UTMS 2007–12 July 26, 2006

$K3$ surfaces with involution, equivariant analytic torsion, and automorphic forms on the moduli space II: a structure theorem

by

Ken-ichi Yoshikawa

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

$K3$ SURFACES WITH INVOLUTION, EQUIVARIANT ANALYTIC TORSION, AND AUTOMORPHIC FORMS ON THE MODULI SPACE II: A STRUCTURE THEOREM

KEN-ICHI YOSHIKAWA

Abstract. In [59], we introduced an invariant of $K3$ surfaces with involution, which we obtained using equivariant analytic torsion. This invariant gives rise to a function on the moduli space of $K3$ surfaces with involution and is expressed as the Petersson norm of an automorphic form characterizing the discriminant locus. In this paper, we study the structure of this automorphic form. Under certain assumption, we prove that the automorphic form is expressed as the product of a certain Borcherds lift and the Igusa form.

Contents

1. Introduction

In this paper, we study the structure of the invariant of $K3$ surfaces with involution introduced in [59]. Let us recall briefly this invariant.

A $K3$ surface with holomorphic involution (X, ι) is called a 2-elementary $K3$ surface if ι acts non-trivially on the holomorphic 2-forms on X. Let $\mathbb{L}_{K3}$ be the $K3$ lattice, i.e., an even unimodular lattice of signature $(3, 19)$, which is isometric to $H^2(X, \mathbf{Z})$ endowed with the cup-product pairing. Let M be a sublattice of $\mathbb{L}_{K3}$ with rank $r(M)$. A 2-elementary $K3$ surface (X, ι) is of type M if the invariant sublattice of $H^2(X, \mathbf{Z})$ with respect to the ι-action is isometric to M. By [43], $M \subset \mathbb{L}_{K3}$ must be a primitive 2-elementary Lorentzian sublattice. The parity of the 2-elementary lattice M is denoted by $\delta(M) \in \{0, 1\}$ (cf. [45]).

Let $M^\perp$ be the orthogonal complement of M in $\mathbb{L}_{K3}$. Let $\Omega_{M^\perp}$ be the period domain for 2-elementary $K3$ surfaces of type M, which is an open subset of a quadric hypersurface of $\mathbf{P}(M^\perp \otimes \mathbf{C})$. We fix a connected component $\Omega^+_{M^\perp}$ of $\Omega_{M^\perp}$, which is isomorphic to a bounded symmetric domain of type IV of dimension $20 - r(M)$. Let $\mathcal{D}_{M^\perp}$ be the discriminant locus of $\Omega^+_{M^\perp}$, which is a reduced divisor on $\Omega^+_{M^\perp}$. Let

$O(M^\perp)$ be the group of isometries of $M^\perp$, which acts properly discontinuously on $\Omega_{M^\perp}$. Let $O^+(M^\perp)$ be the subgroup of $O(M^\perp)$ with index 2 that preserves $\Omega^+_{M^\perp}$. The coarse moduli space of 2-elementary $K3$ surfaces of type M is isomorphic to the analytic space $\mathcal{M}^o_{M^\perp} = (\Omega^+_{M^\perp} \setminus \mathcal{D}_{M^\perp})/O^+(M^\perp)$ via the period map by [49], [13], [45], [16], [59] and Proposition 11.2 below. The period of a 2-elementary $K3$ surface (X,ι) of type M is denoted by $\varpi_M(X,\iota) \in \mathcal{M}^o_{M^\perp}$.

Let (X,ι) be a 2-elementary $K3$ surface of type M. In [59], we introduced a real-valued invariant $\tau_M(X,\iota)$, which we obtained using the equivariant analytic torsion of (X,ι), the analytic torsions of the connected components of X^ι and a certain Bott–Chern secondary class. (See [5], [4], [50] and Sect. 5.) Since $\tau_M(X,\iota)$ depends only on the isomorphism class of (X,ι), we get the function

$$\tau_M \colon \mathcal{M}^o_{M^\perp} \ni \varpi_M(X,\iota) \to \tau_M(X,\iota) \in \mathbf{R}_{>0}.$$

By [59], there exists an automorphic form Φ_M on $\Omega^+_{M^\perp}$ with values in a certain $O^+(M^\perp)$-equivariant holomorphic line bundle on $\Omega^+_{M^\perp}$, such that

$$\tau_M = \|\Phi_M\|^{-\frac{1}{2\nu}}, \qquad \operatorname{div}\Phi_M = \nu\,\mathcal{D}_{M^\perp}, \qquad \nu \in \mathbf{Z}_{>0}.$$

Here $\|\cdot\|$ denotes the Petersson norm. By [59], Φ_M is given by the Borcherds Φ-function [7], [8] when M is exceptional.

The purpose of this paper is to give an explicit formula for τ_M for a class of non-exceptional M. We use two kinds of automorphic forms to express τ_M, i.e., the Borcherds lift $\Psi_{M^\perp}(\cdot, F_{M^\perp})$ and the Igusa form χ_g, which we explain briefly.

In [7], [9], Borcherds developed the theory of automorphic forms with infinite product over domains of type IV. (See also [28].) For an even 2-elementary lattice Λ of signature $(2, r(\Lambda)-2)$, we define the Borcherds lift $\Psi_\Lambda(\cdot, F_\Lambda)$ as follows.

Let A_Λ be the discriminant group of Λ, which is a vector space over $\mathbf{Z}/2\mathbf{Z}$. Let $\mathbf{C}[A_\Lambda]$ be the group ring of A_Λ and let $\rho_\Lambda \colon Mp_2(\mathbf{Z}) \to GL(\mathbf{C}[A_\Lambda])$ be the Weil representation, where $Mp_2(\mathbf{Z})$ is the metaplectic double cover of $SL_2(\mathbf{Z})$. Let $\{\mathbf{e}_\gamma\}_{\gamma\in A_\Lambda}$ be the standard basis of $\mathbf{C}[A_\Lambda]$. Let $\eta(\tau)$ be the Dedekind η-function and set $\eta_{1^{-8}2^84^{-8}}(\tau) = \eta(\tau)^{-8}\eta(2\tau)^8\eta(4\tau)^{-8}$. Let $\theta_{\mathbb{A}_1^+}(\tau)$ be the theta function of the (positive-definite) A_1-lattice. Then $\eta_{1^{-8}2^84^{-8}}(\tau)$ and $\theta_{\mathbb{A}_1^+}(\tau)$ are modular forms for the subgroup $M\Gamma_0(4) \subset Mp_2(\mathbf{Z})$ corresponding to the congruence subgroup $\Gamma_0(4) \subset SL_2(\mathbf{Z})$. Following [10] and [52], we define a $\mathbf{C}[A_\Lambda]$-valued holomorphic function $F_\Lambda(\tau)$ on the complex upper half-plane $\mathfrak{H}$ as

$$F_\Lambda(\tau) = \sum_{g\in M\Gamma_0(4)\backslash Mp_2(\mathbf{Z})} \left\{\eta_{1^{-8}2^84^{-8}}\,\theta_{\mathbb{A}_1^+}^{12-r(\Lambda)}\right\}\Big|_g(\tau)\,\rho_\Lambda(g^{-1})\,\mathbf{e}_0.$$

Here we used the notation $\phi|_g(\tau) = \phi(\frac{a\tau+b}{c\tau+d})(c\tau+d)^{-k}$ for a modular form $\phi(\tau)$ for $M\Gamma_0(4)$ of weight k with certain character and $g = (\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \sqrt{c\tau+d}) \in Mp_2(\mathbf{Z})$. By [10] and [52], $F_\Lambda(\tau)$ is an elliptic modular form for $Mp_2(\mathbf{Z})$ of type ρ_Λ with weight $2 - \frac{r(\Lambda)}{2}$. Then $\Psi_\Lambda(\cdot, F_\Lambda)$ is defined as the Borcherds lift of $F_\Lambda(\tau)$, which is an automorphic form on Ω^+_Λ for $O^+(\Lambda)$ by [9]. The Petersson norm $\|\Psi_{M^\perp}(\cdot, F_{M^\perp})\|^2$ is an $O^+(M^\perp)$-invariant C^∞ function on $\Omega^+_{M^\perp}$.

Recall that the Igusa form of degree g is the Siegel modular form (with character when $g = 1, 2$) on the Siegel upper half-space $\mathfrak{S}_g$ of degree g defined as the product of all even theta constants (cf. [29])

$$\chi_g(\Omega) = \prod_{(a,b)\text{ even}} \theta_{a,b}(\Omega), \qquad \Omega \in \mathfrak{S}_g, \qquad \chi_0 = 1.$$

The Igusa form gives rise to another function on $\mathcal{M}^o_{M^\perp}$ as follows. For a 2-elementary $K3$ surface (X,ι), let X^ι denote the set of fixed points of ι. By [45], X^ι is the disjoint union of (possibly empty) compact Riemann surfaces, whose topological type is determined by M. Let $g(M) \in \mathbf{Z}_{\geq 0}$ denote the total genus of X^ι. The period of X^ι is denoted by $\Omega(X^\iota) \in \mathfrak{S}_{g(M)}/Sp_{2g(M)}(\mathbf{Z})$. By [59], there exist a proper Zariski closed subset $Z \subset \mathcal{D}_{M^\perp}$ and an $O^+(M^\perp)$-equivariant holomorphic map $J_M \colon \Omega_{M^\perp} \setminus Z \to \mathfrak{S}_{g(M)}/Sp_{2g(M)}(\mathbf{Z})$ that induces the map of moduli spaces

$$\mathcal{M}^o_{M^\perp} \ni \varpi_M(X,\iota) \to \Omega(X^\iota) \in \mathfrak{S}_{g(M)}/Sp_{2g(M)}(\mathbf{Z}).$$

Then $J_M^*\|\chi_{g(M)}\|^2$ is an $O^+(M^\perp)$-invariant C^∞ function on $\Omega^o_{M^\perp}$.

The following structure theorem for τ_M is the main result of this paper:

Theorem 1.1. (cf. Theorem 9.1) *Let M be a primitive 2-elementary Lorentzian sublattice of $\mathbb{L}_{K3}$ satisfying the following two conditions* (1), (2)*:*

(1) $11 \leq r(M) \leq 17$ *or* $(r(M),\delta(M)) = (10,1)$*;*
(2) $\chi_{g(M)}(\Omega(X^\iota)) \neq 0$ *for some 2-elementary $K3$ surface (X,ι) of type M.*

Then there exists a constant C_M depending only on the lattice M such that the following identity holds for all 2-elementary $K3$ surface (X,ι) of type M:

$$\tau_M(X,\iota)^{-2^{g(M)+1}(2^{g(M)}+1)} = C_M\, \|\Psi_{M^\perp}(\varpi_M(X,\iota),F_{M^\perp})\|^{2^{g(M)}}\, \|\chi_{g(M)}(\Omega(X^\iota))\|^{16}.$$

After Bruinier [14], Theorem 1.1 may not be surprizing. If $M^\perp$ contains an even unimodular lattice of signature $(2,2)$ as a direct summand and if there is a Siegel modular form S such that $\operatorname{div}(J_M^*S)$ is a Heegner divisor on $\Omega^+_{M^\perp}$, then Φ_M must be the product of a Borcherds lift and J_M^*S by [14, Th. 0.8], because the zero divisor of Φ_M is a Heegner divisor. For most of M with $g(M)=2$, this explains the factorization of τ_M in Theorem 1.1. It is an interesting problem of understanding the geometric meaning of the elliptic modular forms F_Λ and $\eta_{1^{-8}2^84^{-8}}\theta^k_{\mathbb{A}_1^+}$. We remark that the same Borcherds lifts $\Psi_\Lambda(\cdot,F_\Lambda)$ appear in the formulae for the BCOV invariants of certain Calabi–Yau threefolds [18], [60], [63].

There are at least 30 isometry classes of primitive 2-elementary Lorentzian sublattices of $\mathbb{L}_{K3}$ satisfying Conditions (1) and (2) in Theorem 1.1. (See Theorem 9.3 and Remark 9.4.) There is an example of primitive 2-elementary Lorentzian sublattice of $\mathbb{L}_{K3}$ with *rank* 9 for which Theorem 1.1 holds. (See Theorem 9.2.) By Theorem 1.1 and [59, Ths. 8.2 and 8.7], τ_M and Φ_M are determined for 33 isometry classes of M. Notice that the total number of the isometry classes of primitive 2-elementary Lorentzian sublattice of $\mathbb{L}_{K3}$ is 75 by Nikulin [45].

Following [59, Th. 8.7], we shall prove Theorem 1.1 by comparing the $O^+(M^\perp)$-invariant currents $dd^c\log\tau_M$, $dd^c\log\|\Psi_{M^\perp}(\cdot,F_{M^\perp})\|$ and $dd^c\log J_M^*\|\chi^8_{g(M)}\|^2$. (See Sect. 9.) The current $dd^c\log\tau_M$ was computed in [59]. In Sect. 8, the weight and the zero divisor of $\Psi_{M^\perp}(\cdot,F_{M^\perp})$ shall be computed (cf. [9]), from which a formula for $dd^c\log\|\Psi_{M^\perp}(\cdot,F_{M^\perp})\|$ follows. In Sect. 4, the current $dd^c\log J_M^*\|\chi^8_{g(M)}\|^2$ shall be computed. For this purpose, we estimate the number of the irreducible components of the divisor $\mathcal{D}_{M^\perp}/O(M^\perp)$. (See Sect. 11.)

Since the Hartogs principle is used in the proof, we need the assumption $r(M) \leq 17$. In fact, Theorem 1.1 remains valid even if $r(M) \geq 18$. Since the proof of Theorem 1.1 requires an analysis of τ_M near the boundary locus of the Baily-Borel-Satake compactification of $\Omega^+_{M^\perp}/O^+(M^\perp)$ when $r(M) \geq 18$, these cases shall be treated in the forthcoming paper. In Theorem 9.5, we shall prove that $\chi_{g(M)}$ vanishes identically on $J_M(\Omega^+_{M^\perp} \setminus \mathcal{D}_{M^\perp})$ for most of M with $(r(M), \delta(M)) = (10, 0)$, so that Theorem 1.1 does not hold in these cases.

There are some applications of the Borcherds lift $\Psi_\Lambda(\cdot, F_\Lambda)$ to the moduli space of $K3$ surfaces. In [44], [16], the notion of lattice polarized $K3$ surfaces were introduced, which extends the classical notion of polarized $K3$ surfaces to general Lorentzian lattices. Since $\Psi_\Lambda(\cdot, F_\Lambda)$ vanishes exactly on the discriminant locus $\mathcal{D}_\Lambda$ when $r(\Lambda) \leq 12$, we get the following (cf. Corollaries 8.3 and 8.4):

Theorem 1.2. *If $M \subset \mathbb{L}_{K3}$ is a primitive 2-elementary Lorentzian sublattice with $r(M) \geq 10$, then the coarse moduli space of 2-elementary $K3$ surfaces of type M and the coarse moduli space of ample M-polarized $K3$ surfaces are quasi-affine.*

By [45], there are 49 isometry classes of primitive 2-elementary Lorentzian sublattices $M \subset \mathbb{L}_{K3}$ with $r(M) \geq 10$. It is not easy to find a primitive sublattice $\Lambda \subset \mathbb{L}_{K3}$ of signature $(2, r(\Lambda) - 2)$ such that there is an automorphic form on Ω^+_Λ vanishing exactly on $\mathcal{D}_\Lambda$. (See e.g. [7], [8], [10], [11], [25, II], [35], [52]). For example, if the discriminant locus of polarized $K3$ surfaces of degree $2d$ is irreducible, there is *no* automorphic form on the coarse moduli space of polarized $K3$ surfaces of degree $2d$ vanishing exactly on the discriminant locus [37, Sect. 3.3], [46].

This paper is organized as follows. In Sect. 2, we recall some basic definitions and properties of lattices. For a lattice with signature $(2, n)$, the corresponding modular variety is recalled. In Sect. 3, we recall 2-elementary $K3$ surfaces and their moduli spaces, and we study the singular fiber of an ordinary singular family of 2-elementary $K3$ surfaces. In Sect. 4, we study the current $dd^c J_M^* \|\chi^8_{g(M)}\|^2$ and we recall the notion of automorphic forms on $\Omega^+_{M^\perp}$. In Sect. 5, we recall the invariant τ_M. In Sect. 6, we recall Borcherds products. In Sect. 7, we construct the elliptic modular form $F_\Lambda(\tau)$. In Sect. 8, we study the Borcherds lift $\Psi_\Lambda(\cdot, F_\Lambda)$. In Sect. 9, we prove the main theorem. In Sect. 10, we interpret the main theorem into a statement about the equivariant determinant of the Laplacian of real $K3$ surfaces. In Sect. 11, we determine the number of the irreducible components of $\mathcal{D}_{M^\perp}/O(M^\perp)$. In Sect. 12, we study the set of fixed points of a generic 2-elementary $K3$ surface of type M for certain M with $g(M) = 3$.

Warning: In [59], we used the notation Ω_M, $\mathcal{M}_M$, $\mathcal{D}_M$ etc. in stead of $\Omega_{M^\perp}$, $\mathcal{M}_{M^\perp}$, $\mathcal{D}_{M^\perp}$ etc.

Acknowledgements We thank Professors Jean-Michel Bismut, Jean-Benoît Bost, Kai Köhler, Vincent Maillot, Damian Roessler for helpful discussions about the invariant τ_M and the automorphic form Φ_M, in particular their arithmetic nature. The main result of this paper is a refinement of [57, Th. 8.2]. We thank Professor Richard E. Borcherds, from whom we learned how to construct modular forms of type ρ_Λ from scalar-valued modular forms for $\Gamma_0(4)$ (cf. Proposition 7.1, [57, Prop. 8.1]); the construction of $F_\Lambda(\tau)$ is an obvious extension of that of $F_k(\tau)$ in [57, Sect. 8.2]. We also thank Professor Shigeyuki Kondō, who suggested us the proofs of Propositions 12.2 and 12.3 in Sect. 12. The author is partially supported by the Grants-in-Aid for Scientific Research (B) 19340016 and (S) 17104001, JSPS.

2. **Lattices**

A free $\mathbf{Z}$-module of finite rank endowed with a non-degenerate, integral, symmetric bilinear form is called a lattice. We often identify a non-degenerate, integral, symmetric matrix with the corresponding lattice. The rank of a lattice L is denoted by $r(L)$. The signature of L is denoted by $\mathrm{sign}(L) = (b^+(L), b^-(L))$. We define $\sigma(L) := b^+(L) - b^-(L)$. A lattice L is *Lorentzian* if $\mathrm{sign}(L) = (1, r(L) - 1)$. For a lattice $L = (\mathbf{Z}^r, \langle\cdot,\cdot\rangle)$, we define $L(k) := (\mathbf{Z}^r, k\langle\cdot,\cdot\rangle)$.

The group of isometries of L is denoted by $O(L)$. The set of roots of L is defined by $\Delta_L := \{d \in L;\ \langle d, d\rangle = -2\}$. For $d \in \Delta_L$, the corresponding reflection $s_d \in O(L)$ is defined as $s_d(x) := x + \langle x, d\rangle\, d$. The Weyl group of L is defined as the subgroup of $O(L)$ generated by $\{s_d\}_{d\in\Delta_L}$ and is denoted by $W(L)$. We define

$$\Delta'_L := \{d \in \Delta_L,\ d/2 \notin L^\vee\}, \qquad \Delta''_L := \{d \in \Delta_L,\ d/2 \in L^\vee\},$$

which are preserved by $O(L)$. Let $L^\vee = \mathrm{Hom}_{\mathbf{Z}}(L, \mathbf{Z})$ be the dual lattice of L, which is naturally embedded into $L \otimes \mathbf{Q}$. The finite abelian group $A_L := L^\vee/L$ is called the *discriminant group* of L. For $\lambda \in L^\vee$, we write $\bar\lambda := \lambda + L \in A_L$. A lattice L is *unimodular* if $A_L = 0$. A lattice L is *even* if $\langle x, x\rangle \in 2\mathbf{Z}$ for all $x \in L$. A lattice is *odd* if it is not even. For simplicity, we often write x^2 for $\langle x, x\rangle$. A sublattice $M \subset L$ is *primitive* if L/M has no torsion elements. The *level* of an even lattice L is the smallest positive integer l such that $l\,\lambda^2/2 \in \mathbf{Z}$ for all $\lambda \in L^\vee$.

2.1. Discriminant forms

For an even lattice L, the *discriminant form* q_L of A_L is the quadratic form on A_L with values in $\mathbf{Q}/2\mathbf{Z}$ defined as $q_L(\bar l) := l^2 + 2\mathbf{Z}$ for $\bar l \in A_L$. The corresponding bilinear form on A_L with values in $\mathbf{Q}/\mathbf{Z}$ is denoted by b_L. Then $b_L(\bar l, \bar l') = \langle l, l'\rangle + \mathbf{Z}$ for $\bar l, \bar l' \in A_L$. Since $\lambda \in L^\vee$ lies in L if and only if $\langle\lambda, l\rangle \in \mathbf{Z}$ for all $l \in L^\vee$, the bilinear form b_L is non-degenerate, i.e., if $b_L(\gamma, x) \equiv 0 \mod \mathbf{Z}$ for all $x \in A_L$, then $\gamma = 0$ in A_L. We often write γ^2 (resp. $\langle\gamma, \delta\rangle$) for $q_L(\gamma)$ (resp. $b_L(\gamma, \delta)$). The group of automorphisms of A_L preserving q_L and hence b_L is denoted by $O(q_L)$. See [43] for more about discriminant forms.

2.2. 2-elementary lattices

Set $\mathbf{Z}_2 := \mathbf{Z}/2\mathbf{Z}$. An even lattice L is 2*-elementary* if there is an integer $l \in \mathbf{Z}_{\geq 0}$ with $A_L \cong \mathbf{Z}_2^l$. For a 2-elementary lattice L, we set $l(L) := \dim_{\mathbf{Z}_2} A_L$. Then $r(L) \geq l(L)$ and $r(L) \equiv l(L) \mod 2$ by [43, Th. 3.6.2 (2)]. The parity $\delta(L)$ of an even 2-elementary lattice L is defined as

$$\delta(L) := \begin{cases} 0 & \text{if} \quad x^2 \in \mathbf{Z} \ \text{ for all } x \in L^\vee \\ 1 & \text{if} \quad x^2 \notin \mathbf{Z} \ \text{ for some } x \in L^\vee. \end{cases}$$

The triplet $(\mathrm{sign}(L), l(L), \delta(L))$ determines the isometry class of an *indefinite* even 2-elementary lattice L by [43, Th. 3.6.2].

Since A_L is a vector space over $\mathbf{Z}_2$ and since the mapping $A_L \ni \gamma \to \gamma^2 \in \frac{1}{2}\mathbf{Z}/\mathbf{Z} \cong \mathbf{Z}_2$ is $\mathbf{Z}_2$-linear, there exists a unique element $\mathbf{1}_L \in A_L$, called the *characteristic element* of A_L, such that $\langle\gamma, \mathbf{1}_L\rangle \equiv \gamma^2 \mod \mathbf{Z}$ for all $\gamma \in A_L$. By [43, Sect. 3.9 pp.149-150], $\mathbf{1}_L$ satisfies the properties: $g(\mathbf{1}_L) = \mathbf{1}_L$ for all $g \in O(q_L)$; $\mathbf{1}_L = 0$ if and only if $\delta(L) = 0$; if $L = L' \oplus L''$, then $\mathbf{1}_L = \mathbf{1}_{L'} \oplus \mathbf{1}_{L''}$.

Let $\mathbb{U} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and let $\mathbb{A}_1$, $\mathbb{D}_{2k}$, $\mathbb{E}_7$, $\mathbb{E}_8$ be the *negative-definite* Cartan matrix of type A_1, D_{2k}, E_7, E_8 respectively, which are identified with the corresponding even lattices. Then $\mathbb{U}$ and $\mathbb{E}_8$ are unimodular, and $\mathbb{A}_1$, $\mathbb{D}_{2k}$ and $\mathbb{E}_7$ are 2-elementary. Set

$$\mathbb{L}_{K3} := \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8 \oplus \mathbb{E}_8.$$

For a sublattice $\Lambda \subset \mathbb{L}_{K3}$, we define $\Lambda^\perp := \{l \in \mathbb{L}_{K3};\, \langle l, \Lambda\rangle = 0\}$. When $\Lambda \subset \mathbb{L}_{K3}$ is primitive, then $(A_\Lambda, -q_\Lambda) \cong (A_{\Lambda^\perp}, q_{\Lambda^\perp})$ by [43, Cor. 1.6.2].

Let $\mathbb{I}_{1,m}$ be an odd unimodular Lorentzian lattice of rank $m+1$. Then $\mathbb{I}_{1,m}(2)$ is an even 2-elementary Lorentzian lattice of rank $m+1$.

Let $M \subset \mathbb{L}_{K3}$ be a primitive 2-elementary Lorentzian sublattice. Let I_M be the involution on $M \oplus M^\perp$ defined as $I_M(x,y) = (x,-y)$ for $(x,y) \in M \oplus M^\perp$. Then I_M extends uniquely to an involution on $\mathbb{L}_{K3}$ by [43, Cor. 1.5.2]. We define

$$g(M) := \{22 - r(M) - l(M)\}/2, \qquad k(M) := \{r(M) - l(M)\}/2.$$

For $d \in \Delta_{M^\perp}$, the smallest sublattice of $\mathbb{L}_{K3}$ containing M and $\mathbf{Z}d$ is given by

$$[M \perp d] := (M^\perp \cap d^\perp)^\perp.$$

By Lemma 11.3 below, $[M \perp d]$ is again a 2-elementary Lorentzian lattice such that

$$I_{[M\perp d]} = s_d \circ I_M = I_M \circ s_d, \qquad [M \perp d]^\perp = M^\perp \cap d^\perp. \tag{2.1}$$

By e.g. [20, Appendix, Tables 1,2,3], M and $M^\perp$ are expressed as a direct sum of the 2-elementary lattices $\mathbb{A}_1^+$, $\mathbb{A}_1$, $\mathbb{U}$, $\mathbb{U}(2)$, $\mathbb{D}_{2k}$, $\mathbb{E}_7$, $\mathbb{E}_8$, $\mathbb{E}_8(2)$.

2.3. Lorentzian lattices

Let L be a Lorentzian lattice. The set $\mathcal{C}_L := \{v \in L \otimes \mathbf{R};\, v^2 > 0\}$ is called the light cone of L. Since L is Lorentzian, $\mathcal{C}_L$ consists of two connected components, which are written as $\mathcal{C}_L^+$, $\mathcal{C}_L^-$. The closure of $\mathcal{C}_L^\pm$ in $L \otimes \mathbf{R}$ are written as $\overline{\mathcal{C}_L^\pm}$.

For $l \in L \otimes \mathbf{R}$, we set $h_l := \{v \in \mathcal{C}_L^+;\, \langle v, l\rangle = 0\}$. Then $h_l \neq \emptyset$ if and only if $l^2 < 0$. Define $(\mathcal{C}_L^+)^o := \mathcal{C}_L^+ \setminus \bigcup_{d\in\Delta_L} h_d$. By [12, Chap. V], the Weyl group $W(L)$ acts simply transitively on the set of connected components of $(\mathcal{C}_L^+)^o$. Any connected component of $(\mathcal{C}_L^+)^o$ is called a *Weyl chamber* of L.

Let $\mathcal{W}$ be a Weyl chamber of L, so that $(\mathcal{C}_L^+)^o = \amalg_{w\in W(L)} w(\mathcal{W})$. We define $\Delta_L^+ := \{d \in \Delta_L;\, \langle v, d\rangle > 0\,(\forall\, v \in \mathcal{W})\}$. Then $\Delta_L = \Delta_L^+ \amalg (-\Delta_L^+)$ and $\mathcal{W}$ has the expression $\mathcal{W} = \{v \in \mathcal{C}_L^+;\, \langle v, d\rangle > 0\,(\forall\, d \in \Delta_L^+)\}$. A hyperplane $h_d \subset L \otimes \mathbf{R}$, $d \in \Delta_L^+$ is called a *wall* of $\mathcal{W}$ if $\dim(h_d \cap \overline{\mathcal{W}}) = r(L) - 1$, where $\overline{\mathcal{W}}$ is the closure of $\mathcal{W}$ in $L \otimes \mathbf{R}$. We set $\Pi(L, \mathcal{W}) := \{d \in \Delta_L^+;\, h_d \text{ is a wall of } \mathcal{W}\}$, which is the minimal set of roots defining $\mathcal{W}$, i.e.,

$$\mathcal{W} = \{v \in \mathcal{C}_L^+;\, \langle v, d\rangle > 0,\, \forall\, d \in \Pi(L, \mathcal{W})\} \tag{2.2}$$

and any inequality $\langle v, d\rangle > 0$, $d \in \Pi(L, \mathcal{W})$ is essential in (2.2).

A vector $\varrho \in L \otimes \mathbf{Q}$ is called a *Weyl vector* of $(L, \mathcal{W})$ if $\langle \varrho, d\rangle = 1$ for all $d \in \Pi(L, \mathcal{W})$. A Lorentzian lattice does not necessarily have a Weyl vector.

2.4. Lattices of signature $(2, n)$

Let Λ be a lattice with $\mathrm{sign}(\Lambda) = (2, n)$. Define

$$\Omega_\Lambda := \{[x] \in \mathbf{P}(\Lambda \otimes \mathbf{C});\, \langle x, x\rangle = 0,\, \langle x, \bar{x}\rangle > 0\},$$

which has two connected components $\Omega_\Lambda^\pm$. Each of $\Omega_\Lambda^\pm$ is isomorphic to a bounded symmetric domain of type IV of dimension n. On Ω_Λ, acts $O(\Lambda)$ projectively. Set

$$O^+(\Lambda) := \{g \in O(\Lambda);\, g(\Omega_\Lambda^\pm) = \Omega_\Lambda^\pm\},$$

which is a subgroup of $O(\Lambda)$ of index 2 with $\Omega_\Lambda/O(\Lambda) = \Omega_\Lambda^+/O^+(\Lambda)$. Since $O^+(\Lambda)$ is an arithmetic subgroup of $\mathrm{Aut}(\Omega_\Lambda^+)$, $O^+(\Lambda)$ acts properly discontinuously on Ω_Λ^+. In particular, the stabilizer $O^+(\Lambda)_{[\eta]} := \{g \in O^+(\Lambda);\ g \cdot [\eta] = [\eta]\}$ is finite for all $[\eta] \in \Omega_\Lambda^+$, and the quotient

$$\mathcal{M}_\Lambda := \Omega_\Lambda/O(\Lambda) = \Omega_\Lambda^+/O^+(\Lambda)$$

is an analytic space. There exists a compactification $\mathcal{M}_\Lambda^*$ of $\mathcal{M}_\Lambda$, called the Baily–Borel–Satake compactification [1], such that $\mathcal{M}_\Lambda^*$ is an irreducible normal projective variety of dimension n with $\dim(\mathcal{M}_\Lambda^* \setminus \mathcal{M}_\Lambda) = 1$.

For $\lambda \in \Lambda \otimes \mathbf{R}$, set

$$H_\lambda := \{[x] \in \Omega_\Lambda;\ \langle x, \lambda\rangle = 0\}.$$

Then $H_\lambda \neq \emptyset$ if and only if $\lambda^2 < 0$. We define the discriminant locus of Ω_Λ by

$$\mathcal{D}_\Lambda := \sum_{d \in \Delta_\Lambda/\pm 1} H_d,$$

which is a reduced divisor on Ω_Λ. We define the *reduced* divisors $\mathcal{D}'_\Lambda$ and $\mathcal{D}''_\Lambda$ by

$$\mathcal{D}'_\Lambda = \sum_{d \in \Delta'_\Lambda/\pm 1} H_d, \qquad \mathcal{D}''_\Lambda = \sum_{d \in \Delta''_\Lambda/\pm 1} H_d.$$

Since $\Delta_\Lambda = \Delta'_\Lambda \amalg \Delta''_\Lambda$, we have $\mathcal{D}_\Lambda = \mathcal{D}'_\Lambda + \mathcal{D}''_\Lambda$. For $k \in \mathbf{Q}_{<0}$ and $\gamma \in A_\Lambda$ with $\gamma = -\gamma$, we define the Heegner divisor of discriminant (γ, k) as (cf. [14, p.119])

$$\mathcal{H}_\Lambda(\gamma, k) := \frac{1}{2} \sum_{\lambda \in \gamma + \Lambda,\, \lambda^2 = k} H_\lambda = \sum_{\{\lambda \in \gamma + \Lambda,\, \lambda^2 = k\}/\pm 1} H_\lambda.$$

Then $\mathcal{D}_\Lambda$, $\mathcal{D}'_\Lambda$, $\mathcal{D}''_\Lambda$ are linear combinations of Heegner divisors. Notice that our $\mathcal{H}_\Lambda(\gamma, k)$ is the half of $\mathcal{H}_\Lambda(\gamma, k)$ in [14].

Assume that Λ is a primitive 2-elementary sublattice of $\mathbb{L}_{K3}$. We set

$$\Omega_\Lambda^o := \Omega_\Lambda \setminus \mathcal{D}_\Lambda, \qquad \mathcal{M}_\Lambda^o := \Omega_\Lambda^o/O(\Lambda).$$

For $d \in \Delta_\Lambda$, we have

$$H_d \cap \Omega_\Lambda = \Omega_{\Lambda \cap d^\perp} = \Omega_{[\Lambda^\perp \perp d]^\perp}.$$

We define the subsets $H_d^o \subset H_d$ $(d \in \Delta_\Lambda)$ and $\mathcal{D}_\Lambda^o \subset \mathcal{D}_\Lambda$ by

$$H_d^o := \{[\eta] \in \Omega_\Lambda^+;\ O^+(\Lambda)_{[\eta]} = \{\pm 1, \pm s_d\}\}, \qquad \mathcal{D}_\Lambda^o := \sum_{d \in \Delta_\Lambda/\pm 1} H_d^o.$$

If $H_d \neq \emptyset$ (resp. $\mathcal{D}_\Lambda \neq \emptyset$), then H_d^o (resp. $\mathcal{D}_\Lambda^o$) is a non-empty Zariski open subset of $\Omega_{\Lambda \cap d^\perp}$ (resp. $\mathcal{D}_\Lambda$). Since $O(\Lambda)$ preserves $\mathcal{D}_\Lambda$ and $\mathcal{D}_\Lambda^o$, we define

$$\overline{\mathcal{D}}_\Lambda := \mathcal{D}_\Lambda/O(\Lambda), \qquad \overline{\mathcal{D}}_\Lambda^o := \mathcal{D}_\Lambda^o/O(\Lambda) \subset \overline{\mathcal{D}}_\Lambda.$$

Then $\overline{\mathcal{D}}_\Lambda^o \cap \mathrm{Sing}\,\mathcal{M}_\Lambda = \emptyset$ by [59, Prop. 1.9 (5)]. For the number of the irreducible components of $\overline{\mathcal{D}}_\Lambda$, see Corollary 11.16 below.

When $\Lambda = \mathbb{U}(N) \oplus L$, a vector of $\Lambda \otimes \mathbf{C}$ is denoted by (m, n, v), where $m, n \in \mathbf{C}$ and $v \in L \otimes \mathbf{C}$. The tube domain $L \otimes \mathbf{R} + i\,\mathcal{C}_L$ is identified with Ω_Λ via the map

$$L \otimes \mathbf{R} + i\,\mathcal{C}_L \ni z \to [(-z^2/2, 1/N, z)] \in \Omega_\Lambda \subset \mathbf{P}(\Lambda \otimes \mathbf{C}), \qquad z \in L \otimes \mathbf{C} \tag{2.3}$$

by [9, p.542]. The component of Ω_Λ corresponding to $L \otimes \mathbf{R} + i\,\mathcal{C}_L^+$ via the isomorphism (2.3) is written as Ω_Λ^+.

3. $K3$ surfaces with involution

3.1. $K3$ surfaces with involution and their moduli space

A compact, connected, smooth complex surface X is called a $K3$ *surface* if it is simply connected and has trivial canonical bundle K_X. Let X be a $K3$ surface. Then $H^2(X, \mathbf{Z})$ endowed with the cup-product pairing is isometric to the $K3$ lattice $\mathbb{L}_{K3}$. The Picard lattice of X is defined as $\mathrm{Pic}(X) := H^{1,1}(X, \mathbf{R}) \cap H^2(X, \mathbf{Z})$. An isometry of lattices $\alpha\colon H^2(X, \mathbf{Z}) \cong \mathbb{L}_{K3}$ is called a *marking* of X. The pair (X, α) is called a *marked* $K3$ *surface*, whose period is defined as

$$\pi(X, \alpha) := [\alpha(\eta)] \in \mathbf{P}(\mathbb{L}_{K3} \otimes \mathbf{C}), \qquad \eta \in H^0(X, K_X) \setminus \{0\}.$$

Let $M \subset \mathbb{L}_{K3}$ be a primitive 2-elementary Lorentzian sublattice. A $K3$ surface equipped with a holomorphic involution $\iota\colon X \to X$ is called a 2-*elementary* $K3$ *surface of type* M if there exists a marking α of X satisfying

$$\iota^*|_{H^0(X,K_X)} = -1, \qquad \iota^* = \alpha^{-1} \circ I_M \circ \alpha.$$

Equivalently, $\alpha(H^2_+(X, \mathbf{Z})) = M$, where $H^2_{\pm}(X, \mathbf{Z}) := \{l \in H^2(X, \mathbf{Z});\ \iota^* l = \pm l\}$.

Let (X, ι) be a 2-elementary $K3$ surface of type M and let α be a marking with $\theta^* = \alpha^{-1} \circ I_M \circ \alpha$. Since $H^{2,0}(X, \mathbf{C}) \subset H^2_-(X, \mathbf{C})$ and hence $\mathrm{Pic}(X) \supset H^2_+(X, \mathbf{Z})$, we have $\pi(X, \alpha) \in \Omega^o_{M^\perp}$ and $\alpha(\mathrm{Pic}(X)) \supset M$. By [59, Th. 1.8] and Proposition 11.2 below, the $O(M^\perp)$-orbit of $\pi(X, \iota)$ is independent of the choice of a marking α with $\iota^* = \alpha^{-1} I_M \alpha$. The Griffiths period of (X, ι) is defined as the $O(M^\perp)$-orbit

$$\varpi_M(X, \iota) := O(M^\perp) \cdot \pi(X, \alpha) \in \mathcal{M}^o_{M^\perp}.$$

By [49], [13], [45], [16], [59, Th. 1.8] and Proposition 11.2 below, the coarse moduli space of 2-elementary $K3$ surfaces of type M is isomorphic to $\mathcal{M}^o_{M^\perp}$ via the map ϖ_M. In the rest of this paper, we identify the point $\varpi_M(X, \iota) \in \mathcal{M}^o_{M^\perp}$ with the isomorphism class of (X, ι).

For a 2-elementary $K3$ surface (X, ι), set $X^\iota := \{x \in X;\ \iota(x) = x\}$.

Lemma 3.1. *Let (X, ι) be a 2-elementary $K3$ surface of type M.*

1. *If $M \cong \mathbb{U}(2) \oplus \mathbb{E}_8(2)$, then $X^\iota = \emptyset$.*
2. *If $M \cong \mathbb{U} \oplus \mathbb{E}_8(2)$, then X^ι is the disjoint union of two elliptic curves.*
3. *If $M \not\cong \mathbb{U}(2) \oplus \mathbb{E}_8(2), \mathbb{U} \oplus \mathbb{E}_8(2)$, there exist a smooth irreducible curve C of genus $g(M)$ and smooth rational curves $E_1, \ldots, E_{k(M)}$ such that $X^\iota = C \amalg E_1 \amalg \cdots \amalg E_{k(M)}$.*

Proof. See [45, Th. 4.2.2]. $\square$

After Lemma 3.1, a primitive 2-elementary Lorentzian sublattice $M \subset \mathbb{L}_{K3}$ is said to be *exceptional* if $M \cong \mathbb{U}(2) \oplus \mathbb{E}_8(2)$ or $\mathbb{U} \oplus \mathbb{E}_8(2)$.

For $g \geq 0$, let $\mathfrak{S}_g$ be the Siegel upper half-space of degree g. When $g = 1$, $\mathfrak{S}_1$ is the complex upper half-plane. We write $\mathfrak{H}$ for $\mathfrak{S}_1$. Let $Sp_{2g}(\mathbf{Z})$ be the symplectic group of degree $2g$ over $\mathbf{Z}$ and let $\mathcal{A}_g := \mathfrak{S}_g / Sp_{2g}(\mathbf{Z})$ be the Siegel modular variety of degree g, where $Sp_{2g}(\mathbf{Z})$ acts on $\mathfrak{S}_g$ by $\gamma \cdot \Omega := (A\Omega + B)(C\Omega + D)^{-1}$ for $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp_{2g}(\mathbf{Z})$. Then $\mathcal{A}_g$ is a coarse moduli space of principally polarized Abelian varieties of dimension g via the period map. The Satake compactification $\mathcal{A}_g^*$ of $\mathcal{A}_g$ is a normal projective variety that contains $\mathcal{A}_g$ as a dense Zariski open subset. We have the equality of sets $\mathcal{A}_g^* = \mathcal{A}_g \amalg \mathcal{A}_{g-1} \amalg \cdots \amalg \mathcal{A}_0$.

After Lemma 3.1, the Jacobian variety of X^ι is defined as the complex torus $\mathrm{Jac}(X^\iota) := H^1(X^\iota, \mathcal{O}_{X^\iota})/H^1(X^\iota, \mathbf{Z})$, which is equipped with the principal polarization. Hence $\mathrm{Jac}(X^\iota)$ is a principally polarized Abelian variety of dimension $g(M)$, if $M \not\cong \mathbb{U}(2) \oplus \mathbb{E}_8(2)$. When $M \cong \mathbb{U}(2) \oplus \mathbb{E}_8(2)$, one has $\mathrm{Jac}(X^\iota) = \{0\}$. The period of X^ι, i.e., the period of $\mathrm{Jac}(X^\iota)$, is denoted by $\mathit{\Omega}(X^\iota) \in \mathcal{A}_{g(M)}$.

For a 2-elementary $K3$ surface (X, ι), we define

$$\overline{J}^o_M(X, \iota) = \overline{J}^o_M(\varpi_M(X, \iota)) := \mathit{\Omega}(X^\iota) \in \mathcal{A}_{g(M)}.$$

Let $\mathit{\Pi}_{M^\perp} \colon \Omega_{M^\perp} \to \mathcal{M}_{M^\perp}$ be the projection and set $J^o_M := \overline{J}^o_M \circ \mathit{\Pi}_{M^\perp}|_{\Omega^o_{M^\perp}}$. Then J^o_M is an $O(M^\perp)$-equivariant holomorphic map from $\Omega^o_{M^\perp}$ to $\mathcal{A}_{g(M)}$ with respect to the trivial $O(M^\perp)$-action on $\mathcal{A}_{g(M)}$. By [59, Th. 3.3], J^o_M extends to an $O(M^\perp)$-equivariant holomorphic map $J_M \colon \Omega^o_{M^\perp} \cup \mathcal{D}^o_{M^\perp} \to \mathcal{A}^*_{g(M)}$. Let $\overline{J}_M \colon \mathcal{M}^o_{M^\perp} \cup \overline{\mathcal{D}}^o_{M^\perp} \to \mathcal{A}^*_{g(M)}$ denote the corresponding holomorphic extension of $\overline{J}^o_M$.

Proposition 3.2. *The map $\overline{J}_M$ extends to a meromorphic map from $\mathcal{M}^*_{M^\perp}$ to $\mathcal{A}^*_{g(M)}$. When $r(M) \geq 19$, $\overline{J}_M$ extends to a holomorphic map from $\mathcal{M}^*_{M^\perp}$ to $\mathcal{A}^*_{g(M)}$.*

Proof. By the Borel–Kobayashi–Ochiai extension theorem, $\overline{J}_M$ extends to a holomorphic map from $\mathcal{M}^*_{M^\perp} \setminus (\mathrm{Sing}\, \mathcal{M}^*_{M^\perp} \cup \mathrm{Sing}\, \overline{\mathcal{D}}_{M^\perp})$ to $\mathcal{A}^*_{g(M)}$. Since $\mathcal{M}^*_{M^\perp}$ is normal, we get $\dim(\mathrm{Sing}\, \mathcal{M}^*_{M^\perp} \cup \mathrm{Sing}\, \overline{\mathcal{D}}_{M^\perp}) \leq \dim \mathcal{M}^*_{M^\perp} - 2$ when $r(M) \leq 18$, so that $\overline{J}_M$ extends to a meromorphic map from $\mathcal{M}^*_{M^\perp}$ to $\mathcal{A}^*_{g(M)}$ in this case. If $r(M) = 19$, $\mathcal{M}^*_{M^\perp} \setminus \mathcal{M}^o_{M^\perp}$ consists of finite points. The result follows from the Borel–Kobayashi–Ochiai extension theorem. If $r(M) = 20$, the result is trivial. □

3.2. Degenerations of 2-elementary $K3$ surfaces

Let $\mathit{\Delta} \subset \mathbf{C}$ be the unit disc. Let $\mathcal{Z}$ be a smooth complex threefold. Let $p \colon \mathcal{Z} \to \mathit{\Delta}$ be a proper, surjective holomorphic function without critical points on $\mathcal{Z} \setminus p^{-1}(0)$. Let $\iota \colon \mathcal{Z} \to \mathcal{Z}$ be a holomorphic involution preserving the fibers of p. We set $Z_t = p^{-1}(t)$ and $\iota_t = \iota|_{Z_t}$ for $t \in \mathit{\Delta}$. Then $p \colon (\mathcal{Z}, \iota) \to \mathit{\Delta}$ is called an ordinary singular family of 2-elementary $K3$ surfaces of type M if p has a unique, non-degenerate critical point on Z_0 and if (Z_t, ι_t) is a 2-elementary $K3$ surface of type M for all $t \in \mathit{\Delta}^*$. Since Z_0 is a singular $K3$ surface, $\iota_0 \in \mathrm{Aut}(Z_0)$ extends to an anti-symplectic holomorphic involution $\widetilde{\iota}_0$ on the minimal resolution $\widetilde{Z}_0$ of Z_0, i.e., $(\widetilde{\iota}_0)^* = -1$ on $H^0(\widetilde{Z}_0, K_{\widetilde{Z}_0})$.

Theorem 3.3. *Let $d \in \Delta_{M^\perp}$ and let $\overline{H}^o_d := \mathit{\Pi}_{M^\perp}(H^o_d)$ be the image of H^o_d by the natural projection $\mathit{\Pi}_{M^\perp} \colon \Omega_{M^\perp} \to \mathcal{M}_{M^\perp}$. Let $\gamma \colon \mathit{\Delta} \to \mathcal{M}_{M^\perp}$ be a holomorphic curve intersecting $\overline{H}^o_d$ transversally at $\gamma(0)$. Then there exists an ordinary singular family of 2-elementary $K3$ surfaces $p_{\mathcal{Z}} \colon (\mathcal{Z}, \iota) \to \mathit{\Delta}$ of type M with Griffiths period map γ, such that $p_{\mathcal{Z}}$ is projective and such that $(\tilde{Z}_0, \tilde{\iota}_0)$ is a 2-elementary $K3$ surface of type $[M \perp d]$ with Griffiths period $\gamma(0)$.*

Proof. By [59, Th. 2.6], there exists an ordinary singular family of 2-elementary $K3$ surfaces $p_{\mathcal{Z}} \colon (\mathcal{Z}, \iota) \to \mathit{\Delta}$ of type M with Griffiths period map γ such that $p_{\mathcal{Z}}$ is projective. We prove that $(\widetilde{Z}_0, \widetilde{\iota}_0)$ is a 2-elementary $K3$ surface of type $[M \perp d]$.

Let $o_{\mathcal{Z}} \in Z_0$ be the unique critical point of $p_{\mathcal{Z}}$. Let $p_{\mathcal{Y}} \colon (\mathcal{Y}, \iota_{\mathcal{Y}}) \to \mathit{\Delta}$ be the family induced from $p_{\mathcal{Z}} \colon (\mathcal{Z}, \iota) \to \mathit{\Delta}$ by the map $\mathit{\Delta} \ni t \to t^2 \in \mathit{\Delta}$. Then $\mathcal{Y} = \mathcal{Z} \times_{\mathit{\Delta}} \mathit{\Delta}$ and $p_{\mathcal{Y}} = \mathrm{pr}_2$. The projection pr_1 induces an identification between $(Y_t, \iota_{\mathcal{Y}}|_{Y_t})$ and (Z_{t^2}, ι_{t^2}) for all $t \in \mathit{\Delta}$. Since the Picard-Lefschetz transformation for the family of

$K3$ surfaces $p_{\mathcal{Y}}|_{\Delta^*}: \mathcal{Y}|_{\Delta^*} \to \Delta^*$ is trivial, there exists a marking $\beta: R^2(p_{\mathcal{Y}}|_{\Delta^*})_*\mathbf{Z} \cong \mathbb{L}_{K3,\Delta^*}$. Let $o_{\mathcal{Y}}$ be the unique singular point of $\mathcal{Y}$ with $\mathrm{pr}_2(o_{\mathcal{Y}}) = o_{\mathcal{Z}}$. Since $(\mathcal{Y}, o_{\mathcal{Y}})$ is a three-dimensional ordinary double point, there exist two different resolutions $\pi: (\mathcal{X}, E) \to (\mathcal{Y}, o_{\mathcal{Y}})$ and $\pi': (\mathcal{X}', E') \to (\mathcal{Y}, o_{\mathcal{Y}})$. By e.g. [59, Th. 2.1 and Proof of Th. 2.6] and the references therein, the following (i), (ii), (iii), (iv) hold:

(i) Set $p := p_{\mathcal{Y}} \circ \pi$ and $p' := p_{\mathcal{Y}} \circ \pi'$. Then $p: \mathcal{X} \to \Delta$ and $p': \mathcal{X}' \to \Delta$ are simultaneous resolutions of $p_{\mathcal{Y}}: \mathcal{Y} \to \Delta$, and they are smooth families of $K3$ surfaces. The marking β induces a marking α for $p: \mathcal{X} \to \Delta$ and a marking α' for $p': \mathcal{X}' \to \Delta$.

(ii) $E = \pi^{-1}(o_{\mathcal{Y}})$ is a smooth rational curve on X_0, and $E' = (\pi')^{-1}(o_{\mathcal{Y}})$ is a smooth rational curve on X_0'. The marked family $(p': \mathcal{X}' \to \Delta, \alpha')$ is the elementary modification of $(p: \mathcal{X} \to \Delta, \alpha)$ with center E (cf. [59, Sect. 2.1]). Replacing β by $g \circ \beta$, $g \in \Gamma(M) := \{g \in O(\mathbb{L}_{K3});\, gI_M = I_M g\}$ if necessary, we have $d = \alpha(c_1([E]))$.

(iii) Let $e: \mathcal{X} \setminus E \to \mathcal{X}' \setminus E'$ be the isomorphism defined as $e := (\pi')^{-1} \circ \pi$. Then e is an isomorphism of fiber spaces over Δ^* and the isomorphism $e|_{X_0 \setminus E}: X_0 \setminus E \to X_0' \setminus E'$ extends to an isomorphism $\widetilde{e}_0: X_0 \to X_0'$ with

$$\alpha_0 \circ (\widetilde{e}_0)^* \circ (\alpha_0')^{-1} = s_d. \tag{3.1}$$

(iv) There exists an isomorphism $\varphi_{K3}(I_M): \mathcal{X} \to \mathcal{X}'$ of fiber spaces over Δ such that the following diagrams are commutative (cf. [59, Eqs. (1.6), (2.8)]):

(3.2)
$$\begin{array}{ccccc} (\mathcal{X}, E) & \xrightarrow{\pi} & (\mathcal{Y}, o) & \xrightarrow{\mathrm{pr}_1} & (\mathcal{Z}, o) \\ {\scriptstyle \varphi_{K3}(I_M)}\downarrow & & {\scriptstyle \iota_{\mathcal{Y}}}\downarrow & & {\scriptstyle \iota}\downarrow \\ (\mathcal{X}', E') & \xrightarrow{\pi'} & (\mathcal{Y}, o) & \xrightarrow{\mathrm{pr}_1} & (\mathcal{Z}, o) \end{array} \qquad \begin{array}{ccc} R^2p'_*\mathbf{Z} & \xrightarrow{\varphi_{K3}(I_M)^*} & R^2p_*\mathbf{Z} \\ {\scriptstyle \alpha'}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathbb{L}_{K3,\Delta} & \xrightarrow{I_M} & \mathbb{L}_{K3,\Delta} \end{array}$$

We define $\theta := (\widetilde{e}_0)^{-1} \circ \varphi_{K3}(I_M)|_{X_0} \in \mathrm{Aut}(X_0)$. Since $\pi' \circ \widetilde{e}_0 = \pi|_{X_0}$ by (iii) and hence $\pi'|_{X_0' \setminus E'} = (\pi|_{X_0 \setminus E}) \circ (\widetilde{e}_0)^{-1}|_{X_0' \setminus E'}$, we get by the first diagram of (3.2)

$$\begin{aligned} (\pi|_{X_0 \setminus E}) \circ (\theta|_{X_0 \setminus E}) &= (\pi|_{X_0 \setminus E}) \circ (\widetilde{e}_0)^{-1}|_{X_0' \setminus E'} \circ \varphi_{K3}(I_M)|_{X_0 \setminus E} \\ &= (\pi'|_{X_0' \setminus E'}) \circ \varphi_{K3}(I_M)|_{X_0 \setminus E} \\ &= (\iota_{\mathcal{Y}}|_{Y_0 \setminus \{o\}}) \circ \pi|_{X_0 \setminus E}, \end{aligned}$$

which implies that $(\pi|_{X_0}) \circ \theta = (\iota_{\mathcal{Y}})|_{Y_0} \circ (\pi|_{X_0})$. Since X_0 is the minimal resolution of Z_0, i.e., $X_0 = \widetilde{Z}_0$ and since $(Y_0, \iota_{\mathcal{Y}}|_{Y_0}) = (Z_0, \iota_0)$, this last equality implies that θ is the involution on X_0 induced from ι_0. Thus we have $\theta = \widetilde{\iota}_0$.

By (2.1), (3.1) and the second diagram of (3.2), we get

$$\alpha_0\, \theta^*\, \alpha_0^{-1} = \alpha_0 \varphi_{K3}(I_M)^* (\alpha_0')^{-1} \circ \alpha_0' (\widetilde{e}_0^{-1})^* \alpha_0^{-1} = I_M \circ s_d = I_{[M \perp d]}. \tag{3.3}$$

By (3.3), $\theta = \widetilde{\iota}_0$ is an anti-symplectic involution of type $[M \perp d]$. □

Let $\mathcal{C}$ be a smooth complex surface. Let $p: \mathcal{C} \to \Delta$ be a proper, surjective holomorphic function without critical points on $\mathcal{C} \setminus p^{-1}(0)$. Then $p: \mathcal{C} \to \Delta$ is called an ordinary singular family of curves if p has a unique, non-degenerate critical point on the central fiber $p^{-1}(0)$. Notice that an ordinary singular family of curves is *not* necessarily a family of stable curves, since $p^{-1}(0)$ may contain a (-1)-curve of $\mathcal{C}$.

Lemma 3.4. *Let $p: \mathcal{C} \to \Delta$ be an ordinary singular family of curves of genus g and set $C_t := p^{-1}(t)$ for $t \in \Delta$. Let $J: \Delta \setminus \{0\} \to \mathcal{A}_g$ be the holomorphic map* *defined as* $J(t) := \Omega(C_t)$ *for* $t \in \Delta \setminus \{0\}$. *Then* J *extends to a holomorphic map from* Δ *to* $\mathcal{A}_g^*$ *by setting* $J(0) := \Omega(\widehat{C_0})$, *where* $\widehat{C}_0$ *is the normalization of* C_0.

Proof. Since the result is obvious when $g = 0$, we assume $g > 0$. Since J is locally liftable, J extends to a holomorphic map from Δ to $\mathcal{A}_g^*$ by the Borel-Kobayashi-Ochiai extension theorem. Let $C_0 = D \amalg D_1 \amalg \ldots \amalg D_k$ be the decomposition into the connected components of C_0. We may assume that D is singular and that all D_i are smooth. Let $o \in \mathcal{C}$ be the unique critical point of p. Since $\operatorname{Sing} D = \{o\}$, D consists of at most two irreducible components. There are two possible cases: (i) D is irreducible; (ii) D is the join of two smooth curves A and B intersecting transversally at o. When D is stable, the result follows from e.g. [19, Cor. 3.8], [41, Sect. 3 Th. 3]. When D is not stable, then $D = A + B$ and $g(A)g(B) = 0$. In this case, we may assume $g(B) = 0$, i.e., $B \cong \mathbf{P}^1$. Then B is a (-1)-curve on $\mathcal{C}$ by Zariski's lemma [2, Chap. III Lemma 8.2]. Let $\sigma\colon \mathcal{C} \to \overline{\mathcal{C}}$ be the blow-down of B. Then $\overline{p} := p \circ \sigma^{-1}$ extends to a holomorphic function from $\overline{\mathcal{C}}$ to Δ. Since $\overline{p}^{-1}(0) = A \amalg D_1 \amalg \ldots \amalg D_k$ is a smooth reduced divisor of $\overline{\mathcal{C}}$, $\overline{p}\colon \overline{\mathcal{C}} \to \Delta$ is a smooth morphism. Since $J(t) = \Omega(p^{-1}(t)) = \Omega(\overline{p}^{-1}(t))$ for $t \neq 0$ and hence $\lim_{t\to 0} J(t) = \lim_{t\to 0} \Omega(\overline{p}^{-1}(t)) = \Omega(\overline{p}^{-1}(0))$ by the smoothness of $\overline{p}$, we get

$$\lim_{t\to 0} J(t) = \Omega(A \amalg D_1 \amalg \ldots \amalg D_k) = \Omega(A \amalg B \amalg D_1 \amalg \ldots \amalg D_k) = \Omega(\widehat{C_0}),$$

where we used the fact $\operatorname{Jac}(B) = \{0\}$ to get the second equality. □

If $p\colon (\mathcal{Z}, \iota) \to \Delta$ is an ordinary singular family of 2-elementary $K3$ surfaces of type M and if $o \in \mathcal{Z}$ is the unique critical point of p, then there exists a system of coordinates $(\mathcal{U}, (z_1, z_2, z_3))$ centered at o such that

$$\iota(z) = (-z_1, -z_2, -z_3) \quad \text{or} \quad (z_1, z_2, -z_3), \qquad z \in \mathcal{U}. \tag{3.4}$$

If $\iota(z) = (-z_1, -z_2, -z_3)$ on $\mathcal{U}$, ι is said to be of type $(0, 3)$. If $\iota(z) = (z_1, z_2, -z_3)$ on $\mathcal{U}$, ι is said to be of type $(2, 1)$.

Theorem 3.5. *Let* $M \subset \mathbb{L}_{K3}$ *be a primitive* 2*-elementary Lorentzian sublattice. For* $d \in \Delta_{M^\perp}$, *the following identity holds*

$$J_M|_{H_d^o} = J^o_{[M\perp d]}|_{H_d^o}.$$

Proof. Let $\mathfrak{p} \in \overline{H}_d^o$ and let $\gamma\colon \Delta \to \mathcal{M}_{M^\perp}$ be a holomorphic curve intersecting $\overline{H}_d^o$ transversally at $\mathfrak{p} = \gamma(0)$. Let $p_{\mathcal{Z}}\colon (\mathcal{Z}, \iota) \to \Delta$ be an ordinary singular family of 2-elementary $K3$ surfaces of type M with Griffiths period map γ, such that $p_{\mathcal{Z}}$ is projective and such that $(\tilde{Z}_0, \tilde{\iota}_0)$ is a 2-elementary $K3$ surface of type $[M \perp d]$ with Griffiths period $\gamma(0)$ (cf. Theorem 3.3). Let $o \in \mathcal{Z}$ be the unique critical point of $p_{\mathcal{Z}}$. Since $\overline{J}_M(\mathfrak{p}) = \overline{J}_M(\gamma(0)) = \lim_{t\to 0} \overline{J}_M(\gamma(t))$ by the continuity of $\overline{J}_M$ and since $\overline{J}^o_{[M\perp d]}(\mathfrak{p}) = \overline{J}^o_{[M\perp d]}(\widetilde{Z}_0, \widetilde{\iota}_0) = \Omega((\widetilde{Z}_0)^{\widetilde{\iota}_0})$ by Theorem 3.3, it suffices to prove

$$\overline{J}_M(\mathfrak{p}) = \lim_{t\to 0} \overline{J}_M(\gamma(t)) = \Omega((\widetilde{Z}_0)^{\widetilde{\iota}_0}) = \overline{J}^o_{[M\perp d]}(\mathfrak{p}). \tag{3.5}$$

Set $\mathcal{Z}^\iota := \{z \in \mathcal{Z};\ \iota(z) = z\}$.

(Case 1) Assume that ι is of type $(0, 3)$. By [59, Prop. 2.5 (1)], $\mathcal{C} := \mathcal{Z}^\iota \setminus \{o\}$ is a smooth complex surface and $p|_{\mathcal{C}}\colon \mathcal{C} \to \Delta$ is a proper holomorphic submersion. Set $C_t := (p|_{\mathcal{C}})^{-1}(t)$. Then

$$\lim_{t\to 0} \overline{J}^o_M(Z_t, \iota_t) = \lim_{t\to 0} \Omega(C_t) = \Omega(C_0). \tag{3.6}$$

Since $Z_0^{\iota_0} = C_0 \amalg \{o\}$, we get

$$(\widetilde{Z}_0)^{\widetilde{\iota}_0} = C_0 \amalg \mathbf{P}^1, \tag{3.7}$$

which yields that

$$\Omega(C_0) = \Omega((\widetilde{Z}_0)^{\widetilde{\iota}_0}). \tag{3.8}$$

Eq. (3.5) follows from (3.6) and (3.8) in this case.

(Case 2) Assume that ι is of type $(2,1)$. By [59, Prop. 2.5 (2)], $p|_{\mathcal{Z}^\iota} \colon \mathcal{Z}^\iota \to \Delta$ is an ordinary singular family of curves. Let $W \to Z_0^{\iota_0}$ be the normalization. Then

$$\lim_{t\to 0} \overline{J}^o_M(Z_t, \iota_t) = \lim_{t\to 0} \Omega(Z_t^{\iota_t}) = \Omega(W) \in \mathcal{A}^*_{g(M)}, \tag{3.9}$$

where the last equality follows from Lemma 3.4. Since $\widetilde{Z}_0 \to Z_0$ is the blow-up at the ordinary double point o, it follows from the local description (3.4) that $(\widetilde{Z}_0)^{\widetilde{\iota}_0}$ is the proper transform of $Z_0^{\iota_0}$. Hence $(\widetilde{Z}_0)^{\widetilde{\iota}_0}$ is a resolution of the singularity of $Z_0^{\iota_0}$. Namely, we have $W = (\widetilde{Z}_0)^{\widetilde{\iota}_0}$, which together with (3.9), yields (3.5) in this case. Since $\mathfrak{p}$ is an arbitrary point of $\overline{H}^o_d$, we get the result. □

Let us give some applications of Theorem 3.5.

Proposition 3.6. *If $g(M) = 1$ and $d \in \Delta'_{M^\perp}$, then*

$$J_M(H_d^o) = \mathcal{A}_0 = \mathcal{A}_1^* \setminus \mathcal{A}_1.$$

Proof. By Lemma 11.5 below, $g([M \perp d]) = g(M) - 1 = 0$. By Theorem 3.5, we get $J_M(H_d^o) = J^o_{[M\perp d]}(H_d^o) = \mathcal{A}_0 = \mathcal{A}_1^* \setminus \mathcal{A}_1$. □

Proposition 3.7. *If $g(M) = 1$, then*

$$\overline{J^o_M(\Omega^o_{M^\perp})} = \mathcal{A}_1^*.$$

Proof. By Proposition 3.2, $\overline{J}_M$ extends to a meromorphic map from $\overline{\mathcal{M}^*_{M^\perp}}$ to $\mathcal{A}_1^*$. Since $J^o_M(\Omega^o_{M^\perp}) = \overline{J}_M(\mathcal{M}^o_{M^\perp})$ and since $\dim \mathcal{A}_1^* = 1$, we have $\overline{J^o_M(\Omega^o_{M^\perp})} = \mathcal{A}_1^*$ if $\overline{J}^o_M$ is non-constant. We see that $\overline{J}^o_M$ is non-constant.

Since $g(M) = 1$, we get by [45, p.1434, Table 1] or by [20, Appendix, Table 2]

$$M^\perp \cong \mathbb{U} \oplus \mathbb{I}_{1,m-1}(2) \quad (1 \le m \le 10), \quad \mathbb{U}(2) \oplus \mathbb{U}(2) \oplus \mathbb{D}_4, \quad \mathbb{U} \oplus \mathbb{U}(2). \tag{3.10}$$

By (3.10), $\Delta'_{M^\perp} \neq \emptyset$. Let $d \in \Delta'_{M^\perp}$. By Proposition 3.6, we get $J_M(H_d^o) = \mathcal{A}_0 = \mathcal{A}_1^* \setminus \mathcal{A}_1$. Since $J_M(\Omega^o_{M^\perp}) \subset \mathcal{A}_1$, this implies that $\overline{J}_M$ is non-constant. □

Proposition 3.8. *If $g(M) = 1$ and $d \in \Delta''_{M^\perp}$, then*

$$J_M(H_d^o) \subset \mathcal{A}_1.$$

Proof. Since $d \in \Delta''_{M^\perp}$, we get $g([M \perp d]) = g(M) = 1$ by Lemma 11.5 below. By Theorem 3.5, we get $J_M(H_d^o) = J^o_{[M\perp d]}(H_d^o) \subset J^o_{[M\perp d]}(\Omega^o_{[M\perp d]^\perp}) \subset \mathcal{A}_1$. □

Proposition 3.9. *If $g(M) = 2$ and $d \in \Delta'_{M^\perp}$, then*

$$\overline{J_M(H_d^o)} = \mathcal{A}_2^* \setminus \mathcal{A}_2.$$

Proof. By Proposition 11.5 below, $g([M \perp d]) = 1$. By Theorem 3.5, we get

$$\overline{J_M(H_d^o)} = \overline{J^o_{[M\perp d]}(H_d^o)} = \overline{J_{[M\perp d]}(\Omega^o_{[M\perp d]^\perp})} = \mathcal{A}_1^* = \mathcal{A}_2^* \setminus \mathcal{A}_2,$$

where the third equality follows from Proposition 3.7. □

We define the divisor $\mathcal{N}_2 \subset \mathcal{A}_2$ as

$$\mathcal{N}_2 := \{\varOmega(E_1 \times E_2) \in \mathcal{A}_2;\ E_1, E_2 \text{ are elliptic curves}\}.$$

Proposition 3.10. *Let $g(M) = 2$ and $d \in \Delta_{M^\perp}$. Then $J_M(H_d^o) \cap \mathcal{N}_2 \neq \emptyset$ if and only if the following conditions are satisfied:*

$$M \cong \mathbb{I}_{1,8}(2), \qquad d \in \Delta''_{M^\perp}, \qquad d/2 \equiv \mathbf{1}_{M^\perp} \mod M^\perp.$$

In particular, if either $M \not\cong \mathbb{I}_{1,8}(2)$ or $d \notin \Delta''_{M^\perp}$ or $d/2 \not\equiv \mathbf{1}_{M^\perp} \mod M^\perp$, then

$$J_M(H_d^o) \subset \mathcal{A}_2 \setminus \mathcal{N}_2.$$

Proof. Assume $J_M(H_d^o) \cap \mathcal{N}_2 \neq \emptyset$. By Proposition 3.9, $d \in \Delta''_{M^\perp}$. By Theorem 3.5,

$$J_{[M\perp d]}(\Omega^o_{[M\perp d]^\perp}) \cap \mathcal{N}_2 \supset J_{[M\perp d]}(H_d^o) \cap \mathcal{N}_2 = J_M(H_d^o) \cap \mathcal{N}_2 \neq \emptyset.$$

Let (X, ι) be a 2-elementary $K3$ surface of type $[M \perp d]$ such that $J_{[M\perp d]}(X, \iota) \in \mathcal{N}_2$. If $[M \perp d] \not\cong \mathbb{U} \oplus \mathbb{E}_8(2), \mathbb{U}(2) \oplus \mathbb{E}_8(2)$, there exists an irreducible smooth curve C of genus $g([M \perp d])$ with $J_{[M\perp d]}(X, \iota) = \varOmega(C)$ by Lemma 3.1. By $d \in \Delta''_{M^\perp}$ and Lemma 11.5 below, we get $g([M \perp d]) = 2$. However, the period of an irreducible smooth curve of genus 2 lies in $\mathcal{A}_2 \setminus \mathcal{N}_2$. This contradicts the condition $\varOmega(C) \in \mathcal{N}_2$. Thus $[M \perp d] \cong \mathbb{U} \oplus \mathbb{E}_8(2)$ or $[M \perp d] \cong \mathbb{U}(2) \oplus \mathbb{E}_8(2)$. If $[M \perp d] \cong \mathbb{U}(2) \oplus \mathbb{E}_8(2)$, then $C = \emptyset$ by Lemma 3.1 (1), which contradicts the condition $\varOmega(C) \in \mathcal{N}_2$. We get $[M \perp d] \cong \mathbb{U} \oplus \mathbb{E}_8(2)$ and hence $M^\perp \cap d^\perp = [M \perp d]^\perp \cong \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8(2)$. Set $L := \mathbf{Z}\, d \cong \mathbb{A}_1$. Since $d \in \Delta''_{M^\perp}$, we get by (11.4) below

$$M^\perp = (M^\perp \cap d^\perp) \oplus L \cong \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8(2) \oplus L. \tag{3.11}$$

Since $r(M) = 22 - r(M^\perp) = 9$, $l(M) = l(M^\perp) = 9$ and $\delta(M) = \delta(M^\perp) = 1$, we get $M \cong \mathbb{I}_{1,8}(2)$. Since $\delta(M^\perp \cap d^\perp) = \delta(\mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8(2)) = 0$ and hence $\mathbf{1}_{M^\perp \cap d^\perp} = 0$, we deduce from (3.11) that $\mathbf{1}_{M^\perp} = \mathbf{1}_{M^\perp \cap d^\perp} \oplus \mathbf{1}_L = \mathbf{1}_L = d/2$ in $A_{M^\perp}$.

Conversely, assume that $M \cong \mathbb{I}_{1,8}(2)$, $d \in \Delta''_{M^\perp}$, and $d/2 \equiv \mathbf{1}_{M^\perp} \mod M^\perp$. We get the decomposition $M^\perp = (M^\perp \cap d^\perp) \oplus L$ by (11.4) below. Then $r(M^\perp \cap d^\perp) = r(M^\perp) - 1 = 12$ and $l(M^\perp \cap d^\perp) = l(M^\perp) - 1 = 8$. Let us see that $\delta(M^\perp \cap d^\perp) = 0$. Let $x \in (M^\perp \cap d^\perp)^\vee$ and $k \in \mathbf{Z}$. Set $y := x + k(d/2) \in (M^\perp)^\vee$. By the definition of $\mathbf{1}_{M^\perp}$, we get

$$-k/2 = \langle y, d/2 \rangle \equiv \langle y, \mathbf{1}_{M^\perp} \rangle \equiv \langle y, y \rangle \equiv \langle x, x \rangle - k^2/2 \mod \mathbf{Z}.$$

Hence $x^2 \equiv k(k-1)/2 \equiv 0 \mod \mathbf{Z}$. Since $x \in (M^\perp \cap d^\perp)^\vee$ is an arbitrary element, we get $\delta(M^\perp \cap d^\perp) = 0$. Since the isometry class of $M^\perp \cap d^\perp$ is determined by the triplet (r, l, δ) by [43, Th. 3.6.2], we get $M^\perp \cap d^\perp \cong \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8(2)$. By Lemma 3.1 (2) and Theorem 3.5, we get $J_M(H_d^o) \subset \mathcal{N}_2$. This proves the proposition. $\square$

4. Automorphic forms on the period domain

4.1. The Igusa cusp form and its pull-back on $\Omega_{M^\perp}$

Let $\mathcal{F}_g := (\mathfrak{S}_g \times \mathbf{C})/Sp(2g, \mathbf{Z})$ be the Hodge line bundle on $\mathcal{A}_g$, where $Sp(2g, \mathbf{Z})$ acts on $\mathfrak{S}_g \times \mathbf{C}$ as follows: For $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(2g, \mathbf{Z})$ and $(\varOmega, \xi) \in \mathfrak{S}_g \times \mathbf{C}$,

$$\gamma \cdot (\varOmega, \xi) := ((A\varOmega + B)(C\varOmega + D)^{-1}, \det(C\varOmega + D)^k\, \xi).$$

Then $\mathcal{F}_g$ is a holomorphic line bundle on $\mathcal{A}_g$ in the sense of orbifolds. There is an integer $\nu \in \mathbf{N}$ such that $\mathcal{F}_g^\nu$ is a line bundle on $\mathcal{A}_g$ in the ordinary sense. By Baily–Borel–Satake, $\mathcal{F}_g^{m\nu}$ extends uniquely to a very ample line bundle on $\mathcal{A}_g^*$ for $m \gg 0$. In this case, let $\overline{\mathcal{F}}_g^{m\nu}$ denote the holomorphic extension of $\mathcal{F}_g^{m\nu}$ to $\mathcal{A}_g^*$.

An element of $H^0(\mathcal{A}_g, \mathcal{F}_g^k)$ is identified with a Siegel modular form on $\mathfrak{S}_g$ of weight k. For $g > 0$, we define the Igusa form as

$$\chi_g(\Omega) := \prod_{(a,b)\ \mathrm{even}} \theta_{a,b}(\Omega),$$

where $a, b \in \{0, \frac{1}{2}\}^g$ and $\theta_{a,b}(\Omega) := \sum_{n\in\mathbf{Z}^g} \exp\{\pi i^t(n+a)\Omega(n+a) + 2\pi i^t(n+a)b\}$ is the corresponding theta constant. Here (a,b) is *even* if $4^t ab \equiv 0 \mod 2$. When $g = 0$, we define $\chi_0 := 1$. By [29, Lemma 10], χ_g^8 is a Siegel modular form of weight $2^{g+1}(2^g+1)$. Set

$$\theta_{\mathrm{null},g} := \{[\Omega] \in \mathcal{A}_g;\ \chi_g(\Omega) = 0\},$$

which is a reduced divisor on $\mathcal{A}_g$. It is classical that $\mathcal{N}_2 = \theta_{\mathrm{null},2}$. (See e.g. [39, Chap. II, Cors. 3.12, 3.15, 3.17].) In Sect. 9, χ_g^8 shall play the crucial role.

Define the *Petersson metric* $\|\cdot\|_{\mathcal{F}_g}$ on $\mathcal{F}_g$ by

$$\|\xi\|^2_{\mathcal{F}_g}(\Omega) := (\det \operatorname{Im}\Omega)|\xi|^2, \qquad (\Omega, \xi) \in \mathfrak{S}_g \times \mathbf{C}. \tag{4.1}$$

Since χ_g^8 is a Siegel modular form, $\|\chi_g^8\|^2_{\mathcal{F}_g^{2^{g+1}(2^g+1)}}$ is a C^∞ function on $\mathcal{A}_g$ in the sense of orbifolds. For simplicity, we write $\|\chi_g^8\|^2$ for $\|\chi_g^8\|^2_{\mathcal{F}_g^{2^{g+1}(2^g+1)}}$.

Lemma 4.1. *Let $p\colon \mathcal{C} \to \Delta$ be an ordinary singular family of curves of genus g and set $C_t := p^{-1}(t)$ for $t \in \Delta$. Let $\hat{C}_0$ be the normalization of C_0 with genus $g(\hat{C}_0)$.*

(1) *If C_0 is irreducible and $g(\hat{C}_0) = g - 1$, there exists $h(t) \in \mathcal{O}(\Delta)$ such that*

$$\log \|\chi_g(\Omega(C_t))^8\|^2 = 2^{2g-2}\log|t|^2 + \log|h(t)|^2 + O(\log\log|t|^{-1}) \qquad (t \to 0).$$

(2) *If $g = 2$ and if $\hat{C}_0$ is the disjoint union of two elliptic curves, then*

$$\log \|\chi_2(\Omega(C_t))^8\|^2 = 8\log|t|^2 + O(\log\log|t|^{-1}) \qquad (t \to 0).$$

Proof. **(1)** By [19, Cor.3.8], one can write

$$\Omega(C_t) = \left[\frac{\log t}{2\pi i}\Lambda + \psi(t)\right] \in \mathcal{A}_g, \qquad \Lambda = \begin{pmatrix} 1 & {}^t\mathbf{0}_{g-1} \\ \mathbf{0}_{g-1} & O_{g-1}\end{pmatrix} \tag{4.2}$$

where $\mathbf{0}_{g-1}$ is the zero vector of $\mathbf{C}^{g-1}$, O_{g-1} is the $(g-1)\times(g-1)$-zero matrix, and $\psi(t)$ is a holomorphic function on Δ with values in complex symmetric $g \times g$-matrices. If we write $\psi(0) = \left(\begin{smallmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22}\end{smallmatrix}\right)$, then $\psi_{22} \in \mathfrak{S}_{g-1}$.

Write $\Omega = \left(\begin{smallmatrix} z & {}^t\omega \\ \omega & Z\end{smallmatrix}\right)$, where $z \in \mathfrak{H}$, $\omega \in \mathbf{C}^{g-1}$, $Z \in \mathfrak{S}_{g-1}$. We follow [40, p.370, Sect. 3]. Let $a_1 = 1/2$. There is a holomorphic function $f_{a,b}(\zeta, \omega, Z)$ such that

$$\begin{aligned}\theta_{a,b}(\Omega) &= \sum_{n=(n_1,n')\in\mathbf{Z}\times\mathbf{Z}^{g-1}} e^{\pi i(n_1+\frac{1}{2})^2 z + 2\pi i(n_1+\frac{1}{2})^t\omega(n'+a') + \pi i^t(n'+a')Z(n'+a') + 2\pi i^t(n+a)b} \\ &= e^{\pi i z/4} f_{a,b}(e^{2\pi i z}, \omega, Z).\end{aligned}$$

The number of even (a,b) with $a_1 = 1/2$ is given by $2^{2(g-1)}$. Similarly, let $a_1 = 0$. There is a holomorphic function $g_{a,b}(\zeta, \omega, Z)$ such that

$$\begin{aligned}\theta_{a,b}(\Omega) &= \sum_{n=(n_1,n')\in\mathbf{Z}\times\mathbf{Z}^{g-1}} e^{\pi i n_1^2 z + 2\pi i n_1{}^t\omega(n'+a') + \pi i^t(n'+a')Z(n'+a') + 2\pi i^t(n+a)b} \\ &= 1 + e^{\pi i z} g_{a,b}(e^{\pi i z}, \omega, Z).\end{aligned}$$

Hence there is a holomorphic function $F(\zeta,\omega,Z)$ in the variables ζ,ω,Z such that
(4.3)
$$\chi_g(\Omega)^8=\prod_{\text{even}}\theta_{a,b}(\Omega)^8=(e^{\frac{\pi iz}{4}})^{8\cdot 2^{2(g-1)}}F(e^{\pi iz},\omega,Z)=(e^{2\pi iz})^{2^{2g-2}}F(e^{\pi iz},\omega,Z).$$
Since χ_g^8 is a Siegel modular form and hence $\chi_g(\Omega+\Lambda)^8=\chi_g(\Omega)^8$, $F(\zeta,\omega,Z)$ is an even function in ζ. By (4.3), there exists $h(t)\in\mathcal{O}(\Delta)$ such that

$$\chi_g\left(\frac{\log t}{2\pi i}\Lambda+\psi(t)\right)^8=t^{2^{2g-2}}h(t). \tag{4.4}$$

Since $\operatorname{Im}(\frac{\log t}{2\pi i}\Lambda+\psi(t))=-\frac{1}{2\pi}\log|t|\Lambda+\operatorname{Im}\psi(0)+O(|t|)$ with $\psi(0)=\left(\begin{smallmatrix}\psi_{11}&\psi_{12}\\ \psi_{21}&\psi_{22}\end{smallmatrix}\right)$, $\psi_{22}\in\mathfrak{S}_{g-1}$, we get

$$\det\operatorname{Im}\left(\frac{\log t}{2\pi i}\Lambda+\psi(t)\right)=-\frac{\det\operatorname{Im}\psi_{22}}{2\pi}\log|t|+O(1). \tag{4.5}$$

The result follows from (4.2), (4.4), (4.5).
(2) Since $g=2$ and C_0 is reducible, we deduce from [19, Cor.3.8] the existence of a holomorphic map $\psi\colon\Delta\to\mathfrak{S}_2$ with

$$\Omega(C_t)=[\psi(t)],\quad \psi(0)=\begin{pmatrix}\psi_1&0\\0&\psi_2\end{pmatrix},\quad \psi'(0)=\begin{pmatrix}0&a\\a&0\end{pmatrix},\quad \psi_1,\psi_2\in\mathfrak{H},\quad a\neq 0.$$

The result follows from e.g. [58, Eq. (A.24)]. □

Let $\omega_{\mathfrak{S}_g}$ be the $Sp(2g,\mathbf{Z})$-invariant Kähler form on $\mathfrak{S}_g$ defined as

$$\omega_{\mathfrak{S}_g}(\tau):=-dd^c\log\det\operatorname{Im}\tau,\qquad \tau\in\mathfrak{S}_g.$$

Let $\omega_{\mathcal{A}_g}$ be the Kähler form on $\mathcal{A}_g$ in the sense of orbifolds induced from $\omega_{\mathfrak{S}_g}$. The following equation of $(1,1)$-forms on $\mathcal{A}_g$ holds

$$\omega_{\mathcal{A}_g}=c_1(\mathcal{F}_g,\|\cdot\|_{\mathcal{F}_g}).$$

Let $\mathcal{I}(M)\subset\mathbf{Z}$ be the ideal defined as follows: $q\in\mathcal{I}(M)$ if and only if there exists $\overline{\mathcal{F}}^q_{g(M)}\in H^1(\mathcal{A}^*_{g(M)},\mathcal{O}^*_{\mathcal{A}^*_{g(M)}})$ with $\overline{\mathcal{F}}^q_{g(M)}|_{\mathcal{A}_{g(M)}}=\mathcal{F}^q_{g(M)}$.

Let $i\colon\Omega^o_{M^\perp}\cup\mathcal{D}^o_{M^\perp}\hookrightarrow\Omega_{M^\perp}$ be the inclusion. For $q\in\mathcal{I}(M)$, we set

$$\lambda^q_M:=i_*\mathcal{O}_{\Omega^o_{M^\perp}\cup\mathcal{D}^o_{M^\perp}}(J^*_M\overline{\mathcal{F}}^q_{g(M)}).$$

By [59, Lemma 3.6] and Proposition 3.2, the $\mathcal{O}_{\Omega_{M^\perp}}$-module λ^q_M is an invertible sheaf on $\Omega_{M^\perp}$. We identify λ^q_M with the corresponding holomorphic line bundle on $\Omega_{M^\perp}$. By [59, Lemma 3.7] and Proposition 3.2, the $O(M^\perp)$-action on $\lambda^q_M|_{\Omega^o_{M^\perp}\cup\mathcal{D}^o_{M^\perp}}$ induced from the $O(M^\perp)$-equivariant map J_M, extends to the one on λ^q_M. Hence λ^q_M is equipped with the structure of an $O(M^\perp)$-equivariant line bundle on λ^q_M.

Let $\|\cdot\|_{\lambda^q_M}$ be the $O(M^\perp)$-invariant Hermitian metric on $\lambda^q_M|_{\Omega^o_{M^\perp}}$ defined as

$$\|\cdot\|_{\lambda^q_M}:=(J^o_M)^*\|\cdot\|_{\mathcal{F}^q_{g(M)}}.$$

By (4.1), $(J^o_M)^*\omega_{\mathcal{A}_{g(M)}}$ is a C^∞ closed semi-positive $(1,1)$-form on $\Omega^o_{M^\perp}$ such that $q\,(J^o_M)^*\omega_{\mathcal{A}_{g(M)}}=c_1(\lambda^q_M|_{\Omega^o_{M^\perp}},\|\cdot\|_{\lambda^q_M})$. Since $\dim\Omega_{M^\perp}\setminus(\Omega^o_{M^\perp}\cup\mathcal{D}^o_{M^\perp})\leq\dim\Omega_{M^\perp}-2$ when $r(M)\leq 18$, we can define the closed positive $(1,1)$-current $J^*_M\omega_{\mathcal{A}_{g(M)}}$ on $\Omega_{M^\perp}$ as the trivial extension of $(J^o_M)^*\omega_{\mathcal{A}_{g(M)}}$ from $\Omega^o_{M^\perp}$ to $\Omega_{M^\perp}$ by [59, Th. 3.9] and [54, p. 53 Th. 1]. When $r(M)=19$, $(J^o_M)^*\omega_{\mathcal{A}_{g(M)}}$ extends trivially to a closed positive $(1,1)$-current $J^*_M\omega_{\mathcal{A}_{g(M)}}$ on $\Omega_{M^\perp}$, because $(J^o_M)^*\omega_{\mathcal{A}_{g(M)}}$

has Poincaré growth along $\mathcal{D}_{M^\perp}$ by [59, Prop. 3.8]. By [59, Th. 3.13] and [54, p. 53 Th. 1], the Hermitian metric $\|\cdot\|_{\lambda_M^q}$ on $\lambda_M^q|_{\Omega^o_{M^\perp}}$ extends to a singular Hermitian metric on λ_M^q, whose curvature current on $\Omega_{M^\perp}$ is given by

$$c_1(\lambda_M^q, \|\cdot\|_{\lambda_M^q}) = q\, J_M^* \omega_{\mathcal{A}_{g(M)}}. \tag{4.6}$$

Let $\ell \in \mathbf{Z}_{>0}$ be such that $2^{g(M)+1}(2^{g(M)}+1)\ell \in \mathcal{I}(M)$. Then $\mathcal{F}_{g(M)}^{2^{g(M)+1}(2^{g(M)}+1)\ell}$ extends to a holomorphic line bundle on $\mathcal{A}^*_{g(M)}$. Since $\chi_{g(M)}^{8\ell}$ is a holomorphic section of $\mathcal{F}_{g(M)}^{2^{g(M)+1}(2^{g(M)}+1)\ell}$, $J_M^*\chi_{g(M)}^{8\ell}$ is an $O(M^\perp)$-invariant holomorphic section of $\lambda_M^{2^{g(M)+1}(2^{g(M)}+1)\ell}$. If $J_M^o(\Omega^o_{M^\perp}) \not\subset \theta_{\mathrm{null},g(M)}$, we define

$$\mathfrak{D} := \mathrm{div}(J_M^*\chi_{g(M)}^{8\ell}).$$

Since J_M is $O(M^\perp)$-equivariant with respect to the trivial $O(M^\perp)$-action on $\mathcal{A}^*_{g(M)}$, $\mathfrak{D}$ is an $O(M^\perp)$-invariant effective divisor on $\Omega_{M^\perp}$. By [59, Th. 3.13], [54, p. 53 Th. 1] and (4.6), $\log\|J_M^*\chi_{g(M)}^8\|$ lies in $L^1_{\mathrm{loc}}(\Omega_{M^\perp})$ and satisfies the following equation of currents on $\Omega_{M^\perp}$

$$-dd^c \log\|J_M^*\chi_{g(M)}^{8\ell}\|^2 = 2^{g(M)+1}(2^{g(M)}+1)\ell\, J_M^*\omega_{\mathcal{A}_{g(M)}} - \delta_{\mathfrak{D}}. \tag{4.7}$$

Recall that the divisor $\mathcal{D}'_{M^\perp}$ was defined in Sect. 2.4.

Proposition 4.2. *Let $\ell \in \mathbf{Z}_{>0}$ be such that $2^{g(M)+1}(2^{g(M)}+1)\ell \in \mathcal{I}(M)$.*

(1) *Assume $11 \le r(M) \le 17$ or $(r(M), \delta(M)) = (10, 1)$. If $J_M^o(\Omega^o_{M^\perp}) \not\subset \theta_{\mathrm{null},g(M)}$, there exist $a \in \mathbf{Z}_{\ge 0}$ and an $O(M^\perp)$-invariant effective divisor E on $\Omega_{M^\perp}$ such that*

$$\mathrm{div}(J_M^*\chi_{g(M)}^{8\ell}) = 2(2^{2g(M)-2}+a)\ell\, \mathcal{D}'_{M^\perp} + E.$$

In particular, the following equations of currents on $\Omega_{M^\perp}$ holds:

$$-dd^c \log\|J_M^*\chi_{g(M)}^{8\ell}\|^2 = 2^{g(M)+1}(2^{g(M)}+1)\ell\, J_M^*\omega_{\mathcal{A}_{g(M)}} - 2(2^{2g(M)-2}+a)\ell\, \delta_{\mathcal{D}'_{M^\perp}} - \delta_E.$$

(2) *Assume $g(M) = 2$ and $r(M) < 10$, i.e., $M \cong \mathbb{I}_{1,8}(2)$. There exists $a \in \mathbf{Z}_{\ge 0}$ such that the following equation of divisors on $\Omega_{M^\perp}$ holds:*

$$\mathrm{div}(J_M^*\chi_{g(M)}^{8\ell}) = (8+2a)\ell\, \mathcal{D}'_{M^\perp} + 16\ell\, \mathcal{H}_{M^\perp}(\mathbf{1}_{M^\perp}, -1/2).$$

In particular, the following equations of currents on $\Omega_{M^\perp}$ holds:

$$-dd^c \log\|J_M^*\chi_{g(M)}^{8\ell}\|^2 = 40\ell\, J_M^*\omega_{\mathcal{A}_{g(M)}} - (8+2a)\ell\, \delta_{\mathcal{D}'_{M^\perp}} - 16\ell\, \delta_{\mathcal{H}_{M^\perp}(\mathbf{1}_{M^\perp}, -1/2)}.$$

Proof. Since $\mathfrak{D}$ is effective, we can write $\mathfrak{D} = \sum_{d \in \Delta'_{M^\perp}} m(d)\, H_d + E$, where $m(d) \in \mathbf{Z}_{\ge 0}$ and E is an effective divisor on $\Omega_{M^\perp}$ with $\dim(\mathcal{D}'_{M^\perp} \cap E) \le \dim \mathcal{D}'_{M^\perp} - 1$. Since $g(H_d) = H_{g(d)}$ for all $g \in O(M^\perp)$ and $d \in \Delta'_{M^\perp}$, the $O(M^\perp)$-invariance of $\mathfrak{D}$ implies that $m(g(d)) = m(d)$ for all $g \in O(M^\perp)$ and $d \in \Delta'_{M^\perp}$. Since $O(M^\perp)$ acts transitively on $\Delta'_{M^\perp}$ by Proposition 11.15 below, there exists $\alpha \in \mathbf{Z}_{\ge 0}$ with

$$\mathfrak{D} = \alpha\, \mathcal{D}'_{M^\perp} + E. \tag{4.8}$$

Let $d \in \Delta_{M^\perp}$ and $\mathfrak{p} \in \overline{H}^o_d$. Let $\gamma\colon \Delta \to \mathcal{M}_{M^\perp}$ be a holomorphic curve intersecting $\overline{H}^o_d$ transversally at $\gamma(0) = \mathfrak{p}$ such that $\gamma(\Delta^*) \subset \mathcal{M}_M \setminus (\overline{\mathcal{D}}_{M^\perp} \cup \mathfrak{D})$. By Theorem 3.3, there exists an ordinary singular family of 2-elementary $K3$ surfaces $p_{\mathcal{Z}}\colon (\mathcal{Z}, \iota) \to \Delta$ of type M with Griffiths period map γ, such that $(\tilde{Z}_0, \tilde{\iota}_0)$ is a 2-elementary $K3$ surface of type $[M \perp d]$ with Griffiths period $\gamma(0)$.

Since the natural projection $\mathit{\Pi}_{M^\perp}\colon \Omega_{M^\perp} \to \mathcal{M}_{M^\perp}$ is doubly ramified along H_d^o by [59, Prop. 1.9 (4)], there exists a holomorphic curve $c\colon \varDelta \to \Omega_{M^\perp}$ intersecting H_d^o transversally at $c(0) \in H_d^o$ such that $\mathit{\Pi}_{M^\perp}(c(t)) = \gamma(t^2)$. Hence we have

$$J_M(c(t)) = \mathit{\Omega}(Z_{t^2}^{\iota_{t^2}}). \tag{4.9}$$

(1) Assume $d \in \Delta'_{M^\perp}$. If ι is of type $(0,3)$, $g([M \perp d]) = g(M)$ by [59, Prop. 2.5]. Since $g([M \perp d]) = g(M) - 1$ by Lemma 11.5 below, we get a contradiction. Hence ι must be of type $(2,1)$. By [59, Prop. 2.5], $p|_{\mathcal{Z}^\iota}\colon \mathcal{Z}^\iota \to \varDelta$ is an ordinary singular family of curves. By Lemma 4.1 (1), there exists $h(t) \in \mathcal{O}(\varDelta)$ such that

$$\log \|\chi_{g(M)}(\mathit{\Omega}(Z_t^{\iota_t}))^8\|^2 = 2^{2g(M)-2} \log|t|^2 + \log|h(t)|^2 + O(\log\log|t|^{-1}). \tag{4.10}$$

Since $\gamma(\varDelta^*) \cap \mathfrak{D} = \emptyset$ by the choice of γ, $h(t)$ does not vanish identically on $\varDelta$ by (4.4). Let $a \in \mathbf{Z}_{\geq 0}$ be the multiplicity of $h(t)$ at $t = 0$. By (4.9) and (4.10), we get

$$\log \|\chi_{g(M)}(J_M(c(t)))^{8\ell}\|^2 = 2(2^{2g(M)-2} + a)\ell \log|t|^2 + O(\log\log|t|^{-1}), \tag{4.11}$$

which yields that $H_d \subset \operatorname{supp}\mathfrak{D}$ for $d \in \Delta'_{M^\perp}$. Comparing (4.7), (4.8) and (4.11), we get $\alpha = 2(2^{2g(M)-2} + a)\ell$ in (4.8). Since $\mathfrak{D}$ and $\mathcal{D}'_{M^\perp}$ are $O(M^\perp)$-invariant, so is E. This proves (1).

(2) Assume $M \cong \mathbb{I}_{1,8}(2)$, $d \in \Delta''_{M^\perp}$, $d/2 = \mathbf{1}_{M^\perp}$. As was proved in Proposition 3.10, $[M \perp d] \cong \mathbb{U} \oplus \mathbb{E}_8(2)$. If ι is of type $(0,3)$, then $Z_0^{\iota_0}$ is the disjoint union of a smooth curve of genus 2 and an isolated point by [59, Prop. 2.5], which implies that $J_M(\gamma(0)) \in \mathcal{A}_2 \setminus \mathcal{N}_2$. However, we get the contradiction by Theorem 3.5:

$$J_M(\gamma(0)) = J^o_{[M\perp d]}(\gamma(0)) = J^o_{\mathbb{U}\oplus\mathbb{E}_8(2)}(\gamma(0)) \in \mathcal{N}_2,$$

where the last inclusion follows from Lemma 3.1 (2). Hence ι must be of type $(2,1)$.

By [59, Prop. 2.5], $p|_{\mathcal{Z}^\iota}\colon \mathcal{Z}^\iota \to \varDelta$ is an ordinary singular family of curves. Since the normalization of $(Z_0)^{\iota_0}$ is the disjoint union of two elliptic curves by Lemma 3.1 (2) and Theorem 3.5, $(Z_0)^{\iota_0}$ is the join of two elliptic curves intersecting at one point transversally. By Lemma 4.1 (2), we get

$$\log \|\chi_{g(M)}(\mathit{\Omega}(Z_t^{\iota_t}))^8\|^2 = 8 \log|t|^2 + O(\log\log|t|^{-1}) \qquad (t \to 0). \tag{4.12}$$

By (4.9) and (4.12), we get

$$\log \|\chi_{g(M)}(J_M(c(t)))^8\|^2 = 16 \log|t|^2 + O(\log\log|t|^{-1}) \qquad (t \to 0). \tag{4.13}$$

By Lemma 3.1, we get $J_M(\Omega^o_{M^\perp}) = J^o_M(\Omega^o_{M^\perp}) \subset \mathcal{A}_2 \setminus \theta_{\mathrm{null},2}$. By Proposition 3.10, we get $J_M(\bigcup_{d\in\Delta''_{M^\perp},\, d/2\not\equiv\mathbf{1}_{M^\perp}} H_d^o) \subset \mathcal{A}_2 \setminus \theta_{\mathrm{null},2}$. By these two inclusions,

$$J_M\left(\Omega^o_{M^\perp} \cup \bigcup_{d\in\Delta''_{M^\perp},\, d/2\not\equiv\mathbf{1}_{M^\perp}} H_d^o\right) \subset \mathcal{A}_2 \setminus \theta_{\mathrm{null},2},$$

which implies that $J_M^*\chi_2^{8\ell}$ does not vanish on $\Omega^o_{M^\perp} \cup \bigcup_{d\in\Delta''_{M^\perp},\, d/2\not\equiv\mathbf{1}_{M^\perp}} H_d^o$. Hence

$$\begin{aligned}(\Omega^o_{M^\perp} \cup \mathcal{D}^o_{M^\perp}) \cap \mathfrak{D} &\subset (\Omega^o_{M^\perp} \cup \mathcal{D}^o_{M^\perp}) \setminus \left(\Omega^o_{M^\perp} \cup \bigcup_{d\in\Delta''_{M^\perp},\, d/2\not\equiv\mathbf{1}_{M^\perp}} H_d^o\right) \\ &= \mathcal{D}^o_{M^\perp} \setminus \bigcup_{d\in\Delta''_{M^\perp},\, d/2\not\equiv\mathbf{1}_{M^\perp}} H_d^o \\ &\subset \mathcal{D}'_{M^\perp} \cup \mathcal{H}_{M^\perp}(\mathbf{1}_{M^\perp}, -1/2).\end{aligned}$$

Since $\Omega_{M^\perp} \setminus (\Omega^o_{M^\perp} \cup \mathcal{D}^o_{M^\perp})$ is an analytic subset of codimension 2 in $\Omega_{M^\perp}$, we get

$$\mathfrak{D} \subset \mathcal{D}'_{M^\perp} \cup \mathcal{H}_{M^\perp}(\mathbf{1}_{M^\perp}, -1/2). \tag{4.14}$$

The desired formula follows from (4.8), (4.11), (4.13), (4.14). □

Remark 4.3. In the proof of Theorem 9.1, we shall prove $a = E = 0$ under the assumption of Proposition 4.2. When $g(M) = 0$, this is trivial. When $g(M) = 1$, this follows from the inclusion $\mathfrak{D} \subset \mathcal{D}_{M^\perp}$ and the estimate $\log |h(t)| = O(1)$ in Lemma 4.1 (1). When $g(M) = 2$, we get $E = 0$ because $\mathfrak{D} \subset \mathcal{D}_{M^\perp}$.

Remark 4.4. A key in the proof of Proposition 4.2 (1) is the fact that $O(M^\perp)$ acts transitively on $\Delta'_{M^\perp}$. In fact, $O^+(M^\perp)$ acts transitively on $\Delta'_{M^\perp}$. To see this, since $O(M^\perp)$ acts transitively on $\Delta'_{M^\perp}$, it suffices to prove the existences of $d \in \Delta'_{M^\perp}$ and $g \in O(M^\perp) \setminus O^+(M^\perp)$ with $g(d) = d$. By [20, Appendix, Tables 1,2,3], we have $M^\perp = \mathbb{U} \oplus L$ if $g(M) > 0$. Since $\Delta'_{M^\perp} = \emptyset$ when $g(M) = 0$, we may assume $g(M) > 0$ and $M^\perp = \mathbb{U} \oplus L$. Let $d \in \Delta_{\mathbb{U}} \subset \Delta_{M^\perp}$. Then $g = 1_{\mathbb{U}} \oplus -1_L \in O(M^\perp) \setminus O^+(M^\perp)$ by (2.3), and $g(d) = d$.

4.2. Automorphic forms on Ω^+_Λ

Let Λ be a lattice of signature $(2, r(\Lambda) - 2)$. We fix a vector $l_\Lambda \in \Lambda \otimes \mathbf{R}$ with $\langle l_\Lambda, l_\Lambda \rangle \geq 0$, and we set

$$j_\Lambda(\gamma, [\eta]) := \frac{\langle \gamma(\eta), l_\Lambda \rangle}{\langle \eta, l_\Lambda \rangle} \qquad [\eta] \in \Omega^+_\Lambda, \quad \gamma \in O^+(\Lambda).$$

Since $H_{l_\Lambda} = \emptyset$, $j_\Lambda(\gamma, \cdot)$ is a nowhere vanishing holomorphic function on Ω^+_Λ.

Let $\Gamma \subset O^+(\Lambda)$ be a cofinite subgroup. A holomorphic function $f \in \mathcal{O}(\Omega^+_\Lambda)$ is called an *automorphic form on Ω^+_Λ for Γ of weight p* if

$$f(\gamma \cdot [\eta]) = \chi(\gamma)\, j_\Lambda(\gamma, [\eta])^p\, f([\eta]), \qquad [\eta] \in \Omega^+_\Lambda, \quad \gamma \in \Gamma,$$

where $\chi \colon \Gamma \to \mathbf{C}^*$ is a character. For an automorphic form f on Ω^+_Λ for Γ of weight p, the Petersson norm $\|f\|$ is the function on Ω^+_Λ defined as

$$\|f([\eta])\|^2 := K_\Lambda([\eta])^p\, |f([\eta])|^2, \qquad K_\Lambda([\eta]) := \frac{\langle \eta, \bar{\eta} \rangle}{|\langle \eta, l_\Lambda \rangle|^2}.$$

If $r(\Lambda) \geq 5$, then $\|f\|^2$ is a Γ-invariant C^∞ function on Ω^+_Λ, because the group $\Gamma/[\Gamma, \Gamma]$ is finite and Abelian and hence χ is finite in this case.

We also consider automorphic forms on $\Omega^+_{M^\perp}$ with values in the sheaf λ^q_M.

Definition 4.5. Let $M \subset \mathbb{L}_{K3}$ be a primitive 2-elementary Lorentzian sublattice. Let χ be a character of $O^+(M^\perp)$. Let $p, q \in \mathbf{Z}$. Then $\Psi \in H^0(\Omega^+_{M^\perp}, \lambda^q_M)$ is called an automorphic form on $\Omega^+_{M^\perp}$ for $O^+(M^\perp)$ of weight (p, q) if for all $\gamma \in O^+(M^\perp)$,

$$\Psi(\gamma \cdot [\eta]) = \chi(\gamma)\, j_{M^\perp}(\gamma, [\eta])^p\, \gamma(\Psi([\eta])), \qquad [\eta] \in \Omega^+_{M^\perp}.$$

For an automorphic form Ψ on $\Omega^+_{M^\perp}$ for $O^+(M^\perp)$ of weight (p, q), the Petersson norm of Ψ is a C^∞ function on $\Omega^+_{M^\perp}$ defined as

$$\|\Psi([\eta])\|^2 := K_{M^\perp}([\eta])^p \cdot \|\Psi([\eta])\|^2_{\lambda^q_M}, \qquad [\eta] \in \Omega^+_{M^\perp}. \tag{4.15}$$

5. The invariant τ_M of 2-elementary $K3$ surfaces of type M

Let (X,ι) be a 2-elementary $K3$ surface of type M. Identify $\mathbf{Z}_2$ with the subgroup of $\mathrm{Aut}(X)$ generated by ι. Let κ be a $\mathbf{Z}_2$-invariant Kähler form on X. Set $\mathrm{vol}(X,\kappa) := (2\pi)^{-2}\int_X \kappa^2/2!$. Let η be a nowhere vanishing holomorphic 2-form on X. The L^2-norm of η is defined as $\|\eta\|_{L^2}^2 := (2\pi)^{-2}\int_X \eta\wedge\bar\eta$.

Let $\square_{0,q} = 2(\bar\partial+\bar\partial^*)^2$ be the $\bar\partial$-Laplacian acting on C^∞ $(0,q)$-forms on X. Let $\sigma(\square_{0,q})$ be the spectrum of $\square_{0,q}$. For $\lambda\in\sigma(\square_{0,q})$, let $E_{0,q}(\lambda)$ be the eigenspace of $\square_{0,q}$ with respect to the eigenvalue λ. Since $\mathbf{Z}_2$ preserves κ, $E_{0,q}(\lambda)$ is a finite-dimensional unitary representation of $\mathbf{Z}_2$. For $s\in\mathbf{C}$, set

$$\zeta_{0,q}(\iota)(s) := \sum_{\lambda\in\sigma(\square_{0,q})\setminus\{0\}} \mathrm{Tr}\left(\iota|_{E_{0,q}(\lambda)}\right)\lambda^{-s}.$$

Then $\zeta_{0,q}(\iota)(s)$ converges absolutely when $\mathrm{Re}\, s > \dim X$, admits a meromorphic continuation to the complex plane $\mathbf{C}$, and is holomorphic at $s=0$. The *equivariant analytic torsion* of the trivial Hermitian line bundle on (X,κ) is defined as

$$\tau_{\mathbf{Z}_2}(X,\kappa)(\iota) := \exp[-\sum_{q\geq 0}(-1)^q q\,\zeta_{0,q}'(\iota)(0)].$$

We refer to [50], [5], [6], [23], [4], [38], [31] for more about equivariant and non-equivariant analytic torsion.

Let $X^\iota = \sum_i C_i$ be the decomposition of the fixed point set of ι into the connected components. Set $\mathrm{vol}(C_i,\kappa|_{C_i}) := (2\pi)^{-1}\int_{C_i}\kappa|_{C_i}$. Let $c_1(C_i,\kappa|_{C_i})$ be the Chern form of $(TC_i,\kappa|_{C_i})$ and let $\tau(C_i,\kappa|_{C_i})$ be the analytic torsion of the trivial Hermitian line bundle on $(C_i,\kappa|_{C_i})$. We define

$$\begin{aligned}\tau_M(X,\iota) := {}&\mathrm{vol}(X,\kappa)^{\frac{14-r(M)}{4}}\tau_{\mathbf{Z}_2}(X,\kappa)(\iota)\prod_i \mathrm{Vol}(C_i,\kappa|_{C_i})\tau(C_i,\kappa|_{C_i})\\ &\times\exp\left[\frac{1}{8}\int_{C_i}\log\left(\frac{\eta\wedge\bar\eta}{\kappa^2/2!}\cdot\frac{\mathrm{Vol}(X,\kappa)}{\|\eta\|_{L^2}^2}\right)\Big|_{C_i} c_1(C_i,\kappa|_{C_i})\right],\end{aligned}$$

which is independent of the choice of κ by [59, Th. 5.7]. Hence $\tau_M(X,\iota)$ is an invariant of the pair (X,ι), so that τ_M descends to a function on $\mathcal{M}^o_{M^\perp}$.

Theorem 5.1. *If $r(M)\leq 17$, there exist an integer $\nu\in\mathbf{Z}_{>0}$ and an automorphic form Φ_M on $\Omega_{M^\perp}$ for $O^+(M^\perp)$ of weight $(\nu(r(M)-6),4\nu)$ with zero divisor $\nu\,\mathcal{D}_{M^\perp}$ such that for every 2-elementary $K3$ surface (X,ι) of type M,*

$$\tau_M(X,\iota) = \|\Phi_M(\varpi_M(X,\iota))\|^{-\frac{1}{2\nu}}.$$

Proof. The result follows from [59, Main Th.] and Proposition 11.2 below. $\square$

6. Borcherds products

6.1. Modular forms for $Mp_2(\mathbf{Z})$

Recall that $\mathfrak{H}\subset\mathbf{C}$ is the complex upper half-plane. Let $Mp_2(\mathbf{Z})$ be the metaplectic double cover of $SL_2(\mathbf{Z})$ (cf. [10, Sect. 2]), which is generated by the two elements $S := (\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right),\sqrt{\tau})$ and $T := (\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right),1)$. For $\gamma = (\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right),\sqrt{c\tau+d})\in Mp_2(\mathbf{Z})$ and $\tau\in\mathfrak{H}$, we set $j(\gamma,\tau) := \sqrt{c\tau+d}$ and $\gamma\cdot\tau := (a\tau+b)/(c\tau+d)$.

Let M be an even lattice. Let $\mathbf{C}[A_M]$ be the group ring of the discriminant group A_M. Let $\{\mathbf{e}_\gamma\}_{\gamma\in A_M}$ be the standard basis of $\mathbf{C}[A_M]$. The Weil representation

$\rho_M\colon Mp_2(\mathbf{Z}) \to GL(\mathbf{C}[A_M])$ is defined as follows [10, Sect. 2]:

$$\rho_M(T)\,\mathbf{e}_\gamma := e^{\pi i \gamma^2}\mathbf{e}_\gamma, \qquad \rho_M(S)\,\mathbf{e}_\gamma := \frac{i^{-\frac{\sigma(M)}{2}}}{|A_M|^{1/2}} \sum_{\delta\in A_M} e^{-2\pi i\langle\gamma,\delta\rangle}\mathbf{e}_\delta. \tag{6.1}$$

A $\mathbf{C}[A_M]$-valued holomorphic function $F(\tau)$ on $\mathfrak{H}$ is a *modular form of type ρ_M with weight $w \in \frac{1}{2}\mathbf{Z}$* if the following conditions (a), (b) are satisfied:

(a) For $\gamma \in Mp_2(\mathbf{Z})$ and $\tau \in \mathfrak{H}$, $F(\gamma\cdot\tau) = j(\gamma,\tau)^{2w}\,\rho_M(\gamma)\cdot F(\tau)$.

(b) $F(\tau) = \sum_{\gamma\in A_M}\mathbf{e}_\gamma \sum_{k\in\frac{1}{l}\mathbf{Z}} c_\gamma(k)\,e^{2\pi i k\tau}$, where l is the level of M, $c_\gamma(k)\in\mathbf{Z}$ for all $k\in\frac{1}{l}\mathbf{Z}$ and $c_\gamma(k) = 0$ for $k \ll 0$.

By the first condition of (6.1), [14, Eq. (1.4)] and Condition (a), we get

$$c_\gamma(k) = \begin{cases} 0 & \text{if} \quad k \notin \gamma^2/2 + \mathbf{Z} \\ c_{-\gamma}(k) & \text{if} \quad k \in \gamma^2/2 + \mathbf{Z}. \end{cases} \tag{6.2}$$

The group $O(M)$ acts on $\mathbf{C}[A_M]$ by $g(\mathbf{e}_\gamma) := \mathbf{e}_{\bar{g}(\gamma)}$, where $\bar{g} \in O(q_M)$ is the element induced by $g \in O(M)$. For a modular form F of type ρ_M, we define $\mathrm{Aut}(M,F) := \{g \in O(M);\, g(F) = F\}$. Then $\mathrm{Aut}(M,F)$ is a cofinite subgroup of $O(M)$, since $O(q_M)$ is finite and since $\mathrm{Aut}(M,F) \supset \ker\{O(M) \to O(q_M)\}$.

6.2. Borcherds products

Let Λ be an even lattice of signature $(2, r(\Lambda)-2)$ with level l. Assume that Λ is 2-elementary and that $\Lambda = \mathbb{U}(N) \oplus L$. A vector of $\Lambda\otimes\mathbf{Q}$ is denoted by (m,n,v), where $m,n\in\mathbf{Q}$ and $v \in L\otimes\mathbf{Q}$. We write a vector of A_Λ in the same manner. If $F(\tau) = \sum_{\gamma\in A_\Lambda} f_\gamma(\tau)\,\mathbf{e}_\gamma$ is a modular form of type ρ_Λ, then $F(\tau)$ induces a modular form $F|_L(\tau)$ of type ρ_L with the same weight as follows [9, Th. 5.3]:

$$F|_L(\tau) := \sum_{\lambda\in A_L} f_{L+\lambda}(\tau)\,\mathbf{e}_\lambda, \qquad f_{L+\lambda}(\tau) := \sum_{n=0}^{N-1} f_{(\frac{n}{N},0,\bar{\lambda})}(\tau). \tag{6.3}$$

Write $F|_L(\tau) = \sum_{\gamma\in A_L}\mathbf{e}_\gamma \sum_{k\in\frac{\gamma^2}{2}+\mathbf{Z}} c_{L,\gamma}(k)\,e^{2\pi i k\tau}$. By [9, Sect. 6, p.517], $F|_L(\tau)$ induces a chamber structure of $\mathcal{C}_L^+$:

$$(\mathcal{C}_L^+)^0_{F|_L} := \mathcal{C}_L^+ \setminus \bigcup_{\lambda\in L^\vee,\,\lambda^2<0,\,c_{L,\bar{\lambda}}(\lambda^2/2)\neq 0} h_\lambda = \amalg_{\alpha\in A}\mathcal{W}_\alpha, \tag{6.4}$$

where $h_\lambda = \lambda^\perp = \{v \in L\otimes\mathbf{R};\, \langle v,\lambda\rangle = 0\}$ and $\{\mathcal{W}_\alpha\}_{\alpha\in A}$ is the set of connected components of $(\mathcal{C}_L^+)^0_{F|_L}$. Each component $\mathcal{W}_\alpha$ is called a *Weyl chamber* of $F|_L(\tau)$. In general, $\mathcal{W}_\alpha$ is not a Weyl chamber of L in the sense of Sect. 2.3. If $\lambda \in L\otimes\mathbf{R}$ satisfies $\langle\lambda, w\rangle > 0$ for all $w \in \mathcal{W}_\alpha$, we write $\lambda\cdot\mathcal{W}_\alpha > 0$.

Theorem 6.1. *Let $F(\tau) = \sum_{\gamma\in A_\Lambda}\mathbf{e}_\gamma\sum_{k\in\frac{\gamma^2}{2}+\mathbf{Z}} c_\gamma(k)\,e^{2\pi i k\tau}$ be a modular form of type ρ_Λ with weight $\sigma(\Lambda)/2$. Then there exists a meromorphic automorphic form $\Psi_\Lambda(z,F)$ on Ω_Λ^+ for $\mathrm{Aut}(\Lambda,F)\cap O^+(\Lambda)$ of weight $c_0(0)/2$ such that*

$$\mathrm{div}(\Psi_\Lambda(\cdot,F)) = \frac{1}{2}\sum_{\lambda\in\Lambda^\vee,\,\lambda^2<0} c_{\bar{\lambda}}(\lambda^2/2)\,H_\lambda = \sum_{\lambda\in\Lambda^\vee/\pm1,\,\lambda^2<0} c_{\bar{\lambda}}(\lambda^2/2)\,H_\lambda.$$

If $\mathcal{W}$ is a Weyl chamber of $F|_L$, then there exists a vector $\varrho(L,F|_L,\mathcal{W}) \in L\otimes\mathbf{Q}$ such that $\Psi_\Lambda(z,F)$ is expressed as the following infinite product near the cusp under *the identification (2.3): For* $z \in L \otimes \mathbf{R} + i\,\mathcal{W}$ *with* $(\operatorname{Im} z)^2 \gg 0$,

$$\Psi_\Lambda(z,F) = e^{2\pi i \langle \varrho(L,F|_L,\mathcal{W}),z\rangle} \prod_{\lambda\in L^\vee,\,\lambda\cdot\mathcal{W}>0}\ \prod_{n\in\mathbf{Z}/N\mathbf{Z}} (1 - e^{2\pi i(\langle\lambda,z\rangle + \frac{n}{N})})^{c_{(\frac{n}{N},0,\bar\lambda)}(\lambda^2/2)}.$$

Proof. See [9, Th. 13.3], [14, Th. 3.22]. □

The automorphic form $\Psi_\Lambda(z,F)$ is called the *Borcherds product* or the *Borcherds lift* of $F(\tau)$, and the vector $\varrho(L,F|_L,\mathcal{W})$ is called the *Weyl vector* of $\Psi_\Lambda(\cdot,F)$. See [9, Th. 10.4], [10, p.321 Correction] for an explicit formula for $\varrho(L,F|_L,\mathcal{W})$.

7. 2-elementary lattices and elliptic modular forms

Throughout Section 7, we assume that Λ *is an even* 2*-elementary lattice.*

7.1. A construction of modular form of type ρ_Λ for 2-elementary lattices

Set $M\Gamma_0(4) := \{((\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}), \sqrt{c\tau+d}) \in Mp_2(\mathbf{Z});\ c \equiv 0 \mod 4\}$. Let $w \in \frac{1}{2}\mathbf{Z}$ and let $\chi\colon M\Gamma_0(4) \to \mathbf{C}^*$ be a character. A holomorphic function $f(\tau)$ on $\mathfrak{H}$ is a modular form for $M\Gamma_0(4)$ of weight w with character χ if the following (a), (b) are satisfied:

(a) $f(\gamma\cdot\tau) = j(\gamma,\tau)^{2w}\chi(\gamma)\,f(\tau)$ for all $\gamma \in M\Gamma_0(4)$ and $\tau \in \mathfrak{H}$.
(b) $f(\tau) = \sum_{k\in\frac{1}{4}\mathbf{Z}} c(k)\,e^{2\pi i k\tau}$ with $c(k) = 0$ for $k \ll 0$.

Set $q = e^{2\pi i\tau}$ for $\tau \in \mathfrak{H}$. Let $\eta(\tau) = q^{1/24}\prod_{n=1}^\infty(1-q^n)$ be the Dedekind η-function and let

$$\vartheta_2(\tau) = \sum_{n\in\mathbf{Z}} q^{(n+\frac{1}{2})^2/2}, \qquad \vartheta_3(\tau) = \sum_{n\in\mathbf{Z}} q^{n^2/2}, \qquad \vartheta_4(\tau) = \sum_{n\in\mathbf{Z}} (-1)^n q^{n^2/2}$$

be the Jacobi theta functions. Notice that we use the notation $q = e^{2\pi i\tau}$ while $q = e^{\pi i\tau}$ in [15, Chap. 4]. Recall that $\mathbb{A}_1$ is the negative-definite one-dimensional A_1-lattice $\langle -2\rangle$. Set $\mathbb{A}_1^+ := \mathbb{A}_1(-1) = \langle 2\rangle$, which is the positive-definite A_1-lattice. For $d \in \{0, 1/2\}$, let $\theta_{\mathbb{A}_1^+ + d/2}(\tau)$ be the theta function of $\mathbb{A}_1^+$:

$$\theta_{\mathbb{A}_1^+}(\tau) := \vartheta_3(2\tau), \qquad \theta_{\mathbb{A}_1^+ + 1/2}(\tau) := \vartheta_2(2\tau).$$

By [10, Lemma 5.2], there exists a character $\chi_\theta\colon M\Gamma_0(4) \to \{\pm1, \pm i\}$ such that $\theta_{\mathbb{A}_1^+}(\tau)$ is a modular form for $M\Gamma_0(4)$ of weight $1/2$ with character χ_θ.

For $k \in \mathbf{Z}$, define $f_k^{(0)}(\tau), f_k^{(1)}(\tau) \in \mathcal{O}(\mathfrak{H})$ and the series $\{c_k^{(0)}(l)\}_{l\in\mathbf{Z}}$, $\{c_k^{(1)}(l)\}_{l\in\mathbf{Z}+k/4}$ by

$$f_k^{(0)}(\tau) := \frac{\eta(2\tau)^8\,\theta_{\mathbb{A}_1^+}(\tau)^k}{\eta(\tau)^8\eta(4\tau)^8} = \sum_{l\in\mathbf{Z}} c_k^{(0)}(l)\,q^l = q^{-1} + 8 + 2k + O(q),$$

$$f_k^{(1)}(\tau) := -16\frac{\eta(4\tau)^8\,\theta_{\mathbb{A}_1^+ + \frac{1}{2}}(\tau)^k}{\eta(2\tau)^{16}} = \sum_{l\in\frac{k}{4}+\mathbf{Z}} 2c_k^{(1)}(l)\,q^l = -2^{k+4}\,q^{\frac{k}{4}}\{1 + (k+16)q^2 + O(q^4)\}.$$

We define holomorphic functions $g_k^{(i)}(\tau) \in \mathcal{O}(\mathfrak{H})$, $i \in \mathbf{Z}/4\mathbf{Z}$ by

$$g_k^{(i)}(\tau) := \sum_{l\equiv i \bmod 4} c_k^{(0)}(l)\,q^{l/4}.$$

By definition,

$$\sum_{i\in\mathbf{Z}/4\mathbf{Z}} g_k^{(i)}(\tau) = \frac{\eta(\tau/2)^8\,\theta_{\mathbb{A}_1^+}(\tau/4)^k}{\eta(\tau)^8\eta(\tau/4)^8} = f_k^{(0)}(\tau/4).$$

For a modular form $\phi(\tau)$ of weight l for $M\Gamma_0(4)$ and for $g \in Mp_2(\mathbf{Z})$, we define

$$\phi|_g(\tau) := \phi(g \cdot \tau)\, j(g,\tau)^{-2l}.$$

The following key construction of modular forms of type ρ_Λ is due to Borcherds.

Proposition 7.1. *Let $\phi(\tau)$ be a modular form for $M\Gamma_0(4)$ of weight l with character $\chi_\theta^{\sigma(\Lambda)}$ and set*

$$\mathcal{B}_\Lambda[\phi](\tau) := \sum_{g \in M\Gamma_0(4)\backslash Mp_2(\mathbf{Z})} \phi|_g(\tau)\, \rho_\Lambda(g^{-1})\, \mathbf{e}_0.$$

Then $\mathcal{B}_\Lambda[\phi](\tau)$ is independent of the choice of representatives of $M\Gamma_0(4)\backslash Mp_2(\mathbf{Z})$. Moreover, $\mathcal{B}_\Lambda[\phi](\tau)$ is a modular form for $Mp_2(\mathbf{Z})$ of type ρ_Λ with weight l.

Proof. See [52, Th. 6.2]. See also [9, Lemma 2.6], [10, Proof of Lemma 11.1]. $\square$

Lemma 7.2. *The function $f_k^{(0)}(\tau)$ is a modular form for $M\Gamma_0(4)$ of weight $-4+\frac{k}{2}$ with character χ_θ^k.*

Proof. The result follows from [10, Lemma 5.2 and Th. 6.2]. $\square$

Set $Z := S^2 = (-\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), i)$ and $V := S^{-1}T^2S = (\left(\begin{smallmatrix}1&0\\-2&1\end{smallmatrix}\right), \sqrt{-2\tau+1})$.

Lemma 7.3. *The coset $M\Gamma_0(4)\backslash M\Gamma(1)$ is represented by $\{1, S, ST, ST^2, ST^3, V\}$.*

Proof. Since $\#M\Gamma_0(4)\backslash Mp_2(\mathbf{Z}) = 6$ by [53, Prop. 1.43 (1)] and since none of two elements of $\{1, S, ST, ST^2, ST^3, V\}$ represent the same element of $M\Gamma_0(4)\backslash Mp_2(\mathbf{Z})$, we get the result. $\square$

Recall that the characteristic element $\mathbf{1}_\Lambda \in A_\Lambda$ was defined in Sect. 2.2. Define $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbf{C}[A_\Lambda]$ by

$$\mathbf{v}_k := \sum_{\delta \in A_\Lambda,\, \delta^2 \equiv k/2 \bmod 2} \mathbf{e}_\delta.$$

Lemma 7.4. *The following identities hold:*

$$(1) \quad \rho_\Lambda((ST^l)^{-1})\, \mathbf{e}_0 = i^{\frac{\sigma(\Lambda)}{2}} 2^{-\frac{l(\Lambda)}{2}} \sum_{k=0}^{3} i^{-lk}\, \mathbf{v}_k, \qquad (2) \quad \rho_\Lambda(V^{-1})\, \mathbf{e}_0 = \mathbf{e}_{\mathbf{1}_\Lambda}.$$

Proof. **(1)** Since $S^{-1} = SZ^3$ and since $\rho_\Lambda(Z)\, \mathbf{e}_\gamma = i^{-\sigma(\Lambda)}\, \mathbf{e}_{-\gamma}$ by (6.1), we get

$$\rho_\Lambda(S^{-1})\, \mathbf{e}_0 = \rho_\Lambda(S)\, \rho_\Lambda(Z^3)\, \mathbf{e}_0 = i^{\sigma(\Lambda)} \frac{i^{-\frac{\sigma(\Lambda)}{2}}}{|A_\Lambda|^{1/2}} \sum_{\delta \in A_\Lambda} \mathbf{e}_\delta = i^{\frac{\sigma(\Lambda)}{2}} 2^{-\frac{l(\Lambda)}{2}} \sum_{\delta \in A_\Lambda} \mathbf{e}_\delta.$$

This, together with the first equation of (6.1), yields (1).
(2) By [10, p.325 l.16], we get

$$\rho_\Lambda(ST^{-2}S)\, \mathbf{e}_0 = i^{-\sigma(\Lambda)} |A_\Lambda|^{-1} \sum_{\gamma, \delta \in A_\Lambda} e^{2\pi i \{\langle \gamma, \delta \rangle + \gamma^2\}}\, \mathbf{e}_\delta = i^{-\sigma(\Lambda)}\, \mathbf{e}_{\mathbf{1}_\Lambda},$$

where we used the identity $\sum_{\gamma \in A_\Lambda} e^{2\pi i \langle \gamma, \epsilon + \gamma \rangle} = \sum_{\gamma \in A_\Lambda} e^{2\pi i \langle \gamma, \epsilon + \mathbf{1}_\Lambda \rangle} = |A_\Lambda|\, \delta_{\mathbf{1}_\Lambda, \epsilon}$ (cf. [10, Lemma 3.1]) to get the second equality. Since $S^{-1} = S^7 = Z^3 S$, we get

$$\rho_\Lambda(V^{-1})\, \mathbf{e}_0 = \rho_\Lambda(Z)^3 \rho_\Lambda(ST^{-2}S)\, \mathbf{e}_0 = i^{-\sigma(\Lambda)} \rho_\Lambda(Z)^3\, \mathbf{e}_{\mathbf{1}_\Lambda} = i^{-\sigma(\Lambda)} i^{-3\sigma(\Lambda)}\, \mathbf{e}_{\mathbf{1}_\Lambda} = \mathbf{e}_{\mathbf{1}_\Lambda}.$$

This proves (2). $\square$

Lemma 7.5. *The following identities hold:*

$$(1)\quad f^{(0)}_k|_{ST^l}(\tau) = 2^{\frac{8-k}{2}} i^{-\frac{k}{2}} f^{(0)}_k\left(\frac{\tau+l}{4}\right), \qquad (2)\quad f^{(0)}_k|_V(\tau) = f^{(1)}_k(\tau).$$

Proof. We apply [9, Th. 5.1] to the lattice $\mathbb{A}_1^+ = \langle 2\rangle$. Since $A_{\mathbb{A}_1^+} = \langle 2\rangle^\vee/\langle 2\rangle = \{0, \frac{1}{2}\}$, the group ring $\mathbf{C}[A_{\mathbb{A}_1^+}]$ is equipped with the standard basis $\{\mathbf{e}_0, \mathbf{e}_{1/2}\}$. Set $\Theta_{\mathbb{A}_1^+}(\tau) := \theta_{\mathbb{A}_1^+}(\tau)\,\mathbf{e}_0 + \theta_{\mathbb{A}_1^+ + 1/2}(\tau)\,\mathbf{e}_{1/2}$. By [9, Th. 5.1] applied to $\mathbb{A}_1^+$, we get

$$\Theta_{\mathbb{A}_1^+}(g\cdot\tau) = j(g,\tau)\,\rho_{\mathbb{A}_1^+}(g)\Theta_{\mathbb{A}_1^+}(\tau), \qquad g\in Mp_2(\mathbf{Z}). \tag{7.1}$$

By (6.1) and (7.1), we have

$$\begin{aligned}\Theta_{\mathbb{A}_1^+}(ST^l\cdot\tau) &= j(ST^l,\tau)\left\{\tfrac{\mathbf{e}_0+\mathbf{e}_{1/2}}{\sqrt{2i}}\,\theta_{\mathbb{A}_1^+}(\tau) + i^l\,\tfrac{\mathbf{e}_0 - i\mathbf{e}_{1/2}}{\sqrt{2i}}\,\theta_{\mathbb{A}_1^+ + 1/2}(\tau)\right\},\\ \Theta_{\mathbb{A}_1^+}(V\cdot\tau) &= j(V,\tau)\left\{\mathbf{e}_0\,\theta_{\mathbb{A}_1^+ + 1/2}(\tau) + \mathbf{e}_{1/2}\,\theta_{\mathbb{A}_1^+}(\tau)\right\}.\end{aligned}$$

Comparing the coefficients of $\mathbf{e}_0$, we get

$$\theta_{\mathbb{A}_1^+}|_{ST^l}(\tau) = (2i)^{-\frac{1}{2}}\{\theta_{\mathbb{A}_1^+}(\tau) + i^l\theta_{\mathbb{A}_1^+ + 1/2}(\tau)\} = (2i)^{-\frac{1}{2}}\theta_{\mathbb{A}_1^+}\left(\frac{\tau+l}{4}\right), \tag{7.2}$$

$$\theta_{\mathbb{A}_1^+}|_V(\tau) = \theta_{\mathbb{A}_1^+ + 1/2}(\tau). \tag{7.3}$$

Here the second equality of (7.2) is the consequence of the following identity:

$$\theta_{\mathbb{A}_1^+}\left(\frac{\tau+l}{4}\right) = \sum_{n\,\text{even}} e^{2\pi i n^2(\tau+l)/4} + \sum_{n\,\text{odd}} e^{2\pi i n^2(\tau+l)/4} = \theta_{\mathbb{A}_1^+}(\tau) + i^l\theta_{\mathbb{A}_1^+ + 1/2}(\tau).$$

Set $\eta_{1^{-8}2^84^{-8}}(\tau) := \eta(\tau)^{-8}\eta(2\tau)^8\eta(4\tau)^{-8}$, which is a modular form for $M\Gamma_0(4)$ by Lemma 7.2. Since $ST^l = (\left(\begin{smallmatrix}1 & -1\\ 1 & l\end{smallmatrix}\right), \sqrt{\tau+l})$ and since $\eta(-\tau^{-1})^8 = \tau^4\eta(\tau)^8$ by [10, Lemma 6.1], we get

$$\begin{aligned}\eta_{1^{-8}2^84^{-8}}|_{ST^l}(\tau) &= (\tau+l)^{\frac{8}{2}}\,\eta_{1^{-8}2^84^{-8}}\left(-\frac{1}{\tau+l}\right)\\ &= (\tau+l)^4\,\eta\left(-\frac{1}{\tau+l}\right)^{-8}\eta\left(-\frac{2}{\tau+l}\right)^8\eta\left(-\frac{4}{\tau+l}\right)^{-8}\\ &= (\tau+l)^4\,(\tau+l)^{-4}\left(\frac{\tau+l}{2}\right)^4\left(\frac{\tau+l}{4}\right)^{-4}\\ &\quad\times\,\eta(\tau+l)^{-8}\eta\left(\frac{\tau+l}{2}\right)^8\eta\left(\frac{\tau+l}{4}\right)^{-8} = 2^4\eta_{1^{-8}2^84^{-8}}\left(\frac{\tau+l}{4}\right),\end{aligned}$$

which, together with (7.2), yields (1).

Since $V = (\left(\begin{smallmatrix}1 & 0\\ -2 & 1\end{smallmatrix}\right), \sqrt{-2\tau+1})$ and since $\eta_{1^{-8}2^84^{-8}}(\tau)$ has weight -4, we get

$$\begin{aligned}\eta_{1^{-8}2^84^{-8}}|_V(\tau) &= (-2\tau+1)^4\,\eta\left(\frac{\tau}{-2\tau+1}\right)^{-8}\eta\left(\frac{2\tau}{-2\tau+1}\right)^8\eta\left(\frac{4\tau}{-2\tau+1}\right)^{-8}\\ &= (-2\tau+1)^4\left(2-\frac{1}{\tau}\right)^{-4}\left(1-\frac{1}{2\tau}\right)^4\left(\frac{1}{2}-\frac{1}{4\tau}\right)^{-4}\\ &\quad\times\,\eta\left(2-\frac{1}{\tau}\right)^{-8}\eta\left(1-\frac{1}{2\tau}\right)^8\eta\left(\frac{1}{2}-\frac{1}{4\tau}\right)^{-8}\\ &= 2^4\tau^4\,\eta\left(2-\frac{1}{\tau}\right)^{-8}\eta\left(1-\frac{1}{2\tau}\right)^8\eta\left(\frac{1}{2}-\frac{1}{4\tau}\right)^{-8}.\end{aligned}$$

We define $h(\tau) := \eta(\tau+\frac{1}{2})^{-8}\eta(2\tau+1)^{8}\eta(4\tau+2)^{-8}$ for $\tau\in\mathfrak{H}$. Then

$$\eta_{1^{-8}2^{8}4^{-8}}|_V(\tau) = 16\tau^4\, h\left(-\frac{1}{4\tau}\right). \tag{7.4}$$

Set $\zeta := \exp(2\pi i/48)$. Since $h(\tau)$ is equal to

$$\begin{aligned}
&\zeta^{-8+16-32}\{q^{-\frac{8}{24}}\prod_{n=1}^{\infty}(1-(-q)^n)^{-8}\}\{q^{\frac{16}{24}}\prod_{n=1}^{\infty}(1-q^{2n})^{8}\}\{q^{-\frac{32}{24}}\prod_{n=1}^{\infty}(1-q^{4n})^{-8}\}\\
&=-q^{-1}\prod_{n=1}^{\infty}\{(1-q^{2n})^{-8}(1+q^{2n-1})^{-8}\}\cdot(1-q^{2n})^{8}\cdot\{(1-q^{2n})^{-8}(1+q^{2n})^{-8}\}\\
&=-q^{-1}\prod_{n=1}^{\infty}(1-q^{2n})^{-8}(1+q^{2n})^{-8}(1+q^{2n-1})^{-8}
\end{aligned}$$

and since we have the identities $\vartheta_2(2\tau)=2q^{1/4}\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n})^2$ and

$$\vartheta_3(2\tau)=\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1})^2,\qquad \vartheta_4(2\tau)=\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1})^2 \tag{7.5}$$

by [15, p.105, Eqs.(32-36)], we get

$$\vartheta_2(2\tau)^4\vartheta_3(2\tau)^4 = 2^4 q\prod_{n=1}^{\infty}(1-q^{2n})^8(1+q^{2n})^8(1+q^{2n-1})^8 = -2^4 h(\tau)^{-1}. \tag{7.6}$$

By [15, p.104, Eq.(20)], we have

$$\vartheta_2(-\tau^{-1})^4=-\tau^2\vartheta_4(\tau)^4,\qquad \vartheta_3(-\tau^{-1})^4=-\tau^2\vartheta_3(\tau)^4,$$

which, together with (7.6), yield the identity

$$\begin{aligned}
h\left(-\frac{1}{4\tau}\right)&=-2^4\vartheta_2\left(-\frac{1}{2\tau}\right)^{-4}\vartheta_3\left(-\frac{1}{2\tau}\right)^{-4}\\
&=-\tau^{-4}\vartheta_3(2\tau)^{-4}\vartheta_4(2\tau)^{-4}\\
&=-\tau^{-4}\left\{\prod_{n=1}^{\infty}(1-q^{2n})^2(1+q^{2n-1})^2(1-q^{2n-1})^2\right\}^{-4}\\
&=-\tau^{-4}\left\{\prod_{n=1}^{\infty}\frac{(1-q^{2n})(1-q^{4n})(1-q^{4n-2})}{(1-q^{4n})}\right\}^{-8}\\
&=-\tau^{-4}\left\{\frac{\prod_{n=1}^{\infty}(1-q^{2n})^2}{\prod_{n=1}^{\infty}(1-q^{4n})}\right\}^{-8}=-\tau^{-4}\eta(2\tau)^{-16}\eta(4\tau)^{8}.
\end{aligned} \tag{7.7}$$

Here we used (7.5) to get the third equality. We deduce from (7.4), (7.7) that

$$\eta_{1^{-8}2^{8}4^{-8}}|_V(\tau) = -16\,\eta(2\tau)^{-16}\eta(4\tau)^{8}. \tag{7.8}$$

We get (2) from (7.3) and (7.8). □

Definition 7.6. For a 2-elementary lattice Λ, define a $\mathbf{C}[A_\Lambda]$-valued holomorphic function $F_\Lambda(\tau)$ on $\mathfrak{H}$ by

$$F_\Lambda(\tau) := f^{(0)}_{8+\sigma(\Lambda)}(\tau)\,\mathbf{e}_0 + 2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}}\sum_{l=0}^{3} g^{(l)}_{8+\sigma(\Lambda)}(\tau)\,\mathbf{v}_l + f^{(1)}_{8+\sigma(\Lambda)}(\tau)\,\mathbf{e}_{\mathbf{1}_\Lambda}.$$

By the Fourier expansions of $f_k^{(0)}(\tau)$ and $f_k^{(1)}(\tau)$ at $q=0$, we get the following Fourier expansion of $F_\Lambda(\tau)$ at $q=0$:
(7.9)
$$\begin{aligned}F_\Lambda(\tau) = &\left\{q^{-1} + 24 + 2\,\sigma(\Lambda) + O(q)\right\}\mathbf{e}_0 + 2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}}\left\{24 + 2\,\sigma(\Lambda) + O(q)\right\}\mathbf{v}_0\\ &+ O(q^{1/4})\,\mathbf{v}_1 + O(q^{1/2})\,\mathbf{v}_2 + 2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}}\left\{q^{-1/4} + O(q^{3/4})\right\}\mathbf{v}_3\\ &- 2^{12+\sigma(\Lambda)}\,q^{\frac{8+\sigma(\Lambda)}{4}}\left\{1 + (24+\sigma(\Lambda))\,q^2 + O(q^4)\right\}\mathbf{e}_{\mathbf{1}_\Lambda}.\end{aligned}$$

Theorem 7.7. (1) *$F_\Lambda(\tau) = \mathcal{B}_\Lambda[\eta_{1^{-8}2^8 4^{-8}}\theta_{\mathbb{A}_1^+}^{8+\sigma(\Lambda)}](\tau)$. In particular, $F_\Lambda(\tau)$ is a modular form for $Mp_2(\mathbf{Z})$ of type ρ_Λ with weight $\sigma(\Lambda)/2$.*
(2) *The group $O(\Lambda)$ preserves F_Λ, i.e., $\mathrm{Aut}(F_\Lambda, \Lambda) = O(\Lambda)$.*
(3) *If $b^+(\Lambda) \le 2$ and $\sigma(\Lambda) \ge -12$, $F_\Lambda(\tau)$ has integral Fourier coefficients.*

Proof. **(1)** Set $k = 8+\sigma(\Lambda)$ and $\phi(\tau) = f_k^{(0)}(\tau)$ in Proposition 7.1. Since $f_k^{(0)}(\tau)$ is a modular form for $M\Gamma_0(4)$ of weight $(k-8)/2 = \sigma(\Lambda)/2$ with character $\chi_\theta^k = \chi_\theta^{\sigma(\Lambda)}$ by Lemma 7.2, $\mathcal{B}_\Lambda[f_k^{(0)}](\tau)$ is a modular form for $Mp_2(\mathbf{Z})$ of type ρ_Λ with weight $\sigma(\Lambda)/2$ by Proposition 7.1. We prove that $F_\Lambda = \mathcal{B}_\Lambda[f_k^{(0)}]$. Since $k = 8+\sigma(\Lambda)$ and $|A_\Lambda| = 2^{l(\Lambda)}$, we deduce from Lemmas 7.4 (1) and 7.5 (1) that
(7.10)
$$\begin{aligned}\sum_{l=0}^{3} f_k^{(0)}|_{ST^l}(\tau)\,\rho_\Lambda\left((ST^l)^{-1}\right)\mathbf{e}_0 &= \sum_{l=0}^{3} 2^{\frac{8-k}{2}}\,i^{-\frac{k}{2}}\,i^{\frac{\sigma(\Lambda)}{2}}|A_\Lambda|^{-\frac12}\sum_{j=0}^{3} f_k^{(0)}\left(\frac{\tau+l}{4}\right) i^{-lj}\,\mathbf{v}_j\\ &= 2^{\frac{-\sigma(\Lambda)-l(\Lambda)}{2}}\sum_{j=0}^{3}\sum_{l=0}^{3} f_k^{(0)}\left(\frac{\tau+l}{4}\right) i^{-lj}\,\mathbf{v}_j\\ &= 2^{-\frac{\sigma(\Lambda)+l(\Lambda)}{2}}\sum_{j=0}^{3}\sum_{l=0}^{3}\sum_{s\in\mathbf{Z}/4\mathbf{Z}} g_k^{(s)}(\tau+l)\,i^{-lj}\,\mathbf{v}_j.\end{aligned}$$

Recall that $f_k^{(0)}(\tau) = \sum_{n=-1}^{\infty} c_k^{(0)}(n)\,q^n$. Since $g_k^{(s)}(\tau) = \sum_{n\equiv s \bmod 4} c_k^{(0)}(n)\,q^{n/4}$, we get
$$g_k^{(s)}(\tau+l) = \sum_{n\equiv s \bmod 4} c_k^{(0)}(n)\,e^{2\pi i n(\tau+l)/4} = \sum_{n\equiv s \bmod 4} c_k^{(0)}(n)\,i^{sl}\,q^{n/4},$$
which yields that
$$\sum_{l=0}^{3} i^{-jl} g_k^{(s)}(\tau+l) = \sum_{n\equiv s \bmod 4} c_k^{(0)}(n)\sum_{l=0}^{3} i^{(s-j)l}\,q^{n/4} = 4\delta_{js}\,g_k^{(s)}(\tau).$$
Hence we get
$$\sum_{l=0}^{3}\sum_{s\in\mathbf{Z}/4\mathbf{Z}} i^{-jl} g_k^{(s)}(\tau+l) = \sum_{s\in\mathbf{Z}/4\mathbf{Z}} 4\,\delta_{sj}\,g_k^{(s)}(\tau) = 4\,g_k^{(j)}(\tau),$$
which, together with (7.10), yields that
$$\sum_{l=0}^{3} f_k^{(0)}|_{ST^l}(\tau)\cdot\rho_\Lambda\left((ST^l)^{-1}\right)\mathbf{e}_0 = 2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}}\sum_{j=0}^{3} g_k^{(j)}(\tau)\,\mathbf{v}_j. \tag{7.11}$$

Similarly, we get by Lemmas 7.4 (2) and 7.5 (2)

$$f_k^{(0)}|_V(\tau)\,\rho_\Lambda(V^{-1})\,\mathbf{e}_0 = f_k^{(1)}(\tau)\,\mathbf{e}_{\mathbf{1}_\Lambda}. \tag{7.12}$$

By (7.11) and (7.12), we get $F_\Lambda = \mathcal{B}_\Lambda[f_k^{(0)}]$.

(2) Since $g(\mathbf{e}_\gamma) = \mathbf{e}_{\bar{g}(\gamma)}$ for $g \in O(\Lambda)$ and $\gamma \in A_\Lambda$, we get $g(\mathbf{v}_i) = \mathbf{v}_i$ for all $g \in O(\Lambda)$ by the definition of $\mathbf{v}_i$. Since the characteristic vector $\mathbf{1}_\Lambda$ is $O(q_\Lambda)$-invariant, we get $\bar{g}(\mathbf{1}_\Lambda) = \mathbf{1}_\Lambda$ for all $g \in O(\Lambda)$.

(3) Since $f_k^{(0)}(\tau)$, $g_k^{(j)}(\tau)$, $f_k^{(1)}(\tau)$ have integral Fourier coefficients for $k \geq -4$, it suffices to prove by Definition 7.6 that $2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}} \in \mathbf{Z}$ when $b^+(\Lambda) \leq 2$. Since $\sigma(\Lambda) = 2b^+(\Lambda) - r(\Lambda)$, $r(\Lambda) \geq l(\Lambda)$ and $r(\Lambda) \equiv l(\Lambda) \mod 2$, we get $4 - \sigma(\Lambda) - l(\Lambda) = 2(2 - b^+(\Lambda)) + r(\Lambda) - l(\Lambda) \geq 0$ and $4 - \sigma(\Lambda) - l(\Lambda) \equiv 0 \mod 2$. □

Remark 7.8. When $\Lambda = \mathbb{U}^2 \oplus \mathbb{E}_8$, we have $F_{\mathbb{U}^2\oplus\mathbb{E}_8}(\tau) = E_4(\tau)^2/\eta(\tau)^{24}$, where $E_4(\tau)$ is the Eisenstein series of weight 4. By [28, Sects. 3.3 and 4], $F_{\mathbb{U}^2\oplus\mathbb{E}_8}(\tau)$ seems to be closely related with the elliptic genus of a certain vector bundle on a $K3$ surface. Is $F_\Lambda(\tau)$ related with the elliptic genera of some manifolds? The universal factor $\eta(\tau)^{-1}\eta(2\tau)\eta(4\tau)^{-1}$ in $f_k^{(0)}(\tau)$ appears in the definition of elliptic genera, because $R(1) = q^{1/8}\eta(\tau)^{-1}\eta(2\tau)\eta(4\tau)^{-1}$ in [36, p.7 l.7]. Is this coincidence accidental?

7.2. Applications to 2-elementary Lorentzian lattices

Recall that F_Λ induces a modular form $F_\Lambda|_L$ of type ρ_L when $\Lambda = \mathbb{U}(N) \oplus L$ (cf. Sect. 6.2). Since Λ is 2-elementary, $N \in \{1, 2\}$ and L is 2-elementary in this case.

Lemma 7.9. *If $\Lambda = \mathbb{U}(N) \oplus L$, then $F_\Lambda|_L = F_L$.*

Proof. Write $F_\Lambda|_L(\tau) = \sum_{\gamma\in A_L}(F_\Lambda|_L)_\gamma(\tau)\,\mathbf{e}_\gamma$. Since $\mathbf{1}_{\mathbb{U}(N)} = (0,0)$, we get $\mathbf{1}_\Lambda = ((0,0), \mathbf{1}_L)$. Since $((n/N, 0), \gamma)^2 = \gamma^2 \mod 2$ for $\gamma \in A_L$, we get by Definition 7.6 and the definition of $(F_\Lambda|_L)(\tau)$ (cf. (6.3))

(7.13)
$$(F_\Lambda|_L)_\gamma(\tau) = \begin{cases} N\,2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}}\,g_{8+\sigma(\Lambda)}^{(l)}(\tau) & (\gamma \neq 0, \mathbf{1}_L,\ \gamma^2 \equiv \frac{l}{2}) \\ f_{8+\sigma(\Lambda)}^{(0)}(\tau) + N\,2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}}\,g_{8+\sigma(\Lambda)}^{(0)}(\tau) & (\gamma = 0) \\ f_{8+\sigma(\Lambda)}^{(1)}(\tau) + N\,2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}}\,g_{8+\sigma(\Lambda)}^{(\sigma(\Lambda))}(\tau) & (\gamma = \mathbf{1}_L). \end{cases}$$

In the last equality, we used the formula $\mathbf{1}_\Lambda^2 \equiv \frac{\sigma(\Lambda)}{2} \mod 2$, which follows from (6.2), (7.9). If $N = 1$, $A_\Lambda = A_L$ and hence $F_\Lambda|_L = F_\Lambda = F_L$ by Definition 7.6 and (7.13). Assume $N = 2$. Since $\sigma(\Lambda) = \sigma(L)$ and $l(\Lambda) = l(L) + 2$, we get $F_\Lambda|_L = F_L$ by comparing the definition of F_L with (7.13). This proves the lemma. □

Lemma 7.10. *Let L be a 2-elementary Lorentzian lattice. If $r(L) \leq 10$, a subset of $\mathcal{C}_L^+$ is a Weyl chamber of L if and only if it is a Weyl chamber of F_L.*

Proof. Write $F_L(\tau) = \sum_{\gamma\in A_L}\mathbf{e}_\gamma \sum_{k\in\frac{\gamma^2}{2}+\mathbf{Z}} c_{L,\gamma}(k)\,q^k$. By (6.4), it suffices to prove that if $\lambda \in L^\vee$, $\lambda^2 < 0$ and $c_{L,\bar{\lambda}}(\lambda^2/2) \neq 0$, then $h_\lambda = h_d$ for some $d \in \Delta_L$. Since $8 + \sigma(L) \geq 0$, this follows from (7.9). □

Theorem 7.11. *Let L be a 2-elementary Lorentzian lattice with $r(L) \leq 10$ and let $\mathcal{W}$ be a Weyl chamber of L. Then*

$$\langle \varrho(L, F_L, \mathcal{W}), d\rangle = \begin{cases} 1 & \text{if} \quad d \in \Delta'_L \cap \Pi(L, \mathcal{W}) \\ 2^{k(L)} + 1 & \text{if} \quad d \in \Delta''_L \cap \Pi(L, \mathcal{W}). \end{cases}$$

In particular, the following hold:

(1) *If* $\Delta''_L = \emptyset$, *then* $\varrho(L, F_L, \mathcal{W})$ *is a Weyl vector of* $(L, \mathcal{W})$.
(2) *If* $\Delta'_L = \emptyset$, *then* $\varrho(L, F_L, \mathcal{W})/(2^{k(L)} + 1)$ *is a Weyl vector of* $(L, \mathcal{W})$.

Proof. We follow [9, Th. 11.2, Th. 12.1]. By Lemma 7.10, $\mathcal{W}$ is a Weyl chamber of $F_L(\tau)$. Let $d \in \Pi(L, \mathcal{W})$. Then h_d is the wall separating $\mathcal{W}$ and $s_d(\mathcal{W})$. Since $s_d \in W(L)$ acts trivially on A_L, we get by [9, p. 514 l.22 and p. 534 l.22] the identity $s_d(\varrho(L, F_L, \mathcal{W})) = \varrho(L, F_L, s_d(\mathcal{W}))$. Namely, we have

$$\varrho(L, F_L, \mathcal{W}) - \varrho(L, F_L, s_d(\mathcal{W})) = -\langle \varrho(L, F_L, \mathcal{W}), d\rangle\, d. \tag{7.14}$$

Write $F_L(\tau) = \sum_{\gamma \in A_L} \mathbf{e}_\gamma \sum_{k \in \frac{\gamma^2}{2} + \mathbf{Z}} c_{L,\gamma}(k)\, q^k$. By the wall crossing formula of Borcherds [9, Cors. 6.3 and 6.4], we get
(7.15)

$$\begin{aligned}
\varrho(L, F_L, \mathcal{W}) - \varrho(L, F_L, s_d(\mathcal{W})) &= -\sum_{\lambda \in L^\vee,\, h_\lambda = h_d,\, \lambda\cdot\mathcal{W} > 0} c_{L,\bar{\lambda}}(\lambda^2/2)\, \lambda \\
&= \begin{cases} -c_{L,\bar{0}}(-1)\, d & \text{if} \quad d \in \Delta'_L \cap \Pi(L, \mathcal{W}) \\ -c_{L,\frac{\bar{d}}{2}}(-\frac{1}{4})\frac{d}{2} - c_{L,\bar{0}}(-1)\, d & \text{if} \quad d \in \Delta''_L \cap \Pi(L, \mathcal{W}) \end{cases} \\
&= \begin{cases} -d & \text{if} \quad d \in \Delta'_L \cap \Pi(L, \mathcal{W}) \\ -(2^{k(L)} + 1)\, d & \text{if} \quad d \in \Delta''_L \cap \Pi(L, \mathcal{W}), \end{cases}
\end{aligned}$$

where the third equality follows from (7.9). (Since $\Gamma(-1/2) = -2\sqrt{\pi}$, it seems that the minus sign is necessary in the formula for $\Phi_1(v) - \Phi_2(v)$ in [9, Cor. 6.4].) Comparing (7.14) and (7.15), we get $\langle \varrho(L, F_L, \mathcal{W}), d\rangle = 1$ (resp. $2^{k(L)} + 1$) for all $d \in \Delta'_L \cap \Pi(L, \mathcal{W})$ (resp. $d \in \Delta''_L \cap \Pi(L, \mathcal{W})$). This proves the theorem. $\square$

Remark 7.12. By e.g. [20, Appendix, Tables 1,3], the table of primitive 2-elementary Lorentzian sublattices of $\mathbb{L}_{K3}$ with $\Delta''_L = \emptyset$ (resp. $\Delta'_L = \emptyset$) is given as follows:
(i) $\Delta''_L = \emptyset$ if and only if $\delta(\Lambda) = 0$ or $L \cong \mathbb{A}_1^+, \mathbb{A}_1^+ \oplus \mathbb{E}_8, \mathbb{A}_1^+ \oplus \mathbb{E}_8^{\oplus 2}$.
(ii) $\Delta'_L = \emptyset$ if and only if $k(L) = 0$, i.e., $r(L) = l(L)$.
The proof is parallel to those of Propositions 11.6 and 11.10 below.

We give a geometric interpretation of Theorem 7.11.

Theorem 7.13. *Let* (X, ι) *be a* 2*-elementary* $K3$ *surface with* $\mathrm{Pic}(X) = H^2_+(X, \mathbf{Z})$. *If* $r(\mathrm{Pic}(X)) \leq 10$, *there is a nef* $\mathbf{Q}$*-divisor* D_X *on* X *with the following properties:*
(1) $\varphi^* c_1(D_X) = c_1(D_X)$ *for every* $\varphi \in \mathrm{Aut}(X)$.
(2) *For every smooth rational curve* E *on* X,

$$D_X \cdot E = \begin{cases} 1 & \text{if} \quad c_1(E)/2 \notin \mathrm{Pic}(X)^\vee \\ 2^{k(\mathrm{Pic}(X))} + 1 & \text{if} \quad c_1(E)/2 \in \mathrm{Pic}(X)^\vee. \end{cases}$$

Proof. The real vector space $H^{1,1}(X, \mathbf{R})$ endowed with the cup-product pairing is a Lorentzian vector space. Let $\mathcal{C}_X^+ \subset H^{1,1}(X, \mathbf{R})$ be the light cone of $H^{1,1}(X, \mathbf{R})$ containing a Kähler class. Let $\mathcal{K}_X \subset H^{1,1}(X, \mathbf{R})$ be the set of Kähler classes on X. Let $\mathrm{Exc}(X)$ denote the set of smooth rational curves on X and let $W(X) := W(\mathrm{Pic}(X))$ be the Weyl group of $\mathrm{Pic}(X)$. By [42, Remark 3.5 i)],

$$\mathcal{K}_X = \{\kappa \in \mathcal{C}_X^+;\ \langle \kappa, c_1(E)\rangle > 0,\ \forall\, E \in \mathrm{Exc}(X)\}. \tag{7.16}$$

The ample cone of X is defined as $\mathcal{A}_X := \mathcal{K}_X \cap (\mathrm{Pic}(X) \otimes \mathbf{R})$. By (7.16), we get

$$\mathcal{A}_X = \{\kappa \in \mathcal{C}^+_{\mathrm{Pic}(X)};\ \langle \kappa, c_1(E)\rangle > 0,\ \forall\, E \in \mathrm{Exc}(X)\}. \tag{7.17}$$

Since $W(X)$ preserves $\mathrm{Pic}(X)$ and since $\mathcal{K}_X$ is a fundamental domain for the $W(X)$-action on $\mathcal{C}_X^+$ by [2, Chap. VIII, Prop. 3.10], $\mathcal{A}_X$ is a fundamental domain for the $W(X)$-action on $\mathcal{C}^+_{\mathrm{Pic}(X)}$. By [42, Remark 3.5 i)], the minimal set of inequalities defining $\mathcal{A}_X$ is given by $\{\langle\kappa, c_1(E)\rangle > 0\}_{E\in\mathrm{Exc}(X)}$. By comparing (2.2) and (7.17), the set of fundamental roots $\Pi(\mathrm{Pic}(X), \mathcal{A}_X)$ is given by

$$\Pi(\mathrm{Pic}(X), \mathcal{A}_X) = \{c_1(E) \in \Delta_{\mathrm{Pic}(X)};\ E \in \mathrm{Exc}(X)\}. \tag{7.18}$$

Let D_X be a $\mathbf{Q}$-divisor on X such that $c_1(D_X) = \varrho(\mathrm{Pic}(X), F_{\mathrm{Pic}(X)}, \mathcal{A}_X) \in \mathrm{Pic}(X)\otimes\mathbf{Q}$. From Theorem 7.11 and (7.18), (2) follows. We prove that D_X is nef.

Assume $\mathrm{Pic}(X) \not\cong \mathbb{U}\oplus\mathbb{E}_8(2)$, $\mathbb{U}(2)\oplus\mathbb{E}_8(2)$, $\mathbb{I}_{1,9}(2)$. Since $r(\mathrm{Pic}(X)) \leq 10$ and hence $r(\mathrm{Pic}(X)) + l(\mathrm{Pic}(X)) \leq 18$ by this assumption, $W(X)$ is a subgroup of $O(\mathrm{Pic}(X))$ with finite index by [45, Th. 4.4.1]. Since $\langle c_1(D_X), d\rangle > 0$ for all $d \in \Pi(\mathrm{Pic}(X), \mathcal{A}_X)$ by (2), we get $c_1(D_X) \in \overline{\mathcal{C}_X^+}$ by [47, Th. 1.4.3 and (1.4.5)]. Namely, $D_X^2 \geq 0$, which, together with (2), implies that D_X is nef by [42, Sect. 3.5].

Assume $\mathrm{Pic}(X) \cong \mathbb{U}\oplus\mathbb{E}_8(2)$, $\mathbb{U}(2)\oplus\mathbb{E}_8(2)$, $\mathbb{I}_{1,9}(2)$. By [9, Th. 10.4], we get

$$c_1(D_X) = \begin{cases} ((1,0),0) & \text{if} \quad \mathrm{Pic}(X)\cong \mathbb{U}\oplus\mathbb{E}_8(2) \\ 0 & \text{if} \quad \mathrm{Pic}(X)\cong \mathbb{U}(2)\oplus\mathbb{E}_8(2) \\ (\frac{3}{2}, -\frac{1}{2}, \dots, -\frac{1}{2}) & \text{if} \quad \mathrm{Pic}(X)\cong \mathbb{I}_{1,9}(2), \end{cases}$$

which yields that $D_X^2 = 0$. This, together with (2), implies that D_X is nef.

By [9, p. 514 l.22 and p. 534 l.22], we get for all $g \in O(\mathrm{Pic}(X))$

$$g(\varrho(\mathrm{Pic}(X), F_{\mathrm{Pic}(X)}, \mathcal{A}_X)) = \varrho(\mathrm{Pic}(X), F_{\mathrm{Pic}(X)}, g(\mathcal{A}_X)). \tag{7.19}$$

Since $\varphi^*\mathrm{Pic}(X) = \mathrm{Pic}(X)$ and $\varphi(\mathrm{Exc}(X)) = \mathrm{Exc}(X)$ for all $\varphi \in \mathrm{Aut}(X)$, it follows from (7.17) that $\mathrm{Im}\{\mathrm{Aut}(X) \to O(\mathrm{Pic}(X))\}$ preserves $\mathcal{A}_X$, i.e., $\varphi^*\mathcal{A}_X = \mathcal{A}_X$ for all $\varphi \in \mathrm{Aut}(X)$. Hence (1) follows from (7.19). $\square$

Remark 7.14. By an explicit formula for $c_1(D_X) = \varrho(L, F_L, \mathcal{W})$ in [9, Th. 10.4], one can see that $c_1(D_X) \equiv \mathbf{1}_{\mathrm{Pic}(X)} \mod \mathrm{Pic(X)}$ and that D_X is ample if $\mathrm{Pic}(X) \not\cong \mathbb{U}(2)$, $\mathbb{U}\oplus\mathbb{E}_8(2)$, $\mathbb{U}(2)\oplus\mathbb{E}_8(2)$, $\mathbb{I}_{1,9}(2)$. Since we do not use the explicit formula for $c_1(D_X)$ in the rest of this paper, we omit it.

8. Borcherds products for 2-elementary lattices

Throughout this section, we assume that Λ is a 2-elementary lattice with $\mathrm{sign}(\Lambda) = (2, r(\Lambda)-2)$. Recall that the divisors $\mathcal{D}'_\Lambda$ and $\mathcal{D}''_\Lambda$ on Ω_Λ were defined in Sect. 2.4.

Theorem 8.1. *If $r(\Lambda) \leq 12$, the Borcherds lift $\Psi_\Lambda(\cdot, F_\Lambda)$ is a holomorphic automorphic form on Ω_Λ^+ for $O^+(\Lambda)$ with zero divisor*

$$\mathrm{div}(\Psi_\Lambda(\cdot, F_\Lambda)) = \mathcal{D}'_\Lambda + (2^{(r(\Lambda)-l(\Lambda))/2} + 1)\,\mathcal{D}''_\Lambda.$$

The weight $w(\Lambda)$ of $\Psi_\Lambda(\cdot, F_\Lambda)$ is given by the following formula:

$$w(\Lambda) = \begin{cases} (16-r(\Lambda))(2^{(r(\Lambda)-l(\Lambda))/2}+1) - 8(1-\delta(\Lambda)) & (r(\Lambda)=12) \\ (16-r(\Lambda))(2^{(r(\Lambda)-l(\Lambda))/2}+1) & (r(\Lambda)<12). \end{cases}$$

Proof. Since $r(\Lambda) \leq 12$ and $\mathrm{sign}(\Lambda) = (2, r(\Lambda)-2)$, we get $\sigma(\Lambda) = 4 - r(\Lambda)$ and $8+\sigma(\Lambda) \geq 0$. By (7.9), we see that the Fourier coefficients of $F_\Lambda(\tau)$ are non-negative for negative exponents q^α, $\alpha < 0$ and that the coefficient of $\mathbf{e}_{\mathbf{1}_\Lambda}$, i.e., $f^{(1)}_{8+\sigma(\Lambda)}(\tau)$, is regular at $q = 0$. By Theorem 7.7 (2), we get $\mathrm{Aut}(\Lambda, F_\Lambda) = O(\Lambda)$. Write

$F_\Lambda(\tau) = \sum_{\gamma\in A_\Lambda} \mathbf{e}_\gamma \sum_{k\in\frac{\gamma^2}{2}+\mathbf{Z}} c_{\Lambda,\gamma}(k)\, q^k$. By Theorem 6.1 and (7.9), $\Psi_\Lambda(\cdot, F_\Lambda)$ is an automorphic form for $O^+(\Lambda)$ such that

(8.1)

$$\begin{aligned}\operatorname{div}(\Psi_\Lambda(\cdot, F_\Lambda)) &= \sum_{\lambda\in\Lambda^\vee/\pm1,\, \lambda^2<0} c_{\Lambda,\bar\lambda}(\lambda^2/2)\, H_\lambda \\ &= \sum_{\lambda\in\Lambda/\pm1,\, \lambda^2/2=-1} c_{\Lambda,\bar 0}(\lambda^2/2)\, H_\lambda + \sum_{\lambda\in\Lambda^\vee/\pm1,\, \lambda^2/2=-1/4} c_{\Lambda,\bar\lambda}(\lambda^2/2)\, H_\lambda \\ &= \sum_{\lambda\in\Delta_\Lambda/\pm1} H_\lambda + 2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}} \sum_{\lambda\in\Delta''_\Lambda/\pm1} H_\lambda = \mathcal{D}'_\Lambda + (2^{\frac{r(\Lambda)-l(\Lambda)}{2}}+1)\,\mathcal{D}''_\Lambda.\end{aligned}$$

By Theorem 6.1, $w(\Lambda) = c_{\Lambda,\bar 0}(0)/2$. If $r(\Lambda) = 12$ and $\delta(\Lambda) = 0$, then $\mathbf{1}_\Lambda = 0$, which, substituted into (7.9), implies that

(8.2)

$$\begin{aligned}F_\Lambda(\tau) = &\left\{q^{-1} + 24 + 2\sigma(\Lambda) + O(q)\right\} \mathbf{e}_0 + 2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}} \{24 + 2\sigma(\Lambda) + O(q)\}\, \mathbf{v}_0 \\ &+ O(q^{1/4})\, \mathbf{v}_1 + O(q^{1/2})\, \mathbf{v}_2 + 2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}} \left\{q^{-1/4} + O(q^{3/4})\right\} \mathbf{v}_3 \\ &+ \{-16 + O(q)\}\, \mathbf{e}_0.\end{aligned}$$

Since $\mathbf{v}_0$ contains $\mathbf{e}_0$ with multiplicity one and since $\sigma(\Lambda) = 4 - r(\Lambda)$, we deduce from (8.2) that

$$w(\Lambda) = \frac{c_{\Lambda,0}(0)}{2} = 12+\sigma(\Lambda)+2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}}(12+\sigma(\Lambda))-8 = (16-r(\Lambda))(2^{\frac{r(\Lambda)-l(\Lambda)}{2}}+1)-8.$$

This proves the formula for $w(\Lambda)$ when $r(\Lambda) = 12$ and $\delta(\Lambda) = 0$.

If $r(\Lambda) < 12$ or $(r(\Lambda), \delta(\Lambda)) = (12, 1)$, the coefficient of $\mathbf{e}_{\mathbf{1}_\Lambda}$ does not contribute to $c_{\Lambda,0}(0)$ by (7.9), so that

$$w(\Lambda) = \frac{c_{\Lambda,0}(0)}{2} = 12 + \sigma(\Lambda) + 2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}}(12 + \sigma(\Lambda)) = (16 - r(\Lambda))(2^{\frac{r(\Lambda)-l(\Lambda)}{2}} + 1)$$

in this case. This completes the proof of Theorem 8.1. □

Corollary 8.2. *If $r(\Lambda) \leq 12$ and $\Delta''_\Lambda = 0$, then $\operatorname{div}(\Psi_\Lambda(\cdot, F_\Lambda)) = \mathcal{D}_\Lambda$.*

Proof. Since $\Delta''_\Lambda = \emptyset$, the result follows from Theorem 8.1. □

For the table of primitive 2-elementary sublattices $\Lambda \subset \mathbb{L}_{K3}$ with $r(\Lambda) \leq 12$ and $\Delta''_\Lambda = \emptyset$, see Proposition 11.6 below.

Corollary 8.3. *The coarse moduli space of 2-elementary $K3$ surfaces of type M is quasi-affine if $r(M) \geq 10$.*

Proof. Set $\Lambda := M^\perp$. Since $r(M) \geq 10$, we get $r(\Lambda) \leq 12$. A holomorphic automorphic form on Ω_Λ is identified with a holomorphic section of an ample line bundle over $\mathcal{M}^*_\Lambda$ by Baily–Borel–Satake [1]. Hence $\mathcal{M}_\Lambda \setminus \operatorname{div}(\Psi_\Lambda(\cdot, F_\Lambda))$ is quasi-affine. Since $\operatorname{supp}\operatorname{div}(\Psi_\Lambda(\cdot, F_\Lambda)) = \mathcal{D}_\Lambda$ by Theorem 8.1 and hence $\mathcal{M}^o_\Lambda = \mathcal{M}_\Lambda \setminus \operatorname{div}(\Psi_\Lambda(\cdot, F_\Lambda))$, we get the result. □

In [44, Sect. 2], [16, Sects. 1–3], the notion of lattice polarized $K3$ surface was introduced, and their moduli spaces were studied. We follow the definition in [16].

Corollary 8.4. *If M is a primitive 2-elementary Lorentzian sublattice of $\mathbb{L}_{K3}$ with $r(M) \geq 10$, then the coarse moduli space of ample M-polarized $K3$ surfaces is quasi-affine.*

Proof. Set $G_M := \ker\{O^+(M^\perp) \to O(q_{M^\perp})\}$, where $O^+(M^\perp) \to O(q_{M^\perp})$ denotes the natural homomorphism. By [16, p.2607], the coarse moduli space of ample M-polarized $K3$ surfaces is isomorphic to the analytic space $\Omega^o_{M^\perp}/G_M$. By this description, the proof of the corollary is similar to that of Corollary 8.3. $\square$

For the table of isometry classes of primitive 2-elementary Lorentzian sublattices $M \subset \mathbb{L}_{K3}$ with $r(M) \geq 10$, see [20, Appendix, Tables 1,2,3]; there are 49 isometry classes. There are some examples of lattices Λ with $b^+(\Lambda) = 2$ admitting an automorphic form on Ω^+_Λ with zero divisor $\mathcal{D}_\Lambda$. See [7, Sect. 16 Examples 1,2,3], [8], [10, Sect. 12], [11, Examples 2.1, 2.2], [25, II, Th. 5.2.1], [35, Th. 6.4], [52, Sect. 10] etc.

Remark 8.5. By [59, Th. 5.9], there exists a strongly pluri-subharmonic function on $\mathcal{M}^o_{M^\perp}$ if $r(M) > 6$. In particular, $\mathcal{M}^o_{M^\perp}$ contains no complete curves when $r(M) > 6$. The existence of a strongly pluri-subharmonic function on a quasi-projective variety X does *not* necessarily imply the quasi-affineness of X. See [26, p. 232 Example 3.2] for a counter example. If $r(M) > 6$, is $\mathcal{M}^o_{M^\perp}$ quasi-affine?

Theorem 8.6. *When $\Lambda = \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8(2) \oplus \mathbb{A}_1$, the Borcherds lift $\Psi_\Lambda(\cdot, F_\Lambda)$ is a meromorphic automorphic form for $O^+(\Lambda)$ of weight* 15 *with zero divisor*

$$\mathcal{D}'_\Lambda + 5\mathcal{D}''_\Lambda - 8\,\mathcal{H}_\Lambda(\mathbf{1}_\Lambda, -1/2).$$

Proof. We have $r(\Lambda) = 13$, $l(\Lambda) = 9$, $\sigma(\Lambda) = -9$ and $\delta(\Lambda) = 1$. By Theorem 6.1 and (7.9), the weight of $\Psi_\Lambda(\cdot, F_\Lambda)$ is given by $(12+\sigma(\Lambda))(2^{(4-\sigma(\Lambda)-l(\Lambda))/2}+1) = 15$ and the divisor of $\Psi_\Lambda(\cdot, F_\Lambda)$ is given by

$$\mathcal{D}_\Lambda + 2^{\frac{4-\sigma(\Lambda)-l(\Lambda)}{2}}\,\mathcal{D}''_\Lambda - 2^{12+\sigma(\Lambda)}\,\mathcal{H}_\Lambda(\mathbf{1}_\Lambda, -1/2) = \mathcal{D}'_\Lambda + 5\mathcal{D}''_\Lambda - 8\,\mathcal{H}_\Lambda(\mathbf{1}_\Lambda, -1/2),$$

where $-2^{12+\sigma(\Lambda)}\mathcal{H}_\Lambda(\mathbf{1}_\Lambda, -\frac{1}{2})$ comes from the negative coefficient of $q^{\frac{8+\sigma(\Lambda)}{4}}\mathbf{e}_{\mathbf{1}_\Lambda}$ in (7.9). This proves the theorem. $\square$

Assume $\Lambda = \mathbb{U}(N) \oplus L$, where L is a 2-elementary Lorentzian lattice with $r(L) \leq 10$ and $N \in \{1, 2\}$. Hence $r(\Lambda) \leq 12$, and $F_\Lambda|_L = F_L$ by Lemma 7.9. By [9, Th. 13.3], Definition 7.6 and the definitions of $f^{(0)}_k(\tau)$, $f^{(1)}_k(\tau)$ and $g^{(i)}_k(\tau)$, the infinite product for $\Psi_\Lambda(\cdot, F_\Lambda)$ is given explicitly as follows:

$$\begin{aligned}\Psi_\Lambda(z, F_\Lambda) = e^{2\pi i\langle\varrho, z\rangle} &\prod_{\lambda\in L,\, \lambda\cdot\mathcal{W}>0,\, \lambda^2\geq -2} (1 - e^{2\pi i\langle\lambda, z\rangle})^{c^{(0)}_{8+\sigma(\Lambda)}(\lambda^2/2)} \\ &\times \prod_{\lambda\in 2L^\vee,\, \lambda\cdot\mathcal{W}>0,\, \lambda^2\geq -2} (1 - e^{\pi i N\langle\lambda, z\rangle})^{2^{\frac{r(\Lambda)-l(\Lambda)}{2}} c^{(0)}_{8+\sigma(\Lambda)}(\lambda^2/2)} \\ &\times \prod_{\lambda\in(\mathbf{1}_L+L),\, \lambda\cdot\mathcal{W}>0,\, \lambda^2\geq 0} (1 - e^{2\pi i\langle\lambda, z\rangle})^{2c^{(1)}_{8+\sigma(\Lambda)}(\lambda^2/2)},\end{aligned} \tag{8.3}$$

where $\mathcal{W} \subset L \otimes \mathbf{R}$ is a Weyl chamber of L by Lemma 7.10 and $\varrho = \varrho(L, F_L, \mathcal{W}) \in L \otimes \mathbf{Q}$ is the Weyl vector of $(L, F_L, \mathcal{W})$.

Example 8.7. Let $\Lambda = \mathbb{U}(2) \oplus \mathbb{U}(2) \oplus \mathbb{E}_8(2)$. We have $l(\Lambda) = 12$ and $w(\Lambda) = 0$. This Λ admits no primitive embedding into $\mathbb{L}_{K3}$ by [43, Th. 1.12.1]. Since $\mathbf{\Delta}_\Lambda = \emptyset$, we get $\mathcal{D}_\Lambda = \emptyset$, so that $\Psi_\Lambda(\cdot, F_\Lambda)$ is a constant function. This $F_\Lambda(\tau)$ gives an example of non-trivial elliptic modular form for $Mp_2(\mathbf{Z})$ whose Borcherds lift becomes trivial.

Example 8.8. Let $\Lambda = \mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{E}_8(2)$. We have $l(\Lambda) = 10$ and $w(\Lambda) = 4$. Then $\Psi_\Lambda(\cdot, F_\Lambda)$ is the Borcherds Φ-function of dimension 10. See [7, Sect. 15, Example 4], [8], [9, Example 13.7], [22, Sect. 11], [34, Remark 4.7, Th. 7.1], [51], [59, Sect. 8.1] for more about this example and related results.

Example 8.9. Let $\Lambda = \mathbb{U}^2 \oplus \mathbb{E}_8(2)$. We have $l(\Lambda) = 8$ and $w(\Lambda) = 12$. Then $\Psi_\Lambda(\cdot, F_\Lambda) = \Psi_\Lambda(\cdot, \Theta_{\Lambda_{16}^+}(\tau)/\eta(\tau)^{24})$ is the restriction of the Borcherds Φ-function of dimension 26 to Ω_Λ, where $\Theta_{\Lambda_{16}^+}(\tau)$ is the theta function [9, Sect. 4] for the positive-definite 16-dimensional Barnes–Wall lattice Λ_{16}^+. See [59, Sect. 8.2].

Example 8.10. Let $\Lambda = \mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{D}_4^2$. We have $l(\Lambda) = 6$ and $w(\Lambda) = 28$. Kondō [35, Th. 6.4] used $\Psi_\Lambda(\cdot, F_\Lambda)$ in the study of the projective model of the moduli space of 8 points on $\mathbf{P}^1$. By [35, Th. 6.7 and its proof], $\Psi_\Lambda(\cdot, F_\Lambda)^{15}$ is expressed as the product of certain 105 *additive Borcherds lifts* [9, Sect. 14]. See also [22, Sect. 12].

Example 8.11. Let $\Lambda = \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8$. Then $l(\Lambda) = 0$ and $w(\Lambda) = 252$. We get $F_\Lambda(\tau) = E_4(\tau)^2/\eta(\tau)^{24}$, where $E_4(\tau)$ is the Eisenstein series of weight 4. The corresponding Borcherds lift $\Psi_\Lambda(\cdot, F_\Lambda) = \Psi_\Lambda(\cdot, E_4(\tau)^2/\eta(\tau)^{24})$ was introduced by Borcherds [7, Th. 10.1, Sect. 16 Example 1]. By Harvey–Moore [28, Sects. 4 and 5], $\Psi_\Lambda(\cdot, E_4(\tau)^2/\eta(\tau)^{24})$ appears in the formula for the one-loop coupling renormalization [28, Eqs. (4.1), (4.5), (4.16), (4.27)].

Example 8.12. When $\Lambda = \mathbb{U}^2 \oplus \mathbb{D}_4$, $\Psi_\Lambda(\cdot, F_\Lambda)$ coincides with the automorphic form Δ of Freitag–Hermann [21, Th. 11.6]. Notice that the weight of Δ is 72 in our definition (cf. [21, p.250 l.21–l.23]). By [21, Proof of Th. 11.5], $\Psi_\Lambda(\cdot, F_\Lambda)$ is expressed as the product of certain 36 theta functions.

Example 8.13. When $\Lambda = \mathbb{I}_{2,4}(2)$, $\Psi_\Lambda(\cdot, F_\Lambda)$ is the product of all even Freitag theta functions [56], [61, Th. 7.9], so that the structure of $\Psi_{\mathbb{I}_{2,4}(2)}(\cdot, F_{\mathbb{I}_{2,4}(2)})$ is similar to that of $\Psi_{\mathbb{U}\oplus\mathbb{U}(2)\oplus\mathbb{D}_4^2}(\cdot, F_{\mathbb{U}\oplus\mathbb{U}(2)\oplus\mathbb{D}_4^2})$, $\Psi_{\mathbb{U}^2\oplus\mathbb{D}_4}(\cdot, F_{\mathbb{U}^2\oplus\mathbb{D}_4})$. For the corresponding 2-elementary $K3$ surfaces, see [56].

Example 8.14. When $\Lambda = \mathbb{I}_{2,3}(2)$, $\Psi_\Lambda(\cdot, F_\Lambda)$ coincides with the automorphic form Δ_{11} of Gritsenko–Nikulin [25, II, Example 3.4 and Th. 5.2.1]. When $\Lambda = \mathbb{U}^2 \oplus \mathbb{A}_1$, $\Psi_\Lambda(\cdot, F_\Lambda)$ coincides with the automorphic form $\Delta_5^4\Delta_{35}$ of Gritsenko–Nikulin [25, II, Examples 2.4 and 3.9, Th. 5.2.1].

9. An explicit formula for τ_M

Theorem 9.1. *Let M be a primitive 2-elementary Lorentzian sublattice of $\mathbb{L}_{K3}$. Assume that M satisfies the following two conditions:*

(1) $11 \leq r(M) \leq 17$ *or* $(r(M), \delta(M)) = (10, 1)$.
(2) $J_M^o(\Omega_{M^\perp}^o) \not\subset \theta_{\mathrm{null},g(M)}$.

Then there exists a non-zero constant C_M depending only on the lattice M such that for every 2-elementary $K3$ surface (X, ι) of type M,

(9.1)
$$\tau_M(X,\iota)^{-2^{g(M)+1}(2^{g(M)}+1)} = C_M \,\|\Psi_{M^\perp}(\varpi_M(X,\iota), F_{M^\perp})\|^{2^{g(M)}} \|\chi_{g(M)}(\Omega(X^\iota)\|^{16}.$$

Let $\ell \in \mathbf{Z}_{>0}$ be an integer such that $\mathcal{F}_{g(M)}^{2^{g(M)+1}(2^{g(M)}+1)\ell}$ extends to a very ample line bundle on $\mathcal{A}_{g(M)}^*$. We may assume $\nu = 2^{g(M)-1}(2^{g(M)}+1)\ell$ in Theorem 5.1. By Theorem 9.1, we have

$$\Phi_M = C_M^{\ell/2}\, \Psi_{M^\perp}(\cdot, F_{M^\perp})^{2^{g(M)-1}\ell} \otimes J_M^*\chi_{g(M)}^{8\ell}. \tag{9.2}$$

Proof. By our assumption, $r(M^\perp) \leq 12$. When $r(M^\perp) = 12$, we have $\delta(M) = 1$. We set $\Lambda = M^\perp$ in Theorem 8.1. Then we have $16 - r(\Lambda) = r(M) - 6$ and $\frac{r(\Lambda)-l(\Lambda)}{2} = 11 - \frac{r(M)+l(M)}{2} = g(M)$.

Recall that $K_{M^\perp} \in C^\infty(\Omega^+_{M^\perp})$ was defined in Sect. 4.2. Let $\omega_{M^\perp}$ be the Kähler form of the Bergman metric on $\Omega^+_{M^\perp}$, i.e.,

$$\omega_{M^\perp} := -dd^c \log K_{M^\perp}.$$

By [59, Eq. (7.1)], we have the following equation of currents on $\Omega_{M^\perp}$:

$$dd^c \log \tau_M = \frac{r(M)-6}{4}\,\omega_{M^\perp} + J_M^*\omega_{\mathcal{A}_{g(M)}} - \frac{1}{4}\delta_{\mathcal{D}_{M^\perp}}. \tag{9.3}$$

By Theorem 8.1, (4.15) and the Poincaré-Lelong formula, we get

$$\begin{aligned} &-2^{g(M)-1}\,dd^c \log \|\Psi_{M^\perp}(\cdot, F_{M^\perp})\|^2 \\ &= 2^{g(M)-1}(2^{g(M)}+1)(r(M)-6)\,\omega_{M^\perp} - 2^{g(M)-1}\delta_{\mathcal{D}'_{M^\perp}} - 2^{g(M)-1}(2^{g(M)}+1)\delta_{\mathcal{D}''_{M^\perp}}. \end{aligned} \tag{9.4}$$

By Proposition 4.2 (1), there exist $a \in \mathbf{Z}_{\geq 0}$ and an $O^+(M^\perp)$-invariant effective divisor E on $\Omega^+_{M^\perp}$ such that

$$-dd^c \log \|J_M^*\chi^{8\ell}_{g(M)}\|^2 = 2^{g(M)+1}(2^{g(M)}+1)\ell\, J_M^*\omega_{\mathcal{A}_{g(M)}} - 2(2^{2g(M)-2}+a)\ell\,\delta_{\mathcal{D}'_{M^\perp}} - \delta_E. \tag{9.5}$$

By (9.3-5), we get the following equation of currents on $\Omega^+_{M^\perp}$:

$$\begin{aligned} &-dd^c \log\left[\tau_M^{2^{g(M)+1}(2^{g(M)}+1)\ell}\|\Psi_{M^\perp}(\cdot, F_{M^\perp})^{2^{g(M)-1}\ell} \otimes J_M^*\chi^{8\ell}_{g(M)}\|^2\right] \\ &= -2a\ell\,\delta_{\mathcal{D}'_{M^\perp}} - \delta_E. \end{aligned} \tag{9.6}$$

Since $\log \tau_M$, $\log \|\Psi_{M^\perp}(\cdot, F_{M^\perp})\|$ and $\log \|J_M^*\chi^{8\ell}_{g(M)}\|$ are $O^+(M^\perp)$-invariant L^1_{loc}-function on $\Omega^+_{M^\perp}$, we deduce from (9.6) and [59, Th. 3.17] the existences of an integer m and an $O^+(M^\perp)$-invariant meromorphic function φ_M on $\Omega^+_{M^\perp}$ with zero divisor $m(2a\ell\,\mathcal{D}'_{M^\perp} + E)$ such that

$$\tau_M^{2^{g(M)+1}(2^{g(M)}+1)\ell}\|\Psi_{M^\perp}(\cdot, F_{M^\perp})^{2^{g(M)-1}\ell} \otimes J_M^*\chi^{8\ell}_{g(M)}\|^2 = |\varphi_M|^{2/m}. \tag{9.7}$$

Since $\mathcal{M}^*_{M^\perp}$, the Baily–Borel–Satake compactification of $\mathcal{M}_{M^\perp}$, is an irreducible normal projective variety and since $\dim(\mathcal{M}^*_{M^\perp} \setminus \mathcal{M}_{M^\perp}) \leq \dim \mathcal{M}^*_{M^\perp} - 2$ by the condition $r(M) \leq 17$, φ_M descends to a meromorphic function on $\mathcal{M}^*_{M^\perp}$. Since φ_M is a meromorphic function on $\mathcal{M}^*_{M^\perp}$ whose divisor is effective, φ_M must be a constant function on $\mathcal{M}^*_{M^\perp}$. Hence $a = 0$ and $E = 0$. Setting $C_M := |\varphi_M|^{-2/m}$ in (9.7), we get the result. □

Theorem 9.2. *If $M \cong \mathbb{I}_{1,8}(2)$, there exists a non-zero constant C_M depending only on the lattice M such that for every 2-elementary $K3$ surface (X, ι) of type M,*

$$\tau_M(X,\iota)^{-40} = C_M\,\|\Psi_{M^\perp}(\varpi_M(X,\iota), F_{M^\perp})\|^4\|\chi_{g(M)}(\Omega(X^\iota))\|^{16}.$$

Proof. Since $M \cong \mathbb{I}_{1,8}(2)$, we get $M^{\perp} \cong \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8(2) \oplus \mathbb{A}_1$ by e.g. [20, Appendix, Table 2]. By (9.3) and Proposition 4.2 (2), we get

$$\begin{aligned}(9.8)\qquad & dd^c\{-40\ell \log \tau_M - \log \|J_M^* \chi_2^{8\ell}\|^2\} \\ &= \ell \left\{-30\,\omega_{M^\perp} + 10\,\delta_{\mathcal{D}_{M^\perp}} - (8+2a)\,\delta_{\mathcal{D}'_{M^\perp}} - 16\,\delta_{\mathcal{H}_{M^\perp}(\mathbf{1}_{M^\perp},-\frac{1}{2})}\right\} \\ &= \ell \left\{-30\,\omega_{M^\perp} + (2-2a)\delta_{\mathcal{D}'_{M^\perp}} + 10\,\delta_{\mathcal{D}''_{M^\perp}} - 16\,\delta_{\mathcal{H}_{M^\perp}(\mathbf{1}_{M^\perp},-\frac{1}{2})}\right\}.\end{aligned}$$

By (9.8) and [59, Th. 3.17], there is a meromorphic automorphic form φ_M on $\Omega^+_{M^\perp}$ for $O^+(M^\perp)$ of weight 30ℓ with

$$(9.9)\qquad \operatorname{div}\varphi_M = \ell\,\{(2-2a)\mathcal{D}'_{M^\perp} + 10\mathcal{D}''_{M^\perp} - 16\,\mathcal{H}_{M^\perp}(\mathbf{1}_{M^\perp}, -1/2)\}$$

such that

$$(9.10)\qquad 40\ell \log \tau_M + \log \|J_M^* \chi_2^{8\ell}\|^2 = -\log \|\varphi_M\|^2.$$

Since $O^+(M^\perp)/[O^+(M^\perp), O^+(M^\perp)]$ is a finite Abelian group, there exists $\nu \in \mathbf{Z}_{>0}$ such that φ_M^ν and $\Psi_{M^\perp}(\cdot, F_{M^\perp})^{2\nu}$ are automorphic forms with trivial character. By Theorem 8.6 and (9.9), $(\Psi_{M^\perp}(\cdot, F_{M^\perp})^{2\ell}/\varphi_M)^\nu$ is an $O^+(M^\perp)$-invariant meromorphic function on $\Omega^+_{M^\perp}$ with

$$\begin{aligned}\operatorname{div}(\Psi_{M^\perp}(\cdot, F_{M^\perp})^{2\ell}/\varphi_M)^\nu &= \nu\ell\{2\mathcal{D}'_{M^\perp} + 10\mathcal{D}''_{M^\perp} - 16\mathcal{H}_{M^\perp}(\mathbf{1}_{M^\perp}, -1/2)\} \\ &\quad - \nu\ell\{(2-2a)\delta_{\mathcal{D}'_{M^\perp}} + 10\,\delta_{\mathcal{D}''_{M^\perp}} - 16\,\delta_{\mathcal{H}_{M^\perp}(\mathbf{1}_{M^\perp},-\frac{1}{2})}\} \\ &= 2a\nu\ell\,\mathcal{D}'_{M^\perp}.\end{aligned}$$

Since $\operatorname{div}(\Psi_{M^\perp}(\cdot, F_{M^\perp})^{2\ell}/\varphi_M)^\nu$ is an effective divisor on $\Omega^+_{M^\perp}$, the same argument as in the proof of Theorem 9.1 using the Hartogs theorem implies $a = 0$ and hence the existence of a non-zero constant C_M with

$$(9.11)\qquad \varphi_M = C_M^{\ell/2}\,\Psi_{M^\perp}(\cdot, F_{M^\perp})^{2\ell}.$$

By (9.10), (9.11), we get the result. □

Theorem 9.3. *Let $M \subset \mathbb{L}_{K3}$ be a primitive 2-elementary Lorentzian sublattice. Assume that M is non-exceptional and satisfies one of the following two conditions:*

(1) *$g(M) \leq 2$, $9 \leq r(M) \leq 17$*
(2) *$g(M) = 3$, $r(M) \geq 10$, $(r(M), \delta(M)) \neq (10, 0)$.*

Then either M satisfies the Conditions (1) and (2) in Theorem 9.1 or $M \cong \mathbb{I}_{1,8}(2)$. In particular, Eqs. (9.1) *and* (9.2) *hold for these M.*

Proof. **(1)** Let $g(M) \leq 1$ and $10 \leq r(M) \leq 17$. Since M is not exceptional, M satisfies Condition (1) in Theorem 9.1 by [45, p.1434, Table 1]. Since $\mathcal{A}_g \cap \theta_{\mathrm{null},g} = \emptyset$ when $g \in \{0, 1\}$, we get $J_M^o(\Omega^o_{M^\perp}) \subset \mathcal{A}_{g(M)} \setminus \theta_{\mathrm{null},g(M)}$. Hence Condition (2) in Theorem 9.1 holds when $g(M) \leq 1$ and $r(M) \leq 17$.

Let $g(M) = 2$ and $10 \leq r(M) \leq 17$. Since M is not exceptional, $(r(M), \delta(M)) \neq (10, 0)$, so that Condition (1) in Theorem 9.1 holds. Since $M \not\cong \mathbb{U} \oplus \mathbb{E}_8(2)$, we get $J_M^o(\Omega^o_{M^\perp}) \subset \mathcal{A}_2 \setminus \mathcal{N}_2$ by Lemma 3.1. Since $\theta_{\mathrm{null},2} = \mathcal{N}_2$, we get $J_M^o(\Omega^o_{M^\perp}) \subset \mathcal{A}_2 \setminus \theta_{\mathrm{null},2}$. Thus M satisfies Condition (2) in Theorem 9.1 in this case.

Let $g(M) = 2$ and $r(M) = 9$. Then $M \cong \mathbb{I}_{1,8}(2)$ by [45, p.1434, Table 1].

(2) Let $g(M) = 3$, $r(M) \geq 10$ and $(r(M), \delta(M)) \neq (10, 0)$. Then Condition (1) in Theorem 9.1 holds. If Condition (2) in Theorem 9.1 does not hold for some M,

$$(9.12)\qquad J_M^o(\Omega^o_{M^\perp}) \subset \theta_{\mathrm{null},3}.$$

By [20, p.14 Figure 1 and p.23 Table 1], we can write $M^{\perp} \cong \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{D}_4 \oplus \mathbb{A}_1^m$, where $0 \leq m \leq 4$. Let $\{d_1, \ldots, d_m\}$ be the standard basis of $\mathbb{A}_1^m$ whose Gram matrix is $-2 \cdot 1_m$. Then $d_1, \ldots, d_m \in \Delta''_{M^{\perp}}$. We define 2-elementary lattices M_k inductively by $M_{k+1} := [M_k \perp d_{k+1}]$, $M_0 := M$. Then $M_k^{\perp} \cong \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{D}_4 \oplus \mathbb{A}_1^{m-k}$. By Lemma 11.5, we get $g(M_k) = 3$ for all M_k.

Assume $J^o_{M_k}(\Omega^o_{M_k^{\perp}}) \subset \theta_{\mathrm{null},3}$. Since J_{M_k} is a continuous map from $\Omega^o_{M_k^{\perp}} \cup \mathcal{D}^o_{M_k^{\perp}}$ to $\mathcal{A}_3$ and since $\Omega^o_{M_k^{\perp}}$ is dense in $\Omega^o_{M_k^{\perp}} \cup \mathcal{D}^o_{M_k^{\perp}}$, we get by Theorem 3.5 and (9.12)

$$J^o_{M_{k+1}}(\Omega^o_{M_{k+1}^{\perp}}) \subset \overline{J_{M_k}(\mathcal{D}^o_{M_k^{\perp}})} \subset \overline{J^o_{M_k}(\Omega^o_{M_k^{\perp}})} \subset \overline{\theta_{\mathrm{null},3}}.$$

Hence $J^o_{M_{k+1}}(\Omega^o_{M_{k+1}^{\perp}}) \subset \overline{\theta_{\mathrm{null},3}} \cap \mathcal{A}_3 = \theta_{\mathrm{null},3}$. In particular, we get $J^o_{M_m}(\Omega^o_{M_m^{\perp}}) \subset \theta_{\mathrm{null},3}$. Recall that the period of a curve of genus 3 lies in $\theta_{\mathrm{null},3}$ if and only if the curve is hyperelliptic by [29, Lemma 11]. Since $M_m^{\perp} \cong \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{D}_4$ and hence $M_m \cong \mathbb{U} \oplus \mathbb{D}_{12}$, the inclusion $J^o_{\mathbb{U}\oplus\mathbb{D}_{12}}(\Omega^o_{(\mathbb{U}\oplus\mathbb{D}_{12})^{\perp}}) \subset \theta_{\mathrm{null},3}$ implies that the non-rational component of X^{ι} is a hyperelliptic curve of genus 3 for every 2-elementary $K3$ surface (X, ι) of type $\mathbb{U} \oplus \mathbb{D}_{12}$. This contradicts Proposition 12.3 (2) below. Thus, if $g(M) = 3$, $r(M) \geq 10$ and $(r(M), \delta(M)) \neq (10, 0)$, then we never have the inclusion (9.12). Namely, M satisfies Condition (2) in Theorem 9.1. □

Remark 9.4. If M satisfies Condition (1) or (2) in Theorem 9.3, then $M^{\perp}$ is given by the following table by [20, Appendix, Tables 1,2,3]:

(0) If $g(M) = 0$, $M^{\perp}$ is isometric to one of the following 7 lattices:

$$\mathbb{U}(2) \oplus \mathbb{A}_1^+ \oplus \mathbb{A}_1^k \quad (2 \leq k \leq 8).$$

(1) If $g(M) = 1$, then $M^{\perp}$ is isometric to one of the following 9 lattices:

$$\mathbb{U} \oplus \mathbb{A}_1^+ \oplus \mathbb{A}_1^k \quad (2 \leq k \leq 9), \qquad \mathbb{U}(2) \oplus \mathbb{U}(2) \oplus \mathbb{D}_4.$$

(2) If $g(M) = 2$, then $M^{\perp}$ is isometric to one of the following 10 lattices:

$$\mathbb{U} \oplus \mathbb{U} \oplus \mathbb{A}_1^k \quad (1 \leq k \leq 9), \qquad \mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{D}_4.$$

(3) If $g(M) = 3$, then $M^{\perp}$ is isometric to one of the following 5 lattices:

$$\mathbb{U} \oplus \mathbb{U} \oplus \mathbb{D}_4 \oplus \mathbb{A}_1^k \quad (0 \leq k \leq 4).$$

After Theorem 9.3, we conjecture the following: If M satisfies Condition (1) in Theorem 9.1, then M satisfies Condition (2). In particular, if M satisfies Condition (1) in Theorem 9.1, then Eqs. (9.1) and (9.2) hold.

Theorem 9.5. *If $r(M) = 10$, $\delta(M) = 0$ and $M \not\cong \mathbb{U}(2) \oplus \mathbb{E}_8(2)$, then*

$$J^o_M(\Omega^o_{M^{\perp}}) \subset \theta_{\mathrm{null},g(M)}.$$

Proof. Assume $J^o_M(\Omega^o_{M^{\perp}}) \not\subset \theta_{\mathrm{null},g(M)}$. Since $\delta(M^{\perp}) = 0$,

$$\varphi := \Psi_{M^{\perp}}(\cdot, F_{M^{\perp}})^{2^{g(M)-1}(2^{g(M)}+1)\ell} \otimes (J_M^* \chi_{g(M)}^{8\ell})^{2^{g(M)}-1}$$

is an automorphic form on $\Omega^+_{M^{\perp}}$ for $O^+(M^{\perp})$ of weight $2^{g(M)-1}(2^{2g(M)} - 1)\ell(4, 4)$ by Theorem 8.1. Since $J^o_M(\Omega^o_{M^{\perp}}) \not\subset \theta_{\mathrm{null},g(M)}$, we get $\varphi \not\equiv 0$. Recall that $\nu = 2^{g(M)-1}(2^{g(M)} + 1)\ell$ in Theorem 5.1. Since

$$\psi := \varphi / \Phi_M^{2^{g(M)}-1}$$

is an $O^+(M^{\perp})$-invariant meromorphic function on $\Omega^+_{M^{\perp}}$, ψ extends to a meromorphic function on $\mathcal{M}^*_{M^{\perp}}$. We compute the divisor of ψ.

Since $\delta(M^{\perp}) = 0$, we get $\mathcal{D}''_{M^{\perp}} = \emptyset$ by Proposition 11.8 below. Since $r(M) = 10$ and $M \not\cong \mathbb{U}(2) \oplus \mathbb{E}_8(2)$, we get $g(M) > 0$ by Lemma 3.1. By Proposition 4.2 (1) and Theorem 8.1, we get

(9.13)
$$\begin{aligned}\mathrm{div}(\varphi) &= 2^{g(M)-1}(2^{g(M)}+1)\ell\,\mathcal{D}'_{M^{\perp}} + (2^{g(M)}-1)\{2(2^{2g(M)-2}+a)\ell\,\mathcal{D}'_{M^{\perp}} + E\}\\ &= \{2^{g(M)-1}(2^{2g(M)}+1) + 2a\,(2^{g(M)}-1)\}\ell\,\mathcal{D}'_{M^{\perp}} + (2^{g(M)}-1)\,E.\end{aligned}$$

By Theorem 5.1,

$$\mathrm{div}(\Phi_M) = \nu\,\mathcal{D}'_{M^{\perp}} = 2^{g(M)-1}(2^{g(M)}+1)\ell\,\mathcal{D}'_{M^{\perp}}. \tag{9.14}$$

By (9.13) and (9.14), we get

$$\begin{aligned}\mathrm{div}(\psi) &= \mathrm{div}(\varphi) - (2^{g(M)}-1)\,\mathrm{div}(\Phi_M)\\ &= \{2^{g(M)} + 2a\,(2^{g(M)}-1)\}\ell\,\mathcal{D}'_{M^{\perp}} + (2^{g(M)}-1)\,E.\end{aligned} \tag{9.15}$$

Since $\ell \geq 1$, $a \geq 0$ and since E is an effective divisor, $\mathrm{div}(\psi)$ is a *non-zero* effective divisor on $\Omega^{+}_{M^{\perp}}$ by (9.15). This contradicts the fact that ψ descends to a meromorphic function on $\mathcal{M}^{*}_{M^{\perp}}$. Since $J^{o}_{M}(\Omega^{o}_{M^{\perp}}) \not\subset \theta_{\mathrm{null},g(M)}$ yields a contradiction, we get the desired inclusion $J^{o}_{M}(\Omega^{o}_{M^{\perp}}) \subset \theta_{\mathrm{null},g(M)}$. □

Example 9.6. Let $(r(M),\delta(M),g(M)) = (10,0,2)$. Then $M \cong \mathbb{U} \oplus \mathbb{E}_8(2)$. Since $\theta_{\mathrm{null},2} = \mathcal{N}_2$ and $J_M(\Omega^{o}_{M^{\perp}}) \subset \mathcal{N}_2$ by Lemma 3.1 (2), we get $J_M(\Omega^{o}_{M^{\perp}}) \subset \theta_{\mathrm{null},2}$ in this case. This confirms Theorem 9.5 when $g(M) = 2$.

Example 9.7. Let $(r(M),\delta(M),g(M)) = (10,0,3)$. Then $M \cong \mathbb{U}(2) \oplus \mathbb{D}_4 \oplus \mathbb{D}_4$. In Proposition 12.2 below, we shall prove that if (X,ι) is a 2-elementary $K3$ surface of type $M = \mathbb{U}(2) \oplus \mathbb{D}_4 \oplus \mathbb{D}_4$ and if $\mathrm{Pic}(X) = M$, the non-rational component of X^{ι} is hyperelliptic. In this case, $J_M(X,\iota) = \Omega(X^{\iota}) \in \theta_{\mathrm{null},3}$. Since the periods of 2-elementary $K3$ surfaces of type $M = \mathbb{U}(2) \oplus \mathbb{D}_4 \oplus \mathbb{D}_4$ with Picard lattice M form a dense subset of $\mathcal{M}^{o}_{M^{\perp}}$ by e.g. [48, p.411], we get $J_M(\Omega^{o}_{M^{\perp}}) \subset \theta_{\mathrm{null},3}$. This confirms Theorem 9.5 when $g(M) = 3$.

Question 9.8. Is $\mathrm{div}(J^{*}_{M}\chi^{8\ell}_{g(M)})$ a linear combination of Heegner divisors on $\Omega^{+}_{M^{\perp}}$? If it is the case and if $M^{\perp} \cong \mathbb{U}^2 \oplus K$ for some K, $\Phi_M / J^{*}_{M}\chi^{8\ell}_{g(M)}$ will be expressed as a Borcherds product by [14, Th. 0.8]. Is there a Siegel modular form ψ on $\mathfrak{S}_{g(M)}$ such that $\mathrm{div}(J^{*}_{M}\psi)$ is a linear combination of Heegner divisors on $\Omega^{+}_{M^{\perp}}$?

10. An application to real $K3$ surfaces

In this section, we study the equivariant determinant of real $K3$ surfaces. We refer to [17], [62] for more details about real $K3$ surfaces.

The pair of a $K3$ surface and an anti-holomorphic involution is called a real $K3$ surface. Let (Y,σ) be a real $K3$ surface. There exists a primitive 2-elementary Lorentzian sublattice $M \subset \mathbb{L}_{K3}$ and a marking α of Y such that $\alpha\sigma^{*}\alpha^{-1} = I_M$. A holomorphic 2-form η on Y is defined over $\mathbf{R}$ if $\sigma^{*}\eta = \bar{\eta}$. Let γ be a σ-invariant Ricci-flat Kähler metric on Y with volume 1. Let $\Delta_{(Y,\gamma)}$ be the Laplacian of (Y,γ). Since σ preserves γ, $\Delta_{(Y,\gamma)}$ commutes with the σ-action on $C^{\infty}(Y)$. We define $C^{\infty}_{\pm}(Y) := \{f \in C^{\infty}(Y);\, \sigma^{*}f = \pm f\}$, which are preserved by $\Delta_{(Y,\gamma)}$. We set $\Delta_{(Y,\gamma),\pm} := \Delta_{(Y,\gamma)}|_{C^{\infty}_{\pm}(Y)}$. Let $\zeta_{\pm}(Y,\gamma)(s)$ denote the spectral zeta function of

$\Delta_{(Y,\gamma),\pm}$. Then it converges absolutely for $\mathrm{Re}\, s \gg 0$ and extends meromorphically to the complex plane $\mathbf{C}$, and it is holomorphic at $s = 0$. We define

$$\det{}^*_{\mathbf{Z}_2} \Delta_{(Y,\gamma)}(\sigma) := \exp[-\zeta'_+(Y,\gamma)(0) + \zeta'_-(Y,\gamma)(0)].$$

Let $Y(\mathbf{R}) := \{y \in Y;\, \sigma(y) = y\}$ be the set of real points of (Y,σ) and let $Y(\mathbf{R}) = \amalg_i C_i$ be the decomposition into the connected components. Then $Y(\mathbf{R})$ is the disjoint union of oriented two-dimensional manifolds. The Riemannian metric $g|_{Y(\mathbf{R})}$ induces a complex structure on $Y(\mathbf{R})$. The Jacobian variety of $Y(\mathbf{R})$ equipped with this complex structure is denoted by $\mathrm{Jac}(Y(\mathbf{R}), \gamma|_{Y(\mathbf{R})})$. Let $\Delta_{(C_i,\gamma|_{C_i})}$ be the Laplacian of the Riemannian manifold $(C_i, \gamma|_{C_i})$ and let $\zeta(C_i, \gamma|_{C_i})(s)$ denote the spectral zeta function of $\Delta_{(C_i,\gamma|_{C_i})}$. The regularized determinant of $\Delta_{(C_i,\gamma|_{C_i})}$ is defined as

$$\det{}^* \Delta_{(C_i,\gamma|_{C_i})} := \exp\left[-\zeta(C_i, \gamma|_{C_i})'(0)\right].$$

After [62, Def. 4.4], we define

$$\tau(Y,\sigma,\gamma) := \left\{\det{}^*_{\mathbf{Z}_2} \Delta_{(Y,\gamma)}(\sigma)\right\}^{-2} \prod_i \mathrm{Vol}(C_i, \gamma|_{C_i})\,(\det{}^*\Delta_{(C_i,\gamma|_{C_i})})^{-1}.$$

Theorem 10.1. *Let (Y,σ) be a real $K3$ surface and let α be a marking of Y such that $\alpha\sigma^*\alpha^{-1} = I_M$. Let γ be a σ-invariant Ricci-flat Kähler metric on Y with volume 1. Let ω_γ be the Kähler form of γ, and let η_γ be a holomorphic 2-form on Y defined over $\mathbf{R}$ such that $\eta_\gamma \wedge \bar{\eta}_\gamma = 2\omega_\gamma^2$. If M satisfies Conditions (1) and (2) in Theorem 9.1, then the following identity holds:*

$$-4(2^{g(M)}+1)\,\log\tau(Y,\sigma,\gamma) = \log\|\Psi_{M^\perp}(\alpha(\omega_\gamma + \sqrt{-1}\mathrm{Im}\,\eta_\gamma), F_{M^\perp})\|^2 + 2^{(4-g(M))}\,\log\|\chi_{g(M)}(\Omega(Y(\mathbf{R}), \gamma|_{Y(\mathbf{R})}))\|^2 + C'_M,$$

where $C'_M = 2\log C_M$ and ω_γ, η_γ are identified with their cohomology classes.

Proof. The result follows from Theorem 9.1 and [62, Lemma 4.5 Eq. (4.6)]. □

11. The irreducible components of the discriminant locus

In this section, we prove some technical results concerning lattices used in earlier sections and we give a formula for the number of the irreducible components of $\mathcal{D}_{M^\perp}$. We use Nikulin's theory of discriminant forms, for which we refer to [43].

11.1. A proof of the equality $\Gamma_M = O(M^\perp)$

Let M be a primitive sublattice of $\mathbb{L}_{K3}$ and set $H_M := \mathbb{L}_{K3}/(M \oplus M^\perp)$. Since $\mathbb{L}_{K3}$ is unimodular, we get $M \oplus M^\perp \subset \mathbb{L}_{K3} = \mathbb{L}_{K3}^\vee \subset M^\vee \oplus (M^\perp)^\vee$, so that $H_M \subset A_M \oplus A_{M^\perp}$. Let $p_1\colon H_M \to A_M$ and $p_2\colon H_M \to A_{M^\perp}$ be the homomorphism induced by the projections $A_M \oplus A_{M^\perp} \to A_M$ and $A_M \oplus A_{M^\perp} \to A_{M^\perp}$, respectively. By [43, Props. 1.5.1 and 1.6.1], the following are known:

(a) p_1 and p_2 are isomorphisms.
(b) $A_M \cong A_{M^\perp}$ via the isomorphism $\gamma^{\mathbb{L}_{K3}}_{M,M^\perp} := p_2 \circ p_1^{-1}$.
(c) $q_{M^\perp} \circ \gamma^{\mathbb{L}_{K3}}_{M,M^\perp} = -q_M$.

Recall that $g \in O(M^\perp)$ induces $\overline{g} \in O(q_{M^\perp})$. For $g \in O(M^\perp)$, we set $\psi_g := (\gamma^{\mathbb{L}_{K3}}_{M,M^\perp})^{-1} \circ \overline{g} \circ \gamma^{\mathbb{L}_{K3}}_{M,M^\perp}$. Then $\psi_g \in \mathrm{Aut}(A_M)$.

Lemma 11.1. *The automorphism ψ_g preserves q_M, i.e., $\psi_g \in O(q_M)$.*

Proof. For an arbitrary $\overline{m} \in A_M$, we get

$$\begin{aligned} q_M(\psi_g(\overline{m})) &= q_M((\gamma^{\mathbb{L}_{K3}}_{M,M^\perp})^{-1} \circ \overline{g} \circ \gamma^{\mathbb{L}_{K3}}_{M,M^\perp}(\overline{m})) \\ &= -q_{M^\perp}(\overline{g} \circ \gamma^{\mathbb{L}_{K3}}_{M,M^\perp}(\overline{m})) \\ &= -q_{M^\perp}(\gamma^{\mathbb{L}_{K3}}_{M,M^\perp}(\overline{m})) = q_M(\overline{m}), \end{aligned}$$

where the second and the last equalities follow from Condition (c) and the third equality follows from the fact $\overline{g} \in O(q_{M^\perp})$. □

Assume that $M \subset \mathbb{L}_{K3}$ is a primitive 2-elementary Lorentzian sublattice. Recall that the isometry $I_M \in O(\mathbb{L}_{K3})$ was defined in Sect. 2.2. In [59, Sect. 1.4 (c)], we introduced the following subgroup $\Gamma_M \subset O(M^\perp)$:

$$\Gamma_M := \{g|_{M^\perp} \in O(M^\perp);\ g \in O(\mathbb{L}_{K3}),\ g \circ I_M = I_M \circ g\}.$$

Proposition 11.2. *The following equality holds:*

$$\Gamma_M = O(M^\perp).$$

Proof. By the definition of Γ_M, it suffices to prove $O(M^\perp) \subset \Gamma_M$. Let $g \in O(M^\perp)$ be an arbitrary element. Since M is 2-elementary and indefinite, the natural homomorphism $O(M) \to O(q_M)$ is surjective by [43, Th. 3.6.3], which implies the existence of $\Psi_g \in O(M)$ with $\psi_g = \overline{\Psi_g}$. Define $\widetilde{g} := \Psi_g \oplus g \in O(M \oplus M^\perp)$. Then

$$\gamma^{\mathbb{L}_{K3}}_{M,M^\perp} \circ \overline{\Psi_g} = \gamma^{\mathbb{L}_{K3}}_{M,M^\perp} \circ \psi_g = \overline{g} \circ \gamma^{\mathbb{L}_{K3}}_{M,M^\perp}. \tag{11.1}$$

By (11.1) and the criterion of Nikulin [43, Cor. 1.5.2], we get $\widetilde{g} \in O(\mathbb{L}_{K3})$. We have $\widetilde{g} \circ I_M = I_M \circ \widetilde{g}$ on $M \oplus M^\perp$ because for all $(m, n) \in M \oplus M^\perp$,

$$\widetilde{g} \circ I_M(m, n) = \widetilde{g}(m, -n) = (\Psi_g(m), -g(n)) = I_M(\Psi_g(m), g(n)) = I_M \circ \widetilde{g}(m, n).$$

Since $M \oplus M^\perp$ linearly spans $\mathbb{L}_{K3} \otimes \mathbf{Q}$, we have $\widetilde{g} \circ I_M = I_M \circ \widetilde{g}$ in $O(\mathbb{L}_{K3})$. Hence $\widetilde{g} \in \Gamma_M$. This proves the inclusion $O(M^\perp) \subset \Gamma_M$. □

11.2. A formula for $g([M \perp d])$

Lemma 11.3. *Let $d \in \Delta_{M^\perp}$. The smallest primitive 2-elementary Lorentzian sublattice of $\mathbb{L}_{K3}$ containing $M \oplus \mathbf{Z}d$ is given by $[M \perp d] = (M^\perp \cap d^\perp)^\perp$.*

Proof. Set $L := \mathbf{Z}d \cong \mathbb{A}_1$. Then $[M \perp d]$ is the smallest primitive Lorentzian sublattice of $\mathbb{L}_{K3}$ containing $M \oplus L$. Since $M \oplus L \subset [M \perp d] \subset [M \perp d]^\vee \subset M^\vee \oplus L^\vee$ and hence $[M \perp d]/(M \oplus L) \subset [M \perp d]^\vee/(M \oplus L) \subset A_M \oplus A_L \cong \mathbf{Z}_2^{l(M)+1}$, $A_{[M \perp d]} = [M \perp d]^\vee/[M \perp d]$ is a vector space over $\mathbf{Z}_2$. □

Lemma 11.4. *Let $d \in \Delta_{M^\perp}$. Then*

$$l(M^\perp \cap d^\perp) = \begin{cases} l(M^\perp) + 1 & \text{if} \quad d \in \Delta'_{M^\perp} \\ l(M^\perp) - 1 & \text{if} \quad d \in \Delta''_{M^\perp}. \end{cases}$$

Proof. Set $\Lambda := M^\perp$, $N := M^\perp \cap d^\perp$, $L := \mathbf{Z}d \cong \mathbb{A}_1$, and $S := \Lambda/(N \oplus L)$. The inclusions of lattices $N \oplus L \subset \Lambda \subset \Lambda^\vee \subset N^\vee \oplus L^\vee$ yields that

$$S = \Lambda/(N \oplus L) \subset \Lambda^\vee/(N \oplus L) \subset A_N \oplus A_L. \tag{11.2}$$

Let $S^\perp := \{v \in A_N \oplus A_L;\ b_{N \oplus L}(v, s) \equiv 0 \mod \mathbf{Z},\ \forall\, s \in S\}$ be the orthogonal complement of S with respect to the discriminant bilinear form $b_{N \oplus L}$. Since $S^\perp = \Lambda^\vee/(N \oplus L)$, we get $A_{M^\perp} = \Lambda^\vee/\Lambda = S^\perp/S$ by (11.2). Since $b_{N \oplus L}$ is non-degenerate,

$$l(M^\perp) = \dim_{\mathbf{Z}_2} S^\perp/S = l(N) + 1 - 2\dim_{\mathbf{Z}_2} S = l(M^\perp \cap d^\perp) + 1 - 2\dim_{\mathbf{Z}_2} S. \tag{11.3}$$

Let $x \in M^\perp$. Since $x = (x + \frac{\langle x,d\rangle}{2} d) - \frac{\langle x,d\rangle}{2} d$ and since $x + \frac{\langle x,d\rangle}{2} d \in (M^\perp \cap d^\perp) \otimes \mathbf{Q}$ is orthogonal to $\frac{\langle x,d\rangle}{2} d \in \mathbf{Q}\, d$, we get

$$
\begin{aligned}
S = 0 \iff & \quad M^\perp = (M^\perp \cap d^\perp) \oplus \mathbf{Z}\, d \\
\iff & \quad \langle x, d\rangle \equiv 0 \mod 2 \quad (\forall\, x \in M^\perp) \\
\iff & \quad \langle x, d/2\rangle \in \mathbf{Z} \quad (\forall\, x \in M^\perp) \\
\iff & \quad d/2 \in (M^\perp)^\vee \quad \iff \quad d \in \Delta''_{M^\perp}.
\end{aligned}
\tag{11.4}
$$

Let $d \in \Delta''_{M^\perp}$. By (11.4), we get $M^\perp = (M^\perp \cap d^\perp) \oplus \mathbf{Z}d$, which yields that $l(M^\perp \cap d^\perp) = l(M^\perp) - 1$. This proves the assertion when $d \in \Delta''_{M^\perp}$.

Let $d \in \Delta'_{M^\perp}$. We prove $\dim_{\mathbf{Z}_2} S = 1$. Let $p_1 \colon S \to A_L$ and $p_2 \colon S \to A_N$ be the natural projections. Since L and N are *primitive* sublattices of Λ, p_1 and p_2 are injective. If $p_1(S) = 0$, then the injectivity of p_1 implies $S = 0$, which contradicts $\dim_{\mathbf{Z}_2} S > 0$. Hence $p_1(S) \neq 0$. Since $A_L \cong \mathbf{Z}_2$, we get $p_1(S) = A_L$, so that $p_1 \colon S \to A_L$ is an isomorphism. We set $n_d := p_2 \circ p_1^{-1}(d/2) \in A_N$. Then

$$
S = \mathbf{Z}_2(d/2, n_d) \subset A_L \oplus A_N. \tag{11.5}
$$

By (11.5), we get $\dim_{\mathbf{Z}_2} S = 1$ and hence $l(M^\perp \cap d^\perp) = l(M^\perp) + 1$ by (11.3). This proves the assertion when $d \in \Delta'_{M^\perp}$. □

Lemma 11.5. *Let $d \in \Delta_{M^\perp}$. Then*

$$
g([M \perp d]) = \begin{cases} g(M) - 1 & \text{if} \quad d \in \Delta'_{M^\perp} \\ g(M) & \text{if} \quad d \in \Delta''_{M^\perp}. \end{cases}
$$

Proof. Since $r(M^\perp \cap d^\perp) = r(M^\perp) - 1$ and

$$
g(M) = \{r(M^\perp) - l(M^\perp)\}/2, \qquad g([M \perp d]) = \{r(M^\perp \cap d^\perp) - l(M^\perp \cap d^\perp)\}/2,
$$

the result follows from Lemma 11.4. □

11.3. The number of the irreducible components of $\mathcal{D}''_\Lambda/O(\Lambda)$

In Sects. 11.3 and 11.4, we assume that Λ *is a primitive* 2*-elementary sublattice of* $\mathbb{L}_{K3}$ *with* $\operatorname{sign}(\Lambda) = (2, r(\Lambda) - 2)$.

Proposition 11.6. $\Delta''_\Lambda = \emptyset$ *if and only if one of the following* (1) *or* (2) *holds:*

(1) $\delta(\Lambda) = 0$
(2) $(\delta(\Lambda), r(\Lambda), l(\Lambda)) = (1, 2, 2),\ (1, 3, 1),\ (1, 10, 2),\ (1, 11, 1),\ (1, 18, 2),\ (1, 19, 1)$.

Proof. If $\delta(\Lambda) = 0$ and $\Delta''_\Lambda \neq \emptyset$, there exists $d \in \Delta''_\Lambda$. Since $d/2 \in \Lambda^\vee$ and $(d/2)^2 = -1/2 \notin \mathbf{Z}$, we get the contradiction $\delta(\Lambda) = 1$. Hence $\delta(\Lambda) = 0$ implies $\Delta''_\Lambda = \emptyset$.

Assume that $\delta(\Lambda) = 1$ and $\Delta''_\Lambda = \emptyset$. If $t > 0$ in [20, Appendix, Table 2], then we get $\Delta''_\Lambda \neq \emptyset$ because $\mathbb{A}_1$ is a direct summand in this case. If $t = 0$ and $\delta(\Lambda) = 1$, Λ must be isometric to one of the following lattices by [20, Appendix, Table 2]:

$$
(\mathbb{A}_1^+)^{\oplus 2},\ \mathbb{A}_1^+ \oplus \mathbb{U},\ (\mathbb{A}_1^+)^{\oplus 2} \oplus \mathbb{E}_8,\ \mathbb{A}_1^+ \oplus \mathbb{U} \oplus \mathbb{E}_8,\ (\mathbb{A}_1^+)^{\oplus 2} \oplus \mathbb{E}_8^{\oplus 2},\ \mathbb{A}_1^+ \oplus \mathbb{U} \oplus \mathbb{E}_8^{\oplus 2}. \tag{11.6}
$$

We see that $\Delta''_\Lambda = \emptyset$ for the lattices (11.6). Let Λ be one of the above six lattices. Then we can write $\Lambda = (\mathbb{A}_1^+)^{\oplus k} \oplus L$, where $k = 1, 2$ and L is an even unimodular lattice. If $d \in \Delta''_\Lambda$, write $d = (u, v)$ with $u \in (\mathbb{A}_1^+)^{\oplus k}$ and $v \in L$. Since $d/2 \in \Lambda^\vee$, we get $w := v/2 \in L$, so that $d = (u, 2w)$. Since $-2 = d^2 = u^2 + 4w^2 \equiv u^2 \mod 8$, we get $u^2 \equiv 6 \mod 8$. On the other hand, since $a^2 \equiv 0, 2 \mod 8$ for all $a \in \mathbb{A}_1^+$

and since $k = 1, 2$, we get $u^2 \equiv 0, 2, 4 \mod 8$, which contradicts $u^2 \equiv 6 \mod 8$. Hence $\Delta''_\Lambda = \emptyset$ for the lattices (11.6). $\square$

Proposition 11.7. *The following holds:*

$$\#(\Delta''_\Lambda/O(\Lambda)) = \#\{\delta(d^\perp \cap \Lambda) \in \mathbf{Z}_2;\ d \in \Delta''_\Lambda\} \leq 2.$$

Proof. Let $d, d' \in \Delta''_\Lambda$. Then $l(d^\perp \cap \Lambda) = l((d')^\perp \cap \Lambda)$ by Lemma 11.4. We prove that d and d' lie on the same $O(\Lambda)$-orbit if and only if $\delta(d^\perp \cap \Lambda) = \delta((d')^\perp \cap \Lambda)$.

Assume $\delta(d^\perp \cap \Lambda) = \delta((d')^\perp \cap \Lambda)$. By [43, Th. 3.6.2], there exists an isometry $\varphi\colon d^\perp \cap \Lambda \cong (d')^\perp \cap \Lambda$. Since $\Lambda = \mathbf{Z}d \oplus (d^\perp \cap \Lambda) = \mathbf{Z}d' \oplus ((d')^\perp \cap \Lambda)$ by (11.4), we get an element $\gamma \in O(\Lambda)$ with $\gamma(d) = d'$ by defining

$$\gamma\colon \mathbf{Z}d \oplus (d^\perp \cap \Lambda) \ni (\nu\, d, \lambda) \to (\nu\, d', \varphi(\lambda)) \in \mathbf{Z}d' \oplus ((d')^\perp \cap \Lambda).$$

Conversely, assume the existence of $\gamma \in O(\Lambda)$ with $\gamma(d) = d'$. Since $\gamma(d^\perp \cap \Lambda) = (d')^\perp \cap \Lambda$, we get $\delta(d^\perp \cap \Lambda) = \delta((d')^\perp \cap \Lambda)$. This proves the assertion. $\square$

Proposition 11.8. *Set* $N''(\Lambda) := \#(\Delta''_\Lambda/O(\Lambda))$. *Then*

$$N''(\Lambda) = \begin{cases} 0 & \text{if} \quad \delta = 0 \\ 0 & \text{if} \quad \delta = 1,\ (r, l) = (2,2), (3,1), (10,2), (11,1), (18,2), (19,1) \\ 2 & \text{if} \quad \delta = 1,\ (r, l) = (5,3), (5,5), (9,5), (9,7), (13,3), (13,5), \\ & \qquad\qquad\qquad\quad (13,7), (13,9), (17,5) \\ 1 & \text{if} \quad (\delta, r, l) : \text{otherwise} \end{cases}$$

Proof. The first two equalities follow from Proposition 11.6. Set $r = r(\Lambda)$ and $l = l(\Lambda)$. Assume that $\#(\Delta''_\Lambda/O(\Lambda)) = 2$. Since $\Delta''_\Lambda \neq \emptyset$, we get $\delta(\Lambda) = 1$ by Proposition 11.6. By Proposition 11.7 and (11.4), $\#(\Delta''_\Lambda/O(\Lambda)) = 2$ if and only if there exist 2-elementary lattices L, L' of signature $(2, r-3)$ with $(\delta(L), r(L), l(L)) = (1, r-1, l-1)$ and $(\delta(L'), r(L'), l(L')) = (0, r-1, l-1)$ such that $\Lambda \cong L \oplus \mathbb{A}_1 \cong L' \oplus \mathbb{A}_1$. Namely, $(r-1, l-1)$ satisfies the following property:

- **(P)** Both of $(0, r-1, l-1)$ and $(1, r-1, l-1)$ are realized by primitive 2-elementary sublattices of $\mathbb{L}_{K3}$ with signature $(2, r-3)$.

In view of the table [45, p.1434, Table 1] of 2-elementary sublattices of $\mathbb{L}_{K3}$ (cf. Remark 11.9 below), the pair $(r-1, l-1)$ with property **(P)** is one of the following:

$$(4,2), (4,4), (8,4), (8,6), (12,2), (12,4), (12,6), (12,8), (12,10), (16,4), (20,2),$$

so that the possible pairs of $(r(\Lambda), l(\Lambda))$ are given as follows:

$$(5,3), (5,5), (9,5), (9,7), (13,3), (13,5), (13,7), (13,9), (13,11), (17,5), (21,3).$$

By [45, p.1434, Table 1], there are no primitive 2-elementary sublattices of $\mathbb{L}_{K3}$ with invariants $(1, 13, 11)$, $(1, 21, 3)$, and all other triplets (δ, r, l) are realized by primitive 2-elementary sublattices of $\mathbb{L}_{K3}$. This proves the result. $\square$

Remark 11.9. In [45, p.1434, Table 1], the triplets (δ, r, l) are considered for primitive 2-elementary *Lorentzian* sublattices of $\mathbb{L}_{K3}$. To get a table of the triplets (δ, r, l) for primitive 2-elementary sublattices of $\mathbb{L}_{K3}$ with *signature* $(2, r-2)$, we must replace r by $22 - r$ in [45, p.1434, Table 1], because they are always the orthogonal complement of a primitive 2-elementary Lorentzian sublattice of $\mathbb{L}_{K3}$.

11.4. The number of the irreducible components of $\mathcal{D}'_\Lambda/O(\Lambda)$

Proposition 11.10. *$\Delta'_\Lambda = \emptyset$ if and only if $r(\Lambda) = l(\Lambda)$.*

Proof. If $r(\Lambda) = l(\Lambda)$, then $\lambda/2 \in \Lambda^\vee$ for all $\lambda \in \Lambda$, so that $\Delta'_\Lambda = \emptyset$. Conversely, assume $\Delta'_\Lambda = \emptyset$. If Λ contains $\mathbb{U}$, $\mathbb{D}_{2k}$, $\mathbb{E}_7$, $\mathbb{E}_8$, then $\Delta'_\Lambda \neq \emptyset$. This, compared with [20, Appendix, Tables 2,3], yields that Λ must be isometric to $\mathbb{I}_{2,k}(2)$ $(0 \leq k \leq 10)$ or $\mathbb{U}(2) \oplus \mathbb{U}(2)$, so that $r(\Lambda) = l(\Lambda)$. This proves the result. $\square$

Lemma 11.11. *Assume that $\Delta'_\Lambda \neq \emptyset$. If $d \in \Delta'_\Lambda$, then*

$$r(d^\perp \cap \Lambda) = r(\Lambda) - 1, \qquad l(d^\perp \cap \Lambda) = l(\Lambda) + 1, \qquad \delta(d^\perp \cap \Lambda) = 1.$$

In particular, the isometry class of $d^\perp \cap \Lambda$ is independent of the choice of $d \in \Delta'_\Lambda$.

Proof. We set $L := \mathbf{Z}d \cong \mathbb{A}_1$, $N := d^\perp \cap \Lambda$, and $S := \Lambda/(L \oplus N)$. By (11.2), S is a vector space over $\mathbf{Z}_2$ with $S \subset A_L \oplus A_N$. Since $S \neq 0$ by (11.4), we get $l(N) = l(\Lambda) + 1$ by Lemma 11.4. We prove $\delta(N) = 1$.

By (11.5), there exists a unique $n_d \in A_N$ such that $S = \mathbf{Z}_2(d/2, n_d) \subset A_L \oplus A_N$. Since $(d/2, n_d) \in S = \Lambda/(L \oplus N)$, we get $q_L(d/2) + q_N(n_d) \equiv 0 \mod 2\mathbf{Z}$. Namely,

$$q_N(n_d) \equiv -q_L(d/2) \equiv 1/2 \mod 2\mathbf{Z}. \tag{11.7}$$

Since $n_d \in A_N$, we get $\delta(N) = 1$ by (11.7). This proves the lemma. $\square$

If $\Delta'_\Lambda \neq \emptyset$, we define the 2-elementary lattice $\partial\Lambda$ as

$$\partial\Lambda := d^\perp \cap \Lambda, \qquad d \in \Delta'_\Lambda,$$

whose isometry class is independent of the choice of $d \in \Delta'_\Lambda$ by Lemma 11.5. We set

$$B_{\partial\Lambda} := \{v \in A_{\partial\Lambda};\, q_{\partial\Lambda}(v) \equiv 1/2 \mod 2\mathbf{Z}\}.$$

Let $p\colon \mathbb{A}_1^\vee \oplus (\partial\Lambda)^\vee \to A_{\mathbb{A}_1} \oplus A_{\partial\Lambda}$ be the projection. For $\mu \in B_{\partial\Lambda}$, set

$$S_\mu := \mathbf{Z}_2(d/2, \mu) \subset A_{\mathbb{A}_1} \oplus A_{\partial\Lambda}, \qquad \Lambda_\mu := p^{-1}(S_\mu) \subset \mathbb{A}_1^\vee \oplus (\partial\Lambda)^\vee.$$

Then Λ_μ is equipped with the bilinear form induced from the one on $\mathbb{A}_1^\vee \oplus (\partial\Lambda)^\vee$.

Lemma 11.12. *Λ_μ is an even 2-elementary lattice with $\mathrm{sign}(\Lambda_\mu) = (2, r(\Lambda) - 2)$ and $l(\Lambda_\mu) = l(\Lambda)$.*

Proof. Let $S_\mu^\perp \subset A_{\mathbb{A}_1} \oplus A_{\partial\Lambda}$ be the orthogonal complement of S_μ in $A_{\mathbb{A}_1} \oplus A_{\partial\Lambda}$ with respect to the discriminant bilinear form $b_{\mathbb{A}_1} \oplus b_{\partial\Lambda}$. Since S_μ is an isotropic subspace of $A_{\mathbb{A}_1} \oplus A_{\partial\Lambda}$ by the condition $\mu \in B_{\partial\Lambda}$, Λ_μ is an even integral lattice. Since $\Lambda_\mu^\vee = p^{-1}(S_\mu^\perp)$ and since $p\colon \mathbb{A}_1^\vee \oplus (\partial\Lambda)^\vee \to A_{\mathbb{A}_1} \oplus A_{\partial\Lambda}$ is surjective,

$$A_{\Lambda_\mu} = \Lambda_\mu^\vee/\Lambda_\mu = p^{-1}(S_\mu^\perp)/p^{-1}(S_\mu) = S_\mu^\perp/S_\mu \subset (A_{\mathbb{A}_1} \oplus A_{\partial\Lambda})/S_\mu$$

is a vector space over $\mathbf{Z}_2$, so that Λ_μ is 2-elementary. By (11.3) and Lemma 11.11, $l(\Lambda_\mu) = \dim_{\mathbf{Z}_2} A_{\Lambda_\mu} = l(\partial\Lambda) + 1 - 2\dim_{\mathbf{Z}_2} S_\mu = l(\Lambda)$. Since $\mathbb{A}_1 \oplus \partial\Lambda \subset \Lambda_\mu \subset \mathbb{A}_1^\vee \oplus (\partial\Lambda)^\vee$, we get $\mathrm{sign}(\Lambda_\mu) = (2, r(\Lambda) - 2)$. This proves the lemma. $\square$

Lemma 11.13. *Let $\mu \in B_{\partial\Lambda}$. Then*

$$(r(\Lambda_\mu), l(\Lambda_\mu), \delta(\Lambda_\mu)) = \begin{cases} (r(\Lambda), l(\Lambda), 1) & \text{if} \quad \mu \neq \mathbf{1}_{\partial\Lambda} \\ (r(\Lambda), l(\Lambda), 0) & \text{if} \quad \mu = \mathbf{1}_{\partial\Lambda}. \end{cases}$$

Proof. By Lemma 11.12, $r(\Lambda_\mu) = r(\Lambda)$ and $l(\Lambda_\mu) = l(\Lambda)$. It suffices to prove that $\delta(\Lambda_\mu) = 0$ if and only if $\mathbf{1}_{\partial\Lambda} \in B_{\partial\Lambda}$ and $\mu = \mathbf{1}_{\partial\Lambda}$. Let $x = (b\frac{d}{2}, \nu) \in S_\mu^\perp$, $b \in \mathbf{Z}$, $\nu \in A_{\partial\Lambda}$ and let $(\frac{d}{2}, \mu) \in S_\mu$. We get $x \cdot (\frac{d}{2}, \mu) \equiv -\frac{b}{2} + \nu \cdot \mu \equiv 0 \mod \mathbf{Z}$. Since

$$x^2 \equiv -\frac{b^2}{2} + \nu^2 \equiv -\frac{b}{2} + \nu \cdot \mathbf{1}_{\partial\Lambda} \equiv \nu \cdot (\mathbf{1}_{\partial\Lambda} - \mu) \equiv x \cdot (0, \mathbf{1}_{\partial\Lambda} - \mu) \mod \mathbf{Z}$$

for all $x \in S_\mu^\perp$ and since $\Lambda_\mu^\vee = p^{-1}(S_\mu^\perp)$, we get the result as follows:

$$\begin{aligned}\delta(\Lambda_\mu) = 0 \iff & \quad q_{\mathbb{A}_1 \oplus \partial\Lambda}(x) \equiv 0 \mod \mathbf{Z} \quad (\forall\, x \in S_\mu^\perp) \\ \iff & \quad b_{\mathbb{A}_1 \oplus \partial\Lambda}(x, (0, \mathbf{1}_{\partial\Lambda} - \mu)) \equiv 0 \mod \mathbf{Z} \quad (\forall\, x \in S_\mu^\perp) \\ \iff & \quad (0, \mathbf{1}_{\partial\Lambda} - \mu) \in (S_\mu^\perp)^\perp = S_\mu = \mathbf{Z}_2(d/2, \mu) \\ \iff & \quad \mu = \mathbf{1}_{\partial\Lambda}.\end{aligned}$$

This proves the lemma. □

Lemma 11.14. *Let $\mu \in B_{\partial\Lambda}$. Then $\Lambda \cong \Lambda_\mu$ if and only if*

$$\mu \begin{cases} \neq \mathbf{1}_{\partial\Lambda} & \text{if} \quad \delta(\Lambda) = 1 \\ = \mathbf{1}_{\partial\Lambda} & \text{if} \quad \delta(\Lambda) = 0. \end{cases}$$

Proof. Since the isometry class of the indefinite 2-elementary lattice Λ is determined by the triplet $(\mathrm{sign}(\Lambda), l(\Lambda), \delta(\Lambda))$, the result follows from Lemma 11.13. □

Proposition 11.15. *Set $N'(\Lambda) := \#(\Delta'_\Lambda/O(\Lambda))$. Then*

$$N'(\Lambda) = \begin{cases} 0 & \text{if} \quad r(\Lambda) = l(\Lambda) \\ 1 & \text{if} \quad r(\Lambda) > l(\Lambda). \end{cases}$$

Proof. Let $d, d' \in \Delta'_\Lambda$. We set $N = d^\perp \cap \Lambda$, $N' = (d')^\perp \cap \Lambda$, $L = \mathbf{Z}d$ and $L' = \mathbf{Z}d'$. By Lemma 11.11, there exist isometries $\beta\colon N \cong \partial\Lambda$ and $\beta'\colon N' \cong \partial\Lambda$. By (11.5), (11.7), there exist unique $n_d \in A_N$, $n_{d'} \in A_{N'}$ such that $\bar\beta(n_d), \bar\beta'(n_{d'}) \in B_{\partial\Lambda}$ and

$$(11.8) \qquad \Lambda/(L \oplus N) = \mathbf{Z}_2(d/2, n_d), \qquad \Lambda/(L' \oplus N') = \mathbf{Z}_2(d'/2, n_{d'}).$$

By (11.8) and the definition of Λ_μ, $\bar\beta(n_d), \bar\beta'(n_{d'}) \in B_{\partial\Lambda}$ are such that

$$(11.9) \qquad \Lambda_{\bar\beta(n_d)} \cong \Lambda_{\bar\beta'(n_{d'})} \cong \Lambda.$$

By [43, Cor. 1.5.2] and (11.8), there exists $\gamma \in O(\Lambda)$ with $\gamma(d) = d'$ if and only if there exists an isometry $\alpha\colon N \cong N'$ with $\bar\alpha(n_d) = n_{d'}$, where $\bar\alpha\colon (A_N, q_{A_N}) \to (A_{N'}, q_{A_{N'}})$ is the isometry induced by α. Equivalently, d and d' lie on the same $O(\Lambda)$-orbit if and only if there exists $g \in O(\partial\Lambda)$ with $\bar g(\bar\beta(n_d)) = \bar\beta'(n_{d'})$. Since the natural homomorphism $O(\partial\Lambda) \to O(q_{\partial\Lambda})$ is surjective by [43, Th. 3.6.3], d and d' lie on the same $O(\Lambda)$-orbit if and only if $\bar\beta(n_d), \bar\beta'(n_{d'}) \in B_{\partial\Lambda}$ lie on the same $O(q_{\partial\Lambda})$-orbit. This implies that

$$\begin{aligned}\#(\Delta'_\Lambda/O(\Lambda)) &\leq \#\,[\{\mu \in B_{\partial\Lambda};\ \Lambda_\mu \cong \Lambda\}/O(q_{\partial\Lambda})] \\ &= \begin{cases} \#\,[\{\mu \in B_{\partial\Lambda};\ \mu \neq \mathbf{1}_{\partial\Lambda}\}/O(q_{\partial\Lambda})] & \text{if} \quad \delta(\Lambda) = 1 \\ \#\,[\{\mu \in B_{\partial\Lambda};\ \mu = \mathbf{1}_{\partial\Lambda}\}/O(q_{\partial\Lambda})] & \text{if} \quad \delta(\Lambda) = 0 \end{cases} \qquad (11.10) \\ &= 1,\end{aligned}$$

where the first inequality follows from (11.9), the second equality follows from Lemma 11.14 and the last equality follows from [43, Lemma 3.9.1]. The result follows from Proposition 11.10 and (11.10). □

Theorem 11.16. *Set $N(\Lambda) := \#(\Delta_\Lambda/O(\Lambda))$. Then*

$$N(\Lambda) = \begin{cases} 0 & \text{if} \quad (\delta, r, l) = (0,4,4), (1,2,2) \\ 1 & \text{if} \quad \delta = 0,\ r > l \\ 1 & \text{if} \quad \delta = 1,\ r = l,\ r > 2,\ r \neq 5 \\ 1 & \text{if} \quad \delta = 1,\ (r,l) = (3,1), (10,2), (11,1), (18,2), (19,1) \\ 3 & \text{if} \quad \delta = 1,\ (r,l) = (5,3), (9,5), (9,7), (13,3), (13,5), \\ & \qquad\qquad\qquad (13,7), (13,9), (17,5) \\ 2 & \text{if} \quad (\delta, r, l) : \text{otherwise} \end{cases}$$

Proof. Since $N(\Lambda) = N'(\Lambda) + N''(\Lambda)$, the result follows from Propositions 11.8 and 11.15. □

Corollary 11.17. *Let $M \subset \mathbb{L}_{K3}$ be a primitive 2-elementary Lorentzian sublattice. Let $\mathcal{N}(M)$ be the number of the irreducible components of the divisor $\overline{\mathcal{D}}_{M^\perp} = \mathcal{D}_{M^\perp}/O(M^\perp)$ on $\mathcal{M}_{M^\perp}$. Then*

$$\mathcal{N}(M) = \begin{cases} 0 & \text{if} \quad (\delta, r, l) = (0,18,4), (1,20,2) \\ 1 & \text{if} \quad \delta = 0,\ r + l < 22 \\ 1 & \text{if} \quad \delta = 1,\ r + l = 22,\ r < 20,\ r \neq 17 \\ 1 & \text{if} \quad \delta = 1,\ (r,l) = (19,1), (12,2), (11,1), (4,2), (3,1) \\ 3 & \text{if} \quad \delta = 1,\ (r,l) = (17,3), (13,5), (13,7), (9,3), (9,5), \\ & \qquad\qquad\qquad (9,7), (9,9), (5,5) \\ 2 & \text{if} \quad (\delta, r, l) : \text{ otherwise.} \end{cases}$$

Proof. Since $\mathcal{D}_{M^\perp} = \sum_{d \in \Delta_{M^\perp}} H_d$ and since $\overline{H}_d = \overline{H}_{d'}$ if and only if d and d' lie on the same $O(M^\perp)$-orbit, the set of irreducible components of $\overline{\mathcal{D}}_{M^\perp}$ is identified with $\Delta_{M^\perp}/O(M^\perp)$, so that $\mathcal{N}(M) = N(M^\perp)$. We get the result by Theorem 11.16. □

Remark 11.18. Let $M \subset \mathbb{L}_{K3}$ be a primitive 2-elementary Lorentzian sublattice. Let $[M]$ be the isometry class of M, which corresponds to a vertex of the $K3$-graph Γ_{K3} of Finashin-Kharlamov [20, Figure 1]. Comparing Γ_{K3} and Corollary 11.17, we see that $\mathcal{N}(M)$ is exactly the number of edges in Γ_{K3} going out $[M]$. In Γ_{K3}, an odd edge going out $[M]$ corresponds to the set $\Delta'_{M^\perp}/O(M^\perp)$, even non-Wu edge going out $[M]$ corresponds to the $O(M^\perp)$-orbit of a root $d \in \Delta''_{M^\perp}$ with $\delta(d^\perp \cap M^\perp) = 1$, and an even Wu edge going out $[M]$ corresponds to the $O(M^\perp)$-orbit of a root $d \in \Delta''_{M^\perp}$ with $\delta(d^\perp \cap M^\perp) = 0$. By Theorem 3.3, $[M]$ and $[M']$ are connected by an oriented edge from $[M]$ to $[M']$ if and only if there exist $g \in O(\mathbb{L}_{K3})$ and $d \in \Delta_{M^\perp}$ with $g(M') = [M \perp d]$, i.e., $\mathcal{M}_{g(M')^\perp}$ is an irreducible component of $\mathcal{D}_{M^\perp}$.

12. Appendix: Some geometric properties of the set of fixed points

In this section, we prove some geometric properties of the set of fixed points of 2-elementary $K3$ surfaces used in earlier sections. The proof of the main results of this section, Propositions 12.2 and 12.3, have been suggested by S. Kondō.

Let S be a compact complex smooth surface. By a $(-m)$-*curve* of S, we mean a smooth *rational* curve on S with self-intersection number $-m$. For divisors C, D on S, we write $C \sim D$ if C is linearly equivalent to D. For divisors C, D on S, let

$C \cdot D$ denote the intersection number of C and D. Recall that an irreducible divisor D on S is exceptional if $D^2 < 0$. The set of all irreducible exceptional divisors on S is denoted by $\mathrm{Exc}(S)$. For $E \in \mathrm{Exc}(S)$, the blow-down of E is a smooth complex surface if and only if E is a (-1)-curve (cf. [2, Chap. III, Th. 4.2]).

Let E be a (-1)-curve on S and let $\sigma\colon S \to \overline{S}$ be the blow-down of E. Let $C \subset S$ be an irreducible curve and set $\overline{C} = \sigma(C)$. If we set $\mu = \mathrm{mult}_{\sigma(E)}\overline{C}$, then

$$\mu = C \cdot E, \qquad \overline{C}^2 = C^2 + \mu^2 \tag{12.1}$$

by e.g. [27, Chap. V, Prop. 3.6, Cor. 3.7]. Hence $\sigma(\mathrm{Exc}(S)) \supset \mathrm{Exc}(\overline{S})$. If $\mu = C \cdot E = 1$ in (12.1) and if C is smooth, $\overline{C}$ is smooth by the equality $\mathrm{mult}_{\sigma(E)}\overline{C} = 1$. In this case, σ induces an isomorphism from C to $\overline{C}$.

When X is a $K3$ surface, $\mathrm{Pic}(X)$ is identified with $H^1(X, \mathcal{O}_X^*)$, and $\mathrm{Exc}(X)$ is the set of (-2)-curves on X. Recall that for a 2-elementary $K3$ surface (X, ι), there is an inclusion $\mathrm{Pic}(X) \supset H^2_+(X, \mathbf{Z})$.

Lemma 12.1. *Let (X, ι) be a 2-elementary $K3$ surface and set $Y := X/\iota$. Let $p\colon X \to Y$ be the quotient map. If $\mathrm{Pic}(X) = H^2_+(X, \mathbf{Z})$, the following hold:*

1. *If D is a divisor on X, then $D \sim \iota(D)$. In particular, ι preserves every (-2)-curve on X and $\mathrm{Exc}(Y) = p(\mathrm{Exc}(X))$.*
2. *For $E \in \mathrm{Exc}(X)$, regard $\overline{E} := p(E)$ as a reduced divisor on Y. Then*

$$\overline{E}^2 = \begin{cases} -4 & \text{if} \quad E \subset X^\iota \\ -1 & \text{if} \quad E \not\subset X^\iota. \end{cases}$$

Proof. **(1)** Let L be an arbitrary holomorphic line bundle on X. Since $\mathrm{Pic}(X) = H^2_+(X, \mathbf{Z})$ and hence $c_1(L) = \iota^* c_1(L)$, we get $\iota^* L \cong L$. In particular, $D \sim \iota(D)$ for every divisor D on X. If $E \in \mathrm{Exc}(X)$, then $E = \iota(E)$ by [32, Lemma 1.4].

Let $\overline{C} \in \mathrm{Exc}(Y)$. Then $C := p^{-1}(\overline{C})$ is a divisor on X. Assume that C is reducible, and let D be an irreducible component of C. By the irreducibility of $\overline{C}$, $p(D) = p(\iota(D)) = \overline{C}$. Since $p\colon X \to Y$ is a double covering and hence $C = D \cup \iota(D)$, the reducibility of C implies that $D \neq \iota(D)$ and $C = D + \iota(D)$. Since $C^2 = 2\overline{C}^2 < 0$ by the projection formula, we get $0 > C^2 = 2(D^2 + D \cdot \iota(D))$. Since $D \cdot \iota(D) \geq 0$ by the irreducibility of D and $D \neq \iota(D)$, we get $D^2 < 0$. Hence $D \in \mathrm{Exc}(X)$. Since $\iota(D) = D$ by [32, Lemma 1.4], this contradicts the reducibility of C. We get the irreducibility of C. Since $C^2 = 2\overline{C}^2$, we get $\mathrm{Exc}(Y) = p(\mathrm{Exc}(X))$.
(2) Since we have the equation of divisors $p^*\overline{E} = 2E$ (resp. $p^*\overline{E} = E$) if $E \subset X^\iota$ (resp. $E \not\subset X^\iota$), we get the result by the identities $E^2 = -2$ and $(p^*\overline{E})^2 = 2\overline{E}^2$. $\square$

12.1. The case $M \cong \mathbb{U}(2) \oplus \mathbb{D}_4 \oplus \mathbb{D}_4$

Proposition 12.2. *Let (X, ι) be a 2-elementary $K3$ surface of type $\mathbb{U}(2) \oplus \mathbb{D}_4 \oplus \mathbb{D}_4$. Let C be the non-rational irreducible component of X^ι. If $\mathrm{Pic}(X) = H^2_+(X, \mathbf{Z})$, C is isomorphic to a curve of bidegree $(4, 2)$ on $\mathbf{P}^1 \times \mathbf{P}^1$. In particular, C is hyperelliptic.*

Proof. By [35, Props. 2.6 (ii) and 2.9], $\mathrm{Exc}(X)$ consists of 18 (-2)-curves. Let $\mathrm{Exc}(X) = \{S_0, S_1, E_1, \dots, E_8, F_1, \dots, F_8\}$. By [33, p.230], we may assume that

$$E_i \cdot S_0 = F_i \cdot S_1 = 1, \qquad E_i \cdot S_1 = F_i \cdot S_0 = 0, \tag{12.2}$$

$$S_k \cdot S_l = -2\delta_{kl}, \qquad E_i \cdot E_j = F_i \cdot F_j = -2\delta_{ij}, \qquad E_i \cdot F_j = 2\delta_{ij}. \tag{12.3}$$

By (12.2), $E_i \cap S_0$ (resp. $F_i \cap S_1$) consists of a unique point. Since an element of $\mathrm{Exc}(X)$ is ι-invariant by Lemma 12.1 (1), we get $\iota(E_i \cap S_0) = E_i \cap S_0$ and $\iota(F_i \cap S_1) = F_i \cap S_1$. Hence $E_i \cap S_0 \subset X^\iota$ and $F_i \cap S_1 \subset X^\iota$, which yields that $\#(S_0 \cap X^\iota) \geq \#(S_0 \cap \cup_i E_i) = 8$ and $\#(S_1 \cap X^\iota) \geq \#(S_0 \cap \cup_i F_i) = 8$. Recall that a non-trivial holomorphic involution on $\mathbf{P}^1$ has exactly 2 fixed points. Since ι induces an involution on the rational curve S_0 (resp. S_1), we get $\iota|_{S_0} = \mathrm{id}_{S_0}$ and $\iota|_{S_1} = \mathrm{id}_{S_1}$. Thus $S_0 \amalg S_1 \subset X^\iota$. Since $g(M) = 3$ and $k(M) = 2$ for $M = \mathbb{U}(2) \oplus \mathbb{D}_4^2$, there exists by Lemma 3.1 (3) a smooth curve C of genus 3 with $X^\iota = C \amalg S_0 \amalg S_1$. Since $E_i, F_j \not\subset X^\iota$ and since ι preserves each E_i and F_j, ι has exactly 2 fixed points on each of E_i and F_j, i.e., $E_i \cdot X^\iota = F_j \cdot X^\iota = 2$. By (12.2), we get

$$C \cdot E_i = C \cdot F_j = 1 \qquad (1 \leq i, j \leq 8). \tag{12.4}$$

Let Y be the quotient of X by ι and let $p\colon X \to Y = X/\iota$ be the quotient map. Set $\overline{S}_k := p(S_k)$, $\overline{E}_i := p(E_i)$, $\overline{F}_j := p(F_j)$, $\overline{C} := p(C)$. By Lemma 12.1 (1), we get

$$\mathrm{Exc}(Y) = \{\overline{S}_0, \overline{S}_1, \overline{E}_1, \ldots, \overline{E}_8, \overline{F}_1, \ldots, \overline{F}_8\}.$$

Since $S_0 \amalg S_1 \subset X^\iota$ and $E_i, F_j \not\subset X^\iota$, we get by (12.2–4) and Lemma 12.1 (2)

$$\overline{E}_i \cdot \overline{S}_0 = \overline{F}_i \cdot \overline{S}_1 = 1, \qquad \overline{E}_i \cdot \overline{S}_1 = \overline{F}_i \cdot \overline{S}_0 = 0, \tag{12.5}$$

$$\overline{S}_k \cdot \overline{S}_l = -4\delta_{kl}, \qquad \overline{E}_i \cdot \overline{E}_j = \overline{F}_i \cdot \overline{F}_j = -\delta_{ij}, \qquad \overline{E}_i \cdot \overline{F}_j = \delta_{ij}, \tag{12.6}$$

$$\overline{C} \cdot \overline{E}_i = \overline{C} \cdot \overline{F}_j = 1. \tag{12.7}$$

By (12.5–7), the configuration of the curves $\overline{E}_1, \ldots, \overline{F}_8, \overline{S}_0, \overline{S}_1, \overline{C}$ on Y is given as follows in Figure 1, in which a (-1)-curve is denoted by a thick line and the number in the bracket is the self-intersection number.

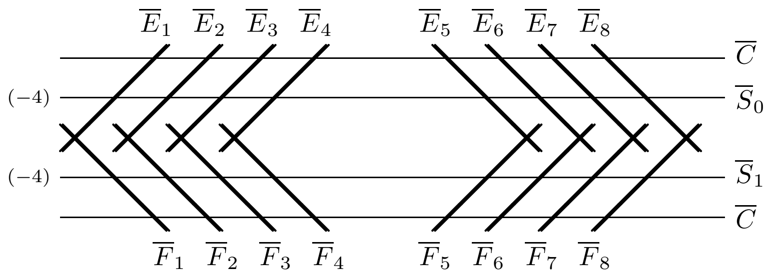

Figure 1

Let $\sigma\colon Y \to Z$ be the blow-down of the mutually disjoint 8 (-1)-curves $\overline{E}_1, \cdots, \overline{E}_4$, $\overline{F}_5, \cdots, \overline{F}_8$. Set $S'_k := \sigma(\overline{S}_k)$, $E'_i := \sigma(\overline{E}_i)$, $F'_j := \sigma(\overline{F}_j)$ and $C' := \sigma(\overline{C})$. By (12.1), we get $\mathrm{Exc}(Z) \subset \sigma(\mathrm{Exc}(Y)) = \{S'_0, S'_1, E'_5, \ldots, E'_8, F'_1, \ldots, F'_4\}$. By Figure 1 and (12.1), the configuration of the curves C', S'_0, S'_1, $E'_5, \ldots, E'_8$, $F'_1, \ldots, F'_4$ on Z is given as follows in Figure 2.

By Figure 2, the self-intersection number of any curve of $\sigma(\mathrm{Exc}(Y))$ is equal to 0, so that $\mathrm{Exc}(Z) = \emptyset$. This, together with the rationality of Z, implies $Z \cong \mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$. Since Z contains a curve with self-intersection number 0, we get $Z \cong \mathbf{P}^1 \times \mathbf{P}^1$. By Figure 2 again, we may assume that S'_k is a divisor of bidegree $(0, 1)$ and that E'_i and F'_j are divisors of bidegree $(1, 0)$. Since $C' \cdot S'_0 = C' \cdot S'_1 = 4$ and $C' \cdot E'_i = C' \cdot F'_j = 2$ by Figure 2, C' is an irreducible curve of bidegree $(4, 2)$ on $\mathbf{P}^1 \times \mathbf{P}^1$. Since the projection $\mathrm{pr}_1\colon \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ induces a double covering

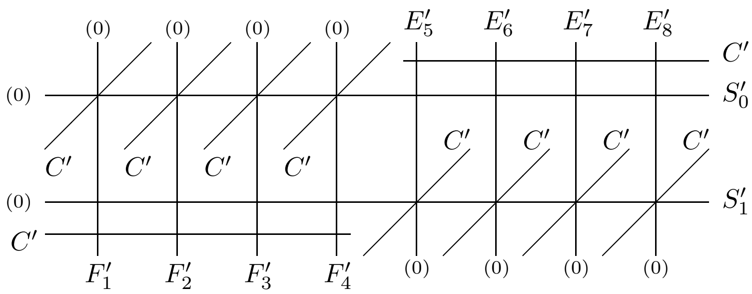

FIGURE 2

from C' to $\mathbf{P}^1$, C' is hyperelliptic. Since $p|_C : C \to \overline{C}$ is an isomorphism by the definition of p and since $\sigma|_{\overline{C}} : \overline{C} \to C'$ is an isomorphism by (12.1) and (12.7), the composition $(\sigma \circ p)|_C$ induces an isomorphism between C and C'. $\square$

12.2. The case $M \cong \mathbb{U} \oplus \mathbb{D}_{12}$

Let $\mathbf{R}_{>0}$ be the set of positive real numbers. For a Lorentzian lattice M, we define $\mathcal{L}(M) := \{[v] \in (M \otimes \mathbf{R} \setminus \{0\})/\mathbf{R}_{>0};\ v^2 > 0\}$ and we identify $\mathcal{L}(M)$ with the hyperboloid $\{v \in M \otimes \mathbf{R};\ v^2 = 1\}$, which is equipped with the Riemannian metric induced from the inner product on M. Then $\mathcal{L}(M)$ consists of two connected components $\mathcal{L}^+(M)$ and $\mathcal{L}^-(M)$, each of which is a hyperbolic space of dimension $r(M) - 1$. When M is the Picard lattice of an algebraic $K3$ surface X, we define $\mathcal{L}^+(X)$ as the component of $\mathcal{L}(\mathrm{Pic}(X))$ containing ample classes.

In the following proposition, which is the key to the proof of Theorem 9.3 (2), we use Kodaira's notation for singular fibers of elliptic fibration, for which we refer to [30], [2, Chap. V Sect. 7].

Proposition 12.3. *Let (X, ι) be a 2-elementary $K3$ surface of type $\mathbb{U} \oplus \mathbb{D}_{12}$. Assume that $\mathrm{Pic}(X) = H^2_+(X, \mathbf{Z})$. Then the following hold:*

(1) *X contains exactly 15 (-2)-curves $\alpha_0, \ldots, \alpha_{14}$. If $\alpha_i \neq \alpha_j$, then $\alpha_i \cdot \alpha_j \in \{0, 1\}$. The dual graph Γ of these 15 (-2)-curves $\mathrm{Exc}(X)$ is given as follows in Figure 3, where each vertex denotes the corresponding (-2)-curve and two vertices corresponding to α_i and α_j are connected by an edge if and only if $\alpha_i \cdot \alpha_j = 1$.*

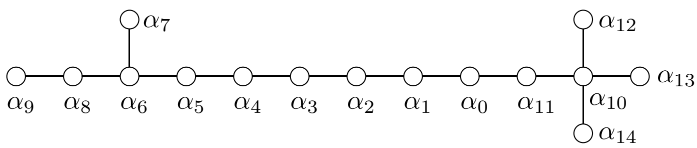

FIGURE 3. The dual graph Γ of $\mathrm{Exc}(X)$

(2) *Let C be the non-rational irreducible component of X^ι. Then C is isomorphic to a smooth plane quartic curve. In particular, C is non-hyperelliptic.*

Proof. **(1)** Since $\mathrm{Pic}(X) \cong \mathbb{U} \oplus \mathbb{E}_8 \oplus \mathbb{D}_4$, the hyperbolic plane $\mathbb{U} \subset \mathrm{Pic}(X)$ defines an elliptic fibration $\pi : X \to \mathbf{P}^1$ with a section α_0 by [32, Lemma 2.1 (i)]. By [32, Lemma 2.2], $\pi : X \to \mathbf{P}^1$ has a singular fiber of type II^* and a singular fiber of type I_0^*. Let $F_{II^*} := \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 5\alpha_5 + 6\alpha_6 + 3\alpha_7 + 4\alpha_8 + 2\alpha_9$ be the singular fiber of type II^* and let $F_{I_0^*} := 2\alpha_{10} + \alpha_{11} + \alpha_{12} + \alpha_{13} + \alpha_{14}$ be the singular fiber of type I_0^*, where $\alpha_1, \ldots, \alpha_{14}$ are (-2)-curves on X. The dual graphs of F_{II^*} and $F_{I_0^*}$ are given by the corresponding subgraphs of Γ by e.g. [2, Chap. 5, Sect. 7, Table 3]. Since α_0 is a section and hence $\alpha_0 \cdot F_{II^*} = 1$, we get

$$\alpha_0 \cdot \alpha_1 = 1, \qquad \alpha_0 \cdot \alpha_i = 0 \quad (2 \leq i \leq 9). \tag{12.8}$$

Similarly, since $\alpha_0 \cdot F_{I_0^*} = 1$, we may assume that

$$\alpha_0 \cdot \alpha_{11} = 1, \qquad \alpha_0 \cdot \alpha_{10} = \alpha_0 \cdot \alpha_{12} = \alpha_0 \cdot \alpha_{13} = \alpha_0 \cdot \alpha_{14} = 0. \tag{12.9}$$

By (12.8), (12.9), the dual graph of $\{\alpha_0, \ldots, \alpha_{14}\}$ is given by Γ as in Figure 3.

Let $W(\Gamma)$ be the discrete subgroup of motions of $\mathcal{L}^+(X)$ generated by the reflections $\{s_{\alpha_i};\, i = 0, \ldots, 14\}$, where $s_{\alpha_i}[v] := [v + (v \cdot \alpha_i)\, c_1(\alpha_i)]$ for $[v] \in \mathcal{L}^+(X)$. Then Γ is the Coxeter diagram of $W(\Gamma)$ (cf. [55, Sect. 1]). Set

$$\mathcal{C} := \{[v] \in (\mathrm{Pic}(X) \otimes \mathbf{R} \setminus \{0\})/\mathbf{R}_{>0};\ v \cdot \alpha_i \geq 0\, (i = 0, \ldots, 14)\},$$

which is a convex polyhedron in the sphere $(\mathrm{Pic}(X) \otimes \mathbf{R} \setminus \{0\})/\mathbf{R}_{>0}$. By [12, Chap. V], $\mathcal{C} \cap \mathcal{L}^+(X)$ is a fundamental domain for the $W(\Gamma)$-action on $\mathcal{L}^+(X)$. Since any maximal extended Dynkin diagram in Γ is either $\widetilde{E}_8 \oplus \widetilde{D}_4$ or $\widetilde{D}_{12}$ and since both of them have the maximal rank 12, Γ satisfies the condition in [55, Th. 2.6 bis.]. By [55, Th. 2.6 bis.], $\mathcal{C} \cap \mathcal{L}^+(X)$ has finite volume. By [55, p.335 l.28], we get

$$\overline{\mathcal{C}} \subset \overline{\mathcal{L}^+(X)}, \tag{12.10}$$

where the closures are considered in the sphere $(\mathrm{Pic}(X) \otimes \mathbf{R} \setminus \{0\})/\mathbf{R}_{>0}$.

If there exists a (-2)-curve $E \notin \{\alpha_0, \ldots, \alpha_{14}\}$, then $E \cdot \alpha_i \geq 0$ for $0 \leq i \leq 14$, so that $c_1(\mathcal{O}_X(E)) \in \overline{\mathcal{L}^+(X)}$ by (12.10). This implies the contradiction $E^2 \geq 0$, because $E \in \mathrm{Exc}(X)$. This proves that $\mathrm{Exc}(X) = \{\alpha_0, \ldots, \alpha_{14}\}$.

(2) By Lemma 12.1 (1), ι preserves the 15 (-2)-curves $\alpha_0, \ldots, \alpha_{14}$. By [32, Lemma 2.3 (ii)], we get $\alpha_0 \subset X^\iota$. If F is a fiber of the elliptic fibration $\pi\colon X \to \mathbf{P}^1$, then $\iota(F) \sim F$ by Lemma 12.1 (1), so that $\iota(F)$ is again a fiber of π. Since $\alpha_0 \subset X^\iota$ and since α_0 is a section of π, we get $F \cap \iota(F) \supset F \cap \alpha_0 \neq \emptyset$. Hence ι preserves the fibers of π. Since the dual graph of F_{II^*} (resp. $F_{I_0^*}$) is $\widetilde{E}_8$ (resp. $\widetilde{D}_4$), we deduce from [32, Lemma 2.3 (i)] that $\alpha_2 \amalg \alpha_4 \amalg \alpha_6 \amalg \alpha_9 \amalg \alpha_{10} \subset X^\iota$. Since $H^2_+(X, \mathbf{Z}) \cong \mathbb{U} \oplus \mathbb{D}_{12}$ and hence $g(H^2_+(X, \mathbf{Z})) = 3$, $k(H^2_+(X, \mathbf{Z})) = 6$, we get by Lemma 3.1 (3)

$$X^\iota = C \amalg \alpha_0 \amalg \alpha_2 \amalg \alpha_4 \amalg \alpha_6 \amalg \alpha_9 \amalg \alpha_{10}, \qquad g(C) = 3. \tag{12.11}$$

Since $\iota(\alpha_i) = \alpha_i$ and $\alpha_i \not\subset X^\iota$ for $i = 1, 3, 5, 7, 8, 11, 12, 13, 14$ by Lemma 12.1 (1) and (12.11), we get $X^\iota \cdot \alpha_i = 2$ for these i. By [32, Lemma 2.3 (i)] and Figure 3, we get

$$C \cdot \alpha_i = 1 \quad (i = 7, 12, 13, 14), \qquad C \cdot \alpha_j = 0 \quad (\text{otherwise}). \tag{12.12}$$

(Step 0) Set $R_0 := X/\iota$, which is a smooth rational surface. Let $p\colon X \to R_0$ be the quotient map. We set $\alpha_i^{(0)} := p(\alpha_i)$ and $C^{(0)} := p(C)$. By Lemma 12.1 (1), we get $\mathrm{Exc}(R_0) = \{\alpha_0^{(0)}, \ldots, \alpha_{14}^{(0)}\}$. By (12.12), we get

$$C^{(0)} \cdot \alpha_i^{(0)} = 1 \quad (i = 7, 12, 13, 14), \qquad C^{(0)} \cdot \alpha_j^{(0)} = 0 \quad (\text{otherwise}). \tag{12.13}$$

The configuration of the curves $C^{(0)}, \alpha_0^{(0)}, \ldots, \alpha_{14}^{(0)}$ is given as in Figure 4 from Figure 3, Lemma 12.1 (2) and (12.13). In the figures below, we use the following convention: a (-1)-curve is denoted by a thick line; the number in the bracket is the self-intersection number; we write α_i (resp. C) for $\alpha_i^{(k)}$ (resp. $C^{(k)}$) for simplicity.

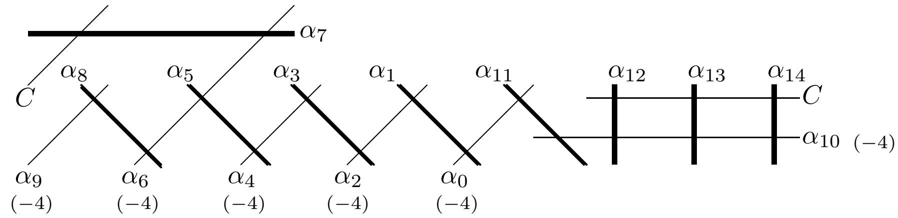

FIGURE 4

(Step 1) Let $\sigma_1\colon R_0 \to R_1$ be the blow-down of the 9 disjoint (-1)-curves $\alpha_i^{(0)}$, where $i = 1, 3, 5, 7, 8, 11, 12, 13, 14$. Set $\alpha_j^{(1)} := \sigma_1(\alpha_j^{(0)})$ for $j = 0, 2, 4, 6, 9, 10$ and $C^{(1)} := \sigma_1(C^{(0)})$. By (12.1) and Figure 4, the configuration of these curves is given as in Figure 5. By Figure 4 and (12.1), we get $\mathrm{Exc}(R_1) = \{\alpha_0^{(1)}, \alpha_2^{(1)}, \alpha_4^{(1)}, \alpha_6^{(1)}, \alpha_9^{(1)}\}$.

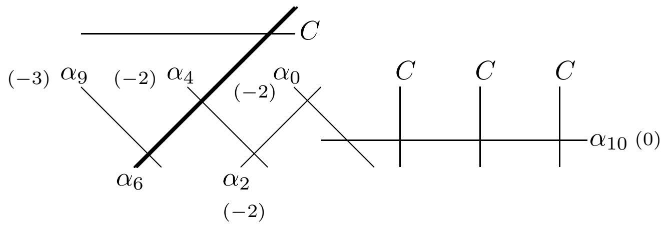

FIGURE 5

(Step 2) Let $\sigma_2\colon R_1 \to R_2$ be the blow-down of the (-1)-curve $\alpha_6^{(1)}$. Set $\alpha_j^{(2)} := \sigma_2(\alpha_j^{(1)})$ for $j = 0, 2, 4, 9, 10$ and $C^{(2)} := \sigma_2(C^{(1)})$. By (12.1) and Figure 5, the configuration of these curves is given as in Figure 6, where $C^{(2)}$ passes through the point $\alpha_9^{(2)} \cap \alpha_4^{(2)}$. By Figure 5 and (12.1), we get $\mathrm{Exc}(R_2) = \{\alpha_0^{(2)}, \alpha_2^{(2)}, \alpha_4^{(2)}, \alpha_9^{(2)}\}$.

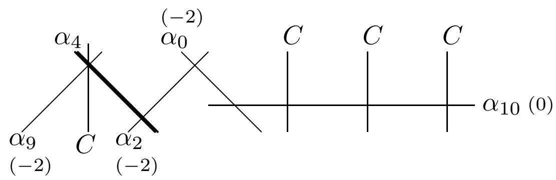

FIGURE 6

(Step 3) Let $\sigma_3\colon R_2 \to R_3$ be the blow-down of the (-1)-curve $\alpha_4^{(2)}$. Set $\alpha_j^{(3)} := \sigma_3(\alpha_j^{(2)})$ for $j = 0, 2, 9, 10$ and $C^{(3)} := \sigma_3(C^{(2)})$. By (12.1) and Figure 6, the configuration of these curves is given as in Figure 7, where $C^{(3)}$ is tangent to $\alpha_9^{(3)}$ of order 2 at $\alpha_9^{(3)} \cap \alpha_2^{(3)}$. By Figure 6 and (12.1), we get $\mathrm{Exc}(R_3) = \{\alpha_0^{(3)}, \alpha_2^{(3)}, \alpha_4^{(3)}\}$.

(Step 4) Let $\sigma_4\colon R_3 \to R_4$ be the blow-down of the (-1)-curve $\alpha_2^{(3)}$. Set $\alpha_j^{(4)} := \sigma_4(\alpha_j^{(3)})$ for $j = 0, 9, 10$ and $C^{(4)} := \sigma_4(C^{(3)})$. By (12.1) and Figure 7, the configuration of these curves is given as in Figure 8, where $C^{(4)}$ is tangent to $\alpha_9^{(4)}$ of order

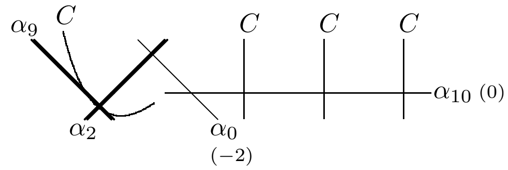

FIGURE 7

3 at $\alpha_9^{(4)} \cap \alpha_0^{(4)}$. By Figure 7 and (12.1), we get $\mathrm{Exc}(R_4) = \{\alpha_0^{(4)}\}$.

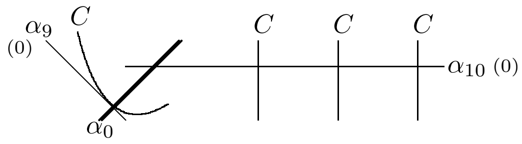

FIGURE 8

(Step 5) Let $\sigma_5 \colon R_4 \to R_5$ be the blow-down of the (-1)-curve $\alpha_0^{(4)}$. Set $\alpha_i^{(5)} := \sigma_5(\alpha_i^{(4)})$ for $i = 9, 10$ and $C^{(5)} := \sigma_5(C^{(4)})$. By (12.1) and Figure 8, the configuration of these curves is given as in Figure 9, where $C^{(5)}$ is tangent to $\alpha_9^{(5)}$ of order 4 at $\alpha_9^{(5)} \cap \alpha_{10}^{(5)}$.

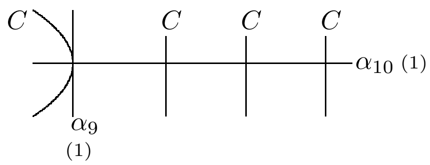

FIGURE 9

(Step 6) Since $\mathrm{Exc}(R_5) = \emptyset$ by Figure 9, R_5 is a rational surface without exceptional curves. Hence $R_5 \cong \mathbf{P}^2$ or $R_5 \cong \mathbf{P}^1 \times \mathbf{P}^1$. Since R_5 contains a curve with self-intersection number 1, we get $R_5 \cong \mathbf{P}^2$. In each blow-down $\sigma_{i+1} \colon R_i \to R_{i+1}$, $C^{(i)}$ intersects transversally the exceptional curves of σ_{i+1}, so that σ_{i+1} induces an isomorphism from $C^{(i)}$ to $C^{(i+1)}$ for all i. Since the composition $\sigma := \sigma_5\sigma_4\sigma_3\sigma_2\sigma_1 p \colon X \to R_5$ induces an isomorphism between C and $C^{(5)}$, C is isomorphic to the smooth plane quartic $C^{(5)} \subset \mathbf{P}^2$. Since the canonical line bundle of C is very ample by the adjunction formula $K_C \cong \mathcal{O}_{\mathbf{P}^2}(1)|_{C^{(5)}}$, C is non-hyperelliptic (cf. [27, Chap. IV, Example 5.2.1]). $\square$

REFERENCES

[1] Baily, W.L., Borel, A. *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. **84** (1966), 442–528.
[2] Barth, W., Peters, C., Van de Ven, A. *Compact Complex Surfaces*, Springer, Berlin (1984)
[3] Beauville, A. *Complex Algebraic Surfaces*, Cambridge University Press, Cambridge (1983)
[4] Bismut, J.-M. *Equivariant immersions and Quillen metrics*, Jour. Differ. Geom. **41** (1995), 53–157.
[5] Bismut, J.-M., Gillet, H., Soulé, C. *Analytic torsion and holomorphic determinant bundles I, II, III*, Commun. Math. Phys. **115** (1988), 49–78, 79–126, 301–351.

[6] Bismut, J.-M., Lebeau, G. *Complex immersions and Quillen metrics*, Publ. Math. IHES **74** (1991), 1–297.

[7] Borcherds, R.E. *Automorphic forms on $O_{s+2,s}(\mathbf{R})$ and infinite products*, Invent. Math. **120** (1995), 161–213.

[8] ——— *The moduli space of Enriques surfaces and the fake monster Lie superalgebra*, Topology **35** (1996), 699–711.

[9] ——— *Automorphic forms with singularities on Grassmanians*, Invent. Math. **132** (1998), 491-562.

[10] ——— *Reflection groups of Lorentzian lattices*, Duke Math. Jour. **104** (2000), 319–366.

[11] Borcherds, R.E., Katzarkov, L., Pantev, T., Shepherd-Barron, N.I. *Families of $K3$ surfaces*, Jour. Algebr. Geom. **7** (1998), 183–193.

[12] Bourbaki, N. *Groupes et algèbres de Lie, Chap. 4-6*, Hermann, Paris (1968)

[13] Burns, D., Rapoport, M. *On the Torelli problems for Kählerian $K3$ surfaces*, Ann. Sci. Éc. Norm. Supér. IV Sér. **8** (1975), 235–274.

[14] Bruinier, J.H. *Borcherds Products on $O(2, l)$ and Chern classes of Heegner Divisors*, Lecture Notes Math. **1780** Springer, Berline (2002)

[15] Conway, J.H., Sloan, N.J.A. *Sphere Packings, Lattices and Groups*, Springer, Berlin (1999)

[16] Dolgachev, I. *Mirror symmetry for lattice polarized $K3$ surfaces*, Jour. Math. Sci. **81** (1996), 2599–2630.

[17] Degtyarev, A., Itenberg, I., Kharlamov, V. *Real Enriques Surfaces*, Lecture Notes Math. **1746** Springer, Berline (2000)

[18] Fang, H., Lu, Z., Yoshikawa, K.-I. *Analytic torsion for Calabi–Yau threefolds*, preprint, arXiv: math.DG/0601411 (2006)

[19] Fay, J.D. *Theta functions on Riemann Surfaces*, Lecture Notes in Math. **352** Springer, Berline (1973)

[20] Finashin, S., Kharlamov, V. *Deformation classes of real four-dimensional cubic hypersurfaces*, preprint, arXiv:math.AG/0607137 (2006)

[21] Freitag, E., Hermann, C.F. *Some modular varieties of low dimension*, Adv. Math. **152** (2000), 203–287.

[22] Freitag, E., Salvati Manni, R. *Modular forms for the even unimodular lattice of signature* $(2, 10)$, preprint, arXiv:math.AG/0511433 (2005)

[23] Gillet, H., Soulé, C. *An arithmetic Riemann-Roch theorem*, Invent. Math. **110** (1992), 473–543.

[24] Gritsenko, V.A., Nikulin, V.V. *Siegel automorphic form corrections of some Lorentzian Kac–Moody Lie algebras*, Amer. Jour. Math. **119** (1997), 181–224.

[25] ——— *Automorphic forms and Lorentzian Kac–Moody algebras I, II*, Internar. Jour. Math. **9** (1998), 153–199, 201–275.

[26] Hartshorne, R. *Ample subvarieties of algebraic varieties*, Lecture Notes in Math. **156** Springer, Berlin (1970)

[27] ——— *Algebraic Geometry*, Springer, Berlin (1977)

[28] Harvey, J., Moore, G. *Algebras, BPS states, and strings*, Nuclear Phys. **B 463** (1996), 315–368.

[29] Igusa, J-I. *Modular forms and projective invariants*, Amer. Jour. Math. **89** (1967), 817–855.

[30] Kodaira, K. *On compact complex analytic surfaces II, III*, Ann. Math. **77** (1963), 563–626, **78** (1963), 1–40.

[31] Köhler, K., Roessler, D. *A fixed point formula of Lefschetz type in Arakelov geometry I*, Invent. Math. **145** (2001), 333–396.

[32] Kondō, S. *Automorphisms of algebraic $K3$ surfaces which act trivially on Picard groups*, Jour. Math. Soc. Japan **44** (1992), 75–98.

[33] ——— *A complex hyperbolic structure for the moduli space of curves of genus three*, J. reine angew. Math. **525** (2000), 219–232.

[34] ——— *The moduli space of Enriques surfaces and Borcherds products*, Jour. Algebr. Geom. **11** (2002), 601–627.

[35] ——— *The moduli space of 8 points on $\mathbf{P}^1$ and automorphic forms*, Algebraic Geometry, ed. by J.H. Keum and S. Kondō, Contemporary Math. **422** A.M.S. (2007), 89–106.

[36] Landweber, P.S. *Elliptic genera: an introductory overview*, Elliptic Curves and Modular Forms in Algebraic Topology, ed. by P.S. Landweber, Lecture Notes in Math. **1326** Springer, Berline (1988), 1–10.

[37] Looijenga, E. *Compactifications defined by arrangements, II*, Duke Math. Jour. **119** (2003), 527–588.
[38] Ma, X. *Submersions and equivariant Quillen metrics*, Ann. Inst. Fourier **50** (2000), 1539–1588.
[39] Mumford, D. *Tata Lectures on Theta I*, Progress in Math. **28** Birkhäuser, Boston (1983)
[40] ——— *On the Kodaira dimension of the Siegel modular variety*, Algebraic Geometry — Open Problems, Lecture Notes in Math. **997** Springer, Berline (1983), 348–375.
[41] Namikawa, Y. *On the canonical holomorphic map from the moduli space of stable curves to the Igusa compactification*, Nagoya Math. Jour. **52** (1973), 197–259.
[42] ——— *Periods of Enriques surfaces*, Math. Ann. **270** (1985), 201–222.
[43] Nikulin, V.V. *Integral Symmetric bilinear forms and some of their applications*, Math. USSR Izv. **14** (1980), 103–167.
[44] ——— *Finite automorphism groups of Kähler $K3$ surfaces*, Trans. Moscow Math. Soc. **38** (1980), 71–135.
[45] ——— *Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections*, Jour. Soviet Math. **22** (1983), 1401–1476.
[46] ——— *A remark on discriminants for moduli of $K3$ surfaces as sets of zeros of automorphic forms*, Jour. Math. Sci. **81** (1996), 2738–2743.
[47] ——— *Reflection groups in Lobachevsky spaces and an identity for the denominator of Lorentzian Kac–Moody algebras*, Izv. Math. **60** (1996), 305–334.
[48] Oguiso, K. *Local families of $K3$ surfaces and applications*, Jour. Algebr. Geom. **12** (2003), 405–433.
[49] Piatetskii-Shapiro, I., Shafarevich, I.R. *A Torelli theorem for algebraic surfaces of type $K3$*, Math. USSR Izv. **35** (1971), 530–572.
[50] Ray, D.B., Singer, I.M. *Analytic torsion for complex manifolds*, Ann. Math. **98** (1973), 154–177.
[51] Scheithauer, N. *The fake monster superalgebras*, Adv. Math. **151** (2000), 226–269.
[52] ——— *On the classification of automorphic products and generalized Kac–Moody algebras*, Invent. Math. **164** (2006), 641–678.
[53] Shimura, G. *Introduction to the Arithmetic Theory of Automorphic Functions*, Iwanami Shoten Publ., Tokyo (1971)
[54] Siu, Y.-T. *Analyticity of sets associated to Lelong numbers and the extension of closed positive currents*, Invent. Math. **27** (1974), 53–156.
[55] Vinberg, E.B. *Some arithmetic discrete groups in Lobachevskii spaces*, Discrete subgroups of Lie groups and applications to moduli, Tata-Oxford (1975), 323–348.
[56] Yoshida, M. *Hypergeometric Functions, My Love*, Aspects of Math. Vieweg, Braunschweig (1997)
[57] Yoshikawa, K.-I. *$K3$ surfaces with involution and analytic torsion*, unpublished version of [59], math.AG/9808129v3
[58] ——— *Discriminant of theta divisors and Quillen metrics*, Jour. Differ. Geom. **52** (1999), 73–115.
[59] ——— *$K3$ surfaces with involution, equivariant analytic torsion, and automorphic forms on the moduli space*, Invent. Math. **156** (2004), 53–117.
[60] ——— *Analytic Torsion and an invariant of Calabi–Yau threefold*, Differential Geometry and Physics, Nankai Tracts in Math. **10**, World Scientific (2006), 480–489.
[61] ——— *Discriminant of certain $K3$ surfaces*, Representation Theory and Automorphic Forms, ed. by T. Kobayashi, W. Schmid, J.-H. Yang, Progress in Math. **255** Birkhäuser, Boston (2007) to appear
[62] ——— *Real $K3$ surfaces without real points, equivariant determinant of the Laplacian, and the Borcherds Φ-function*, Math. Zeit. DOI: 10.1007/s00209-007-0170-z
[63] ——— *Calabi–Yau threefolds of Borcea–Voisin, analytic torsion, and Borcherds products*, in preparation

Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Tokyo 153-8914, JAPAN
E-mail address: yosikawa@ms.u-tokyo.ac.jp

UTMS

2007–2 Tomoya Takeuchi and Masahiro Yamamoto: *Tikhonov regularization by a reproducing kernel Hilbert space for the Cauchy problem for an elliptic equation.*

2007–3 Takayuki Oda: *The standard* $(\mathfrak{g}, K)$*-modules of* $Sp(2, R)$ *I – The case of principal series –.*

2007–4 Masatoshi Iida and Takayuki Oda: *Harish-Chandra expansion of the matrix coefficients of* P_J *Principal series representation of* $Sp(2, \mathbb{R})$.

2007–5 Yutaka Matsui and Kiyoshi Takeuchi: *Microlocal study of Lefschetz fixed point formulas for higher-dimensional fixed point sets.*

2007–6 Shumin Li and Masahiro Yamamoto: *Lipschitz stability in an inverse hyperbolic problem with impulsive forces.*

2007–7 Tadashi Miyazaki: *The* $(\mathfrak{g}, K)$*-module structures of principal series representations of* $Sp(3, \mathbb{R})$.

2007–8 Wuqing Ning: *On stability of an inverse spectral problem for a nonsymmetric differential operator.*

2007–9 Tomohiro Yamazaki and Masahiro Yamamoto: *Inverse problems for vibrating systems of first order.*

2007–10 Yasufumi Osajima: *General asymptotics of Wiener functionals and application to mathematical finance.*

2007–11 Shuichi Iida: *Adiabatic limits of* η*-invariants and the meyer functions.*

2007–12 Ken-ichi Yoshikawa: $K3$ *surfaces with involution, equivariant analytic torsion, and automorphic forms on the moduli space II: a structure theorem.*

The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012

UTMS

2007–3 Takayuki Oda: *The standard* $(\mathfrak{g}, K)$*-modules of* $Sp(2, R)$ *I – The case of principal series –.*

2007–4 Masatoshi Iida and Takayuki Oda: *Harish-Chandra expansion of the matrix coefficients of* P_J *Principal series representation of* $Sp(2, \mathbb{R})$.

2007–5 Yutaka Matsui and Kiyoshi Takeuchi: *Microlocal study of Lefschetz fixed point formulas for higher-dimensional fixed point sets.*

2007–6 Shumin Li and Masahiro Yamamoto: *Lipschitz stability in an inverse hyperbolic problem with impulsive forces.*

2007–7 Tadashi Miyazaki: *The* $(\mathfrak{g}, K)$*-module structures of principal series representations of* $Sp(3, \mathbb{R})$.

2007–8 Wuqing Ning: *On stability of an inverse spectral problem for a nonsymmetric differential operator.*

2007–9 Tomohiro Yamazaki and Masahiro Yamamoto: *Inverse problems for vibrating systems of first order.*

2007–10 Yasufumi Osajima: *General asymptotics of Wiener functionals and application to mathematical finance.*

2007–11 Shuichi Iida: *Adiabatic limits of* η*-invariants and the meyer functions.*

2007–12 Ken-ichi Yoshikawa: $K3$ *surfaces with involution, equivariant analytic torsion, and automorphic forms on the moduli space II: a structure theorem.*

2007–13 M. Bellassoued, D. Jellali and M. Yamamoto: *Stability estimate for the hyperbolic inverse boundary value problem by local Dirichlet-to-Neumann map.*

The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012