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A nontrivial algebraic cycle in the Jacobian variety of the Fermat sextic

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A NONTRIVIAL ALGEBRAIC CYCLE IN THE JACOBIAN VARIETY OF THE FERMAT SEXTIC

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Abstract. We compute some value of the harmonic volume for the Fermat sextic. Using this computation, we prove that some special algebraic cycle in the Jacobian variety of the Fermat sextic is not algebraically equivalent to zero.

1. Introduction

B. Harris [5] defined the harmonic volume for the compact Riemann surface X of genus $g \geq 3$, using Chen's iterated integrals [2]. Let $J(X)$ be the Jacobian variety of X. By the Abel-Jacobi map $X \to J(X)$, X is embedded in $J(X)$. By a consideration of the special harmonic volume, Harris [6] proved that the algebraic cycle $F(4) - F(4)^-$ is not algebraically equivalent to zero in $J(F(4))$. Here, $F(4)$ is the Fermat quartic, which is a compact Riemann surface of genus 3. Ceresa [1] showed that the algebraic cycle $X - X^-$ is not algebraically equivalent to zero in $J(X)$ for a generic X. We know few explicit nontrivial examples except for $F(4)$. Harris [7] used the special feature of $F(4)$ that its normalized period matrix has entries in a discrete subring of $\mathbb{C}$. The Fermat sextic $F(6)$ has the same feature. We use this and prove

Theorem 4.3. *Let $F(6)$ be the Fermat sextic. Then, the algebraic cycle $F(6) - F(6)^-$ is not algebraically equivalent to zero in $J(F(6))$.*

We compute iterated integrals with some common base point of $F(6)$. This is a similar computation of Tretkoff and Tretkoff [10]. In order to compute the Poincaré dual of $F(6)$, we use the result of Kamata [8] for the intersection number of the first integral homology class of the Fermat curves. It is difficult to apply Harris' method to other Fermat curves. We [9] proved the same fact as the Klein quartic, but we did not use the above special feature.

Now we describe the contents of this paper briefly. In §2, we recall the definition and fundamental properties of the harmonic volume and algebraic cycle in $J(X)$. §3 is devoted to the computation of iterated integrals of the Fermat curves. In the latter half of this section, we prove that iterated integrals on those curves are represented by some special values of the generalized hypergeometric function ${}_3F_2$. It was introduced in [9] but not proved. In §4, we prove Main Theorem, using the numerical calculation by the MATHEMATICA program.

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2. Harmonic volumes and algebraic cycles

Let R be a discrete subring of $\mathbb{C}$. We suppose that all the entries of the period matrix of the compact Riemann surface X can be reduced to elements of R. Harris [7] pointed out that we may replace $\mathbb{Z}[\sqrt{-1}]$ for the Fermat quartic in Harris' method in [6] with R. We recall the harmonic volume for such X as follows. Let $H_R^{1,0}$ denote the space of homolophic 1-forms on X with R-periods. It is a g-dimensional $\mathbb{C}$-vector space. We choose a basis $\{K_1, K_2, \ldots, K_{2g}\}$ of the first integral homology group $H_1(X;\mathbb{Z})$ of X.

Definition 2.1 ([7])**.** The harmonic volume is defined to be the homomorphism $(H_R^{1,0})^{\otimes_R 3} \to \mathbb{C}/R$ by

$$I_R(\omega_1 \otimes \omega_2 \otimes \omega_3) = \sum_{r=1}^{2g} a_r \int_{C_r} \omega_1\omega_2 \mod R.$$

Here $\omega_1 \otimes \omega_2 \otimes \omega_3$ is an element of $(H_R^{1,0})^{\otimes_R 3}$, C_r is a loop in X at the fixed base point x_0 whose homology class is K_r, and the Poincaré dual of ω_k is equal to $\sum_{r=1}^{2g} a_r K_r$ ($a_r \in \mathbb{C}$). The integral $\int_{C_r} \omega_1\omega_2$ is Chen's iterated integral [2], that is, $\int_{C_r} \omega_1\omega_2 = \int_{0 \le t_1 \le t_2 \le 1} f_i(t_1) f_j(t_2) dt_1 dt_2$ for $C_r^*\omega_i = f_i(t)dt$, $i = 1, 2$, where t is the coordinate in the unit interval $[0,1]$.

We remark that I_R dose not depend on the choice of the base point x_0. It is a modified version of the original harmonic volume I. See Harris [5] for I.

Let $J = J(X)$ be the Jacobian variety of X. By the Abel-Jacobi map $X \to J(X)$, X is embedded in $J(X)$. The algebraic 1-cycle $X - X^-$ in $J(X)$ is homologous to zero. Here we denote by X^- the image of X under the multiplication map by -1. We recall the relation between the harmonic volume and algebraic 1-cycle $X - X^-$ in J. We say the algebraic cycle $X - X^-$ is *algebraically equivalent to zero in* J if there exists a topological 3-chain W such that $\partial W = X - X^-$ and W lies on S, where S is an algebraic (or complex analytic) subset of J of complex dimension 2 (Harris [7]). The chain W is unique up to 3-cycles. Harris proved the key theorem.

Theorem 2.2 (Section 2.7 in [7])**.** *If the algebraic cycle $X - X^-$ is algebraically equivalent to zero in J, then $2I_R(\omega) \equiv 0$ modulo R for each $\omega \in (H_R^{1,0})^{\otimes_R 3}$.*

See Harris [6, 7] for details. In §4, we find some element $\omega \in (H_R^{1,0})^{\otimes_R 3}$ such that $2I_R(\omega) \not\equiv 0$ modulo R for the Fermat sextic.

3. Iterated integrals of the Fermat curves

In this section we compute iterated integrals of the Fermat sextic. Let $H^{1,0}$ denote the space of holomorphic 1-forms on X We choose a basis $\{\omega_1, \omega_2, \ldots, \omega_g\}$ of $H^{1,0}$. Let γ be a loop in X at some base point. We remark that the iterated integral $\int_\gamma \omega_i\omega_j$ depends on the choice of the base points and is invariant under homotopy relative a fixed base point. This iterated integral and the quadratic period defined by Gunning [4] are essentially same except for the sign.

For $N \in \mathbb{Z}_{\geq 3}$, let $F(N) = \{(X : Y : Z) \in \mathbb{C}P^2; X^N + Y^N = Z^N\}$ denote the Fermat curve of degree N, which is a compact Riemann surface of genus $(N-1)(N-2)/2$. Let x and y denote X/Z and Y/Z respectively. The equation $X^N + Y^N = Z^N$ induces $x^N + y^N = 1$. Using this coordinate $(x, y) \in F(N)$, the holomorphic map $\pi : F(N) \to \mathbb{C}P^1$ is defined by $\pi(x, y) = x$. It is clear that π is an N-sheeted covering $F(N) \to \mathbb{C}P^1$, branched over N branch points $\{\zeta_N^i\}_{i=0,1,\ldots,N-1} \subset \mathbb{C}P^1$. Here ζ_N denotes $\exp(2\pi\sqrt{-1}/N)$. Holomorphic automorphisms α and β of $F(N)$ are defined by $\alpha(X : Y : Z) = (\zeta_N X : Y : Z)$ and $\beta(X : Y : Z) = (X : \zeta_N Y : Z)$ respectively. We have that $\alpha\beta = \beta\alpha$ and the subgroup of the holomorphic automorphisms of $F(N)$ which is generated by α and β is isomorphic to $(\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})$. Let P_i and Q_i denote $\alpha^i(1, 0)$ and $\beta^i(0, 1)$, $i = 0, 1, \ldots, N-1$ respectively. We define a simply connected domain Ω by $\mathbb{C} \setminus \bigcup_{j=0}^{j=N-1}\{t\zeta^j; |t| \geq 1, t \in \mathbb{R}\}$. Then $\pi^{-1}(\Omega)$ consists of N path-connected components and we denote by Ω_i a connected component of $\pi^{-1}(\Omega)$ which contains Q_i, $i = 0, 1, \ldots, N-1$. Let γ_0 be a path $[0, 1] \ni t \mapsto (t, \sqrt[N]{1-t^N}) \in F(N)$, where $\sqrt[N]{1-t^N}$ is a real nonnegative analytic function on $[0, 1]$. A loop in $F(N)$ is defined by

$$\kappa_0 = \gamma_0 \cdot (\beta\gamma_0)^{-1} \cdot (\alpha\beta\gamma_0) \cdot (\alpha\gamma_0)^{-1},$$

where the product $\ell_1 \cdot \ell_2$ indicates that we traverse ℓ_1 first, then ℓ_2. We consider a loop $\alpha^i\beta^j\kappa_0$ as an element of the first homology group $H_1(F(N); \mathbb{Z})$ of $F(N)$. Kamata obtained the following lemma for the intersection number of $H_1(F(N); \mathbb{Z})$.

Lemma 3.1 (Section 5 in [8]). *We have*

$$\begin{cases} (\kappa_0, \alpha\kappa_0) = & 1 & = -(\alpha\kappa_0, \kappa_0) \\ (\kappa_0, \beta\kappa_0) = & 1 & = -(\beta\kappa_0, \kappa_0) \\ (\kappa_0, \alpha\beta\kappa_0) = & -1 & = -(\alpha\beta\kappa_0, \kappa_0) \\ (\kappa_0, \alpha\beta^{-1}\kappa_0) = & 0 & = (\alpha\beta^{-1}\kappa_0, \kappa_0). \end{cases}$$

From this lemma, it is to show

Proposition 3.2 (Section 5 in [8]). *We have* $\{\alpha^i\beta^j\kappa_0\}_{i=0,1,\dots,N-3,j=0,1,\dots,N-2}$ *is a basis of* $H_1(F(N);\mathbb{Z})$.

Remark 3.3. Intersection matrix of $\{\alpha^i\beta^j\kappa_0\}_{i=0,1,\dots,N-3,j=0,1,\dots,N-2}$ is given by K in case (i) in [8].

It is a known fact that $\{\omega'_{r,s} = x^{r-1}y^{s-1}dx/y^{N-1}\}_{r,s\geq 1, r+s\leq N-1}$ is a basis of $H^{1,0}$ of $F(N)$. It is clear that

$$\int_{\alpha^i\beta^j\gamma_0}\omega'_{r,s} = \zeta_N^{ir+js}\int_{\gamma_0}\omega'_{r,s} = \zeta_N^{ir+js}\frac{B(r/N,s/N)}{N}.$$

The integral of $\omega'_{r,s}$ along $\alpha^i\beta^j\kappa_0$ is obtained as follows.

Proposition 3.4 (Appendix in [3]). *We have*

$$\int_{\alpha^i\beta^j\kappa_0}\omega'_{r,s} = B(r/N,s/N)(1-\zeta_N^r)(1-\zeta_N^s)\zeta_N^{ir+js}/N.$$

We denote the 1-form $N\omega'_{r,s}/B^N_{r,s}$ by $\omega_{r,s}$. Here, $B^N_{r,s} = B(r/N,s/N)$. This implies $\int_{\alpha^i\beta^j\kappa_0}\omega_{r,s} \in \mathbb{Z}[\zeta_N]$.

Let $f_{r,s}$ be a real 1-form on $[0,1]$ defined by $\gamma_0^*\omega'_{r,s} = t^{r-1}\left(\sqrt[N]{1-t^N}\right)^{s-N}dt$ for $r,s\geq 1, r+s\leq N-1$. The iterated integral $\int_{\gamma_0}\omega_{r,s}\omega_{l,m} = N^2\int_\gamma f_{r,s}f_{l,m}\Big/(B^N_{r,s}B^N_{l,m})$ is denoted by $x_{r,s,l,m}$. Iterated integrals of $\omega_{r,s}$ along the loop $\alpha^i\beta^j\kappa_0$ can be computed.

Lemma 3.5. *We consider* $\alpha^i\beta^j\kappa_0$ *as a loop at the base point* Q_j. *Then the iterated integral* $\int_{\alpha^i\beta^j\kappa_0}\omega_{r,s}\omega_{l,m}$ *is given by*

$$\zeta_N^{i(r+l)+j(s+m)}\left\{(1-\zeta_N^{r+l})(1-\zeta_N^{s+m})x_{r,s,l,m} + (1-\zeta_N^s)(\zeta_N^{r+l}+\zeta_N^{l+m}-\zeta_N^m-\zeta_N^l)\right\}.$$

Proof. It is clear that $\int_{\alpha^i\beta^j\kappa_0}\omega_{r,s}\omega_{l,m} = \zeta_N^{i(r+l)+j(s+m)}\int_{\kappa_0}\omega_{r,s}\omega_{l,m}$. We have only to compute $\int_{\kappa_0}\omega_{r,s}\omega_{l,m}$. We denote $\left(\int_{\ell_1}+\int_{\ell_2}\right)\omega_{r,s}\omega_{l,m} = \int_{\ell_1}\omega_{r,s}\omega_{l,m} + \int_{\ell_2}\omega_{r,s}\omega_{l,m}$ only here.

Proposition 3.4, the equation $\int_{\gamma_0}\omega_{r,s} = 1$, and

$$\int_{\gamma_0}\omega_{r,s}\omega_{l,m} + \int_{\gamma_0^{-1}}\omega_{r,s}\omega_{l,m} + \int_{\gamma_0}\omega_{r,s}\int_{\gamma_0^{-1}}\omega_{l,m} = \int_{\gamma_0\cdot\gamma_0^{-1}}\omega_{r,s}\omega_{l,m} = 0$$

give us the equation

$$\int_{\kappa_0}\omega_{r,s}\omega_{l,m} = \left(\int_{\gamma_0}+\int_{(\beta\gamma_0)^{-1}}+\int_{\alpha\beta\gamma_0}+\int_{(\alpha\gamma_0)^{-1}}\right)\omega_{r,s}\omega_{l,m}$$

$$+\int_{\gamma_0}\omega_{r,s}\left(\int_{(\beta\gamma_0)^{-1}}+\int_{\alpha\beta\gamma_0}+\int_{(\alpha\gamma_0)^{-1}}\right)\omega_{l,m}+\int_{(\beta\gamma_0)^{-1}}\omega_{r,s}\left(\int_{\alpha\beta\gamma_0}+\int_{(\alpha\gamma_0)^{-1}}\right)\omega_{l,m}$$
$$+\int_{\alpha\beta\gamma_0}\omega_{r,s}\int_{(\alpha\gamma_0)^{-1}}\omega_{l,m}$$
$$=\left(\int_{\gamma_0}+\zeta_N^{s+m}\int_{\gamma_0^{-1}}+\zeta_N^{r+s+l+m}\int_{\gamma_0}+\zeta_N^{r+l}\int_{\gamma_0^{-1}}\right)\omega_{r,s}\omega_{l,m}$$
$$+\int_{\gamma_0}\omega_{r,s}\left(-\zeta_N^{m}\int_{\gamma_0}+\zeta_N^{l+m}\int_{\gamma_0}-\zeta_N^{l}\int_{\gamma_0}\right)\omega_{l,m}-\zeta_N^{s}\int_{\gamma_0}\omega_{r,s}\left(\zeta_N^{l+m}\int_{\gamma_0}-\zeta_N^{l}\int_{\gamma_0}\right)\omega_{l,m}$$
$$-\left(\zeta_N^{r+s}\int_{\gamma_0}\omega_{r,s}\right)\zeta_N^{l}\int_{\gamma_0}\omega_{l,m}$$
$$=\left\{(1+\zeta_N^{r+s+l+m})\int_{\gamma_0}-(\zeta_N^{s+m}+\zeta_N^{r+l})\int_{\gamma_0}\right\}\omega_{r,s}\omega_{l,m}+(\zeta_N^{s+m}+\zeta_N^{r+l})\int_{\gamma_0}\omega_{r,s}\int_{\gamma_0}\omega_{l,m}$$
$$-\zeta_N^{m}+\zeta_N^{l+m}-\zeta_N^{l}-\zeta_N^{s+l+m}+\zeta_N^{s+l}-\zeta_N^{r+s+l}$$
$$=(1-\zeta_N^{r+l})(1-\zeta_N^{s+m})\int_{\gamma_0}\omega_{r,s}\omega_{l,m}+(1-\zeta_N^{s})(\zeta_N^{r+l}+\zeta_N^{l+m}-\zeta_N^{m}-\zeta_N^{l}). \qquad \square$$

We define a path γ_j by $\gamma_0\cdot(\beta^j\gamma_0)^{-1}$. Let $\gamma_{i,j}$ denote the loop $\gamma_j\cdot(\alpha^i\beta^j\kappa_0)\cdot\gamma_j^{-1}$. Using the above lemma, we have iterated integrals of $\omega_{r,s}$ along the loop $\gamma_{i,j}$ at the common base point Q_0.

Theorem 3.6. *The iterated integral* $\int_{\gamma_{i,j}}\omega_{r,s}\omega_{l,m}$ *is given by*

$$\zeta_N^{i(r+l)+j(s+m)}\{(1-\zeta_N^{l+r})(1-\zeta_N^{m+s})x_{r,s,l,m}+(1-\zeta_N^{s})(\zeta_N^{l+r}+\zeta_N^{l+m}-\zeta_N^{m}-\zeta_N^{l})\}$$
$$+(1-\zeta_N^{js})(1-\zeta_N^{l})(1-\zeta_N^{m})\zeta_N^{il+jm}-(1-\zeta_N^{jm})(1-\zeta_N^{r})(1-\zeta_N^{s})\zeta_N^{ir+js}.$$

Tretkoff and Tretkoff [10] computed the quadratic periods with another base point by similar computation.

Proof. We have

$$\int_{\gamma_{i,j}}\omega_{r,s}\omega_{l,m}=\int_{\alpha^i\beta^j\kappa_0}\omega_{r,s}\omega_{l,m}+\int_{\gamma_j}\omega_{r,s}\int_{\alpha^i\beta^j\kappa_0}\omega_{l,m}-\int_{\alpha^i\beta^j\kappa_0}\omega_{r,s}\int_{\gamma_j}\omega_{l,m}.$$

From this equation and Lemma 3.5, the result follows. $\square$

For the numerical calculation of $x_{r,s,l,m}$, we recall the generalized hypergeometric function ${}_3F_2$. Let $\Gamma(\tau)$ denote the gamma function $\int_0^\infty e^{-t}t^{\tau-1}dt$ for $\tau>0$. We define (α,n) by $\Gamma(\alpha+n)/\Gamma(\alpha)$ for $n\in\mathbb{Z}_{\geq 0}$. For $x\in\{z\in\mathbb{C};|z|<1\}$ and $\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2>-1$, the generalized hypergeometric function ${}_3F_2$ is defined by

$${}_3F_2\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3\\ \beta_1,\beta_2\end{matrix};x\right)=\sum_{n=0}^{\infty}\frac{(\alpha_1,n)(\alpha_2,n)(\alpha_3,n)}{(\beta_1,n)(\beta_2,n)(1,n)}x^n.$$

Proposition 3.7. *Let* Δ *be a* 1*-simplex* $\{(u,v)\in\mathbb{R}^2; 0\le v\le 1, 0\le u\le v\}$*. If* $a,b,p,q>0, b<1$*, then we have*

$$\int_\Delta u^{a-1}(1-u)^{b-1}v^{p-1}(1-v)^{q-1}dudv = \frac{B(a+p,q)}{a}\lim_{\substack{t\to 1-0\\ t\in\mathbb{R}}} {}_3F_2\left(\begin{matrix} a, 1-b, a+p \\ 1+a, a+p+q\end{matrix}; t\right).$$

Proof. Using the equation

$$\begin{aligned}\int_0^v u^{a-1}(1-u)^{b-1}du &= \int_0^v \sum_{n=0}^\infty u^{a-1}\binom{b-1}{n}(-u)^n du\\ &= \sum_{n=0}^\infty \frac{(1-b,n)}{(1,n)}\int_0^v u^{n+a-1}du,\end{aligned}$$

we compute as follows:

$$\begin{aligned}&\int_0^1 v^{p-1}(1-v)^{q-1}\int_0^v u^{a-1}(1-u)^{b-1}dudv\\ =&\int_0^1 v^{p-1}(1-v)^{q-1}\sum_{n=0}^\infty \frac{(1-b,n)}{(1,n)}\int_0^v u^{n+a-1}du\\ =&\sum_{n=0}^\infty\int_0^1 v^{a+p+n-1}(1-v)^{q-1}\frac{(1-b,n)}{(1,n)}\frac{1}{a+n}dv\\ =&\sum_{n=0}^\infty B(a+p+n,q)\frac{(1-b,n)}{(1,n)}\frac{1}{a+n}\\ =&\sum_{n=0}^\infty \frac{\Gamma(a+p+n)\Gamma(q)}{\Gamma(a+p+q+n)}\frac{(1-b,n)}{(1,n)}\frac{1}{a+n}\\ =&\frac{\Gamma(a+p)\Gamma(q)}{a\Gamma(a+p+q)}\sum_{n=0}^\infty\frac{a}{a+n}\frac{\Gamma(a+p+q)}{\Gamma(a+p+q+n)}\frac{\Gamma(a+p+n)}{\Gamma(a+p)}\frac{(1-b,n)}{(1,n)}\\ =&\frac{B(a+p,q)}{a}\lim_{\substack{t\to 1-0\\ t\in\mathbb{R}}} {}_3F_2\left(\begin{matrix} a, 1-b, a+p \\ 1+a, a+p+q\end{matrix}; t\right).\end{aligned}$$

□

From this proposition, we have

Lemma 3.8.

$$x_{r,s,l,m} = \frac{N^2\int_\gamma f_{r,s}f_{l,m}}{B^N_{r,s}B^N_{l,m}} = \frac{NB^N_{r+l,m}}{rB^N_{r,s}B^N_{l,m}}\lim_{\substack{t\to 1-0\\ t\in\mathbb{R}}} {}_3F_2\left(\begin{matrix} r/N, 1-s/N, (r+l)/N \\ 1+r/N, (r+l+m)/N\end{matrix}; t\right).$$

4. A nontrivial algebraic cycle in $J(F(6))$

In this section, we consider only the case $N = 6$. We compute some value of the harmonic volume for the Fermat sextic $F(6)$. This tells the nontriviality of the algebraic cycle $F(6) - F(6)^-$ in $J(F(6))$. We have the genus of $F(6)$ is equal to 10 and $\{\omega_{r,s}\}_{r,s\geq 1, r+s\leq 5}$ is a basis of $H^{1,0}$ of $F(6)$. For the rest of this paper, we denote $\zeta = \zeta_6$ and $R = \mathbb{Z}[\zeta]$. Proposition 3.2 gives that a set of loops $\{\gamma_{0,0}, \gamma_{0,1}, \ldots, \gamma_{0,4}, \gamma_{1,0}, \gamma_{1,1}, \ldots, \gamma_{1,4}, \gamma_{2,0} \ldots, \gamma_{3,0}, \gamma_{3,1}, \ldots, \gamma_{3,4}\}$ may be considered as a basis of the integral homology group $H_1(F(6);\mathbb{Z})$ of $F(6)$. Let P.D.: $H^1(F(6);\mathbb{C}) \to H_1(F(6);\mathbb{C})$ be the Poincaré dual.

Lemma 4.1. *Let $L_{i,k}$ be a linear combination $\sum_{n=0}^{5}\zeta^{nk}\gamma_{i,n}$ in $H_1(F(6);\mathbb{C})$. Then we have*

$$\mathrm{P.D.}(\omega_{1,1}) = \frac{1}{122}\big\{(60-13\zeta)L_{0,1}-(15-49\zeta)L_{1,1}-(43-51\zeta)L_{2,1}-(50-21\zeta)L_{3,1}\big\}.$$

Proof. Since $\beta_*(\gamma_{i,j}) = \gamma_{i,j+1}$ as a homology class, we obtain

$$\beta_* L_{i,k} = \zeta^{-k}L_{i,k}.$$

We have

$$\beta_*(\mathrm{P.D.}(\omega_{1,1})) = \mathrm{P.D.}((\beta^{-1})^*\omega_{1,1}) = \zeta^5\mathrm{P.D.}(\omega_{1,1}).$$

Since $\beta_* L_{i,1} = \zeta^5 L_{i,1}$, there exist constants $\lambda_0, \ldots, \lambda_3 \in \mathbb{C}$ such that $\mathrm{P.D.}(\omega_{1,1}) = \sum_{i=0}^{3}\lambda_i L_{i,1}$. The result follows from Proposition 3.4 and the equations

$$\begin{aligned}
\int_{\gamma_{0,0}}\omega_{1,1} = (\mathrm{P.D.}(\omega_{1,1}), \gamma_{0,0}) &= \sum_{i=0}^{3}\lambda_i(L_{i,1},\gamma_{0,0}) = (\zeta^5-\zeta)\lambda_0 + (\zeta-1)\lambda_1,\\
\int_{\gamma_{1,0}}\omega_{1,1} = (\mathrm{P.D.}(\omega_{1,1}), \gamma_{1,0}) &= \sum_{i=0}^{3}\lambda_i(L_{i,1},\gamma_{1,0}) = (1-\zeta)\lambda_0 + (\zeta^5-\zeta)\lambda_1 + (\zeta-1)\lambda_2,\\
\int_{\gamma_{2,0}}\omega_{1,1} = (\mathrm{P.D.}(\omega_{1,1}), \gamma_{2,0}) &= \sum_{i=0}^{3}\lambda_i(L_{i,1},\gamma_{2,0}) = (1-\zeta)\lambda_1 + (\zeta^5-\zeta)\lambda_2 + (\zeta-1)\lambda_3,\\
\int_{\gamma_{3,0}}\omega_{1,1} = (\mathrm{P.D.}(\omega_{1,1}), \gamma_{3,0}) &= \sum_{i=0}^{3}\lambda_i(L_{i,1},\gamma_{3,0}) = (1-\zeta)\lambda_2 + (\zeta^5-\zeta)\lambda_3.
\end{aligned}$$

□

Let $\int_{L_{i,k}}\omega_{r,s}\omega_{l,m}$ denote $\sum_{n=0}^{5}\zeta^{nk}\int_{\gamma_{i,n}}\omega_{r,s}\omega_{l,m}$.

Lemma 4.2. *For $i = 0, \ldots, 3$, we have*

$$\int_{L_{i,1}}\omega_{1,2}\omega_{1,3} = 6\left\{\zeta^{2i}(1+\zeta)(x_{1,2,1,3}-1) - \zeta^i\right\}.$$

Proof. By Theorem 3.6, it is easy to compute

$$\int_{\gamma_{i,n}} \omega_{1,2}\omega_{1,3} = \zeta^{2i-1}(1+\zeta)(x_{1,2,1,3}-1)+2(1-\zeta^{2n})\zeta^{i+3n}(1-\zeta)-(1-\zeta^{3n})\zeta^{i+2n}(1-2\zeta).$$

Using this equation, we obtain the result in a straightforward way. □

Theorem 4.3. *Let $F(6)$ be the Fermat sextic. Then, the cycle $F(6) - F(6)^-$ is not algebraically equivalent to zero in $J(F(6))$.*

Proof. By the definition of the harmonic volume I_R, we have

$$I_R(\omega_{1,2} \otimes \omega_{1,3} \otimes \omega_{1,1}) \equiv \sum_{i=0}^{3} \lambda_i \int_{L_{i,1}} \omega_{1,2}\omega_{1,3} \bmod R.$$

Using Lemma 4.1 and 4.2, we obtain

$$2I_R(\omega_{1,2} \otimes \omega_{1,3} \otimes \omega_{1,1}) \equiv \frac{6}{61}\big\{(42 - 3\zeta)x_{1,2,1,3} - 95 + 46\zeta\big\} \bmod R,$$

and denote it by α. By Lemma 3.8 and the numerical calculation (Figure 1 in Appendix), we obtain the value

$$2\Re(\alpha) \equiv \frac{6}{61}(81x_{1,2,1,3} - 144) \equiv 0.74286 \pm 1 \times 10^{-5} \bmod \mathbb{Z}.$$

The result follows from Theorem 2.2 and the lemma

$$2\Re(\alpha) \notin \mathbb{Z} \Rightarrow \alpha \notin \mathbb{Z}[\zeta].$$

□

5. Appendix

We introduce the MATHEMATICA program [11] in the proof of Theorem 4.3.

```
x[r_, s_, l_, m_] := (6 * Beta[(r + l) / 6, m / 6]) / (r * Beta[r / 6, s / 6] * Beta[l / 6, m / 6]) *
  HypergeometricPFQ[{r / 6, 1 - s / 6, (r + l) / 6}, {1 + r / 6, (r + l + m) / 6}, 1]
N[2 * FullSimplify[6 / 61 (81 * x[1, 2, 1, 3] - 144)], 20] + 22
```

Figure 1. A numerical calculation program in the proof of Theorem 4.3

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