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A NEW WEAK APPROXIMATION SCHEME OF STOCHASTIC DIFFERENTIAL EQUATIONS BY USING THE RUNGE-KUTTA METHOD

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1. Introduction

A number of studies on numerical calculations of stochastic differential equations (SDEs) have been carried out as there is a great demand for it in various fields such as mathematical finance. It is shown in [11], [15], [16], and [19] that the new higher order scheme introduced by Kusuoka in [8] and [10] does extremely faster calculation in application to some finance problems. Lyons and Victoir extensively developed the scheme in [13] by using the notion of free Lie algebra.

In this paper, we successfully construct in Theorem 1.1 and Corollary 1.1 a new implementation method of the new higher order scheme of weak approximation. The point in the algorithm is that the approximation operator can be considered to be composition of solutions of ODEs when ω is given. The concrete ODEs are constructed by Theorem 1.2 and can be approximated by the Runge-Kutta method for ODEs by Theorem 4.1. We should note that another higher-order weak approximation method is introduced in [17]. Although this algorithm and the new method which we are going to present in this paper are based on the same scheme ([8] [10][13]) and have many common features, algorithms themselves are completely different and the diversity is not trivial.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. We define $B^0(t)$ as t and $(B^1(t), \dots, B^d(t))$ as the d-dimensional standard Brownian Motion. $C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$ denotes the set of $\mathbb{R}^N$-valued infinitely differentiable functions defined over $\mathbb{R}^N$ whose derivatives are all bounded. Our interest is in weak approximation, that is to say, approximation of $(P_t f)(x) = E[f(X(1, x))]$ where $f \in C_b^\infty(\mathbb{R}^N; \mathbb{R})$ and $X(t, x)$ is a solution to the Stratonovich stochastic integral equation

$$X(t, x) = x + \sum_{i=0}^{d} \int_0^t V_i(X(s, x)) \circ dB^i(s), \tag{1.1}$$

where $V_i \in C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$, $i = 0, \dots, d$. $V_i \in C_b^\infty\left(\mathbb{R}^N; \mathbb{R}^N\right)$ is regarded as a vector field in the following way:

$$V_i f(x) = \sum_{j=1}^{N} V_i^j(x) \frac{\partial f}{\partial x_j}(x), \quad \text{for } f \in C_b^\infty(\mathbb{R}^N; \mathbb{R}).$$

Let $A = \{v_0, v_1, \dots, v_d\}$, $d \geq 1$ be an alphabet and A^* denote the set of all words consisting of the elements of A. The empty word 1 is the identity of A^*. For

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$u = v_{i_1} \cdots v_{i_n} \in A^*$, $i_k \in \{0, 1, \ldots, d\}$, $|u|$ and $\|u\|$ are defined by $|u| = n$ and $\|u\| = |u| + \mathrm{card}(\{k \,|\, i_k = 0\})$, respectively, where $\mathrm{card}(S)$ denotes the cardinality of a set S. A^*_m and $A^*_{\leq m}$ denote $\{w \in A^* |\ |w| = m\}$ and $\{w \in A^* |\ |w| \leq m\}$, respectively. Let $\mathbb{R}\langle A\rangle$ be the $\mathbb{R}$-coefficient free algebra with basis A^* and $\mathbb{R}\langle\langle A\rangle\rangle$ be the set of all $\mathbb{R}$-coefficient formal series with basis A^*. Then, $\mathbb{R}\langle A\rangle$ is a sub $\mathbb{R}$-algebra of $\mathbb{R}\langle\langle A\rangle\rangle$. We call an element of $\mathbb{R}\langle A\rangle$ a non-commutative polynomial. Let $\mathbb{R}\langle A\rangle_m = \{P \in \mathbb{R}\langle A\rangle \,|\, (P, w) = 0, \text{ if } \|w\| \neq m\}$. $P \in \mathbb{R}\langle\langle A\rangle\rangle$ is written as

$$P = \sum_{w \in A^*} (P, w)\, w \quad \text{or} \quad \sum_{w \in A^*} a_w w,$$

where $(P, w) = a_w \in \mathbb{R}$ denotes the coefficient of w. The algebra structure is defined as usual, that is to say,

$$\left(\sum_{w \in A^*} a_w w\right)\left(\sum_{w \in A^*} b_w w\right) = \sum_{\substack{w = uv \\ w \in A^*}} a_u b_v w.$$

The Lie bracket is defined as $[x, y] = xy - yx$ for $x, y \in \mathbb{R}\langle\langle A\rangle\rangle$. For $w = v_{i_1} \cdots v_{i_n} \in A^*$, $\mathrm{r}(w)$ denotes $[v_{i_1}, [v_{i_2}, [\ldots, [v_{i_{n-1}}, v_{i_n}] \ldots]]]$. We define $\mathcal{L}_{\mathbb{R}}(A)$ as the set of Lie polynomials in $\mathbb{R}\langle A\rangle$ and $\mathcal{L}_{\mathbb{R}}((A))$ as the set of Lie series. For $m \in \mathbb{Z}_{\geq 0}$, let j_m be a map defined as follows:

$$j_m\left(\sum_{w \in A^*} a_w w\right) = \sum_{\|w\| \leq m} a_w w.$$

For arbitrary $P, Q \in \mathbb{R}\langle A\rangle$, the inner product $\langle P, Q\rangle$ is defined as follows:

$$\langle P, Q\rangle = \sum_{w \in A^*} (P, w)(Q, w).$$

Also we let $\|P\|_2 = (\langle P, P\rangle)^{1/2}$ for $P \in \mathbb{R}\langle A\rangle$. For $P \in \mathbb{R}\langle\langle A\rangle\rangle$ such that $(P, 1) = 0$, we can define $\exp(P)$ as $1 + \sum_{k=1}^{\infty} P^k/k!$. Also, $\log(Q)$ can be defined as $\sum_{k=1}^{\infty} (-1)^{k-1}(Q - 1)^k/k$ for $Q \in \mathbb{R}\langle\langle A\rangle\rangle$ if $(Q, 1) = 1$. The following relations hold:

$$\log(\exp(P)) = P \quad \text{and} \quad \exp(\log(Q)) = Q.$$

By the natural identification $\mathbb{R}\langle\langle A\rangle\rangle \approx \mathbb{R}^{\infty}$, we can induce the direct product topology into $\mathbb{R}\langle\langle A\rangle\rangle$. $\mathbb{R}\langle\langle A\rangle\rangle$ becomes a Polish space by the topology. Also we can consider its Borel σ-algebra $\mathcal{B}(\mathbb{R}\langle\langle A\rangle\rangle)$, $\mathbb{R}\langle\langle A\rangle\rangle$-valued random variables, their expectations, and other notions as usual.

Let Φ be a homomorphism between $\mathbb{R}\langle A\rangle$ and the $\mathbb{R}$-algebra which consists of smooth differential operators over $\mathbb{R}^N$ such that

$$\begin{aligned} &\Phi(1) = \mathrm{Id}, \\ &\Phi(v_{i_1} \cdots v_{i_n}) = V_{i_1} \cdots V_{i_n},\ i_1, \ldots, i_n \in \{0, 1, \ldots, d\}. \end{aligned} \tag{1.2}$$

Also, for $s \in \mathbb{R}_{>0}$, $\Psi_s : \mathbb{R}\langle\langle A\rangle\rangle \longrightarrow \mathbb{R}\langle\langle A\rangle\rangle$ is defined as follows:

$$\Psi_s\left(\sum_{m=0}^{\infty} P_m\right) = \sum_{m=0}^{\infty} s^{m/2} P_m, \text{ where } P_m \in \mathbb{R}\langle A\rangle_m.$$

For a smooth vector field V, i. e. an element of $C_b^\infty\left(\mathbb{R}^N;\mathbb{R}^N\right)$, $\exp\left(V\right)\left(x\right)$ denotes the solution at time 1 of the ordinary differential equation

$$\frac{dz_t}{dt} = V\left(z_t\right), \quad z_0 = x.$$

We also define $\|V\|_{C^n}$ for $V \in C_b^\infty(\mathbb{R}^N;\mathbb{R}^N)$ as follows:

$$\begin{aligned} \|V\| &= \sup_x |V(x)| \\ \left\|V^{(n)}\right\| &= \sup_x \left\{\left|V_{(x)}^{(n)}(U_1, U_2, \ldots, U_n)\right|;\ \|U_i\| = 1, i = 1, \ldots, n\right\} \\ \|V\|_{C^n} &= \sum_{i=0}^{n} \left\|V^{(i)}\right\| \end{aligned}$$

Here $V^{(k)}$ denotes the k-th order total differential of V, that is,

$$V_{(x)}^{(n)}\left(U_1, U_2, \ldots, U_n\right) = \sum_{i=1}^{N}\sum_{j_1=1}^{N}\cdots\sum_{j_n=1}^{N} \frac{\partial^n V_i}{\partial x_{j_1}\cdots\partial x_{j_n}}(x) U_1^{j_1}\cdots U_n^{j_n} e_i$$

where e_i denotes an N-dimensional unit vector and U_k^j is the j-th component of U_k.

Definition 1.1. *A map g from $C_b^\infty(\mathbb{R}^N;\mathbb{R}^N)$ to the set of all maps from $\mathbb{R}^N$ to $\mathbb{R}^N$ is called an integration scheme of order m if there exists a positive constant C_m such that*

$$\left|g(W)(x) - \exp\left(W\right)(x)\right| \leq C_m \|W\|_{C^{m+1}}^{m+1} \tag{1.3}$$

for all $W \in C_b^\infty(\mathbb{R}^N;\mathbb{R}^N)$ and $x \in \mathbb{R}^N$. Here C_m depends only on m and g. Let $\mathcal{IS}(m)$ be the set of all integration schemes of order m.

Notation 1.1. *For $z_1, z_2 \in \mathcal{L}_{\mathbb{R}}((A))$, we define $z_2 \mathbin{\text{H}} z_1$ as $\log(\exp(z_2)\exp(z_1))$. Then from the definition, for $z_1, z_2, z_3 \in \mathcal{L}_{\mathbb{R}}((A))$,*

$$\left(z_1 \mathbin{\text{H}} z_2\right) \mathbin{\text{H}} z_3 = \log\left(\exp(z_1)\exp(z_2)\exp(z_3)\right) = z_1 \mathbin{\text{H}} \left(z_2 \mathbin{\text{H}} z_3\right),$$

and so we can write for $z_1, \ldots, z_n \in \mathcal{L}_{\mathbb{R}}((A))$

$$z_1 \mathbin{\text{H}} z_2 \mathbin{\text{H}} \cdots \mathbin{\text{H}} z_n = \log\left(\exp(z_1)\cdots\exp(z_n)\right). \tag{1.4}$$

The followings are the main results of our study.

Theorem 1.1. *Let $m \geq 1$, $n \geq 2$, and $Z_1, \ldots, Z_n$ be $\mathcal{L}_{\mathbb{R}}((A))$-valued random variables. Assume that $Z_1, \ldots, Z_n$ satisfy the followings:*

$$Z_i = j_m Z_i \quad i = 1, \ldots, n, \tag{1.5}$$

$$E\left[\left\|j_m Z_i\right\|_2\right] < \infty, \tag{1.6}$$

$$E\left[\exp\left(a\sum_{j=1}^{n}\left\|\Phi\Psi_s\left(Z_j\right)\right\|_{C^{m+1}}\right)\right] < \infty \quad \text{for any } a > 0. \tag{1.7}$$

Then for arbitrary $g_1, \ldots, g_n \in \mathcal{IS}(m)$, there exists a positive constant C such that

$$\left\|g_1\left(\Phi\left(\Psi_s\left(Z_1\right)\right)\right) \circ \cdots \circ g_n\left(\Phi\left(\Psi_s\left(Z_n\right)\right)\right)(x) - \exp\left(\Phi\left(\Psi_s\left(j_m\left(Z_n \mathbin{\text{H}} \cdots \mathbin{\text{H}} Z_1\right)\right)\right)\right)(x)\right\|_{L^p} \leq Cs^{(m+1)/2}. \tag{1.8}$$

Here for functions f and g, $f \circ g(x)$ denotes $f\left(g(x)\right)$ as usual.

Let $\left\{S^i_j\right\}_{i=1,\dots,d,\ j=1,\dots,n}$ be a set of $\mathbb{R}$-valued normally distributed random variable and c_j's and $R_{jj'}$ be real numbers such that

$$\sum_{j=1}^{n} c_j = 1, \quad E\left[S^i_j\right] = 0, \quad \text{and} \quad E\left[S^i_j S^{i'}_{j'}\right] = R_{jj'}\delta_{ii'} \tag{1.9}$$

for $j, j' = 1, \dots, n$, and $i, i' = 1, \dots, d$,. Here our interest is in finding a set of random variables $\left\{Z_j = c_j v_0 + \sum_{i=1}^{d} S^i_j v_i \ : \ j = 1, \dots, n\right\}$, such that

$$E\left[j_m\left(\exp\left(Z_1\right)\cdots\exp\left(Z_n\right)\right)\right] = j_m\left(\exp\left(v_0 + \frac{1}{2}\sum_{i=1}^{d} v_i^2\right)\right). \tag{1.10}$$

Theorem 1.2. *For $m = 5$, $n = 2$, Z_j as above can be constructed if and only if*

$$\begin{aligned} c_1 &= \frac{\mp\sqrt{2\left(2u-1\right)}}{2}, \quad c_2 = 1 \pm \frac{\sqrt{2\left(2u-1\right)}}{2}, \quad R_{11} = u \\ R_{22} &= 1 + u \pm \sqrt{2\left(2u-1\right)}, \quad R_{12} = -u \mp \frac{\sqrt{2\left(2u-1\right)}}{2} \end{aligned} \tag{1.11}$$

for some $u \geq 1/2$.

Remark 1.1. *We can show that in the case where $m = 7$, $n = 3$ there is no solution to* (1.9) *and* (1.10).

The M-stage Runge-Kutta method of order m in the sense of [2] can be written as follows: for $W \in C_b^\infty\left(\mathbb{R}^N; \mathbb{R}^N\right)$,

$$\begin{aligned} Y_i\left(W, s\right) &= y + s\sum_{j=1}^{M} a_{ij} W\left(Y_j(W, s)\right), \\ Y(y; W, s) &= y + s\sum_{i=1}^{M} b_i W\left(Y_i(W, s)\right), \end{aligned} \tag{1.12}$$

where $A = \left(a_{ij}\right)_{i,j=1,\dots,M}$ with $a_{ij} \in \mathbb{R}$ and $b = {}^t(b_1, \dots, b_M) \in \mathbb{R}^M$ satisfy (4.2) in Section 4. (1.12) gives the m-th order approximation of an ODE

$$\frac{d}{dt} y(t) = W\left(y(t)\right), \quad y(0) = y. \tag{1.13}$$

Let $g_{(m)}(W)(y)$ be $Y(y; W, 1)$. Then $g_{(m)}$ belongs to $\mathcal{IS}(m)$, which is Theorem 4.1 in Section 4.

Corollary 1.1. *Let Z_j's $j = 1, \dots, n$, be $\mathcal{L}_{\mathbb{R}}((A))$-valued random variables as in Theorem 1.2 and define linear operators $Q_{(s)}$, $s \in (0, 1]$ by*

$$\left(Q_{(s)} f\right)(x) = E\left[f\left(g_{(m)}\left(\Phi\Psi_s\left(Z_1\right)\right) \circ \cdots \circ g_{(m)}\left(\Phi\Psi_s\left(Z_n\right)\right)(x)\right)\right]. \tag{1.14}$$

Then for $f \in C_b^\infty(\mathbb{R}^N; \mathbb{R})$,

$$\left\|P_s f - Q_{(s)} f\right\|_\infty \leq C s^{(m+1)/2} \left\|\mathrm{grad}(f)\right\|_\infty. \tag{1.15}$$

where C is a positive constant and $s \in (0, 1]$.

Remark 1.2. *Kusuoka has shown the following results in* [10]*:*

(1) *For a Lipschitz continuous function f, the inequality* (1.15) *still holds.*

(2) *The Romberg extrapolation can be applied to this algorithm.*

2. Proof of Theorem 1.1

In order to prove Theorem 1.1, we provide some lemmas first.

Proposition 2.1.

(1) *For any $V \in C_b^\infty(\mathbb{R}^N;\mathbb{R})$, $f \in C^\infty(\mathbb{R}^N;\mathbb{R})$, $x \in \mathbb{R}^N$ and $n \geq 1$,*

$$f(\exp(tV)(x)) = \sum_{k=0}^{n} \frac{t^k}{k!}\left(V^k f\right)(x) + \int_0^t \frac{(t-s)^n}{n!}\left(V^{n+1} f\right)(\exp(sV)(x))\,ds. \tag{2.1}$$

(2) *For all $z \in \mathcal{L}_{\mathbb{R}}((A))$, and $n, m \geq 1$,*

$$\left| f\left(\exp\left(\Phi(j_m z)\right)(x)\right) - \sum_{k=0}^{n} \frac{1}{k!}\left(\Phi\left((j_m z)^k\right) f\right)(x)\right| \leq \frac{1}{(n+1)!}\left\|\Phi\left((j_m z)^{n+1} f\right)\right\|_\infty. \tag{2.2}$$

Proof. Since we have

$$\frac{d}{dt} f(\exp(tV)(x)) = Vf\left(\exp(tV)(x)\right),$$

from the Taylor expansion and by integration by parts we obtain (2.1). (2.2) can be derived from (2.1). □

Lemma 2.1. *For all $n \geq 1$, there exists a constant $C_n > 0$ such that for all $z \in \mathcal{L}_{\mathbb{R}}((A))$ and $f \in C^\infty(\mathbb{R}^N;\mathbb{R})$,*

$$\left\|\Phi(j_n z) f\right\|_\infty \leq C_n \left\|j_n z\right\|_2 \left\|\mathrm{grad}(f)\right\|_{C^{n-1}} \tag{2.3}$$

Proof. Let p_m be a map such that

$$p_m \ : \ \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha \longmapsto \sum_{|\alpha|=m} a_\alpha D^\alpha$$

where $a_\alpha \in C_b^\infty(\mathbb{R}^N;\mathbb{R}^N)$ and α is a multi-index. Then we have

$$\Phi(w) = \sum_{i=1}^{|w|} p_i(\Phi(w)),$$

because $|w| \geq 1$. Since there exists a constant $C_{w,i} > 0$ such that

$$\left\|p_i(\Phi(w)) f\right\|_\infty \leq C_{w,i} \sup_{\substack{\alpha \in (\mathbb{Z}_{\geq 0})^N \\ |\alpha| = i-1}} \left\|D^\alpha(\mathrm{grad}(f))\right\|_\infty,$$

we can obtain

$$\begin{aligned}
\left\|\Phi(j_n z)f\right\|_\infty &\le \sum_{\substack{w\in A^* \\ 1\le\|w\|\le n}} \left\|\Phi(w)f\right\|_\infty |\langle z,w\rangle| \\
&\le \sum_{\substack{w\in A^* \\ 1\le\|w\|\le n}} \sum_{i=1}^{|w|} C_{w,i}\, |\langle z,w\rangle| \sup_{\substack{\alpha\in(\mathbb{Z}_{\ge 0})^N \\ |\alpha|=i-1}} \left\|D^\alpha(\mathrm{grad}(f))\right\|_\infty \\
&\le C_n \left\|j_n z\right\|_2 \sup_{\substack{\alpha\in(\mathbb{Z}_{\ge 0})^N \\ |\alpha|=i-1}} \left\|D^\alpha(\mathrm{grad}(f))\right\|_\infty \\
&\le C_n \left\|j_n z\right\|_2 \left\|\mathrm{grad}(f)\right\|_{C^{n-1}}
\end{aligned}$$

where $C_n = \sup_{\substack{w\in A^* \\ 1\le\|w\|\le n}} (\sum_{i=1}^{|w|} C_{w,i})$. □

For simplification of notation, we let $\Phi_s(y)$ denote $\Phi(\Psi_s(y))$ for an element $y \in \mathcal{L}_{\mathbb{R}}((A))$ in the following part.

Lemma 2.2. *For $z_1,\dots,z_n \in \mathcal{L}_{\mathbb{R}}((A))$, there exists a constant $C_{m,n} > 0$ such that*

(2.4)
$$\begin{aligned}
&\left| f\left(\exp\left(\Phi_s\left((j_m z_n)\,\mathsf{H}\cdots\mathsf{H}\,(j_m z_1)\right)\right)(x)\right) - \left(\Phi_s\left(j_m \exp\left((j_m z_n)\,\mathsf{H}\cdots\mathsf{H}\,(j_m z_1)\right)\right) f\right)(x)\right| \\
&\qquad \le C_{m,n} s^{(m+1)/2} \left(1 + \sum_{i=1}^n \left\|j_m z_i\right\|_2\right)^{m+1} \left\|\mathrm{grad}(f)\right\|_{C^{m(m+1)-1}}.
\end{aligned}$$

Proof. From the fact that

$$j_m\left(\exp\left(j_m z\right)\right) = \sum_{k=0}^m \frac{1}{k!}\left(j_m z\right)^k - \sum_{k=2}^m \frac{1}{k!}\left(j_{m(m+1)} - j_m\right)\left(\left(j_m z\right)^k\right),$$

and (2.2) in Proposition 2.1,

$$\begin{aligned}
(2.5)\quad &\left| f\left(\exp\left(\Phi(j_m z)\right)(x)\right) - \left(\Phi\left(j_m\left(\exp(j_m z)\right)\right) f\right)(x)\right| \\
&\le \frac{1}{(m+1)!}\left\|\Phi\left(\left(j_m z\right)^{m+1}\right) f\right\|_\infty + \left|\sum_{k=2}^m \frac{1}{k!}\left(\Phi\left(\left(j_{m(m+1)} - j_m\right)\left(\left(j_m z\right)^k\right)\right) f\right)(x)\right|.
\end{aligned}$$

Since for $z \in \mathcal{L}_{\mathbb{R}}((A))$,

$$\left(j_m z\right)^{m+1} = \left(j_{m(m+1)} - j_m\right)\left(j_m z\right)^{m+1},$$

we can derive the followings by applying Lemma 2.1:

$$\begin{aligned}
&\left| f\left(\exp\left(\Phi\left(j_m z\right)\right)(x)\right) - \left(\Phi\left(j_m\left(\exp\left(j_m z\right)\right)\right) f\right)(x)\right| \\
(2.6)\qquad &\le \sum_{k=2}^{m+1} \frac{1}{k!}\left\|\Phi\left(\left(j_{m(m+1)} - j_m\right)\left(\left(j_m z\right)^k\right)\right) f\right\|_\infty \\
&\le C_m \sum_{k=2}^{m+1}\left\|\Phi\left(\left(j_{m(m+1)} - j_m\right)\left(j_m z\right)^k\right)\right\|_2 \left\|\mathrm{grad}(f)\right\|_{C^{m(m+1)-1}}
\end{aligned}$$

where C_m is a positive constant.

Taking $z_n \mathrm{H} \cdots \mathrm{H} z_1$ as z above and evaluating by

$$\sum_{k=1}^{m+1} \left\| \left(j_{m(m+1)} - j_m \right) \left(j_m z_n \mathrm{H} \cdots \mathrm{H} z_1 \right)^k \right\|_2 \leq C_{m,n} \left(1 + \sum_{i=1}^{n} \left\| j_m z_i \right\|_2 \right)^{m+1},$$

we obtain (2.4). □

Lemma 2.3. *For $z_1, \ldots, z_n \in \mathcal{L}_{\mathbb{R}}((A))$, there exists a constant $C > 0$ such that*

(2.7)
$$\left| f\left(\exp\left(\Phi_s\left(j_m z_1\right)\right) \circ \cdots \circ \exp\left(\Phi_s\left(j_m z_n\right)\right)(x)\right) - \left(\Phi_s\left(j_m \exp\left(\left(j_m z_n\right) \mathrm{H} \cdots \mathrm{H} \left(j_m z_1\right)\right)\right) f\right)(x) \right|$$
$$\leq C s^{(m+1)/2} \sum_{i=1}^{n} \left(1 + \left\| j_m z_i \right\|_2\right)^{m+1} \left\| \mathrm{grad}(f) \right\|_{C^{m(m+n)-1}}.$$

Proof. We prove the lemma by induction on n. When $n = 1$, (2.4) and (2.7) are equivalent. Assume that (2.7) holds for n. Then

$$\begin{aligned}
&\left| f\left(\exp\left(\Phi_s\left(j_m z_1\right)\right) \circ \cdots \circ \exp\left(\Phi_s\left(j_m z_{n+1}\right)\right)(x)\right) - \left(\Phi_s\left(j_m \exp\left(\left(j_m z_{n+1}\right) \mathrm{H} \cdots \mathrm{H} \left(j_m z_1\right)\right)\right) f\right)(x) \right| \\
&\leq \left| f\left(\exp\left(\Phi_s\left(j_m z_1\right)\right) \circ \cdots \circ \exp\left(\Phi_s\left(j_m z_{n+1}\right)\right)(x)\right) \right. \\
&\qquad \left. - \left(\Phi_s\left(j_m \exp\left(\left(j_m z_n\right) \mathrm{H} \cdots \mathrm{H} \left(j_m z_1\right)\right)\right) f\right)\left(\exp\left(\Phi_s\left(j_m z_{n+1}\right)\right)(x)\right) \right| \\
&\quad + \left| \left(\Phi_s\left(j_m \exp\left(\left(j_m z_n\right) \mathrm{H} \cdots \mathrm{H} \left(j_m z_1\right)\right)\right) f\right)\left(\exp\left(\Phi_s\left(j_m z_{n+1}\right)\right)(x)\right) \right. \\
&\qquad \left. - \left(\Phi_s\left(j_m \exp\left(\left(j_m z_{n+1}\right) \mathrm{H} \cdots \mathrm{H} \left(j_m z_1\right)\right)\right) f\right)(x) \right|.
\end{aligned}$$

Substituting $\Phi_s\left(j_m \exp\left(\left(j_m z_n\right) \mathrm{H} \cdots \mathrm{H} \left(j_m z_1\right)\right)\right) f$ into f in (2.4), we can derive evaluation of the second term on the right-hand side. As a result, we obtain

$$\begin{aligned}
&\left| f\left(\exp\left(\Phi_s\left(j_m z_1\right)\right) \circ \cdots \circ \exp\left(\Phi_s\left(j_m z_{n+1}\right)\right)(x)\right) - \left(\Phi_s\left(j_m \exp\left(\left(j_m z_{n+1}\right) \mathrm{H} \cdots \mathrm{H} \left(j_m z_1\right)\right)\right) f\right)(x) \right| \\
&\leq C_1 s^{(m+1)/2} \left(\sum_{i=1}^{n} \left(1 + \left\| j_m z_i \right\|_2\right)^{m+1} + \left(1 + \left\| j_m z_{n+1} \right\|_2\right)^{m+1} \right) \left\| \mathrm{grad}(f) \right\|_{C^{m(m+n+1)-1}},
\end{aligned}$$

where $C_1 > 0$ is a constant and the statement holds in the case of $n + 1$. □

Lemma 2.4. *For all $m \geq 1$, there exists a constant $C_{m,n} > 0$ such that for all $s \in (0, 1]$, $z_1, \ldots, z_n \in \mathcal{L}_{\mathbb{R}}((A))$, and $f \in C^{\infty}(\mathbb{R}^N; \mathbb{R})$*

(2.8)
$$\left| f\left(\exp\left(\Phi_s\left(j_m z_1\right)\right) \circ \cdots \circ \exp\left(\Phi_s\left(j_m z_n\right)\right)(x)\right) - f\left(\exp\left(\Phi_s\left(j_m \left(\left(j_m z_n\right) \mathrm{H} \cdots \mathrm{H} (j_m z_1)\right)\right)\right)(x)\right) \right|$$
$$\leq C_{m,n} s^{(m+1)/2} \sum_{i=1}^{n} \left(1 + \left\| j_m z_i \right\|_2\right)^{m+1} \left\| \mathrm{grad}(f) \right\|_{C^{m(m+n)-1}}.$$

Proof. We have

(2.9)
$$\begin{aligned}
&\left| f\left(\exp\left(\Phi_s\left(j_m z_1\right)\right) \circ \cdots \circ \exp\left(\Phi_s\left(j_m z_n\right)\right)(x)\right) - f\left(\exp\left(\Phi_s\left(j_m \left(\left(j_m z_n\right) \mathrm{H} \cdots \mathrm{H} (j_m z_1)\right)\right)\right)(x)\right) \right| \\
&\leq \left| f\left(\exp\left(\Phi_s\left(j_m z_1\right)\right) \circ \cdots \circ \exp\left(\Phi_s\left(j_m z_n\right)\right)(x)\right) - \left(\Phi_s\left(j_m \exp\left(\left(j_m z_n\right) \mathrm{H} \cdots \mathrm{H} \left(j_m z_1\right)\right)\right) f\right)(x) \right| \\
&\quad + \left| f\left(\exp\left(\Phi_s\left(j_m \left(\left(j_m z_n\right) \mathrm{H} \cdots \mathrm{H} \left(j_m z_1\right)\right)\right)\right)(x)\right) - \left(\Phi_s\left(j_m \exp\left(\left(j_m z_n\right) \mathrm{H} \cdots \mathrm{H} \left(j_m z_1\right)\right)\right) f\right)(x) \right|.
\end{aligned}$$

From Lemmas 2.2 and 2.3, (2.8) can be derived. □

Lemma 2.5. *Let $Z_1, \dots, Z_n$ be $\mathcal{L}_{\mathbb{R}}((A))$-valued random variables such that for $m \geq 1$, $E\left[\left\|j_m Z_i\right\|_2\right] < \infty$, $i = 1, \dots, n$. Then, for $p \in [1, \infty)$, there exists a constant $C_{m,n} > 0$ such that for any $s \in (0,1]$ and $x \in \mathbb{R}^N$,*

(2.10)
$$\left\|\exp\left(\Phi_s\left(j_m Z_1\right)\right) \circ \cdots \circ \exp\left(\Phi_s\left(j_m Z_n\right)\right)(x) - \exp\left(\Phi_s\left(j_m\left(\left(j_m Z_n\right) \text{H} \cdots \text{H}\left(j_m Z_1\right)\right)\right)\right)(x)\right\|_{L^p} \leq C_{m,n} s^{(m+1)/2}.$$

Proof. If for $i \in \{1, \dots, N\}$, $f((x^1, \dots, x^N)) = x^i$, then $\left\|\mathrm{grad}(f)\right\|_{C^{m(m+n)-1}} = 1$ for all $m \geq 1$. Therefore, applying Lemma 2.4 for such f, we obtain (2.10). □

Proposition 2.2. *For $g \in \mathcal{IS}(m)$ and $W \in C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$, there exists a constant $C > 0$ such that*

$$\left|g(W)(x) - g(W)(y)\right| \leq C\left\|W\right\|_{C^{m+1}}^{m+1} + |x - y| \exp\left(\|W\|_{C^1}\right). \tag{2.11}$$

Proof. Since from the Gronwall's inequality we have

$$\left|\exp\left(W\right)(x) - \exp\left(W\right)(y)\right| \left|x - y\right| \exp\left(\|W\|_{C^1}\right),$$

(2.11) can be derived. □

Now we give the proof of Theorem 1.1.

(2.12)
$$\begin{aligned}
&\left\|g_1\left(\Phi_s\left(Z_1\right)\right) \circ \cdots \circ g_n\left(\Phi_s\left(Z_n\right)\right)(x) - \exp\left(\Phi_s\left(j_m\left(Z_n \text{H} \cdots \text{H} Z_1\right)\right)\right)(x)\right\|_{L^p} \\
&\leq \left\|\exp\left(\Phi_s\left(Z_1\right)\right) \circ \cdots \circ \exp\left(\Phi_s\left(Z_n\right)\right)(x) - \exp\left(\Phi_s\left(j_m\left(Z_n \text{H} \cdots \text{H} Z_1\right)\right)\right)(x)\right\|_{L^p} \\
&\quad + \left\|g_1\left(\Phi_s\left(Z_1\right)\right) \circ \cdots \circ g_n\left(\Phi_s\left(Z_n\right)\right)(x) - \exp\left(\Phi_s\left(Z_1\right)\right) \circ \cdots \circ \exp\left(\Phi_s\left(Z_n\right)\right)(x)\right\|_{L^p}.
\end{aligned}$$

Since $g_i \in \mathcal{IS}(m)$ and Z_i satisfies (1.7), we have for some $C_1 > 0$,

$$\left\|g_n\left(\Phi_s\left(Z_n\right)\right)(x) - \exp\left(\Phi_s\left(Z_n\right)\right)(x)\right\|_{L^p} \leq \left\|C_m\left\|\Phi_s\left(Z_n\right)\right\|_{C^{m+1}}^{m+1}\right\|_{L^p} \leq C_1 s^{(m+1)/2}. \tag{2.13}$$

From this fact and Proposition 2.2, there exists a constant $C_3 > 0$ such that

$$\begin{aligned}
&\left\|g_{n-1}\left(\Phi_s\left(Z_{n-1}\right)\right) \circ g_n\left(\Phi_s\left(Z_n\right)\right)(x) - \exp\left(\Phi_s\left(Z_{n-1}\right)\right) \circ \exp\left(\Phi_s\left(Z_n\right)\right)(x)\right\|_{L^p} \\
&\leq \left\|g_{n-1}\left(\Phi_s\left(Z_{n-1}\right)\right) \circ \exp\left(\Phi_s\left(Z_n\right)\right)(x) - \exp\left(\Phi_s\left(Z_{n-1}\right)\right) \circ \exp\left(\Phi_s\left(Z_n\right)\right)(x)\right\|_{L^p} \\
&\quad + \left\|g_{n-1}\left(\Phi_s\left(Z_{n-1}\right)\right) \circ g_n\left(\Phi_s\left(Z_n\right)\right)(x) - g_{n-1}\left(\Phi_s\left(Z_{n-1}\right)\right) \circ \exp\left(\Phi_s\left(Z_n\right)\right)(x)\right\|_{L^p} \\
&\leq \left\|C_m\left\|\Phi_s\left(Z_{n-1}\right)\right\|_{C^{m+1}}^{m+1}\right\|_{L^p} \\
&\qquad + \left\|C_2\left\|\Phi_s\left(Z_{n-1}\right)\right\|_{C^{m+1}}^{m+1} + \left|g_n\left(\Phi_s\left(Z_n\right)\right)(x) - \exp\left(\Phi_s\left(Z_n\right)\right)(x)\right| \exp\left(\left\|\Phi_s\left(Z_{n-1}\right)\right\|_{C^1}\right)\right\|_{L^p} \\
&\leq C_3 s^{(m+1)/2}.
\end{aligned}$$

where C_2 is a positive constant. Inductively,

(2.14)
$$\left\|g_1\left(\Phi_s\left(Z_1\right)\right) \circ \cdots \circ g_n\left(\Phi_s\left(Z_n\right)\right)(x) - \exp\left(\Phi_s\left(Z_1\right)\right) \circ \cdots \circ \exp\left(\Phi_s\left(Z_n\right)\right)(x)\right\|_{L^p} \leq C_4 s^{(m+1)/2}$$

where $C_4 > 0$. From (2.14) and Lemma 2.5, (1.8) can be shown.

3. Construction of the $\mathcal{L}_{\mathbb{R}}((A))$-valued random variables Z_j's

We introduce some notations first so as to obtain simple representation of the coefficient $C(w)$ of each $w = v_{i_1}v_{i_2}\cdots v_{i_\ell}$, in $E\left[\exp(Z_1)\cdots\exp(Z_n)\right]$ where $i_j \in \{0,1,\dots,d\}$, $j=1,\dots,\ell$, and Z_j's are $\mathcal{L}_{\mathbb{R}}((A))$-valued random variables constructed with Gaussian random variables satisfying (1.9).

For $(i_1,\dots,i_\ell)\in\{0,1,\dots,d\}^\ell$, we define $(\bar{i}_0,\bar{i}_1,\dots,\bar{i}_{\ell'})$ as follows:

$$\ell' = \begin{cases} \mathrm{card}(\{i_1,\dots,i_\ell\}) & \text{if } \{k \,|\, i_k = 0\} = \emptyset, \\ \mathrm{card}(\{i_1,\dots,i_\ell\}) - 1 & \text{otherwise,} \end{cases}$$
$$\bar{i}_0 = 0, \quad \{\bar{i}_1,\dots,\bar{i}_{\ell'}\} = \{i_1,\dots,i_\ell\}\setminus\{0\}.$$

For such $(\bar{i}_0,\dots\bar{i}_{\ell'})$, we also define $m_r\in\mathbb{N}$, $r=0,1,\dots,\ell'$ by $m_r = \mathrm{card}\left(\left\{j \,\middle|\, \bar{i}_r = i_j\right\}\right)$.

For $\ell,n\in\mathbb{N}$, let $\mathcal{K}_\ell(n) = \left\{(k_1,\dots,k_n)\in\mathbb{Z}^n_{\geq 0} \,\middle|\, k_1+\cdots+k_n = \ell\right\}$. For a set of indexed variables $\left\{X^i_j\right\}_{i,j\in\mathbb{Z}}$ and two sequences of integers $i_1i_2\dots i_a$ and $j_1j_2\dots j_a$, we denote by $X^{i_1i_2\dots i_a}_{j_1j_2\dots j_a}$ the product $\prod_{k=1}^a X^{i_k}_{j_k}$. A sequence of indexed letters $i_1i_2\dots i_a$ is frequently denoted by $i\langle 1,2,\dots,a\rangle$ through this section. Using these notations, we can write as follows:

$$X^{i_1}_{j_1}X^{i_2}_{j_2}\cdots X^{i_a}_{j_a} = X^{i_1i_2\dots i_a}_{j_1j_2\dots j_a} = X^{i\langle 1,2,\dots,a\rangle}_{j\langle 1,2,\dots,a\rangle}.$$

Also, $X^{i\langle 1,\dots,\ell\rangle}_{j\langle 1^{k_1},2^{k_2},\cdots,n^{k_n}\rangle}$ means $\underbrace{X^{i_1}_{j_1}\cdots X^{i_{k_1}}_{j_1}}_{k_1}\underbrace{X^{i_{k_1+1}}_{j_2}\cdots X^{i_{k_1+k_2}}_{j_2}}_{k_2}\cdots\underbrace{X^{i_{k_1+\cdots+k_{n-1}+1}}_{j_n}\cdots X^{i_{k_1+\cdots+k_n}}_{j_n}}_{k_n}$.

For $(k_1,\dots,k_{2m})\in\mathbb{N}^{2m}$ with $k_1<k_2<\cdots<k_{2m}$, we define a set of maps $\mathfrak{T}(k_1,\dots,k_{2m})$ by the statement that $T\in\mathfrak{T}(k_1,\dots,k_{2m})$ is equivalent to the following conditions:

(i) T is a bijection from $\{1,\dots,m\}\times\{1,2\}$ to $\{k_1,k_2,\dots,k_{2m}\}$
(ii) $T(i,1)<T(i,2)$ for all $i\in\{1,\dots,m\}$
(iii) $T(i_1,1)<T(i_2,1)$ if $i_1<i_2$.

Lemma 3.1. *Let $\left\{S^i_j\right\}_{i=1,\dots,d,\ j=1,\dots,n}$ be the set of Gaussian random variables satisfying* (1.9) *and m be an integer satisfying $m\leq d$, then*

$$E\left[S^{ii\dots i}_{j\langle 1,2,\dots,2m\rangle}\right] = \sum_{T\in\mathfrak{T}(1,2,\dots,2m)}\prod_{i=1}^m R[j](T(i,1),T(i,2)) \tag{3.1}$$

where $R[j]$ denotes the $(2m)\times(2m)$ matrix whose (a,b) component is $R_{j_aj_b}$ and $R[j](a,b)$ denotes the (a,b) component of $R[j]$, that is, $R[j](a,b) = R_{j_aj_b}$.

This lemma is proved later. Let

$$R[j\,;\,k,2m] = \sum_{T\in\mathfrak{T}(k+1,\dots,k+2m)}\prod_{i=1}^m R[j](T(i,1),T(i,2)).$$

Theorem 3.1. *If m_r is even for any $r\in\{1,\dots,\ell'\}$, then*

$$C(w) = \sum_{k\in\mathcal{K}_\ell(n)}\frac{1}{k!}\prod_{p=1}^{m_0}c_{j_p}\prod_{q=1}^{\ell'}R[j\,;\,\tilde{m}(q-1),m_q], \tag{3.2}$$

otherwise $C(w) = 0$ *where* $\tilde{m}(q) = \sum_{r=0}^{q} m_r$, *for* $k = (k_1, \ldots, k_n)$, $k!$ *denotes* $k_1! \cdots k_n!$ *and* $(j_1, \ldots, j_\ell)$ *is a sequence defined for each* $(k_1, \ldots, k_n)$ *such that*

$$\left(\prod_{p=1}^{m_0} S_{j_p}^{\bar{i}_0}\right)\left(\prod_{p=1}^{m_1} S_{j_{\tilde{m}(0)+p}}^{\bar{i}_1}\right)\cdots\left(\prod_{p=1}^{m_{\ell'}} S_{j_{\tilde{m}(\ell'-1)+p}}^{\bar{i}_{\ell'}}\right) = S_{1^{k_1}2^{k_2}\cdots n^{k_n}}^{i\langle 1,\ldots,\ell\rangle}. \tag{3.3}$$

Proof. For the case in which m_r is odd for some $r \in \{1, \ldots, \ell'\}$, (3.2) is directly derived from (1.9).

We therefore consider the other case, that is, $m_r = 2m'_r$ for all $r \in \{1, \ldots, \ell'\}$. By the Taylor expansion of $\exp(Z_1) \cdots \exp(Z_n)$, we have for $w = v_{i_1} \ldots v_{i_\ell}$

$$C(w) = E\left[\sum_{k=(k_1,\ldots,k_n)\in\mathcal{K}_\ell(n)} \frac{1}{k!} S_1^{i_1} \cdots S_1^{i_{k_1}} S_2^{i_{k_1+1}} \cdots S_2^{i_{k_1+k_2}} \cdots S_n^{i_{k_1+\cdots+k_{n-1}+1}} \cdots S_n^{i_{k_1+\cdots+k_n}}\right]. \tag{3.4}$$

By (3.3) and the definition of $\left\{S_j^i\right\}$ in (1.9), (3.4) becomes

$$C(w) = \sum_{k=(k_1,\ldots,k_n)\in\mathcal{K}_\ell(n)} \frac{1}{k!}\left(\prod_{p=1}^{m_0} c_{j_p}\right) E\left[\prod_{p=1}^{m_1} S_{j_{\tilde{m}(0)+p}}^{\bar{i}_1}\right]\cdots E\left[\prod_{p=1}^{m_{\ell'}} S_{j_{\tilde{m}(\ell'-1)+p}}^{\bar{i}_{\ell'}}\right]. \tag{3.5}$$

Applying Lemma 3.1 to each $E\left[\left(\prod_{p=1}^{m_r} S_{j_{\tilde{m}(r-1)+p}}^{\bar{i}_r}\right)\right]$, $r = 1, \ldots, \ell'$, we obtain (3.2). □

Proof of Lemma 3.1. Let S be an $\mathbb{R}^{2m}$-valued random variable defined by

$$S = \left(S_{j_1}^i, S_{j_2}^i, \ldots, S_{j_{2m}}^i\right).$$

Let $\varphi_S(z)$ be the characteristic function of S, that is,

$$\varphi_S(z) = E\left[\exp\left(\sqrt{-1}\langle S, z\rangle\right)\right] \tag{3.6}$$

where $z = (z_1, \ldots, z_{2m}) \in \mathbb{R}^{2m}$ and $\langle S, z\rangle$ denotes the inner product of S and z. Because S_j^i's are normal random variables satisfying (1.9), we also have

$$\varphi_S(z) = \exp\left(-\frac{1}{2}{}^t z R[j] z\right). \tag{3.7}$$

From (3.6),

$$\begin{aligned}
\left.\frac{\partial^{2m}\varphi_S(z)}{\partial z_1 \partial z_2 \ldots \partial z_{2m}}\right|_{z=0} &= \left.\frac{\partial^{2m}}{\partial z_1 \partial z_2 \ldots \partial z_{2m}} \sum_{l=0}^{\infty} \frac{\left(\sqrt{-1}\right)^l}{l!} E\left[(\langle S, z\rangle)^l\right]\right|_{z=0} \\
&= \left.\frac{(-1)^m}{(2m)!} \frac{\partial^{2m}}{\partial z_1 \partial z_2 \ldots \partial z_{2m}} E\left[(\langle S, z\rangle)^{2m}\right]\right|_{z=0} \\
&= (-1)^m E\left[S_{j\langle 1,2,\ldots,2m\rangle}^{ii\ldots i}\right].
\end{aligned} \tag{3.8}$$

We also have

$$
\begin{aligned}
\left.\frac{\partial^{2m}\varphi_S(z)}{\partial z_1\partial z_2\ldots\partial z_{2m}}\right|_{z=0} &= \left.\frac{\partial^{2m}}{\partial z_1\partial z_2\ldots\partial z_{2m}}\sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{1}{2}\sum_{1\le h,i\le 2m}R[j](h,i)z_hz_i\right)^k\right|_{z=0}\\
&=\left.\frac{1}{m!}\,\frac{\partial^{2m}}{\partial z_1\partial z_2\ldots\partial z_{2m}}\left(-\sum_{1\le h<i\le 2m}R[j](h,i)z_hz_i\right)^m\right|_{z=0}\\
&=\frac{(-1)^m}{m!}\sum_{T\in\mathfrak{T}(1,\ldots,2m)}m!\prod_{i=1}^{m}R[j](T(i,1),T(i,2))\\
&=(-1)^m\sum_{T\in\mathfrak{T}(1,\ldots,2m)}\prod_{i=1}^{m}R[j](T(i,1),T(i,2))
\end{aligned}
\tag{3.9}
$$

from (3.7). The lemma is proved by (3.8) and (3.9). □

On the other hand, the value of the coefficient of each w in $j_m\left(\exp\left(v_0+(1/2)\sum_{i=1}^d v_i^2\right)\right)$ can be obtained by the following proposition.

Proposition 3.1. *Let $A^0=\{v_0, v_1v_1, v_2v_2,\ldots,v_dv_d\}\subset A^*$. Then*

$$
\exp\left(v_0+\frac{1}{2}\sum_{i=1}^{d}v_i^2\right)=\sum_{\substack{w=w_1\cdots w_l\\ w_1,\ldots,w_l\in A^0}}\frac{1}{2^{|w|-l}l!}w.
\tag{3.10}
$$

Therefore, taking S^i_j's to equate (3.2) with (3.10) for $w=v_{i_1}v_{i_2}\cdots v_{i_l}$ with $\|w\|\le m$, we can construct Z_j's.

For $m=5$, we take $n=2$ to have solvable simultaneous equations which are actually become the following five:

$$
\begin{aligned}
&c_1+c_2=1,\quad \frac{1}{2}(c_1R_{11}+c_2R_{22})+R_{12}=\frac{1}{2},\\
&\frac{1}{6}(c_1R_{11}+c_2R_{22})+\frac{1}{2}c_1(R_{12}+R_{22})=\frac{1}{4},\\
&\frac{1}{6}(c_1R_{11}+c_2R_{22})+\frac{1}{2}c_2(R_{11}+R_{22})=\frac{1}{4},\\
&\frac{1}{24}(R_{11}^2+R_{22}^2)+\frac{1}{6}R_{12}(R_{11}+R_{22})+\frac{1}{4}R_{11}R_{22}=\frac{1}{8}.
\end{aligned}
\tag{3.11}
$$

The solution is (1.11). Since we let $\{S^i_j\}_{i=1,\ldots,d,j=1,\ldots,n}$ be the Gaussian system, such random variables can be definitely constructed.

Remark 3.1. *If we let $m=5$, then n has to be two at least.*

4. The Runge-Kutta method

We begin by briefly introducing the tree theory following [2], [3] and [1]. For details of the Runge-Kutta method, see [2], [3], and [18].

All trees introduced here are called directed or rooted trees in the literature listed above.

Definition 4.1. *A labelled tree $\mathbf{t}$ is a pair of finite sets $(V(\mathbf{t}),E(\mathbf{t}))$ which satisfies the following conditions:*

(1) *$V(\mathbf{t}) \subset \mathbb{Z}$ and $E(\mathbf{t}) \subset \{(x, y) : x, y \in V(\mathbf{t})$ and $x \neq y\}$.*
(2) *If $(x, y) \in E(\mathbf{t})$ then $x < y$.*

(3) *For each $x \in V(\mathbf{t})$, if $(x, y) \in E(\mathbf{t})$ and $(x', y) \in E(\mathbf{t})$, then $x = x'$.*

(4) *For any two different elements $x, y \in V(\mathbf{t})$, one of the followings holds:*
   (i) *There exists a path from x to y,*
   (ii) *There exists a path from y to x,*
   (iii) *For some $z \in V(\mathbf{t}) \setminus \{x, y\}$, there exist paths z to x and z to y.*

   Here a path from p_1 to p_l is a sequence $(p_1, p_2), (p_2, p_3), \ldots, (p_{l-1}, p_l)$ of elements of $E(\mathbf{t})$ such that $p_i \neq p_j$ if $i \neq j$.

An element of $V(\mathbf{t})$ is called a vertex of $\mathbf{t}$ and of $E(\mathbf{t})$ is called an edge of $\mathbf{t}$.

A particular labelled tree τ is the one with $\mathrm{card}\,(V(\tau)) = 1$. For only τ, $E(\tau)$ is allowed to be empty.

For a labelled tree $\mathbf{t} = (V(\mathbf{t}), E(\mathbf{t}))$, let $\mathbf{r}(\mathbf{t})$ be $\mathrm{card}\,(V(\mathbf{t}))$. We define $\mathbf{T}$ as the set ot all labelled trees.

Proposition 4.1. *For each $\mathbf{t} = (V(\mathbf{t}), E(\mathbf{t}))$, there exists a unique vertex $r \in V(\mathbf{t})$ such that for any $x \in V(\mathbf{t}) \setminus \{r\}$, there is a path from r to x.*

Such a vertex r is called the root of $\mathbf{t}$. τ consists of only the root.

Definition 4.2. *For $\mathbf{t}_i = (V(\mathbf{t}_i), E(\mathbf{t}_i)) \in \mathbf{T}$, $i = 1, \ldots, n$, such that $V(\mathbf{t}_i) \cap V\left(\mathbf{t}_j\right) = \emptyset$ if $i \neq j$, $[\mathbf{t}_1 \cdots \mathbf{t}_n]$ is defined as $\mathbf{t} = (V(\mathbf{t}), E(\mathbf{t}))$ such that*

$$\begin{aligned} V(\mathbf{t}) &= \{r\} \cup V(\mathbf{t}_1) \cup \cdots \cup V(\mathbf{t}_n) \\ E(\mathbf{t}) &= \{(r, r_1), \ldots, (r, r_n)\} \cup E(\mathbf{t}_1) \cup \cdots \cup E(\mathbf{t}_n) \end{aligned}$$

where r_i, $i = 1, \ldots, n$ denotes $\mathbf{t}_i$'s root and $r = \min\{r_1, \ldots, r_n\} - 1$.

Remark 4.1. *For $\mathbf{t}_1, \ldots, \mathbf{t}_n \in \mathbf{T}$,*

$$[\mathbf{t}_1 \cdots \mathbf{t}_n] = \left[\mathbf{t}_{\varpi(1)} \cdots \mathbf{t}_{\varpi(n)}\right]$$

for any permutation $\varpi \in \mathfrak{S}_n$.

Definition 4.3. *For $\mathbf{t}_i = (V(\mathbf{t}_i), E(\mathbf{t}_i)) \in \mathbf{T}$, $i = 1, 2$, $\mathbf{t}_1$ and $\mathbf{t}_2$ are isomorphic, written $\mathbf{t}_1 \approx \mathbf{t}_2$, if there exists a bijection $\varpi : V(\mathbf{t}_1) \longrightarrow V(\mathbf{t}_2)$ such that $(x, y) \in E(\mathbf{t}_1)$ if and only if $(\varpi(x), \varpi(y)) \in E(\mathbf{t}_2)$.*

In particular, when $\mathbf{t}_1 \approx \mathbf{t}_2$ and $V(\mathbf{t}_1) = V(\mathbf{t}_2)$, that is, ϖ is a permutation, we say that $\mathbf{t}_1$ and $\mathbf{t}_2$ are equivalent and write $\mathbf{t}_1 \sim \mathbf{t}_2$.

Proposition 4.2. *Both $\approx$ and $\sim$ are equivalence relations.*

Proposition 4.3. *If for $i = 1, \ldots, n$, $\mathbf{t}_i \in \mathbf{T}$ and $\mathbf{u}_i \in \mathbf{T}$ are isomorphic, then $[\mathbf{t}_1 \cdots \mathbf{t}_n]$ and $[\mathbf{u}_1 \cdots \mathbf{u}_n]$ are also isomorphic.*

Definition 4.4. *We define $T = \mathbf{T}/\approx$. An element $t \in T$ is called a non-labelled tree. For a labelled tree $\mathbf{t} \in \mathbf{T}$, $|\mathbf{t}|$ denotes the corresponding non-labelled tree $t \in T$.*

Proposition 4.4. *For $\mathbf{t}_i$, $\mathbf{t}'_i \in \mathbf{T}$, $i = 1, \ldots, n$, if $|\mathbf{t}_i| = |\mathbf{t}'_i|$, then*

$$|[\mathbf{t}_1 \cdots \mathbf{t}_n]| = \left|\left[\mathbf{t}'_1 \cdots \mathbf{t}'_n\right]\right|.$$

By virtue of Proposition 4.4, we can define a non-labelled tree $t = [t_1 \cdots t_n]$ for $t_1, \ldots, t_n \in T$ as $|[\mathbf{t}_1 \cdots \mathbf{t}_n]|$ where $\mathbf{t}_i \in \mathbf{T}$ is a representative element of $t_i \in T$. We let $\tau = |\tau|$ for all $i = 1, \ldots, n$.

Proposition 4.5. *For $t \in T \setminus \{\tau\}$, there exist $t_1, \ldots, t_n \in T$ such that $t = [t_1 \cdots t_n]$.*

Moreover, if $t = \left[t'_1 \cdots t'_{n'}\right]$, then $n = n'$ and there exists a permutation $\varpi \in \mathfrak{S}_n$ such that $t_i = t'_{\varpi(i)}$.

If for $t = [t_1 \cdots t_n] \in T$, there are $u_1, \ldots, u_l \in T$ such that for any t_i there exists u_j such that $t_i = u_j$ and that $u_k \neq u_j$ if $k \neq j$, t is written as $[u_1^{m_1} \cdots u_l^{m_l}]$ where $m_j = \mathrm{card}\left(\{t_i : u_j = t_i\}\right)$.

In order to determine $A = (a_{ij})_{i,j=1,\ldots,M}$ and $b = {}^t(b_1, \ldots, b_M)$ where a_{ij}'s and b_i's are *Rset*-valued coefficients appearing in (1.12), we define some functions on T.

Definition 4.5. *For $t = (V(t), E(t)) \in T$,*

$$\begin{aligned}
\alpha(t) &= \mathrm{card}\left(\{\mathbf{u} \in \mathbf{T} : \ \mathbf{u} \sim \mathbf{t} \textit{ where } \mathbf{t} \in \mathbf{T} \textit{ is a representative element of } t\}\right)\\
r(t) &= \mathrm{card}\left(V(t)\right)\\
\sigma(t) &= \begin{cases} 1 & \textit{if } t = \tau \\ \prod_{i=1}^{l} m_i! \sigma(t_i)^{m_i} & \textit{if } t = \left[t_1^{m_i} \cdots t_l^{m_l}\right],\ l \geq 1 \end{cases}\\
\zeta(t; \bar{A}) &= {}^t(\zeta_i(t; \bar{A}))_{i=1,\ldots,M+1} = \bar{A}\bar{\zeta}(t)
\end{aligned}$$

where $\bar{A} = \begin{pmatrix} A \\ {}^t b \end{pmatrix}$ *and*

$$\bar{\zeta}(t) = {}^t(\bar{\zeta}_i(t))_{i=1,\ldots,M} = \begin{cases} \mathbf{1}_M & \textit{if } t = \tau \\ {}^t\!\left(\prod_{j=1}^{l} \zeta_i\left(t_j; \bar{A}\right)\right)_{i=1,\ldots,M} & \textit{if } t = [t_1 \cdots t_l],\ l \geq 1. \end{cases}$$

Also, we define the elementary differentials D for an $\mathbb{R}^N$-valued function $W \in C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$ as follows:

(4.1)
$$D(W, t)(\cdot) = \begin{cases} W(\cdot) & \textit{if } t = \tau, \\ W^{(l)}(\cdot)\left(D(W, t_1)(\cdot), D(W, t_2)(\cdot), \ldots, D(W, t_l)(\cdot)\right) & \textit{if } t = [t_1 t_2 \cdots t_l],\ l \geq 1. \end{cases}$$

a_{ij}'s and b_i's for the Runge-Kutta method (1.12) of order m satisfy that for $t \in T$ with $r(t) \leq m$

$$\frac{\alpha(t)}{r(t)!} = \frac{\zeta(t; \bar{A})}{\sigma(t)} \tag{4.2}$$

because the following evaluations for the solution to (1.13) and the Runge-Kutta method (1.12) can be shown to hold:

$$\left| \exp(sW)(y) - \left(y + \sum_{\substack{t \in T \\ r(t) \leq m}} \frac{s^{r(t)}}{r(t)!} \alpha(t) D(W, t)(y) \right) \right| \leq C_{m+1} s^{m+1} \|W\|_{C^{m+1}}^{m+1} \tag{4.3}$$

and

$$\left| Y(y;W,s) - \left(y + \sum_{\substack{t\in T \\ r(t)\le m}} \frac{s^{r(t)}}{\sigma(t)} \zeta(t;\bar{A}) D(W,t)\,(y) \right) \right| \le C'_{m+1} s^{m+1} \|W\|_{C^{m+1}}^{m+1} \tag{4.4}$$

respectively where C_{m+1} and C'_{m+1} are both positive constants.

We recall that $g_{(m)}$ denotes the m-th order Runge-Kutta method with $s = 1$ as in section 1. The following theorem confirms that the m-th order Runge-Kutta method belongs to $\mathcal{IS}(m)$.

Theorem 4.1. *$g_{(m)} \in \mathcal{IS}(m)$.*

Proof. As we let $g_{(m)}(W)(y) = Y(y;W,1)$, from (4.3), (4.4), and (4.2)

$$\begin{aligned} \left| g_{(m)}(W)(y) - \exp(W)(y) \right| \le & \left| g_{(m)}(W)(y) - \left(y + \sum_{\substack{t\in T \\ r(t)\le m}} \frac{\zeta(t;\bar{A})}{\sigma(t)} D(W,t)\,(y) \right) \right| \\ & + \left| \exp(W)(y) - \left(y + \sum_{\substack{t\in T \\ r(t)\le m}} \frac{\alpha(t)}{r(t)!} D(W,t)\,(y) \right) \right| \\ \le & C_{m+1} \|W\|_{C^{m+1}}^{m+1}. \end{aligned}$$

□

5. The new simulation scheme and Corollary 1.1

Corollary 1.1 indicates the new implementation method of the new higher order scheme proposed by Kusuoka in [8], [9], and [10]. Corollary 1.1 can be proved by Theorem 1.1 and 4.1 and a theorem in [10].

This implementation method seems to be distinguished mainly for two advantages. One is that the approximation operator can be obtained by numerical calculations if the Runge-Kutta method is applied to calculation of each $\exp(Z_j)$ while the tediousness in symbolical calculations of the operator might be an obstacle for practical application, which can be seen in [11], [16], and [19]. The other is that the partial sampling problem discussed in [11] and [16] can be conquered by using quasi-Monte Carlo methods. More precisely, the following two points make effective the use of the Low-Discrepancy sequences, which are essential to quasi-Monte Carlo methods([14]):

- S^i_j's can be taken to be continuous random variables in this implementation
- the scheme itself is characterized by the need of the much less number of discretization of time, which leads to reduction of the number of dimensions of the numerical integration.

In this paper, we assume that the SDE (1.1) satisfies the following condition, **UFG**:

UFG: There exist an integer l and $\varphi_{u,u'} \in C_b^\infty(\mathbb{R}^N;\mathbb{R})$ which satisfy

$$\Phi(\mathrm{r}(u)) = \sum_{u'\in A^*_{\le l}\setminus\{1,v_0\}} \varphi_{u,u'} \Phi(\mathrm{r}(u')) \tag{5.1}$$

for any $u \in A^* \setminus \{1, v_0\}$.

6. Application

We give a numerical example in this section in order to illustrate the implementation method proposed in Corollary 1.1, comparing with some existing schemes.

6.1. **Simulation.** Let $X(t,x)$ be a diffusion process defined by (1.1). The most popular scheme of first order is the Euler-Maruyama scheme. It is shown in [7] and [22] that for an arbitrary C^4 function f

$$\left\| E\left[f\left(X_1^{(\mathrm{EM}),n} \right) \right] - E\left[f\left(X(1,x) \right) \right] \right\| \leq C_f \frac{1}{n} \tag{6.1}$$

where $X_1^{(\mathrm{EM}),n}$ denotes the Euler-Maruyama scheme approximating $X(t,x)$.

Construction of higher order scheme is based on the higher order stochastic Taylor formula ([4][7]). When the vector fields $\{V_i\}_{i=0}^d$ commute, higher-order schemes can be easily simplified to a direct product of one-dimensional problem as seen in [7]. Contrastingly, for non-commutative $\{V_i\}_{i=0}^d$, acquisition of all iterated integrals of Brownian motion is required, which is very demanding. This is done in [8][12][20] [21] and [11] and generalized as the cubature method on Wiener space ([13]).

Once a pth-order scheme $\{X_{k/n}^{(\mathrm{ord}\,p),n}\}_{k=0,\ldots,n}$ is obtained and expanded with some constant K_f as

$$E\left[f\left(X_1^{(\mathrm{ord}\,p),n} \right) \right] - E\left[f\left(X(1,x) \right) \right] = K_f \frac{1}{n^p} + O\left(\frac{1}{n^{p+1}} \right), \tag{6.2}$$

the $(p+1)$th-order scheme can be derived as

$$\frac{2^p}{2^p - 1} E\left[f\left(X_1^{(\mathrm{ord}\,p),2n} \right) \right] - \frac{1}{2^p - 1} E\left[f\left(X_1^{(\mathrm{ord}\,p),n} \right) \right]. \tag{6.3}$$

This boosting method is called Romberg extrapolation and is shown to become applicable to the Euler-Maruyama scheme under some conditions ([22]).

Simulation approach is to be necessarily followed by numerical calculation of $E\left[f\left(X_1^{(\mathrm{ord}\,p),n} \right) \right]$. However, when $n \times d$ is large, it is practically impossible to proceed the integration by using trapezoidal formula and so we fall back on the Monte Carlo or quasi-Monte Carlo method ([14]). Here we only introduce remarks on each method. For details, see [17].

Remark 6.1. *As long as we use the Monte Carlo method for numerical approximation of $E[f(X(1,x))]$, the number of sample points needed to attain a given accuracy is independent of the number of the dimensions of integration, namely both the number n of partitions and the order p of the approximation scheme.*

Remark 6.2. *In contrast to the Monte Carlo case, the number of sample points needed for the quasi-Monte Carlo method for numerical approximation of $E[f(X(1,x))]$ heavily depends on the number of the dimensions of integration. The smaller the number of the dimensions, the less the number of samples are needed.*

6.2. **The algorithm and competitors.**

6.2.1. *The algorithm of the new method.* We take the algorithm which is proposed in Theorem 1.2 and Corollary 1.1 with $u = 3/4$. From Corollary 1.1, we can implement the second order algorithm with numerical approximation of $\exp(Z_i)$'s of at least fifth-order Runge-Kutta method because the order m for an integration scheme attained by Z_1 and Z_2 is five and so the order of the new implementation method becomes two. As a result of the same argument it can be shown that at least seventh-order explicit Runge-Kutta method has to be applied to approximation of $\exp(Z_i)$'s when we boost the new method to the third order by Romberg extrapolation. Details of these Runge-Kutta algorithms used here are given in Appendix.

6.2.2. *Competitive schemes.* Although there are a lot of studies on acceleration of Monte Carlo methods ([6]), we choose by the following reasons only the crude Euler-Maruyama scheme and the algorithm introduced in [17], which we will refer as N-V method in this paper, both with and without Romberg extrapolation as competitors:

(i) Only these two schemes can be recognized to be comparable to the new method in that they are model-independent.
(ii) Almost all variance reduction techniques and dimension reduction techniques which we can apply to the Euler-Maruyama scheme are also applicable to the new method.

6.3. **Numerical results.** We provide an example on financial option pricing in the following part of this paper.

6.3.1. *Asian option under the Heston model.* We consider an Asian call option written on an asset having the price process under the Heston model which is known as a two-factor stochastic volatility model. Comparison with the N-V method is to be given as well from the result shown in [17].

Non-commutativity of this example should be of note here.
Let Y_1 be the price process of an asset following the Heston model:

$$\begin{aligned} Y_1(t,x) =& x_1 + \int_0^t \mu Y_1(s,x)\,ds + \int_0^t Y_1(s,x)\sqrt{Y_2(s,x)}\,dB^1(s), \\ Y_2(t,x) =& x_2 + \int_0^t \alpha\,(\theta - Y_2(s,x))\,ds \\ &+ \int_0^t \beta\sqrt{Y_2(s,x)}\left(\rho\,dB^1(s) + \sqrt{1-\rho^2}\,dB^2(s)\right), \end{aligned} \tag{6.4}$$

where $x = (x_1, x_2) \in (\mathbb{R}_{>0})^2$, $(B^1(t), B^2(t))$ is a two-dimensional standard Brownian motion, $-1 \le \rho \le 1$, and α, θ, μ are some positive coefficients such that $2\alpha\theta - \beta^2 > 0$ to ensure the existence and uniqueness of a solution to the SDE ([5]). Then the payoff of Asian call option on this asset with maturity T and strike K is $\max(Y_3(T,x)/T - K, 0)$ where

$$Y_3(t,x) = \int_0^t Y_1(s,x)\,ds. \tag{6.5}$$

Hence, the price of this option becomes $D \times E\,[\max(Y_3(T,x)/T - K,\ 0)]$ where D is an appropriate discount factor that we do not focus on in this experiment. We set

$T = 1$, $K = 1.05$, $\mu = 0.05$, $\alpha = 2.0$, $\beta = 0.1$, $\theta = 0.09$, $\rho = 0$, and $(x_1, x_2) = (1.0, 0.09)$ and take

$$E\left[\max\left(Y_3(T,x)/T - K,\ 0\right)\right] = 6.0473534496 \times 10^{-2}$$

which is obtained by the new method with Romberg extrapolation and the quasi-Monte Carlo with $n = 96 + 48$, and $M = 8 \times 10^8$ where M denotes the number of sample points.

Let $Y(t,x) = {}^t(Y_1(t,x), Y_2(t,x), Y_3(t,x))$. Transformation of the SDEs (6.4) and (6.5) gives the following Stratonovich-form SDEs:

$$Y(t,x) = \sum_{i=0}^{2} \int_0^t V_i(Y(s,x)) \circ dB^i(s), \tag{6.6}$$

where

$$\begin{aligned} V_0\left({}^t(y_1,y_2,y_3)\right) &= {}^t\!\left(y_1\left(\mu - \frac{y_2}{2} - \frac{\rho\beta}{4}\right),\ \alpha(\theta - y_2) - \frac{\beta^2}{4},\ y_1\right) \\ V_1\left({}^t(y_1,y_2,y_3)\right) &= {}^t\!\left(y_1\sqrt{y_2},\ \rho\beta\sqrt{y_2},\ 0\right) \\ V_2\left({}^t(y_1,y_2,y_3)\right) &= {}^t\!\left(0,\ \beta\sqrt{(1-\rho^2)\,y_2},\ 0\right). \end{aligned} \tag{6.7}$$

6.3.2. *The dimensions of integrations.* As we mentioned in Remarks 6.1 and 6.2, the dimensions of integrations included in these methods have an effect on the quasi-Monte Carlo. The relation among d: the number of factors, n: the number of partitions, and the dimensions of integration of each method can be summarized as in Table 1.

TABLE 1. # of dimensions involved in each method.

Method	Num. of dim.
Euler-Maruyama	dn
N-V	$n + dn$ (n-Bernoulli and $(d \times n)$-Gaussian)
New Method	$2dn$

6.3.3. *Discretization Error.* The relation between discretization error and the number of partitions of each algorithm is plotted in Figure 6.1. We can observe from this figure that for 10^{-4} accuracy the new method with Romberg extrapolation takes the minimum number of partitions $n = 1 + 2$ whereas $n = 16$ for the Euler-Maruyama scheme with the extrapolation. Even without the extrapolation, the new method attains that accuracy with $n = 10$ while the Euler-Maruyama scheme takes $n = 2000$. Also, it can be said that the N-V method shows a little worse performance than the new method.

6.3.4. *Integration Error.* Looking at Figure 6.2, we can compare convergence errors of respective methods for each number of sample points, M. For Monte Carlo case, 2σ of 10 batches is taken as convergence error while for the quasi-Monte Carlo method, absolute difference from the value to be convergent is considered. For 10^{-4} accuracy with 95% confidence level (2σ), $M = 10^8$ is taken for the Monte Carlo method. On the other hand, if we apply the quasi-Monte Carlo method instead,

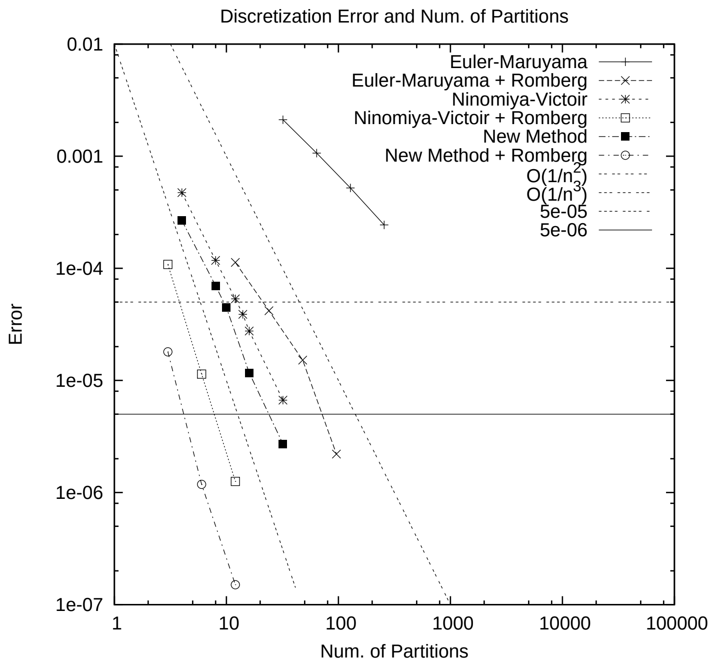

FIGURE 6.1. Error coming from the discretization

the new method and the N-V method require $M = 2 \times 10^5$ sample points, though $M = 5 \times 10^6$ has to be taken for the Euler-Maruyama scheme.

TABLE 2. #Partitions, #Samples, Dimension, and CPU time required for accuracy of 10^{-4}.

Method	#Part.	Dim.	#Samples	CPU time (sec)
E-M + MC	2000	4000	10^8	1.72×10^5
E-M + Romb. + QMC	16 + 8	48	5×10^6	1.27×10^2
N-V + QMC	16	32 + 16	2×10^5	4.38
N-V + Romb. + QMC	4 + 2	12 + 6	2×10^5	1.76
New Method + QMC	10	40	2×10^5	3.4
New Method + Romb. + QMC	2 + 1	12	2×10^5	1.2

6.3.5. *Overall performance comparison.* The number of partitions, the number of samples, and the amount of computation time required for 10^{-4} accuracy for each method are summarized in Table 2. CPU used in this experiment is Athlon 64 3800+ by AMD.

Since the amount of time to do calculation for each sample point is proportional to the number of partitions, the consumed time for calculation as a whole is proportional both to the number of partitions and to the number of samples. Therefore, we can easily guess from this table how it varies depending on the change in the number of partitions or the number of samples.

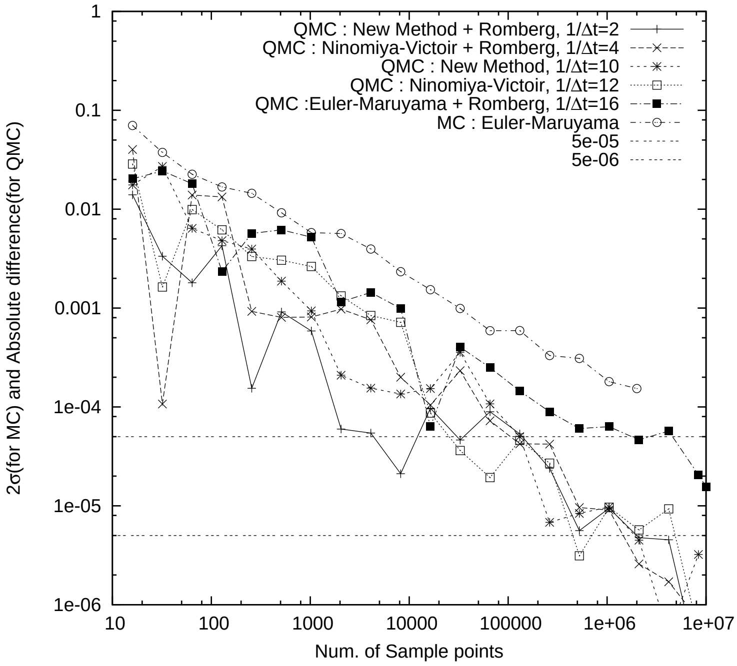

FIGURE 6.2. Convergence Error from quasi-Monte Carlo and Monte Carlo

From this table, we can see that the speed of the new method is approximately 100 times faster than that of the Euler-Maruyama scheme when Romberg extrapolation and quasi-Monte Carlo are applied to each. Even when the extrapolation is not applied, the new method dose more or less 37 times faster calculation than the Euler-Maruyama scheme with Romberg extrapolation and quasi-Monte Carlo does.

Lastly, Remarks 6.1 and 6.2 should be emphasized to recall that the advantage of the new method is deeply related to the property of the quasi-Monte Carlo method.

APPENDIX: THE FIFTH-ORDER AND THE SEVENTH-ORDER RUNGE-KUTTA ALGORITHMS

We give the concrete algorithms of the explicit fifth and seventh order Runge-Kutta methods applied in subsection 6.2. The fifth order method is taken from [2] as follows:

$$a_{21} = \frac{2}{5}, \quad a_{31} = \frac{11}{64}, \quad a_{32} = \frac{5}{64}, \quad a_{43} = \frac{1}{2}, \quad a_{51} = \frac{3}{64}, \quad a_{52} = -\frac{15}{64},$$
$$a_{53} = \frac{3}{8}, \quad a_{54} = \frac{9}{16}, \quad a_{62} = \frac{5}{7}, \quad a_{63} = \frac{6}{7}, \quad a_{64} = -\frac{12}{7}, \quad a_{65} = \frac{8}{7},$$
$$a_{ij} = 0 \quad \text{otherwise},$$
$$b = \begin{pmatrix} \frac{7}{90} & 0 & \frac{32}{90} & \frac{12}{90} & \frac{32}{90} & \frac{7}{90} \end{pmatrix}.$$

The seventh order method is taken from [3] as follows:

$$a_{21}=\frac{1}{6},\quad a_{32}=\frac{1}{3},\quad a_{41}=\frac{1}{8},\quad a_{43}=\frac{3}{8},\quad a_{51}=\frac{148}{1331},\quad a_{53}=\frac{150}{1331},\quad a_{54}=-\frac{56}{1331},$$
$$a_{61}=-\frac{404}{243},\quad a_{63}=-\frac{170}{27},\quad a_{64}=\frac{4024}{1701},\quad a_{65}=\frac{10648}{1701},\quad a_{71}=\frac{2466}{2401},\quad a_{73}=\frac{1242}{343},$$
$$a_{74}=-\frac{19176}{16807},\quad a_{75}=-\frac{51909}{16807},\quad a_{76}=\frac{1053}{2401},\quad a_{81}=\frac{5}{154},\quad a_{84}=\frac{96}{539},\quad a_{85}=-\frac{1815}{20384},$$
$$a_{86}=-\frac{405}{2464},\quad a_{87}=\frac{49}{1144},\quad a_{91}=-\frac{113}{32},\quad a_{93}=-\frac{195}{22},\quad a_{94}=\frac{32}{7},\quad a_{95}=\frac{29403}{3584},$$
$$a_{96}=-\frac{729}{512},\quad a_{97}=\frac{1029}{1408},\quad a_{98}=\frac{21}{16},\quad a_{ij}=0\quad\text{otherwise},$$
$$b=\begin{pmatrix}0 & 0 & 0 & \frac{32}{105} & \frac{1771561}{6289920} & \frac{243}{1560} & \frac{16807}{74880} & \frac{77}{1440} & \frac{11}{70}\end{pmatrix}.$$

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