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REGULARITY OF THE TIKHONOV REGULARIZED SOLUTIONS AND THE EXACT SOLUTION

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Abstract. We discuss an operator equation $Kf = g$ where X, Y are reflexive Banach spaces of functions in bounded domains $\Omega \subset \mathbb{R}^N$ and Ω_0, and $K : X \longrightarrow Y$ has no continuous inverse. Let V and V_1 be another reflexive Banach spaces of functions in Ω and $D \subset \Omega$ respectively such that the embedding $V \subset X$ is continuous. In order to stably reconstruct g by noise data g_δ with $\|g - g_\delta\|_Y \le \delta$: noise level, we consider the Tikhonov regularization: Minimize $\|Kf - g_\delta\|^2_Y + \alpha\|f\|^2_V$. We prove that if $\alpha = c_0\delta^2$ with a constant $c_0 > 0$ and the V_1-norms of the regularized solutions f_δ are bounded uniformly in δ, then the exact solution f is in V_1. This property can be applied to the determination of non-smooth points of a function f for example in the case of $X = L^2(\Omega)$, $V = H^\ell(\Omega)$ and $V_1 = H^\ell(D)$ with $\ell \in \mathbb{N}$ and small ball D.

§1. Introduction.

In terms of an operator equation, we can describe inverse problems for partial differential equations or integral equations such as determination of coefficients. Throughout this paper, let X, Y be reflexive Banach spaces and let $K : X \longrightarrow Y$ an injective continuous operator. Then the inverse problem is given by an operator equation:

$$Kf = g.$$

We note that K may be nonlinear. Henceforth we assume that $V \subset X$ is another Banach reflexive space and the embedding $V \longrightarrow X$ is continuous, V is dense in

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X. We assume that

$$Kf_0 = g_0, \qquad f_0 \in X. \tag{1.1}$$

We discuss a stable reconstruction scheme from noisy data g_δ:

$$\|g_0 - g_\delta\|_Y \leqq \delta. \tag{1.2}$$

Here $\delta > 0$ is a noise level and we assume that g_0 cannot be known but g_δ is available data with known noise level. Then our task is to establish a stable reconstruction method satisfying the following requirements:

(1) We can stably find f_δ from g_δ.

(2) $f_\delta \longrightarrow f_0$ in a suitable norm as $\delta \longrightarrow 0$.

For it, the Tikhonov regularization is widely used. We set

$$F_\alpha(f, g) = \|Kf - g\|_Y^2 + \alpha\|f\|_V^2. \tag{1.3}$$

Here $\alpha > 0$ is a parameter which we have to choose appropriately, and α is called a regularizing parameter. Then under suitable assumptions (see Lemmata 2.1 and 3.1 below) we can prove that there exists a minimizer f_δ of $F_\alpha(f, g_\delta)$ over a suitable admissible set and that f_δ satisfies the above requirements. As for the Tikhonov regularization, we have many works and here we refer only to books by Banks and Kunisch [2], Baumeister [3], Engl, Hanke and Neubauer [4], Groetsch [5], Hofmann [6], Kirsch [7], Kress [8], Tikhonov and Arsenin [9].

In order to guarantee the convergence of f_δ to f_0 with concrete convergence rate, the choice of regularizing parameter α is important and in general we have to assume extra regularity on f_0 which is called a source condition. Such a source condition means that f_0 should belong to some subspace e.g., V of X. In the case where $V = X$ is a Hilbert space, we refer to [5] as one typical source condition. The determination of interfaces, non-smoothness or discontinuities is a practically very demanded class of inverse problems and the edge detection is one typical example. In those cases, we are mainly concerned with a not smooth exact solution f_0.

The main purpose of this paper is to prove the asymptotic behaviour of the regularized solutions for a non-smooth exact solution f_0. More precisely, we prove that if the exact solution is not in a space V_1 of locally smoother functions, then the V_1-norms of the regularized solutions blow up. The property can be applied in order to detect irregular points of a function as the exact solution, and in a forthcoming paper, we will develop a numerical method for detecting irregular points of a function. In the case where the solution of $Kf = g$ is a numerical differentiation, more detailed studies and applications to the edge detection problem, are done in Wan, Wang and Yamamoto [10], Wang, Jia and Cheng [11].

§2. Main result - linear case.

In this section, we assume that K is a linear operator. Hence a Banach space V is called to be uniformly convex if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) \in (0, 1)$ such that $\|x\|_V \leqq 1$, $\|y\|_V \leqq 1$ and $\|x - y\|_V \geq \varepsilon$ imply $\|x + y\|_V \leqq 2(1 - \delta)$ (e.g., Adams [1], Yosida [12]).

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. Then it is known (e.g., Adams [1]) that $L^p(\Omega)$ and the Sobolev spaces $W^{\ell,p}(\Omega)$ with $1 < p < \infty$ and $\ell \in \mathbb{N}$ are uniformly convex. As is directly checked, every Hilbert space is uniformly convex.

Lemma 2.1. *Let $\alpha > 0$ and let the embedding $V \longrightarrow X$ be compact.*

(i) There exists a minimizer to $F(f, g)$ over V for an arbitrarily given $g \in Y$.

(ii) Moreover let V be uniformly convex. Then the minimizer is unique.

The lemma is well known, but for convenience we will give the proof in Appendix I. For the existence of a minimizer, we need not assume the linearity of K.

For example as X and V, we can set $X = L^2(\Omega)$ and $V = H^\ell(\Omega)$ with $\ell \in \mathbb{N}$.

Our aim is the behaviour of the regularized solutions f_δ if the exact solution f_0 is not in V. Moreover for an effective reconstruction of non-smooth points of $f_0 \notin V$, we will observe the V-norms of f_δ locally in Ω. For it, we introduce subspaces $V_1 = V_1(D)$ and $X_1 = X_1(D)$ as follows. Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and let X and Y be reflexive Banach spaces of functions defined in bounded domains Ω and Ω_0 respectively such that $C_0^\infty(\Omega) \subset V \subset X$ topologically. We arbitrarily choose a domain $D \subset \Omega$ such that the boundary ∂D is smooth. Let $X_1 = X_1(D)$ and $V_1 = V_1(D)$ be reflexive Banach spaces such that

the weak convergence in V_1 implies the convergence in

$(C_0^\infty(D))'$ (i.e., in the distribution sense in D) and $V_1 \subset X_1$ topologically.

Moreover the restriction Rf of a function in Ω to D, is defined by $Rf = f_{|D}$. We assume that

$$RX \subset X_1, \qquad RV \subset V_1.$$

Henceforth $f \in X$ and $f \in V$ are regarded as $f \in X_1$ and $f \in V_1$ by means of R, respectively.

Example. Let $X = L^2(\Omega)$, $V = H^\ell(\Omega)$, $X_1 = L^2(D)$ and $V_1 = H^\ell(D)$ with $\ell \in \mathbb{N}$. Then if $f_n \longrightarrow f_0$ weakly in V_1 then $f_n \longrightarrow f_0$ in $(C_0^\infty(D))'$.

Let f_δ^α be a minimizer of $F_\alpha(f, g_\delta)$ over V. We note by Lemma 2.1 (i) that f_δ^α exists and by Lemma 2.1 (ii) that f_δ^α is unique if K is linear and V is uniformly convex.

Theorem 2.1. *We assume that the embedding $V \longrightarrow X$ is compact and that*

$$\lim_{n\to\infty} Kf_n = Kf \text{ in } Y \text{ implies } \lim_{n\to\infty} f_n = f \text{ in } (C_0^\infty(\Omega))'. \tag{2.1}$$

Let $c_0 > 0$ be arbitrarily fixed. If $\sup_{\delta>0} \|f_\delta^{c_0\delta^2}\|_{V_1} < \infty$, then $f_0 \in V_1$.

Proof. We set $f_\delta = f_\delta^{c_0\delta^2}$. For $f_0 \in X$ and $n \in \mathbb{N}$, by the density of V in X, we can find $h_n \in V$ such that

$$\|h_n - f_0\|_X \leqq \frac{1}{n}. \tag{2.2}$$

Then we may assume that

$$\sup_{n\in\mathbb{N}} \|h_n\|_V = \infty. \tag{2.3}$$

In fact, let $\sup_{n\in\mathbb{N}} \|h_n\|_V < \infty$. Then, by the reflexiveness of V, there exists a subsequence h_n and $\widetilde{h} \in V$ such that $h_n \longrightarrow \widetilde{h}$ weakly in V. On the other hand, we see from (2.2) that $h_n \longrightarrow f_0$ strongly in X. Therefore $f_0 = \widetilde{h} \in V$ and we have already proved the theorem.

We assume (2.3).

$$\delta_n = \frac{1}{\|h_n\|_V^2}, \qquad n \in \mathbb{N}. \tag{2.4}$$

Then we see that $\lim_{n\to\infty} \delta_n = 0$.

Since $\sup_{n\in\mathbb{N}} \|f_{\delta_n}\|_{V_1} < \infty$, by the reflexiveness of V_1, we can choose a subsequence f_{δ_n} and $\widetilde{f} \in V_1$ such that

$$f_{\delta_n} \longrightarrow \widetilde{f} \quad \text{weakly in } V_1.$$

By the assumption on the weak convergence in V_1, we see that

$$f_{\delta_n} \longrightarrow \widetilde{f} \quad \text{in } (C_0^\infty(D))'. \tag{2.5}$$

On the other hand, by the definition and (2.4), we have

$$\|Kf_{\delta_n} - g_{\delta_n}\|_Y^2 + c_0\delta_n^2\|f_{\delta_n}\|_V^2 \leqq \|Kh_n - g_{\delta_n}\|_Y^2 + c_0\delta_n^2\|h_n\|_V^2$$
$$\leqq (\|Kh_n - Kf_0\|_Y + \|Kf_0 - g_{\delta_n}\|_Y)^2 + c_0\delta_n.$$

Hence

$$\|Kf_{\delta_n} - g_{\delta_n}\|_Y \leqq \frac{\|K\|}{n} + \delta_n + \sqrt{c_0\delta_n}$$

by (1.2). Therefore

$$\|Kf_{\delta_n} - Kf_0\|_Y \leqq \|Kf_{\delta_n} - g_{\delta_n}\|_Y + \|g_{\delta_n} - Kf_0\|_Y$$
$$\leqq \frac{\|K\|}{n} + \delta_n + \sqrt{c_0\delta_n} + \delta_n \longrightarrow 0$$

as $n \longrightarrow \infty$. Hence $Kf_{\delta_n} \longrightarrow Kf_0$ in Y. By assumption (2.1), it follows that $f_{\delta_n} \longrightarrow f_0$ in $(C_0^\infty(\Omega))'$. Note that $C_0^\infty(D) \subset C_0^\infty(\Omega)$ topologically, by the 0-extension. Therefore if $f_{\delta_n} \longrightarrow f_0$ in $(C_0^\infty(\Omega))'$, then $f_{\delta_n} \longrightarrow f_0$ in $(C_0^\infty(D))'$. Since the limit of a sequence in $(C_0^\infty(D))'$, is unique, it follows from (2.5) that $f_0 = \widetilde{f}$. By $\widetilde{f} \in V_1$, we see that $f_0 \in V_1$. The proof is complete.

In the case of $V_1 = V$, we have a sharper result.

Proposition 2.1. *Let the embedding $V \longrightarrow X$ be compact. Then $f_0 \in V$ if and only if* $\sup_{\delta>0} \|f_\delta^{c_0\delta^2}\|_V < \infty$.

Here we note that we need not assume (2.1) and the linearity of K, but only the density of V in X.

Proof. (i) Let $f_0 \in V$. By the definition of $f_\delta \equiv f_\delta^{c_0\delta^2}$, we have

$$\|Kf_\delta - g_\delta\|_Y^2 + c_0\delta^2\|f_\delta\|_V^2 \leqq \|Kf_0 - g_\delta\|_Y^2 + c_0\delta^2\|f_0\|_V^2 \leqq \delta^2 + c_0\delta^2\|f_0\|_V^2.$$

Hence $\|f_\delta\|_V^2 \leqq \frac{1}{c_0} + \|f_0\|_V^2$. Thus the the proof of the "only if" part is complete.

(ii) Since $\sup_{\in\mathbb{N}} \|f_{\delta_n}\|_V < \infty$, by the reflexiveness of V, we can choose a subsequence f_{δ_n} and $\widetilde{f} \in V$ such that

$$f_{\delta_n} \longrightarrow \widetilde{f} \quad \text{weakly in } V.$$

Hence, since the embedding $V \longrightarrow X$ is compact, we see that

$$f_{\delta_n} \longrightarrow \widetilde{f} \quad \text{strongly in } X. \tag{2.6}$$

In the same way as the proof of Theorem 2.1, we can choose $h_n \in V$ and $\delta_n > 0$, $n \in \mathbb{N}$, and we can prove $Kf_{\delta_n} \longrightarrow Kf_0$ in Y. By (2.6) and the continuity of K, we have $Kf_{\delta_n} \longrightarrow K\widetilde{f}$ in Y. Consequently $Kf_0 = K\widetilde{f}$. Since K is injective and $\widetilde{f} \in V$, the proof is complete.

Scheme for finding D where f_0 is not smooth, that is, $f_0 \notin V_1(D)$.

(i) Find a minimizer f_δ of $F_\alpha(f, g_\delta)$ over V.

(ii) If $\sup_{\delta>0} \|f_\delta\|_V < \infty$, then $f_0 \in V$. We can stop.

(iii) If $\sup_{\delta>0} \|f_\delta\|_V = \infty$, then $f_0 \notin V$ by Proposition 2.1.

We start a localization process of singular points of f_0 to proceed to (iv).

(iv) For $D \subset \Omega$, if $\sup_{\delta>0} \|f_\delta\|_{V_1(D)} = \infty$, then it is possible that D may contain points x_0 where $f_0 \notin V_1(\mathcal{U}(x_0))$. Here $\mathcal{U}(x_0)$ is a neighbourhood of $x_0 \in \Omega$.

For the scheme, we can replace the minimizer f_δ by a quasi-minimizer $\widetilde{f}_\delta$ satisfying

$$\|K\widetilde{f}_\delta - g_\delta\|_Y^2 + \alpha\|\widetilde{f}_\delta\|_V^2 \leqq \inf_{f\in V} \|Kf - g_\delta\|_Y^2 + \alpha\|f\|_V^2 + \varepsilon$$

with a fixed small $\varepsilon > 0$. In a forthcoming paper, we will develop a numerical method on the basis of the above scheme.

Since the converse of Theorem 2.1 is not true, we note that in (iv) we may still have $f_0 \in V_1(D)$. If we assume that $f \in V$ if and only if $f \in V_1(D)$ for any subdomain $D \subset \Omega$, then $f_0 \notin V$ means that there exists some $D \subset \Omega$ such that $f_0 \notin V_1(D)$. Therefore $\sup_{\delta>0} \|f_\delta\|_{V_1(D)} = \infty$. Thus the above step (iv) can be considered effective for finding such D.

Condition (2.1) means that the topology in X induced by K^{-1} is stronger than $(C_0^\infty(\Omega))'$. That is, we define a norm $\|f\|_{X_{-1}}$ by $\|f\|_{X_{-1}} = \|Kf\|_Y$. Since K is injective, $\|\cdot\|_{X_{-1}}$ defines a norm in X. Then the topology generated by $\|\cdot\|_{X_{-1}}$ is stronger than $(C_0^\infty(\Omega))'$. It is usually considered that the distribution $(C_0^\infty(\Omega))'$ gives a weak topology, so that condition (2.1) seems generous. However (2.1) is not satisfied by a backward heat equation.

Backward heat problems. We consider

$$\begin{cases} \partial_t u(x,t) = \partial_x^2 u(x,t), \qquad 0 < x < \pi,\ t > 0, \\ u(0,t) = u(\pi,t) = 0, \qquad t > 0, \\ u(x,0) = f(x), \qquad 0 < x < \pi. \end{cases}$$

We set $X = Y = L^2(0,\pi)$ and

$$(Kf)(x) = \sum_{n=1}^\infty (f,\varphi_n)e^{-n^2T}\varphi_n(x), \quad f \in X,$$

where

$$\varphi_n(x) = \sqrt{\frac{2}{\pi}}\sin nx, \quad 0 < x < \pi, \qquad (f,\varphi_n) = \int_0^\pi f(x)\varphi_n(x)dx.$$

The backward heat problem of determining $f(x) = u(x,0)$, $0 < x < \pi$ by $g(x) = u(x,T)$, $0 < x < \pi$, is described by an operator equation $Kf = g$. Let $f_n = e^{\frac{n^2T}{2}}\varphi_n$, $n \in \mathbb{N}$. Then $\lim_{n\to\infty}\|f_n\|_{X_{-1}} = 0$, but f_n do not converge to 0 in $(C_0^\infty(0,\pi))'$. In fact, $\|f_n\|_{X_{-1}} = e^{-\frac{n^2T}{2}} \longrightarrow 0$ as $n \longrightarrow \infty$. Let $\psi \in C_0^\infty(0,\pi)$ be not identically zero and let us assume that $\lim_{n\to\infty}(f_n,\psi) = 0$. Then

$$\psi = \sum_{n=1}^\infty a_n\varphi_n \quad \text{in } L^2(0,\pi). \tag{2.7}$$

Choosing a bounded neighbourhood U in $\mathbb{C}$ of $[0, \pi]$, we have

$$|\varphi_n(z)| \leqq C_1 e^{C_2 n}, \qquad n \in \mathbb{N},\ z \in U. \tag{2.8}$$

By $\lim_{n\to\infty}(f_n, \psi) = 0$, we have $\lim_{n\to\infty} a_n e^{\frac{n^2 T}{2}} = 0$. Therefore

$$|a_n| \leqq C_3 e^{-\frac{n^2 T}{2}}, \qquad n \in \mathbb{N}. \tag{2.9}$$

By (2.8) and (2.9) the series (2.7) is convergent uniformly in $z \in U \subset \mathbb{C}$, so that ψ is holomorphic in U. By $\psi \in C_0^\infty(0, \pi)$, the unicity theorem yields that $\psi = 0$ in U. This contradicts that $\psi \not\equiv 0$, and so f_n, $n \in \mathbb{N}$ do not converge to 0 in $(C_0^\infty(0, \pi))'$.

Thus in the next section, we give a criterion for $f_0 \not\in V_1$ without (2.1) for non-linear K.

§3. Main result - nonlinear case.

We treat general K which may be nonlinear. However, in order to prove the corresponding result to Theorem 2.1, we have to modify the regularization scheme. We consider a minimization problem:

$$\inf_{f \in V, \|f\|_X \leqq M} \|Kf - g\|_Y^2 + \alpha \|f\|_V^2 \tag{3.1}$$

for a fixed constant $M > 0$. In other words, we will consider the Tikhonov functional over a bounded set in X. From a numerical viewpoint, this extra constraint of the X-boundedness can be expected not to be a serious inconvenience, but we have to take extra cares of the constraint $\|f\|_X \leqq M$ in the numerical implementation.

We can prove the existence of a minimizer of problem (3.1) by the same manner as Lemma 2.1 (i) and for convenience the proof is given in Appendix II.

Lemma 3.1. *Let K be compact. Then there exists a minimizer of (3.1).*

Here we do not know the uniqueness of a minimizer.

We assume that X and Y are reflexive Banach spaces of functions defined in bounded domains Ω and Ω_0 with smooth boundaries respectively, and $V \subset X$ topologically. Moreover let V_1 and X_1 be reflexive Banach spaces of functions in a subdomain $D \subset \Omega$ and $V_1 \subset X_1$ topologically. We set $F_\alpha(f,g) = \|Kf - g\|_Y^2 + \alpha\|f\|_V^2$.

Now we are ready to show our main result for general K.

Theorem 3.1. *We assume that K is compact and that the weak convergence in X implies the weak convergence in X_1. Let $c_0 > 0$ be an arbitrarily fixed constant and let $f_\delta^{c_0\delta^2}$ be a minimizer of $F_{c_0\delta^2}(f, g_\delta)$ over $\{f \in V;\ \|f\|_X \leqq M\}$. If $\sup_{\delta>0} \|f_\delta^{c_0\delta^2}\|_{V_1} < \infty$, then $f_0 \in V_1$.*

Example. Let $X = L^2(\Omega)$ and $X_1 = L^2(D)$. If $f_n \longrightarrow f$ weakly in X, then $f_n \longrightarrow f$ weakly in X_1. Then the assumption in Theorem 3.1 is satisfied. In fact, we have $\lim_{n\to\infty} \int_\Omega f_n \varphi dx = \int_\Omega f\varphi dx$ for any $\varphi \in L^2(\Omega)$. In particular, the limit holds for any $\varphi \in L^2(\Omega)$ with $\varphi_{|\Omega\setminus\overline{D}} = 0$, which means that $f_n \longrightarrow f$ weakly in $X_1 = L^2(D)$.

Proof. In the same way as the proof of Theorem 2.1, we can find $h_n \in V$ satisfying (2.2) and (2.3). We choose $\delta_n > 0$ defined by (2.4) and set $f_{\delta_n} = f_{\delta_n}^{c_0\delta_n^2}$, $n \in \mathbb{N}$. The reflexiveness of V_1 yields a subsequence $\{f_{\delta_n}\}_{n\in\mathbb{N}}$ and $\widetilde{f} \in V_1$ such that

$$f_{\delta_n} \longrightarrow \widetilde{f} \quad \text{weakly in } V_1. \tag{3.2}$$

Moreover, in the same manner as Theorem 2.1, we can prove

$$Kf_{\delta_n} \longrightarrow Kf_0 \quad \text{weakly in } Y. \tag{3.3}$$

Here we note that for nonlinar K we can still obtain that $\lim_{n\to\infty} \|Kh_n - Kf_0\|_Y = 0$ by $\lim_{n\to\infty} \|h_n - f_0\|_X = 0$ by (2.2). On the other hand, by $\|f_{\delta_n}\|_X \leqq M$ for

$n \in \mathbb{N}$, we can extract a subsequence, denoted by the same notations, such that $f_{\delta_n} \longrightarrow \widetilde{f_0}$ weakly in X. By the compactness of K, we see that $Kf_{\delta_n} \longrightarrow K\widetilde{f_0}$ in Y. Hence (3.3) yields $Kf_0 = K\widetilde{f_0}$, so that $f_0 = \widetilde{f_0}$ by the injectivity of K. Consequently $f_{\delta_n} \longrightarrow f_0$ weakly in X. By the assumption of the theorem, we see that $f_{\delta_n} \longrightarrow f_0$ weakly in X_1. On the other hand, since $V_1 \subset X_1$ topologically, (3.2) implies that $f_{\delta_n} \longrightarrow \widetilde{f} \in V_1$ weakly in X_1. Hence $f_0 = \widetilde{f} \in V_1$. Thus the proof of the theorem is complete.

Appendix I. Proof of Lemma 2.1.

Let us set $\mu = \inf_{f\in V} F_\alpha(f,g) = \inf_{f\in V} \|Kf - g\|_Y^2 + \alpha\|f\|_V^2$.

Proof of (i). We can choose a sequence $f_n \in V$ such that $\lim_{n\to\infty} F_\alpha(f_n, g) = \mu$. We see that $\alpha\|f_n\|_V^2$ is bounded, and by $\alpha > 0$ and the reflexiveness of V, we can extract a subsequence of f_n, $n \in \mathbb{N}$, which is denoted again by the same notations, such that f_n converge to some $\widetilde{f}$ weakly in V, and so strongly in X by the compactness of the embedding $V \longrightarrow X$. Since K is continuous, we see that $\lim_{n\to\infty} Kf_n = K\widetilde{f}$ in Y. Moreover the weak convergence in V yields $\|\widetilde{f}\|_V \leqq \liminf_{n\to\infty} \|f_n\|_V$ (e.g., Section 1 in Chapter V in Yosida [12]). Therefore

$$\mu = \lim_{n\to\infty} F_\alpha(f_n, g) \geq \|K\widetilde{f} - g\|_Y^2 + \alpha\|\widetilde{f}\|_V^2.$$

Noting that $\mu = \inf_{f\in V} F_\alpha(f,g)$, we see that $F_\alpha(\widetilde{f}, g) = \mu$. This means the existence of a minimizer.

Proof of (ii). Assume contrarily that there exist two minimizers f_1 and f_2:

$$\|Kf_1 - g\|_Y^2 + \alpha\|f_1\|_V^2 = \|Kf_2 - g\|_Y^2 + \alpha\|f_2\|_V^2 = \mu \tag{1}$$

and

$$\|f_1 - f_2\|_V \geq \varepsilon \tag{2}$$

with some $\varepsilon > 0$. First, in a Banach space Z, by the triangle inequality we can directly verify that

$$\begin{aligned} & s\|x\|_Z^2 + (1-s)\|y\|_Z^2 - \|sx + (1-s)y\|_Z^2 \\ & \geq s(1-s)(\|x\|_Z - \|y\|_Z)^2 \geq 0, \quad 0 \leqq s \leqq 1,\, x, y \in Z. \end{aligned} \tag{3}$$

Next, by (3) with $s = \frac{1}{2}$, we have

$$\begin{aligned} & F_\alpha\left(\frac{f_1+f_2}{2}, g\right) = \left\| K\left(\frac{f_1+f_2}{2}\right) - g\right\|_Y^2 + \alpha \left\|\frac{f_1+f_2}{2}\right\|_V^2 \\ = & \left\| \frac{1}{2}(Kf_1 - g) + \frac{1}{2}(Kf_2 - g)\right\|_Y^2 + \alpha\left\|\frac{1}{2}f_1 + \frac{1}{2}f_2\right\|_V^2 \\ \leqq & \frac{1}{2}\left(\|Kf_1 - g\|_Y^2 + \alpha\|f_1\|_V^2\right) + \frac{1}{2}\left(\|Kf_2 - g\|_Y^2 + \alpha\|f_2\|_V^2\right) = \mu. \end{aligned}$$

Since μ is the minimum, we have

$$\begin{aligned} & \mu = \left\| K\left(\frac{f_1+f_2}{2}\right) - g\right\|_Y^2 + \alpha\left\|\frac{f_1+f_2}{2}\right\|_V^2 \\ = & \frac{1}{2}\left(\|Kf_1 - g\|_Y^2 + \alpha\|f_1\|_V^2\right) + \frac{1}{2}\left(\|Kf_2 - g\|_Y^2 + \alpha\|f_2\|_V^2\right), \end{aligned}$$

that is,

$$\begin{aligned} & \left\{\frac{1}{2}\|Kf_1 - g\|_Y^2 + \frac{1}{2}\|Kf_2 - g\|_Y^2 - \left\|K\left(\frac{f_1+f_2}{2}\right) - g\right\|_Y^2\right\} \\ + & \alpha\left\{\frac{1}{2}\|f_1\|_V^2 + \frac{1}{2}\|f_1\|_V^2 - \left\|\frac{f_1+f_2}{2}\right\|_V^2\right\} = 0. \end{aligned}$$

By (3), the two terms within the brackets are non-negative, so that

$$\frac{1}{2}\|f_1\|_V^2 + \frac{1}{2}\|f_2\|_V^2 = \left\|\frac{f_1+f_2}{2}\right\|_V^2.$$

By (3) with $s = \frac{1}{2}$, we have $\frac{1}{4}(\|f_1\|_V - \|f_2\|_V)^2 = 0$, that is, $\beta = \|f_1\|_V = \|f_2\|_V$. By $f_1 \neq f_2$, we have $\beta \neq 0$.

On the other hand, let $\beta > 0$. Then, for any $\varepsilon > 0$, there exists $\delta = \delta(\beta, \varepsilon) \in (0, 1)$ such that $\|x\|_V = \|y\|_V = \beta$ and $\|x - y\|_V \geq \varepsilon$ imply $\left\|\frac{x+y}{2}\right\|_V \leqq (1-\delta)\beta$.

In fact, by setting $x_1 = \frac{x}{\beta}$ and $y_1 = \frac{y}{\beta}$, the definition of the uniform convexity yields the conclusion.

Hence we see that

$$\left\| \frac{f_1 + f_2}{2} \right\|_V < \beta = \|f_1\|_V = \|f_2\|_V. \tag{4}$$

Moreover

$$\left\| K\left(\frac{f_1 + f_2}{2}\right) - g \right\|_Y^2 \leqq \frac{1}{2}\|Kf_1 - g\|_Y^2 + \frac{1}{2}\|Kf_2 - g\|_Y^2 \tag{5}$$

again by (3). Therefore (4) and (5) yield

$$\begin{aligned} &\left\| K\left(\frac{f_1 + f_2}{2}\right) - g \right\|_Y^2 + \alpha \left\| \frac{f_1 + f_2}{2} \right\|_V^2 \\ <&\frac{1}{2}(\|Kf_1 - g\|_Y^2 + \alpha\|f_1\|_V^2) + \frac{1}{2}(\|Kf_2 - g\|_Y^2 + \alpha\|f_2\|_V^2) = \mu. \end{aligned}$$

This contradicts that μ is the minimum of $F_\alpha(f, g)$ over $f \in V$. Thus the proof of (ii) is complete.

Appendix II. Proof of Lemma 3.1.

Let $f_n \in V$, $\|f_n\|_X \leqq M$ be a minimizing sequence. That is, $\lim_{n\to\infty} F_\alpha(f_n, g) = \mu \equiv \inf_{f\in V, \|f\|_X \leqq M} F_\alpha(f, g)$. Moreover $\{\|f_n\|_V\}_{n\in\mathbb{N}}$ is a bounded sequence, so that the reflexiveness of V implies that there exists a subsequence of $\{f_n\}_{n\in\mathbb{N}}$, denoted by the same notations, so that $f_n \longrightarrow \widetilde{f}$ weakly in V and $\|\widetilde{f}\|_V \leqq \liminf_{n\to\infty} \|f_n\|_V$ (e.g., [12]). Moreover, by the reflexiveness of X and $\|f_n\|_X \leqq M$, we can again extract a subsequence, denoted again by $\{f_n\}_{n\in\mathbb{N}}$, such that $f_n \longrightarrow f^0$ weakly in X. Since the embedding $V \longrightarrow X$ is continuous, we see that the weak convergence in V implies the weak convergence in X. Therefore $f_n \longrightarrow \widetilde{f}$ weakly in V means that $f_n \longrightarrow \widetilde{f}$ weakly in X. Hence $f^0 = \widetilde{f} \in V$. Since K is compact, it follows that $Kf_n \longrightarrow K\widetilde{f}$ in Y. Therefore $\mu = \liminf_{n\to\infty} F_\alpha(f_n, g) \geq F_\alpha(\widetilde{f}, g)$. Furthermore

$\|\widetilde{f}\|_X \leqq \liminf_{n\to\infty} \|f_n\|_X \leqq M$ by the property of the weak convergence (e.g., [12]). Since $\widetilde{f} \in V$, $\|\widetilde{f}\|_X \leqq M$ and μ is the minimum, we see that $F_\alpha(\widetilde{f}, g) = \mu$. That is, $\widetilde{f}$ is a minimizer. Thus the proof of the lemma is complete.

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