assume Condition (III).

In order to state transformation formulae, which are basic tools for our inverse problem, we prove the following lemma. Until the end of section 2, we will consider general $n \in \mathbf{N}$, not necessarily $n = 2$.

Lemma 2.4 *Assume that $P(x)$ and $Q(x)$ are $2n\times 2n$-matrix functions whose elements are in $C^1[0,1]$. Let $a_{k,\ell}(x), b_{k,\ell}(x), 1 \le k,\ell \le n$ be real valued functions. Let $h_k, 1\le k\le n$ be constants and $|h_k| \ne 1, 1 \le k \le n$. Moreover we set*

$$\Omega = \Big\{(y,x) \in \mathbf{R}^2 \ ;\ 0 < y < x < 1\Big\}$$

Then there exists a unique solution $K(y,x) \in \{C^1(\bar\Omega)\}^{2n\times 2n}$ to

$$B_{2n}\frac{\partial K}{\partial x}(y,x) + Q(x)K(y,x) = K(y,x)P(y) - \frac{\partial K}{\partial y}(y,x)B_{2n} \qquad \text{in } \Omega \tag{2.3}$$

$$\begin{cases} K_{k,\ell+n}(0,x) = -h_k K_{k,\ell}(0,x) \\ K_{k+n,\ell+n}(0,x) = -h_k K_{k+n,\ell}(0,x) \end{cases}, \qquad k,\ell = 1,2,\cdots,n, \qquad 0 \le x \le 1 \tag{2.4}$$

$$\begin{cases} K_{k,\ell+n}(x,x) - K_{k+n,\ell}(x,x) = a_{k,\ell}(x) \\ K_{k,\ell}(x,x) - K_{k+n,\ell+n}(x,x) = b_{k,\ell}(x) \end{cases} \qquad k,\ell = 1,2,\cdots,n, \qquad 0 \le x \le 1. \tag{2.5}$$

The proof is given in Appendix A.

Transformation formulae are given as follows. Let $P(x),\ Q(x)$ be fixed $2n \times 2n$ -matrix functions with $C^1[0,1]$-elements. Here, let $2n \times 2n$-matrix function $R(x)$

$$R(x) = \begin{pmatrix} R^1(x) & R^2(x) \\ R^2(x) & R^1(x) \end{pmatrix}$$

with $n \times n$ matrix functions $R^j(x)$, $j = 1, 2$, satisfy system of $2n^2$ ordinary differential equations:

$$\begin{cases} (B_{2n}R'(x) + Q(x)R(x) - R(x)P(x))_{k,\ell} + (B_{2n}R'(x) + Q(x)R(x) - R(x)P(x))_{k+n,\ell+n} = 0 \\ (B_{2n}R'(x) + Q(x)R(x) - R(x)P(x))_{k,\ell+n} + (B_{2n}R'(x) + Q(x)R(x) - R(x)P(x))_{k+n,\ell} = 0, \end{cases} \tag{2.6}$$

$$0 < x < 1, \quad k, \ell = 1, 2, \cdots, n$$

and

$$R(0) = E_{2n}.$$

By a classical theory of ordinary differential equations, we can prove that there exists a unique solution $R = R(x)$ to this system of ordinary differential equations.

Theorem 2.5 (Transformation formula in the stationary case)

Let $\tau_1, \tau_2, \cdots, \tau_n \in \mathbf{R}$ and let $K = K(y, x)$ be the solution to (2.3), (2.4) *and*

$$\begin{cases} K_{k,\ell+n}(x,x) - K_{k+n,\ell}(x,x) = (B_{2n}R'(x) + Q(x)R(x) - R(x)P(x))_{k,\ell} \\ K_{k,\ell}(x,x) - K_{k+n,\ell+n}(x,x) = (B_{2n}R'(x) + Q(x)R(x) - R(x)P(x))_{k,\ell+n} \end{cases}, \quad k, \ell = 1, 2, \cdots, n. \tag{2.7}$$

Assume that $\phi(x,\lambda) = \begin{pmatrix} \phi_1(x,\lambda) \\ \vdots \\ \phi_{2n}(x,\lambda) \end{pmatrix}$ *and* $\psi(x,\lambda) = \begin{pmatrix} \psi_1(x,\lambda) \\ \vdots \\ \psi_{2n}(x,\lambda) \end{pmatrix}$ *are $\mathbf{R}^{2n}$-valued functions and satisfy*

$$\begin{cases} B_{2n}\frac{d\phi}{dx} + P(x)\phi = \lambda\phi, \qquad 0 < x < 1 \\ \phi_1(0,\lambda) = \tau_1, \quad \cdots, \quad \phi_n(0,\lambda) = \tau_n \\ \phi_{n+1}(0,\lambda) = h_1\tau_1, \quad \cdots, \quad \phi_{2n}(0,\lambda) = h_n\tau_n \end{cases} \tag{2.8}$$

$$\begin{cases} B_{2n}\frac{d\psi}{dx} + Q(x)\psi = \lambda\psi, \qquad 0 < x < 1 \\ \psi_1(0,\lambda) = \tau_1, \quad \cdots, \quad \psi_n(0,\lambda) = \tau_n \\ \psi_{n+1}(0,\lambda) = h_1\tau_1, \quad \cdots, \quad \psi_{2n}(0,\lambda) = h_n\tau_n. \end{cases} \tag{2.9}$$

Then,

$$\psi(x,\lambda) = R(x)\phi(x,\lambda) + \int_0^x K(y,x)\phi(y,\lambda)dy, \qquad 0 < x < 1. \tag{2.10}$$

The proof of Theorem 2.5 is given in Appendix B.

Next we consider the following Cauchy problems:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = B_{2n}\frac{\partial u}{\partial x}(t,x) + P(x)u(t,x), \qquad x > 0, \quad -T+x < t < T-x \\ u_\ell(t,0) = \omega_\ell(t), \quad u_{n+\ell}(t,0) = h_\ell\omega_\ell(t), \quad \ell = 1,2,\cdots,n \end{cases} \tag{2.11}$$

and

$$\begin{cases} \frac{\partial \widetilde{u}}{\partial t}(t,x) = B_{2n}\frac{\partial \widetilde{u}}{\partial x}(t,x) + Q(x)\widetilde{u}(t,x), \qquad x > 0, \quad -T+x < t < T-x \\ \widetilde{u}_\ell(t,0) = \omega_\ell(t), \quad \widetilde{u}_{n+\ell}(t,0) = h_\ell\omega_\ell(t), \quad \ell = 1,2,\cdots,n \end{cases} \tag{2.12}$$

for given $\omega_\ell \in C^1[-T,T]$.

We can prove the transformation formula for these Cauchy problems.

Theorem 2.6 *Between the solution to* (2.11) *and the solution to* (2.12)*, the following relation holds :*

$$\widetilde{u}(t,x) = R(x)u(t,x) + \int_0^x K(y,x)u(t,y)dy, \qquad x > 0, \quad -T+x < t < T-x,$$

where $R(x)$ and $K(y,x)$ are defined in Theorem 2.5.

Theorem 2.6 can be proved similarly to Theorem 2.5, by verifying that the right hand side satisfies (2.12) and using the uniqueness for the Cauchy problem (2.12). We omit the proof.

3 The proof of Theorem 2.1

Step 1. We shall prove the following lemma.

Lemma 3.1 *The spectrum $\sigma(A_P)$ consists entirely of countable isolated eigenvalues with finite algebraic multiplicities.*

Proof of Lemma 3.1. Let $U(x,\lambda) = (U_{k,\ell}(x,\lambda))_{k,\ell=1,2,3,4}$ be the solution to

$$\begin{cases} B_4\frac{dU}{dx} + P(x)U = \lambda U, & 0 < x < 1 \\ U(0,\lambda) = E_4. \end{cases} \tag{3.1}$$

We set

$$\widetilde{h} = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \quad \widetilde{H} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix},$$

$$B_4^{(0)} = \begin{pmatrix} -\widetilde{h} & E_2 \\ 0 & 0 \end{pmatrix}, \quad B_4^{(1)} = \begin{pmatrix} 0 & 0 \\ -\widetilde{H} & E_2 \end{pmatrix}.$$

We note that 0 means a zero matrix whose sizes may change line by line and for example, in the above, 0 means the 2×2 zero matrix. Then we have

$$\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} \in D(A_P)$$

if and only if

$$\gamma \in \{H^1(0,1)\}^4, \quad B_4^{(0)}\gamma(0) + B_4^{(1)}\gamma(1) = 0.$$

For given $f \in \left\{L^2(0,1)\right\}^4$, let us consider the following equation:

$$\left(B_4\frac{d}{dx} + P(x) - \lambda\right)\gamma = f.$$

By the variation of constants, a general solution to this equation is

$$\gamma(x,\lambda) = U(x,\lambda)\eta + U(x,\lambda)\int_0^x U(y,\lambda)^{-1}B_4 f(y)dy,$$

where $U(x,\lambda)$ is the fundamental solution and $\eta \in \mathbf{C}^4$ is arbitrary. In order to satisfy the condition $\gamma \in D(A_P)$ for fixed λ, we choose η such that

$$B_4^{(0)}\gamma(0) + B_4^{(1)}\gamma(1) = 0,$$

that is to say,

$$(B_4^{(0)} + B_4^{(1)}U(1,\lambda))\eta + B_4^{(1)}U(1,\lambda)\int_0^1 U(y,\lambda)^{-1}B_4 f(y)dy = 0.$$

If $\det(B_4^{(0)} + B_4^{(1)}U(1,\lambda)) \neq 0$, then

$$\eta = -(B_4^{(0)} + B_4^{(1)}U(1,\lambda))^{-1}B_4^{(1)}U(1,\lambda)\int_0^1 U(y,\lambda)^{-1}B_4 f(y)dy$$

satisfies this condition. Moreover we can write

$$\begin{aligned}\gamma(x,\lambda) = &-U(x,\lambda)(B_4^{(0)} + B_4^{(1)}U(1,\lambda))^{-1}B_4^{(1)}U(1,\lambda)\int_0^1 U(y,\lambda)^{-1}B_4 f(y)dy \\ &+ U(x,\lambda)\int_0^x U(y,\lambda)^{-1}B_4 f(y)dy.\end{aligned}$$

Therefore, if $\det(B_4^{(0)} + B_4^{(1)}U(1,\lambda_0)) \neq 0$ for some $\lambda_0 \in \mathbf{C}$, then $(A_P - \lambda_0)^{-1}$ is a compact operator from $\{L^2(0,1)\}^4$ to itself. By Kato [3], this implies that $\sigma(A_P)$ consists of isolated eigenvalues with finite algebraic multiplicities. Hence it is sufficient to show that there exists $\lambda_0 \in \mathbf{C}$ such that $\det(B_4^{(0)} + B_4^{(1)}U(1,\lambda_0)) \neq 0$.

Since

$$U_0(x,\lambda) = \begin{pmatrix} E_2\cosh\lambda x & E_2\sinh\lambda x \\ E_2\sinh\lambda x & E_2\cosh\lambda x \end{pmatrix}$$

is the solution to

$$\begin{cases} B_4\frac{d}{dx}U_0(x,\lambda) = \lambda U_0(x,\lambda), & 0 < x < 1 \\ U_0(0,\lambda) = E_4, \end{cases}$$

by the transformation formula, we can write

$$\begin{aligned}U(x,\lambda) = R(x)U_0(x,\lambda) &+ \int_0^x K^{(1)}(y,x)\begin{pmatrix} E_2\cosh\lambda y & 0 \\ E_2\sinh\lambda y & 0 \end{pmatrix}dy \\ &+ \int_0^x K^{(2)}(y,x)\begin{pmatrix} 0 & E_2\sinh\lambda y \\ 0 & E_2\cosh\lambda y \end{pmatrix}dy. \qquad (3.2)\end{aligned}$$

Here, we recall that the 4×4-matrix

$$R(x) = \begin{pmatrix} R^1(x) & R^2(x) \\ R^2(x) & R^1(x) \end{pmatrix}$$

with 2×2-matrix functions R^j, $j = 1, 2$, satisfies

$$\begin{cases} (B_4R'(x) + P(x)R(x))_{k,\ell} + (B_4R'(x) + P(x)R(x))_{k+2,\ell+2} = 0 \\ (B_4R'(x) + P(x)R(x))_{k,\ell+2} + (B_4R'(x) + P(x)R(x))_{k+2,\ell} = 0, \end{cases} \qquad k, \ell = 1, 2, \quad 0 \le x \le 1$$

and $R(0) = E_4$. Let $K^{(1)}$ be the solution to

$$\begin{cases} B_4 \frac{\partial K^{(1)}}{\partial x}(y, x) + P(x)K^{(1)}(y, x) = -\frac{\partial K^{(1)}}{\partial y}(y, x)B_4 \qquad in\ \Omega \\ K^{(1)}_{k,\ell+2}(0, x) = 0, \quad k = 1, 2, 3, 4, \quad \ell = 1, 2 \\ K^{(1)}_{k,\ell+2}(x, x) - K^{(1)}_{k+2,\ell}(x, x) = [B_4R'(x) + P(x)R(x)]_{k,\ell}, \quad 0 \le x \le 1 \quad k, \ell = 1, 2 \\ K^{(1)}_{k,\ell}(x, x) - K^{(1)}_{k+2,\ell+2}(x, x) = [B_4R'(x) + P(x)R(x)]_{k,\ell+2}, \quad 0 \le x \le 1 \quad k, \ell = 1, 2, \end{cases} \tag{3.3}$$

and $K^{(2)}$ be the solution to

$$\begin{cases} B_4 \frac{\partial K^{(2)}}{\partial x}(y, x) + P(x)K^{(2)}(y, x) = -\frac{\partial K^{(2)}}{\partial y}(y, x)B_4 \qquad in\ \Omega \\ K^{(2)}_{k,\ell}(0, x) = 0, \quad k = 1, 2, 3, 4, \quad \ell = 1, 2 \\ K^{(2)}_{k,\ell+2}(x, x) - K^{(2)}_{k+2,\ell}(x, x) = [B_4R'(x) + P(x)R(x)]_{k,\ell}, \quad 0 \le x \le 1 \quad k, \ell = 1, 2 \\ K^{(2)}_{k,\ell}(x, x) - K^{(2)}_{k+2,\ell+2}(x, x) = [B_4R'(x) + P(x)R(x)]_{k,\ell+2}, \quad 0 \le x \le 1 \quad k, \ell = 1, 2. \end{cases} \tag{3.4}$$

We can prove by a usual method of characteristics that $K^{(1)}$ and $K^{(2)}$ exist uniquely.

Let us consider the second term on the right hand side of (3.2). By integration by parts, we obtain

$$\int_0^x K^{(1)}(y, x) \begin{pmatrix} E_2 \cosh \lambda y & 0 \\ E_2 \sinh \lambda y & 0 \end{pmatrix} dy = \frac{1}{\lambda} \int_0^x K^{(1)}(y, x) \frac{d}{dy} \begin{pmatrix} E_2 \sinh \lambda y & 0 \\ E_2 \cosh \lambda y & 0 \end{pmatrix} dy$$
$$= \frac{1}{\lambda} \left\{ \left[K^{(1)}(y, x) \begin{pmatrix} E_2 \sinh \lambda y & 0 \\ E_2 \cosh \lambda y & 0 \end{pmatrix} \right]_{y=0}^{y=x} - \int_0^x \frac{\partial}{\partial y} K^{(1)}(y, x) \begin{pmatrix} E_2 \sinh \lambda y & 0 \\ E_2 \cosh \lambda y & 0 \end{pmatrix} dy \right\}.$$

Therefore, for any $C > 0$, there exists a constant $C_0 > 0$, which is dependent on C and is independent of λ, such that

$$\sup_{0 \le x \le 1} \left| \int_0^x K^{(1)}(y, x) \begin{pmatrix} E_2 \cosh \lambda y & 0 \\ E_2 \sinh \lambda y & 0 \end{pmatrix} dy \right| \le \frac{C_0}{|\lambda|} \qquad \text{if } |Re\lambda| \le C.$$

Here, for a 4×4-matrix M, we define a matrix norm $|M|$ by

$$|M| = \max_{k,\ell=1,2,3,4} |M_{k,\ell}|.$$

Similarly, we can verify that there exists a constant $C_0 = C_0(C) > 0$ such that

$$\sup_{0 \le x \le 1} \left| \int_0^x K^{(1)}(y, x) \begin{pmatrix} E_2 \cosh \lambda y & 0 \\ E_2 \sinh \lambda y & 0 \end{pmatrix} dy + \int_0^x K^{(2)}(y, x) \begin{pmatrix} 0 & E_2 \sinh \lambda y \\ 0 & E_2 \cosh \lambda y \end{pmatrix} dy \right| \le \frac{C_0}{|\lambda|} \tag{3.5}$$

if $|\operatorname{Re} \lambda| \le C$.

Setting $\lambda = \beta + 2m\pi i$ with $\beta \in \mathbf{C}$ and $m \in \mathbf{Z}$, we can write

$$\det(B_4^{(0)} + B_4^{(1)} U(1,\lambda)) = \det\left(B_4^{(0)} + B_4^{(1)} R(1) \begin{pmatrix} E_2 \cosh\beta & E_2 \sinh\beta \\ E_2 \sinh\beta & E_2 \cosh\beta \end{pmatrix} \right) + O\left(\frac{1}{|m|}\right).$$

Let us calculate

$$B_4^{(0)} + B_4^{(1)} R(1) \begin{pmatrix} E_2 \cosh\beta & E_2 \sinh\beta \\ E_2 \sinh\beta & E_2 \cosh\beta \end{pmatrix}.$$

By the definition of $R(x)$, we can write

$$\begin{aligned} R'_{k,\ell}(x) &= -\frac{1}{2} \sum_{m=1}^{2} (P_{k,m+2}(x) + P_{k+2,m}(x)) R_{m,\ell}(x) \\ &\quad - \frac{1}{2} \sum_{m=1}^{2} (P_{k,m}(x) + P_{k+2,m+2}(x)) R_{m,\ell+2}(x), \qquad k,\ell = 1,2 \\ R'_{k,\ell+2}(x) &= -\frac{1}{2} \sum_{m=1}^{2} (P_{k,m}(x) + P_{k+2,m+2}(x)) R_{m,\ell}(x) \\ &\quad - \frac{1}{2} \sum_{m=1}^{2} (P_{k,m+2}(x) + P_{k+2,m}(x)) R_{m,\ell+2}(x), \qquad k,\ell = 1,2 \\ R_{k,\ell}(0) &= \delta_{k\ell}, \quad k,\ell = 1,2, \quad R_{k,\ell+2}(0) = 0, \quad k,\ell = 1,2. \end{aligned}$$

Here and henceforth we set $\delta_{kk} = 1$ and $\delta_{k\ell} = 0$ if $k \neq \ell$. Setting

$$\begin{cases} r_{k,\ell}(x) = R_{k,\ell}(x) + R_{k,\ell+2}(x) \\ \widetilde{r}_{k,\ell}(x) = R_{k,\ell}(x) - R_{k,\ell+2}(x), \end{cases} \qquad k,\ell = 1,2$$

we can reduce the preceding differential equation into

$$\begin{cases} r'_{k,\ell}(x) + \sum_{m=1}^{2} \theta^P_{k,m}(x) r_{m,\ell}(x) = 0 \\ \widetilde{r}'_{k,\ell}(x) + \sum_{m=1}^{2} \widetilde{\theta}^P_{k,m}(x) \widetilde{r}_{m,\ell}(x) = 0 \\ r_{k,\ell}(0) = \widetilde{r}_{k,\ell}(0) = \delta_{k\ell}, \end{cases} \qquad k,\ell = 1,2.$$

Recalling the definition of $G(\theta^P)(x)$ and $G(\widetilde{\theta}^P)(x)$, we can write

$$R(x) = \frac{1}{2} \begin{pmatrix} G(\theta^P)(x) + G(\widetilde{\theta}^P)(x) & G(\theta^P)(x) - G(\widetilde{\theta}^P)(x) \\ G(\theta^P)(x) - G(\widetilde{\theta}^P)(x) & G(\theta^P)(x) + G(\widetilde{\theta}^P)(x) \end{pmatrix}. \tag{3.6}$$

Hence

$$
\begin{aligned}
&B_4^{(0)} + B_4^{(1)} R(1) \begin{pmatrix} E_2 \cosh\beta & E_2 \sinh\beta \\ E_2 \sinh\beta & E_2 \cosh\beta \end{pmatrix} \\
&= \begin{pmatrix} -\widetilde{h} & E_2 \\ 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 \\ -\widetilde{H} & E_2 \end{pmatrix} \begin{pmatrix} e^{\beta} G(\theta^P)(1) + e^{-\beta} G(\widetilde{\theta}^P)(1) & e^{\beta} G(\theta^P)(1) - e^{-\beta} G(\widetilde{\theta}^P)(1) \\ e^{\beta} G(\theta^P)(1) - e^{-\beta} G(\widetilde{\theta^P})(1) & e^{\beta} G(\theta^P)(1) + e^{-\beta} G(\widetilde{\theta^P})(1) \end{pmatrix} \\
&= \begin{pmatrix} -\widetilde{h} & E_2 \\ \frac{e^{\beta}(E_2 - \widetilde{H}) G(\theta^P)(1) - e^{-\beta}(E_2 + \widetilde{H}) G(\widetilde{\theta^P})(1)}{2} & \frac{e^{\beta}(E_2 - \widetilde{H}) G(\theta^P)(1) + e^{-\beta}(E_2 + \widetilde{H}) G(\widetilde{\theta}^P)(1)}{2} \end{pmatrix}.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&\det \left(B_4^{(0)} + B_4^{(1)} R(1) \begin{pmatrix} E_2 \cosh\beta & E_2 \sinh\beta \\ E_2 \sinh\beta & E_2 \cosh\beta \end{pmatrix} \right) \\
&= \frac{1}{2} \det \begin{pmatrix} 0 & E_2 \\ e^{\beta} \left\{ (E_2 - \widetilde{H}) G(\theta^P)(1)(E_2 + \widetilde{h}) - e^{-2\beta}(E_2 + \widetilde{H}) G(\widetilde{\theta}^P)(1)(E_2 - \widetilde{h}) \right\} & * \end{pmatrix}.
\end{aligned}
$$

This determinant is not zero if and only if

$$
\det \left(e^{\beta} \left\{ (E_2 - \widetilde{H}) G(\theta^P)(1)(E_2 + \widetilde{h}) - e^{-2\beta}(E_2 + \widetilde{H}) G(\widetilde{\theta}^P)(1)(E_2 - \widetilde{h}) \right\} \right) \neq 0.
$$

By $h_j \neq \pm 1$ and $H_j \neq \pm 1$ for $j = 1, 2$, $\det G(\theta^P)(1) \neq 0$ and the continuity of the determinant, for sufficiently large $\operatorname{Re}\beta > 0$, the preceding determinant is not equal to zero. Here we used that $\det G(\theta^P)(1) \neq 0$. In fact, for $y \in (0, 1)$, by $G(\theta)(x; y)$ we denote the solution to (1.15) such that $G(\theta)(y; y) = E_2$. Then the uniqueness of the initial value problem for (1.15) yields $G(\theta)(x; y) G(\theta)(y; x) = E_2$, which implies det $G(\theta)(x; y) \neq 0$ for any $x, y \in (0, 1)$. Since $G(\theta^P)(1) = G(\theta^P)(1; 0)$ by the definition, we have $\det G(\theta^P)(1) \neq 0$.

Consequently we can choose sufficiently large $|m|$ and sufficiently large $\operatorname{Re}\beta > 0$ such that

$$
\det(B_4^{(0)} + B_4^{(1)} U(1, \lambda)) \neq 0
$$

for $\lambda \neq \beta + 2m\pi i$.

Therefore, the proof of Lemma 3.1 is completed. $\square$

Step 2. Let a 4×2-matrix function $\phi(x, \lambda)$ be the solution to the following equations:

$$\begin{cases} B_4 \frac{d}{dx}\phi(x,\lambda) + P(x)\phi(x,\lambda) = \lambda\phi(x,\lambda), & 0 < x < 1 \\ \phi(0,\lambda) = \begin{pmatrix} E_2 \\ \widetilde{h} \end{pmatrix}. \end{cases}$$

Then, $\lambda \in \mathbf{C}$ is an eigenvalue of A_P if and only if the determinant of

$$\Phi(\lambda) = \begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} \phi(1,\lambda)$$

is equal to zero. Henceforth we call $\det \Phi(\lambda)$ the characteristic function for A_P. In fact, if ψ is an eigenfunction of A_P, then we can choose $(c_1, c_2) \neq (0, 0)$ such that

$$\psi(x,\lambda) = c_1\phi_1(x,\lambda) + c_2\phi_2(x,\lambda),$$

where ϕ_ℓ is the ℓ-th column vector of $\phi(x,\lambda)$, $\ell = 1, 2$. Since

$$\begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} \psi(1,\lambda) = 0,$$

we have

$$\begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} \psi(1,\lambda) = \begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} c_1\phi_1(1,\lambda) + \begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} c_2\phi_2(1,\lambda) = 0.$$

Hence

$$\left\{ \begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} \phi_1(1,\lambda) \, , \, \begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} \phi_2(1,\lambda) \right\}$$

is linearly dependent, so that $\det \Phi(\lambda) = 0$ follows.

Conversely, if

$$\begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} \phi_\ell(1,\lambda), \qquad \ell = 1, 2$$

are linearly dependent, then there exists $(c_1, c_2) \neq (0, 0)$ such that

$$\begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} (c_1\phi_1(1,\lambda) + c_2\phi_2(1,\lambda)) = 0.$$

Then $\psi(x,\lambda) = c_1\phi_1(x,\lambda) + c_2\phi_2(x,\lambda)$ is an eigenfunction of A_P, that is, λ is an eigenvalue of A_P. Thus we have proved that λ is an eigenvalue of A_P if and only if $\det \Phi(\lambda) = 0$.

Moreover we can prove

Lemma 3.2 *The algebraic multiplicity of an eigenvalue λ_0 is equal to the multiplicity of λ_0 as zero of $\det \Phi(\lambda)$.*

The proof is given in Appendix C.

Let us calculate $\Phi(\lambda)$. Using the transformation formula, we have

$$\begin{aligned}
\phi(x,\lambda) = R(x) &\begin{pmatrix} \cosh \lambda x + h_1 \sinh \lambda x & 0 \\ 0 & \cosh \lambda x + h_2 \sinh \lambda x \\ \sinh \lambda x + h_1 \cosh \lambda x & 0 \\ 0 & \sinh \lambda x + h_2 \cosh \lambda x \end{pmatrix} \\
&+ \int_0^x K^{(1)}(y,x) \begin{pmatrix} \cosh \lambda y & 0 \\ 0 & \cosh \lambda y \\ \sinh \lambda y & 0 \\ 0 & \sinh \lambda y \end{pmatrix} dy \\
&+ \int_0^x K^{(2)}(y,x) \begin{pmatrix} h_1 \sinh \lambda y & 0 \\ 0 & h_2 \sinh \lambda y \\ h_1 \cosh \lambda y & 0 \\ 0 & h_2 \cosh \lambda y \end{pmatrix} dy. \qquad (3.7)
\end{aligned}$$

Here $K^{(1)}(y,x)$ and $K^{(2)}(y,x)$ are defined by (3.3) and (3.4).

For simplicity, by $\widetilde{\phi}(\lambda)$ we denote the integral terms on the right hand side of (3.7) with $x = 1$. Setting $h_\ell = \tanh \mu_\ell$, we can write

$$\phi(1,\lambda) = R(1) \begin{pmatrix} \cosh(\lambda+\mu_1) & 0 \\ 0 & \cosh(\lambda+\mu_2) \\ \sinh(\lambda+\mu_1) & 0 \\ 0 & \sinh(\lambda+\mu_2) \end{pmatrix} \begin{pmatrix} \frac{1}{\cosh \mu_1} & 0 \\ 0 & \frac{1}{\cosh \mu_2} \end{pmatrix} + \widetilde{\phi}(\lambda). \qquad (3.8)$$

By (3.6), we have

$$\phi(1,\lambda) = \frac{1}{2} \begin{pmatrix} (A_{k,\ell} e^{\lambda+\mu_\ell} + B_{k,\ell} e^{-\lambda-\mu_\ell})_{k,\ell=1,2} \\ (A_{k,\ell} e^{\lambda+\mu_\ell} - B_{k,\ell} e^{-\lambda-\mu_\ell})_{k,\ell=1,2} \end{pmatrix} \begin{pmatrix} \frac{1}{\cosh \mu_1} & 0 \\ 0 & \frac{1}{\cosh \mu_2} \end{pmatrix} + \widetilde{\phi}(\lambda),$$

where $G(\theta^P)(1) = (A_{k,\ell})_{k,\ell=1,2}$, $\ G(\widetilde{\theta}^P)(1) = (B_{k,\ell})_{k,\ell=1,2}$.

Multiplying $\begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix}$ from the left, we obtain

$$\Phi(\lambda) = \frac{1}{2}\begin{pmatrix} \frac{1}{\cosh \nu_1} & 0 \\ 0 & \frac{1}{\cosh \nu_2} \end{pmatrix} (A_{k,\ell}e^{\lambda+\mu_\ell+\nu_k} - B_{k,\ell}e^{-(\lambda+\mu_\ell+\nu_k)})_{k,\ell=1,2}$$
$$\times \begin{pmatrix} \frac{1}{\cosh \mu_1} & 0 \\ 0 & \frac{1}{\cosh \mu_2} \end{pmatrix} + \widetilde{\Phi}(\lambda) \equiv \Phi_0(\lambda) + \widetilde{\Phi}(\lambda), \tag{3.9}$$

where $H_\ell = -\tanh \nu_\ell$ and $\widetilde{\Phi}(\lambda) = \begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} \widetilde{\phi}(\lambda)$.

Let us calculate $\widetilde{\Phi}(\lambda)$. By integration by parts,

$$\widetilde{\Phi}(\lambda) = \begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} \frac{1}{\lambda}$$
$$\times \left[K^{(1)}(y,1) \begin{pmatrix} \sinh \lambda y & 0 \\ 0 & \sinh \lambda y \\ \cosh \lambda y & 0 \\ 0 & \cosh \lambda y \end{pmatrix} + K^{(2)}(y,1) \begin{pmatrix} h_1 \cosh \lambda y & 0 \\ 0 & h_2 \cosh \lambda y \\ h_1 \sinh \lambda y & 0 \\ 0 & h_2 \sinh \lambda y \end{pmatrix} \right]_{y=0}^{y=1}$$
$$- \begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} \frac{1}{\lambda} \int_0^1 \frac{\partial}{\partial y} K^{(1)}(y,1) \begin{pmatrix} \sinh \lambda y & 0 \\ 0 & \sinh \lambda y \\ \cosh \lambda y & 0 \\ 0 & \cosh \lambda y \end{pmatrix} dy$$
$$- \begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} \frac{1}{\lambda} \int_0^1 \frac{\partial}{\partial y} K^{(2)}(y,1) \begin{pmatrix} h_1 \cosh \lambda y & 0 \\ 0 & h_2 \cosh \lambda y \\ h_1 \sinh \lambda y & 0 \\ 0 & h_2 \sinh \lambda y \end{pmatrix} dy.$$

Hence

$$|\widetilde{\Phi}(\lambda)| \le \frac{C}{|\lambda|} e^{|Re\lambda|}, \tag{3.10}$$

where we recall that $|\widetilde{\Phi}(\lambda)|$ denotes the matrix norm of $\widetilde{\Phi}(\lambda)$ and $C > 0$ is a positive constant which is independent of λ.

We show that there exists a positive constant K satisfying

$$|\operatorname{Re} \lambda| \le K \qquad \text{for any } \lambda \in \sigma(A_P). \tag{3.11}$$

If not, then there exists a sequence $\{\lambda_m\}_{m \in \mathbf{N}} \subset \sigma(A_P)$ such that $\lim_{m\to\infty} |\operatorname{Re} \lambda_m| = \infty$. Without loss of generality, we suppose that there exists a subsequence $\{\lambda_{j_m}\}_{m\in\mathbf{N}} \subset \{\lambda_m\}_{m\in\mathbf{N}}$ satisfying $\lim_{m\to\infty} \operatorname{Re} \lambda_{j_m} =$

∞. Since λ_{j_m} are eigenvalues, by (3.9) we have

$$
\begin{aligned}
0 &= |\det \Phi(\lambda_{j_m})| \\
&= \left|\det\left(\frac{1}{2}\begin{pmatrix} \frac{1}{\cosh \nu_1} & 0 \\ 0 & \frac{1}{\cosh \nu_2} \end{pmatrix} (A_{k,\ell}e^{\lambda_{j_m}+\mu_\ell+\nu_k} - B_{k,\ell}e^{-(\lambda_{j_m}+\mu_\ell+\nu_k)})_{k,\ell=1,2} \right.\right. \\
&\qquad \left.\left. \times \begin{pmatrix} \frac{1}{\cosh \mu_1} & 0 \\ 0 & \frac{1}{\cosh \mu_2} \end{pmatrix} + \widetilde{\Phi}(\lambda_{j_m})\right)\right|.
\end{aligned}
$$

Here, using (3.10), we have

$$
\begin{aligned}
&0 = |\det \Phi(\lambda_{j_m})| = \\
&\left|\det\left(\frac{1}{2}\begin{pmatrix} \frac{1}{\cosh \nu_1} & 0 \\ 0 & \frac{1}{\cosh \nu_2} \end{pmatrix} (A_{k,\ell}e^{i\,\mathrm{Im}\,\lambda_{j_m}+\mu_\ell+\nu_k} - B_{k,\ell}e^{-2\,\mathrm{Re}\,\lambda_{j_m}-i\,\mathrm{Im}\,\lambda_{j_m}-\mu_\ell-\nu_k})_{k,\ell=1,2} \right.\right. \\
&\qquad \left.\left. \times \begin{pmatrix} \frac{1}{\cosh \mu_1} & 0 \\ 0 & \frac{1}{\cosh \mu_2} \end{pmatrix} + \varepsilon_m\right)\right| e^{2\,\mathrm{Re}\,\lambda_{j_m}}
\end{aligned}
$$

where $\lim_{|m|\to\infty} |\varepsilon_m| = 0$. Here we have $\det G(\theta^P)(1) \ne 0$, which is derived at the end of Step 1. Hence, since $\det(A_{k,\ell}e^{\mu_\ell+\nu_k})_{k,\ell=1,2} = e^{\mu_1+\mu_2+\nu_1+\nu_2}\ \det\ G(\theta^P)(1) \ne 0$. Then taking $|m| \to \infty$, the right hand side tends to ∞, because of the continuity of the determinant. Thus this yields a contradiction and the proof of (3.11) is completed.

Step 3. We choose sufficiently large $K > 0$ satisfying (3.11) and

$$
\left|\frac{1}{2}\,\mathrm{Re}\log\alpha_j\right| < K, \qquad j = 1, 2.
$$

We further choose $K > 0$ large enough, so that

$$
|\det \Phi(\lambda) - \det \Phi_0(\lambda)| < |\det \Phi_0(\lambda)|
$$

for all λ with $|\,\mathrm{Re}\,\lambda| = K$. Here we recall taht $\Phi_0 = \Phi - \widetilde{\Phi}$. It is possible because (3.10) holds and $\det(A_{k,\ell}e^{\mu_\ell+\nu_k})_{k,\ell=1,2} \ne 0$, $\det(B_{k,\ell}e^{-\mu_\ell-\nu_k})_{k,\ell=1,2} \ne 0$ in (3.9).

Then we set

$$
K_m = \left\{\lambda\ ;\ -K-1 < \mathrm{Re}\,\lambda < K+1,\ \ \widetilde{\alpha} + m\pi - \frac{\pi}{2} < \mathrm{Im}\,\lambda < \widetilde{\alpha} + m\pi + \frac{\pi}{2}\right\}, \qquad m \in \mathbf{Z},
$$

using the constant $\widetilde{\alpha}$ defined in the statement of Theorem 2.1.

Now we will prove the following Assertion :

There exists $N \in \mathbf{N}$ *such that in* K_m *there are exactly* 2 *zeros of* $\det \Phi$ *by taking the algebraic multiplicities into consideration for* $|m| \geq N$.

Noting

$$K_m = \{\lambda + m\pi i \; ; \; \lambda \in K_0\},$$

and

$$\Phi_0(\lambda) = \begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} \cosh(\lambda+\mu_1) & 0 \\ 0 & \cosh(\lambda+\mu_2) \\ \sinh(\lambda+\mu_1) & 0 \\ 0 & \sinh(\lambda+\mu_2) \end{pmatrix} \begin{pmatrix} \frac{1}{\cosh\mu_1} & 0 \\ 0 & \frac{1}{\cosh\mu_2} \end{pmatrix},$$

by definition (3.9) of Φ_0, we have

$$\min_{\lambda \in \partial K_m} |\det \Phi_0(\lambda)| = \min_{\lambda \in \partial K_0} |\det \Phi_0(\lambda)| \equiv L.$$

For sufficiently large $N \in \mathbf{N}$, we have

$$\sup_{\lambda \in \partial K_m} |\det \Phi(\lambda) - \det \Phi_0(\lambda)| < L, \qquad N \leq |m|$$

by (3.10) and the linearity of the determinant in each column. Therefore

$$|\det\ \Phi(\lambda) - \det\ \Phi_0(\lambda)| < |\det\ \Phi_0(\lambda)|, \qquad \text{on} \quad \lambda \in \partial K_m. \tag{3.12}$$

On the other hand,

$$
\begin{aligned}
&\det \Phi_0(\lambda) = 0 \\
\Longleftrightarrow\ &\det \left(\frac{1}{2} \begin{pmatrix} \frac{1}{\cosh \nu_1} & 0 \\ 0 & \frac{1}{\cosh \nu_2} \end{pmatrix} (A_{k,\ell} e^{\lambda+\mu_\ell+\nu_k} - B_{k,\ell} e^{-(\lambda+\mu_\ell+\nu_k)})_{k,\ell=1,2} \right. \\
&\qquad \left. \times \begin{pmatrix} \frac{1}{\cosh \mu_1} & 0 \\ 0 & \frac{1}{\cosh \mu_2} \end{pmatrix} \right) = 0 \\
\Longleftrightarrow\ &\det \left((A_{k,\ell} e^{\lambda+\mu_\ell+\nu_k} - B_{k,\ell} e^{-(\lambda+\mu_\ell+\nu_k)})_{k,\ell=1,2} \right) = 0 \\
\Longleftrightarrow\ &\det \left(e^{2\lambda} \begin{pmatrix} e^{\nu_1} & 0 \\ 0 & e^{\nu_2} \end{pmatrix} G(\theta^P)(1) \begin{pmatrix} e^{\mu_1} & 0 \\ 0 & e^{\mu_2} \end{pmatrix} \right. \\
&\quad \left. - \begin{pmatrix} e^{-\nu_1} & 0 \\ 0 & e^{-\nu_2} \end{pmatrix} G(\widetilde{\theta}^P)(1) \begin{pmatrix} e^{-\mu_1} & 0 \\ 0 & e^{-\mu_2} \end{pmatrix} \right) = 0 \\
\Longleftrightarrow\ &\det \left(e^{2\lambda} E_2 - \begin{pmatrix} e^{-2\nu_1} & 0 \\ 0 & e^{-2\nu_2} \end{pmatrix} G(\widetilde{\theta}^P)(1) \begin{pmatrix} e^{-2\mu_1} & 0 \\ 0 & e^{-2\mu_2} \end{pmatrix} G(\theta^P)(1)^{-1} \right) = 0.
\end{aligned}
$$

Therefore, from the definition of α_1 and α_2, the zeros of $\det \Phi_0$ are

$$\frac{1}{2} \log \alpha_j + m\pi i, \qquad m \in \mathbf{Z}.$$

By the Rouché theorem, all K_m contains exactly 2 zeros of $\det \Phi$ by taking into consideration the multiplicities. Thus the proof of Assertion is completed.

Setting

$$K^{(0)} \equiv \left\{ \lambda\ ;\ -K-1 < \operatorname{Re} \lambda < K+1,\ \ \widetilde{\alpha} - N\pi + \frac{\pi}{2} < \operatorname{Im} \lambda < \widetilde{\alpha} + N\pi - \frac{\pi}{2} \right\},$$

we have

$$|\det \Phi(\lambda) - \det \Phi_0(\lambda)| < |\det \Phi_0(\lambda)| \quad on \ \ \partial K^{(0)},$$

by (3.12). Hence, since $\det \Phi_0(\lambda) = 0$ possesses $2(2N-1)$ zeros in $K^{(0)}$, the Rouché theorem yields that $K^{(0)}$ contains exactly $2(2N-1)$ zeros of $\det \Phi$ by taking into consideration the multiplicities.

According to the argument of this step, in terms of Lemma 3.2, we see:

There exists $N \in \mathbf{N}$ such that K_m contains exactly 2 eigenvalues of A_P for all $|m| \geq N$ and $K^{(0)}$ contains $2(2N-1)$ eigenvalues of A_P by taking into consideration the algebraic multiplicities.

Step 4. We will show the asymptotic behaviour of the eigenvalues. Here let $N \leq |m|$. We note that two zeros of $\det \Phi$ are included in K_m with the multiplicities. Now we consider $\det\, \Phi(\lambda) = 0$ in K_m. By (3.9) and (3.10), using the linearity of the determinant in each column, we see that $\det \Phi(\lambda) = 0$, $\lambda \in K_m$, is rewritten as

$$\det\left((A_{k,\ell} e^{\lambda+\mu_\ell+\nu_k} - B_{k,\ell} e^{-(\lambda+\mu_\ell+\nu_k)})_{k,\ell=1,2} \right) = O\left(\frac{1}{|m|}\right), \quad \lambda \in K_m.$$

By $|\mathrm{Re}\lambda| < K+1$, we can rewrite the left hand side to obtain

$$\det\left\{ e^{2\lambda} E_2 - \begin{pmatrix} e^{-2\nu_1} & 0 \\ 0 & e^{-2\nu_2} \end{pmatrix} G(\widetilde{\theta}^P)(1) \begin{pmatrix} e^{-2\mu_1} & 0 \\ 0 & e^{-2\mu_2} \end{pmatrix} G(\theta^P)(1)^{-1} \right\} = O\left(\frac{1}{|m|}\right), \quad \lambda \in K_m.$$

We rewrite this equation as

$$(e^{2\lambda})^2 + a_1 e^{2\lambda} + a_0 = O\left(\frac{1}{|m|}\right), \quad \lambda \in K_m, \tag{3.13}$$

where a_1 and a_0 are constants. That is, λ is a root of

$$e^{4\lambda} + a_1 e^{2\lambda} + a_0 + \kappa_m = 0, \quad \kappa_m = O\left(\frac{1}{|m|}\right).$$

We set $\zeta_m = \frac{1}{2}\log \alpha_1 + m\pi i \in K_m$. Then $\alpha_1 = e^{2\zeta_m}$ and by the definition of α_1, we have

$$\det\left\{ e^{2\zeta_m} E_2 - \begin{pmatrix} e^{-2\nu_1} & 0 \\ 0 & e^{-2\nu_2} \end{pmatrix} G(\widetilde{\theta}^P)(1) \begin{pmatrix} e^{-2\mu_1} & 0 \\ 0 & e^{-2\mu_2} \end{pmatrix} G(\theta^P)(1)^{-1} \right\} = 0,$$

that is, ζ_m is a root of the equation in λ :

$$e^{4\lambda} + a_1 e^{2\lambda} + a_0 = 0.$$

Using the Rouché theorem, we will estimate the difference between ζ_m and a root of (3.13). First, for sufficiently large $|m|$, we consider a circle S_{ζ_m, r_m} centred at ζ_m with radius r_m. For large $|m|$, we will find

r_m such that

$$|\kappa_m| < |e^{4\lambda} + a_1 e^{2\lambda} + a_0| \quad on \quad S_{\zeta_m, r_m}. \tag{3.14}$$

We set $\rho(\lambda) = e^{4\lambda} + a_1 e^{2\lambda} + a_0$ and $\eta = a_1 + 2\alpha_1$. Let us calculate $|\rho(\lambda)|$ under $|\lambda - \zeta_m| = r_m$. By $\rho(\zeta_m) = 0$, we have

$$|e^{4\lambda} + a_1 e^{2\lambda} + a_0| = |\{(e^{2\lambda} - \alpha_1) + \alpha_1\}^2 + a_1\{(e^{2\lambda} - \alpha_1) + \alpha_1\} + a_0|$$

$$= |(e^{2\lambda} - \alpha_1)^2 + 2(e^{2\lambda} - \alpha_1)\alpha_1 + a_1(e^{2\lambda} - \alpha_1)| = |(e^{2\lambda} - \alpha_1)^2 + \eta(e^{2\lambda} - \alpha_1)|.$$

Case 1: (1.12) possesses distinct roots α_1 and α_2.

Then $\eta \neq 0$ and we have

$$|\rho(\lambda)| = |e^{2\lambda} - \alpha_1| \left|(e^{2\lambda} - \alpha_1) + \eta\right| \geq C_0 r \left|(e^{2\lambda} - \alpha_1) + \eta\right| \quad \text{on} \quad S_{\zeta_m, r}.$$

At the last inequality, we used $\zeta_m = \frac{1}{2}\log\alpha_1 + m\pi i$ and $|e^{2\lambda} - \alpha_1| = |\alpha_1||e^{2(\lambda - \zeta_m)} - 1|$. Taking sufficiently small $d < 1$, by $|\eta| > 0$ we can estimate

$$\left|(e^{2\lambda} - \alpha_1) + \eta\right| \geq |\eta| - C_0 r \geq C > 0 \quad on \quad S_{\zeta_m, r}, \ for\ all \quad r < d,$$

where d and C are dependent on a_j, α_1, and independent of m. Hence

$$|\rho(\lambda)| \geq Cr.$$

Therefore, since $|\kappa_m| = O\left(\frac{1}{|m|}\right)$, for sufficiently large $C' > 0$, we set $r_m = \frac{C'}{|m|}$, so that $Cr_m \geq |\kappa_m|$, that is, (3.14) holds on $S_{\lambda_m^{(1)}, r_m}$.

Moreover $\rho(\lambda)$ possesses a unique zero in $\{\lambda;\ |\lambda - \zeta_m| < r_m\}$ for sufficiently large $|m|$. Applying the Rouché theorem, in terms of (3.14), we see that $e^{4\lambda} + a_1 e^{2\lambda} + a_0 + \kappa_m = 0$ possesses a unique zero denoted by $\lambda_{1,m}$ in $\{\lambda;\ |\lambda - \zeta_m| < r_m\}$ and

$$\lambda_{1,m} = \frac{1}{2}\log\alpha_1 + m\pi i + O\left(\frac{1}{|m|}\right).$$

For α_2, we can argue similarly. Thus the proof of (2.1) is completed in Case 1.

Case 2: (1.12) possesses the multiple root $\alpha_1 = \alpha_2$. Then $\eta = 0$, and

$$|\rho(\lambda)| = |e^{2\lambda} - e^{2\zeta_m}|^2 \geq C_0^2 r^2 \quad on \quad S_{\zeta_m,r}$$

and for sufficiently large $|m|$, the function $\rho(\lambda)$ possesses exactly two zeros in $\{\lambda\ ;\ |\lambda - \zeta_m| < r_m\}$ including the multiplicity. Choosing $r_m = \frac{C'}{\sqrt{|m|}}$ with large $C' > 0$, we can argue similarly to Case 1, in terms of the Rouché theorem to see that $e^{4\lambda} + a_1 e^{2\lambda} + a_0 + \kappa_m = 0$ possesses two zeros $\lambda_{1,m}$ and $\lambda_{2,m}$ in $\{\lambda;\ |\lambda - \zeta_m| < r_m\}$ by taking into consideration the multiplicities, and

$$|\lambda_{j,m} - \zeta_m| = O\left(\frac{1}{\sqrt{|m|}}\right), \qquad j = 1, 2$$

as $|m| \to \infty$. Thus the proof of Theorem 2.1 is completed. $\square$

4 The proof of Theorem 2.3

In this section, we prove Theorem 2.3. For this, we apply the Bari theorem (e.g., Gohberg and Kreĭn [2]).

Let α_1, α_2 be the solutions to (1.12). Because of the assumption $\alpha_1 \neq \alpha_2$, for sufficiently large $|m|$, we see that

$$K_m = \left\{\lambda\ ;\ -K-1 < \operatorname{Re}\lambda < K+1,\ \ \widetilde{\alpha} + m\pi - \frac{\pi}{2} < \operatorname{Im}\lambda < \widetilde{\alpha} + m\pi + \frac{\pi}{2}\right\}$$

contains two eigenvalues each of whose algebraic multiplicity is one.

Let us set $\beta_j = \frac{1}{2}\log\alpha_j$, $j = 1, 2$. Now we prove that

$$\operatorname{rank}\begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} \cosh(\beta_1 + \mu_1) & 0 \\ 0 & \cosh(\beta_1 + \mu_2) \\ \sinh(\beta_1 + \mu_1) & 0 \\ 0 & \sinh(\beta_1 + \mu_2) \end{pmatrix} = 1.$$

By (3.6), this ranks is equal to

$$\operatorname{rank}\left\{e^{2\beta_1}E_2 - \begin{pmatrix} e^{-2\nu_1} & 0 \\ 0 & e^{-2\nu_2} \end{pmatrix} G(\widetilde{\theta}^P)(1) \begin{pmatrix} e^{-2\mu_1} & 0 \\ 0 & e^{-2\mu_2} \end{pmatrix} G(\theta^P)(1)^{-1}\right\}.$$

By the assumption that $\alpha_1 = e^{2\beta_1}$ is the solution to (1.12), the rank is not equal to 2. We assume

$$\operatorname{rank}\left\{e^{2\beta_1}E_2 - \begin{pmatrix} e^{-2\nu_1} & 0 \\ 0 & e^{-2\nu_2} \end{pmatrix} G(\widetilde{\theta}^P)(1) \begin{pmatrix} e^{-2\mu_1} & 0 \\ 0 & e^{-2\mu_2} \end{pmatrix} G(\theta^P)(1)^{-1}\right\} = 0.$$

Then because each column of this matrix is equal to 0, we have

$$\frac{d}{d\alpha}\left[\det\left\{\alpha E_2 - \begin{pmatrix} e^{-2\nu_1} & 0 \\ 0 & e^{-2\nu_2} \end{pmatrix} G(\widetilde{\theta}^P)(1) \begin{pmatrix} e^{-2\mu_1} & 0 \\ 0 & e^{-2\mu_2} \end{pmatrix} G(\theta^P)(1)^{-1}\right\}\right]\Bigg|_{\alpha=e^{2\beta_1}} = 0.$$

This contradicts the assumption that quadratic equation (1.12) has distinct roots. Therefore we obtain

$$\operatorname{rank}\begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} \cosh(\beta_1+\mu_1) & 0 \\ 0 & \cosh(\beta_1+\mu_2) \\ \sinh(\beta_1+\mu_1) & 0 \\ 0 & \sinh(\beta_1+\mu_2) \end{pmatrix} = 1.$$

Then there exists $(c_1, c_2) \neq (0,0)$ such that

$$\begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} \cosh(\beta_1+\mu_1) & 0 \\ 0 & \cosh(\beta_1+\mu_2) \\ \sinh(\beta_1+\mu_1) & 0 \\ 0 & \sinh(\beta_1+\mu_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Without loss of generality, we can assume that

$$\begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} 0 \\ \cosh(\beta_1+\mu_2) \\ 0 \\ \sinh(\beta_1+\mu_2) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.1}$$

Then we have $c_1 \neq 0$.

Similarly, we can take $(d_1, d_2) \neq (0,0)$ such that

$$\begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} \cosh(\beta_2+\mu_1) & 0 \\ 0 & \cosh(\beta_2+\mu_2) \\ \sinh(\beta_2+\mu_1) & 0 \\ 0 & \sinh(\beta_2+\mu_2) \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and we can assume that

$$\begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} 0 \\ \cosh(\beta_2+\mu_2) \\ 0 \\ \sinh(\beta_2+\mu_2) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

without loss of generality. Then we can directly verify that $d_1 \neq 0$.

By $S(x)$, we denote a 4×4 matrix

$$S(x) = R(x) \begin{pmatrix} c_1 \cosh(\beta_1 x+\mu_1) & d_1 \cosh(\beta_2 x+\mu_1) & c_1 \sinh(\beta_1 x+\mu_1) & d_1 \sinh(\beta_2 x+\mu_1) \\ c_2 \cosh(\beta_1 x+\mu_2) & d_2 \cosh(\beta_2 x+\mu_2) & c_2 \sinh(\beta_1 x+\mu_2) & d_2 \sinh(\beta_2 x+\mu_2) \\ c_1 \sinh(\beta_1 x+\mu_1) & d_1 \sinh(\beta_2 x+\mu_1) & c_1 \cosh(\beta_1 x+\mu_1) & d_1 \cosh(\beta_2 x+\mu_1) \\ c_2 \sinh(\beta_1 x+\mu_2) & d_2 \sinh(\beta_2 x+\mu_2) & c_2 \cosh(\beta_1 x+\mu_2) & d_2 \cosh(\beta_2 x+\mu_2) \end{pmatrix}.$$

Since the property of the determinant yields

$$\det \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \det \begin{pmatrix} A-B & B-A \\ B & A \end{pmatrix}$$
$$= \det \begin{pmatrix} A-B & 0 \\ B & A+B \end{pmatrix} = \det(A-B)\det(A+B)$$

for 2×2-matrices A, B, we have

$$\det S(x) = \det R(x) \det \begin{pmatrix} c_1 \exp(\beta_1 x+\mu_1) & d_1 \exp(\beta_2 x+\mu_1) \\ c_2 \exp(\beta_1 x+\mu_2) & d_2 \exp(\beta_2 x+\mu_2) \end{pmatrix}$$
$$\times \det \begin{pmatrix} c_1 \exp(-\beta_1 x-\mu_1) & d_1 \exp(-\beta_2 x-\mu_1) \\ c_2 \exp(-\beta_1 x-\mu_2) & d_2 \exp(-\beta_2 x-\mu_2) \end{pmatrix}.$$

If $c_1 d_2 - c_2 d_1 \neq 0$, then the inverse matrix $S^{-1}(x)$ exists. We will prove $c_1 d_2 - c_2 d_1 \neq 0$. If not, then we can take a constant γ such that

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \gamma \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Hence β_1 and β_2 are the solution to the following equation in λ:

$$\begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} \cosh(\lambda+\mu_1) & 0 \\ 0 & \cosh(\lambda+\mu_2) \\ \sinh(\lambda+\mu_1) & 0 \\ 0 & \sinh(\lambda+\mu_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0, \tag{4.2}$$

which implies

$$\begin{cases} d_{1,1}e^{\lambda} + d_{1,2}e^{-\lambda} = 0 \\ d_{2,1}e^{\lambda} + d_{2,2}e^{-\lambda} = 0 \end{cases}$$

with some $d_{k,\ell} \in \mathbf{C}$, $k, \ell = 1, 2$. Then there exists $d_{k,\ell} \neq 0$. Otherwise all $\lambda \in \mathbf{C}$ is the solution to (4.2), which means that $\Phi_0(\lambda) = 0$ for all $\lambda \in \mathbf{C}$. This is a contradiction.

Therefore dividing some $d_{k,\ell} \neq 0$, we obtain $e^{2\beta_1} = e^{2\beta_2}$. Hence $2\beta_1 - 2\beta_2 = 2k\pi i$ with some $k \in \mathbf{Z}$, that is,

$$\log \alpha_1 = \log \alpha_2 + 2k\pi i.$$

This contradicts that $\alpha_1 \neq \alpha_2$. Thus we proved that $c_1 d_2 - c_2 d_1 \neq 0$.

By the definition of S, we have

$$S(x)\begin{pmatrix} \cos m\pi x \\ 0 \\ i \sin m\pi x \\ 0 \end{pmatrix} = R(x)\begin{pmatrix} c_1 \cosh(\beta_1 x + m\pi i x + \mu_1) \\ c_2 \cosh(\beta_1 x + m\pi i x + \mu_2) \\ c_1 \sinh(\beta_1 x + m\pi i x + \mu_1) \\ c_2 \sinh(\beta_1 x + m\pi i x + \mu_2) \end{pmatrix},$$
$$S(x)\begin{pmatrix} 0 \\ \cos m\pi x \\ 0 \\ i \sin m\pi x \end{pmatrix} = R(x)\begin{pmatrix} d_1 \cosh(\beta_2 x + m\pi i x + \mu_1) \\ d_2 \cosh(\beta_2 x + m\pi i x + \mu_2) \\ d_1 \sinh(\beta_2 x + m\pi i x + \mu_1) \\ d_2 \sinh(\beta_2 x + m\pi i x + \mu_2) \end{pmatrix}.$$

We set

$$e_{1,m} := R(x)\begin{pmatrix} c_1 \cosh(\beta_1 x + m\pi i x + \mu_1) \\ c_2 \cosh(\beta_1 x + m\pi i x + \mu_2) \\ c_1 \sinh(\beta_1 x + m\pi i x + \mu_1) \\ c_2 \sinh(\beta_1 x + m\pi i x + \mu_2) \end{pmatrix}, \quad e_{2,m} := R(x)\begin{pmatrix} d_1 \cosh(\beta_2 x + m\pi i x + \mu_1) \\ d_2 \cosh(\beta_2 x + m\pi i x + \mu_2) \\ d_1 \sinh(\beta_2 x + m\pi i x + \mu_1) \\ d_2 \sinh(\beta_2 x + m\pi i x + \mu_2) \end{pmatrix}.$$

Since $S(x)$ is invertible and

$$\left\{ \begin{pmatrix} \cos m\pi x \\ 0 \\ i \sin m\pi x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \cos m\pi x \\ 0 \\ i \sin m\pi x \end{pmatrix} \right\}_{m \in \mathbf{Z}}$$

is a Riesz basis in $\{L^2(0,1)\}^4$, we see that $\{e_{1,m}, e_{2,m}\}_{m \in \mathbf{Z}}$ is a Riesz basis in $\{L_2(0,1)\}^4$ (e.g., Gohberg and Kreĭn [2]).

We can write an eigenfunction corresponding to $\lambda_{1,m}$ as

$$\begin{aligned} & R(x)\begin{pmatrix} \cosh(\lambda_{1,m}x+\mu_1) & 0 \\ 0 & \cosh(\lambda_{1,m}x+\mu_2) \\ \sinh(\lambda_{1,m}x+\mu_1) & 0 \\ 0 & \sinh(\lambda_{1,m}x+\mu_2) \end{pmatrix}\begin{pmatrix} \widetilde{c}_1^{(m)} \\ \widetilde{c}_2^{(m)} \end{pmatrix} \\ & \quad + \begin{pmatrix} \widetilde{\phi}_1(\lambda_{1,m},x) & \widetilde{\phi}_2(\lambda_{1,m},x) \end{pmatrix}\begin{pmatrix} \widetilde{c}_1^{(m)} \\ \widetilde{c}_2^{(m)} \end{pmatrix}. \end{aligned} \tag{4.3}$$

from (3.7). Here, $\widetilde{\phi}_1(\lambda_{1,m},x), \widetilde{\phi}_2(\lambda_{1,m},x),\ m\in\mathbf{Z}$ correspond to the integral terms on (3.7) and $\widetilde{c}_1^{(m)}, \widetilde{c}_2^{(m)}, m\in\mathbf{Z}$ are constants such that

$$\begin{aligned} & \begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} R(1)\begin{pmatrix} \cosh(\lambda_{1,m}+\mu_1) & 0 \\ 0 & \cosh(\lambda_{1,m}+\mu_2) \\ \sinh(\lambda_{1,m}+\mu_1) & 0 \\ 0 & \sinh(\lambda_{1,m}+\mu_2) \end{pmatrix}\begin{pmatrix} \widetilde{c}_1^{(m)} \\ \widetilde{c}_2^{(m)} \end{pmatrix} \\ & \quad + \begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix}\begin{pmatrix} \widetilde{\phi}_1(\lambda_{1,m},1) & \widetilde{\phi}_2(\lambda_{1,m},1) \end{pmatrix}\begin{pmatrix} \widetilde{c}_1^{(m)} \\ \widetilde{c}_2^{(m)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{4.4}$$

Such $\widetilde{c}_1^{(m)}, \widetilde{c}_2^{(m)},\ m\in\mathbf{Z}$ exist because $\lambda_{1,m}$ are eigenvalues. By (2.1) and (3.10), we choose $C>0$ such that

$$|\widetilde{\phi}_k(\lambda_{1,m},x)| \le \frac{C}{|m|}, \quad m\in\mathbf{Z},\ 0<x<1. \tag{4.5}$$

Now we prove that for sufficiently large $|m|$, we can take $\widetilde{c}_1^{(m)}, \widetilde{c}_2^{(m)}$ such that $\widetilde{c}_1^{(m)} = c_1$ and $\widetilde{c}_2^{(m)} - c_2 = O\left(\frac{1}{|m|}\right)$.

Because we assume (4.1), we have

$$\begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} R(1)\begin{pmatrix} 0 \\ \cosh\left(\lambda_{1,m}+\mu_2\right) \\ 0 \\ \sinh\left(\lambda_{1,m}+\mu_2\right) \end{pmatrix} \ne \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.6}$$

for sufficiently large $|m|$. Then, by (4.4) and (4.5), we have $\widetilde{c}_1^{(m)} \ne 0$ for sufficiently large $|m|$. Multiplying $\widetilde{c}_1^{(m)}, \widetilde{c}_2^{(m)}$ with $\frac{c_1}{\widetilde{c}_1^{(m)}}$, we can take $(c_1, \widetilde{c}_2^{(m)})$ as $(\widetilde{c}_1^{(m)}, \widetilde{c}_2^{(m)})$.

Now let us prove $c_2 - \widetilde{c}_2^{(m)} = O\left(\frac{1}{|m|}\right)$. For this purpose, we will first prove that $\widetilde{c}_2^{(m)} = O(1)$. Equation

(4.4) yields

$$\begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} \cosh(\lambda_{1,m}+\mu_1) & 0 \\ 0 & \cosh(\lambda_{1,m}+\mu_2) \\ \sinh(\lambda_{1,m}+\mu_1) & 0 \\ 0 & \sinh(\lambda_{1,m}+\mu_2) \end{pmatrix} \begin{pmatrix} c_1 \\ \widetilde{c}_2^{(m)} \end{pmatrix}$$
$$+ \begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} \begin{pmatrix} \widetilde{\phi}_1(\lambda_{1,m},1) & \widetilde{\phi}_2(\lambda_{1,m},1) \end{pmatrix} \begin{pmatrix} c_1 \\ \widetilde{c}_2^{(m)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{4.7}$$

that is,

$$c_1 \begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} \cosh(\lambda_{1,m}+\mu_1) \\ 0 \\ \sinh(\lambda_{1,m}+\mu_1) \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} \widetilde{\phi}_1(\lambda_{1,m},1)$$
$$+\widetilde{c}_2^{(m)} \begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} 0 \\ \cosh(\lambda_{1,m}+\mu_2) \\ 0 \\ \sinh(\lambda_{1,m}+\mu_2) \end{pmatrix} + \widetilde{c}_2^{(m)} \begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} \widetilde{\phi}_2(\lambda_{1,m},1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By using (4.6) and $\widetilde{\phi}_k(\lambda_{1,m},1) = O\left(\frac{1}{|m|}\right)$, we obtain $\widetilde{c}_2^{(m)} = O(1)$.

We will estimate $c_2 - \widetilde{c}_2^{(m)}$. Because of (4.7) and $\lambda_{1,m} = \beta_1 + m\pi i + \delta_m$ with $\delta_m = O\left(\frac{1}{|m|}\right)$, we have

$$\begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} \cosh(\beta_1+\mu_1) & 0 \\ 0 & \cosh(\beta_1+\mu_2) \\ \sinh(\beta_1+\mu_1) & 0 \\ 0 & \sinh(\beta_1+\mu_2) \end{pmatrix} \cosh\delta_m \begin{pmatrix} c_1 \\ \widetilde{c}_2^{(m)} \end{pmatrix}$$
$$+ \begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} \sinh(\beta_1+\mu_1) & 0 \\ 0 & \sinh(\beta_1+\mu_2) \\ \cosh(\beta_1+\mu_1) & 0 \\ 0 & \cosh(\beta_1+\mu_2) \end{pmatrix} \sinh\delta_m \begin{pmatrix} c_1 \\ \widetilde{c}_2^{(m)} \end{pmatrix}$$
$$+ \begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} \begin{pmatrix} \widetilde{\phi}_1(\lambda_{1,m},1) & \widetilde{\phi}_2(\lambda_{1,m},1) \end{pmatrix} \begin{pmatrix} c_1 \\ \widetilde{c}_2^{(m)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

From this equation, we subtract the following:

$$\begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} \cosh(\beta_1+\mu_1) & 0 \\ 0 & \cosh(\beta_1+\mu_2) \\ \sinh(\beta_1+\mu_1) & 0 \\ 0 & \sinh(\beta_1+\mu_2) \end{pmatrix} \cosh\delta_m \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which follows from (4.2). Then, since the second and the third terms on the left hand side are bounded by

$O\left(\frac{1}{|m|}\right)$ in terms of $\delta_m = O\left(\frac{1}{|m|}\right)$ and (4.5), we obtain

$$\cosh\delta_m \begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} 0 \\ \cosh(\beta_1+\mu_2) \\ 0 \\ \sinh(\beta_1+\mu_2) \end{pmatrix} (c_2 - \widetilde{c}_2^{(m)}) = O\left(\frac{1}{|m|}\right).$$

Therefore, by (4.1) we have

$$c_2 - \widetilde{c}_2^{(m)} = O\left(\frac{1}{|m|}\right).$$

Thus for sufficiently large $|m|$, we can choose an eigenfunction $f_{1,m}$ corresponding to $\lambda_{1,m}$ such that

$$f_{1,m}(x) = R(x) \begin{pmatrix} \cosh(\lambda_{1,m}x+\mu_1) & 0 \\ 0 & \cosh(\lambda_{1,m}x+\mu_2) \\ \sinh(\lambda_{1,m}x+\mu_1) & 0 \\ 0 & \sinh(\lambda_{1,m}x+\mu_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 + O\left(\frac{1}{|m|}\right) \end{pmatrix} + O\left(\frac{1}{|m|}\right). \tag{4.8}$$

For the eigenvalue $\lambda_{2,m}$ for sufficiently large $|m|$, we can argue similarly and can choose an eigenfunction $f_{2,m}$ corresponding to $\lambda_{2,m}$ such that

$$f_{2,m}(x) = R(x) \begin{pmatrix} \cosh(\lambda_{2,m}x+\mu_1) & 0 \\ 0 & \cosh(\lambda_{2,m}x+\mu_2) \\ \sinh(\lambda_{2,m}x+\mu_1) & 0 \\ 0 & \sinh(\lambda_{2,m}x+\mu_2) \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 + O\left(\frac{1}{|m|}\right) \end{pmatrix} + O\left(\frac{1}{|m|}\right). \tag{4.9}$$

Supplementing root vectors to $f_{j,m}$, $j = 1, 2$ for sufficiently large $|m|$, we can obtain the totality of all the root vectors which can be denoted by $\{f_{j,m}\}_{j=1,2,\ m\in\mathbf{Z}}$ without fear of confusion such that

$$|e_{j,m}(x) - f_{j,m}(x)| = O\left(\frac{1}{|m|}\right), \qquad 0 < x < 1.$$

Therefore we have

$$\sum_{j=1,2}\sum_{m\in\mathbf{Z}} ||e_{j,m} - f_{j,m}||^2_{\{L_2(0,1)\}^4} < \infty.$$

If $\{f_{j,m}\}_{j=1,2,m\in\mathbf{Z}}$ is linearly independent, then we can complete the proof of the theorem by the Bari theorem (e.g., [2]). Let us prove the linear independence of $\{f_{j,m}\}_{j=1,2,m\in\mathbf{Z}}$. For this purpose, we renumber the eigenvalues of A_P and the root vectors $\{f_{j,m}\}_{j=1,2,m\in\mathbf{Z}}$ as follows. In terms of Theorem 2.1, we number the eigenvalues $\{\lambda_{j,m}\}_{j=1,2,m\in\mathbf{Z}}$ as

$$\sigma(A_P) = \{\mu_k\}_{k\in\mathbf{Z}} \cup \{\nu_\ell\}_{1\le\ell\le N},$$

where $\mu_k, k \in \mathbf{Z}$ are the eigenvalues with algebraic multiplicity one, ν_ℓ, $1 \le \ell \le N$ are the eigenvalues with algebraic multiplicity $\chi_\ell \ge 2$ and

$$\mu_{k_1} \ne \mu_{k_2}, \quad \nu_{\ell_1} \ne \nu_{\ell_2}, \quad k_1 \ne k_2, \ell_1 \ne \ell_2.$$

We renumber the root vectors $\{f_{j,m}\}_{j=1,2,m\in\mathbf{Z}}$ as

$$\{f_{j,m}\}_{j=1,2,m\in\mathbf{Z}} = \{g_k\}_{k\in\mathbf{Z}} \cup \{h_{\ell,j}\}_{1\le\ell\le N,1\le j\le\chi_\ell},$$

where g_k is an eigenfunction corresponding to the eigenvalue μ_k, and $\{h_{\ell,j}\}_{1\le j\le\chi_\ell}$ is a basis of $\{\phi;\, (A_P - \nu_\ell)^k\phi = 0 \,\text{for some}\ k \in \mathbf{N}\}$.

Now we verify that

$$\sum_{\ell=1,2,\cdots,N,\ j=1,2,\cdots,\chi_\ell} \alpha_{\ell,j} h_{\ell,j} + \sum_{k\in\mathbf{Z}} \beta_k g_k = 0, \quad \alpha_{\ell,j}, \beta_k \in \mathbf{C} \tag{4.10}$$

implies $\alpha_{\ell,j} = 0$, $1 \le \ell \le N$, $1 \le j \le \chi_\ell$ and $\beta_k = 0$, $k \in \mathbf{Z}$. We define

$$P_k = \frac{1}{2\pi i}\int_{\Gamma_k} (\mu - A_P)^{-1} d\mu, \quad k \in \mathbf{Z}$$

where $\Gamma_k,\ k \in \mathbf{Z}$ is a sufficiently small circle centred at μ_k including no other points of $\sigma(A_P)$. By Theorem 2.1, such Γ_k exists. Then

$$P_k g_k = g_k,\ P_k g_{k_1} = 0,\ P_k h_{\ell,j} = 0 \quad \text{if } k \ne k_1,\ 1 \le \ell \le N,\ 1 \le j \le \chi_\ell$$

hold (e.g., Kato [3]). Applying P_k to (4.10), we have $\beta_k = 0,\ k \in \mathbf{Z}$. Since $\{h_{\ell,j}\}_{1\le\ell\le N,1\le j\le\chi_\ell}$ is a linearly independent system, we obtain $\alpha_{\ell,j} = 0$, $1 \le \ell \le N$, $1 \le j \le \chi_\ell$. Thus the proof of Theorem 2.3 is completed.

5 The proof of Theorem 1.4

We denote the adjoint operator of A_P by A_P^*. We can easily see that

$$(A_P^* u)(x) = -B_4 \frac{du}{dx}(x) + P^t(x)u(x) \quad 0 < x < 1$$

$$D(A_P^*) = \{u \in \{H^1(0,1)\}^4 \ ; \ u_{\ell+n}(0) + h_\ell u_\ell(0) = 0, \quad u_{\ell+n}(1) + H_\ell u_\ell(1) = 0, \quad \ell = 1,2\}.$$

Here P^t denotes the transpose matrix of P. By Theorem 2.1, we can number all the eigenvalues of A_P as $\{\lambda_m\}_{|m|\le N-1} \cup \{\lambda_{j,m}\}_{|m|\ge N, j=1,2}$ such that the algebraic multiplicity of $\lambda_{j,m}$ is one for $|m| \ge N$ and $j = 1,2$, and the value $\lambda_m, \ |m| \le N-1$ appears as many times as its algebraic multiplicity. According to the numbering of the eigenvalues, we number the eigenvectors and the associated root vectors. That is, in the case $|m| \ge N$, for $j = 1,2$ we choose an eigenvector $f_{j,m}$ of A_P for $\lambda_{j,m}$ satisfying (4.8) and (4.9). We note that by Condition (III) an eigenvector is determined uniquely up to multiples. Furthermore we know (e.g., [3]) that $\overline{\sigma(A_P)} = \sigma(A_P^*)$ and the algebraic multiplicity of $\overline{\lambda} \in \sigma(A_P^*)$ is equal to the one of $\lambda \in \sigma(A_P)$. By $g_{j,m}$, $j = 1,2, |m| \ge N$, we denote an eigenvector of A_P^* for $\overline{\lambda_{j,m}}$ such that

$$(f_{j,m}, g_{j,m})_{\{L^2(0,1)\}^4} \ne 0.$$

In fact, $g_{j,m}$ is orthogonal to $\{\phi \in \{L^2(0,1)\}^4; \ (A_P - \lambda)^k \phi = 0 \text{ for some } k \in \mathbf{N}\}$ for any eigenvalue λ of A_P which is different from $\lambda_{j,m}$ (e.g., [3]). Therefore if $(f_{j,m}, g_{j,m})_{\{L^2(0,1)\}^4} = 0$, then Theorem 2.3 implies that $g_{j,m} = 0$, which is impossible. Hence, for any $a \in \{L^2(0,1)\}^4$, we can set

$$\alpha_{j,m} = \frac{(a, g_{j,m})_{\{L^2(0,1)\}^4}}{(f_{j,m}, g_{j,m})_{\{L^2(0,1)\}^4}}.$$

Moreover we put

$$\theta_{j,m}(t) = \alpha_{j,m} e^{\lambda_{j,m} t}, \ |m| \ge N, \ j = 1,2.$$

In the case $|m| \le N-1$, the eigenvalue λ_m appears χ_m-times according to its algebraic multiplicity χ_m : $\lambda_q = ... = \lambda_{q+\chi_m-1}$. Then by f_q we denote a corresponding eigenvector, and by $f_{q+\ell}(x)$, $1 \le \ell \le \chi_m - 1$, a

Jordan chain of the associated root vectors. That is, $f_{q+\ell}$, $1 \le \ell \le \chi_m - 1$, satisfy $(A_P - \lambda_q)f_{q+\ell} = f_{q+\ell-1}$. We denote by $g_{q+\chi_m-1}$ an eigenvector of the adjoint operator A_P^* for the eigenvalue $\overline{\lambda_q}$, and by $g_{q+\chi_m-\ell}$, $2 \le \ell \le \chi_m$ we denote a Jordan chain of associated root vectors. Here, $(A_P - \lambda_q)g_{q+\chi_m-\ell} = g_{q+\chi_m-\ell+1}$ for $\ell = 2, 3, \cdots, \chi_m$. Then we can prove (e.g., Propositions 2.2 and 2.3 in [7]) that $(f_{q+\ell}, g_{q+\ell})_{\{L^2(0,1)\}^4} \ne 0$, $0 \le \ell \le \chi_m - 1$. Thus for any $a \in \{L^2(0,1)\}^4$, we can set

$$\gamma_{q+\ell} = \frac{(a, g_{q+\ell})_{\{L^2(0,1)\}^4}}{(f_{q+\ell}, g_{q+\ell})_{\{L^2(0,1)\}^4}}, \qquad 0 \le \ell \le \chi_m - 1$$

and

$$\theta_{q+\ell}(t) = e^{\lambda_q t} \left(\sum_{k=0}^{\chi_m-\ell-1} \frac{t^k}{k!} \gamma_{q+\ell+k} \right), \ 0 \le \ell \le \chi_m - 1.$$

Then we renumber $\{f_{j,m}\}_{|m|\ge N, j=1,2}$, $f_{q+\ell}$, $\{\theta_{j,m}\}_{|m|\ge N, j=1,2}$, $\theta_{q+\ell}$, $\{g_{j,m}\}_{|m|\ge N, j=1,2}$, $g_{q+\ell}$ with $0 \le j \le \chi_m - 1$ as $\{f_m\}_{m\in\mathbf{Z}}$, $\{\theta_m\}_{m\in\mathbf{Z}}$ and $\{g_m\}_{m\in\mathbf{Z}}$.

In terms of θ_m and f_m, we can prove an expansion of the solution to the initial value/boundary value problem (1.9). The proof is done by arguments similar to Appendix in [7] and Proposition 2.2 in [11], and is omitted.

Propositon 5.1 *Let $a \in \{C^3[0,1]\}^4 \cap D(A^2)$ and $u_{P,a}$ satisfy* (1.9). *Then*

$$u(t,x) = \sum_{m\in\mathbf{Z}} \theta_m(t) f_m(x),$$

where the series converges absolutely and uniformly in $-T \le t \le T$ and $0 \le x \le 1$.

Now we proceed

Proof of Theorem 1.4. The "if" part is directly proved. In fact, by (1.19) and (1.20), we see that $K = 0$ satisfies (2.3), (2.4) and (2.7), so that $\widetilde{u}(t,x) = R(x)u(t,x)$ satisfies (2.12) with some $\omega_1(t)$ and $\omega_2(t)$ by Theorem 2.6. In terms of (1.18) and (1.21), we can conclude that $(Q,b) \in M_T(P,a)$.

Proof of "only if" part. Let us recall that $u_{P,a}$ is the solution to (1.9) with coefficient matrix P and initial value a. Let us suppose that $u_{P,a}(t,0) = u_{Q,b}(t,0)$ and $u_{P,a}(t,1) = u_{Q,b}(t,1)$ for $-T \le t \le T$. Then it follows from Theorem 2.6 that for $-T+1 \le t \le T-1$

$$u_{Q,b}(t,1) = u_{P,a}(t,1) = R(1)u_{P,a}(t,1) + \int_0^1 K(y,1)u_{P,a}(t,y)dy.$$

We recall that

$$R(1) = \begin{pmatrix} R^1(1) & R^2(1) \\ R^2(1) & R^1(1) \end{pmatrix}, \qquad \widetilde{H} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix},$$

where R^1, R^2 are 2×2 matrices. For simplicity, we set

$$u(t,x) = u_{P,a}(t,x) = \begin{pmatrix} u_1(t,x) \\ u_2(t,x) \\ u_3(t,x) \\ u_4(t,x) \end{pmatrix}, \qquad K(y,1) = \begin{pmatrix} K_1(y,1) \\ K_2(y,1) \end{pmatrix}$$

where $K_1(y,1)$ and $K_2(y,1)$ are 2×4-matrices. Then it follows from $u_{Q,b}(t,1) = u(t,1)$ and $u_{\ell+2}(t,1) = H_\ell u_\ell(t,1)$, $\ell = 1,2$ that

$$\begin{aligned}
(E_2 - R^1(1) - R^2(1)\widetilde{H}) \begin{pmatrix} u_1(t,1) \\ u_2(t,1) \end{pmatrix} &= \int_0^1 K_1(y,1)u(t,y)dy, \\
(\widetilde{H} - R^2(1) - R^1(1)\widetilde{H}) \begin{pmatrix} u_1(t,1) \\ u_2(t,1) \end{pmatrix} &= \int_0^1 K_2(y,1)u(t,y)dy.
\end{aligned}$$

By Proposition 5.1, we have

$$\begin{aligned}
&(E_2 - R^1(1) - R^2(1)\widetilde{H}) \sum_{m\in\mathbf{Z}} \theta_m(t) \begin{pmatrix} f_m^1(1) \\ f_m^2(1) \end{pmatrix} \\
&= \int_0^1 K_1(y,1) \sum_{m\in\mathbf{Z}} \theta_m(t) f_m(y)dy, \\
&(\widetilde{H} - R^2(1) - R^1(1)\widetilde{H}) \sum_{m\in\mathbf{Z}} \theta_m(t) \begin{pmatrix} f_m^1(1) \\ f_m^2(1) \end{pmatrix} \\
&= \int_0^1 K_2(y,1) \sum_{m\in\mathbf{Z}} \theta_m(t) f_m(y)dy
\end{aligned}$$

for $-T+1 \le t \le T-1$. Here f_m^ℓ, $\ell = 1,2$ is the ℓ-th component of f_m. Since the series on the right hand converge uniformly by Proposition 5.1, we can change orders of summation and integration :

$$\sum_{m\in\mathbf{Z}} \theta_m(t)\left\{(E_2 - R^1(1) - R^2(1)\widetilde{H})\begin{pmatrix} f_m^1(1) \\ f_m^2(1)\end{pmatrix} - \int_0^1 K_1(y,1)f_m(y)dy\right\} = 0$$
$$\sum_{m\in\mathbf{Z}} \theta_m(t)\left\{(\widetilde{H} - R^2(1) - R^1(1)\widetilde{H})\begin{pmatrix} f_m^1(1) \\ f_m^2(1)\end{pmatrix} - \int_0^1 K_2(y,1)f_m(y)dy\right\} = 0$$

for $-1 \le t \le 1$. Here we used that $-T+1 \le t \le T-1$ and $T \ge 2$ implies $-1 \le t \le 1$.

We can prove that for the system $\mathcal{S} = \{\theta_m\}_{m\in\mathbf{Z}}$, there exists another system $\widetilde{\mathcal{S}} \subset L^2(-1,1)$ such that for any $\varphi \in \mathcal{S}$, we can choose a unique $\widetilde{\varphi} \in \widetilde{\mathcal{S}}$ satisfying $(\varphi,\psi)_{L^2(-1,1)} = 0$ if and only if $\psi \in \widetilde{\mathcal{S}} \setminus \{\widetilde{\varphi}\}$. The proof is based on Theorem 1.1.1 in Sedletskii [9], and see Appendix C in [11] for the proof. Taking the scalar products in $L^2(-1,1)$ with all $\psi \in \widetilde{\mathcal{S}}$, we can obtain

$$(E_2 - R^1(1) - R^2(1)\widetilde{H})\begin{pmatrix} f_m^1(1) \\ f_m^2(1)\end{pmatrix} - \int_0^1 K_1(y,1)f_m(y)dy = 0, \quad m \in \mathbf{Z},$$
$$(\widetilde{H} - R^2(1) - R^1(1)\widetilde{H})\begin{pmatrix} f_m^1(1) \\ f_m^2(1)\end{pmatrix} - \int_0^1 K_2(y,1)f_m(y)dy = 0, \quad m \in \mathbf{Z}.$$

Here for sufficiently large $|m|$, as f_m, we see that $f_{j,m} = \begin{pmatrix} f_{j,m}^1 \\ f_{j,m}^2 \\ f_{j,m}^3 \\ f_{j,m}^4 \end{pmatrix}$, $j = 1,2$, are two linearly independent eigenvectors corresponding to the eigenvalue $\lambda_{j,m}$. We will prove that

$$\left\{\lim_{m\to\infty}\begin{pmatrix} f_{1,m}^1(1) \\ f_{1,m}^2(1)\end{pmatrix}, \quad \lim_{m\to\infty}\begin{pmatrix} f_{2,m}^1(1) \\ f_{2,m}^2(1)\end{pmatrix}\right\}$$

is linearly independent. In order to prove this, it is sufficient to prove that

$$\left\{\lim_{m\to\infty}\begin{pmatrix} f_{1,m}^1(1) \\ f_{1,m}^2(1) \\ f_{1,m}^3(1) \\ f_{1,m}^4(1)\end{pmatrix}, \quad \lim_{m\to\infty}\begin{pmatrix} f_{2,m}^1(1) \\ f_{2,m}^2(1) \\ f_{2,m}^3(1) \\ f_{2,m}^4(1)\end{pmatrix}\right\}$$

is linearly independent because of

$$f_{j,m}^3(1) = H_1 f_{j,m}^1(1), \quad f_{j,m}^4(1) = H_2 f_{j,m}^2(1), \quad j = 1,2.$$

By (4.8) and (4.9), we have

$$\lim_{m\to+\infty}\begin{pmatrix} f_{1,m}^1(1) \\ f_{1,m}^2(1) \\ f_{1,m}^3(1) \\ f_{1,m}^4(1)\end{pmatrix} = R(1)\begin{pmatrix} c_1\cosh(\beta_1+\mu_1) \\ c_2\cosh(\beta_1+\mu_2) \\ c_1\sinh(\beta_1+\mu_1) \\ c_2\sinh(\beta_1+\mu_2)\end{pmatrix},$$

$$\lim_{m\to+\infty}\begin{pmatrix} f_{2,m}^1(1) \\ f_{2,m}^2(1) \\ f_{2,m}^3(1) \\ f_{2,m}^4(1)\end{pmatrix} = R(1)\begin{pmatrix} d_1\cosh(\beta_2+\mu_1) \\ d_2\cosh(\beta_2+\mu_2) \\ d_1\sinh(\beta_2+\mu_1) \\ d_2\sinh(\beta_2+\mu_2)\end{pmatrix}.$$

Since $R^{-1}(1)$ exists and $c_1d_2 - c_2d_1 \neq 0$ which is proved for (4.2), we can verify that

$$\left\{ R(1)\begin{pmatrix} c_1\cosh(\beta_1+\mu_1) \\ c_2\cosh(\beta_1+\mu_2) \\ c_1\sinh(\beta_1+\mu_1) \\ c_2\sinh(\beta_1+\mu_2)\end{pmatrix}, \quad R(1)\begin{pmatrix} d_1\cosh(\beta_2+\mu_1) \\ d_2\cosh(\beta_2+\mu_2) \\ d_1\sinh(\beta_2+\mu_1) \\ d_2\sinh(\beta_2+\mu_2)\end{pmatrix}. \right\}$$

is linearly independent. Thus

$$\left\{ \lim_{m\to\infty}\begin{pmatrix} f_{1,m}^1(1) \\ f_{1,m}^2(1)\end{pmatrix}, \quad \lim_{m\to\infty}\begin{pmatrix} f_{2,m}^1(1) \\ f_{2,m}^2(1)\end{pmatrix} \right\}$$

is linearly independent.

Furthermore, from Riemann-Lebesgue lemma, we have

$$\lim_{m\to\infty}\int_0^1 K_\ell(y,1)f_m(y)dy = 0, \quad \ell = 1,2.$$

Therefore, we obtain

$$E_2 - R^1(1) - R^2(1)\widetilde{H} = 0, \quad \widetilde{H} - R^2(1) - R^1(1)\widetilde{H} = 0 \tag{5.1}$$

and

$$\int_0^1 K_\ell(y,1)f_m(y)dy = 0, \quad \ell = 1,2, \quad m \in \mathbf{Z}.$$

Since $\{f_m\}_{m\in\mathbf{Z}}$ forms a Riesz basis, it follows that

$$K_1(y,1) = K_2(y,1) = 0, \quad 0 \le y \le 1. \tag{5.2}$$

Therefore, using a characteristic method, we can prove the uniqueness in the problem (2.3) - (2.4) with (5.2) (e.g., [10], [12]), and obtain

$$K(y,x)=0, \quad 0 \le y \le x \le 1.$$

Consequently, we obtain (1.19), (1.20) and (1.21). Since $H_\ell \neq \pm 1$, we can directly derive $R^1(1)=E_2$ and $R^2(1)=0$ from (5.1). Thus we obtain (1.18), and the proof of Theorem 1.4 is completed.

A Proof of Lemma 2.4

We set

$$\begin{cases} L^{(1)}_{k,\ell}(y,x) = K_{k,\ell}(y,x) - K_{k+n,\ell+n}(y,x) \\ L^{(1)}_{k,\ell+n}(y,x) = K_{k,\ell+n}(y,x) - K_{k+n,\ell}(y,x) \\ L^{(2)}_{k,\ell}(y,x) = K_{k,\ell}(y,x) + K_{k+n,\ell+n}(y,x) \\ L^{(2)}_{k,\ell+n}(y,x) = K_{k,\ell+n}(y,x) + K_{k+n,\ell}(y,x), \end{cases} \quad k,\ell = 1,2,\cdots,n$$

and

$$f_{k,\ell}(y,x) = (K(y,x)P(x) - Q(x)K(y,x))_{k,\ell}, \qquad k,\ell = 1,2,\cdots,2n.$$

From (2.3), we obtain

$$\begin{cases} \frac{\partial}{\partial x}K_{k+n,\ell} + \frac{\partial}{\partial y}K_{k,\ell+n} = f_{k,\ell} \\ \frac{\partial}{\partial x}K_{k+n,\ell+n} + \frac{\partial}{\partial y}K_{k,\ell} = f_{k,\ell+n} \\ \frac{\partial}{\partial x}K_{k,\ell} + \frac{\partial}{\partial y}K_{k+n,\ell+n} = f_{k+n,\ell} \\ \frac{\partial}{\partial x}K_{k,\ell+n} + \frac{\partial}{\partial y}K_{k+n,\ell} = f_{k+n,\ell+n}, \end{cases} \quad \text{in } \Omega, \quad k,\ell = 1,2,\cdots,n.$$

Hence we obtain the following system for $k,\ell = 1,2,\cdots,n$:

$$\frac{\partial}{\partial x}L^{(1)}_{k,\ell} - \frac{\partial}{\partial y}L^{(1)}_{k,\ell} = \widetilde{f}_{k,\ell} \equiv f_{k+n,\ell} - f_{k,\ell+n} \tag{A.1}$$

$$\frac{\partial}{\partial x}L^{(1)}_{k,\ell+n} - \frac{\partial}{\partial y}L^{(1)}_{k,\ell+n} = \widetilde{f}_{k,\ell+n} \equiv f_{k+n,\ell+n} - f_{k,\ell} \tag{A.2}$$

$$\frac{\partial}{\partial x}L^{(2)}_{k,\ell} + \frac{\partial}{\partial y}L^{(2)}_{k,\ell} = \widetilde{f}_{k+n,\ell} \equiv f_{k,\ell+n} + f_{k+n,\ell} \tag{A.3}$$

$$\frac{\partial}{\partial x}L^{(2)}_{k,\ell+n} + \frac{\partial}{\partial y}L^{(2)}_{k,\ell+n} = \widetilde{f}_{k+n,\ell+n} \equiv f_{k+n,\ell+n} + f_{k,\ell}. \tag{A.4}$$

By (2.5), we have

$$L^{(1)}_{k,\ell}(x,x) = b_{k,\ell}(x), \quad 0 < x < 1, \quad k,\ell = 1,2,\cdots,n. \tag{A.5}$$

$$L^{(1)}_{k,\ell+n}(x,x) = a_{k,\ell}(x), \quad 0 < x < 1, \quad k,\ell = 1,2,\cdots,n. \tag{A.6}$$

Moreover from (2.4), we have

$$\begin{cases} L^{(2)}_{k,\ell}(0,x) = K_{k,\ell}(0,x) + K_{k+n,\ell+n}(0,x) = K_{k,\ell}(0,x) - h_k K_{k+n,\ell}(0,x) \\ L^{(2)}_{k,\ell+n}(0,x) = K_{k,\ell+n}(0,x) + K_{k+n,\ell}(0,x) = -h_k K_{k,\ell}(0,x) + K_{k+n,\ell}(0,x). \end{cases}$$

Since

$$\begin{cases} L^{(1)}_{k,\ell}(0,x) = K_{k,\ell}(0,x) - K_{k+n,\ell+n}(0,x) = K_{k,\ell}(0,x) + h_k K_{k+n,\ell}(0,x) \\ L^{(1)}_{k,\ell+n}(0,x) = K_{k,\ell+n}(0,x) - K_{k+n,\ell}(0,x) = -h_k K_{k,\ell}(0,x) - K_{k+n,\ell}(0,x), \end{cases}$$

we have

$$\begin{cases} K_{k,\ell}(0,x) = \frac{1}{1-h_k^2} L^{(1)}_{k,\ell}(0,x) + \frac{h_k}{1-h_k^2} L^{(1)}_{k,\ell+n}(0,x) \\ K_{k+n,\ell}(0,x) = -\frac{h_k}{1-h_k^2} L^{(1)}_{k,\ell}(0,x) - \frac{1}{1-h_k^2} L^{(1)}_{k,\ell+n}(0,x). \end{cases}$$

Consequently we have

$$L^{(2)}_{k,\ell}(0,x) = \frac{1+h_k^2}{1-h_k^2} L^{(1)}_{k,\ell}(0,x) + \frac{2h_k}{1-h_k^2} L^{(1)}_{k,\ell+n}(0,x) \tag{A.7}$$

$$L^{(2)}_{k,\ell+n}(0,x) = -\frac{2h_k}{1-h_k^2} L^{(1)}_{k,\ell}(0,x) - \frac{1+h_k^2}{1-h_k^2} L^{(1)}_{k,\ell+n}(0,x). \tag{A.8}$$

Here, we introduce the other variables

$$\begin{cases} u = \frac{x+y}{2} \\ v = \frac{x-y}{2}. \end{cases}$$

Then, we integrate $(A.1)$ and $(A.2)$ for v with $(A.5)$ and $(A.6)$ and we have

$$L^{(1)}_{k,\ell}(y,x) = \int_{\frac{x+y}{2}}^{x} \widetilde{f}_{k,\ell}(-\xi + x + y, \xi) d\xi + b_{k,\ell}\left(\frac{x+y}{2}\right), \quad (y,x) \in \Omega, \quad 1 \le k,\ell \le n \tag{A.9}$$

$$L^{(1)}_{k,\ell+n}(y,x) = \int_{\frac{x+y}{2}}^{x} \widetilde{f}_{k,\ell+n}(-\xi + x + y, \xi) d\xi + a_{k,\ell}\left(\frac{x+y}{2}\right), \quad (y,x) \in \Omega, \quad 1 \le k,\ell \le n. \tag{A.10}$$

Integrating $(A.3)$ and $(A.4)$ for u, we have

$$L^{(2)}_{k,\ell}(y,x) = \int_{x-y}^{x} \widetilde{f}_{k+n,\ell}(\xi - x + y, \xi) d\xi + L^{(2)}_{k,\ell}(0, x-y), \quad (y,x) \in \Omega, 1 \le k,\ell \le n, \tag{A.11}$$

$$L^{(2)}_{k,\ell+n}(y,x) = \int_{x-y}^{x} \widetilde{f}_{k,\ell+n}(\xi - x + y, \xi) d\xi + L^{(2)}_{k,\ell+n}(0, x-y), \quad (y,x) \in \Omega, 1 \le k,\ell \le n. \tag{A.12}$$

By $(A.7)$ - $(A.10)$, we have

$$\begin{aligned}
L^{(2)}_{k,\ell}(y,x) &= \int_{x-y}^{x} \widetilde{f}_{k+n,\ell}(\xi - x + y, \xi)d\xi \\
&+ \int_{\frac{x-y}{2}}^{x-y} \left\{ g_k \widetilde{f}_{k,\ell}(-\xi + x - y, \xi) + \widetilde{g_k} \widetilde{f}_{k,\ell+n}(-\xi + x - y, \xi) \right\} d\xi \\
&+ g_k b_{k,\ell}\left(\frac{x-y}{2}\right) + \widetilde{g_k} a_{k,\ell}\left(\frac{x-y}{2}\right), \qquad \text{(A.13)}\\
L^{(2)}_{k,\ell+n}(y,x) &= \int_{x-y}^{x} \widetilde{f}_{k,\ell+n}(\xi - x + y, \xi)d\xi \\
&+ \int_{\frac{x-y}{2}}^{x-y} \left\{ -\widetilde{g_k} \widetilde{f}_{k,\ell}(-\xi + x - y, \xi) - g_k \widetilde{f}_{k,\ell+n}(-\xi + x - y, \xi) \right\} d\xi \\
&- \widetilde{g_k} b_{k,\ell}\left(\frac{x-y}{2}\right) - g_k a_{k,\ell}\left(\frac{x-y}{2}\right) \qquad \text{(A.14)}
\end{aligned}$$

for $(y,x) \in \Omega$ and $k, \ell = 1, 2, \cdots, n$. Here we set

$$g_k = \frac{1+h_k^2}{1-h_k^2}, \qquad \widetilde{g_k} = \frac{2h_k}{1-h_k^2}.$$

Therefore we obtain Volterra integral eqations $(A.9), (A.10), (A.13)$ and $(A.14)$ of the second kind, which are equivalent to (2.3) - (2.5). Using the iteration method, we can complete the proof. $\square$

B Proof of Theorem 2.5

According to the general theory of the ordinary differential equation, equation (2.9) possesses a unique solution in $\{C^1[0,1]\}^{2n}$. Let us denote the right hand side of (2.10) by $\widetilde{\psi}(x,\lambda)$. Hence it suffices to verify that $\widetilde{\psi}$ satisfies (2.9). Clearly, initial conditions of (2.9) are satisfied.

We have

$$\begin{aligned}
&B_{2n}\frac{d\widetilde{\psi}}{dx}(x,\lambda) + Q(x)\widetilde{\psi}(x,\lambda) - \lambda\widetilde{\psi}(x,\lambda) \\
&= B_{2n}R(x)\frac{d\phi}{dx}(x,\lambda) + \left\{B_{2n}R'(x) + B_{2n}K(x,x) + Q(x)R(x)\right\}\phi(x,\lambda) - \lambda R(x)\phi(x,\lambda) \\
&+ \int_0^x B_{2n}\frac{\partial K}{\partial x}(y,x)\phi(y,\lambda)dy + (Q(x)-\lambda)\int_0^x K(y,x)\phi(y,\lambda)dy.
\end{aligned}$$

Using (2.3) in Lemma 2.4 and (2.8), we obtain by integration by parts,

$$
\begin{aligned}
& B_{2n}\frac{d\widetilde{\psi}}{dx}(x,\lambda)+Q(x)\widetilde{\psi}(x,\lambda)-\lambda\widetilde{\psi}(x,\lambda) \\
& \quad = B_{2n}R(x)\frac{d\phi}{dx}(x,\lambda)+\left\{B_{2n}R'(x)+B_{2n}K(x,x)+Q(x)R(x)\right\}\phi(x,\lambda)-\lambda R(x)\phi(x,\lambda) \\
& \quad + K(0,x)B_{2n}\phi(0,\lambda)-K(x,x)B_{2n}\phi(x,\lambda).
\end{aligned}
$$

Here (2.4) and the condition in (2.8) at $x=0$, yield

$$
K(0,x)B_{2n}\phi(0,\lambda)=0.
$$

Hence

$$
\begin{aligned}
& B_{2n}\frac{d\widetilde{\psi}}{dx}(x,\lambda)+Q(x)\widetilde{\psi}(x,\lambda)-\lambda\widetilde{\psi}(x,\lambda) \\
& = B_{2n}R(x)\frac{d\phi}{dx}(x,\lambda)-\lambda R(x)\phi(x,\lambda)+\left\{B_{2n}R'(x)+Q(x)R(x)-(K(x,x)B_{2n}-B_{2n}K(x,x))\right\}\phi(x,\lambda).
\end{aligned}
$$

By the differential equation in (2.8) and $R=\begin{pmatrix} R^1 & R^2 \\ R^2 & R^1 \end{pmatrix}$, we have

$$
B_{2n}R(x)\frac{d\phi}{dx}(x,\lambda)=R(x)B_{2n}\frac{d\phi}{dx}(x,\lambda)=R(x)(-P(x)+\lambda)\phi(x,\lambda),
$$

so that

$$
\begin{aligned}
& B_{2n}\frac{d\widetilde{\psi}}{dx}(x,\lambda)+Q(x)\widetilde{\psi}(x,\lambda)-\lambda\widetilde{\psi}(x,\lambda) \\
& \quad = \left\{B_{2n}R'(x)+Q(x)R(x)-R(x)P(x)-(K(x,x)B_{2n}-B_{2n}K(x,x))\right\}\phi(x,\lambda).
\end{aligned}
$$

By (2.6) and (2.7), we can directly verify that the right hand side of this equation is zero. Then $\widetilde{\psi}(x,\lambda)=\psi(x,\lambda)$. Thus the proof is completed. $\square$

C Proof of Lemma 3.2

We set

$$
f_j(\lambda)=\begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix}\phi_j(1,\lambda), \qquad j=1,2.
$$

Here, $\phi_j(x,\lambda)$, $j=1,2$ satisfies

$$\begin{cases} B_4\frac{d\phi_j}{dx}(x,\lambda)+P(x)\phi_j(x,\lambda)=\lambda\phi_j(x,\lambda), \\ \phi_1(0,\lambda)=\begin{pmatrix}1\\0\\h_1\\0\end{pmatrix}, \qquad \phi_2(0,\lambda)=\begin{pmatrix}0\\1\\0\\h_2\end{pmatrix}. \end{cases}$$

Then we have

$$\det\Phi(\lambda)=\det\begin{pmatrix} f_1(\lambda) & f_2(\lambda)\end{pmatrix}.$$

Let $\ell_0\in\mathbf{N}\cup\{0\}$ be the smallest number in

$$\left\{\ell\in\mathbf{N}\cup\{0\}\ ;\ \mathrm{rank}\begin{pmatrix}\frac{d^\ell f_1}{d\lambda^\ell}(\lambda_0) & \frac{d^\ell f_2}{d\lambda^\ell}(\lambda_0)\end{pmatrix}\neq 0\right\}.$$

We consider two cases separetely ; **Case I**: $\ell_0\ge 1$ and **Case II**: $\ell_0=0$.

If $\ell_0=0$, then we can argue similarly to the Case I-B stated below. Thus we argue only for the Case I.

Case I: Let $\ell_0\ge 1$. Then

$$\frac{d^\ell f_1}{d\lambda^\ell}(\lambda_0)=\frac{d^\ell f_2}{d\lambda^\ell}(\lambda_0)=\begin{pmatrix}0\\0\end{pmatrix}, \qquad 0\le\ell\le\ell_0-1. \tag{C.1}$$

In particular, we have $f_1(\lambda_0)=f_2(\lambda_0)=0$, that is, $(-\widetilde{H}\quad E_2)\phi_j(1,\lambda_0)=0$, $j=1,2$, which means that ϕ_j, $j=1,2$ satisfies the boundary condition at $x=1$ in (1.10). Therefore $\phi_1,\phi_2\in D(A_P)$.

Let us define $\{\phi^{(j,\ell)}(x)\}_{j=1,2,\ell=1,2,\cdots,\ell_0}$ as follows:

$$\begin{cases}\phi^{(1,1)}(x)=\phi_1(x,\lambda_0)\\ \phi^{(1,2)}(x)=\frac{1}{1!}\frac{\partial\phi_1}{\partial\lambda}(x,\lambda_0)\\ \phi^{(1,3)}(x)=\frac{1}{2!}\frac{\partial^2\phi_1}{\partial\lambda^2}(x,\lambda_0)\\ \vdots\\ \phi^{(1,\ell_0)}(x)=\frac{1}{(\ell_0-1)!}\frac{\partial^{\ell_0-1}\phi_1}{\partial\lambda^{\ell_0-1}}(x,\lambda_0),\end{cases} \qquad \begin{cases}\phi^{(2,1)}(x)=\phi_2(x,\lambda_0)\\ \phi^{(2,2)}(x)=\frac{1}{1!}\frac{\partial\phi_2}{\partial\lambda}(x,\lambda_0)\\ \phi^{(2,3)}(x)=\frac{1}{2!}\frac{\partial^2\phi_2}{\partial\lambda^2}(x,\lambda_0)\\ \vdots\\ \phi^{(2,\ell_0)}(x)=\frac{1}{(\ell_0-1)!}\frac{\partial^{\ell_0-1}\phi_2}{\partial\lambda^{\ell_0-1}}(x,\lambda_0).\end{cases} \tag{C.2}$$

Now by (C.1) we can easily check that $\phi^{(j,\ell)}\in D(A_P)$ for all $j=1,2$ and $\ell=1,2,\cdots,\ell_0$. Moreover

$$(A_P-\lambda_0)\phi^{(j,\ell)}=\phi^{(j,\ell-1)}$$

holds for all $j=1,2$ and $\ell=1,2,\cdots,\ell_0$ where $\phi^{(j,0)}=0$, $j=1,2$. This fact is checked by differentiating the equation

$$A_P\phi_j(x,\lambda) = \lambda\phi_j(x,\lambda)$$

with respect to λ successively.

By $(C.2)$, we can check that $\{\phi^{(j,\ell)}(x)\}_{j=1,2,\ell=1,2,\cdots,\ell_0}$ is a linearly independent system. In fact, let $\sum_{\ell=1}^{\ell_0}\sum_{j=1,2} a_{j,\ell}\phi^{(j,\ell)} = 0$. Applying $(A_P - \lambda_0)$ successively and using $(A_P - \lambda_0)\phi^{(j,\ell)} = \phi^{(j,\ell-1)}$ for $2 \le \ell \le \ell_0$ and $(A_P - \lambda_0)\phi^{(j,1)} = 0$, we see by the linear independence of $\phi^{(1,1)}$ and $\phi^{(2,1)}$ that $a_{j,\ell} = 0$.

Therefore, the algebraic multiplicity of the eigenvalue λ_0 is at least $2\ell_0$.

Moreover, by $(C.1)$ and the linearity of the determinant, we have

$$\frac{d^\ell}{d\lambda^\ell}\det\Phi(\lambda)\bigg|_{\lambda=\lambda_0} = 0, \quad 0 \le \ell \le 2\ell_0 - 1.$$

for $\ell_0 \ge 1$. Hence, for $\ell_0 \ge 1$, the multiplicity of a zero λ_0 of $\det\Phi(\lambda)$ is at least $2\ell_0$.

Therefore, we proved that the algebraic multiplicitiy of λ_0 and the multiplicity of a zero λ_0 of $\det\Phi(\lambda)$ are at least $2\ell_0$.

We separately discuss the following two cases :

Case I-A: The case of

$$\operatorname{rank}\begin{pmatrix} \frac{d^{\ell_0}f_1}{d\lambda^{\ell_0}}(\lambda_0) & \frac{d^{\ell_0}f_2}{d\lambda^{\ell_0}}(\lambda_0) \end{pmatrix} = 2.$$

Case I-B: The case of

$$\operatorname{rank}\begin{pmatrix} \frac{d^{\ell_0}f_1}{d\lambda^{\ell_0}}(\lambda_0) & \frac{d^{\ell_0}f_2}{d\lambda^{\ell_0}}(\lambda_0) \end{pmatrix} = 1.$$

Case I-A: Let

$$\operatorname{rank}\begin{pmatrix} \frac{d^{\ell_0}f_1}{d\lambda^{\ell_0}}(\lambda_0) & \frac{d^{\ell_0}f_2}{d\lambda^{\ell_0}}(\lambda_0) \end{pmatrix} = 2.$$

We will prove that the algebraic multiplicity of the eigenvalue λ_0 is $2\ell_0$.

The set of all the solutions to

$$\begin{cases} (A_P - \lambda_0)\phi(x) = \sum_{j=1,2,\ \ell=1,2,\cdots,\ell_0} a_{j,\ell}\phi^{(j,\ell)}(x) \\ \begin{pmatrix} -\widetilde{h} & E_2 \end{pmatrix} \phi(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{cases}$$

with given $a_{j,\ell} \in \mathbf{C}$, is written as

$$\Bigg\{ \sum_{j=1,2,\ \ell=1,2,\cdots,\ell_0-1} a_{j,\ell}\phi^{(j,\ell+1)}(x) + a_{1,\ell_0}\frac{1}{\ell_0!}\frac{\partial^{\ell_0}\phi_1}{\partial\lambda^{\ell_0}}(x,\lambda_0) \\ + a_{2,\ell_0}\frac{1}{\ell_0!}\frac{\partial^{\ell_0}\phi_2}{\partial\lambda^{\ell_0}}(x,\lambda_0) + b_1\phi_1(x,\lambda_0) + b_2\phi_2(x,\lambda_0)\ ;\ b_1, b_2 \in \mathbf{C} \Bigg\}. \tag{C.3}$$

Then there exists a solution to

$$\begin{cases} (A_P - \lambda_0)\phi(x) = \sum_{j=1,2,\ \ell=1,2,\cdots,\ell_0} a_{j,\ell}\phi^{(j,\ell)}(x) \\ \begin{pmatrix} -\widetilde{h} & E_2 \end{pmatrix} \phi(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} \phi(1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{cases} \tag{C.4}$$

if and only if

$$a_{1,\ell_0}\frac{d^{\ell_0}f_1}{d\lambda^{\ell_0}}(\lambda_0) + a_{2,\ell_0}\frac{d^{\ell_0}f_2}{d\lambda^{\ell_0}}(\lambda_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Because of

$$\operatorname{rank}\begin{pmatrix} \frac{d^{\ell_0}f_1}{d\lambda^{\ell_0}}(\lambda_0) & \frac{d^{\ell_0}f_2}{d\lambda^{\ell_0}}(\lambda_0) \end{pmatrix} = 2,$$

this condition holds if and only if $a_{1,\ell_0} = a_{2,\ell_0} = 0$. Therefore for $j = 1, 2$, there exist no solutions to $(A_P - \lambda_0)\phi = \phi^{(j,\ell_0)}$. Hence the Jordan block corresponding $\phi^{(j,1)}$ is of size $\ell_0 \times \ell_0$, and the algebraic multiplicities of λ_0 is $2\ell_0$.

Because of $(C.1)$ and the linearity of the determinant, we have

$$\frac{d^{2\ell_0}}{d\lambda^{2\ell_0}}\det\Phi(\lambda)\bigg|_{\lambda=\lambda_0} = \frac{(2\ell_0)!}{(\ell_0!)^2}\det\begin{pmatrix} \frac{d^{\ell_0}f_1}{d\lambda^{\ell_0}}(\lambda_0) & \frac{d^{\ell_0}f_2}{d\lambda^{\ell_0}}(\lambda_0) \end{pmatrix} \neq 0,$$

that is, the multiplicity of the zero λ_0 of $\det\Phi(\lambda)$ is $2\ell_0$. Therefore, the algebraic multiplicity of λ_0 is equal to the multiplicity of a zero λ_0 of $\det\Phi(\lambda)$.

Case I-B: Let

$$\operatorname{rank}\begin{pmatrix} \frac{d^{\ell_0} f_1}{d\lambda^{\ell_0}}(\lambda_0) & \frac{d^{\ell_0} f_2}{d\lambda^{\ell_0}}(\lambda_0) \end{pmatrix} = 1.$$

Without the loss of generality, we assume that there exists some $c \in \mathbf{C}$ such that

$$\frac{d^{\ell_0} f_1}{d\lambda^{\ell_0}}(\lambda_0) = c\frac{d^{\ell_0} f_2}{d\lambda^{\ell_0}}(\lambda_0) \tag{C.5}$$

and we assume that

$$\frac{d^{\ell_0} f_2}{d\lambda^{\ell_0}}(\lambda_0) \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{C.6}$$

Now we define $\{\widetilde{\phi}^{(j,\ell)}(x)\}_{j=1,2,\ \ell=1,2,\cdots,\ell_0}$ as follows:

$$\begin{cases} \widetilde{\phi}^{(1,1)}(x) = \phi^{(1,1)}(x) - c\phi^{(2,1)}(x) \\ \widetilde{\phi}^{(1,2)}(x) = \phi^{(1,2)}(x) - c\phi^{(2,2)}(x) \\ \vdots \\ \widetilde{\phi}^{(1,\ell_0)}(x) = \phi^{(1,\ell_0)}(x) - c\phi^{(2,\ell_0)}(x), \end{cases} \qquad \begin{cases} \widetilde{\phi}^{(2,1)}(x) = \phi^{(2,1)}(x) \\ \widetilde{\phi}^{(2,2)}(x) = \phi^{(2,2)}(x) \\ \vdots \\ \widetilde{\phi}^{(2,\ell_0)}(x) = \phi^{(2,\ell_0)}. \end{cases}$$

We can easily check that $(A_P - \lambda_0)\widetilde{\phi}^{(j,\ell)} = \widetilde{\phi}^{(j,\ell-1)}$ for $j = 1,2$ and $\ell \in \{1,2,\cdots,\ell_0\}$ and that $\widetilde{\phi}^{(j,\ell)} \in D(A_P)$ for all $j = 1,2$ and $\ell \in \{1,2,\cdots,\ell_0\}$. Here we set $\widetilde{\phi}^{(j,0)} = 0,\ j = 1,2$.

We set

$$\widetilde{f}_j(\lambda) = \begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix} \widetilde{\phi}^{(j,1)}(1,\lambda), \quad j = 1,2.$$

Then

$$\det\begin{pmatrix} \widetilde{f}_1(\lambda) & \widetilde{f}_2(\lambda) \end{pmatrix} = \det\begin{pmatrix} f_1(\lambda) & f_2(\lambda) \end{pmatrix} = \det \Phi(\lambda).$$

Because

$$\frac{d^\ell f_j}{d\lambda^\ell}(\lambda_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad j = 1,2,\ \ell = 1,2,\cdots,\ell_0 - 1, \tag{C.7}$$

we obtain

$$\frac{d^\ell \widetilde{f}_j}{d\lambda^\ell}(\lambda_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad j = 1,2,\ \ell = 1,2,\cdots,\ell_0 - 1. \tag{C.8}$$

By the definition of $\widetilde{\phi}^{(1,1)}$ and $\widetilde{\phi}^{(2,1)}$, and from $(C.5)$ and $(C.6)$, we have

$$\frac{d^{\ell_0}\widetilde{f}_1}{d\lambda^{\ell_0}}(\lambda_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \frac{d^{\ell_0}\widetilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{C.9}$$

Therefore, we obtain

$$\frac{d^{2\ell_0}}{d\lambda^{2\ell_0}} \det \Phi(\lambda)\Big|_{\lambda=\lambda_0} = 0,$$

that is, the multiplicity of a zero λ_0 of $\det \Phi(\lambda)$ is at least $2\ell_0 + 1$. Now we define $\widetilde{\phi}^{(1,\ell_0+1)}(x)$ by

$$\widetilde{\phi}^{(1,\ell_0+1)}(x) = \frac{1}{\ell_0!}\left(\frac{\partial^{\ell_0}\phi_1}{\partial\lambda^{\ell_0}}(x,\lambda_0) - c\frac{\partial^{\ell_0}\phi_2}{\partial\lambda^{\ell_0}}(x,\lambda_0)\right).$$

Then $\widetilde{\phi}^{(1,\ell_0+1)} \in D(A_P)$ and $(A_P - \lambda_0)\widetilde{\phi}^{(1,\ell_0+1)} = \widetilde{\phi}^{(1,\ell_0)}$.

According to the following respective cases, we proceed:

Case I-B-a: the multiplicity of a zero λ_0 of $\det \Phi(\lambda)$ is $2\ell_0 + 1$.

Case I-B-b: the multiplicity of a zero λ_0 of $\det \Phi(\lambda)$ is $2\ell_0 + \ell_1$ with $\ell_1 \geq 2$.

Case I-B-a: Let the multiplicity of a zero λ_0 of $\det \Phi(\lambda)$ be $2\ell_0 + 1$, that is,

$$\frac{d^{2\ell_0+1}\Phi}{d\lambda^{2\ell_0+1}}(\lambda_0) \neq 0.$$

Then by (C.8) and (C.9), we have

$$\det\begin{pmatrix} \frac{d^{\ell_0+1}\widetilde{f}_1}{d\lambda^{\ell_0+1}}(\lambda_0) & \frac{d^{\ell_0}\widetilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) \end{pmatrix} \neq 0. \tag{C.10}$$

The set of all the solutions to

$$\begin{cases} (A_P - \lambda_0)\phi(x) = \sum_{\ell=1,2,\cdots,\ell_0+1} a_{1,\ell}\widetilde{\phi}^{(1,\ell)}(x) + \sum_{\ell=1,2,\cdots,\ell_0} a_{2,\ell}\widetilde{\phi}^{(2,\ell)}(x) \\ \begin{pmatrix} -\widetilde{h} & E_2 \end{pmatrix}\phi(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{cases}$$

with $a_{1,\ell} \in \mathbf{C},\ \ell \in \{1, 2, \cdots, \ell_0 + 1\}$ and $a_{2,\ell} \in \mathbf{C},\ \ell \in \{1, 2, \cdots, \ell_0\}$, is written as

$$
\begin{aligned}
\Bigg\{ & \sum_{\ell=1,2,\cdots,\ell_0} a_{1,\ell}\phi^{(1,\ell+1)}(x) + \sum_{\ell=1,2,\cdots,\ell_0-1} a_{2,\ell}\phi^{(2,\ell+1)}(x) \\
& + a_{1,\ell_0+1}\frac{1}{(\ell_0+1)!}\left(\frac{\partial^{\ell_0+1}\phi_1}{\partial\lambda^{\ell_0+1}}(x,\lambda_0) - c\frac{\partial^{\ell_0+1}\phi_2}{\partial\lambda^{\ell_0+1}}(x,\lambda_0)\right) \\
& + a_{2,\ell_0}\frac{1}{\ell_0!}\frac{\partial^{\ell_0}\phi_2}{\partial\lambda^{\ell_0}}(x,\lambda_0) + b_1\phi_1(x,\lambda_0) + b_2\phi_2(x,\lambda_0)\ ;\ b_1, b_2 \in \mathbf{C}\Bigg\}.
\end{aligned}
\tag{C.11}
$$

Then there exists a solution to

$$
\begin{cases}
(A_P - \lambda_0)\phi(x) = \sum_{\ell=1,2,\cdots,\ell_0+1} a_{1,\ell}\widetilde{\phi}^{(1,\ell)}(x) + \sum_{\ell=1,2,\cdots,\ell_0} a_{2,\ell}\widetilde{\phi}^{(2,\ell)}(x) \\
\begin{pmatrix} -\widetilde{h} & E_2 \end{pmatrix}\phi(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix}\phi(1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{cases}
\tag{C.12}
$$

if and only if

$$
a_{1,\ell_0+1}\frac{1}{(\ell_0+1)!}\frac{d^{\ell_0+1}\widetilde{f}_1}{d\lambda^{\ell_0+1}}(\lambda_0) + a_{2,\ell_0}\frac{1}{\ell_0!}\frac{d^{\ell_0}\widetilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

By $(C.10)$, this condition holds if and only if $a_{1,\ell_0+1} = a_{2,\ell_0} = 0$. Therefore, by an argument similar to Case I-A, the algebraic multiplicities of λ_0 is $2\ell_0 + 1$. Hence we see that the algebraic multiplicity of λ_0 is $2\ell_0 + 1$ which is equal to the multiplicity of a zero λ_0 of $\det \Phi(\lambda)$.

Case I-B-b: Let the multiplicity of a zero λ_0 of $\det \Phi(\lambda)$ be $2\ell_0+\ell_1$ with $\ell_1 \geq 2$. Let us define $c_1, c_2, \cdots, c_{\ell_1-1}$ as follows.

(a):The definition of c_1.

We define c_1 by

$$
\frac{1}{\ell_0!}c_1\frac{d^{\ell_0}\widetilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) = \frac{1}{(\ell_0+1)!}\frac{d^{\ell_0+1}\widetilde{f}_1}{d\lambda^{\ell_0+1}}(\lambda_0).
\tag{C.13}
$$

Such c_1 exists. In fact, since $\frac{d^{2\ell_0+1}}{d\lambda^{2\ell_0+1}}\det\Phi(\lambda)\Big|_{\lambda=\lambda_0} = 0$, by means of (C.8) and (C.9), we have

$$
\det \begin{pmatrix} \frac{d^{\ell_0+1}\widetilde{f}_1}{d\lambda^{\ell_0+1}}(\lambda_0) & \frac{d^{\ell_0}\widetilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) \end{pmatrix} = 0.
$$

Recalling

$$\frac{d^{\ell_0}\widetilde{f_2}}{d\lambda^{\ell_0}}(\lambda_0) \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from $(C.9)$ we can obtain c_1 such that $(C.13)$ holds.

(b): The definition of $c_2, c_3, \cdots, c_{\ell_1-1}$**.**

We define $c_2, c_3, \cdots, c_{\ell_1-1}$ in an inductive way as follows. For $k = 2, 3, ..., \ell_1 - 2$, assume that $c_1, c_2, \cdots, c_{k-1}$ are already defined. Then we will define c_k such that

$$\frac{1}{\ell_0!}c_k\frac{d^{\ell_0}\widetilde{f_2}}{d\lambda^{\ell_0}}(\lambda_0) = \frac{1}{(\ell_0+k)!}\frac{d^{\ell_0+k}\widetilde{f_1}}{d\lambda^{\ell_0+k}}(\lambda_0) - \sum_{q=1}^{k-1}\frac{c_q}{(\ell_0+k-q)!}\frac{d^{\ell_0+k-q}\widetilde{f_2}}{d\lambda^{\ell_0+k-q}}(\lambda_0) \tag{C.14}$$

holds. Now we prove that such c_k exists and that there does not exist c_{ℓ_1} such that

$$\frac{1}{\ell_0!}c_{\ell_1}\frac{d^{\ell_0}\widetilde{f_2}}{d\lambda^{\ell_0}}(\lambda_0) = \frac{1}{(\ell_0+\ell_1)!}\frac{d^{\ell_0+\ell_1}\widetilde{f_1}}{d\lambda^{\ell_0+\ell_1}}(\lambda_0) - \sum_{q=1}^{\ell_1-1}\frac{c_q}{(\ell_0+\ell_1-q)!}\frac{d^{\ell_0+\ell_1-q}\widetilde{f_2}}{d\lambda^{\ell_0+\ell_1-q}}(\lambda_0). \tag{C.15}$$

Let us calculate

$$\begin{aligned}
&\frac{d^{2\ell_0+k}}{d\lambda^{2\ell_0+k}}\det\Phi(\lambda)\bigg|_{\lambda=\lambda_0} \\
&= \sum_{q=0}^{2\ell_0+k}\frac{(2\ell_0+k)!}{q!(2\ell_0+k-q)!}\det\begin{pmatrix} \frac{d^q\widetilde{f_1}}{d\lambda^q}(\lambda_0) & \frac{d^{2\ell_0+k-q}\widetilde{f_2}}{d\lambda^{2\ell_0+k-q}}(\lambda_0) \end{pmatrix} \\
&= \sum_{q=\ell_0+1}^{\ell_0+k}\frac{(2\ell_0+k)!}{q!(2\ell_0+k-q)!}\det\begin{pmatrix} \frac{d^q\widetilde{f_1}}{d\lambda^q}(\lambda_0) & \frac{d^{2\ell_0+k-q}\widetilde{f_2}}{d\lambda^{2\ell_0+k-q}}(\lambda_0) \end{pmatrix} \\
&= \sum_{q=1}^{k}\frac{(2\ell_0+k)!}{(\ell_0+q)!(\ell_0+k-q)!}\det\begin{pmatrix} \frac{d^{\ell_0+q}\widetilde{f_1}}{d\lambda^{\ell_0+q}}(\lambda_0) & \frac{d^{\ell_0+k-q}\widetilde{f_2}}{d\lambda^{\ell_0+k-q}}(\lambda_0) \end{pmatrix}.
\end{aligned}$$

Here we used $(C.8)$ and $(C.9)$.

Now we eliminate $\frac{d^{\ell_0+q}\widetilde{f}_1}{d\lambda^{\ell_0+q}}(\lambda_0)$ for $q \in \{1, 2, \cdots, k-1\}$ by $(C.14)$. Using $(C.14)$, we have

$$
\begin{aligned}
&\frac{d^{2\ell_0+k}}{d\lambda^{2\ell_0+k}} \det \Phi(\lambda)\bigg|_{\lambda=\lambda_0} \\
&= \sum_{q=1}^{k-1} \frac{(2\ell_0+k)!}{(\ell_0+q)!(\ell_0+k-q)!}(\ell_0+q)! \det \begin{pmatrix} \sum_{p=1}^{q} \frac{c_p}{(\ell_0+q-p)!} \frac{d^{\ell_0+q-p}\widetilde{f}_2}{d\lambda^{\ell_0+q-p}}(\lambda_0) & \frac{d^{\ell_0+k-q}\widetilde{f}_2}{d\lambda^{\ell_0+k-q}}(\lambda_0) \end{pmatrix} \\
&\quad + \frac{(2\ell_0+k)!}{(\ell_0+k)!\ell_0!} \det \begin{pmatrix} \frac{d^{\ell_0+k}\widetilde{f}_1}{d\lambda^{\ell_0+k}}(\lambda_0) & \frac{d^{\ell_0}\widetilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) \end{pmatrix} \\
&= (2\ell_0+k)! \sum_{q=1}^{k-1} \sum_{p=1}^{q} c_p \det \begin{pmatrix} \frac{1}{(\ell_0+q-p)!} \frac{d^{\ell_0+q-p}\widetilde{f}_2}{d\lambda^{\ell_0+q-p}}(\lambda_0) & \frac{1}{(\ell_0+k-q)!} \frac{d^{\ell_0+k-q}\widetilde{f}_2}{d\lambda^{\ell_0+k-q}}(\lambda_0) \end{pmatrix} \\
&\quad + \frac{(2\ell_0+k)!}{(\ell_0+k)!\ell_0!} \det \begin{pmatrix} \frac{d^{\ell_0+k}\widetilde{f}_1}{d\lambda^{\ell_0+k}}(\lambda_0) & \frac{d^{\ell_0}\widetilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) \end{pmatrix}.
\end{aligned}
$$

Then changing orders of the summations, we have

$$
\begin{aligned}
&\frac{d^{2\ell_0+k}}{d\lambda^{2\ell_0+k}} \det \Phi(\lambda)\bigg|_{\lambda=\lambda_0} \\
&= (2\ell_0+k)! \sum_{p=1}^{k-1} c_p \sum_{q=p}^{k-1} \det \begin{pmatrix} \frac{1}{(\ell_0+q-p)!} \frac{d^{\ell_0+q-p}\widetilde{f}_2}{d\lambda^{\ell_0+q-p}}(\lambda_0) & \frac{1}{(\ell_0+k-q)!} \frac{d^{\ell_0+k-q}\widetilde{f}_2}{d\lambda^{\ell_0+k-q}}(\lambda_0) \end{pmatrix} \\
&\quad + \frac{(2\ell_0+k)!}{(\ell_0+k)!\ell_0!} \det \begin{pmatrix} \frac{d^{\ell_0+k}\widetilde{f}_1}{d\lambda^{\ell_0+k}}(\lambda_0) & \frac{d^{\ell_0}\widetilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) \end{pmatrix} \\
&= (2\ell_0+k)! \sum_{p=1}^{k-1} c_p \sum_{q=0}^{k-p-1} \det \begin{pmatrix} \frac{1}{(\ell_0+q)!} \frac{d^{\ell_0+q}\widetilde{f}_2}{d\lambda^{\ell_0+q}}(\lambda_0) & \frac{1}{(\ell_0+k-p-q)!} \frac{d^{\ell_0+k-p-q}\widetilde{f}_2}{d\lambda^{\ell_0+k-p-q}}(\lambda_0) \end{pmatrix} \\
&\quad + \frac{(2\ell_0+k)!}{(\ell_0+k)!\ell_0!} \det \begin{pmatrix} \frac{d^{\ell_0+k}\widetilde{f}_1}{d\lambda^{\ell_0+k}}(\lambda_0) & \frac{d^{\ell_0}\widetilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) \end{pmatrix}.
\end{aligned}
$$

Now we prove that

$$
\begin{aligned}
&\sum_{q=0}^{k-p-1} \det \begin{pmatrix} \frac{1}{(\ell_0+q)!} \frac{d^{\ell_0+q}\widetilde{f}_2}{d\lambda^{\ell_0+q}}(\lambda_0) & \frac{1}{(\ell_0+k-p-q)!} \frac{d^{\ell_0+k-p-q}\widetilde{f}_2}{d\lambda^{\ell_0+k-p-q}}(\lambda_0) \end{pmatrix} \\
&= \det \begin{pmatrix} \frac{1}{\ell_0!} \frac{d^{\ell_0}\widetilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) & \frac{1}{(\ell_0+k-p)!} \frac{d^{\ell_0+k-p}\widetilde{f}_2}{d\lambda^{\ell_0+k-p}}(\lambda_0) \end{pmatrix}.
\end{aligned}
$$

In fact, let $b_q = \frac{1}{(\ell_0+q)!} \frac{d^{\ell_0+q}\widetilde{f}_2}{d\lambda^{\ell_0+q}}(\lambda_0)$ for $q = 1, 2, \cdots, k-p-1$. Let $k-p-1$ be odd. Then the left hand side of the above equation is

$$
\begin{aligned}
&\sum_{q=0}^{k-p-1} \det\begin{pmatrix} b_q & b_{k-p-q} \end{pmatrix} \\
&= \det\begin{pmatrix} b_0 & b_{k-p} \end{pmatrix} + \det\begin{pmatrix} b_1 & b_{k-p-1} \end{pmatrix} + \det\begin{pmatrix} b_2 & b_{k-p-2} \end{pmatrix} + \cdots + \det\begin{pmatrix} b_{k-p-1} & b_1 \end{pmatrix} \\
&= \det\begin{pmatrix} b_0 & b_{k-p} \end{pmatrix} + \Big\{ \det\begin{pmatrix} b_1 & b_{k-p-1} \end{pmatrix} + \det\begin{pmatrix} b_{k-p-1} & b_1 \end{pmatrix} \Big\} \\
&\quad + \Big\{ \det\begin{pmatrix} b_2 & b_{k-p-2} \end{pmatrix} + \det\begin{pmatrix} b_{k-p-2} & b_2 \end{pmatrix} \Big\} + \cdots \\
&\quad + \Big\{ \det\begin{pmatrix} b_{\frac{k-p}{2}-1} & b_{\frac{k-p}{2}+1} \end{pmatrix} + \det\begin{pmatrix} b_{\frac{k-p}{2}+1} & b_{\frac{k-p}{2}-1} \end{pmatrix} \Big\} + \det\begin{pmatrix} b_{\frac{k-p}{2}} & b_{\frac{k-p}{2}} \end{pmatrix} \\
&= \det\begin{pmatrix} b_0 & b_{k-p} \end{pmatrix} = \det\begin{pmatrix} \frac{1}{\ell_0!}\frac{d^{\ell_0}\widetilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) & \frac{1}{(\ell_0+k-p)!}\frac{d^{\ell_0+k-p}\widetilde{f}_2}{d\lambda^{\ell_0+k-p}}(\lambda_0) \end{pmatrix}.
\end{aligned}
$$

For even $k-p-1$, the argument is similar.

Therefore, we obtain

$$
\begin{aligned}
&\frac{d^{2\ell_0+k}}{d\lambda^{2\ell_0+k}} \det\Phi(\lambda)\Big|_{\lambda=\lambda_0} \\
&= (2\ell_0+k)! \sum_{p=1}^{k-1} c_p \det\begin{pmatrix} \frac{1}{\ell_0!}\frac{d^{\ell_0}\widetilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) & \frac{1}{(\ell_0+k-p)!}\frac{d^{\ell_0+k-p}\widetilde{f}_2}{d\lambda^{\ell_0+k-p}}(\lambda_0) \end{pmatrix} \\
&\quad + \frac{(2\ell_0+k)!}{(\ell_0+k)!\ell_0!} \det\begin{pmatrix} \frac{d^{\ell_0+k}\widetilde{f}_1}{d\lambda^{\ell_0+k}}(\lambda_0) & \frac{d^{\ell_0}\widetilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) \end{pmatrix} \\
&= (2\ell_0+k)! \det\begin{pmatrix} \frac{1}{(\ell_0+k)!}\frac{d^{\ell_0+k}\widetilde{f}_1}{d\lambda^{\ell_0+k}}(\lambda_0) - \sum_{p=1}^{k-1}\frac{c_p}{(\ell_0+k-p)!}\frac{d^{\ell_0+k-p}\widetilde{f}_2}{d\lambda^{\ell_0+k-p}}(\lambda_0) & \frac{1}{\ell_0!}\frac{d^{\ell_0}\widetilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) \end{pmatrix}. \quad \text{(C.16)}
\end{aligned}
$$

Now, since λ_0 is a zero of $\det\Phi(\lambda)$ with multiplicity $2\ell_0+\ell_1$, we have

$$
\begin{cases}
\frac{d^{2\ell_0+k}}{d\lambda^{2\ell_0+k}} \det\Phi(\lambda)\Big|_{\lambda=\lambda_0} = 0 & \text{if } 1 \le k \le \ell_1 - 1 \\
\frac{d^{2\ell_0+k}}{d\lambda^{2\ell_0+k}} \det\Phi(\lambda)\Big|_{\lambda=\lambda_0} \neq 0 & \text{if } k = \ell_1.
\end{cases}
$$

Then there exists c_k satisfying $(C.14)$ for $k = 2, 3, \cdots, \ell_1 - 1$, and there is no c_{ℓ_1} satisfying $(C.15)$.

Now we define $\widetilde{\phi}^{(1,\ell_0+2)}, \widetilde{\phi}^{(1,\ell_0+3)}, \cdots, \widetilde{\phi}^{(1,\ell_0+\ell_1)}$ as follows:

$$
\begin{aligned}
\widetilde{\phi}^{(1,\ell_0+k)}(x) &= \frac{1}{(\ell_0+k-1)!}\left(\frac{d^{\ell_0+k-1}\phi_1}{d\lambda^{\ell_0+k-1}}(x,\lambda_0) - c\frac{d^{\ell_0+k-1}\phi_2}{d\lambda^{\ell_0+k-1}}(x,\lambda_0)\right) \\
&\quad - \sum_{q=1}^{k-1} \frac{c_q}{(\ell_0+k-1-q)!}\frac{d^{\ell_0+k-1-q}\phi_2}{d\lambda^{\ell_0+k-1-q}}(x,\lambda_0)
\end{aligned}
$$

for $k \in \{2,3,\cdots,\ell_1\}$. Then, for all $k \in \{2,3,\cdots,\ell_1\}$, we have $\widetilde{\phi}^{(1,\ell_0+k)} \in D(A_P)$ and

$$
\begin{aligned}
&(A_P-\lambda_0)\widetilde{\phi}^{(1,\ell_0+k)}(x)\\
&=\widetilde{\phi}^{(1,\ell_0+k-1)}(x)+\left(\text{a linear combination of } \phi_2(x,\lambda_0), \frac{\partial \phi_2}{\partial \lambda}(x,\lambda_0),\cdots,\frac{\partial^{\ell_0-1}\phi_2}{\partial \lambda^{\ell_0-1}}(x,\lambda_0)\right).
\end{aligned}
$$

We define $\zeta^{(1,1)}(x), \zeta^{(1,2)}(x), \cdots, \zeta^{(1,\ell_0+\ell_1)}(x), \zeta^{(2,1)}, \zeta^{(2,2)}, \cdots, \zeta^{(2,\ell_0)}$ by

$$
\left\{
\begin{array}{lcl}
\zeta^{(1,\ell_0+\ell_1)}(x) & = & \widetilde{\phi}^{(1,\ell_0+\ell_1)}(x)\\
\zeta^{(1,\ell_0+\ell_1-1)}(x) & = & (A_P-\lambda_0)\widetilde{\phi}^{(1,\ell_0+\ell_1)}(x)\\
\zeta^{(1,\ell_0+\ell_1-2)}(x) & = & (A_P-\lambda_0)^2\widetilde{\phi}^{(1,\ell_0+\ell_1)}(x)\\
\vdots & & \\
\zeta^{(1,\ell_0+1)}(x) & = & (A_P-\lambda_0)^{\ell_1-1}\widetilde{\phi}^{(1,\ell_0+\ell_1)}(x)\\
\zeta^{(1,\ell_0)}(x) & = & (A_P-\lambda_0)^{\ell_1}\widetilde{\phi}^{(1,\ell_0+\ell_1)}(x)\\
\zeta^{(1,\ell_0-1)}(x) & = & (A_P-\lambda_0)^{\ell_0+1}\widetilde{\phi}^{(1,\ell_0+\ell_1)}(x)\\
\vdots & & \\
\zeta^{(1,1)}(x) & = & (A_P-\lambda_0)^{\ell_0+\ell_1-1}\widetilde{\phi}^{(1,\ell_0+\ell_1)}(x),
\end{array}
\right.
$$

$$
\left\{
\begin{array}{lcl}
\zeta^{(2,\ell_0)}(x) & = & \widetilde{\phi}^{(2,\ell_0)}(x)\\
\zeta^{(2,\ell_0-1)}(x) & = & \widetilde{\phi}^{(2,\ell_0-1)}(x)\\
\vdots & & \\
\zeta^{(2,1)}(x) & = & \widetilde{\phi}^{(2,1)}(x).
\end{array}
\right.
$$

Then we can see that $\{\zeta^{(1,\ell)}(x)\}_{\ell=1,2,\cdots,\ell_0+\ell_1} \cup \{\zeta^{(2,\ell)}\}_{\ell=1,2,\cdots,\ell_0}$ is a linearly independent system.

For fixed $a_{1,\ell} \in \mathbf{C},\ j \in \{1,2,\cdots,\ell_0+\ell_1\}$ and $a_{2,\ell} \in \mathbf{C},\ \ell \in \{1,2,\cdots,\ell_0\}$, the set of all the solutions to

$$
\left\{
\begin{array}{l}
(A_P-\lambda_0)\phi(x) = \sum_{\ell=1,2,\cdots,\ell_0+\ell_1} a_{1,\ell}\zeta^{(1,\ell)}(x) + \sum_{\ell=1,2,\cdots,\ell_0} a_{2,\ell}\zeta^{(2,\ell)}(x)\\
\begin{pmatrix} -\widetilde{h} & E_2 \end{pmatrix}\phi(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{array}
\right.
$$

is written as

$$
\begin{aligned}
\Bigg\{ & \sum_{\ell=1,2,\cdots,\ell_0+\ell_1-1} a_{1,\ell}\zeta^{(1,\ell+1)}(x) + \sum_{\ell=1,2,\cdots,\ell_0-1} a_{2,\ell}\zeta^{(2,\ell+1)}(x)\\
&+ a_{1,\ell_0+\ell_1}\left[\frac{1}{(\ell_0+\ell_1)!}\left(\frac{\partial^{\ell_0+\ell_1}\phi_1}{\partial\lambda^{\ell_0+\ell_1}}(x,\lambda_0) - c\frac{\partial^{\ell_0+\ell_1}\phi_2}{\partial\lambda^{\ell_0+\ell_1}}(x,\lambda_0)\right)\right.\\
&\left. - \sum_{q=1}^{j_1-1}\frac{c_q}{(\ell_0+\ell_1-q)!}\frac{\partial^{\ell_0+\ell_1-q}\phi_2}{\partial\lambda^{\ell_0+\ell_1-q}}(x,\lambda_0)\right]\\
&+ a_{2,\ell_0}\frac{1}{\ell_0!}\frac{\partial^{\ell_0}\phi_2}{\partial\lambda^{\ell_0}}(x,\lambda_0) + b_1\phi_1(x,\lambda_0) + b_2\phi_2(x,\lambda_0)\ ;\ b_1,b_2\in\mathbf{C}\Bigg\}. \qquad \text{(C.17)}
\end{aligned}
$$

Then there exists a solution to

$$\left\{\begin{array}{l}(A_P-\lambda_0)\phi(x)=\sum_{\ell=1,2,\cdots,\ell_0+\ell_1} a_{1,\ell}\widetilde{\phi}^{(1,\ell)}(x)+\sum_{\ell=1,2,\cdots,\ell_0} a_{2,\ell}\widetilde{\phi}^{(2,\ell)}(x) \\ \begin{pmatrix} -\widetilde{h} & E_2 \end{pmatrix}\phi(0)=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -\widetilde{H} & E_2 \end{pmatrix}\phi(1)=\begin{pmatrix} 0 \\ 0 \end{pmatrix}\end{array}\right. \tag{C.18}$$

if and only if

$$\begin{aligned} a_{1,\ell_0+\ell_1}&\left(\frac{1}{(\ell_0+\ell_1)!}\frac{d^{\ell_0+\ell_1}\widetilde{f}_1}{d\lambda^{\ell_0+\ell_1}}(\lambda_0)-\sum_{q=1}^{\ell_1-1}\frac{c_q}{(\ell_0+\ell_1-q)!}\frac{d^{\ell_0+\ell_1-q}\widetilde{f}_2}{d\lambda^{\ell_0+\ell_1-q}}(\lambda_0)\right) \\ &+a_{2,\ell_0}\frac{1}{\ell_0!}\frac{d^{\ell_0}\widetilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0)=\begin{pmatrix} 0 \\ 0 \end{pmatrix}.\end{aligned}$$

Because there is no c_{ℓ_1} satisfying $(C.15)$, this condition holds if and only if $a_{1,\ell_0+\ell_1}=a_{2,\ell_0}=0$.

Therefore, if the solution to $(C.18)$ exists, then it is in the space spanned by $\{\zeta^{(1,\ell)}(x)\}_{\ell=1,2,\cdots,\ell_0+\ell_1}\cup\{\zeta^{(2,\ell)}\}_{\ell=1,2,\cdots,\ell_0}$. Hence, by an argument similar to Case I-A, the algebraic multiplicity of an eigenvalue λ_0 of A_P is $2\ell_0+\ell_1$. This is equal to the multiplicity of a zero λ_0 of $\det\Phi(\lambda)$.

Thus in all the cases, we have seen that the algebraic multiplicity of λ_0 is equal to the multiplicity of a zero λ_0 of $\det\Phi(\lambda)$, that is, the proof is completed. $\square$

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