UTMS 2007–21 November 30, 2007

Calculus of principal series Whittaker functions on $GL(3,C)$

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CALCULUS OF PRINCIPAL SERIES WHITTAKER FUNCTIONS ON $GL(3, \mathbf{C})$

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ABSTRACT: In this paper, we discuss the Whittaker functions for the non-spherical principal series representations of $GL(3, \mathbf{C})$. In particular, we give explicit formulas for these functions.

1. Introduction

The global Whittaker function of automorphic representations on $GL(n)$ is utilized to have automorphic L-functions, as can be seen in the theory developed by Jacquet, Piatetski-Shapiro, and Shalika (*cf.* [2]). The basic parts of local investigations are to handle the unramified p-adic cases and the archimedean cases.

Compared with the explicit formula of Whittaker functions for unramified principal series representations of $GL(n)$ over p-adic fields ([20]), the history to have explicit integral expressions of Whittaker functions for principal series representations of $GL(n)$ over the archimedean fields $\mathbf{R}$ and $\mathbf{C}$ is more involved and longer. The classical case $GL(2)$ is found in the literature of automorphic forms such as Jacquet-Langlands [12] and Weil [28]. The beginning works beyond this point seems to be those of Vinogradov-Tahtajan [25], Proskurin [18], and Bump [1]. They obtained the explicit integral formula of archimedean Whittaker functions for class one principal series representations of $GL(3)$ by evaluating the Jacquet's integral ([11]) or by solving the differential equations for them. And further investigation of the class one Whittaker functions is developed gradually by the papers of Stade and Ishii (*cf.* [21], [22], [23], [9], [10]).

Contrary to the class one case refereed above, explicit integral formulas of the archimedean Whittaker functions for non-spherical principal series representations begin to be investigated rather recently (Manabe-Ishii-Oda [13]). The reason of this delay is not clear. But it is true that the discussion of non-spherical cases which is not a trivial extension of the spherical cases is more time-demanding and requires some new ideas.

In this paper, we discuss the Whittaker functions with minimal K-types belonging to general principal series representations of $GL(3, \mathbf{C})$. We need two new ideas in this paper. One is the use of *Gelfand-Zelevinsky basis* of simple K-modules in order to treat the Whittaker functions which is vector-valued different from the spherical cases. This basis is defined in the paper [3] and is recognized as the (classical limit of) dual of canonical basis in quantum groups investigated by Kashiwara and Lusztig. The other is the use of *Dirac-Schmid operators* in our constructions of differential equations satisfied by the Whittaker functions. These operators are elements in $U(\mathfrak{g}_{\mathbb{C}})$ defined by the injectors of the minimal K-type τ into the tensor product $\mathfrak{p}_{\mathbb{C}} \otimes \tau$ and their explicit descriptions require the Clebsch-Gordan coefficients.

The main results are explicit formulas for Whittaker functions in section 7. In more detail, the results are explicit formulas for the *secondary* Whittaker functions, two equivalent integral representations for the *primary* Whittaker function, and the factorization theorem of the primary function by the secondaries. These formulas are a natural extension of the class one case and can be handled easily like as those for class one functions. We expect that our results are applicable for deeper investigation of automorphic forms on $GL(3)$.

Also, based on the explicit formula for the primary function, we derive an inductive procedure to write Whittaker functions on $GL(3,\mathbf{C})$ by these on $GL(2,\mathbf{C})$ in section 8, which we call a *propagation formula*. This is an analogue of the formula in the real cases by Ishii-Stade [10] and Hina-Ishii-Oda [5]. It seems not only to show the similarity between the real and the complex cases in the non-class one situations but also to give a hint on a basis of $\mathfrak{gl}_n$-modules which is suitable for an explicit description of general principal series Whittaker functions on $GL(n,\mathbf{C})$.

2. Preliminaries

2.1. **Groups and algebras.** Let $G = GL(3,\mathbf{C})$ be the complex general linear group of degree 3. We view G as a real reductive group and denote the imaginary unit by J; $J^2 = -1$. The center Z_G of G is $\{ru1_3 \,|\, r \in \mathbf{R}_{>0}, u \in U(1)\} \simeq \mathbf{C}^\times$. Here 1_3 is the unit matrix of degree 3. For a Cartan involution $\theta(g) = {}^t\bar{g}^{-1}$, $g \in G$ of G, its fixed part $K = \{g \in G \,|\, \theta(g) = g\} = U(3)$, the unitary group of degree 3, is a maximal compact subgroup of G.

Let $\mathfrak{g} = \mathfrak{gl}(3,\mathbf{C})$ be the Lie algebra of G. If we denote the differential of θ again by θ, then we have $\theta(X) = -{}^t\bar{X}$ for $X \in \mathfrak{g}$. Let $\mathfrak{k}$ and $\mathfrak{p}$ be the $+1$ and the -1 eigenspaces of θ in $\mathfrak{g}$, respectively. Then $\mathfrak{k} = \mathfrak{u}(3)$ is the Lie algebra of K and $\mathfrak{g}$ has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

In general for a Lie algebra $\mathfrak{l}$, its complexification is denoted by $\mathfrak{l}_{\mathbb{C}}$. For $1 \le i,j \le 3$, let E_{ij} (resp. E'_{ij}) in $\mathfrak{g}$ be the matrix unit with its (i,j)-entry 1 (resp. J) and the remaining entries 0. Moreover put $H_{ij} = E_{ii} - E_{jj}$, $H'_{ij} = E'_{ii} - E'_{jj}$, $I_3 = E_{11} + E_{22} + E_{33}$, and $I'_3 = E'_{11} + E'_{22} + E'_{33}$. Then we have $\mathfrak{k} = Z_{\mathfrak{k}} \oplus \mathfrak{k}_0$ and $\mathfrak{p} = Z_{\mathfrak{p}} \oplus \mathfrak{p}_0$ with

$$Z_{\mathfrak{k}} = \mathbf{R}I'_3, \quad \mathfrak{k}_0 = \mathbf{R}H'_{12} \oplus \mathbf{R}H'_{23} \oplus \{\oplus_{i<j}\mathbf{R}(E_{ij} - E_{ji})\} \oplus \left\{\oplus_{i<j}\mathbf{R}(E'_{ij} + E'_{ji})\right\},$$

and

$$Z_{\mathfrak{p}} = \mathbf{R}I_3, \quad \mathfrak{p}_0 = \mathbf{R}H_{12} \oplus \mathbf{R}H_{23} \oplus \{\oplus_{i<j}\mathbf{R}(E_{ij} + E_{ji})\} \oplus \left\{\oplus_{i<j}\mathbf{R}(E'_{ij} - E'_{ji})\right\}.$$

In the complexification $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_{\mathbb{C}}$, we use the following symbols.

$$I_3^{\mathfrak{k}} = -\sqrt{-1}I'_3, \quad H_{ij}^{\mathfrak{k}} = \sqrt{-1}H'_{ij}, \quad E_{ij}^{\mathfrak{k}} = \frac{1}{2}\left\{(E_{ij} - E_{ji}) - \sqrt{-1}\left(E'_{ij} + E'_{ji}\right)\right\}$$

in $\mathfrak{k}_{\mathbb{C}}$ and

$$I_3^{\mathfrak{p}} = I_3, \quad H_{ij}^{\mathfrak{p}} = H_{ij}, \quad E_{ij}^{\mathfrak{p}} = \frac{1}{2}\left\{(E_{ij} + E_{ji}) - \sqrt{-1}\left(E'_{ij} - E'_{ji}\right)\right\}$$

in $\mathfrak{p}_{\mathbb{C}}$.

Put $\mathfrak{a} = Z_{\mathfrak{p}} \oplus \mathbf{R}H_{12} \oplus \mathbf{R}H_{23}$. Then $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{p}$. Also if we put $\mathfrak{n} = \oplus_{i<j}(\mathbf{R}E_{ij} \oplus \mathbf{R}E'_{ij})$, then $\mathfrak{n}$ is the direct sum of the all positive restricted

root spaces with respect to $(\mathfrak{g},\mathfrak{a})$ and we have an Iwasawa decomposition $\mathfrak{g}=\mathfrak{n}\oplus\mathfrak{a}\oplus\mathfrak{k}$ of $\mathfrak{g}$. Moreover, we have an Iwasawa decomposition $G=NAK$ of G, where A and N is the analytic subgroup with Lie algebra $\mathfrak{a}$ and $\mathfrak{n}$, respectively, that is,

$$A=\{\mathrm{diag}\,(a_1,a_2,a_3)\in G \mid a_i\in\mathbf{R}_{>0},\ i=1,2,3\},$$

$$N=\left\{\begin{pmatrix}1 & x_1 & x_2\\ 0 & 1 & x_3\\ 0 & 0 & 1\end{pmatrix}\in G \;\middle|\; x_i\in\mathbf{C},\ i=1,2,3\right\}.$$

Consider the centralizer M of A in K;

$$\begin{aligned} M &= \{k\in K \mid kak^{-1}=a,\ a\in A\}\\ &= \{\mathrm{diag}\,(u_1,u_2,u_3)\mid u_i\in U(1),\ i=1,2,3\}\simeq U(1)^3. \end{aligned}$$

Then the upper triangular subgroup $P=NAM$ is a minimal parabolic subgroup of G and the right hand side gives its Langlands decomposition. Namely, N is the unipotent radical of P and AM is a Levi subgroup whose split component is A.

2.2. **Representations of K.** According to the theory of highest weight, the equivalence classes of irreducible continuous representations of the maximal compact subgroup $K=U(3)$ of G are parameterized by the set of highest weights

$$\Lambda=\{\mu=(\mu_1,\mu_2,\mu_3)|\mu_i\in\mathbf{Z},\mu_1\geq\mu_2\geq\mu_3\}.$$

The representation of K corresponding to a highest weight $\mu\in\Lambda$ is denoted by (τ_μ,V_μ). The dimension of V_μ is given by the Weyl dimension formula (*cf.* [27], Theorem 2.4.1.6).

Lemma 2.1.

$$\dim_{\mathbf{C}} V_\mu=\frac{1}{2}(\mu_1-\mu_2+1)(\mu_2-\mu_3+1)(\mu_1-\mu_3+2).$$

In the following, we often use the symbol $\mathbf{e}_i$ for $1\leq i\leq 3$ which means the unit vector of degree 3 with its i-th component 1 and the remaining component 0 in order to write an element in $\mathbf{Z}^3$.

2.3. **Principal series representations of G.** The (irreducible) characters of $M\simeq U(1)^3$ are exhausted by

$$\sigma_{\mathbf{n}}(\mathrm{diag}(u_1,u_2,u_3))=u_1^{n_1}u_2^{n_2}u_3^{n_3},\quad \mathbf{n}=(n_1,n_2,n_3)\in\mathbf{Z}^3.$$

Since the Lie algebra $\mathfrak{a}$ of A has a system of generators consisting of diagonal matrix units $\{E_{ii}|i=1,2,3\}$, each linear form $\nu\in\mathrm{Hom}_{\mathbf{R}}(\mathfrak{a},\mathbf{C})$ can be identified with the complex vector $(\nu_1,\nu_2,\nu_3)\in\mathbf{C}^3$ of degree 3 via $\nu_i=\nu(E_{ii})$ for $1\leq i\leq 3$. The adjoint action of A on the Lie algebra $\mathfrak{n}$ of N induces the action $e^{2\rho}$ on the top degree wedge product $\wedge^6_{\mathbf{R}}\mathfrak{n}$. Here ρ is the half-sum of the positive restricted roots, i.e.,

$$e^\rho(\mathrm{diag}\,(a_1,a_2,a_3))=\left(\frac{a_1}{a_3}\right)^2,\quad \mathrm{diag}\,(a_1,a_2,a_3)\in A.$$

Let us take a character $\sigma_{\mathbf{n}}$ of M parameterized by $\mathbf{n}=(n_1,n_2,n_3)\in\mathbf{Z}^3$ and an element ν in $\mathfrak{a}^*_{\mathbb{C}}$ identified with $(\nu_1,\nu_2,\nu_3)\in\mathbf{C}^3$. Then the induced representation

$$\pi=\pi(\nu,\sigma_{\mathbf{n}})=\mathrm{Ind}_P^G(1_N\otimes e^{\nu+\rho}\otimes\sigma_{\mathbf{n}})$$

of G from the parabolic subgroup $P = NAM$ is called the *principal series representation* of G. The representation π is a Hilbert representation (i.e., a Banach representation on a Hilbert space) with the representation space

$$L^2_{(M,\sigma_{\mathbf{n}})}(K) = \{f \in L^2(K) \mid f(mk) = \sigma_{\mathbf{n}}(m)f(k), \quad m \in M, \ k \in K\},$$

and the action of G on $L^2_{(M,\sigma_{\mathbf{n}})}(K)$ is given by

$$(\pi(x)f)(k) = a(kx)^{\nu+\rho} f(\kappa(kx)), \ \ k \in K, \ x \in G.$$

Here $g = n(g)a(g)\kappa(g) \in G$ is the Iwasawa decomposition of $g \in G$. If we put $\tilde{\nu} = \nu_1 + \nu_2 + \nu_3$ and $\tilde{n} = n_1 + n_2 + n_3$, the central character of π is given by

$$Z_G \ni ru1_3 \mapsto r^{\tilde{\nu}} u^{\tilde{n}}, \quad r \in \mathbf{R}_{>0}, \ u \in U(1).$$

The K-types of the principal series representation $\pi = \pi(\nu, \sigma_{\mathbf{n}})$ are understood via the right K-action on $L^2_{(M,\sigma_{\mathbf{n}})}(K)$. A standard argument using the Frobenius reciprocity for induced representations leads the following proposition.

Proposition 2.2. *Let $\pi = \pi(\nu, \sigma_{\mathbf{n}})$ be a principal series representation with data $(\nu, \sigma_{\mathbf{n}})$. A necessary and sufficient condition for a representation τ_μ of K corresponding to a highest weight $\mu = (\mu_1, \mu_2, \mu_3) \in \Lambda$ to be a constituent of the restriction $\pi|_K$ of π to K is that the convex closure of the subset*

$$\{(\mu_i, \mu_j, \mu_k) \in \mathbf{Z}^3 \mid (i,j,k) \text{ are permutations of } (1,2,3) \ \}$$

in $\mathbf{R}^3$ contains the point $\mathbf{n} = (n_1, n_2, n_3)$. In particular, if $\mathbf{m} = (n_a, n_b, n_c)$ is the dominant permutation of $\mathbf{n}$ (namely $n_a \geq n_b \geq n_c$), then the representation $\tau_{\mathbf{m}}$ is the minimal K-type of π and occurs with multiplicity one in $\pi|_K$.

2.4. **Unitary characters of N.** Since a set $\{E_{ij}, E'_{ij} \,|\, 1 \leq i < j \leq 3\}$ gives a system of generators of $\mathfrak{n}$, a non-degenerate character $\eta = \eta_{c_1,c_2}$ of N can be specified as

$$\begin{aligned} \eta(E_{12}) &= 2\pi\sqrt{-1}\mathrm{Re}\,(c_1), \quad \eta(E_{23}) = 2\pi\sqrt{-1}\mathrm{Re}\,(c_2), \\ \eta(E'_{12}) &= 2\pi\sqrt{-1}\mathrm{Im}\,(c_1), \quad \eta(E'_{23}) = 2\pi\sqrt{-1}\mathrm{Im}\,(c_2), \end{aligned}$$

with two non-zero complex numbers $c_1, c_2 \in \mathbf{C}^\times$. Then we have

$$\eta \begin{pmatrix} 1 & x_1 & x_2 \\ & 1 & x_3 \\ & & 1 \end{pmatrix} = \exp\left(2\pi\sqrt{-1}\mathrm{Re}\,(\bar{c}_1 x_1 + \bar{c}_2 x_3)\right), \ \ x_i \in \mathbf{C}.$$

3. Whittaker functions

For a finite dimensional representation (τ, V_τ) of K and a non-degenerate character η of N, we denote by $C^\infty_{\eta,\tau}(N\backslash G/K)$ the space consisting of smooth functions $\varphi : G \to V_\tau$ satisfying the condition

$$\varphi(ngk) = \eta(n)\tau(k)^{-1}\varphi(g), \quad (n,g,k) \in N \times G \times K.$$

Then the function $\varphi \in C^\infty_{\eta,\tau}(N\backslash G/K)$ is determined by its restriction $\varphi|_A$ to A, because of the Iwasawa decomposition $G = NAK$ of G. Moreover, let $C^\infty\mathrm{Ind}_N^G(\eta)$

be the representation of G induced from η as C^∞-induction. Here the representation space of $C^\infty\mathrm{Ind}_N^G(\eta)$ is

$$C^\infty_\eta(N\backslash G)=\{\varphi\in C^\infty(G)\,|\,\varphi(ng)=\eta(n)\varphi(g),\ (n,g)\in N\times G\},$$

on which G acts via right translation.

If we denote by (τ^*,V_{τ^*}) the contragradient representation of (τ,V_τ) and by $\langle\cdot,\cdot\rangle$ the canonical bilinear form on $V_{\tau^*}\times V_\tau$, then the relation

$$\iota(v^*)(g)=\langle v^*,F^{[\iota]}(g)\rangle,\quad v^*\in V_{\tau^*},\ g\in G,$$

defines an association from $\iota\in\mathrm{Hom}_K(\tau^*,C^\infty\mathrm{Ind}_N^G(\eta))$ to $F^{[\iota]}\in C^\infty_{\eta,\tau}(N\backslash G/K)$, which gives an isomorphism $\mathrm{Hom}_K(\tau^*,C^\infty\mathrm{Ind}_N^G(\eta))\cong C^\infty_{\eta,\tau}(N\backslash G/K)$.

For an (irreducible) admissible representation (π,H_π) of G, we choose a K-type (τ^*,V_{τ^*}) in π which occurs with multiplicity one and fix an injective K-homomorphism $i\in\mathrm{Hom}_K(\tau^*,\pi|_K)$. Let

$$\mathcal{I}_{\eta,\pi}=\mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(\pi,C^\infty\mathrm{Ind}_N^G(\eta))$$

be the intertwining space between $(\mathfrak{g}_\mathbb{C},K)$-modules π and $C^\infty\mathrm{Ind}_N^G(\eta)$ consisting of all K-finite vectors. For each $T\in\mathcal{I}_{\eta,\pi}$, we define an element $T_i\in C^\infty_{\eta,\tau}(N\backslash G/K)$ by

$$T(i(v^*))(g)=\langle v^*,T_i(g)\rangle,\ \ v^*\in V_{\tau^*},\ \ g\in G.$$

Then we call the subspace

$$\mathrm{Wh}(\pi,\eta,\tau)=\bigcup_{i\in\mathrm{Hom}_K(\tau^*,\pi|_K)}\{T_i\in C^\infty_{\eta,\tau}(N\backslash G/K)\,|\,T\in\mathcal{I}_{\eta,\pi}\}$$

of $C^\infty_{\eta,\tau}(N\backslash G/K)$ the *space of Whittaker functions with respect to* (π,η,τ). Moreover, we denote by $\mathcal{I}^\circ_{\eta,\pi}$ the subspace of $\mathcal{I}_{\eta,\pi}$ consisting of the intertwining operators whose images in $C^\infty_\eta(N\backslash G)$ are moderate growth functions ([26] §8.1) and define the subspace

$$\mathrm{Wh}(\pi,\eta,\tau)^{\mathrm{mod}}=\bigcup_{i\in\mathrm{Hom}_K(\tau^*,\pi|_K)}\big\{T_i\in C^\infty_{\eta,\tau}(N\backslash G/K)\,\big|\,T\in\mathcal{I}^\circ_{\eta,\pi}\big\},$$

of $\mathrm{Wh}(\pi,\eta,\tau)$. An element in $\mathrm{Wh}(\pi,\eta,\tau)^{\mathrm{mod}}$ is called a *Whittaker function of moderate growth.*

4. A small $U(3)$ machine

4.1. **Gelfand-Zelevinsky basis.** Let (τ_μ,V_μ) be an irreducible representation of $K=U(3)$ associated with a highest weight $\mu=(\mu_1,\mu_2,\mu_3)\in\Lambda$. The representation space V_μ of τ_μ has a *Gelfand-Zelevinsky basis* (or a *proper basis*) defined and studied in the paper of Gelfand and Zelevinsky [3]. This basis can be parameterized by the set $G(\mu)$ of G-patterns belonging to μ as well as the Gelfand-Tsetlin basis. Here a G-pattern $M\in G(\mu)$ belonging to μ is a triangle

$$M=\begin{pmatrix}\mu_1\ \mu_2\ \mu_3\\ \alpha_1\ \alpha_2\\ \beta\end{pmatrix}$$

consisting of 6 integers satisfying the inequalities

$$\mu_1\ge\alpha_1\ge\mu_2\ge\alpha_2\ge\mu_3\ \text{and}\ \alpha_1\ge\beta\ge\alpha_2.$$

A Gelfand-Zelevinsky basis for $\mathfrak{gl}_3$ has the ambiguity of scalar multiples. In the paper [3], a normalization of this basis was defined and the explicit action of $\mathfrak{gl}_3$ on them was given. We denote this normalized Gelfand-Zelevinsky basis by $\{f(M)\}_{M\in G(\mu)}$ and call it the *GZ-basis* simply.

In order to describe the explicit action of $\mathfrak{k}_{\mathbb{C}}$ on the GZ-basis, we introduce some notations for G-patterns. For a G-pattern $M = \left(\begin{smallmatrix} \mu_1 & \mu_2 & \mu_3 \\ \alpha_1 & \alpha_2 \\ \beta \end{smallmatrix}\right) \in G(\mu)$ belonging to $\mu \in \Lambda$ and a triangular array $I = \left(\begin{smallmatrix} i_{13} & i_{23} & i_{33} \\ i_{12} & i_{22} \\ i_{11} \end{smallmatrix}\right)$ of integers, we define the shift $M(I)$ of M by I as

$$M(I) = \left(\begin{smallmatrix} \mu_1+i_{13} & \mu_2+i_{23} & \mu_3+i_{33} \\ \alpha_1+i_{12} & \alpha_2+i_{22} \\ \beta+i_{11} \end{smallmatrix}\right).$$

If the vector $(i_{13}\ i_{23}\ i_{33})$ is zero, we omit the top row of I, that is, $M(I)$ is written as $M\left(\begin{smallmatrix} i_{12} & i_{22} \\ i_{11} \end{smallmatrix}\right)$. We use a convenient symbol $M[k]$ defined by $M\left(\begin{smallmatrix} k & -k \\ 0 \end{smallmatrix}\right)$. Put

$$\delta(M) = \alpha_1 + \alpha_2 - \mu_2 - \beta,$$

and define the characteristic functions $\chi_+^{(i)}(M)$ and $\chi_-^{(i)}(M)$ of the sets $\{M \mid \delta(M) > i\}$ and $\{M \mid \delta(M) < -i\}$, respectively. If $i = 0$, we write $\chi_\pm^{(0)}(M)$ simply by $\chi_\pm(M)$. Moreover, we introduce 'piecewise-linear' functions $C_1(M)$ and $\bar{C}_1(M)$ by

$$C_1(M) = \mathrm{Min}\{\beta - \alpha_2, \alpha_1 - \mu_2\} = \begin{cases} \beta - \alpha_2, & \text{if } \delta(M) \geq 0 \\ \alpha_1 - \mu_2, & \text{if } \delta(M) \leq 0 \end{cases},$$

$$\bar{C}_1(M) = \mathrm{Min}\{\mu_2 - \alpha_2, \alpha_1 - \beta\} = \begin{cases} \mu_2 - \alpha_2, & \text{if } \delta(M) \geq 0 \\ \alpha_1 - \beta, & \text{if } \delta(M) \leq 0 \end{cases},$$

and put $C_2(M) = C_1(M)\bar{C}_1(M)$. Also we define the functions

$$\begin{aligned} D(M) &= -\mu_1 + \alpha_1 - \delta(M), \\ E(M) &= \bar{C}_1(M)\left\{\mu_1 - \mu_3 + 1 - C_1(M)\right\}, \\ F(M) &= -C_2(M) \\ &\quad -\chi_-(M)\left\{(\mu_1 - \alpha_1)(\alpha_2 - \mu_3) - (\mu_1 - \mu_3 + 1)\delta(M)\right\}, \end{aligned}$$

and its duals

$$\begin{aligned} \bar{D}(M) &= -\alpha_2 + \mu_3 + \delta(M), \\ \bar{E}(M) &= C_1(M)\left\{\mu_1 - \mu_3 + 1 - \bar{C}_1(M)\right\}, \\ \bar{F}(M) &= -C_2(M) \\ &\quad -\chi_+(M)\left\{(\mu_1 - \alpha_1)(\alpha_2 - \mu_3) + (\mu_1 - \mu_3 + 1)\delta(M)\right\}. \end{aligned}$$

Lemma 4.1. *Let V_μ be an irreducible finite dimensional representation of $\mathfrak{k}_{\mathbb{C}}$ corresponding to a highest weight $\mu \in \Lambda$ and $\{f(M)\}_{M\in G(\mu)}$ be the GZ-basis of V_μ. If we take the subalgebra consisting of diagonal matrices as the Cartan subalgebra, the actions of the elements $E_{ii}^{\mathfrak{k}}$ in the Cartan subalgebra and the simple root vectors $E_{ij}^{\mathfrak{k}}$ on $\{f(M)\}_{M\in G(\mu)}$ are given as follows.*

$$E_{ii}^{\mathfrak{k}} f(M) = w_i f(M),$$

$$
\begin{aligned}
E^{\mathfrak{k}}_{12}f(M) &= (\alpha_1-\beta)f\left(M\left(\begin{smallmatrix}0\ 0\\ 1\end{smallmatrix}\right)\right)\\
&\quad +\chi_+(M)(\mu_2-\alpha_2)f\left(M\left(\begin{smallmatrix}0\ 0\\ 1\end{smallmatrix}\right)[-1]\right),\\
E^{\mathfrak{k}}_{21}f(M) &= (\beta-\alpha_2)f\left(M\left(\begin{smallmatrix}0\ 0\\ -1\end{smallmatrix}\right)\right)\\
&\quad +\chi_-(M)(\alpha_1-\mu_2)f\left(M\left(\begin{smallmatrix}0\ 0\\ -1\end{smallmatrix}\right)[-1]\right),\\
E^{\mathfrak{k}}_{23}f(M) &= (\mu_1-\alpha_1)f\left(M\left(\begin{smallmatrix}1\ 0\\ 0\end{smallmatrix}\right)\right)\\
&\quad +\chi_-(M)\left\{\mu_1-\alpha_1-\delta(M)\right\}f\left(M\left(\begin{smallmatrix}1\ 0\\ 0\end{smallmatrix}\right)[-1]\right),\\
E^{\mathfrak{k}}_{32}f(M) &= (\alpha_2-\mu_3)f\left(M\left(\begin{smallmatrix}0\ -1\\ 0\end{smallmatrix}\right)\right)\\
&\quad +\chi_+(M)\left\{\alpha_2-\mu_3+\delta(M)\right\}f\left(M\left(\begin{smallmatrix}0\ -1\\ 0\end{smallmatrix}\right)[-1]\right),\\
E^{\mathfrak{k}}_{13}f(M) &= (\mu_1-\alpha_1)f\left(M\left(\begin{smallmatrix}1\ 0\\ 1\end{smallmatrix}\right)\right)\\
&\quad -\bar{C}_1(M)f\left(M\left(\begin{smallmatrix}1\ 0\\ 1\end{smallmatrix}\right)[-1]\right),\\
E^{\mathfrak{k}}_{31}f(M) &= -(\alpha_2-\mu_3)f\left(M\left(\begin{smallmatrix}0\ -1\\ -1\end{smallmatrix}\right)\right)\\
&\quad +C_1(M)f\left(M\left(\begin{smallmatrix}0\ -1\\ -1\end{smallmatrix}\right)[-1]\right).
\end{aligned}
$$

Here, $(w_1,w_2,w_3)=(\beta,\alpha_1+\alpha_2-\beta,m_1+m_2+m_3-\alpha_1-\alpha_2)$ is the weight of the vector $f(M)$ associated with a G-pattern $M=\left(\begin{smallmatrix}m_1\ m_2\ m_3\\ \alpha_1\ \alpha_2\\ \beta\end{smallmatrix}\right)$, and we promise the corresponding vector $f(M')$ is zero if a shift M' of M appearing in the above formulas violates the conditions of G-patterns.

4.2. **$\mathfrak{p}_{\mathbb{C}}$ as a K-module.** Let $\mathfrak{p}=Z_{\mathfrak{p}}\oplus\mathfrak{p}_0$ be the (-1)-eigenspace for the Cartan involution θ in $\mathfrak{g}$ as explained in §2.1. It is well known that the complexification $\mathfrak{p}_{\mathbb{C}}$ of $\mathfrak{p}$ is a K-module via the adjoint action and has the irreducible decomposition $\mathfrak{p}_{\mathbb{C}}=Z_{\mathfrak{p},\mathbb{C}}\oplus\mathfrak{p}_{0,\mathbb{C}}$, where $Z_{\mathfrak{p},\mathbb{C}}$ and $\mathfrak{p}_{0,\mathbb{C}}$ are isomorphic to the trivial representation $V_{(0,0,0)}$ and the 8 dimensional representation $V_{\mathbf{e}_1-\mathbf{e}_3}$ corresponding to the highest weight $\mathbf{e}_1-\mathbf{e}_3$, respectively. The correspondence between the GZ-basis $\{f(M)\}_{M\in G(\mathbf{e}_1-\mathbf{e}_3)}$ of $V_{\mathbf{e}_1-\mathbf{e}_3}$ and the elements in $\mathfrak{p}_{0,\mathbb{C}}$ is given by the following lemma.

Lemma 4.2. *We have an isomorphism $V_{\mathbf{e}_1-\mathbf{e}_3} \simeq \mathfrak{p}_{0,\mathbb{C}}$ by the following correspondence between their basis.*

$$f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\ 0 \\ 1 \end{pmatrix} \leftrightarrow E^{\mathfrak{p}}_{13}, \quad f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\ {-1} \\ 1 \end{pmatrix} \leftrightarrow -E^{\mathfrak{p}}_{12}, \quad f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\ 0 \\ 0 \end{pmatrix} \leftrightarrow E^{\mathfrak{p}}_{23},$$

$$f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\ {-1} \\ 0 \end{pmatrix} \leftrightarrow \frac{1}{3}(H^{\mathfrak{p}}_{12}+H^{\mathfrak{p}}_{13}) = \frac{1}{3}(2H^{\mathfrak{p}}_{12}+H^{\mathfrak{p}}_{23}),$$

$$f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 0\ 0 \\ 0 \end{pmatrix} \leftrightarrow \frac{1}{3}(H^{\mathfrak{p}}_{31}+2H^{\mathfrak{p}}_{32}) = -\frac{1}{3}(H^{\mathfrak{p}}_{12}+H^{\mathfrak{p}}_{23}),$$

$$f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 0\ {-1} \\ 0 \end{pmatrix} \leftrightarrow -E^{\mathfrak{p}}_{32}, \quad f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\ {-1} \\ -1 \end{pmatrix} \leftrightarrow E^{\mathfrak{p}}_{21}, \quad f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 0\ {-1} \\ -1 \end{pmatrix} \leftrightarrow E^{\mathfrak{p}}_{31}.$$

Proof. We can find the following table of the adjoint action of $\mathfrak{k}_{\mathbb{C}}$ on the elements in $\mathfrak{p}_{0,\mathbb{C}}$ by direct computation. Comparing this with the action of the simple root vectors of $\mathfrak{k}_{\mathbb{C}}$ on the GZ-basis $\{f(M)\}_{M\in G(\mathbf{e}_1-\mathbf{e}_3)}$ of $V_{\mathbf{e}_1-\mathbf{e}_3}$ in Lemma 4.1, we have the assertion. □

	$E^{\mathfrak{k}}_{11}$	$E^{\mathfrak{k}}_{22}$	$E^{\mathfrak{k}}_{33}$	$E^{\mathfrak{k}}_{12}$	$E^{\mathfrak{k}}_{21}$	$E^{\mathfrak{k}}_{23}$	$E^{\mathfrak{k}}_{32}$	$E^{\mathfrak{k}}_{13}$	$E^{\mathfrak{k}}_{31}$
$H^{\mathfrak{p}}_{12}$	0	0	0	$-2E^{\mathfrak{p}}_{12}$	$2E^{\mathfrak{p}}_{21}$	$E^{\mathfrak{p}}_{23}$	$-E^{\mathfrak{p}}_{32}$	$-E^{\mathfrak{p}}_{13}$	$E^{\mathfrak{p}}_{31}$
$H^{\mathfrak{p}}_{23}$	0	0	0	$E^{\mathfrak{p}}_{12}$	$-E^{\mathfrak{p}}_{21}$	$-2E^{\mathfrak{p}}_{23}$	$2E^{\mathfrak{p}}_{32}$	$-E^{\mathfrak{p}}_{13}$	$E^{\mathfrak{p}}_{31}$
$E^{\mathfrak{p}}_{12}$	$E^{\mathfrak{p}}_{12}$	$-E^{\mathfrak{p}}_{12}$	0	0	$H^{\mathfrak{p}}_{21}$	$-E^{\mathfrak{p}}_{13}$	0	0	$E^{\mathfrak{p}}_{32}$
$E^{\mathfrak{p}}_{21}$	$-E^{\mathfrak{p}}_{21}$	$E^{\mathfrak{p}}_{21}$	0	$H^{\mathfrak{p}}_{12}$	0	0	$E^{\mathfrak{p}}_{31}$	$-E^{\mathfrak{p}}_{23}$	0
$E^{\mathfrak{p}}_{23}$	0	$E^{\mathfrak{p}}_{23}$	$-E^{\mathfrak{p}}_{23}$	$E^{\mathfrak{p}}_{13}$	0	0	$H^{\mathfrak{p}}_{32}$	0	$-E^{\mathfrak{p}}_{21}$
$E^{\mathfrak{p}}_{32}$	0	$-E^{\mathfrak{p}}_{32}$	$E^{\mathfrak{p}}_{32}$	0	$-E^{\mathfrak{p}}_{31}$	$H^{\mathfrak{p}}_{23}$	0	$E^{\mathfrak{p}}_{12}$	0
$E^{\mathfrak{p}}_{13}$	$E^{\mathfrak{p}}_{13}$	0	$-E^{\mathfrak{p}}_{13}$	0	$E^{\mathfrak{p}}_{23}$	0	$-E^{\mathfrak{p}}_{12}$	0	$H^{\mathfrak{p}}_{31}$
$E^{\mathfrak{p}}_{31}$	$-E^{\mathfrak{p}}_{31}$	0	$E^{\mathfrak{p}}_{31}$	$-E^{\mathfrak{p}}_{32}$	0	$E^{\mathfrak{p}}_{21}$	0	$H^{\mathfrak{p}}_{13}$	0

TABLE 1. The adjoint actions of $E^{\mathfrak{k}}_{ij}$ on $\mathfrak{p}_{0,\mathbb{C}}$.

By the isomorphism in Lemma 4.2, we identify the tensor product $\mathfrak{p}_{0,\mathbb{C}} \otimes V_\mu$ with $V_{\mathbf{e}_1-\mathbf{e}_3} \otimes V_\mu$ for a general irreducible representation V_μ of K.

4.3. **Injectors and their Clebsch-Gordan coefficients.** For the 8 dimensional representation $(\tau_{\mathbf{e}_1-\mathbf{e}_3}, V_{\mathbf{e}_1-\mathbf{e}_3})$ of $K = U(3)$, we consider the tensor product with a general irreducible representation (τ_μ, V_μ) associated with a highest weight $\mu = (\mu_1, \mu_2, \mu_3) \in \Lambda$. The tensor product $V_{\mathbf{e}_1-\mathbf{e}_3} \otimes V_\mu$ has the following irreducible decomposition.

$$V_{\mathbf{e}_1-\mathbf{e}_3} \otimes V_\mu \simeq \left(\oplus_{i\neq j} V_{\mu+\mathbf{e}_i-\mathbf{e}_j}\right) \oplus V_\mu^{\oplus 2}.$$

Here, if the weight $\mu + \mathbf{e}_i - \mathbf{e}_j$ is not dominant, the corresponding irreducible component $V_{\mu+\mathbf{e}_i-\mathbf{e}_j}$ does not appear, and if either $\mu_1 = \mu_2$ or $\mu_2 = \mu_3$ holds, the irreducible component V_μ occurs with multiplicity free in $V_{\mathbf{e}_1-\mathbf{e}_3} \otimes V_\mu$. Among others, the intertwining space $\mathrm{Hom}\,(V_{\mathbf{e}_1-\mathbf{e}_3} \otimes V_\mu,\ V_\mu)$ has dimension 2 if $\mu_1 > \mu_2 > \mu_3$. An explicit description for projectors from $V_{\mathbf{e}_1-\mathbf{e}_3} \otimes V_\mu$ into its irreducible components with respect to the GZ-basis are given in our previous paper [7]. Now we give an explicit formula of injectors from the irreducible component V_μ into $V_{\mathbf{e}_1-\mathbf{e}_3} \otimes V_\mu$. The injectors we construct here are based on the following lemma given in our previous paper [7], Lemma 3.9.

Lemma 4.3. *Let $L^{(1)} = \begin{pmatrix} \mu_1\,\mu_2\,\mu_3 \\ \mu_1\,\mu_2 \\ \mu_1 \end{pmatrix} \in G(\mu)$ be the G-pattern giving the highest weight vector $f(L^{(1)})$ in V_μ, and let us define two vectors $v_1 = v_1^{(1)}$ and $v_2 = v_2^{(1)}$ in $V_{\mathbf{e}_1-\mathbf{e}_3} \otimes V_\mu$ by the formulas*

$$\begin{aligned} v_1 &= f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 0\,0 \\ 0 \end{pmatrix} \otimes f\left(L^{(1)}\right) - f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\,0 \\ 0 \end{pmatrix} \otimes f\left(L^{(1)}\begin{pmatrix} 0\ {-1} \\ 0 \end{pmatrix}\right) \\ &\quad + f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\,0 \\ 1 \end{pmatrix} \otimes f\left(L^{(1)}\begin{pmatrix} 0\ {-1} \\ -1 \end{pmatrix}\right), \\ v_2 &= f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\ {-1} \\ 0 \end{pmatrix} \otimes f\left(L^{(1)}\right) - f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\ {-1} \\ 1 \end{pmatrix} \otimes f\left(L^{(1)}\begin{pmatrix} 0\,0 \\ -1 \end{pmatrix}\right) \\ &\quad + f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\,0 \\ 1 \end{pmatrix} \otimes f\left(L^{(1)}\begin{pmatrix} -1\,0 \\ -1 \end{pmatrix}\right). \end{aligned}$$

Then, if $\mu_1 > \mu_2 > \mu_3$, each of v_1 and v_2 respectively generates a representation isomorphic to V_μ in $V_{\mathbf{e}_1-\mathbf{e}_3} \otimes V_\mu$ and gives the highest weight vector in each space. If $\mu_1 = \mu_2$ (resp. $\mu_2 = \mu_3$), then the vector v_2 (resp. v_1) is not valid.

Since each of the vectors v_1 and v_2 defined in the above lemma is a highest weight vector for a representation isomorphic to V_μ in $V_{\mathbf{e}_1-\mathbf{e}_3} \otimes V_\mu$, two injectors which map the highest weight vector $f(L^{(1)})$ in V_μ into v_1 and v_2 can be constructed. The following theorem which is the main theorem in this subsection gives an explicit description for such injectors.

Theorem 4.4. *Let $M = \begin{pmatrix} \mu_1\,\mu_2\,\mu_3 \\ \alpha_1\,\alpha_2 \\ \beta \end{pmatrix} \in G(\mu)$ be a G-pattern belonging to μ. Then, for $i = 1, 2$, the following formulas give injective K-homomorphisms ι_i from V_μ into $V_{\mathbf{e}_1-\mathbf{e}_3} \otimes V_\mu$ satisfying $\iota_i(f(L^{(1)})) = v_i$ with the G-pattern $L^{(1)} = \begin{pmatrix} \mu_1\,\mu_2\,\mu_3 \\ \mu_1\,\mu_2 \\ \mu_1 \end{pmatrix}$.*

1.

$$\begin{aligned} &\iota_1\left((\mu_1-\mu_3+1)(\mu_2-\mu_3)f(M)\right) \\ = \;& f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 0\,-1 \\ -1 \end{pmatrix} \otimes \Big\{ -(\mu_1-\alpha_1)(\alpha_2-\mu_3) f\left(M\begin{pmatrix} 1\,0 \\ 1 \end{pmatrix}\right) \\ &\qquad + E(M) f\left(M\begin{pmatrix} 1\,0 \\ 1 \end{pmatrix}[-1]\right) \Big\} \\ &+ f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 0\,-1 \\ 0 \end{pmatrix} \otimes \Big\{ (\mu_1-\alpha_1)(\alpha_2-\mu_3) f\left(M\begin{pmatrix} 1\,0 \\ 0 \end{pmatrix}\right) \\ &\qquad - F(M) f\left(M\begin{pmatrix} 1\,0 \\ 0 \end{pmatrix}[-1]\right) + \chi_-(M) C_2(M) f\left(M\begin{pmatrix} 1\,0 \\ 0 \end{pmatrix}[-2]\right) \Big\} \\ &+ f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\,-1 \\ -1 \end{pmatrix} \otimes \Big\{ (\mu_1-\alpha_1)(\alpha_2-\mu_3) f\left(M\begin{pmatrix} 1\,-1 \\ 1 \end{pmatrix}\right) \\ &\qquad - \Big[E(M) \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad -\chi_+(M)(\mu_1-\alpha_1)(\alpha_2-\mu_3)\Big] f\left(M\left(\begin{smallmatrix} 1 & -1 \\ & 1 \end{smallmatrix}\right)[-1]\right) \\
&\qquad -\chi_+(M)E(M)f\left(M\left(\begin{smallmatrix} 1 & -1 \\ & 1 \end{smallmatrix}\right)[-2]\right)\Bigg\} \\
&+f\left(\begin{smallmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\ -1 \\ 0 \end{smallmatrix}\right)\otimes\Bigg\{-2(\mu_1-\alpha_1)(\alpha_2-\mu_3)f\left(M\left(\begin{smallmatrix} 1 & -1 \\ & 0 \end{smallmatrix}\right)\right) \\
&\qquad +\Big[-(\mu_1-\alpha_1)(\alpha_2-\mu_3)-C_2(M)+E(M) \\
&\qquad\qquad +\chi_-(M)\delta(M)(\mu_1-\mu_3+1)\Big] f\left(M\left(\begin{smallmatrix} 1 & -1 \\ & 0 \end{smallmatrix}\right)[-1]\right) \\
&\qquad -C_2(M)f\left(M\left(\begin{smallmatrix} 1 & -1 \\ & 0 \end{smallmatrix}\right)[-2]\right)\Bigg\} \\
&+f\left(\begin{smallmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 0\,0 \\ 0 \end{smallmatrix}\right)\otimes\Bigg\{-(\mu_1-\alpha_1)(\alpha_2-\mu_3)f\left(M\left(\begin{smallmatrix} 1 & -1 \\ & 0 \end{smallmatrix}\right)\right) \\
&\qquad +\Big[-(\mu_1-\alpha_1)(\alpha_2-\mu_3)-C_2(M) \\
&\qquad\qquad +(\alpha_1-\mu_3+1)(\alpha_2-\mu_3) \\
&\qquad\qquad +\chi_-(M)\delta(M)(\mu_1-\mu_3+1)\Big] f\left(M\left(\begin{smallmatrix} 1 & -1 \\ & 0 \end{smallmatrix}\right)[-1]\right) \\
&\qquad -2C_2(M)f\left(M\left(\begin{smallmatrix} 1 & -1 \\ & 0 \end{smallmatrix}\right)[-2]\right)\Bigg\} \\
&+f\left(\begin{smallmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\ -1 \\ 1 \end{smallmatrix}\right)\otimes\Bigg\{(\mu_1-\alpha_1)(\alpha_2-\mu_3)f\left(M\left(\begin{smallmatrix} 1 & -1 \\ & -1 \end{smallmatrix}\right)\right) \\
&\qquad -F(M)f\left(M\left(\begin{smallmatrix} 1 & -1 \\ & -1 \end{smallmatrix}\right)[-1]\right) \\
&\qquad +\chi_-(M)C_2(M)f\left(M\left(\begin{smallmatrix} 1 & -1 \\ & -1 \end{smallmatrix}\right)[-2]\right)\Bigg\} \\
&+f\left(\begin{smallmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\,0 \\ 0 \end{smallmatrix}\right)\otimes\Bigg\{-(\alpha_1-\mu_3+1)(\alpha_2-\mu_3)f\left(M\left(\begin{smallmatrix} 0 & -1 \\ & 0 \end{smallmatrix}\right)\right) \\
&\qquad +\Big[C_2(M) \\
&\qquad\qquad -\chi_+(M)(\alpha_1-\mu_3+1)(\alpha_2-\mu_3)\Big] f\left(M\left(\begin{smallmatrix} 0 & -1 \\ & 0 \end{smallmatrix}\right)[-1]\right) \\
&\qquad +\chi_+(M)C_2(M)f\left(M\left(\begin{smallmatrix} 0 & -1 \\ & 0 \end{smallmatrix}\right)[-2]\right)\Bigg\} \\
&+f\left(\begin{smallmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\,0 \\ 1 \end{smallmatrix}\right)\otimes\Bigg\{(\alpha_1-\mu_3+1)(\alpha_2-\mu_3)f\left(M\left(\begin{smallmatrix} 0 & -1 \\ & -1 \end{smallmatrix}\right)\right) \\
&\qquad -C_2(M)f\left(M\left(\begin{smallmatrix} 0 & -1 \\ & -1 \end{smallmatrix}\right)[-1]\right)\Bigg\}.
\end{aligned}
$$

2.

$$\iota_2\left((\mu_1-\mu_3+1)(\mu_1-\mu_2)f(M)\right)$$

$$
\begin{aligned}
= \;& f\left(\begin{smallmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 0\;-1 \\ -1 \end{smallmatrix}\right) \otimes \Big\{(\mu_1-\alpha_1)(\mu_1-\alpha_2+1) f\left(M\left(\begin{smallmatrix} 1\,0 \\ 1 \end{smallmatrix}\right)\right) \\
&\qquad -C_2(M) f\left(M\left(\begin{smallmatrix} 1\,0 \\ 1 \end{smallmatrix}\right)[-1]\right)\Big\} \\
&+ f\left(\begin{smallmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 0\;-1 \\ 0 \end{smallmatrix}\right) \otimes \Big\{-(\mu_1-\alpha_1)(\mu_1-\alpha_2+1) f\left(M\left(\begin{smallmatrix} 1\,0 \\ 0 \end{smallmatrix}\right)\right) \\
&\qquad +\Big[C_2(M) \\
&\qquad\quad -\chi_-(M)(\mu_1-\alpha_1)(\mu_1-\alpha_2+1)\Big] f\left(M\left(\begin{smallmatrix} 1\,0 \\ 0 \end{smallmatrix}\right)[-1]\right) \\
&\qquad +\chi_-(M)C_2(M) f\left(M\left(\begin{smallmatrix} 1\,0 \\ 0 \end{smallmatrix}\right)[-2]\right)\Big\} \\
&+ f\left(\begin{smallmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\;-1 \\ -1 \end{smallmatrix}\right) \otimes \Big\{(\mu_1-\alpha_1)(\alpha_2-\mu_3) f\left(M\left(\begin{smallmatrix} 1\;-1 \\ 1 \end{smallmatrix}\right)\right) \\
&\qquad -\bar{F}(M) f\left(M\left(\begin{smallmatrix} 1\;-1 \\ 1 \end{smallmatrix}\right)[-1]\right) \\
&\qquad +\chi_+(M)C_2(M) f\left(M\left(\begin{smallmatrix} 1\;-1 \\ 1 \end{smallmatrix}\right)[-2]\right)\Big\} \\
&+ f\left(\begin{smallmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\;-1 \\ 0 \end{smallmatrix}\right) \otimes \Big\{-2(\mu_1-\alpha_1)(\alpha_2-\mu_3) f\left(M\left(\begin{smallmatrix} 1\;-1 \\ 0 \end{smallmatrix}\right)\right) \\
&\qquad +\Big[-(\mu_1-\alpha_1)(\alpha_2-\mu_3)-C_2(M)+\bar{E}(M) \\
&\qquad\quad -\chi_+(M)\delta(M)(\mu_1-\mu_3+1)\Big] f\left(M\left(\begin{smallmatrix} 1\;-1 \\ 0 \end{smallmatrix}\right)[-1]\right) \\
&\qquad -C_2(M) f\left(M\left(\begin{smallmatrix} 1\;-1 \\ 0 \end{smallmatrix}\right)[-2]\right)\Big\} \\
&+ f\left(\begin{smallmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 0\,0 \\ 0 \end{smallmatrix}\right) \otimes \Big\{-(\mu_1-\alpha_1)(\alpha_2-\mu_3) f\left(M\left(\begin{smallmatrix} 1\;-1 \\ 0 \end{smallmatrix}\right)\right) \\
&\qquad +\Big[-(\mu_1-\alpha_1)(\alpha_2-\mu_3)-C_2(M) \\
&\qquad\quad +(\mu_1-\alpha_2+1)(\mu_1-\alpha_1) \\
&\qquad\quad -\chi_+(M)\delta(M)(\mu_1-\mu_3+1)\Big] f\left(M\left(\begin{smallmatrix} 1\;-1 \\ 0 \end{smallmatrix}\right)[-1]\right) \\
&\qquad -2C_2(M) f\left(M\left(\begin{smallmatrix} 1\;-1 \\ 0 \end{smallmatrix}\right)[-2]\right)\Big\} \\
&+ f\left(\begin{smallmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\;-1 \\ 1 \end{smallmatrix}\right) \otimes \Big\{(\mu_1-\alpha_1)(\alpha_2-\mu_3) f\left(M\left(\begin{smallmatrix} 1\;-1 \\ -1 \end{smallmatrix}\right)\right) \\
&\qquad +\Big[-\bar{E}(M) \\
&\qquad\quad +\chi_-(M)(\mu_1-\alpha_1)(\alpha_2-\mu_3)\Big] f\left(M\left(\begin{smallmatrix} 1\;-1 \\ -1 \end{smallmatrix}\right)[-1]\right)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad -\chi_-(M)\bar{E}(M)f\left(M\left(\begin{smallmatrix}1\ -1\\ -1\end{smallmatrix}\right)[-2]\right)\Bigg\}\\
&+f\left(\begin{smallmatrix}\mathbf{e}_1-\mathbf{e}_3\\ 1\,0\\ 0\end{smallmatrix}\right)\otimes\Bigg\{(\mu_1-\alpha_1)(\alpha_2-\mu_3)f\left(M\left(\begin{smallmatrix}0\ -1\\ 0\end{smallmatrix}\right)\right)\\
&\qquad\qquad -\bar{F}(M)f\left(M\left(\begin{smallmatrix}0\ -1\\ 0\end{smallmatrix}\right)[-1]\right)\\
&\qquad\qquad +\chi_+(M)C_2(M)f\left(M\left(\begin{smallmatrix}0\ -1\\ 0\end{smallmatrix}\right)[-2]\right)\Bigg\}\\
&+f\left(\begin{smallmatrix}\mathbf{e}_1-\mathbf{e}_3\\ 1\,0\\ 1\end{smallmatrix}\right)\otimes\Bigg\{-(\mu_1-\alpha_1)(\alpha_2-\mu_3)f\left(M\left(\begin{smallmatrix}0\ -1\\ -1\end{smallmatrix}\right)\right)\\
&\qquad\qquad +\bar{E}(M)f\left(M\left(\begin{smallmatrix}0\ -1\\ -1\end{smallmatrix}\right)[-1]\right)\Bigg\}.
\end{aligned}
$$

Proof. If we put $M = L^{(1)}$ in the formula, then we have $\iota_i(f(L^{(1)})) = v_i$ for each i. Thus, to prove the formulas, it suffices to check that each of these injectors ι_i gives a $\mathfrak{gl}_3$-homomorphism. An essential part which we should check is to confirm the commutativity $\iota_i \cdot E_{kl} = E_{kl} \cdot \iota_i$ with the simple root vectors E_{kl} for $|k-l| = 1$. This is done by a direct but a long computation. We leave it for the reader. □

A generic irreducible representation τ_μ corresponding to a highest weight $\mu = (\mu_1, \mu_2, \mu_3) \in \Lambda$ has the 6 extremal weight vectors. Here an extremal weight means a weight given by permutations of μ. Each extremal weight vector is annihilated by the action of three different simple root vectors. If we evaluate the formulas in Theorem 4.4 at the G-patterns which give the extremal weight vectors in V_μ, all extremal vectors in $\iota_i(V_\mu)$ are obtained. The explicit description of the five extremal weight vectors except the highest weight vector in $\iota_i(V_\mu)$ is given as follows.

Corollary 4.5. *For $i = 1, 2$, let ι_i be the injectors $V_\mu \to V_{\mathbf{e}_1-\mathbf{e}_3} \otimes V_\mu$ defined in the above lemma.*

1. *For the G-pattern $L^{(2)} = \left(\begin{smallmatrix}\mu_1\,\mu_2\,\mu_3\\ \mu_1\,\mu_2\\ \mu_2\end{smallmatrix}\right) \in G(\mu)$ giving the extremal weight vector $f(L^{(2)})$ of weight (μ_2, μ_1, μ_3) in V_μ, we have*

$$
\begin{aligned}
v_1^{(2)} = \iota_1\left(f\left(L^{(2)}\right)\right) &= f\left(\begin{smallmatrix}\mathbf{e}_1-\mathbf{e}_3\\ 0\,0\\ 0\end{smallmatrix}\right)\otimes f\left(L^{(2)}\right) + f\left(\begin{smallmatrix}\mathbf{e}_1-\mathbf{e}_3\\ 1\,0\\ 1\end{smallmatrix}\right)\otimes f\left(L^{(2)}\left(\begin{smallmatrix}0\ -1\\ -1\end{smallmatrix}\right)\right)\\
&\quad -f\left(\begin{smallmatrix}\mathbf{e}_1-\mathbf{e}_3\\ 1\,0\\ 0\end{smallmatrix}\right)\otimes\left\{f\left(L^{(2)}\left(\begin{smallmatrix}0\ -1\\ 0\end{smallmatrix}\right)\right)+f\left(L^{(2)}\left(\begin{smallmatrix}-1\,0\\ 0\end{smallmatrix}\right)\right)\right\},\\
v_2^{(2)} = \iota_2\left(f\left(L^{(2)}\right)\right) &= f\left(\begin{smallmatrix}\mathbf{e}_1-\mathbf{e}_3\\ 1\ -1\\ -1\end{smallmatrix}\right)\otimes f\left(L^{(2)}\left(\begin{smallmatrix}0\,0\\ 1\end{smallmatrix}\right)\right)\\
&\quad -\left\{f\left(\begin{smallmatrix}\mathbf{e}_1-\mathbf{e}_3\\ 1\ -1\\ 0\end{smallmatrix}\right)+f\left(\begin{smallmatrix}\mathbf{e}_1-\mathbf{e}_3\\ 0\,0\\ 0\end{smallmatrix}\right)\right\}\otimes f\left(L^{(2)}\right)\\
&\quad +f\left(\begin{smallmatrix}\mathbf{e}_1-\mathbf{e}_3\\ 1\,0\\ 0\end{smallmatrix}\right)\otimes f\left(L^{(2)}\left(\begin{smallmatrix}-1\,0\\ 0\end{smallmatrix}\right)\right).
\end{aligned}
$$

2. *For the G-pattern* $L^{(3)}=\begin{pmatrix} \mu_1\,\mu_2\,\mu_3 \\ \mu_1\,\mu_3 \\ \mu_1 \end{pmatrix}\in G(\mu)$ *giving the extremal weight vector* $f(L^{(3)})$ *of weight* (μ_1,μ_3,μ_2) *in* V_μ*, we have*

$$\begin{aligned} v_1^{(3)}=\iota_1(f(L^{(3)})) &= f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 0\;-1 \\ 0 \end{pmatrix}\otimes f(L^{(3)}\begin{pmatrix} 0\,1 \\ 0 \end{pmatrix}) \\ &\quad -\left\{ f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\;-1 \\ 0 \end{pmatrix}+f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 0\,0 \\ 0 \end{pmatrix}\right\}\otimes f(L^{(3)}) \\ &\quad +f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\;-1 \\ 1 \end{pmatrix}\otimes f\left(L^{(3)}\begin{pmatrix} 0\,0 \\ -1 \end{pmatrix}\right), \\ v_2^{(3)}=\iota_2(f(L^{(3)})) &= f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\;-1 \\ 0 \end{pmatrix}\otimes f(L^{(3)})+f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\,0 \\ 1 \end{pmatrix}\otimes f\left(L^{(3)}\begin{pmatrix} -1\,0 \\ -1 \end{pmatrix}\right) \\ &\quad -f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\;-1 \\ 1 \end{pmatrix}\otimes\left\{ f\left(L^{(3)}\begin{pmatrix} 0\,0 \\ -1 \end{pmatrix}\right)+f\left(L^{(3)}\begin{pmatrix} -1\,1 \\ -1 \end{pmatrix}\right)\right\}. \end{aligned}$$

3. *For the G-pattern* $L^{(4)}=\begin{pmatrix} \mu_1\,\mu_2\,\mu_3 \\ \mu_1\,\mu_3 \\ \mu_3 \end{pmatrix}\in G(\mu)$ *giving the extremal weight vector* $f(L^{(4)})$ *of weight* (μ_3,μ_1,μ_2) *in* V_μ*, we have*

$$\begin{aligned} v_1^{(4)}=\iota_1(f(L^{(4)})) &= f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 0\;-1 \\ -1 \end{pmatrix}\otimes f\left(L^{(4)}\begin{pmatrix} 0\,1 \\ 1 \end{pmatrix}\right)+f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\;-1 \\ 0 \end{pmatrix}\otimes f\left(L^{(4)}\right) \\ &\quad -f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\;-1 \\ -1 \end{pmatrix}\otimes\left\{ f\left(L^{(4)}\begin{pmatrix} 0\,0 \\ 1 \end{pmatrix}\right)+f\left(L^{(4)}\begin{pmatrix} -1\,1 \\ 1 \end{pmatrix}\right)\right\}, \\ v_2^{(4)}=\iota_2(f(L^{(4)})) &= f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\;-1 \\ -1 \end{pmatrix}\otimes f(L^{(4)}\begin{pmatrix} 0\,0 \\ 1 \end{pmatrix}) \\ &\quad -\left\{ f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\;-1 \\ 0 \end{pmatrix}+f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 0\,0 \\ 0 \end{pmatrix}\right\}\otimes f(L^{(4)}) \\ &\quad +f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\,0 \\ 0 \end{pmatrix}\otimes f\left(L^{(4)}\begin{pmatrix} -1\,0 \\ 0 \end{pmatrix}\right). \end{aligned}$$

4. *For the G-pattern* $L^{(5)}=\begin{pmatrix} \mu_1\,\mu_2\,\mu_3 \\ \mu_2\,\mu_3 \\ \mu_2 \end{pmatrix}\in G(\mu)$ *giving the extremal weight vector* $f(L^{(5)})$ *of weight* (μ_2,μ_3,μ_1) *in* V_μ*, we have*

$$\begin{aligned} v_1^{(5)}=\iota_1(f(L^{(5)})) &= f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 0\;-1 \\ 0 \end{pmatrix}\otimes f(L^{(5)}\begin{pmatrix} 0\,1 \\ 0 \end{pmatrix}) \\ &\quad -\left\{ f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\;-1 \\ 0 \end{pmatrix}+f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 0\,0 \\ 0 \end{pmatrix}\right\}\otimes f(L^{(5)}) \\ &\quad +f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\;-1 \\ 1 \end{pmatrix}\otimes f\left(L^{(5)}\begin{pmatrix} 0\,0 \\ -1 \end{pmatrix}\right), \\ v_2^{(5)}=\iota_2(f(L^{(5)})) &= f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 0\;-1 \\ -1 \end{pmatrix}\otimes f\left(L^{(5)}\begin{pmatrix} 1\,0 \\ 1 \end{pmatrix}\right)+f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 0\,0 \\ 0 \end{pmatrix}\otimes f\left(L^{(5)}\right) \end{aligned}$$

$$-f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 0\ -1 \\ 0 \end{pmatrix} \otimes \left\{ f\left(L^{(5)}\begin{pmatrix} 1\,0 \\ 0 \end{pmatrix}\right) + f\left(L^{(5)}\begin{pmatrix} 0\,1 \\ 0 \end{pmatrix}\right)\right\}.$$

5. *For the G-pattern* $L^{(6)} = \begin{pmatrix} \mu_1\,\mu_2\,\mu_3 \\ \mu_2\,\mu_3 \\ \mu_3 \end{pmatrix} \in G(\mu)$ *giving the lowest weight vector* $f(L^{(6)})$ *of weight* (μ_3, μ_2, μ_1) *in* V_μ*, we have*

$$\begin{aligned} v_1^{(6)} = \iota_1(f(L^{(6)})) &= f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\ -1 \\ 0 \end{pmatrix} \otimes f(L^{(6)}) - f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 1\ -1 \\ -1 \end{pmatrix} \otimes f\left(L^{(6)}\begin{pmatrix} 0\,0 \\ 1 \end{pmatrix}\right) \\ &\quad + f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 0\ -1 \\ -1 \end{pmatrix} \otimes f\left(L^{(6)}\begin{pmatrix} 0\,1 \\ 1 \end{pmatrix}\right), \\ v_2^{(6)} = \iota_2(f(L^{(6)})) &= f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 0\,0 \\ 0 \end{pmatrix} \otimes f(L^{(6)}) - f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 0\ -1 \\ 0 \end{pmatrix} \otimes f\left(L^{(6)}\begin{pmatrix} 1\,0 \\ 0 \end{pmatrix}\right) \\ &\quad + f\begin{pmatrix} \mathbf{e}_1-\mathbf{e}_3 \\ 0\ -1 \\ -1 \end{pmatrix} \otimes f\left(L^{(6)}\begin{pmatrix} 1\,0 \\ 1 \end{pmatrix}\right). \end{aligned}$$

4.4. **The realization of τ_μ in $L^2(K)$.** Let (τ_μ, V_μ) be a representations of K associated with a highest weight $\mu = (\mu_1, \mu_2, \mu_3) \in \Lambda$. In this subsection, we give a natural construction of τ_μ of K in $L^2(K)$. To do this, it suffices to investigate $\tau_{(p,0,-q)}$ with $p = \mu_1 - \mu_2$ and $q = \mu_2 - \mu_3$ instead of τ_μ, since there is an isomorphism

$$\tau_\mu \simeq \tau_{(p,0,-q)} \otimes \det{}^{\mu_2}.$$

Here $\det{}^{\mu_2} = \tau_{(\mu_2,\mu_2,\mu_2)}$ is the character of $K = U(3)$ given by $X \mapsto (\det X)^{\mu_2}$.

First, we remark the following lemma which is easy to prove, say, utilizing the harmonic polynomial model (*cf.* [24] for example).

Lemma 4.6.

$$\tau_{(p,0,0)} \otimes \tau_{(0,0,-q)} \simeq \bigoplus_{i=0}^{\min\{p,q\}} \tau_{(p-i,0,-q+i)}.$$

In particular, $\tau_{(p,0,-q)}$ *occurs in* $\tau_{(p,0,0)} \otimes \tau_{(0,0,-q)}$ *with multiplicity one.*

Now we give a natural construction of the representation $\tau_{(p,0,-q)}$ in $L^2(K)$. In the tautological representation

$$K \ni k \mapsto s(k) = (s_{ij}(k))_{1\le i,j\le 3} \in U(3) \subset G$$

of K, we can consider each of the matrix coefficients s_{ij} as a L^2-function on K. Then, for each fixed $1 \le i \le 3$, the set $\{s_{i1}, s_{i2}, s_{i3}\}$ of the matrix coefficients generates a representation isomorphic to $\tau_{\mathbf{e}_1}$ in $L^2_{(M,\sigma_{\mathbf{e}_i})}(K)$. The correspondence to the GZ-basis is given as follows.

$$s_{i1} \leftrightarrow f\begin{pmatrix} \mathbf{e}_1 \\ 1\,0 \\ 1 \end{pmatrix}, \quad s_{i2} \leftrightarrow f\begin{pmatrix} \mathbf{e}_1 \\ 1\,0 \\ 0 \end{pmatrix}, \quad s_{i3} \leftrightarrow f\begin{pmatrix} \mathbf{e}_1 \\ 0\,0 \\ 0 \end{pmatrix}.$$

	$E_{12}^{\mathfrak{k}}$	$E_{21}^{\mathfrak{k}}$	$E_{23}^{\mathfrak{k}}$	$E_{32}^{\mathfrak{k}}$	$E_{13}^{\mathfrak{k}}$	$E_{31}^{\mathfrak{k}}$
s_{i1}	0	s_{i2}	0	0	0	s_{i3}
s_{i2}	s_{i1}	0	0	s_{i3}	0	0
s_{i3}	0	0	s_{i2}	0	s_{i1}	0

TABLE 2. Actions of the simple root vectors $E^{\mathfrak{k}}_{ij}$ on $\{s_{ij}(k)\}$.

Similarly, if we consider the matrix coefficients $\overline{s_{ij}}$ of the representation

$$K \ni k \mapsto \overline{s(k)} = (\overline{s_{ij}(k)})_{1\le i,j\le 3} \in U(3) \subset G,$$

as a L^2-function on K, the set $\{\overline{s_{i1}}, \overline{s_{i2}}, \overline{s_{i3}}\}$ generates a representation isomorphic to $\tau_{-\mathbf{e}_3}$ in $L^2_{(M,\sigma_{-\mathbf{e}_i})}(K)$ for each fixed i. The correspondence to the GZ-basis is given by

$$\overline{s_{i3}} \leftrightarrow f\begin{pmatrix} \begin{smallmatrix} -\mathbf{e}_3 \\ 0\,0 \\ 0 \end{smallmatrix} \end{pmatrix}, \quad -\overline{s_{i2}} \leftrightarrow f\begin{pmatrix} \begin{smallmatrix} -\mathbf{e}_3 \\ 0\,-1 \\ 0 \end{smallmatrix} \end{pmatrix}, \quad \overline{s_{i1}} \leftrightarrow f\begin{pmatrix} \begin{smallmatrix} -\mathbf{e}_3 \\ 0\,-1 \\ -1 \end{smallmatrix} \end{pmatrix}.$$

	$E^{\mathfrak{k}}_{12}$	$E^{\mathfrak{k}}_{21}$	$E^{\mathfrak{k}}_{23}$	$E^{\mathfrak{k}}_{32}$	$E^{\mathfrak{k}}_{13}$	$E^{\mathfrak{k}}_{31}$
$\overline{s_{i3}}$	0	0	0	$-\overline{s_{i2}}$	0	$-\overline{s_{i1}}$
$\overline{s_{i2}}$	0	$-\overline{s_{i1}}$	$-\overline{s_{i3}}$	0	0	0
$\overline{s_{i1}}$	$-\overline{s_{i2}}$	0	0	0	$-\overline{s_{i3}}$	0

TABLE 3. Actions of the simple root vectors $E^{\mathfrak{k}}_{ij}$ on $\{\overline{s_{ij}(k)}\}$.

Since we have the isomorphisms $\tau_{(p,0,0)} \simeq \mathrm{Sym}^p\tau_{\mathbf{e}_1}$ and $\tau_{(0,0,-q)} \simeq \mathrm{Sym}^q\tau_{-\mathbf{e}_3}$, the facts discussed above lead the following lemma immediately.

Lemma 4.7. *Let $p, q \in \mathbf{Z}_{\ge 0}$.*

1. *For each fixed $1 \le i \le 3$, the function $s_{i1}^p \in L^2_{(M,\sigma_{p\mathbf{e}_i})}(K)$ generates a representation isomorphic to $\tau_{(p,0,0)}$ by its right translations and becomes its highest weight vector.*
2. *For each fixed $1 \le i \le 3$, the function $\overline{s_{i3}}^q \in L^2_{(M,\sigma_{-q\mathbf{e}_i})}(K)$ generates a representation isomorphic to $\tau_{(0,0,-q)}$ by its right translations and becomes its highest weight vector.*
3. *For each fixed $1 \le i, j \le 3$ such that $i \ne j$, the function $s_{i1}^p\overline{s_{j3}}^q \in L^2_{(M,\sigma_{p\mathbf{e}_i-q\mathbf{e}_j})}(K)$ generates a representation isomorphic to $\tau_{(p,0,-q)}$ by its right translations and becomes its highest weight vector.*

In the above realization, the highest weight vector $f(L^{(1)})$ in $V_{(p,0,-q)}$ corresponds to the function $s_{i1}^p\overline{s_{j3}}^q$ in $L^2_{(M,\sigma_{p\mathbf{e}_i-q\mathbf{e}_j})}(K)$. The next lemma gives the correspondence between the extremal weight vectors $f(L^{(k)})$ for $1 \le k \le 6$ in $V_{(p,0,-q)}$ and the functions in $L^2_{(M,\sigma_{p\mathbf{e}_i-q\mathbf{e}_j})}(K)$ together with their neighbors.

Lemma 4.8. *Let $\mu = (p,0,-q)$, and for $1 \le k \le 6$ let $L^{(k)}$ be the G-patterns belonging to the highest weight μ defined in Lemma 4.3 and Lemma 4.5 which give the extremal vectors in V_μ. In the above embedding of V_μ in $L^2_{(M,\sigma_{p\mathbf{e}_i-q\mathbf{e}_j})}(K)$, we have the following correspondence with the GZ-basis.*

1.

$$f\left(L^{(1)}\right) \leftrightarrow s_{i1}^p\overline{s_{j3}}^q, \qquad f\left(L^{(1)}\begin{pmatrix}\begin{smallmatrix} 0\,0 \\ -1 \end{smallmatrix}\end{pmatrix}\right) \leftrightarrow s_{i1}^{p-1}s_{i2}\overline{s_{j3}}^q,$$

$$f\left(L^{(1)}\begin{pmatrix}\begin{smallmatrix} 0\,-1 \\ 0 \end{smallmatrix}\end{pmatrix}\right) \leftrightarrow -s_{i1}^p\overline{s_{j3}}^{q-1}\overline{s_{j2}}, \qquad f\left(L^{(1)}\begin{pmatrix}\begin{smallmatrix} 0\,-1 \\ -1 \end{smallmatrix}\end{pmatrix}\right) \leftrightarrow s_{i1}^p\overline{s_{j1}}\,\overline{s_{j3}}^{q-1},$$

$$f\left(L^{(1)}\begin{pmatrix}\begin{smallmatrix} -1\,0 \\ -1 \end{smallmatrix}\end{pmatrix}\right) \leftrightarrow s_{i1}^{p-1}s_{i3}\overline{s_{j3}}^q.$$

2.

$$f\left(L^{(2)}\right) \leftrightarrow s_{i2}^{p}\overline{s_{j3}}^{q}, \qquad f\left(L^{(2)}\begin{pmatrix} 0\;0 \\ 1 \end{pmatrix}\right) \leftrightarrow s_{i2}^{p-1}s_{i1}\overline{s_{j3}}^{q},$$
$$f\left(L^{(2)}\begin{pmatrix} 0\;-1 \\ -1 \end{pmatrix}\right) \leftrightarrow s_{i2}^{p}\overline{s_{j3}}^{q-1}\overline{s_{j1}}, \qquad f\left(L^{(2)}\begin{pmatrix} 0\;-1 \\ 0 \end{pmatrix}\right) \leftrightarrow s_{i2}^{p-1}s_{i1}\overline{s_{j3}}^{q-1}\,\overline{s_{j1}},$$
$$f\left(L^{(2)}\begin{pmatrix} -1\;0 \\ 0 \end{pmatrix}\right) \leftrightarrow s_{i2}^{p-1}s_{i3}\overline{s_{j3}}^{q}.$$

3.

$$f\left(L^{(3)}\right) \leftrightarrow s_{i1}^{p}(-\overline{s_{j2}})^{q}, \qquad f\left(L^{(3)}\begin{pmatrix} 0\;1 \\ 0 \end{pmatrix}\right) \leftrightarrow s_{i1}^{p}(-\overline{s_{j2}})^{q-1}\overline{s_{j3}},$$
$$f\left(L^{(3)}\begin{pmatrix} -1\;0 \\ -1 \end{pmatrix}\right) \leftrightarrow s_{i1}^{p-1}s_{i3}(-\overline{s_{j2}})^{q}, \qquad f\left(L^{(3)}\begin{pmatrix} 0\;0 \\ -1 \end{pmatrix}\right) \leftrightarrow s_{i1}^{p}(-\overline{s_{j2}})^{q-1}\overline{s_{j1}},$$
$$f\left(L^{(3)}\begin{pmatrix} -1\;1 \\ -1 \end{pmatrix}\right) \leftrightarrow s_{i1}^{p-1}s_{i3}(-\overline{s_{j2}})^{q-1}\overline{s_{j3}}.$$

4.

$$f\left(L^{(4)}\right) \leftrightarrow s_{i2}^{p}\overline{s_{j1}}^{q}, \qquad f\left(L^{(4)}\begin{pmatrix} 0\;1 \\ 1 \end{pmatrix}\right) \leftrightarrow s_{i2}^{p}\overline{s_{j1}}^{q-1}\overline{s_{j3}},$$
$$f\left(L^{(4)}\begin{pmatrix} -1\;0 \\ 0 \end{pmatrix}\right) \leftrightarrow s_{i2}^{p-1}s_{i3}\overline{s_{j1}}^{q}, \qquad f\left(L^{(4)}\begin{pmatrix} 0\;0 \\ 1 \end{pmatrix}\right) \leftrightarrow s_{i2}^{p-1}s_{i1}\overline{s_{j1}}^{q},$$
$$f\left(L^{(4)}\begin{pmatrix} -1\;1 \\ 1 \end{pmatrix}\right) \leftrightarrow s_{i2}^{p-1}s_{i3}\overline{s_{j1}}^{q-1}\overline{s_{j3}}.$$

5.

$$f\left(L^{(5)}\right) \leftrightarrow s_{i3}^{p}(-\overline{s_{j2}})^{q}, \qquad f\left(L^{(5)}\begin{pmatrix} 1\;0 \\ 1 \end{pmatrix}\right) \leftrightarrow s_{i3}^{p-1}s_{i1}(-\overline{s_{j2}})^{q},$$
$$f\left(L^{(5)}\begin{pmatrix} 0\;0 \\ -1 \end{pmatrix}\right) \leftrightarrow s_{i3}^{p}(-\overline{s_{j2}})^{q-1}\overline{s_{j1}}, \qquad f\left(L^{(5)}\begin{pmatrix} 1\;0 \\ 0 \end{pmatrix}\right) \leftrightarrow s_{i3}^{p-1}s_{i1}(-\overline{s_{j2}})^{q-1}\overline{s_{j1}},$$
$$f\left(L^{(5)}\begin{pmatrix} 0\;1 \\ 0 \end{pmatrix}\right) \leftrightarrow s_{i3}^{p}(-\overline{s_{j2}})^{q-1}\overline{s_{j3}}.$$

6.

$$f\left(L^{(6)}\right) \leftrightarrow s_{i3}^{p}\overline{s_{j1}}^{q}, \qquad f\left(L^{(6)}\begin{pmatrix} 1\;0 \\ 0 \end{pmatrix}\right) \leftrightarrow s_{i3}^{p-1}s_{i2}\overline{s_{j1}}^{q},$$
$$f\left(L^{(6)}\begin{pmatrix} 0\;0 \\ 1 \end{pmatrix}\right) \leftrightarrow s_{i3}^{p}\overline{s_{j1}}^{q-1}(-\overline{s_{j2}}), \qquad f\left(L^{(6)}\begin{pmatrix} 1\;0 \\ 1 \end{pmatrix}\right) \leftrightarrow s_{i3}^{p-1}s_{i1}\overline{s_{j1}}^{q},$$
$$f\left(L^{(6)}\begin{pmatrix} 0\;1 \\ 1 \end{pmatrix}\right) \leftrightarrow s_{i3}^{p}\overline{s_{j1}}^{q-1}\overline{s_{j3}}.$$

Proof. First we prove the correspondence in the assertion 1. From Lemma 4.1, we have

$$E_{21}^{\mathfrak{k}}f(L^{(1)}) = p\cdot f\left(L^{(1)}\begin{pmatrix} 0\;0 \\ -1 \end{pmatrix}\right), \quad E_{32}^{\mathfrak{k}}f(L^{(1)}) = q\cdot f\left(L^{(1)}\begin{pmatrix} 0\;-1 \\ 0 \end{pmatrix}\right).$$

On the other hand, by using the actions given in tables 2 and 3 we obtain

$$E_{21}^{\mathfrak{k}}(s_{i1}^{p}\overline{s_{j3}}^{q}) = p\cdot s_{i1}^{p-1}s_{i2}\overline{s_{j3}}^{q}, \quad E_{32}^{\mathfrak{k}}(s_{i1}^{p}\overline{s_{j3}}^{q}) = -q\cdot s_{i1}^{p}\overline{s_{j3}}^{q-1}\overline{s_{j2}}.$$

These give the second and the third correspondences in the assertion 1. The fourth and the fifth one are obtained by the equations

$$\begin{aligned} E_{31}^{\mathfrak{k}}f(L^{(1)}) &= -q\cdot f\left(L^{(1)}\begin{pmatrix} 0\;-1 \\ -1 \end{pmatrix}\right) + p\cdot f\left(L^{(1)}\begin{pmatrix} -1\;0 \\ -1 \end{pmatrix}\right), \\ E_{32}^{\mathfrak{k}}f\left(L^{(1)}\begin{pmatrix} 0\;0 \\ -1 \end{pmatrix}\right) &= -q\cdot f\left(L^{(1)}\begin{pmatrix} 0\;-1 \\ -1 \end{pmatrix}\right) + (-q+1)\cdot f\left(L^{(1)}\begin{pmatrix} -1\;0 \\ -1 \end{pmatrix}\right), \end{aligned}$$

from Lemma 4.1 and

$$E_{31}^{\mathfrak{k}}(s_{i1}^{p}\overline{s_{j3}}^{q}) = -q\cdot s_{i1}^{p}\overline{s_{j1}}\,\overline{s_{j3}}^{q-1} + p\cdot s_{i1}^{p-1}s_{i3}\overline{s_{j3}}^{q},$$

$$\begin{aligned} E^{\mathfrak{k}}_{32}(s_{i1}^{p-1}s_{i2}\overline{s_{j3}}^q) &= s_{i1}^{p-1}s_{i3}\overline{s_{j3}}^q + q\cdot s_{i1}^{p-1}s_{i2}\overline{s_{j2}}\,\overline{s_{j3}}^{q-1} \\ &= -q\cdot s_{i1}^{p}\overline{s_{j1}}\,\overline{s_{j3}}^{q-1} + (-q+1)\cdot s_{i1}^{p-1}s_{i3}\overline{s_{j3}}^q, \end{aligned}$$

from tables 2, 3 together with the relation

$$s_{i1}\overline{s_{j1}} + s_{i2}\overline{s_{j2}} + s_{i3}\overline{s_{j3}} = 0, \quad i \neq j,$$

which comes from the unitarity.

The correspondences given in the other assertions are obtained similarly, if the correspondences for the extremal weight vectors are given. Lemma 4.1 gives the following relations between the extremal weight vectors in V_μ.

$$\begin{array}{ll} \left(E^{\mathfrak{k}}_{21}\right)^p f\left(L^{(1)}\right) = p!f\left(L^{(2)}\right), & \left(E^{\mathfrak{k}}_{32}\right)^q f\left(L^{(1)}\right) = q!f\left(L^{(3)}\right), \\ \left(E^{\mathfrak{k}}_{31}\right)^q f\left(L^{(2)}\right) = q!(-1)^q f\left(L^{(4)}\right), & \left(E^{\mathfrak{k}}_{31}\right)^p f\left(L^{(3)}\right) = p!f\left(L^{(5)}\right), \\ \left(E^{\mathfrak{k}}_{32}\right)^p f\left(L^{(4)}\right) = p!f\left(L^{(6)}\right), & \left(E^{\mathfrak{k}}_{21}\right)^q f\left(L^{(5)}\right) = q!f\left(L^{(6)}\right). \end{array}$$

By considering the corresponding actions in $L^2_{(M,\sigma_{p\mathbf{e}_i-q\mathbf{e}_j})}(K)$, we have the correspondences for the extremal weight vectors in the assertions 2 to 5. □

5. $(\mathfrak{g}_\mathbb{C}, K)$-module structure

Let $\pi = \pi(\nu, \sigma_\mathbf{n})$ be an irreducible principal series representation with data $\nu = (\nu_1, \nu_2, \nu_3)$ and $\mathbf{n} = (n_1, n_2, n_3)$, and let $\tau^* = \tau_\mathbf{m}$ be the minimal K-type of π. Here $\mathbf{m} = (m_1, m_2, m_3) \in \Lambda$ is the dominant permutation of $\mathbf{n}$. In this subsection, we explain some equations for weight vectors in the minimal K-type $\tau_\mathbf{m}$ of π, which are determined from $(\mathfrak{g}_\mathbb{C}, K)$-module structure of π. Although we need only a partial result here, we can describe the whole $(\mathfrak{g}_\mathbb{C}, K)$-module structure of the principal series representation as in the case of $Sp(2,\mathbf{R})$ ([16]), $Sp(3,\mathbf{R})$ ([14]), and $SL(3,\mathbf{R})$ ([15]).

5.1. **Differential equations for generators of $Z(\mathfrak{g}_\mathbb{C})$.** For a Lie algebra $\mathfrak{l}$ over $\mathbf{C}$, let us denote the universal enveloping algebra of $\mathfrak{l}$ by $U(\mathfrak{l})$.

It is well known that an element $\mathcal{C}$ in the center $Z(\mathfrak{g}_\mathbb{C})$ of $U(\mathfrak{g}_\mathbb{C})$ acts as a scalar on the K-finite vectors in π. Thus, if we take an injection $j \in \mathrm{Hom}_K(\tau_\mathbf{m}, \pi|_K)$, then each element of the GZ-basis $\{f(M)\}_{M\in G(\mathbf{m})}$ of $V_\mathbf{m}$ satisfies the equation

$$\mathcal{C}\cdot j(f(M)) = \chi_\mathcal{C} j(f(M)), \tag{1}$$

for a scalar $\chi_\mathcal{C}$.

Now we construct a set of generators of $Z(\mathfrak{g}_\mathbb{C})$. To do this, we use the Capelli elements in $U(\mathfrak{g})$ given in the following lemma (*cf.* [8] §11).

Lemma 5.1. *Define three elements*

$$\begin{aligned} Cp_{1,\mathbb{R}} &= I_3, \\ Cp_{2,\mathbb{R}} &= (E_{11}-1)E_{22} + E_{22}(E_{33}+1) + (E_{11}-1)(E_{33}+1) \\ &\quad -E_{23}E_{32} - E_{13}E_{31} - E_{12}E_{21}, \\ Cp_{3,\mathbb{R}} &= (E_{11}-1)E_{22}(E_{33}+1) + E_{12}E_{23}E_{31} + E_{13}E_{21}E_{32} \\ &\quad -(E_{11}-1)E_{23}E_{32} - E_{13}E_{22}E_{31} - E_{12}E_{21}(E_{33}+1). \end{aligned}$$

in $U(\mathfrak{g})$. Then the set $\{Cp_{k,\mathbb{R}} \,|\, 1 \le k \le 3\}$ is a system of independent generators of $Z(\mathfrak{g})$.

The complexification $\mathfrak{g}_{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}$ can be identified with $\mathfrak{g} \oplus \mathfrak{g}$ in such way that $X \in \mathfrak{g}_{\mathbb{C}}$ corresponds to the element $X \oplus \bar{X}$, where $\bar{X}$ is the complex conjugate of X. Hence the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ of $\mathfrak{g}_{\mathbb{C}}$ is isomorphic to $U(\mathfrak{g}) \otimes_{\mathbb{C}} U(\mathfrak{g})$. From this identification and Lemma 5.1, we have the following lemma which gives a set of generators of $Z(\mathfrak{g}_{\mathbb{C}})$.

Lemma 5.2. *For $1 \le k \le 3$, put $Cp_k^{(1)} = Cp_{k,\mathbb{R}} \otimes 1$ and $Cp_k^{(2)} = 1 \otimes Cp_{k,\mathbb{R}}$ in $U(\mathfrak{g}) \otimes_{\mathbb{C}} U(\mathfrak{g})$. Then the set $\{Cp_k^{(i)} \,|1 \le i \le 2,\, 1 \le k \le 3\}$ gives a system of independent generators of $Z(\mathfrak{g}_{\mathbb{C}})$, considered as a subalgebra of $U(\mathfrak{g}) \otimes_{\mathbb{C}} U(\mathfrak{g})$.*

For the generators $Cp_k^{(i)}$ given in Lemma 5.2, we give their expression as the elements in $U(\mathfrak{g}_{\mathbb{C}})$.

Lemma 5.3. *As the elements in $U(\mathfrak{g}_{\mathbb{C}})$, the generators $Cp_k^{(i)}$ of $Z_(\mathfrak{g}_{\mathbb{C}})$ are given as follows.*

$$
\begin{aligned}
Cp_1^{(1)} &= \frac{1}{2}\left(I_3^{\mathfrak{p}} + I_3^{\mathfrak{k}}\right), \\
Cp_1^{(2)} &= \frac{1}{2}\left(I_3^{\mathfrak{p}} - I_3^{\mathfrak{k}}\right), \\
Cp_2^{(1)} &= \frac{1}{4}\Big\{\left(E_{11}^{\mathfrak{p}} + E_{11}^{\mathfrak{k}} - 2\right)\left(E_{22}^{\mathfrak{p}} + E_{22}^{\mathfrak{k}}\right) + \left(E_{22}^{\mathfrak{p}} + E_{22}^{\mathfrak{k}}\right)\left(E_{33}^{\mathfrak{p}} + E_{33}^{\mathfrak{k}} + 2\right) \\
&\quad + \left(E_{11}^{\mathfrak{p}} + E_{11}^{\mathfrak{k}} - 2\right)\left(E_{33}^{\mathfrak{p}} + E_{33}^{\mathfrak{k}} + 2\right) - \left(E_{23}^{\mathfrak{p}} + E_{23}^{\mathfrak{k}}\right)\left(E_{32}^{\mathfrak{p}} + E_{32}^{\mathfrak{k}}\right) \\
&\quad - \left(E_{13}^{\mathfrak{p}} + E_{13}^{\mathfrak{k}}\right)\left(E_{31}^{\mathfrak{p}} + E_{31}^{\mathfrak{k}}\right) - \left(E_{12}^{\mathfrak{p}} + E_{12}^{\mathfrak{k}}\right)\left(E_{21}^{\mathfrak{p}} + E_{21}^{\mathfrak{k}}\right)\Big\}, \\
Cp_2^{(2)} &= \frac{1}{4}\Big\{\left(E_{11}^{\mathfrak{p}} - E_{11}^{\mathfrak{k}} - 2\right)\left(E_{22}^{\mathfrak{p}} - E_{22}^{\mathfrak{k}}\right) + \left(E_{22}^{\mathfrak{p}} - E_{22}^{\mathfrak{k}}\right)\left(E_{33}^{\mathfrak{p}} - E_{33}^{\mathfrak{k}} + 2\right) \\
&\quad + \left(E_{11}^{\mathfrak{p}} - E_{11}^{\mathfrak{k}} - 2\right)\left(E_{33}^{\mathfrak{p}} - E_{33}^{\mathfrak{k}} + 2\right) - \left(E_{32}^{\mathfrak{p}} - E_{32}^{\mathfrak{k}}\right)\left(E_{23}^{\mathfrak{p}} - E_{23}^{\mathfrak{k}}\right) \\
&\quad - \left(E_{31}^{\mathfrak{p}} - E_{31}^{\mathfrak{k}}\right)\left(E_{13}^{\mathfrak{p}} - E_{13}^{\mathfrak{k}}\right) - \left(E_{21}^{\mathfrak{p}} - E_{21}^{\mathfrak{k}}\right)\left(E_{12}^{\mathfrak{p}} - E_{12}^{\mathfrak{k}}\right)\Big\}, \\
Cp_3^{(1)} &= \frac{1}{8}\Big\{\left(E_{11}^{\mathfrak{p}} + E_{11}^{\mathfrak{k}} - 2\right)\left(E_{22}^{\mathfrak{p}} + E_{22}^{\mathfrak{k}}\right)\left(E_{33}^{\mathfrak{p}} + E_{33}^{\mathfrak{k}} + 2\right) \\
&\quad + \left(E_{12}^{\mathfrak{p}} + E_{12}^{\mathfrak{k}}\right)\left(E_{23}^{\mathfrak{p}} + E_{23}^{\mathfrak{k}}\right)\left(E_{31}^{\mathfrak{p}} + E_{31}^{\mathfrak{k}}\right) \\
&\quad + \left(E_{13}^{\mathfrak{p}} + E_{13}^{\mathfrak{k}}\right)\left(E_{21}^{\mathfrak{p}} + E_{21}^{\mathfrak{k}}\right)\left(E_{32}^{\mathfrak{p}} + E_{32}^{\mathfrak{k}}\right) \\
&\quad - \left(E_{11}^{\mathfrak{p}} + E_{11}^{\mathfrak{k}} - 2\right)\left(E_{23}^{\mathfrak{p}} + E_{23}^{\mathfrak{k}}\right)\left(E_{32}^{\mathfrak{p}} + E_{32}^{\mathfrak{k}}\right) \\
&\quad - \left(E_{13}^{\mathfrak{p}} + E_{13}^{\mathfrak{k}}\right)\left(E_{22}^{\mathfrak{p}} + E_{22}^{\mathfrak{k}}\right)\left(E_{31}^{\mathfrak{p}} + E_{31}^{\mathfrak{k}}\right) \\
&\quad - \left(E_{12}^{\mathfrak{p}} + E_{12}^{\mathfrak{k}}\right)\left(E_{21}^{\mathfrak{p}} + E_{21}^{\mathfrak{k}}\right)\left(E_{33}^{\mathfrak{p}} + E_{33}^{\mathfrak{k}} + 2\right)\Big\}, \\
Cp_3^{(2)} &= \frac{1}{8}\Big\{\left(E_{11}^{\mathfrak{p}} - E_{11}^{\mathfrak{k}} - 2\right)\left(E_{22}^{\mathfrak{p}} - E_{22}^{\mathfrak{k}}\right)\left(E_{33}^{\mathfrak{p}} - E_{33}^{\mathfrak{k}} + 2\right) \\
&\quad + \left(E_{21}^{\mathfrak{p}} - E_{21}^{\mathfrak{k}}\right)\left(E_{32}^{\mathfrak{p}} - E_{32}^{\mathfrak{k}}\right)\left(E_{13}^{\mathfrak{p}} - E_{13}^{\mathfrak{k}}\right) \\
&\quad + \left(E_{31}^{\mathfrak{p}} - E_{31}^{\mathfrak{k}}\right)\left(E_{12}^{\mathfrak{p}} - E_{12}^{\mathfrak{k}}\right)\left(E_{23}^{\mathfrak{p}} - E_{23}^{\mathfrak{k}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \left(E^{\mathfrak{p}}_{11} - E^{\mathfrak{k}}_{11} - 2\right) \left(E^{\mathfrak{p}}_{32} - E^{\mathfrak{k}}_{32}\right) \left(E^{\mathfrak{p}}_{23} - E^{\mathfrak{k}}_{23}\right)\\
&- \left(E^{\mathfrak{p}}_{31} - E^{\mathfrak{k}}_{31}\right) \left(E^{\mathfrak{p}}_{22} - E^{\mathfrak{k}}_{22}\right) \left(E^{\mathfrak{p}}_{13} - E^{\mathfrak{k}}_{13}\right)\\
&- \left(E^{\mathfrak{p}}_{21} - E^{\mathfrak{k}}_{21}\right) \left(E^{\mathfrak{p}}_{12} - E^{\mathfrak{k}}_{12}\right) \left(E^{\mathfrak{p}}_{33} - E^{\mathfrak{k}}_{33} + 2\right)\Big\}.
\end{aligned}
$$

Proof. From the definition, the elements $E^{\mathfrak{k}}_{ij}$ and $E^{\mathfrak{p}}_{ij}$ in $\mathfrak{g}_{\mathbb{C}}$ correspond to the elements $E_{ij} \oplus (-E_{ji})$ and $E_{ij} \oplus E_{ji}$ in $\mathfrak{g} \oplus \mathfrak{g}$, respectively. Therefore, we have the correspondence between $E^{\mathfrak{p}}_{ij} + E^{\mathfrak{k}}_{ij}$ (resp. $E^{\mathfrak{p}}_{ij} - E^{\mathfrak{k}}_{ij}$) and $2E_{ij} \oplus 0$ (resp. $0 \oplus 2E_{ji}$). The assertion can be obtained from the above correspondences and the definition of the generators $Cp^{(i)}_k$ by direct computation. □

For each $\mathcal{C} = Cp^{(i)}_k$, the scalar value $\chi_{\mathcal{C}}$ in the equation (1) can be obtained by considering the evaluation of the left hand side at the identity.

Lemma 5.4.

$$
\begin{aligned}
\chi_{Cp^{(1)}_1} &= \frac{1}{2} \sum_{1\le i\le 3} (\nu_i + n_i), & \chi_{Cp^{(2)}_1} &= \frac{1}{2} \sum_{1\le i\le 3} (\nu_i - n_i),\\
\chi_{Cp^{(1)}_2} &= \frac{1}{4} \sum_{1\le i<j\le 3} (\nu_i + n_i)(\nu_j + n_j), & \chi_{Cp^{(2)}_2} &= \frac{1}{4} \sum_{1\le i<j\le 3} (\nu_i - n_i)(\nu_j - n_j),\\
\chi_{Cp^{(1)}_3} &= \frac{1}{8} \prod_{1\le i\le 3} (\nu_i + n_i), & \chi_{Cp^{(2)}_3} &= \frac{1}{8} \prod_{1\le i\le 3} (\nu_i - n_i).
\end{aligned}
$$

Proof. We evaluate the actions of $Cp^{(i)}_k$ on the representation space $L^2_{(M,\sigma_{\mathbf{n}})}(K)$ of $\pi = \pi(\nu, \sigma_{\mathbf{n}})$ at the identity using their expressions in Lemma 5.3. Then the elements $E^{\mathfrak{p}}_{11}$, $E^{\mathfrak{p}}_{22}$, and $E^{\mathfrak{p}}_{33}$ in $\mathfrak{a}$ act by the scalar $\nu_1 + 2$, ν_2, and $\nu_3 - 2$, respectively. Also, the elements $E^{\mathfrak{k}}_{11}$, $E^{\mathfrak{k}}_{22}$, and $E^{\mathfrak{k}}_{33}$ in $\mathfrak{m}_{\mathbb{C}}$ act by the scalar n_1, n_2, and n_3, respectively. Moreover, $E^{\mathfrak{p}}_{ij} + E^{\mathfrak{k}}_{ij} = E_{ij} - \sqrt{-1}E'_{ij}$ and $E^{\mathfrak{p}}_{ji} - E^{\mathfrak{k}}_{ji} = E_{ij} + \sqrt{-1}E'_{ij}$ belong to $\mathfrak{n}_{\mathbb{C}}$ for $i < j$ and thus their actions are zero. From the above facts, the eigen-values $\chi_{Cp^{(i)}_k}$ can be calculated as in the assertion. □

5.2. **The Dirac-Schmid eigen-equations.** For $i = 1, 2$, let ι_i be the injectors from $V_{\mathbf{m}}$ into $\mathfrak{p}_{0,\mathbb{C}} \otimes V_{\mathbf{m}} \simeq V_{\mathbf{e}_1-\mathbf{e}_3} \otimes V_{\mathbf{m}}$ defined in Lemma 4.4, and fix an injection $j \in \mathrm{Hom}_K(\tau_{\mathbf{m}}, \pi|_K)$. Since $\tau_{\mathbf{m}}$ occurs with multiplicity one in $\pi|_K$, the composition

$$
V_{\mathbf{m}} \xrightarrow{\iota_i} \mathfrak{p}_{0,\mathbb{C}} \otimes V_{\mathbf{m}} \xrightarrow{\alpha} \pi(\mathfrak{p}_{0,\mathbb{C}})j(V_{\mathbf{m}}) \subset L^2_{(M,\sigma_{\mathbf{n}})}(K)
$$

is a scalar multiple of j, where α is the evaluation map. Thus, if we write

$$
\iota_i(f(M)) = \sum_{M'\in G(\mathbf{m})} X^{(i)}_{M,M'} \otimes f(M'), \;\; X^{(i)}_{M,M'} \in \mathfrak{p}_{0,\mathbb{C}},
$$

for the GZ-basis $\{f(M)\}_{M\in G(\mathbf{m})}$ in $V_{\mathbf{m}}$, then we have the following system of equations

$$
\sum_{M'\in G(\mathbf{m})} X^{(i)}_{M,M'} \cdot j(f(M')) = \lambda_i j(f(M)), \quad M \in G(\mathbf{m}) \tag{2}
$$

for a scalar λ_i. We call this system of equations (2) the *Dirac-Schmid eigen-equations.* Here if $m_1 = m_2$ (resp. $m_2 = m_3$) then the Dirac Schmid eigen-equation (2) for $i = 2$ (resp. $i = 1$) is not valid.

The scalar values λ_i in the Dirac-Schmid eigen-equations (2) are given as follows.

Lemma 5.5. *If* $\mathbf{m} = (n_a, n_b, n_c)$*, then we have*

$$\lambda_1 = \nu_c - \frac{1}{3}\tilde{\nu}, \quad \lambda_2 = \nu_a - \frac{1}{3}\tilde{\nu}.$$

Here $\tilde{\nu} = \nu_1 + \nu_2 + \nu_3$.

Proof. Let us assume that $n_1 > n_2 > n_3$, i.e. $\mathbf{m} = \mathbf{n}$. Then the Dirac-Schmid equation (2) for $i = 1$ and the G-pattern $M = L^{(1)}$ becomes the following equation in $L^2_{(M,\sigma_{\mathbf{m}})}(K)$.

$$\begin{aligned} & \frac{1}{3}\left(H^{\mathfrak{p}}_{31} + H^{\mathfrak{p}}_{32}\right)\left(s_{11}^p \overline{s_{33}}^q \det(S)^{n_2}\right)(k) \\ & -E^{\mathfrak{p}}_{23}\left(-s_{11}^p \overline{s_{32}}\,\overline{s_{33}}^{q-1}\det(S)^{n_2}\right)(k) + E^{\mathfrak{p}}_{13}\left(s_{11}^p \overline{s_{31}}\,\overline{s_{33}}^{q-1}\det(S)^{n_2}\right)(k) \\ = \; & \lambda_1\left(s_{11}^p \overline{s_{33}}^q \det(S)^{n_2}\right)(k), \end{aligned}$$

with $p = n_a - n_b = n_1 - n_2$ and $q = n_b - n_c = n_2 - n_3$. Here we use the identification of $\mathfrak{p}_{0,\mathbb{C}}$ with $V_{\mathbf{e}_1-\mathbf{e}_3}$ in Lemma 4.2, the expression of the highest weight vector $v_1 = \iota_1\left(f\left(L^{(1)}\right)\right)$ in Lemma 4.3, and the correspondence between the GZ-basis of $V_{\mathbf{m}}$ and $L^2_{(M,\sigma_{\mathbf{n}})}(K)$ in Lemma 4.8. If we evaluate this equation at the identity $k = e$ after computing the actions of the elements in $U(\mathfrak{g}_{\mathbb{C}})$, we obtain $\lambda_1 = \nu_3 - \frac{1}{3}\tilde{\nu}$. Similarly we have $\lambda_2 = \nu_1 - \frac{1}{3}\tilde{\nu}$.

For the other cases of $\mathbf{m}$, the scalar-values λ_i can be obtained by evaluating the Dirac-Schmid equation (2) at the G-patterns corresponding to the other extremal weight vectors as in the following table. □

(a, b, c)	$(1, 2, 3)$	$(2, 1, 3)$	$(1, 3, 2)$	$(2, 3, 1)$	$(3, 1, 2)$	$(3, 2, 1)$
G-pattern M	$L^{(1)}$	$L^{(2)}$	$L^{(3)}$	$L^{(4)}$	$L^{(5)}$	$L^{(6)}$

TABLE 4.

For our later computation, we define λ_3 by the relation $\lambda_1 + \lambda_2 + \lambda_3 = 0$, that is, $\lambda_3 = \nu_b - \frac{1}{3}\tilde{\nu}$ if $\mathbf{m} = (n_a, n_b, n_c)$.

6. Whittaker realization

Let $\pi = \pi(\nu, \sigma_{\mathbf{n}})$ be an irreducible principal series representation with data $\nu = (\nu_1, \nu_2, \nu_3)$ and $\mathbf{n} = (n_1, n_2, n_3)$, and let $\tau^* = \tau_{\mathbf{m}}$ associated to the dominant permutation $\mathbf{m} = (m_1, m_2, m_3) \in \Lambda$ of $\mathbf{n}$ be the minimal K-type of π, as in the previous section. Moreover let $\eta = \eta_{c_1,c_2}$ be a non-degenerate unitary character of N specified by the parameters c_1 and c_2. In this section, we write the Whittaker realization, i.e. the realization in the space $C^\infty_\eta(N\backslash G)$, of the equations (1) and (2) explicitly.

6.1. **Preliminaries.** A Whittaker function $\phi \in \mathrm{Wh}(\pi, \eta, \tau) \subset C^\infty_{\eta,\tau}(N\backslash G/K)$ is expressed as

$$T(j(v^*))(g) = \langle v^*, \phi(g)\rangle, \quad v^* \in V_{\tau^*}, \;\; g \in G,$$

with an intertwining operator $T \in \mathcal{I}_{\pi,\eta}$ and an injective K-homomorphism $j \in \mathrm{Hom}_K(\tau^*, \pi|_K)$, by definition. Now, for each G-pattern $M \in G(\mathbf{m})$ belonging to $\mathbf{m}$, we define a function $\phi(M)$ in $C^\infty_\eta(N\backslash G)$ by taking the element $f(M)$ of the GZ-basis $\{f(M)\}_{M\in G(\mathbf{m})}$ for $V_{\mathbf{m}}$ as $v^*\in V_{\tau^*}=V_{\mathbf{m}}$ in the above equation, that is,

$$\phi(M;g)=T(j(f(M)))(g)=\langle f(M),\phi(g)\rangle,\quad g\in G.$$

We call this function $\phi(M)$ the *M-component of a Whittaker function ϕ*.

Whittaker functions are determined by its A-radial parts (i.e. its restriction to A) because of the Iwasawa decomposition of G. Moreover, the values of Whittaker functions on the center Z_G of G are given by the central character of π, i.e.,

$$\phi(rug)=r^{\tilde\nu}u^{\tilde n}\phi(g),\quad \phi\in \mathrm{Wh}(\pi,\eta,\tau),\ \ r\in\mathbf{R}_{>0},\ \ u\in U(1),\ \ g\in G.$$

Therefore, we can describe Whittaker functions as functions of two variables with the coordinates

$$y_1=\frac{a_1}{a_2},\quad y_2=\frac{a_2}{a_3}$$

for $\mathrm{diag}\,(a_1,a_2,a_3)=a_3\cdot\mathrm{diag}\,(y_1y_2,y_2,1)\in A$, which correspond to simple roots of $(\mathfrak{a},\mathfrak{g})$. Also, we denote the Euler operator with respect to y_i by $\partial_i=y_i\dfrac{\partial}{\partial y_i}$.

6.2. **Differential equations.** Let $\phi\in\mathrm{Wh}(\pi,\eta,\tau)$ be a Whittaker function determined by an intertwining operator $T\in\mathcal{I}_{\pi,\tau}$ and an injection $j\in\mathrm{Hom}_K(\tau^*,\pi|_K)$ and $\phi(M)$ be its M-component. For each $M\in G(\mathbf{m})$, we consider the image of both side of the equation (1) by T;

$$T(\mathcal{C}\cdot j(f(M)))(g)=T(\chi_{\mathcal{C}}j(f(M)))(g),\quad g\in G.$$

Then the intertwining property of T leads the differential equation

$$\mathcal{C}\phi(M;y)=\chi_{\mathcal{C}}\phi(M;y),\quad y=(y_1,y_2). \tag{3}$$

for the A-radial part of $\phi(M)$. Similarly, the Dirac-Schmid eigen-equation (2) leads the differential equation

$$\sum_{M'\in G(\mathbf{m})}X^{(i)}_{M,M'}\phi(M';y)=\lambda_i\phi(M;y),\quad y=(y_1,y_2). \tag{4}$$

In this subsection, we write these equations (3) and (4), explicitly.

First, we observe the following fundamental lemmas.

Lemma 6.1. *Let $f\in C^\infty_{\eta,\tau}(N\backslash G/K)$. For $X\in U(\mathfrak{k}_{\mathbb{C}})$, $Y\in U(\mathfrak{n}_{\mathbb{C}})$, $Z\in U(\mathfrak{a}_{\mathbb{C}})$, and $a\in A$, we have* $(\mathrm{Ad}\,(a^{-1})Y)ZXf(a)=\eta(Y)\tau(-X)(Zf)(a)$.

Lemma 6.2. *Let $\phi=\phi(y)\in\mathrm{Wh}(\pi,\eta,\tau)|_A$.*

1. *The actions of elements $H^{\mathfrak{p}}_{12}$, $H^{\mathfrak{p}}_{23}$, and $I^{\mathfrak{p}}_3$ in $\mathfrak{a}_{\mathbb{C}}$ on ϕ are the following differentials.*

$$H^{\mathfrak{p}}_{12}\phi=(2\partial_1-\partial_2)\phi,\ \ H^{\mathfrak{p}}_{23}\phi=(-\partial_1+2\partial_2)\phi,\ \ I^{\mathfrak{p}}_3\phi=\tilde\nu\phi.$$

Thus, for $E^{\mathfrak{p}}_{ii}$ we have

$$E^{\mathfrak{p}}_{11}\phi=\left(\partial_1+\frac{\tilde\nu}{3}\right)\phi,\ \ E^{\mathfrak{p}}_{22}\phi\left(-\partial_1+\partial_2+\frac{\tilde\nu}{3}\right)\phi,\ \ E^{\mathfrak{p}}_{33}\phi=\left(-\partial_2+\frac{\tilde\nu}{3}\right)\phi.$$

2. *The actions of elements $E^{\mathfrak{p}}_{ij}+E^{\mathfrak{k}}_{ij}$ and $E^{\mathfrak{p}}_{ji}-E^{\mathfrak{k}}_{ji}$ with $i<j$ in $\mathfrak{n}_{\mathbb{C}}$ on ϕ are the following multiplications.*
$$\left(E^{\mathfrak{p}}_{21}-E^{\mathfrak{k}}_{21}\right)\phi=2\pi\sqrt{-1}c_1y_1\phi,\ \ \left(E^{\mathfrak{p}}_{12}+E^{\mathfrak{k}}_{12}\right)\phi=2\pi\sqrt{-1}\bar{c}_1y_1\phi,$$
$$\left(E^{\mathfrak{p}}_{32}-E^{\mathfrak{k}}_{32}\right)\phi=2\pi\sqrt{-1}c_2y_2\phi,\ \ \left(E^{\mathfrak{p}}_{23}+E^{\mathfrak{k}}_{23}\right)\phi=2\pi\sqrt{-1}\bar{c}_2y_2\phi,$$
and $\left(E^{\mathfrak{p}}_{31}-E^{\mathfrak{k}}_{31}\right)\phi=\left(E^{\mathfrak{p}}_{13}+E^{\mathfrak{k}}_{13}\right)\phi=0.$

The proof is omitted (*cf.* [13]).

By using the above lemmas together with Lemma 4.1 which gives the actions of elements $E^{\mathfrak{k}}_{ij}$ in $\mathfrak{k}_{\mathbb{C}}$, the following explicit description of the equation (3) is obtained from Lemma 5.3 and Lemma 5.4.

Proposition 6.3. *Let $\phi(M)$ be the M-component of a Whittaker function $\phi\in\mathrm{Wh}(\pi,\eta,\tau)$ and put $\phi(M;y)=y_1^2y_2^2\tilde{\phi}(M;y)$. Then the differential equations (3) for the Capelli elements $\mathcal{C}=Cp^{(i)}_k$ with $k=2,3$ and $i=1,2$ are given as follows: Let $(w_1,w_2,w_3)=(\beta,\alpha_1+\alpha_2-\beta,m_1+m_2+m_3-\alpha_1-\alpha_2)$ be the weight of a G-pattern $M=\begin{pmatrix} m_1\,m_2\,m_3 \\ \alpha_1\,\alpha_2 \\ \beta\end{pmatrix}$.*

1. *For $\mathcal{C}=Cp^{(1)}_2$, we have*
$$\begin{aligned}&\Big[\Big(\partial_1+\frac{\tilde{\nu}}{3}+w_1\Big)\Big(-\partial_1+\partial_2+\frac{\tilde{\nu}}{3}+w_2\Big)\\&\quad+\Big(-\partial_1+\partial_2+\frac{\tilde{\nu}}{3}+w_2\Big)\Big(-\partial_2+\frac{\tilde{\nu}}{3}+w_3\Big)\\&\quad+\Big(\partial_1+\frac{\tilde{\nu}}{3}+w_1\Big)\Big(-\partial_2+\frac{\tilde{\nu}}{3}+w_3\Big)\\&\quad-\left(2\pi\sqrt{-1}\right)^2\left(|c_1|^2y_1^2+|c_2|^2y_2^2\right)-\sum_{1\le i<j\le 3}(\nu_i+n_i)(\nu_j+n_j)\Big]\tilde{\phi}(M;y)\\&-4\pi\sqrt{-1}\bar{c}_2y_2\Big\{(\alpha_2-m_3)\,\tilde{\phi}\Big(M\begin{pmatrix}0\,-1\\0\end{pmatrix};y\Big)\\&\qquad\qquad+\chi_+(M)\,(\alpha_2-m_3+\delta(M))\,\tilde{\phi}\Big(M\begin{pmatrix}-1\,0\\0\end{pmatrix};y\Big)\Big\}\\&-4\pi\sqrt{-1}\bar{c}_1y_1\Big\{(\beta-\alpha_2)\tilde{\phi}\Big(M\begin{pmatrix}0\,0\\-1\end{pmatrix};y\Big)\\&\qquad\qquad+\chi_-(M)(\alpha_1-m_2)\tilde{\phi}\Big(M\begin{pmatrix}-1\,1\\-1\end{pmatrix};y\Big)\Big\}=0.\end{aligned}$$

2. *For $\mathcal{C}=Cp^{(2)}_2$, we have*
$$\begin{aligned}&\Big[\Big(\partial_1+\frac{\tilde{\nu}}{3}-w_1\Big)\Big(-\partial_1+\partial_2+\frac{\tilde{\nu}}{3}-w_2\Big)\\&\quad+\Big(-\partial_1+\partial_2+\frac{\tilde{\nu}}{3}-w_2\Big)\Big(-\partial_2+\frac{\tilde{\nu}}{3}-w_3\Big)\end{aligned}$$

$$
\begin{aligned}
&+\left(\partial_1+\frac{\tilde{\nu}}{3}-w_1\right)\left(-\partial_2+\frac{\tilde{\nu}}{3}-w_3\right)\\
&-\left(2\pi\sqrt{-1}\right)^2\left(|c_1|^2y_1^2+|c_2|^2y_2^2\right)-\sum_{1\le i<j\le 3}(\nu_i-n_i)(\nu_j-n_j)\Bigg]\tilde{\phi}(M;y)\\
&+4\pi\sqrt{-1}c_2y_2\Bigg\{(m_1-\alpha_1)\tilde{\phi}\left(M\left(\begin{smallmatrix} & 1 & 0\\ & & 0\end{smallmatrix}\right);y\right)\\
&\qquad+\chi_-(M)(m_1-\alpha_1-\delta(M))\tilde{\phi}\left(M\left(\begin{smallmatrix} & 0 & 1\\ & & 0\end{smallmatrix}\right);y\right)\Bigg\}\\
&+4\pi\sqrt{-1}c_1y_1\Bigg\{(\alpha_1-\beta)\tilde{\phi}\left(M\left(\begin{smallmatrix} & 0 & 0\\ & & 1\end{smallmatrix}\right);y\right)\\
&\qquad+\chi_+(M)(m_2-\alpha_2)\tilde{\phi}\left(M\left(\begin{smallmatrix} & -1 & 1\\ & & 1\end{smallmatrix}\right);y\right)\Bigg\}=0.
\end{aligned}
$$

3. *For* $\mathcal{C}=Cp_3^{(1)}$, *we have*

$$
\begin{aligned}
&\Bigg[\left(\partial_1+\frac{\tilde{\nu}}{3}+w_1\right)\left(-\partial_1+\partial_2+\frac{\tilde{\nu}}{3}+w_2\right)\left(-\partial_2+\frac{\tilde{\nu}}{3}+w_3\right)\\
&\quad-\left(2\pi\sqrt{-1}|c_2|y_2\right)^2\left(\partial_1+\frac{\tilde{\nu}}{3}+w_1\right)-\left(2\pi\sqrt{-1}|c_1|y_1\right)^2\left(-\partial_2+\frac{\tilde{\nu}}{3}+w_3\right)\\
&\quad-\prod_{1\le i\le 3}(\nu_i+n_i)\Bigg]\tilde{\phi}(M;y)\\
&+2\cdot 2\pi\sqrt{-1}\bar{c}_1y_1\cdot 2\pi\sqrt{-1}\bar{c}_2y_2\\
&\qquad\Bigg\{-(\alpha_2-m_3)\tilde{\phi}\left(M\left(\begin{smallmatrix} & 0 & -1\\ & & -1\end{smallmatrix}\right);y\right)+C_1(M)\tilde{\phi}\left(M\left(\begin{smallmatrix} & -1 & 0\\ & & -1\end{smallmatrix}\right);y\right)\Bigg\}\\
&-2\cdot 2\pi\sqrt{-1}\bar{c}_2y_2\left(\partial_1+\frac{\tilde{\nu}}{3}+w_1\right)\\
&\qquad\Bigg\{(\alpha_2-m_3)\tilde{\phi}\left(M\left(\begin{smallmatrix} & 0 & -1\\ & & 0\end{smallmatrix}\right);y\right)+\chi_+(M)(\alpha_2-m_3+\delta(M))\tilde{\phi}\left(M\left(\begin{smallmatrix} & -1 & 0\\ & & 0\end{smallmatrix}\right);y\right)\Bigg\}\\
&-2\cdot 2\pi\sqrt{-1}\bar{c}_1y_1\left(-\partial_2+\frac{\tilde{\nu}}{3}+w_3\right)\\
&\qquad\Bigg\{(\beta-\alpha_2)\tilde{\phi}\left(M\left(\begin{smallmatrix} & 0 & 0\\ & & -1\end{smallmatrix}\right);y\right)+\chi_-(M)(\alpha_1-m_2)\tilde{\phi}\left(M\left(\begin{smallmatrix} & -1 & 1\\ & & -1\end{smallmatrix}\right);y\right)\Bigg\}=0.
\end{aligned}
$$

4. *For* $\mathcal{C}=Cp_3^{(2)}$, *we have*

$$
\begin{aligned}
&\Bigg[\left(\partial_1+\frac{\tilde{\nu}}{3}-w_1\right)\left(-\partial_1+\partial_2+\frac{\tilde{\nu}}{3}-w_2\right)\left(-\partial_2+\frac{\tilde{\nu}}{3}-w_3\right)\\
&\quad-\left(2\pi\sqrt{-1}|c_2|y_2\right)^2\left(\partial_1+\frac{\tilde{\nu}}{3}-w_1\right)-\left(2\pi\sqrt{-1}|c_1|y_1\right)^2\left(-\partial_2+\frac{\tilde{\nu}}{3}-w_3\right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \prod_{1\le i\le 3}(\nu_i - n_i)\Big]\tilde{\phi}(M;y) \\
&-2\cdot 2\pi\sqrt{-1}c_1y_1\cdot 2\pi\sqrt{-1}c_2y_2 \\
&\qquad \left\{(m_1-\alpha_1)\tilde{\phi}\left(M\left(\begin{smallmatrix}1\,0\\1\end{smallmatrix}\right);y\right) - \bar{C}_1(M)\tilde{\phi}\left(M\left(\begin{smallmatrix}0\,1\\1\end{smallmatrix}\right);y\right)\right\} \\
&+2\cdot 2\pi\sqrt{-1}c_2y_2\left(\partial_1+\frac{\tilde{\nu}}{3}-w_1\right) \\
&\qquad \left\{(m_1-\alpha_1)\tilde{\phi}\left(M\left(\begin{smallmatrix}1\,0\\0\end{smallmatrix}\right);y\right) + \chi_-(M)(m_1-\alpha_1-\delta(M))\tilde{\phi}\left(M\left(\begin{smallmatrix}0\,1\\0\end{smallmatrix}\right);y\right)\right\} \\
&+2\cdot 2\pi\sqrt{-1}c_1y_1\left(-\partial_2+\frac{\tilde{\nu}}{3}-w_3\right) \\
&\qquad \left\{(\alpha_1-\beta)\tilde{\phi}\left(M\left(\begin{smallmatrix}0\,0\\1\end{smallmatrix}\right);y\right) + \chi_+(M)(m_2-\alpha_2)\tilde{\phi}\left(M\left(\begin{smallmatrix}-1\,1\\1\end{smallmatrix}\right);y\right)\right\} = 0.
\end{aligned}
$$

If we evaluate the above equations from $Cp_k^{(2)}$ with $k=2,3$ at the G-pattern $L^{(1)} = \left(\begin{smallmatrix} m_1\, m_2\, m_3 \\ m_1\, m_2 \\ m_1\end{smallmatrix}\right)$ associated with the highest weight vector $f(L^{(1)})$ in $V_{\mathbf{m}}$, we obtain the following system of differential equations for the $L^{(1)}$-component of Whittaker functions.

Corollary 6.4. *Let $\phi(L^{(1)})$ be the $L^{(1)}$-component of a Whittaker function $\phi \in$ $\mathrm{Wh}(\pi,\eta,\tau)$. Then the function $\tilde{\phi}(L^{(1)}) = y_1^{-2}y_2^{-2}\phi(L^{(1)})$ satisfies the following two differential equations.*

1.

$$
\begin{aligned}
&\Big[\partial_1^2+\partial_2^2-\partial_1\partial_2-p(\partial_1-\lambda_2)-q(\partial_2+\lambda_1) \\
&\quad +(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1) \\
&\quad +\left(2\pi\sqrt{-1}\right)^2\left(|c_1|^2y_1^2+|c_2|^2y_2^2\right)\Big]\tilde{\phi}(L^{(1)};y)=0.
\end{aligned}
$$

Here $p = m_1-m_2$ and $q = m_2-m_3$.

2.

$$
\begin{aligned}
&\Big[\left(\partial_1+\frac{\tilde{\nu}}{3}-m_1\right)\left(-\partial_1+\partial_2+\frac{\tilde{\nu}}{3}-m_2\right)\left(-\partial_2+\frac{\tilde{\nu}}{3}-m_3\right) \\
&\quad -\left(2\pi\sqrt{-1}|c_2|y_2\right)^2\left(\partial_1+\frac{\tilde{\nu}}{3}-m_1\right)-\left(2\pi\sqrt{-1}|c_1|y_1\right)^2\left(-\partial_2+\frac{\tilde{\nu}}{3}-m_3\right) \\
&\quad -\left(\lambda_2+\frac{\tilde{\nu}}{3}-m_1\right)\left(\lambda_3+\frac{\tilde{\nu}}{3}-m_2\right)\left(\lambda_1+\frac{\tilde{\nu}}{3}-m_3\right)\Big]\tilde{\phi}(L^{(1)};y)=0.
\end{aligned}
$$

Proof. In the equations 2 and 4 in the above proposition evaluated at $M = L^{(1)}$, all terms in the left hand side except the $L^{(1)}$-component $\tilde{\phi}(L^{(1)})$ vanish, since the highest weight vector $f(L^{(1)})$ in $V_{\mathbf{m}}$ satisfies $E_{ij}^{\mathfrak{k}}f(L^{(1)}) = 0$ with $i<j$. Then direct

computation leads the equations in the corollary. Here we remark the equations

$$\begin{aligned}
&\sum_{i<j}\left(\frac{\tilde{\nu}}{3}-m_i\right)\left(\frac{\tilde{\nu}}{3}-m_j\right)-\sum_{i<j}(\nu_i-n_i)(\nu_j-n_j)\\
=\ &-(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)-p\lambda_2+q\lambda_1,
\end{aligned}$$

and

$$\prod_i(\nu_i-n_i)=\left(\lambda_2-m_1+\frac{\tilde{\nu}}{3}\right)\left(\lambda_3-m_2+\frac{\tilde{\nu}}{3}\right)\left(\lambda_1-m_3+\frac{\tilde{\nu}}{3}\right),$$

which can be shown by using the definition of λ_i in Lemma 5.5. $\square$

Similarly to the equation (3), we can describe the explicit form of the Dirac-Schmid eigen-equation (4) for each G-pattern M. However we need only the following partial result in our later discussion.

Proposition 6.5. *Let $\phi(M)$ be the M-component of a Whittaker function $\phi\in\mathrm{Wh}(\pi,\eta,\tau)$ and put $\phi(M;y)=y_1^2y_2^2\tilde{\phi}(M;y)$.*

1. *If $m_2\neq m_3$, the Dirac-Schmid eigen-equation (4) for $i=1$ at $M=L^{(1)}\left(\begin{smallmatrix}0\ 0\\ -k\end{smallmatrix}\right)$ with $0\le k\le p=m_1-m_2$ is given by*
$$\begin{aligned}
&(\partial_2+\lambda_1)\tilde{\phi}(M;y)\\
=\ &-2\pi\sqrt{-1}\bar{c}_2y_2\left\{\tilde{\phi}\left(M\left(\begin{smallmatrix}0\ -1\\ 0\end{smallmatrix}\right);y\right)+\chi_+(M)\tilde{\phi}\left(M\left(\begin{smallmatrix}-1\ 0\\ 0\end{smallmatrix}\right);y\right)\right\}.
\end{aligned}$$
2. *If $m_1\neq m_2$, the Dirac-Schmid eigen-equation (4) for $i=2$ at $M=L^{(1)}\left(\begin{smallmatrix}0\ -k\\ 0\end{smallmatrix}\right)$ with $0\le k\le q=m_2-m_3$ is given by*
$$\begin{aligned}
&(\partial_1-\lambda_2)\tilde{\phi}(M;y)\\
=\ &-2\pi\sqrt{-1}\bar{c}_1y_1\left\{\tilde{\phi}\left(M\left(\begin{smallmatrix}0\ 0\\ -1\end{smallmatrix}\right);y\right)+\chi_-(M)\tilde{\phi}\left(M\left(\begin{smallmatrix}-1\ 1\\ -1\end{smallmatrix}\right);y\right)\right\}.
\end{aligned}$$

Proof. Assume $m_2\neq m_3$. If we evaluate the formula 1 of the injector ι_1 in Theorem 4.4 at $M=L^{(1)}\left(\begin{smallmatrix}0\ 0\\ -k\end{smallmatrix}\right)$, then we have

$$\begin{aligned}
\iota_1(f(M))\ =\ &f\left(\begin{smallmatrix}\mathbf{e}_1-\mathbf{e}_3\\ 0\ 0\\ 0\end{smallmatrix}\right)\otimes f(M)\\
&-f\left(\begin{smallmatrix}\mathbf{e}_1-\mathbf{e}_3\\ 1\ 0\\ 0\end{smallmatrix}\right)\otimes\left\{f\left(M\left(\begin{smallmatrix}0\ -1\\ 0\end{smallmatrix}\right)\right)+\chi_+(M)f\left(M\left(\begin{smallmatrix}-1\ 0\\ 0\end{smallmatrix}\right)\right)\right\}\\
&+f\left(\begin{smallmatrix}\mathbf{e}_1-\mathbf{e}_3\\ 1\ 0\\ 1\end{smallmatrix}\right)\otimes f\left(M\left(\begin{smallmatrix}0\ -1\\ -1\end{smallmatrix}\right)\right).
\end{aligned}$$

By using the correspondence between $V_{\mathbf{e}_1-\mathbf{e}_3}$ and $\mathfrak{p}_{0,\mathbb{C}}$ in Lemma 4.2 and the fundamental lemmas on the actions of $U(\mathfrak{g}_{\mathbb{C}})$ on the space of Whittaker functions given in the top of this subsection, the above injection formula leads the following equation for the M-components of a Whittaker function $\phi\in\mathrm{Wh}(\pi,\eta,\tau)$.

$$\begin{aligned}
&(\partial_2+\lambda_1)\,\phi(M;y)\\
=\ &-\left(2\phi\sqrt{-1}\bar{c}_2y_2-E_{23}^{\mathfrak{k}}\right)\left\{\phi\left(M\left(\begin{smallmatrix}0\ -1\\ 0\end{smallmatrix}\right);y\right)+\chi_+(M)\phi\left(M\left(\begin{smallmatrix}-1\ 0\\ 0\end{smallmatrix}\right);y\right)\right\}
\end{aligned}$$

$$-E^{\mathfrak{k}}_{13}\phi\left(M\left(\begin{smallmatrix}0\ -1\\ -1\end{smallmatrix}\right);y\right).$$

Thus we have the equation in the assertion 1, because of the equations

$$E^{\mathfrak{k}}_{23}\left\{f\left(M\left(\begin{smallmatrix}0\ -1\\ 0\end{smallmatrix}\right)\right)+\chi_{+}(M)f\left(M\left(\begin{smallmatrix}-1\,0\\ 0\end{smallmatrix}\right)\right)\right\}=-E^{\mathfrak{k}}_{13}f\left(M\left(\begin{smallmatrix}0\ -1\\ -1\end{smallmatrix}\right)\right)=f(M),$$

which are obtained from Lemma 4.1. The equation in 2 can be shown similarly. □

7. Explicit formulas

Let us take $\pi=\pi(\nu,\sigma_{\mathbf{n}})$, $\tau^{*}=\tau_{\mathbf{m}}$, and $\eta=\eta_{c_1,c_2}$ as in the previous section. In this section, we discuss explicit descriptions for the Whittaker functions with respect to (π,η,τ) which is our main theme in this paper.

7.1. **Preliminaries.** Let $\phi\in\mathrm{Wh}(\pi,\eta,\tau)$ be a Whittaker function with respect to (π,η,τ). Then the set $\{\phi(M)\}_{M\in G(\mathbf{m})}$ of M-components of ϕ satisfies the system of equations (3) and (4) in §6.2. Before studying explicit formulas, we observe the following lemma concerning this system of equations.

Lemma 7.1. *A Whittaker function $\phi\in\mathrm{Wh}(\pi,\eta,\tau)$ is determined by its $L^{(1)}$-component $\phi(L^{(1)})$. That is, all M-components $\phi(M)$ of $\phi\in\mathrm{Wh}(\pi,\eta,\tau)$ are uniquely determined from $\phi(L^{(1)})$ by the equations in Proposition 6.3 and Proposition 6.5.*

Proof. To prove this assertion, we may give an effective procedure for determining all M-components $\tilde{\phi}(M)$ from $\tilde{\phi}(L^{(1)})$. In the following, we promise that $\tilde{\phi}(M')$ means zero if M' violates the conditions of G-patterns.

First, we can find the components $\tilde{\phi}\left(L^{(1)}\left(\begin{smallmatrix}0\ -1\\ 0\end{smallmatrix}\right)\right)$ and $\tilde{\phi}\left(L^{(1)}\left(\begin{smallmatrix}0\ 0\\ -1\end{smallmatrix}\right)\right)$ from the equations in Proposition 6.5 for $k=0$;

$$\begin{aligned}(\partial_2+\lambda_1)\tilde{\phi}(L^{(1)};y) &= -2\pi\sqrt{-1}\bar{c}_2y_2\tilde{\phi}\left(L^{(1)}\left(\begin{smallmatrix}0\ -1\\ 0\end{smallmatrix}\right);y\right),\\ (\partial_1-\lambda_2)\tilde{\phi}(L^{(1)};y) &= -2\pi\sqrt{-1}\bar{c}_1y_1\tilde{\phi}\left(L^{(1)}\left(\begin{smallmatrix}0\ 0\\ -1\end{smallmatrix}\right);y\right).\end{aligned}$$

Next let us take $1\le k\le m_1-m_2$ and assume that the components $\tilde{\phi}\left(L^{(1)}\left(\begin{smallmatrix}0\ 0\\ -i\end{smallmatrix}\right)\right)$, $\tilde{\phi}\left(L^{(1)}\left(\begin{smallmatrix}0\ -1\\ -i\end{smallmatrix}\right)\right)$, and $\tilde{\phi}\left(L^{(1)}\left(\begin{smallmatrix}-1\,0\\ -i\end{smallmatrix}\right)\right)$ for $0\le i\le k-1$ are all known. Then, in the equation 1 of Proposition 6.3 evaluated for $M=L^{(1)}\left(\begin{smallmatrix}0\ 0\\ -k+1\end{smallmatrix}\right)$, the only unknown function is $\tilde{\phi}\left(L^{(1)}\left(\begin{smallmatrix}0\ 0\\ -k\end{smallmatrix}\right)\right)$ with the coefficient $-4\pi\sqrt{-1}\bar{c}_1y_1(m_1-m_2-k+1)$. Thus the $L^{(1)}\left(\begin{smallmatrix}0\ 0\\ -k\end{smallmatrix}\right)$-component is determined. Moreover the equation 3 in Proposition 6.3 for $M=L^{(1)}\left(\begin{smallmatrix}0\ 0\\ -k+1\end{smallmatrix}\right)$ and the equation 1 in Proposition 6.5 for $M=L^{(1)}\left(\begin{smallmatrix}0\ 0\\ -k\end{smallmatrix}\right)$ have the unknown terms

$$\begin{aligned}&2\cdot 2\pi\sqrt{-1}\bar{c}_1y_1\cdot 2\pi\sqrt{-1}\bar{c}_2y_2\\ &\times\left\{-(m_2-m_3)\tilde{\phi}\left(L^{(1)}\left(\begin{smallmatrix}0\ -1\\ -k\end{smallmatrix}\right)\right)+(m_1-m_2-k+1)\tilde{\phi}\left(L^{(1)}\left(\begin{smallmatrix}-1\,0\\ -k\end{smallmatrix}\right)\right)\right\},\end{aligned}$$

and

$$-2\pi\sqrt{-1}\bar{c}_2 y_2 \left\{ \tilde{\phi}\left(L^{(1)}\left(\begin{smallmatrix} 0 & -1 \\ & -k \end{smallmatrix}\right)\right) + \tilde{\phi}\left(L^{(1)}\left(\begin{smallmatrix} -1 & 0 \\ & -k \end{smallmatrix}\right)\right)\right\},$$

respectively, and thus these two unknown components are determined from these two equations. Similarly, for fixed $1 \le k \le m_2 - m_3$, if the components $\tilde{\phi}\left(L^{(1)}\left(\begin{smallmatrix} 0 & -i \\ & 0 \end{smallmatrix}\right)\right)$, $\tilde{\phi}\left(L^{(1)}\left(\begin{smallmatrix} 0 & -i \\ & -1 \end{smallmatrix}\right)\right)$, and $\tilde{\phi}\left(L^{(1)}\left(\begin{smallmatrix} -1 & -i+1 \\ & -1 \end{smallmatrix}\right)\right)$ for $0 \le i \le k-1$ are all given, then the three components $\tilde{\phi}\left(L^{(1)}\left(\begin{smallmatrix} 0 & -k \\ & 0 \end{smallmatrix}\right)\right)$, $\tilde{\phi}\left(L^{(1)}\left(\begin{smallmatrix} 0 & -k \\ & -1 \end{smallmatrix}\right)\right)$, and $\tilde{\phi}\left(L^{(1)}\left(\begin{smallmatrix} -1 & -k+1 \\ & -1 \end{smallmatrix}\right)\right)$ can be determined from the equations 1 and 3 in Proposition 6.3 and the equation 2 in Proposition 6.5. Therefore the M-components $\tilde{\phi}(M)$ corresponding to the weights $(m_1 - i, m_2 + i - j, m_3 + j)$ for $0 \le i \le m_1 - m_2$ and $j = 0, 1$ and $(m_1 - j, m_2 - i + j, m_3 + i)$ for $0 \le i \le m_2 - m_3$ and $j = 0, 1$ can be determined.

To determine the remaining M-components, we need only the equations 1 and 3 in Proposition 6.3. This process is done one by one from the larger pair $(w_1 - w_3, |\delta(M)|)$ in lexicographical order, where (w_1, w_2, w_3) is the weight corresponding to G-pattern M. We leave the details for the reader. □

The proof of this lemma shows that all M-components $\phi(M)$ of a Whittaker function ϕ are moderate growth functions if and only if $\phi(L^{(1)})$ is. Thus a Whittaker function is in the space $\mathrm{Wh}(\pi, \eta, \tau)^{\mathrm{mod}}$ if and only if its $L^{(1)}$-component is a moderate growth function.

7.2. **The highest weight components of Whittaker functions.** According to Lemma 7.1 in the previous subsection, we may consider their $L^{(1)}$-components in order to determine Whittaker functions, which satisfy the holonomic system of partial differential equations in Corollary 6.4. In this subsection, we describe the space of solutions for this holonomic system explicitly.

The holonomic system of partial differential equations in Corollary 6.4 has regular singularities along 2 divisors $y_1 = 0$ and $y_2 = 0$ which are of simple normal crossing at $(y_1, y_2) = (0, 0)$, in the sense of [17]. First, we consider the power series solutions of this system at the point $(y_1, y_2) = (0, 0)$, which give the $L^{(1)}$-components of the *secondary Whittaker functions* with respect to (π, η, τ). For a power series

$$(5) \qquad (\pi|c_1|y_1)^{\gamma_1}(\pi|c_2|y_2)^{\gamma_2} \sum_{k,l=0}^{\infty} c^{\gamma}_{k,l} (\pi|c_1|y_1)^k (\pi|c_2|y_2)^l, \quad \gamma = (\gamma_1, \gamma_2) \in \mathbf{C}^2,$$

with a characteristic index $\gamma = (\gamma_1, \gamma_2)$, it is easy to see that the holonomic system in Corollary 6.4 can be translated into the following system of difference equations for the coefficients $\{c^{\gamma}_{k,l}\}$.

Lemma 7.2. *The power series (5) satisfies the holonomic system in Corollary 6.4 if and only if the coefficients $\{c^{\gamma}_{k,l}\}$ satisfy the following system of difference equations.*

1.

$$\begin{aligned} &\big\{(\gamma_1 + k)^2 + (\gamma_2 + l)^2 - (\gamma_1 + k)(\gamma_2 + l) \\ &\quad -p(\gamma_1 + k - \lambda_2) - q(\gamma_2 + l + \lambda_1) + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)\big\} c^{\gamma}_{k,l} \\ &-4c^{\gamma}_{k-2,l} - 4c^{\gamma}_{k,l-2} = 0, \end{aligned}$$

2.

$$\begin{aligned}&\Big\{\left(\gamma_1+k+\frac{\tilde{\nu}}{3}-m_1\right)\left(-\gamma_1+\gamma_2-k+l+\frac{\tilde{\nu}}{3}-m_2\right)\left(-\gamma_2-l+\frac{\tilde{\nu}}{3}-m_3\right)\\&\quad-\left(\lambda_2+\frac{\tilde{\nu}}{3}-m_1\right)\left(\lambda_3+\frac{\tilde{\nu}}{3}-m_2\right)\left(\lambda_1+\frac{\tilde{\nu}}{3}-m_3\right)\Big\}c^{\gamma}_{k,l}\\&\quad+4\left(\gamma_1+k+\frac{\tilde{\nu}}{3}-m_1\right)c^{\gamma}_{k,l-2}+4\left(-\gamma_2-l+\frac{\tilde{\nu}}{3}-m_3\right)c^{\gamma}_{k-2,l}=0.\end{aligned}$$

Here we understand $c^{\gamma}_{k,l}=0$ if $k<0$ or $l<0$.

Observe that all coefficients $c^{\gamma}_{k,l}$ are determined inductively from an initial non-zero coefficients $c^{\gamma}_{0,0}$ by the first difference equation in Lemma 7.2. The characteristic indices γ can be found by putting $k=l=0$ in the equations in Lemma 7.2.

Lemma 7.3. *The set of characteristic indices $\{\gamma^{(i)}=(\gamma_1^{(i)},\gamma_2^{(i)})\,|1\le i\le 6\}$ of the holonomic system of partial differential equations in Corollary 6.4 at $(y_1,y_2)=(0,0)$ is given as follows.*

$$\begin{array}{ll}\gamma^{(1)}=(\lambda_2,-\lambda_1), & \gamma^{(2)}=(\lambda_3+p,-\lambda_1),\\ \gamma^{(3)}=(\lambda_2,-\lambda_3+q), & \gamma^{(4)}=(\lambda_1+p+q,-\lambda_3+q),\\ \gamma^{(5)}=(\lambda_3+p,-\lambda_2+p+q), & \gamma^{(6)}=(\lambda_1+p+q,-\lambda_2+p+q).\end{array}$$

Now, for each $1\le i\le 6$, we define the coefficients $\{C^{(i)}_{k,l}\}_{k,l\ge 0}$ by

$$C^{(i)}_{k,l}=\begin{cases}\dfrac{4(-1)^{k'+l'}}{k'!\cdot l'!}\Gamma\left[\begin{array}{c}\frac{a_1}{2}-k',\ \frac{a_2}{2}-k',\ \frac{a_3}{2}-l',\ \frac{a_4}{2}-l'\\ \frac{b}{2}-k'-l'\end{array}\right] & \text{if } (k,l)=(2k',2l'),\\ 0 & \text{otherwise,}\end{cases}$$

with the parameters

$$a_1=a_3=b=-\gamma_1^{(i)}-\gamma_2^{(i)}+p+q,\ \ a_2=-2\gamma_1^{(i)}+\gamma_2^{(i)}+p,\ \ a_4=\gamma_1^{(i)}-2\gamma_2^{(i)}+q.$$

Here we use the notation

$$\Gamma\left[\begin{array}{c}a_1,\dots,a_r\\ b_1,\dots,b_s\end{array}\right]=\prod_{i=1}^{r}\Gamma(a_i)\Big/\prod_{i=1}^{s}\Gamma(b_i).$$

Since $\Gamma(x+1)=x\Gamma(x)$ for $x\notin\mathbf{Z}_{\le 0}$, we have the relations

$$\begin{aligned}C^{(i)}_{k-2,l}&=C^{(i)}_{k,l}\times(-k')\left(\frac{a_1}{2}-k'\right)\left(\frac{a_2}{2}-k'\right)\left(\frac{b}{2}-k'-l'\right)^{-1},\\ C^{(i)}_{k,l-2}&=C^{(i)}_{k,l}\times(-l')\left(\frac{a_3}{2}-l'\right)\left(\frac{a_4}{2}-l'\right)\left(\frac{b}{2}-k'-l'\right)^{-1},\end{aligned}$$

if $(k,l)=(2k',2l')$, and thus,

$$4\left(C^{(i)}_{k-2,l}+C^{(i)}_{k,l-2}\right)=C^{(i)}_{k,l}\left(k^2-kl+l^2-a_2k-a_4l\right).$$

This identity shows that the coefficients $\{C^{(i)}_{k,l}\}$ satisfy the first difference equations for $\gamma = \gamma^{(i)}$ in Lemma 7.2. Therefore we can state the following proposition on an explicit formula for the $L^{(1)}$-components of secondary Whittaker functions with respect to (π,η,τ).

Proposition 7.4. *For each $1 \le i \le 6$, we define the function $\tilde{\varphi}^{(i)}_3(L^{(1)};y)$ by the power series (5) with the above coefficients $\{C^{(i)}_{k,l}\}$, that is,*

$$\tilde{\varphi}^{(i)}_3(L^{(1)};y) = (\pi|c_1|y_1)^{\gamma^{(i)}_1}(\pi|c_2|y_2)^{\gamma^{(i)}_2}\sum_{k',l'=0,}^{\infty} C^{(i)}_{2k',2l'}(\pi|c_1|y_1)^{2k'}(\pi|c_2|y_2)^{2l'}.$$

Then the set $\{\tilde{\varphi}^{(i)}_3(L^{(1)})\}$ gives the complete system of linearly independent solutions for the holonomic system of differential equations in Corollary 6.4 at $y=(0,0)$.

Next, we consider a solution with moderate growth property for the holonomic system of partial differential equations in Corollary 6.4. As we mentioned in the previous subsection, a Whittaker function ϕ is of moderate growth if and only if its $L^{(1)}$-component $\phi(L^{(1)})$ is. Therefore the local multiplicity one theorem for Whittaker model (*cf.* [19], [26]) tells that the holonomic system in Corollary 6.4 has a solution of moderate growth unique up to scalar multiples, which gives the $L^{(1)}$-component of the *primary Whittaker function.* Here we give two integral expressions of this unique solution of moderate growth.

Proposition 7.5. 1. *Put*

$$\tilde{\varphi}^{\mathrm{mod}}_3(L^{(1)};y) = \frac{1}{(2\pi\sqrt{-1})^2}\int_{s_1}\int_{s_2} V_3(L^{(1)};s_1,s_2)(\pi|c_1|y_1)^{-s_1}(\pi|c_2|y_2)^{-s_2}ds_1ds_2.$$

Here

$$V_3(L^{(1)};s_1,s_2) = \Gamma\left[\begin{array}{c}\frac{s_1+\lambda_1+p+q}{2},\ \frac{s_1+\lambda_2}{2},\ \frac{s_1+\lambda_3+p}{2},\ \frac{s_2-\lambda_1}{2},\ \frac{s_2-\lambda_2+p+q}{2},\ \frac{s_2-\lambda_3+q}{2}\\ \frac{s_1+s_2+p+q}{2}\end{array}\right],$$

and the paths of integrations are the vertical lines from $\mathrm{Re}\, s_i - \sqrt{-1}\infty$ to $\mathrm{Re}\, s_i + \sqrt{-1}\infty$ with large enough real parts. Then, up to scalar multiples, the function $\tilde{\varphi}^{\mathrm{mod}}_3(L^{(1)})$ gives a unique solution with moderate growth property for the holonomic system of partial differential equations in Corollary 6.4.

2. *The function $\tilde{\varphi}^{\mathrm{mod}}_3(L^{(1)})$ has the following integral expression of Euler type.*

$$\begin{aligned}\tilde{\varphi}^{\mathrm{mod}}_3(L^{(1)};y) &= 2^4(\pi|c_1|y_1)^{\frac{-\lambda_3+p+q}{2}}(\pi|c_2|y_2)^{\frac{\lambda_3+p+q}{2}}\\ &\times\int_0^\infty K_A\left(2\pi|c_1|y_1\sqrt{1+\frac{1}{v}}\right)K_A\left(2\pi|c_2|y_2\sqrt{1+v}\right)v^B\frac{dv}{v}.\end{aligned}$$

Here $K_\nu(z)$ is the modified Bessel function of the second kind and the parameters A and B are given by

$$A = \frac{\lambda_1-\lambda_2+p+q}{2},\quad B = \frac{3\lambda_3+p-q}{4}.$$

3. *The function* $\tilde{\varphi}_3^{\mathrm{mod}}(L^{(1)})$ *has the following factorization by the power series* $\tilde{\varphi}_3^{(i)}(L^{(1)})$ *defined in Proposition 7.4.*

$$\tilde{\varphi}_3^{\mathrm{mod}}(L^{(1)};y) = \sum_{i=1}^{6} \tilde{\varphi}_3^{(i)}(L^{(1)};y).$$

Proof. The Stirling formula for the gamma function shows that the double Mellin-Barnes integral defining the function $\tilde{\varphi}_3^{\mathrm{mod}}(L^{(1)})$ converges absolutely and also defines a moderate growth function of y. The second assertion follows from Lemma 7.1 in the paper [13]. Moving the integration paths in the definition of $\tilde{\varphi}_3^{\mathrm{mod}}(L^{(1)})$ to the left, we have the third assertion after the standard residue calculus. The factorization in the third assertion means that the function $\tilde{\varphi}_3^{\mathrm{mod}}(L^{(1)})$ satisfies the holonomic system in Corollary 6.4. Therefore, $\tilde{\varphi}_3^{\mathrm{mod}}(L^{(1)})$ gives a unique solution with moderate growth property for the system, up to scalar multiples. □

7.3. **Explicit formulas of Whittaker functions.** As we asserted in Lemma 7.1, all M-components of a Whittaker function are determined from its $L^{(1)}$-component whose explicit formulas are given in the previous subsection. In this subsection, we give explicit formulas for the whole components of Whittaker functions with respect to (π,η,τ). For simplicity, we assume $c_1 = c_2 = \sqrt{-1}$ in the following discussion.

First, we consider the power series solutions of the holonomic system of differential equations (3) and (4) at $(y_1,y_2) = (0,0)$, which we call the secondary Whittaker functions. That is, we give a family $\{\tilde{\phi}(M;y)\}_{M\in G(\mathbf{m})}$ of power series

$$\tilde{\phi}(M;y) = (\pi y_1)^{\gamma_1(M)}(\pi y_2)^{\gamma_2(M)} \sum_{k,l=0}^{\infty} c_{k,l}^{\gamma(M)}(\pi y_1)^k(\pi y_2)^l, \tag{6}$$

with a characteristic index $\gamma(M) = (\gamma_1(M),\gamma_2(M)) \in \mathbf{C}^2$ satisfying the differential equations in Proposition 6.3 and Propositions 6.5.

Now, for each G-pattern $M = \begin{pmatrix} m_1\, m_2\, m_3 \\ \alpha_1\, \alpha_2 \\ \beta \end{pmatrix} \in G(\mathbf{m})$ and each $1 \le i \le 6$, we define the characteristic index $\gamma^{(i)}(M) = (\gamma_1^{(i)}(M),\gamma_2^{(i)}(M))$ and the set of coefficients $\{C_{k,l}^{(i)}(M)\}_{k,l\ge 0}$ as follows. Put

$$\begin{aligned}
\zeta_1^{(1)}(M) &= \lambda_1 - m_3 + \beta, & \zeta_1^{(2)}(M) &= -\lambda_1 + m_1 - \beta - \delta(M),\\
\zeta_2^{(1)}(M) &= \lambda_2 + m_1 - \beta, & \zeta_2^{(2)}(M) &= -\lambda_2 - m_3 + \beta + \delta(M),\\
\zeta_3^{(1)}(M) &= \lambda_3 + \alpha_1 - \alpha_2 - |\delta(M)|, & \zeta_3^{(2)}(M) &= -\lambda_3 + m_1 - m_3 - \alpha_1 + \alpha_2.
\end{aligned}$$

Then we define $\gamma^{(i)}(M) = (\zeta_{u_i}^{(1)}(M), \zeta_{v_i}^{(2)}(M))$ with the index (u_i,v_i) given in the following table.

i	1	2	3	4	5	6
(u_i,v_i)	$(2,1)$	$(3,1)$	$(2,3)$	$(1,3)$	$(3,2)$	$(1,2)$

TABLE 5. Index (u_i,v_i) in $\gamma^{(i)}(M)$

Here we observe that $\gamma^{(i)}(L^{(1)})=\gamma^{(i)}$ is the characteristic index given in Lemma 7.3. Moreover we define the coefficients $\{C^{(i)}_{k,l}(M)\}_{k,l\geq 0}$ by

$$C^{(i)}_{k,l}(M)=\begin{cases}\dfrac{4(-1)^{k'+l'}}{k'!\cdot l'!}\Gamma\left[\begin{array}{c}\frac{a_1}{2}-k',\ \frac{a_2}{2}-k',\ \frac{a_3}{2}-l',\ \frac{a_4}{2}-l'\\ \frac{b}{2}-k'-l'\end{array}\right] & \text{if } (k,l)=(2k',2l'),\\ 0 & \text{otherwise,}\end{cases}$$

with the parameters given by

$$\begin{aligned}\{a_1,a_2\} &= \{\zeta^{(1)}_u(M)-\zeta^{(1)}_{u_i}(M)\,|\,1\leq u\leq 3,\ \ u\neq u_i\},\\ \{a_3,a_4\} &= \{\zeta^{(2)}_v(M)-\zeta^{(2)}_{v_i}(M)\,|\,1\leq v\leq 3,\ \ v\neq v_i\},\end{aligned}$$

and $b=-\zeta^{(1)}_{u_i}(M)-\zeta^{(2)}_{v_i}(M)+\zeta^{(1)}_3(M)+\zeta^{(2)}_3(M)$. We write the power series with the characteristic index $\gamma^{(i)}(M)$ and the coefficients $\{C^{(i)}_{k,l}(M)\}_{k,l\geq 0}$ defined above by $\tilde{\varphi}^{(i)}_3(M;y)$, i.e.

$$\tilde{\varphi}^{(i)}_3(M;y)=(\pi y_1)^{\gamma^{(i)}_1(M)}(\pi y_2)^{\gamma^{(i)}_2(M)}\sum_{k',l'=0,}^{\infty}C^{(i)}_{2k',2l'}(M)(\pi y_1)^{2k'}(\pi y_2)^{2l'}.$$

When $M=L^{(1)}$, this power series coincides with the one (for $c_1=c_2=\sqrt{-1}$) defined in Proposition 7.4.

Theorem 7.6. *Let $\pi=\pi(\nu,\sigma_{\mathbf{n}})$ be an irreducible principal series representation with data $\nu=(\nu_1,\nu_2,\nu_3)$ and $\mathbf{n}=(n_1,n_2,n_3)$, and let $\tau^*=\tau_{\mathbf{m}}$ associated to the dominant permutation $\mathbf{m}=(m_1,m_2,m_3)\in\Lambda$ of $\mathbf{n}$ be the minimal K-type of π. Moreover let η be a non-degenerate unitary character of N specified by the parameters $c_1=c_2=\sqrt{-1}$. For each $1\leq i\leq 6$, let $\varphi^{(i)}_3\in \mathrm{Wh}(\pi,\eta,\tau)$ be the secondary Whittaker function whose $L^{(1)}$-component is $\varphi^{(i)}_3(L^{(1)})=y_1^2y_2^2\tilde{\varphi}^{(i)}_3(L^{(1)})$ defined in Proposition 7.4. Then, for each G-pattern M, the M-component of $\varphi^{(i)}_3$ is $\varphi^{(i)}_3(M)=y_1^2y_2^2\tilde{\varphi}^{(i)}_3(M)$.*

Proof. We can obtain this assertion similarly to Proposition 7.4, that is, by showing directly for each $1\leq i\leq 6$ the set $\{C^{(i)}_{k,l}(M)\}$ satisfies the difference equations for the coefficients $\{c^{\gamma(M)}_{k,l}\}$ of the power series (6) which is equivalent with Proposition 6.3 and 6.5.

In the case of $\delta(M)>0$ and $\gamma(M)=\gamma^{(1)}(M)$, since

$$\gamma^{(1)}\left(M\left(\begin{smallmatrix}0 & -1\\ & 0\end{smallmatrix}\right)\right)=\gamma^{(1)}\left(M\left(\begin{smallmatrix}-1 & 0\\ & 0\end{smallmatrix}\right)\right)=\left(\gamma^{(1)}_1(M),\gamma^{(1)}_2(M)+1\right),$$

$$\gamma^{(1)}\left(M\left(\begin{smallmatrix}0 & 0\\ & -1\end{smallmatrix}\right)\right)=\left(\gamma^{(1)}_1(M)+1,\gamma^{(1)}_2(M)\right),$$

the difference equation for $\{c^{\gamma(M)}_{k,l}\}$ equivalent to the equation 1 in Proposition 6.3 is given by

$$\Bigg[\left(\gamma_1(M)+k+\frac{\tilde{\nu}}{3}+w_1\right)\left(-\gamma_1(M)+\gamma_2(M)-k+l+\frac{\tilde{\nu}}{3}+w_2\right)$$

$$
\begin{aligned}
&+\left(-\gamma_1(M)+\gamma_2(M)-k+l+\frac{\tilde{\nu}}{3}+w_2\right)\left(-\gamma_2(M)-l+\frac{\tilde{\nu}}{3}+w_3\right)\\
&+\left(\gamma_1(M)+k+\frac{\tilde{\nu}}{3}+w_1\right)\left(-\gamma_2(M)-l+\frac{\tilde{\nu}}{3}+w_3\right)\\
&-\left.\sum_{1\le i<j\le 3}(\nu_i+n_i)(\nu_j+n_j)\right] c_{k,l}^{\gamma(M)}\\
&+4c_{k-2,l}^{\gamma(M)}+4c_{k,l-2}^{\gamma(M)}\\
&-4\left(\alpha_2-m_3\right)c_{k,l-2}^{\gamma\left(M\left(\begin{smallmatrix}0 & -1\\ & 0\end{smallmatrix}\right)\right)}-4\left(\alpha_2-m_3+\delta(M)\right)c_{k,l-2}^{\gamma\left(M\left(\begin{smallmatrix}-1 & 0\\ & 0\end{smallmatrix}\right)\right)}\\
&-4(\beta-\alpha_2)c_{k-2,l}^{\gamma\left(M\left(\begin{smallmatrix}0 & 0\\ & -1\end{smallmatrix}\right)\right)}=0.
\end{aligned}
$$

Then direct computation shows that the coefficients $\{C_{k.l}^{(1)}(M)\}$ satisfy the above difference equation by using the relations

$$
\begin{aligned}
C_{k-2,l}^{(1)}(M)&=-k'\left(\frac{a_1}{2}-k'\right)\left(\frac{a_2}{2}-k'\right)\left(\frac{b}{2}-k'-l'\right)^{-1}C_{k,l}^{(1)}(M),\\
C_{k,l-2}^{(1)}(M)&=-l'\left(\frac{a_3}{2}-l'\right)\left(\frac{a_4}{2}-l'\right)\left(\frac{b}{2}-k'-l'\right)^{-1}C_{k,l}^{(1)}(M),\\
C_{k,l-2}^{(1)}\left(M\left(\begin{smallmatrix}0 & -1\\ & 0\end{smallmatrix}\right)\right)&=-l'\left(\frac{a_2}{2}-k'\right)\left(\frac{b}{2}-k'-l'\right)^{-1}C_{k,l}^{(1)}(M),\\
C_{k,l-2}^{(1)}\left(M\left(\begin{smallmatrix}-1 & 0\\ & 0\end{smallmatrix}\right)\right)&=-l'\left(\frac{a_4}{2}-k'\right)\left(\frac{b}{2}-k'-l'\right)^{-1}C_{k,l}^{(1)}(M),\\
C_{k-2,l}^{(1)}\left(M\left(\begin{smallmatrix}0 & 0\\ & -1\end{smallmatrix}\right)\right)&=-k'C_{k,l}^{(1)}(M),
\end{aligned}
$$

if $(k,l)=(2k',2l')$, where

$$
\begin{aligned}
a_1&=\zeta_1^{(1)}(M)-\zeta_2^{(1)}(M), \quad a_2=\zeta_3^{(1)}(M)-\zeta_2^{(1)}(M),\\
a_3&=\zeta_2^{(2)}(M)-\zeta_1^{(2)}(M), \quad a_4=\zeta_3^{(2)}(M)-\zeta_1^{(2)}(M),
\end{aligned}
$$

and $b=-\zeta_2^{(1)}(M)-\zeta_1^{(2)}(M)+\zeta_3^{(1)}(M)+\zeta_3^{(2)}(M)$. The other cases can be shown similarly and we omit their detail. □

Finally, we state our main result for the primary Whittaker functions with respect to (π,η,τ), i.e. the unique solution of moderate growth for the holonomic system of differential equations (3) and (4). If we write such a solution by $\phi\in \mathrm{Wh}(\pi,\eta,\tau)^{\mathrm{mod}}$, then a family $\{\tilde{\phi}(M;y)\}_{M\in G(\mathbf{m})}$ consisting of all M-components $\phi(M;y)=y_1^2y_2^2\tilde{\phi}(M;y)$ of ϕ is the unique solution of moderate growth for the differential equations in Proposition 6.3 and Propositions 6.5. Also, ϕ is given by a linear combination of the six secondary Whittaker functions $\varphi^{(i)}$ in Theorem 7.6.

The following theorem can be seen by the same way as the proof of Proposition 7.5.

Theorem 7.7. *Let $\pi = \pi(\nu, \sigma_{\mathbf{n}})$, $\tau^* = \tau_{\mathbf{m}}$, and η be the representations as in Theorem 7.6. Moreover let $\varphi_3^{\mathrm{mod}} \in \mathrm{Wh}(\pi, \eta, \tau)^{\mathrm{mod}}$ be the primary Whittaker function whose $L^{(1)}$-component is $\varphi_3^{\mathrm{mod}}(L^{(1)}) = y_1^2 y_2^2 \tilde{\varphi}_3^{\mathrm{mod}}(L^{(1)})$ defined in Proposition 7.5. Then, for each G-pattern $M \in G(\mathbf{m})$ we have the following assertions on the M-component $\varphi_3^{\mathrm{mod}}(M) = y_1^2 y_2^2 \tilde{\varphi}_3^{\mathrm{mod}}(M)$ of φ_3^{mod}.*

1. *The function $\tilde{\varphi}_3^{\mathrm{mod}}(M)$ has the following integral expressions:*

$$\begin{aligned}
\tilde{\varphi}_3^{\mathrm{mod}}(M; y) &= \frac{1}{(2\pi\sqrt{-1})^2} \int_{s_1} \int_{s_2} V_3(M; s_1, s_2)(\pi y_1)^{-s_1} (\pi y_2)^{-s_2} ds_1 ds_2 \\
&= 2^4 (\pi y_1)^{\frac{-\lambda_3 + m_1 - m_3}{2}} (\pi y_2)^{\frac{\lambda_3 + m_1 - m_3}{2}} \\
&\quad \times \int_0^\infty K_A\left(2\pi y_1 \sqrt{1 + \frac{1}{v}}\right) K_{A+\delta(M)}\left(2\pi y_2 \sqrt{1+v}\right) v^B (1+v)^C \frac{dv}{v}.
\end{aligned}$$

Here, in the integral of Mellin-Barnes type, the paths of integrations are the vertical lines from $\mathrm{Re}\, s_i - \sqrt{-1}\infty$ to $\mathrm{Re}\, s_i + \sqrt{-1}\infty$ with large enough real parts and the integrand $V_3(M; s_1, s_2)$ is defined by

$$V_3(M; s_1, s_2) = \Gamma\left[\begin{array}{c} \frac{s_1 + \zeta_1^{(1)}(M)}{2}, \frac{s_1 + \zeta_2^{(1)}(M)}{2}, \frac{s_1 + \zeta_3^{(1)}(M)}{2}, \frac{s_2 + \zeta_1^{(2)}(M)}{2}, \frac{s_2 + \zeta_2^{(2)}(M)}{2}, \frac{s_2 + \zeta_3^{(2)}(M)}{2} \\ \frac{s_1 + s_2 + \zeta_3^{(1)}(M) + \zeta_3^{(2)}(M)}{2} \end{array}\right]$$

Also, in the integral of Euler type, the parameters A, B and C are given by

$$A = \frac{\zeta_1^{(1)}(M) - \zeta_2^{(1)}(M)}{2}, \quad B = \frac{2\zeta_3^{(1)}(M) - \zeta_1^{(1)}(M) - \zeta_2^{(1)}(M)}{4},$$

and $C = \dfrac{|\delta(M)|}{2}$.

2. *The function $\tilde{\varphi}_3^{\mathrm{mod}}(M)$ has the following factorization by the power series $\tilde{\varphi}_3^{(i)}(M)$.*

$$\tilde{\varphi}_3^{\mathrm{mod}}(M; y) = \sum_{i=1}^{6} \tilde{\varphi}_3^{(i)}(M; y).$$

8. Propagation formula

Based on our main result in the previous section, we give here an expression of Whittaker functions on $GL(3,\mathbf{C})$ in terms of those on $GL(2,\mathbf{C})$, which we call a *propagation formula.* This is an analogous formula in the class one case obtained by Ishii-Stade [10].

8.1. **Principal series Whittaker functions on $GL(2,\mathbf{C})$.**

In this subsection, we derive an explicit formula of principal series Whittaker functions on $GL(2,\mathbf{C})$ by similar computation to the case of $GL(3,\mathbf{C})$.

Let $G' = GL(2,\mathbf{C})$ be the complex general linear group of degree 2 and $G' = N'A'K'$ be its Iwasawa decomposition, where $K' = U(2)$ is a maximal compact subgroup of G' and

$$A' = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \middle| \, a_i \in \mathbf{R}_{>0}, i = 1, 2 \right\}, \quad N' = \left\{ n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| \, x \in \mathbf{C} \right\}.$$

The center $Z_{G'}$ of G' is $\{ru1_2 \,|\, r \in \mathbf{R}_{>0}, u \in U(1)\} \simeq \mathbf{C}^\times$. The upper triangular subgroup of G' is $P' = N'A'M'$, where M' is the centralizer of A' in K' given by

$$M' = \left\{ \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \,\middle|\, u_i \in U(1),\, i = 1, 2 \right\} \simeq U(1)^2.$$

Next, we recall the representations of K', G', and N' which we need in order to describe the Whittaker functions. We can parameterize the equivalence classes of irreducible continuous representations of $K' = U(2)$ by the set of highest weights

$$\Lambda' = \{\mu' = (\mu'_1, \mu'_2) \,|\, \mu' \in \mathbf{Z}^2, \mu'_1 \geq \mu'_2\}.$$

The representation space $V_{\mu'}$ of the representation $\tau_{\mu'}$ associated with $\mu' = (\mu'_1, \mu'_2) \in \Lambda'$ has the (normalized) GZ-basis $\{f'(M')\}_{M' \in G(\mu')}$ as in the case of $U(3)$. Here

$$G(\mu') = \left\{ M' = \left(\begin{smallmatrix} \mu'_1 \, \mu'_2 \\ \alpha' \end{smallmatrix} \right) \,\middle|\, \alpha' \in \mathbf{Z},\;\; \mu'_1 \geq \alpha' \geq \mu'_2 \right\}.$$

The explicit action of the complexification $\mathfrak{k}'_{\mathbb{C}}$ of the Lie algebra $\mathfrak{k}'$ of K' on the GZ-basis is given as follows. Let us put

$$E^{\mathfrak{k}'}_{ij} = \frac{1}{2} \left\{ (E_{ij} - E_{ji}) - \sqrt{-1} \left(E'_{ij} + E'_{ji} \right) \right\},$$

for the matrix unit E_{ij} (resp. E'_{ij}) with its (i, j)-entry 1 (resp. J) and the remaining entries 0. Then

$$\begin{aligned} E^{\mathfrak{k}'}_{ii} f'(M') &= w'_i f'(M), \qquad i = 1, 2, \\ E^{\mathfrak{k}'}_{12} f'(M') &= (\mu'_1 - \alpha') f'(M'(1)), \\ E^{\mathfrak{k}'}_{21} f'(M') &= (\alpha' - \mu'_2) f'(M'(-1)). \end{aligned}$$

Here $(w'_1, w'_2) = (\alpha', \mu'_1 + \mu'_2 - \alpha')$ is the weight of vector $f'(M')$ associated with a G-pattern $M' = \left(\begin{smallmatrix} \mu'_1 \, \mu'_2 \\ \alpha' \end{smallmatrix} \right)$ and $M'(i) = \left(\begin{smallmatrix} \mu'_1 \, \mu'_2 \\ \alpha' + i \end{smallmatrix} \right)$. Moreover, we promise the corresponding vector $f'(M')$ is zero if $M'(i)$ appearing in the above formulas violates the conditions of G-patterns. A principal series representation

$$\pi' = \pi'(\nu', \sigma_{\mathbf{n}'}) = \mathrm{Ind}_{P'}^{G'}(1_{N'} \otimes e^{\nu' + \rho'} \otimes \sigma_{\mathbf{n}'}),$$

of G' with data $\nu' = (\nu'_1, \nu'_2) \in \mathbf{C}^2$ and $\mathbf{n}' = (n'_1, n'_2) \in \mathbf{Z}^2$ induced from the minimal parabolic subgroup $P' = N'A'M'$ is defined similarly to the case of $GL(3, \mathbf{C})$. Here, the half-sum ρ' of the positive restricted roots is given by

$$e^{\rho'}(\mathrm{diag}\,(a_1, a_2)) = \frac{a_1}{a_2}, \quad \mathrm{diag}\,(a_1, a_2) \in A'.$$

The central character of π' is

$$Z_{G'} \ni ru1_2 \mapsto r^{\tilde{\nu}'} u^{\tilde{n}'}, \quad r \in \mathbf{R}_{>0},\; u \in U(1),$$

with $\tilde{\nu}' = \nu'_1 + \nu'_2$ and $\tilde{n}' = n'_1 + n'_2$, and the minimal K'-type of π' is the representation $(\tau_{\mathbf{m}'}, V_{\mathbf{m}'})$ associated with the dominant permutation $\mathbf{m}' = (m'_1, m'_2) \in \Lambda'$ of $\mathbf{n}'$. Finally, we take a non-degenerate character η' of N' defined by

$$\eta'(n(x)) = \exp\left(2\pi\sqrt{-1}\mathrm{Im}\,(x)\right).$$

As in the case of $GL(3,\mathbf{C})$, for each element $\mathcal{C}$ in the center $Z(\mathfrak{g}'_{\mathbb{C}})$ of the universal enveloping algebra of $\mathfrak{g}'_{\mathbb{C}}$ each M'-component $\phi(M')$ of a Whittaker function $\phi \in \mathrm{Wh}(\pi',\eta',\tau')$ satisfies a differential equation

$$\mathcal{C}\phi(M') = \chi_{\mathcal{C}}\phi(M') \tag{7}$$

with an eigenvalue $\chi_{\mathcal{C}}$. We can give the following explicit description of the differential equation (7) in terms of the coordinate

$$y = \frac{a_1}{a_2}, \quad \text{for } \mathrm{diag}\,(a_1,a_2) = a_2 \cdot \mathrm{diag}\,(y,1) \in A',$$

by computations similar to the case of $GL(3,\mathbf{C})$.

Proposition 8.1. *Let $\phi(M')$ be the M'-component of a Whittaker function $\phi \in \mathrm{Wh}(\pi',\eta',\tau')$ and put $\phi(M') = y\tilde{\phi}(M')$. Then the differential equations (7) for the Capelli elements of $\mathfrak{gl}_2$ are given as follows: Let us denote the Euler operator with respect to y by $\partial = y\dfrac{d}{dy}$ and put $(w'_1,w'_2) = (\alpha', m'_1+m'_2-\alpha')$ be the weight of a G-pattern $M' = \left(\begin{smallmatrix} m'_1 & m'_2 \\ \alpha' & \end{smallmatrix}\right)$.*

1.

$$\begin{aligned}&\Big[\Big(\partial+\frac{\tilde{\nu}'}{2}+w'_1\Big)\Big(-\partial+\frac{\tilde{\nu}'}{2}+w'_2\Big)\\ &\qquad -\big(2\pi\sqrt{-1}\big)^2y^2-(\nu'_1+n'_1)(\nu'_2+n'_2)\Big]\tilde{\phi}(M';y)\\ &-4\pi y\,(\alpha'-m'_2)\,\tilde{\phi}\,(M'(-1);y)=0.\end{aligned}$$

2.

$$\begin{aligned}&\Big[\Big(\partial+\frac{\tilde{\nu}'}{2}-w'_1\Big)\Big(-\partial+\frac{\tilde{\nu}'}{2}-w'_2\Big)\\ &\qquad -\big(2\pi\sqrt{-1}\big)^2y^2-(\nu'_1-n'_1)(\nu'_2-n'_2)\Big]\tilde{\phi}(M';y)\\ &-4\pi y\,(m'_1-\alpha')\,\tilde{\phi}\,(M'(1);y)=0.\end{aligned}$$

In particular, the second equation at the G-pattern $L' = \left(\begin{smallmatrix} m'_1 & m'_2 \\ m'_1 & \end{smallmatrix}\right)$ associated with the highest weight vector $f'(L')$ in $V_{\mathbf{m}'}$ gives the following differential equation for $\tilde{\phi}(L')$.

$$\begin{aligned}&\Big[\Big(\partial+\frac{\tilde{\nu}'}{2}-m'_1\Big)\Big(-\partial+\frac{\tilde{\nu}'}{2}-m'_2\Big)\\ &\qquad -\big(2\pi\sqrt{-1}\big)^2y^2-(\nu'_1-n'_1)(\nu'_2-n'_2)\Big]\tilde{\phi}(L';y)=0.\end{aligned}$$

If we put

$$\lambda'_1 = \nu'_b - \frac{\tilde{\nu}'}{2}, \;\; \lambda'_2 = \nu'_a - \frac{\tilde{\nu}'}{2}$$

for $\mathbf{m}' = (n'_a, n'_b)$, then we have the relations $\lambda'_1 + \lambda'_2 = 0$ and

$$(\nu'_1 \pm n'_1)(\nu'_2 \pm n'_2) = \left(\lambda'_2 + \frac{\tilde{\nu}'}{2} \pm m'_1\right)\left(\lambda'_1 + \frac{\tilde{\nu}'}{2} \pm m'_2\right),$$

and thus we can write the above equation for $\tilde{\phi}(L')$ as

$$\left[\partial^2 - (m'_1 - m'_2)(\partial + \lambda'_1) + \lambda'_1\lambda'_2 + \left(2\pi\sqrt{-1}\right)^2 y^2\right]\tilde{\phi}(L'; y) = 0.$$

As solutions for the differential equations in Proposition 8.1, explicit formulas of the M'-components of Whittaker functions are given in the next theorem.

Theorem 8.2. *Let $\pi' = \pi(\nu', \sigma_{\mathbf{n}'})$ be an irreducible principal series representation with data $\nu' = (\nu'_1, \nu'_2)$ and $\mathbf{n}' = (n'_1, n'_2)$, and let $(\tau')^* = \tau_{\mathbf{m}'}$ associated to the dominant permutation $\mathbf{m}' = (m'_1, m'_2) \in \Lambda'$ of $\mathbf{n}'$ be the minimal K-type of π. Moreover let η' be a non-degenerate unitary character of N defined above.*

1. *For each G-pattern $M' = \begin{pmatrix} m'_1\, m'_2 \\ \alpha' \end{pmatrix}$ and $i = 1, 2$, we put*

$$\gamma^{(i)}(M') = \begin{cases} \lambda'_2 + m'_1 - \alpha', & i = 1 \\ \lambda'_1 - m'_2 + \alpha', & i = 2 \end{cases},$$

and define the coefficients $\{C^{(i)}_{2k}(M')\}_{k \ge 0}$ by

$$C^{(i)}_{2k}(M') = \frac{2(-1)^k}{k!}\Gamma\left(\frac{a'}{2} - k\right),$$

with the parameter $a' = (-1)^i\left(\gamma^{(1)}(M') - \gamma^{(2)}(M')\right)$. Then the power series

$$\begin{aligned}\tilde{\varphi}^{(i)}_2(M'; y) &= (\pi y)^{\gamma^{(i)}(M')}\sum_{k=0}^{\infty} C^{(i)}_{2k}(M')(\pi y)^{2k} \\ &= 2\pi\left(\sin\frac{a'\pi}{2}\right)^{-1}(\pi y)^{\frac{m'_1 - m'_2}{2}} I_{-\frac{a'}{2}}(2\pi y),\end{aligned}$$

for $i = 1, 2$ give the complete system of linearly independent solutions at $y = 0$ for the equations in Proposition 8.1. Here $I_\nu(z)$ is the modified Bessel function of the first kind.

2. *Let $\tilde{\varphi}^{\mathrm{mod}}_2(M')$ be the unique (up to constant multiples) solution with the moderate growth property for the differential equations in Proposition 8.1. Then we have*

$$\tilde{\varphi}^{\mathrm{mod}}_2(M'; y) = \frac{1}{2\pi\sqrt{-1}}\int_s V_2(M'; s)(\pi y)^{-s} ds = 4(\pi y)^A K_B(2\pi y).$$

Here, the path of integration is the vertical line from $\mathrm{Re}\, s - \sqrt{-1}\infty$ *to* $\mathrm{Re}\, s + \sqrt{-1}\infty$ *with enough large real part and the integrand $V_2(M'; s)$ is defined by*

$$V_2(M'; s) = \Gamma\left[\frac{s + \gamma^{(1)}(M')}{2}, \frac{s + \gamma^{(2)}(M')}{2}\right],$$

and the parameters A and B are given by

$$A = \frac{m'_1 - m'_2}{2}, \quad B = \frac{\lambda'_1 - \lambda'_2 + w'_1 - w'_2}{2}.$$

3. *The function* $\tilde{\varphi}_2^{\rm mod}(M')$ *has the factorization*

$$\tilde{\varphi}_2^{\rm mod}(M';y) = \sum_{i=1}^{2} \tilde{\varphi}_2^{(i)}(M';y).$$

8.2. **Integral formulas.** Here we recall some integral formulas which are fundamental to derive our propagation formula (see [4] for example).

The modified Bessel function $K_\nu(z)$ of the second kind has several integral expressions. Among them, we need two expressions: One is the integral expression of Mellin-Barnes type

$$K_\nu(z) = \frac{1}{4}\cdot\frac{1}{2\pi\sqrt{-1}}\int_s \Gamma\left[\frac{s+\nu}{2},\frac{s-\nu}{2}\right]\left(\frac{z}{2}\right)^{-s} ds.$$

Here, the path of integration is the vertical line from $\mathrm{Re}\, s-\sqrt{-1}\infty$ to $\mathrm{Re}\, s+\sqrt{-1}\infty$ with enough large real part. Another is that of Euler type

$$K_\nu(z) = \frac{1}{2}\int_0^\infty \exp\left(\frac{-z(t+t^{-1})}{2}\right) t^\nu \frac{dt}{t},$$

which is valid only for $\mathrm{Re}\, z>0$.

Also we need the following integral formula so-called Barnes' lemma

$$\frac{1}{2\pi\sqrt{-1}}\int_z \Gamma\left[z+a,z+b,-z+c,-z+d\right]dz = \Gamma\left[\begin{array}{c} a+c,a+d,b+c,b+d \\ a+b+c+d\end{array}\right].$$

Here the path of integration is the vertical line from $\mathrm{Re}\, z-\sqrt{-1}\infty$ to $\mathrm{Re}\, z+\sqrt{-1}\infty$ with enough large real part.

8.3. **Propagation formula.** Let $\pi=\pi(\nu,\sigma_{\mathbf{n}})$ be an irreducible principal series representation of $G=GL(3,\mathbf{C})$ with data $\nu=(\nu_1,\nu_2,\nu_3)\in\mathbf{C}^3$ and $\mathbf{n}=(n_1,n_2,n_3)\in\mathbf{Z}^3$ and let η be a non-degenerate unitary character of N specified by the parameters $c_1=c_2=\sqrt{-1}$ as in §7.3. For simplicity, we assume that the parameter $\mathbf{n}$ satisfies the regularity condition

$$n_1\ge n_2\ge n_3.$$

Then the minimal K-type of π is $(\tau_{\mathbf{m}},V_{\mathbf{m}})=(\tau_{\mathbf{n}},V_{\mathbf{n}})$.

Let $\varphi_3^{\rm mod}\in \mathrm{Wh}(\pi,\eta,\tau)^{\rm mod}$ be the primary Whittaker function with the M-components $\varphi_3^{\rm mod}(M) = y_1^2y_2^2\tilde{\varphi}_3^{\rm mod}(M)$ for each G-pattern $M=\begin{pmatrix} m_1\, m_2\, m_3 \\ \alpha_1\, \alpha_2 \\ \beta\end{pmatrix}\in G(\mathbf{m})$ given in Theorem 7.7. Under the regularity condition on $\mathbf{n}$, we have the parameters

$$(\lambda_1,\lambda_2,\lambda_3)=\left(\nu_3-\frac{\tilde{\nu}}{3},\nu_1-\frac{\tilde{\nu}}{3},\nu_2-\frac{\tilde{\nu}}{3}\right).$$

Theorem 8.3. *Let π, $\tau^*=\tau_{\mathbf{m}}$, and η be as above. The integrand $V_3(M;s_1,s_2)$ in the Mellin-Barnes type integral expression for the M-component $\tilde{\varphi}_3^{\rm mod}(M)$ in Theorem 7.7 has the following expression.*

$$V_3(M;s_1,s_2) \;=\; \Gamma\left[\frac{s_1+\zeta_j^{(1)}(M)}{2},\frac{s_2+\zeta_j^{(2)}(M)}{2}\right]$$

$$\times \frac{1}{2\pi\sqrt{-1}} \int_z \Gamma\left[\frac{z+s_1+\mu_1}{2}, \frac{z+s_2+\mu_2}{2}\right] V_2(M';-z)dz,$$

where $V_2(M';s)$ is the integrand of the integral expression of $\tilde{\varphi}_2^{\mathrm{mod}}(M')$ in Theorem 8.2 for a triple $(\pi'(\nu',\sigma_{\mathbf{n}'}),\eta',\tau_{\mathbf{m}'})$ and a G-pattern $M' \in G(\mathbf{m}')$ and the path of integration is the vertical line from $\mathrm{Re}\, z-\sqrt{-1}\infty$ to $\mathrm{Re}\, z+\sqrt{-1}\infty$ with large enough real part. The parameters and the representations are given as follows.

1. *If $\delta(M) \geq 0$, we have*

$$j=2,\ \ \mu_1 = -\frac{\lambda_2}{2}+\beta-\alpha_2,\ \ \mu_2 = \frac{\lambda_2}{2}+m_1-\alpha_1,$$
$$\nu' = (\nu_2,\nu_3),\ \ \mathbf{n}'=\mathbf{m}'=(m_2,m_3),\ \ M' = \begin{pmatrix} m_2\, m_3 \\ \alpha_2 \end{pmatrix}.$$

2. *If $\delta(M) \leq 0$, we have*

$$j=1,\ \ \mu_1 = -\frac{\lambda_1}{2}+\alpha_1-\beta,\ \ \mu_2 = \frac{\lambda_1}{2}+\alpha_2-m_3,$$
$$\nu' = (\nu_1,\nu_2),\ \ \mathbf{n}'=\mathbf{m}'=(m_1,m_2),\ \ M' = \begin{pmatrix} m_1\, m_2 \\ \alpha_1 \end{pmatrix}.$$

Proof. Assume $\delta(M) \geq 0$. Then, since $\zeta_1^{(1)}(M)+\zeta_1^{(2)}(M) = \zeta_3^{(1)}(M)+\zeta_3^{(2)}(M)$, Barnes' lemma leads the equation

$$\begin{aligned} V_3(M;s_1,s_2) &= \Gamma\left[\frac{s_1+\zeta_2^{(1)}(M)}{2}, \frac{s_2+\zeta_2^{(2)}(M)}{2}\right] \\ &\quad \times \frac{1}{2\pi\sqrt{-1}} \int_z \Gamma\left[\frac{z+s_1+\mu_1}{2}, \frac{z+s_2+\mu_2}{2}, \frac{-z+\mu_3}{2}, \frac{-z+\mu_4}{2}\right] dz, \end{aligned}$$

where the parameters μ_1 and μ_2 are given in the assertion of theorem and μ_3 and μ_4 are

$$\mu_3 = \frac{-\nu_2+\nu_3}{2}+\alpha_2-m_3, \quad \mu_4 = \frac{\nu_2-\nu_3}{2}-\alpha_2+m_2.$$

Here we use the relations $\lambda_1+\dfrac{\lambda_2}{2} = \dfrac{-\nu_2+\nu_3}{2}$ and $\lambda_3+\dfrac{\lambda_2}{2} = \dfrac{\nu_2-\nu_3}{2}$.

In the case of $\delta(M) \leq 0$, the relation $\zeta_2^{(1)}(M)+\zeta_2^{(2)}(M) = \zeta_3^{(1)}(M)+\zeta_3^{(2)}(M)$ brings the assertion by similar computation. □

Corollary 8.4. *We have the following expression of $\tilde{\varphi}_3^{\mathrm{mod}}(M)$.*

$$\begin{aligned} &\tilde{\varphi}_3^{\mathrm{mod}}(M;y) \\ = &\ \frac{2^4}{2\pi\sqrt{-1}} \int_z (\pi y_1)^{\frac{z}{2}+a_1} K_{-\frac{z}{2}+A_1}(2\pi y_1)(\pi y_2)^{\frac{z}{2}+a_2} K_{\frac{z}{2}-A_2}(2\pi y_2) V_2(M';-z)dz. \end{aligned}$$

Here

$$a_k = \frac{1}{2}\left\{\zeta_j^{(k)}(M)+\mu_k\right\}, \quad A_k = \zeta_j^{(k)}(M)-a_k, \quad k=1,2,$$

and the parameters and the representations are given in Theorem 8.3.

Proof. Using the first integral expression of $K_\nu(z)$ of Mellin-Barnes type in the previous subsection, we can get the corollary from Theorem 8.3 together with the integral expression of Mellin-Barnes type for $\tilde{\varphi}_3^{\mathrm{mod}}(M)$ in Theorem 7.7. □

Corollary 8.5. *We have the following expression of $\tilde{\varphi}_3^{\rm mod}(M)$.*

$$\begin{aligned}\tilde{\varphi}_3^{\rm mod}(M;y) &= 4\pi^{a_1+a_2}y_1^{a_1+A_1}y_2^{a_2-A_2}\int_0^\infty\int_0^\infty \exp\left(-\pi\left(y_1^2u_1+\frac{1}{u_1}+y_2^2u_2+\frac{1}{u_2}\right)\right)\\ &\quad\times u_1^{A_1}u_2^{-A_2}\tilde{\varphi}_2^{\rm mod}\left(M';y_2\sqrt{\frac{u_2}{u_1}}\right)\frac{du_1}{u_1}\frac{du_2}{u_2}.\end{aligned}$$

Here the parameters and the representations are given in Theorem 8.3.

Proof. By applying the second integral expression of $K_\nu(z)$ in the previous subsection to the expression of $\tilde{\varphi}_3^{\rm mod}(M)$ in Corollary 8.4, we have

$$\begin{aligned}&\tilde{\varphi}_3^{\rm mod}(M;y)\\ =\ &\frac{4}{2\pi\sqrt{-1}}\int_0^\infty\int_0^\infty\int_z \exp\left(-\pi y_1\left(u_1+\frac{1}{u_1}\right)-\pi y_2\left(u_2+\frac{1}{u_2}\right)\right)u_1^{A_1}u_2^{-A_2}\\ &\times(\pi y_1)^{a_1}(\pi y_2)^{a_2}\left(\pi^2y_1y_2\frac{u_2}{u_1}\right)^{\frac{z}{2}}V_2(M';-z)\frac{du_1}{u_1}\frac{du_2}{u_2}dz.\end{aligned}$$

Then we can get the assertion by the substitutions $u_1\to u_1y_1$, $u_2\to u_2y_2$, and $z\to -z$ in the above integrals. □

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