UTMS 2007–8 June 1, 2007

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Abstract

In this paper, we consider the stability of an inverse spectral problem for a nonsymmetric ordinary differential operator. We give an estimate for deviation in the coefficients of this operator when the spectral data perturb. Our result shows that if two spectral data are sufficiently close to one another, then the corresponding two differential operators must be close each other in the sense of C^{ϑ}-norm ($\vartheta = 0, 1$).

1 Introduction

In this paper, we prove a stability result for an inverse spectral problem for a system:

$$\begin{cases} B\dfrac{\mathrm{d}\varphi}{\mathrm{d}x}(x) + P(x)\varphi(x) = \lambda\varphi(x), \qquad 0 < x < 1, \\ \varphi^{(2)}(0)\cosh\mu - \varphi^{(1)}(0)\sinh\mu = \varphi^{(2)}(1)\cosh\nu + \varphi^{(1)}(1)\sinh\nu = 0, \end{cases} \tag{1.1}$$

where $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\varphi = \begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix} \in (C^1[0,1])^2$, $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \in (C^1[0,1])^4$ with complex-valued components, and the constants $\mu, \nu \in \mathbb{C}$. Physically, problem (1.1) can describe proper vibrations for various phenomena (cf. [2], [8], [11]), etc. On the other hand, this eigenvalue problem can also generalize the Sturm-Liouville problem ([9]) and the one-dimensional Dirac equation ([6]).

Now we define a differential operator $A_{P,\mu,\nu}$ in $\left(L^2(0,1)\right)^2$ by

$$A_{P,\mu,\nu} = \mathcal{A}_P := B\frac{\mathrm{d}}{\mathrm{d}x} + P$$

with the domain

$$\begin{aligned} D(A_{P,\mu,\nu}) = \{\varphi \in \left(H^1(0,1)\right)^2 : \; & \varphi^{(2)}(0)\cosh\mu - \varphi^{(1)}(0)\sinh\mu = 0, \\ & \varphi^{(2)}(1)\cosh\nu + \varphi^{(1)}(1)\sinh\nu = 0\}. \end{aligned}$$

*The author was supported by the Scholarship of Japanese Government and is supported by JSPS Fellowship P05297.

This operator $A_{P,\mu,\nu}$ is nonsymmetric and the corresponding inverse problem is more difficult. In [6] the authors discussed an inverse problem of determining a coefficient matrix by data on spectra, which are called the spectral characteristics, and proved a uniqueness theorem and a reconstruction formula for (1.1). For the reconstruction procedure a nature question comes out: if there is a perturbation in the spectral characteristics, then how much deviation in the reconstructed matrices will appear. For clarity, we give a description of the spectral characteristics $S(P,\mu,\nu)$ as follows. The spectrum of $A_{P,\mu,\nu}$ is $\sigma(A_{P,\mu,\nu}) = \{\lambda^i\}_{1\le i\le N} \bigcup \{\lambda_n\}_{n\in\mathbb{Z}}$, where the algebraic multiplicity of λ_n is 1 (i.e. λ_n is simple) and $m_i \ge 2$ is the algebraic multiplicity of eigenvalue λ^i, and ρ^i (respectively ρ_n) are the scalar products of the corresponding root vectors (respectively eigenvectors) of $A_{P,\mu,\nu}$ and those of the adjoint operator $A^*_{P,\mu,\nu}$ of $A_{P,\mu,\nu}$. Moreover, $\boldsymbol{\alpha}^i = (\alpha^i_1, \cdots, \alpha^i_{m_i-1})$ are $m_i - 1$-dimensional constant vectors whose components are determined by the root vectors of $A^*_{P,\mu,\nu}$ which are orthogonal to those of $A_{P,\mu,\nu}$ in $(L^2(0,1))^2$. The definition of $\rho^i, \boldsymbol{\alpha}^i$ is constructive, and the detailed description can be found in [5] and [6]. Then $\left\{\lambda^i, m_i, \rho^i, \boldsymbol{\alpha}^i\right\}_{1\le i\le N} \bigcup \{\lambda_n, \rho_n\}_{n\in\mathbb{Z}}$ is called *the spectral characteristics*.

We first give the Gateaux derivatives for P which has been obtained as solution of an inverse spectral problem, and then apply these Gateaux derivatives to obtain stability result. Here and henceforth, for convenience, we use the same symbols as in [6].

For eigenvalue problem (1.1), Trooshin and Yamamoto [8] showed that the eigenvalues have an asymptotic behavior

$$\lambda_n = \frac{1}{2}\int_0^1 (p_{11}+p_{22})(s)\mathrm{d}s - \mu - \nu + n\pi\sqrt{-1} + O\left(\frac{1}{|n|}\right) \tag{1.2}$$

as $|n| \to \infty$, and the set of all the root vectors of $A_{P,\mu,\nu}$ is a Riesz basis in $(L^2(0,1))^2$. For simplicity, throughout this paper we assume that matrix-valued functions $P \in (C^1[0,1])^4$ under consideration satisfy $\theta_0 := \frac{1}{2}\int_0^1 \mathrm{tr}P(s)\mathrm{d}s = 0$, where $\mathrm{tr}P$ denotes the trace of P. In this case the asymptotic forms for the eigenvalues and the norming constants are given as

$$\lambda_n = -\mu - \nu + n\pi\sqrt{-1} + \kappa_n, \quad \rho_n = 1 + \zeta_n \tag{1.3}$$

with

$$\kappa_n = O\left(\frac{1}{|n|}\right), \quad \zeta_n = O\left(\frac{1}{|n|}\right). \tag{1.4}$$

If $\theta_0 \neq 0$, then the asymptotic form for the norming constants ρ_n remains the same while $\lambda_n = \theta_0 - \mu - \nu + n\pi\sqrt{-1} + \kappa_n$. However, if we put $\widetilde{\lambda}_n = \lambda_n - \theta_0$ and $\widetilde{P} = P - \theta_0 E$ where E denotes the 2×2 unit matrix, then, by replacing P, λ_n by $\widetilde{P}$ and $\widetilde{\lambda}_n$ respectively, the same argument will be effective as before, and a similar result can be obtained except that $\theta_0 = \lim_{n\to\infty}(\lambda_n + \mu + \nu - n\pi\sqrt{-1})$ should be appended into the spectral data.

The method used here is similar to that of [4] which discussed the classical inverse Sturm-Liouville problem, but some difficulties have to be overcome. First, in our case it is more complicated to obtain the Gateaux derivatives. Second, in our case, the relation between the reconstructed matrix-valued function and the continuously differentiable matrix-valued function which contains the useful information of the spectral characteristics is nonlinear, while in [4] it is linear. Third, due to the non-symmetry of the operator under consideration, we have to take the adjoint system of (1.1) into consideration simultaneously. One of our result shows that the maximum norm of deviations in the reconstructed matrices can be estimated by an l^1-norm of differences between two sequences of spectral characteristics. Our result coincides with [10] where under special conditions, the stability problem for the inverse spectral problem of determining the matrix-valued coefficient P from two spectra (cf. [2] and [9]) was studied by means of the contraction principle. As for the other results of stability on one dimensional inverse Sturm-Liouville problems, we refer to [3] and [7]. For a multidimensional case we refer to [1].

This paper consists of four sections. In Section 2, we will prove the analyticity in components of spectral characteristics of the solutions of the Gel'fand-Levitan equation. In Section 3 we will calculate Gateaux derivatives and give their bound estimates. In Section 4 we obtain the main results on stability.

2 The analyticity of the solution of the Gel'fand-Levitan equation

First we introduce some notations. Let

$$\Omega = \{(x,y) \in (0,1)^2 : 0 < y < x < 1\}.$$

For $P = (p_{ij})_{1\le 1,j\le 2} \in (C^1[0,1])^4$ we define

$$\|P\|_{C^\vartheta} = \max\left\{ \max_{1\le i,j\le 2} \max_{0\le x\le 1} |p_{ij}(x)|\,,\ \max_{1\le i,j\le 2} \max_{0\le x\le 1} \left| \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\vartheta p_{ij}(x) \right| \right\},$$

where $\vartheta = 0,\ 1$ and we set $(\frac{\mathrm{d}}{\mathrm{d}x})^0 f = f$. Henceforth for a scalar or vector continuously differentiable function we define $\|\cdot\|_{C^\vartheta}$ $(\vartheta = 0,1)$ similarly. As stated in Section 1, we take the same symbols as in [6]. In particular we set $p_{11} = p_1,\ p_{12} = p_2,\ p_{21} = u,\ p_{22} = v$, where $u,\ v$ are given functions. Next we show

Lemma 2.1 *Let the matrix-valued function $\widetilde{F}(x,y;z;\aleph)$ be in $\left(C^1(\overline{\Omega})\right)^4$ and in $\left(C^1(\overline{(0,1)^2\backslash\Omega})\right)^4$ as a function of x and y, where z is a complex parameter and $\aleph$ denotes two complex parameters. If $\widetilde{F}(x,y;z;\aleph)$ is analytic in a neighbourhood of the origin with respect to z with $\widetilde{F}(x,y;0;\aleph) = 0$, and $M(x,y;z;\aleph)$ is the unique solution to a Fredholm equation*

$$\widetilde{F}(x,y;z;\aleph) + M(x,y;z;\aleph) + \int_0^x M(x,\tau;z;\aleph)\widetilde{F}(\tau,y;z;\aleph)\mathrm{d}\tau = 0,\ \ (x,y) \in \overline{\Omega}, \tag{2.1}$$

then $M(x,y;z;\aleph)$ is analytic in some neighbourhood of the origin with respect to z with $M(x,y;0;\aleph) = 0$.

Proof. By the assumption we can express $\widetilde{F}(x,y;z;\aleph)$ as

$$\widetilde{F}(x,y;z;\aleph) = \sum_{n=1}^\infty a_n(x,y;\aleph)z^n,\ \text{ where } a_n(x,y;\aleph) \in \left(C^1\left(\overline{\Omega}\right)\right)^4, \in \left(C^1(\overline{(0,1)^2\backslash\Omega})\right)^4. \tag{2.2}$$

Put $\widetilde{M}(x,y;z;\aleph)=\sum_{n=1}^{\infty} b_n(x,y;\aleph)z^n$, where $b_n(x,y;\aleph)$ $(n\geq 1)$ satisfy

$$\left\{\begin{array}{ll} b_1(x,y;\aleph)=-a_1(x,y;\aleph), & n=1,\\ b_n(x,y;\aleph)=-a_n(x,y;\aleph)-\displaystyle\int_0^x\sum_{k=1}^{n-1} b_k(x,\tau;\aleph)a_{n-k}(\tau,y;\aleph)\mathrm{d}\tau, & n\geq 2.\end{array}\right. \tag{2.3}$$

In view of (2.3), we can see that $\{b_n(x,y;\aleph)\}_{n\geq 1}$ can be uniquely determined by $\{a_n(x,y;\aleph)\}_{n\geq 1}$. For any $a=(a_{ij})_{1\leq i,j\leq 2}\in\left(C^1\left(\overline{\Omega}\right)\right)^4$ we let

$$\| a \|_{\left(C(\overline{\Omega})\right)^4}=\max_{1\leq i,j\leq 2}\max_{(x,y)\in\overline{\Omega}}|a_{ij}(x,y)|\,.$$

Then for $n\geq 2$, we have

$$\begin{aligned}\| b_n z^n \|_{\left(C(\overline{\Omega})\right)^4}&\leq\| a_n z^n \|_{\left(C(\overline{\Omega})\right)^4}+\| \sum_{k=1}^{n-1} b_k z^k a_{n-k}z^{n-k} \|_{\left(C(\overline{\Omega})\right)^4}\\ &\leq\| a_n z^n \|_{\left(C(\overline{\Omega})\right)^4}+2\sum_{k=1}^{n-1}\| b_k z^k \|_{\left(C(\overline{\Omega})\right)^4}\| a_{n-k}z^{n-k} \|_{\left(C(\overline{\Omega})\right)^4}\,.\end{aligned}$$

Hence, summation over n from 2 to ∞ yields that

$$\sum_{n=2}^{\infty}\| b_n z^n \|_{\left(C(\overline{\Omega})\right)^4}\leq\sum_{n=2}^{\infty}\| a_n z^n \|_{\left(C(\overline{\Omega})\right)^4}+2\sum_{n=1}^{\infty}\| b_n z^n \|_{\left(C(\overline{\Omega})\right)^4}\sum_{n=1}^{\infty}\| a_n z^n \|_{\left(C(\overline{\Omega})\right)^4}\,.$$

Since $\| b_1 z \|_{\left(C(\overline{\Omega})\right)^4}=\| a_1 z \|_{\left(C(\overline{\Omega})\right)^4}$, we have

$$\sum_{n=1}^{\infty}\| b_n z^n \|_{\left(C(\overline{\Omega})\right)^4}\leq\frac{\displaystyle\sum_{n=1}^{\infty}\| a_n z^n \|_{\left(C(\overline{\Omega})\right)^4}}{1-2\displaystyle\sum_{n=1}^{\infty}\| a_n z^n \|_{\left(C(\overline{\Omega})\right)^4}}<\infty$$

when $|z|$ is sufficiently small such that $\sum_{n=1}^{\infty}\| a_n z^n \|_{\left(C(\overline{\Omega})\right)^4}<\frac{1}{2}$. Therefore, the infinite sum $\widetilde{M}(x,y;z;\aleph)$ is convergent absolutely and uniformly. Hence $\widetilde{M}(x,y;z;\aleph)$ is analytic in a properly small neighborhood of the origin with respect to z, which leads to the following equality

$$\sum_{n=2}^{\infty}\int_0^x\sum_{k=1}^{n-1} b_k(x,\tau;\aleph)a_{n-k}(\tau,y;\aleph)z^n\mathrm{d}\tau=\int_0^x\sum_{n=2}^{\infty}\sum_{k=1}^{n-1} b_k(x,\tau;\aleph)a_{n-k}(\tau,y;\aleph)z^n\mathrm{d}\tau. \tag{2.4}$$

Consequently, multiplying (2.3) by z^n and then summing over n from 1 to ∞, we see by (2.4) that $\widetilde{M}(x,y;z;\aleph)$ satisfies the following equation

$$\widetilde{F}(x,y;z;\aleph)+\widetilde{M}(x,y;z;\aleph)+\int_0^x\widetilde{M}(x,\tau;z;\aleph)\widetilde{F}(\tau,y;z;\aleph)\mathrm{d}\tau=0.$$

By the assumption, we obtain $M(x,y;z;\aleph)=\widetilde{M}(x,y;z;\aleph)$. Thus the proof of Lemma 2.1 is completed. $\square$

Next we apply Lemma 2.1 to the Gel'fand-Levitan equation ([6]) for the nonsymmetric system (1.1) in the following five cases, where only one spectral datum perturbs. Before doing that, we first give some remarks. We only consider a stability problem for $P=\begin{pmatrix} p_1 & p_2 \\ u & v \end{pmatrix}$ and $P_0=\begin{pmatrix} p_1^0 & p_2^0 \\ u & v \end{pmatrix}$ corresponding to the following spectral characteristic respectively:

$$S(P,\mu,\nu)=\left\{\lambda^i,m_i,\rho^i,\boldsymbol{\alpha}^i\right\}_{1\le i\le N}\bigcup\left\{\lambda_n,\rho_n\right\}_{n\in\mathbb{Z}}$$

and

$$S(P_0,\mu,\nu)=\left\{\lambda_0^i,m_i,\rho_0^i,\boldsymbol{\alpha}_0^i\right\}_{1\le i\le N}\bigcup\left\{\lambda_n^0,\rho_n^0\right\}_{n\in\mathbb{Z}}.$$

That is, throughout this paper, we assume that the number N of non-simple eigenvalues λ^i is same as the one of λ_0^i and the multiplicities m^i are same.

Remark 2.1. The direct problem for (1.1) is unstable, namely, it may happen that the deviation between P and P_0 is sufficiently small, while the deviation between two spectral characteristics is large. For example, take $P_0=\begin{pmatrix} 2\pi & 0 \\ 0 & 0 \end{pmatrix}$ and $P=\begin{pmatrix} 2\pi+\epsilon & 0 \\ 0 & 0 \end{pmatrix}$ where ϵ is a sufficiently small real number. In $S(P_0,0,0)$ there is one eigenvalue with algebraic multiplicity 2, while in $S(P,0,0)$ each eigenvalue is simple.

For our stability problem, we restrict ourselves to the case in which there is only a small perturbation in the spectral characteristics. Therefore, we will not consider the cases in which the algebraic multiplicities of eigenvalues for $A_{P,\mu,\nu}$ and $A_{P_0,\mu,\nu}$ do not coincide. Now let us give some notations. For $\lambda\in\mathbb{C}$, let $S(x,\lambda)$ and $S^*(x,\overline{\lambda})$ satisfy the following initial value problems respectively:

$$\begin{cases} \left(\mathcal{A}_{P_0}-\lambda\right)S=0, \\ S(0,\lambda)=\xi, \end{cases}$$

$$\begin{cases} \left(\mathcal{A}_{P_0}^*-\overline{\lambda}\right)S^*=0, \\ S^*(0,\overline{\lambda})=\eta, \end{cases}$$

where

$$\xi=\begin{pmatrix} \cosh\mu \\ \sinh\mu \end{pmatrix},\quad \eta=\begin{pmatrix} \cosh\overline{\mu} \\ -\sinh\overline{\mu} \end{pmatrix}.$$

Let $S_{(j)}(x,\lambda)$, $S_{(j)}^*(x,\overline{\lambda})$ and $\widetilde{S}_{(j)}^*(x,\overline{\lambda})$ $(1\le j\le m_i)$ satisfy the following initial value problems respectively:

$$\begin{cases} \left(\mathcal{A}_{P_0}-\lambda\right)S_{(1)}=0,\ \left(\mathcal{A}_{P_0}-\lambda\right)S_{(j)}=S_{(j-1)},\ 2\le j\le m_i, \\ S_{(j)}(0,\lambda)=\xi,\ 1\le j\le m_i, \end{cases}$$

$$\begin{cases} \left(\mathcal{A}_{P_0}^*-\overline{\lambda}\right)S_{(m_i)}^*=0,\ \left(\mathcal{A}_{P_0}^*-\overline{\lambda}\right)S_{(j)}^*=S_{(j+1)}^*,\ 1\le j\le m_i-1, \\ S_{(m_i)}^*(0,\overline{\lambda})=\eta,\ S_{(j)}^*(0,\overline{\lambda})=\overline{\alpha_j^i}\eta,\ 1\le j\le m_i-1, \end{cases} \tag{2.5}$$

$$\begin{cases} \left(\mathcal{A}^*_{P_0} - \overline{\lambda}\right) \widetilde{S}^*_{(m_i)} = 0, \ \left(\mathcal{A}^*_{P_0} - \overline{\lambda}\right) \widetilde{S}^*_{(j)} = \widetilde{S}^*_{(j+1)}, \ 1 \le j \le m_i - 1, \\ \widetilde{S}^*_{(m_i)}(0, \overline{\lambda}) = \eta, \ \widetilde{S}^*_{(j)}(0, \overline{\lambda}) = \overline{(\alpha^i_j)_0}\eta, \ 1 \le j \le m_i - 1. \end{cases} \tag{2.6}$$

Here we should note that $S^*_{(m_i)}(x, \overline{\lambda}) = \widetilde{S}^*_{(m_i)}(x, \overline{\lambda})$ and hence

$$\rho^i_0 = \left(S_{(m_i)}(\cdot, \lambda^i_0), \widetilde{S}^*_{(m_i)}(\cdot, \overline{\lambda^i_0})\right)_{(L^2(0,1))^2} = \left(S_{(m_i)}(\cdot, \lambda^i_0), S^*_{(m_i)}(\cdot, \overline{\lambda^i_0})\right)_{(L^2(0,1))^2}.$$

Thus we see that $\sigma^i = \rho^i_0$ and hence in Theorem 3 of [6]

$$\widetilde{F}(x, y) = F(x, y) + \sum_{i=1}^{N} \sum_{j=1}^{m_i} \frac{1}{\rho^i_0} \left(\overline{S^*_{(j)}(x, \overline{\lambda^i_0})} - \overline{\widetilde{S}^*_{(j)}(x, \overline{\lambda^i_0})}\right) \left(S_{(j)}(y, \lambda^i_0)\right)^T, \tag{2.7}$$

where

$$F(x, y) = \frac{\partial^2 f}{\partial x \partial y}(x, y), \tag{2.8}$$

and $f(x, y)$ is defined by:

$$\begin{aligned} f(x, y) \quad &= \quad \sum_{i=1}^{N} \sum_{j=1}^{m_i} \left\{ \frac{\overline{C^*_{(j)}(x, \overline{\lambda^i})} C^T_{(j)}(y, \lambda^i)}{\rho^i} - \frac{\overline{C^*_{(j)}(x, \overline{\lambda^i_0})} C^T_{(j)}(y, \lambda^i_0)}{\rho^i_0} \right\} \\ &\quad + \sum_{n \in \mathbb{Z}} \left\{ \frac{\overline{C^*(x, \overline{\lambda_n})} C^T(y, \lambda_n)}{\rho_n} - \frac{\overline{C^*(x, \overline{\lambda^0_n})} C^T(y, \lambda^0_n)}{\rho^0_n} \right\}. \end{aligned} \tag{2.9}$$

Here we set

$$C^*(x, \overline{\lambda}) = \int_0^x S^*(t, \overline{\lambda}) \mathrm{d}t, \quad C^*_{(j)}(x, \overline{\lambda}) = \int_0^x S^*_{(j)}(t, \overline{\lambda}) \mathrm{d}t,$$
$$C(y, \lambda) = \int_0^y S(t, \lambda) \mathrm{d}t, \quad C_{(j)}(y, \lambda) = \int_0^y S_{(j)}(t, \lambda) \mathrm{d}t.$$

Case 1: Let m be an integer. Let the spectral characteristics of P and P_0 be

$$S(P, \mu, \nu) = \left\{\lambda^i_0, m_i, \rho^i_0, \boldsymbol{\alpha}^i_0\right\}_{1 \le i \le N} \bigcup \left\{\lambda^0_n, \rho^0_n\right\}_{n \in \mathbb{Z}, n \ne m} \bigcup \left\{\lambda_m, \rho^0_m\right\},$$
$$S(P_0, \mu, \nu) = \left\{\lambda^i_0, m_i, \rho^i_0, \boldsymbol{\alpha}^i_0\right\}_{1 \le i \le N} \bigcup \left\{\lambda^0_n, \rho^0_n\right\}_{n \in \mathbb{Z}, n \ne m} \bigcup \left\{\lambda^0_m, \rho^0_m\right\}.$$

In this case, by (2.7)–(2.9) $\widetilde{F}(x, y)$ possesses the following form:

$$\widetilde{F}(x, y) = \frac{1}{\rho^0_m} \left(\overline{S^*(x, \overline{\lambda_m})} S^T(y, \lambda_m) - \overline{S^*(x, \overline{\lambda^0_m})} S^T(y, \lambda^0_m)\right).$$

Note that

$$\begin{aligned} &\overline{S^*(x, \overline{\lambda_m})} S^T(y, \lambda_m) - \overline{S^*(x, \overline{\lambda^0_m})} S^T(y, \lambda^0_m) \\ &= \left(\overline{S^*(x, \overline{\lambda_m})} - \overline{S^*(x, \overline{\lambda^0_m})}\right) \left(S^T(y, \lambda_m) - S^T(y, \lambda^0_m)\right) \\ &\quad + \left(\overline{S^*(x, \overline{\lambda_m})} - \overline{S^*(x, \overline{\lambda^0_m})}\right) S^T(y, \lambda^0_m) + \overline{S^*(x, \overline{\lambda^0_m})}\left(S^T(y, \lambda_m) - S^T(y, \lambda^0_m)\right). \end{aligned}$$

Let $z=\lambda_m-\lambda_m^0$. Consequently, if we prove that

$$\begin{array}{c}\text{both } \overline{S^*(x,\overline{\lambda_m})}-\overline{S^*(x,\overline{\lambda_m^0})} \text{ and } S(x,\lambda_m)-S(x,\lambda_m^0) \text{ are analytic in a} \\ \text{neighborhood of the origin with respect to } z \text{ and vanish at } z=0,\end{array} \tag{2.10}$$

then we can rewrite $\widetilde{F}(x,y)$ by $\widetilde{F}(x,y;z;\aleph)$ where $\aleph=\{\rho_m^0,\lambda_m^0\}$, and see that $\widetilde{F}(x,y;z;\aleph)$ satisfies the assumption of analyticity with respect to z in Lemma 2.1.

Now let us prove (2.10). First let us give the definition of transformation operator $X(0,P,\mu)$. Let

$$\theta_1(x)=\frac{1}{2}\int_0^x(p_2+u)(s)\mathrm{d}s,\ \theta_2(x)=\frac{1}{2}\int_0^x(p_1+v)(s)\mathrm{d}s.$$

Set

$$R(0,P)(x)=e^{-\theta_1(x)}\begin{pmatrix}\cosh\theta_2(x) & -\sinh\theta_2(x)\\ -\sinh\theta_2(x) & \cosh\theta_2(x)\end{pmatrix}.$$

Now we define a *transformation operator* $X(0,P,\mu)$ on $\left(H^1(0,1)\right)^2$ by

$$\left(X(0,P,\mu)w\right)(x)=R(0,P)(x)w(x)+\int_0^x K(0,P,\mu)(x,y)w(y)\mathrm{d}y, \tag{2.11}$$

where $K(0,P,\mu)(x,y)\in\left(C^1\left(\overline{\Omega}\right)\right)^4$ is the unique solution to the following system:

$$\begin{cases} B\dfrac{\partial K(0,P,\mu)}{\partial x}(x,y)+\dfrac{\partial K(0,P,\mu)}{\partial y}(x,y)B+P(x)K(0,P,\mu)(x,y)=0,\ (x,y)\in\Omega,\\ K(0,P,\mu)(x,0)B\xi=0,\\ K(0,P,\mu)(x,x)B-BK(0,P,\mu)(x,x)=BR'(0,P)(x)+P(x)R(0,P)(x).\end{cases}$$

By Lemma 3.3 of [6], we have the following transformation formulae

$$S^*(x,\overline{\lambda})=X(0,\overline{-P_0^T},-\overline{\mu})e^*(x,\overline{\lambda}),\quad S(x,\lambda)=X(0,P_0,\mu)e(x,\lambda),$$

where $e^*(x,\overline{\lambda})=\begin{pmatrix}\cosh(\overline{\lambda}x+\overline{\mu})\\ -\sinh(\overline{\lambda}x+\overline{\mu})\end{pmatrix}$, $e(x,\lambda)=\begin{pmatrix}\cosh(\lambda x+\mu)\\ \sinh(\lambda x+\mu)\end{pmatrix}$. By the regularity of the hyperbolic functions cosh and sinh, it is easy to see that both $\overline{e^*(x,\overline{\lambda_m})}-\overline{e^*(x,\overline{\lambda_m^0})}$ and $e(x,\lambda_m)-e(x,\lambda_m^0)$ are analytic in a neighborhood of the origin with respect to z and vanish at $z=0$. Therefore, by the linearity of the transformation operator $X(\cdot,\cdot,\cdot)$, we have the assertion.

Case 2: Let m be an integer. Let the spectral characteristics of P and P_0 be

$$S(P,\mu,\nu)=\left\{\lambda_0^i,m_i,\rho_0^i,\boldsymbol{\alpha}_0^i\right\}_{1\le i\le N}\bigcup\left\{\lambda_n^0,\rho_n^0\right\}_{n\in\mathbb{Z},n\neq m}\bigcup\left\{\lambda_m^0,\rho_m\right\},$$

$$S(P_0,\mu,\nu)=\left\{\lambda_0^i,m_i,\rho_0^i,\boldsymbol{\alpha}_0^i\right\}_{1\le i\le N}\bigcup\left\{\lambda_n^0,\rho_n^0\right\}_{n\in\mathbb{Z},n\neq m}\bigcup\left\{\lambda_m^0,\rho_m^0\right\}.$$

If we let $z=\rho_m-\rho_m^0$, then we can easily prove that

$$\widetilde{F}(x,y)=\widetilde{F}(x,y;z;\aleph)=(\frac{1}{\rho_m}-\frac{1}{\rho_m^0})\overline{S^*(x,\overline{\lambda_m^0})}S^T(y,\lambda_m^0)$$

where $\aleph=\{\rho_m^0,\lambda_m^0\}$ satisfies the assumption of analyticity with respect to z in Lemma 2.1.

Case 3: Let i be a positive integer with $1 \le i \le N$. Let the spectral characteristics of P and P_0 be

$$S(P,\mu,\nu) = \{\lambda_0^k, m_k, \rho_0^k, \boldsymbol{\alpha}_0^k\}_{1\le k\le N, k\ne i} \bigcup \{\lambda^i, m_i, \rho_0^i, \boldsymbol{\alpha}_0^i\} \bigcup \{\lambda_n^0, \rho_n^0\}_{n\in\mathbb{Z}},$$

$$S(P_0,\mu,\nu) = \{\lambda_0^k, m_k, \rho_0^k, \boldsymbol{\alpha}_0^k\}_{1\le k\le N, k\ne i} \bigcup \{\lambda_0^i, m_i, \rho_0^i, \boldsymbol{\alpha}_0^i\} \bigcup \{\lambda_n^0, \rho_n^0\}_{n\in\mathbb{Z}}.$$

In this case we have by (2.7)–(2.9) that

$$\widetilde{F}(x,y) = \frac{1}{\rho_0^i}\sum_{j=1}^{m_i}\Big(\overline{S_{(j)}^*(x,\overline{\lambda^i})}S_{(j)}^T(y,\lambda^i) - \overline{S_{(j)}^*(x,\overline{\lambda_0^i})}S_{(j)}^T(y,\lambda_0^i)\Big).$$

Let $z = \lambda^i - \lambda_0^i$. Similarly to Case 1, in order to prove that

$$\widetilde{F}(x,y) = \widetilde{F}(x,y;z;\aleph) = \frac{1}{\rho_0^i}\sum_{j=1}^{m_i}\Big(\overline{S_{(j)}^*(x,\overline{\lambda^i})}S_{(j)}^T(y,\lambda^i) - \overline{S_{(j)}^*(x,\overline{\lambda_0^i})}S_{(j)}^T(y,\lambda_0^i)\Big)$$

where $\aleph = \{\rho_0^i, \lambda_0^i\}$, satisfies the assumption of analyticity with respect to z in Lemma 2.1, it is sufficient to show that for $1 \le j \le m_i$ both $\overline{S_{(j)}^*(x,\overline{\lambda^i})} - \overline{S_{(j)}^*(x,\overline{\lambda_0^i})}$ and $S_{(j)}(x,\lambda^i) - S_{(j)}(x,\lambda_0^i)$ are analytic with respect to z and vanish at $z=0$.

First, by Lemma 3.3 (ii) in [6],

$$S_{(j)}(x,\lambda^i) = X(0,P_0,\mu)e_{(j)}(x,\lambda^i), \quad S_{(j)}^*(x,\overline{\lambda^i}) = X(0,-\overline{P_0^T},-\overline{\mu})e_{(j)}^*(x,\overline{\lambda^i}).$$

Here $e_{(j)}(x,\lambda^i)$ and $e_{(j)}^*(x,\overline{\lambda^i})$ are given as follows:

$$e_{(j)}(x,\lambda) = \begin{pmatrix} \displaystyle\sum_{k=0}^{j-1}\frac{x^k}{k!}\gamma_k(x,\lambda,\mu) \\ \displaystyle\sum_{k=0}^{j-1}\frac{x^k}{k!}\delta_k(x,\lambda,\mu) \end{pmatrix}, \quad e_{(j)}^*(x,\overline{\lambda}) = \begin{pmatrix} \displaystyle\sum_{k=j}^{m_i}\overline{\alpha_k^i}\frac{x^{k-j}}{(k-j)!}\gamma_{k-j}(x,\overline{\lambda},\overline{\mu}) \\ \displaystyle -\sum_{k=j}^{m_i}\overline{\alpha_k^i}\frac{x^{k-j}}{(k-j)!}\delta_{k-j}(x,\overline{\lambda},\overline{\mu}) \end{pmatrix}$$

where $\alpha_{m_i}^i = 1$,

$$\gamma_k(x,\lambda,\mu) = \begin{cases} \cosh(\lambda x+\mu), & k \ \text{even} \\ \sinh(\lambda x+\mu), & k \ \text{odd} \end{cases}, \quad \delta_k(x,\lambda,\mu) = \begin{cases} \sinh(\lambda x+\mu), & k \ \text{even} \\ \cosh(\lambda x+\mu), & k \ \text{odd} \end{cases}.$$

By an argument similar to Case 1, the desired result follows from these explicit expressions.

Case 4: Let i be a positive integer with $1 \le i \le N$. Let the spectral characteristics of P and P_0 be

$$S(P,\mu,\nu) = \{\lambda_0^k, m_k, \rho_0^k, \boldsymbol{\alpha}_0^k\}_{1\le k\le N, k\ne i} \bigcup \{\lambda_0^i, m_i, \rho^i, \boldsymbol{\alpha}_0^i\} \bigcup \{\lambda_n^0, \rho_n^0\}_{n\in\mathbb{Z}},$$

$$S(P_0,\mu,\nu) = \{\lambda_0^k, m_k, \rho_0^k, \boldsymbol{\alpha}_0^k\}_{1\le k\le N, k\ne i} \bigcup \{\lambda_0^i, m_i, \rho_0^i, \boldsymbol{\alpha}_0^i\} \bigcup \{\lambda_n^0, \rho_n^0\}_{n\in\mathbb{Z}}.$$

In this case, if we let $z = \rho^i - \rho_0^i$, then similarly to Case 2 the desired result follows since

$$\widetilde{F}(x,y) = \widetilde{F}(x,y;z;\aleph) = (\frac{1}{\rho^i} - \frac{1}{\rho_0^i}) \overline{\sum_{j=1}^{m_i} S^*_{(j)}(x,\overline{\lambda_0^i})} S^T_{(j)}(y,\lambda_0^i),$$

where $\aleph = \{\rho_0^i, \lambda_0^i\}$.

Case 5: Let i, j be positive integers with $1 \le j \le m_i, 1 \le i \le N$. Let the spectral characteristics of P and P_0 be

$$\begin{aligned} & S(P,\mu,\nu) \\ & = \{\lambda_0^i, m_i, \rho_0^i, ((\alpha_1^i)_0, \ldots, \alpha_j^i, \ldots, (\alpha_{m_i-1}^i)_0)\} \\ & \bigcup \{\lambda_0^k, m_k, \rho_0^k, \boldsymbol{\alpha}_0^k\}_{1\le k\le N, k\ne i} \bigcup \{\lambda_n^0, \rho_n^0\}_{n\in\mathbb{Z}}, \end{aligned}$$

$$\begin{aligned} & S(P_0,\mu,\nu) \\ & = \left\{\lambda_0^i, m_i, \rho_0^i, \left((\alpha_1^i)_0, \ldots, (\alpha_j^i)_0, \ldots, (\alpha_{m_i-1}^i)_0\right)\right\} \\ & \bigcup \{\lambda_0^k, m_k, \rho_0^k, \boldsymbol{\alpha}_0^k\}_{1\le k\le N, k\ne i} \bigcup \{\lambda_n^0, \rho_n^0\}_{n\in\mathbb{Z}}. \end{aligned}$$

Let $z = \alpha_j^i - (\alpha_j^i)_0$. In this case we have

$$\widetilde{F}(x,y) = \widetilde{F}(x,y;z;\aleph) = \frac{1}{\rho_0^i}\left(\overline{S^*_{(j)}(x,\overline{\lambda_0^i})} - \overline{\widetilde{S}^*_{(j)}(x,\overline{\lambda_0^i})}\right) S^T_{(j)}(y,\lambda_0^i),$$

where $\aleph = \{\rho_0^i, \lambda_0^i\}$. From the definitions of $S^*_{(j)}(x,\overline{\lambda_0^i})$ and $\widetilde{S}^*_{(j)}(x,\overline{\lambda_0^i})$ (see (2.5) and (2.6)), we have

$$\begin{cases} (\mathcal{A}^*_{P_0} - \overline{\lambda_0^i})\left(S^*_{(j)}(x,\overline{\lambda_0^i}) - \widetilde{S}^*_{(j)}(x,\overline{\lambda_0^i})\right) = 0, \\ S^*_{(j)}(0,\overline{\lambda_0^i}) - \widetilde{S}^*_{(j)}(0,\overline{\lambda_0^i}) = \left(\overline{\alpha_j^i} - \overline{(\alpha_j^i)_0}\right)\eta \end{cases}$$

and hence by the transformation formulae

$$S^*_{(j)}(x,\overline{\lambda_0^i}) - \widetilde{S}^*_{(j)}(x,\overline{\lambda_0^i}) = X(0,-\overline{P_0^T},-\overline{\mu})\left(\overline{\alpha_j^i} - \overline{(\alpha_j^i)_0}\right) e^*(x,\overline{\lambda_0^i}).$$

Consequently the desired result follows.

Remark 2.2. In Cases 1 and 2, $\aleph = \{\rho_m^0, \lambda_m^0\}$, and in the rest three cases $\aleph = \{\rho_0^i, \lambda_0^i\}$. Moreover, corresponding to each case above, we list $a_1(x,y;\aleph)$ as follows:

$$\begin{aligned} (1) & \frac{1}{\rho_m^0}\overline{\left[X(0,\overline{-P_0^T},-\overline{\mu})\begin{pmatrix} x\sinh(\overline{\lambda_m^0}x+\overline{\mu}) \\ -x\cosh(\overline{\lambda_m^0}x+\overline{\mu}) \end{pmatrix}\right]} S^T(y,\lambda_m^0) \\ & +\frac{1}{\rho_m^0}\overline{S^*(x,\overline{\lambda_m^0})}\left[X(0,P_0,\mu)\begin{pmatrix} y\sinh(\lambda_m^0 y+\mu) \\ y\cosh(\lambda_m^0 y+\mu) \end{pmatrix}\right]^T; \\ (2) & -\frac{1}{(\rho_m^0)^2}\overline{S^*(x,\overline{\lambda_m^0})} S^T(y,\lambda_m^0); \end{aligned}$$

$$(3)\,\frac{1}{\rho_0^i}\sum_{j=1}^{m_i}\overline{\left[X(0,-\overline{P_0^T},-\overline{\mu})\begin{pmatrix}\displaystyle\sum_{k=j}^{m_i}\overline{\alpha_k^i}\frac{x^{k-j+1}}{(k-j)!}\delta_{k-j}(x,\overline{\lambda_0^i},\overline{\mu})\\ \displaystyle-\sum_{k=j}^{m_i}\overline{\alpha_k^i}\frac{x^{k-j+1}}{(k-j)!}\gamma_{k-j}(x,\overline{\lambda_0^i},\overline{\mu})\end{pmatrix}\right]}S_{(j)}^T(y,\lambda_0^i)$$

$$+\frac{1}{\rho_0^i}\sum_{j=1}^{m_i}\overline{S^*_{(j)}(x,\overline{\lambda_0^i})}\left[X(0,P_0,\mu)\begin{pmatrix}\displaystyle\sum_{k=0}^{j-1}\frac{y^{k+1}}{k!}\delta_k(y,\lambda_0^i,\mu)\\ \displaystyle\sum_{k=0}^{j-1}\frac{y^{k+1}}{k!}\gamma_k(y,\lambda_0^i,\mu)\end{pmatrix}\right]^T;$$

$$(4)\,-\frac{1}{\left(\rho_0^i\right)^2}\sum_{j=1}^{m_i}\overline{S^*_{(j)}(x,\overline{\lambda_0^i})}S^T_{(j)}(y,\lambda_0^i);$$

$$(5)\,\frac{1}{\rho_0^i}\overline{\left[X(0,-\overline{P_0^T},-\overline{\mu})\begin{pmatrix}\cosh(\overline{\lambda_0^i}x+\overline{\mu})\\ -\sinh(\overline{\lambda_0^i}x+\overline{\mu})\end{pmatrix}\right]}S^T_{(j)}(y,\lambda_0^i).$$

3 The Gateaux derivatives and bounds estimates

First we give the Gateaux Derivatives of p_1 and p_2 in each case of Section 2. Let $M(x,y)=(M_{kl})_{1\le k,l\le 2}\in\left(C^1(\overline{\Omega})\right)^4$ be the solution of the Gel'fand-Levitan equation (see (2.27) in [6]). Put

$$L(x)=2(M_{12}-M_{21})(x,x),\ \ N(x)=2(M_{11}-M_{22})(x,x) \tag{3.1}$$

and

$$\omega(x)=\int_0^x(p_1-p_1^0)(s)\mathrm{d}s.$$

By Theorem 3 of [6],

$$L(x)=(v(x)-p_1(x))\cosh\omega(x)+(p_2(x)-u(x))\sinh\omega(x)+p_1^0(x)-v(x), \tag{3.2}$$

$$N(x)=(v(x)-p_1(x))\sin\omega(x)+(p_2(x)-u(x))\cosh\omega(x)+u(x)-p_2^0(x). \tag{3.3}$$

Eliminating $p_2(x)$ in (3.2) and (3.3), we have

$$\begin{aligned}&L(x)\cosh\omega(x)-N(x)\sin\omega(x)\\ &=v(x)-p_1(x)+\left(p_1^0x)-v(x)\right)\cosh\omega(x)-\left(u(x)-p_2^0(x)\right)\sinh\omega(x).\end{aligned} \tag{3.4}$$

In each case of Section 2, we may reset

$$L=L(x;z;\aleph),\quad N=N(x;z;\aleph),$$

$$p_1=p_1(x;z;\aleph),\quad p_2=p_2(x;z;\aleph).$$

It is known in [6] that the spectral characteristics $S(P_0,\mu,\nu)$ can uniquely determine p_1^0,p_2^0 if u,v are given a prior. Therefore, in each case of Section 2, if we let h denote the unperturbed spectral datum (for example, we let $h=\lambda_m^0$ in Case 1), then we can set

$$p_1^0=p_1^0(x;h),\quad p_2^0=p_2^0(x;h),$$

where for simplicity we have omitted the dependence of other spectral data in $S(P_0, \mu, \nu)$.

Remark 3.1. By Theorem 1 of [6], it holds that $p_1(x; 0; \aleph) = p_1^0(x; h)$ and $p_2(x; 0; \aleph) = p_2^0(x; h)$.

Since (3.4) can be rewritten into

$$\omega' := \frac{\mathrm{d}\omega}{\mathrm{d}x} = \big(p_1^0 - v - L\big)\cosh\omega + \big(N - u + p_2^0\big)\sinh\omega + v - p_1^0, \tag{3.5}$$

by the perturbation theory of differential equations containing parameters (see, e.g., [12]), the obtained results in Section 2 show that $\omega = \omega(x; z; \aleph)$ is analytic at the point $z = 0$ with respect to z, and so is $p_1(x; z; \aleph) - p_1^0(x; h)$ by (3.5). Moreover, in view of (3.2) and (3.3),

$$p_2 - u = \big(N - u + p_2^0\big)\cosh\omega + \big(p_1^0 - v - L\big)\sinh\omega. \tag{3.6}$$

Then $p_2 = p_2(x; z; \aleph)$ is also analytic at $z = 0$ with respect to z.

Now let

$$z = \varepsilon\gamma.$$

By the above discussion, we can obtain the Gateaux derivative of p_1 merely by taking the partial derivative of (3.4) with respect to ε and then setting $\varepsilon = 0$. The computation procedure will be shown as follows.

$$\begin{aligned} &\frac{\partial L}{\partial\varepsilon}\cosh\omega + L\frac{\partial\omega}{\partial\varepsilon}\sinh\omega - \frac{\partial N}{\partial\varepsilon}\sinh\omega - N\frac{\partial\omega}{\partial\varepsilon}\cosh\omega \\ = &-\frac{\partial p_1}{\partial\varepsilon} + (p_1^0 - v)\frac{\partial\omega}{\partial\varepsilon}\sinh\omega - (u - p_2^0)\frac{\partial\omega}{\partial\varepsilon}\cosh\omega. \end{aligned}$$

Then, noting that $\omega|_{\varepsilon=0} = N|_{\varepsilon=0} = 0$ by Remark 3.1 and (3.3), we have

$$\left.\frac{\partial L}{\partial\varepsilon}\right|_{\varepsilon=0} = -\left.\frac{\partial p_1}{\partial\varepsilon}\right|_{\varepsilon=0} - g\int_0^x \left.\frac{\partial p_1}{\partial\varepsilon}(s; \varepsilon\gamma; \aleph)\right|_{\varepsilon=0} \mathrm{d}s, \tag{3.7}$$

where we set

$$g = u - p_2^0.$$

Solving (3.7) gives

$$\int_0^x \left.\frac{\partial p_1}{\partial\varepsilon}(s; \varepsilon\gamma; \aleph)\right|_{\varepsilon=0} \mathrm{d}s = -\int_0^x \left.\frac{\partial L}{\partial\varepsilon}(s; \varepsilon\gamma; \aleph)\right|_{\varepsilon=0} \exp\left(-\int_s^x g(\tau)\mathrm{d}\tau\right)\mathrm{d}s. \tag{3.8}$$

If we set

$$\left.\frac{\partial p_1}{\partial\varepsilon}(x; \varepsilon\gamma; \aleph)\right|_{\varepsilon=0} = dp_1(x; \gamma, \aleph) \text{ and } \left.\frac{\partial L}{\partial\varepsilon}(x; \varepsilon\gamma; \aleph)\right|_{\varepsilon=0} = dL(x; \gamma, \aleph),$$

then we have by (3.8) that

$$dp_1(x; \gamma, \aleph) = -dL(x; \gamma, \aleph) + g(x)\int_0^x dL(s; \gamma, \aleph)\exp\left(-\int_s^x g(\tau)\mathrm{d}\tau\right)\mathrm{d}s. \tag{3.9}$$

Similarly, by (3.6) the Gateaux derivative of p_2 is

$$dp_2(x;\gamma,\aleph) = dN(x;\gamma,\aleph) + (p_1^0 - v)(x)\int_0^x dL(s;\gamma,\aleph)\exp\left(-\int_s^x g(\tau)\mathrm{d}\tau\right)\mathrm{d}s. \tag{3.10}$$

On the other hand, by Lemma 2.1, we can find that

$$\left.\frac{\partial M(x,y;z;\aleph)}{\partial z}\right|_{z=0} = b_1(x,y;\aleph) \in \left(C^1\left(\overline{\Omega}\right)\right)^4.$$

Now let us set

$$b = (b_{kl}(x;\aleph))_{1\le k,l\le 2} := b_1(x,x;\aleph).$$

Hence in view of (3.1) we see that

$$dL(x;\gamma,\aleph) = 2\gamma(b_{12} - b_{21})(x;\aleph), \quad dN(x;\gamma,\aleph) = 2\gamma(b_{11} - b_{22})(x;\aleph). \tag{3.11}$$

Moreover let

$$\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}.$$

Then by (3.9), (3.10) and (3.11) we have

$$d\mathbf{p}(x;\gamma,\aleph) = \gamma\mathbf{q}(x;\aleph), \tag{3.12}$$

where $\mathbf{q}(x;\aleph) = \begin{pmatrix} q_1(x;\aleph) \\ q_2(x;\aleph) \end{pmatrix}$ with

$$\begin{aligned} q_1(x;\aleph) = &-2(b_{12} - b_{21})(x;\aleph) \\ &+g(x)\int_0^x 2(b_{12} - b_{21})(s;\aleph)\exp\left(-\int_s^x g(\tau)\mathrm{d}\tau\right)\mathrm{d}s, \end{aligned} \tag{3.13}$$

$$\begin{aligned} q_2(x;\aleph) = &\, 2(b_{11} - b_{22})(x;\aleph) \\ &-(p_1^0 - v)(x)\int_0^x 2(b_{12} - b_{21})(s;\aleph)\exp\left(-\int_s^x g(\tau)\mathrm{d}\tau\right)\mathrm{d}s. \end{aligned} \tag{3.14}$$

Obviously, $\mathbf{q}$ is C^1-differentiable with respect to x on $[0,1]$.

Second, we will show the bounds estimates about the Gateaux Derivatives obtained above. It should be noted that $\omega = O(|z|)$ by Remark 3.1 and the analyticity of ω. We point out that here and henceforth the bound O always depends only on $\|P_0\|_{C^1}$ and $\|P\|_{C^1}$. Furthermore, for $|z|$ sufficiently small, by (3.4) and the expansions

$$\cosh\omega = 1 + \frac{\omega^2}{2} + O(|\omega|^3), \quad \sinh\omega = \omega + O(|\omega|^3),$$

we see that

$$L(1+\frac{\omega^2}{2}) - N\omega + O(|\omega|^3) = v - p_1 + (p_1^0 - v)(1+\frac{\omega^2}{2}) - g\omega + O(|\omega|^3),$$

i.e.,

$$L + \omega' + (g - N)\omega + (L - p_1^0 + v)\frac{\omega^2}{2} + O(|\omega|^3) = 0. \tag{3.15}$$

Note that $L = 2(b_{12} - b_{21})z + O(|z|^2)$, $N = O(|z|)$, and $L - p_1^0 + v$ is bounded. Therefore it follows from (3.15) that

$$2(b_{12} - b_{21})z + \omega' + g\omega + O(|z|^2) = 0. \tag{3.16}$$

On the other hand, putting $\gamma = 1$ in (3.7), we see that

$$2(b_{12} - b_{21})(x; \aleph) = \frac{\partial L}{\partial z}(x; z; \aleph)\bigg|_{z=0} = -q_1(x; \aleph) - g(x)\int_0^x q_1(s; \aleph)\mathrm{d}s.$$

Substituting this equality into (3.16) and setting

$$G(x; z; \aleph) = \omega'(x; z; \aleph) - zq_1(x; \aleph),$$

we obtain

$$G(x; z; \aleph) + g(x)\int_0^x G(s; z; \aleph)\mathrm{d}s + O(|z|^2) = 0.$$

Then we see by Gronwall's inequality that there exists a constant $C > 0$ such that

$$|G(x; z; \aleph)| \leq C|z|^2 \exp(x\|g\|_{C^0}), \text{ i.e., } \|G\|_{C^0} = O(|z|^2).$$

Moreover, if one differentiates (3.4) with respect to x, then similarly he obtains that

$$G'(x; z; \aleph) + g(x)G(x; z; \aleph) + g'(x)\int_0^x G(s; z; \aleph)\mathrm{d}s + O(|z|^2) = 0,$$

which implies by the above result $\|G\|_{C^0} = O(|z|^2)$ that

$$\|G\|_{C^1} = \|p_1(\cdot; z; \aleph) - p_1^0(\cdot; h) - zq_1(\cdot; \aleph)\|_{C^1} = O(|z|^2).$$

As for p_2, by (3.6) and the argument similar to p_1, we can obtain

$$\| p_2(\cdot; z; \aleph) - p_2^0(\cdot; h) - zq_2(\cdot; \aleph) \|_{C^1} = O(|z|^2).$$

In view of (3.12), these results can be put together as follows:

$$\begin{aligned} &\| \mathbf{p}(\cdot; \varepsilon\gamma; \aleph) - \mathbf{p}^0(\cdot; h) - \varepsilon d\mathbf{p}(\cdot; \gamma, \aleph) \|_{C^1} \\ =&\| \mathbf{p}(\cdot; z; \aleph) - \mathbf{p}^0(\cdot; h) - z\mathbf{q}(\cdot; \aleph) \|_{C^1} \\ =& O(|\varepsilon\gamma|^2) = O(|z|^2). \end{aligned} \tag{3.17}$$

4 Stability results

Now we apply the Gateaux derivatives established in Section 3 to obtain the stability results in the inverse spectral problem as follows.

Theorem 4.1 *Let P_0 and P be in a bounded set in $(C^1[0,1])^4$ with $\int_0^1 \mathrm{tr}P_0(s)\mathrm{d}s = \int_0^1 \mathrm{tr}P(s)\mathrm{d}s = 0$. Let the associated spectral characteristics be*

$$S(P_0,\mu,\nu) = \left\{\lambda_0^i, m_i, \rho_0^i, \boldsymbol{\alpha}_0^i\right\}_{1\le i\le N} \bigcup \left\{\lambda_n^0, \rho_n^0\right\}_{n\in\mathbb{Z}}$$

and

$$S(P,\mu,\nu) = \left\{\lambda^i, m_i, \rho^i, \boldsymbol{\alpha}^i\right\}_{1\le i\le N} \bigcup \left\{\lambda_n, \rho_n\right\}_{n\in\mathbb{Z}}.$$

Then, when $\|S(P,\mu,\nu) - S(P_0,\mu,\nu)\|$ is sufficiently small, there exists a constant $C>0$ such that for $\vartheta = 0,1$

$$\|\boldsymbol{p}-\boldsymbol{p}_0\|_{C^\vartheta} \le C\left\{\sum_{i=1}^N (|\delta_i|+|\tau_i|) + \sum_{i=1}^N\sum_{j=1}^{m_i-1}|\theta_{ij}| + \sum_{n\in\mathbb{Z}}(|n|+1)^\vartheta(|\gamma_n|+|\nu_n|)\right\},$$

where $\delta_i = \lambda^i - \lambda_0^i,\ \tau_i = \rho^i - \rho_0^i,\ \theta_{ij} = \alpha_j^i - (\alpha_j^i)_0,\ \gamma_n = \lambda_n - \lambda_n^0,\ \nu_n = \rho_n - \rho_n^0$, and

$$\begin{aligned}&\|S(P,\mu,\nu) - S(P_0,\mu,\nu)\|\\ &= \left\{\sum_{i=1}^N (|\delta_i|^2+|\tau_i|^2) + \sum_{i=1}^N\sum_{j=1}^{m_i-1}|\theta_{ij}|^2 + \sum_{n\in\mathbb{Z}}(|\gamma_n|^2+|\nu_n|^2)\right\}^{1/2}.\end{aligned}$$

Proof. Let Ξ be the set of all spectral characteristics $S(P,\mu,\nu)$ in which κ_n, ζ_n have the asymptotic behavior (1.4). Now we define the map $\mathbb{M}:\ \Xi \to \left(C^1[0,1]\right)^2$ with $\mathbb{M}[S(P,\mu,\nu)] = \boldsymbol{p}$. We set

$$S^0(P,\mu,\nu) = S(P,\mu,\nu)$$

and for $1\le i\le N$

$$S^i(P,\mu,\nu) = \left\{\lambda_0^k, m_k, \rho^k, \boldsymbol{\alpha}^k\right\}_{1\le k\le i} \bigcup \left\{\lambda^k, m_k, \rho^k, \boldsymbol{\alpha}^k\right\}_{i+1\le k\le N} \bigcup \left\{\lambda_n, \rho_n\right\}_{n\in\mathbb{Z}}.$$

We set

$$S_{-\infty}(P,\mu,\nu) = S^N(P,\mu,\nu) = \left\{\lambda_0^k, m_k, \rho^k, \boldsymbol{\alpha}^k\right\}_{1\le k\le N} \bigcup \left\{\lambda_n, \rho_n\right\}_{n\in\mathbb{Z}}$$

and for $n\in\mathbb{Z}$

$$S_n(P,\mu,\nu) = \left\{\lambda_0^i, m_i, \rho^i, \boldsymbol{\alpha}^i\right\}_{1\le i\le N} \bigcup \left\{\lambda_k^0, \rho_k\right\}_{k\le n} \bigcup \left\{\lambda_k, \rho_k\right\}_{k>n}.$$

Moreover we set

$$\widetilde{S}^0(P,\mu,\nu) = S_\infty(P,\mu,\nu) := \left\{\lambda_0^i, m_i, \rho^i, \boldsymbol{\alpha}^i\right\}_{1\le i\le N} \bigcup \left\{\lambda_n^0, \rho_n\right\}_{n\in\mathbb{Z}}$$

and for $1\le i\le N$

$$\widetilde{S}^i(P,\mu,\nu) = \left\{\lambda_0^k, m_k, \rho_0^k, \boldsymbol{\alpha}^k\right\}_{1\le k\le i} \bigcup \left\{\lambda_0^k, m_k, \rho^k, \boldsymbol{\alpha}^k\right\}_{i+1\le k\le N} \bigcup \left\{\lambda_n^0, \rho_n\right\}_{n\in\mathbb{Z}}.$$

Finally, we set for $1\le j\le m_i-1, 1\le i\le N$

$$\begin{aligned}&\widetilde{S}_j^i(P,\mu,\nu)\\ &= \left\{\lambda_0^k, m_k, \rho_0^k, \boldsymbol{\alpha}_0^k\right\}_{1\le k\le i-1} \bigcup \left\{\lambda_0^i, m_i, \rho_0^i, \left(\left(\alpha_1^i\right)_0, \dots, \left(\alpha_j^i\right)_0, \alpha_{j+1}^i, \dots, \alpha_{m_i-1}^i\right)\right\}\\ &\quad \bigcup \left\{\lambda_0^k, m_k, \rho_0^k, \boldsymbol{\alpha}^k\right\}_{i+1\le k\le N} \bigcup \left\{\lambda_n^0, \rho_n\right\}_{n\in\mathbb{Z}}\end{aligned}$$

and for $n \in \mathbb{Z}$

$$\widetilde{S}_n(P,\mu,\nu) = \left\{\lambda_0^i, m_i, \rho_0^i, \boldsymbol{\alpha}_0^i\right\}_{1\le i\le N} \bigcup \left\{\lambda_k^0, \rho_k^0\right\}_{k\le n} \bigcup \left\{\lambda_k^0, \rho_k\right\}_{k>n}.$$

Note the formal relation

$$\begin{aligned}
&\mathbb{M}[S(P,\mu,\nu)] - \mathbb{M}[S(P_0,\mu,\nu)] \\
&= \sum_{i=0}^{N-1} \left(\mathbb{M}[S^i(P,\mu,\nu)] - \mathbb{M}[S^{i+1}(P,\mu,\nu)]\right) + \sum_{n\in\mathbb{Z}} \left(\mathbb{M}[S_n(P,\mu,\nu)] - \mathbb{M}[S_{n+1}(P,\mu,\nu)]\right) \\
&\quad + \sum_{i=0}^{N-1} \Big(\mathbb{M}[\widetilde{S}^i(P,\mu,\nu)] - \mathbb{M}[\widetilde{S}^{i+1}(P,\mu,\nu)]\Big) \\
&\quad + \sum_{i=0}^{N} \sum_{j=1}^{m_i-1} \Big(\mathbb{M}[\widetilde{S}_j^i(P,\mu,\nu)] - \mathbb{M}[\widetilde{S}_{j+1}^i(P,\mu,\nu)]\Big) \\
&\quad + \sum_{n\in\mathbb{Z}} \Big(\mathbb{M}[\widetilde{S}_n(P,\mu,\nu)] - \mathbb{M}[\widetilde{S}_{n+1}(P,\mu,\nu)]\Big),
\end{aligned} \tag{4.1}$$

where we have set $m_0 = 2$, $\widetilde{S}_1^0(P,\mu,\nu) = \widetilde{S}^N(P,\mu,\nu)$, $\widetilde{S}_{m_i}^i(P,\mu,\nu) = \widetilde{S}_1^{i+1}(P,\mu,\nu)$ $(0 \le i \le N-1)$, $\widetilde{S}_{m_N}^N(P,\mu,\nu) = \widetilde{S}_{-\infty}(P,\mu,\nu) = \widetilde{S}_{m_N-1}^N(P,\mu,\nu)$ and $\widetilde{S}_\infty(P,\mu,\nu) = S(P_0,\mu,\nu)$. Then, for each term in parentheses in (4.1), applying the Gateaux derivatives established in Section 3 (see (3.17)), we can prove by (4.1) and the triangle inequality that

$$\Big\|\mathbb{M}[S(P,\mu,\nu)] - \mathbb{M}[S(P_0,\mu,\nu)] - \Theta(\cdot)\Big\|_{\mathbb{X}} = O\left(\|S(P,\mu,\nu) - S(P_0,\mu,\nu)\|^2\right), \tag{4.2}$$

where $\mathbb{X} = C^0$ or C^1,

$$\begin{aligned}
\Theta(\cdot) = &\sum_{i=1}^{N} \delta_i \boldsymbol{q}(\cdot;\{\rho^i,\lambda_0^i\}) + \sum_{n\in\mathbb{Z}} \gamma_n \boldsymbol{q}(\cdot;\{\rho_n,\lambda_n^0\}) + \sum_{i=1}^{N} \tau_i \boldsymbol{q}(\cdot;\{\rho_0^i,\lambda_0^i\}) \\
&+ \sum_{i=1}^{N} \sum_{j=1}^{m_i-1} \theta_{ij} \boldsymbol{q}\left(\cdot;\{\rho_0^i,\lambda_0^i\}\right) + \sum_{n\in\mathbb{Z}} \nu_n \boldsymbol{q}(\cdot;\{\rho_n^0,\lambda_n^0\}).
\end{aligned} \tag{4.3}$$

Since $\|S(P,\mu,\nu) - S(P_0,\mu,\nu)\|$ is sufficiently small, there exists a constant $C_1 > 0$ such that

$$\begin{aligned}
&\|S(P,\mu,\nu) - S(P_0,\mu,\nu)\|^2 \\
&\le C_1 \left\{ \sum_{i=1}^{N} (|\delta_i| + |\tau_i|) + \sum_{i=1}^{N} \sum_{j=1}^{m_i-1} |\theta_{ij}| + \sum_{n\in\mathbb{Z}} (|\gamma_n| + |\nu_n|) \right\}.
\end{aligned}$$

Therefore, to prove Theorem 4.1, by (4.2) and the triangle inequality it is sufficient to show that there exists a constant $C_2 > 0$ such that for $\vartheta = 0, 1$

$$\begin{aligned}
&\|\Theta(\cdot)\|_{C^\vartheta} \\
&\le C_2 \left\{ \sum_{i=1}^{N} (|\delta_i| + |\tau_i|) + \sum_{i=1}^{N} \sum_{j=1}^{m_i-1} |\theta_{ij}| + \sum_{n\in\mathbb{Z}} (|n|+1)^\vartheta (|\gamma_n| + |\nu_n|) \right\}.
\end{aligned} \tag{4.4}$$

By (1.2), (2.3) and Remark 2.2 we can show that there exists a constant $C_3 > 0$ such that $\|b(\cdot;\aleph)\|_{C^0} \le C_3$ for each case of Section 2. And corresponding to Case 1 and Case 2, since $\lambda_n = O(|n|)$, we can show that

$$\|b(\cdot;\{\rho_n,\lambda_n^0\})\|_{C^1} \le C_4(|n|+1), \quad \|b(\cdot;\{\rho_n^0,\lambda_n^0\})\|_{C^1} \le C_4(|n|+1),$$

where the constant $C_4 > 0$ is independent of n. Consequently, in view of (3.13) and (3.14), we can choose a constant $C_5 > 0$ independent of n such that

$$\|\boldsymbol{q}(\cdot;\aleph)\|_{C^0} \le C_5 \text{ for each } \aleph \text{ as in (4.3)}, \tag{4.5}$$

and

$$\|\boldsymbol{q}(\cdot;\{\rho_n,\lambda_n^0\})\|_{C^1} \le C_5(|n|+1), \quad \|\boldsymbol{q}(\cdot;\{\rho_n^0,\lambda_n^0\})\|_{C^1} \le C_5(|n|+1). \tag{4.6}$$

Then, by (4.3), (4.5) and (4.6) we can prove (4.4), since the number of the eigenvalues with algebraic multiplicities≥ 2 is finite. Thus we complete the proof of Theorem 4.1. $\Box$

Now we can give the stability result for the general case without the restriction $\theta_0 = 0$ as follows:

Theorem 4.2 *Let the assumptions be same as in Theorem 4.1 except for* $\int_0^1 \mathrm{tr}P_0(s)\mathrm{d}s = \int_0^1 \mathrm{tr}P(s)\mathrm{d}s = 0$. *Then a constant* $C > 0$ *exists such that for* $\vartheta = 0, 1$

$$\begin{aligned}&\|\boldsymbol{p} - \boldsymbol{p}_0\|_{C^\vartheta}\\ &\le |\lim_{k\to\infty}(\lambda_k - \lambda_k^0)|\\ &\quad + C\left\{\sum_{i=1}^N(|\widetilde{\delta}_i| + |\tau_i|) + \sum_{i=1}^N\sum_{j=1}^{m_i-1}|\theta_{ij}| + \sum_{n\in\mathbb{Z}}(|n|+1)^\vartheta(|\widetilde{\gamma}_n| + |\nu_n|)\right\},\end{aligned}$$

where $\widetilde{\delta}_i = \lambda^i - \lambda_0^i - \lim_{k\to\infty}(\lambda_k - \lambda_k^0)$, $\widetilde{\gamma}_n = \lambda_n - \lambda_n^0 - \lim_{k\to\infty}(\lambda_k - \lambda_k^0)$.

Remark 4.1. In fact, the term $|\lim_{k\to\infty}(\lambda_k - \lambda_k^0)|$ can be thrown away in the estimate for $\|p_2 - p_2^0\|_{C^\vartheta}$.

Proof. As mentioned in Section 1, Theorem 4.1 holds for $\widetilde{P_0} = P_0 - \frac{1}{2}\int_0^1 \mathrm{tr}P_0(s)\mathrm{d}sE$, $\widetilde{P} = P - \frac{1}{2}\int_0^1 \mathrm{tr}P(s)\mathrm{d}sE$ and $\widetilde{\lambda}_n^0 = \lambda_n^0 - \frac{1}{2}\int_0^1 \mathrm{tr}P_0(s)\mathrm{d}s$, $\widetilde{\lambda}_n = \lambda_n - \frac{1}{2}\int_0^1 \mathrm{tr}P(s)\mathrm{d}s$, i.e.,

$$\|\widetilde{P} - \widetilde{P}_0\|_{C^\vartheta} \le C\left\{\sum_{i=1}^N(|\widetilde{\delta}_i| + \tau_i|) + \sum_{i=1}^N\sum_{j=1}^{m_i-1}|\theta_{ij}| + \sum_{n\in\mathbb{Z}}(|n|+1)^\vartheta(|\widetilde{\gamma_n}| + |\nu_n|)\right\}.$$

Noting that $\frac{1}{2}\int_0^1 \mathrm{tr}P(s)\mathrm{d}s - \frac{1}{2}\int_0^1 \mathrm{tr}P_0(s)\mathrm{d}s = \lim_{k\to\infty}(\lambda_k - \lambda_k^0)$ by (1.2), we complete the proof of Theorem 4.2 by the triangle inequality. $\Box$

Acknowledgements

The author heartily thanks Professor Masahiro Yamamoto for helpful suggestion and great comments.

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