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by

Yasufumi Osajima

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

The Asymptotic Expansion Formula of Implied Volatility for Dynamic SABR Model and FX Hybrid Model

Yasufumi OSAJIMA *
Mitsubishi UFJ Securities Co., Ltd.
and
Graduate School of Mathematical Sciences
The University of Tokyo

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Abstract

The author considers SABR (stochastic-$\alpha\beta\rho$) model which is a two factor stochastic volatility model and give an asymptotic expansion formula of implied volatilities for this model. His approach is based on infinite dimensional analysis on the Malliavin calculus and large deviation. Furthermore, he applies the approach to a foreign exchange model where interest rates and the FX volatilities are stochastic and gives an asymptotic expansion formula of implied volatilities of foreign exchange options.

1 Introduction

In financial markets, the Black-Scholes formula [2] has been widely used to price European options. However, several assumptions are inconsistent with real markets. In the Black-Schloes model, assets are assumed to follow constant volatility log-normal processes. It is a common practice to quote option prices in terms of 'implied volatility'. Given a price, the implied volatility is determined for each call option as the unique value of the volatility parameter for which the Black-Scholes formula agrees with that price. In the original Black-Scholes model, the implied volatility must be constant independent of strike rate. But in real financial markets such as foreign exchange options and stock index options, observed implied volatilities depend on strike rate. They are the lowest for at-the-money options and progressively higher as an option moves in the money or out of the money. These phenomena are called 'volatility smile'.

There are two well-known models to explain these phenomena. The first class of models are called local volatility models for which the volatility is assumed to depend on time and the spot price of the underlying (cf. Dupire [5]). The second class of models are stochastic volatility models such as Hull-White [9] and Heston [8] studied.

In the present paper, we consider a mixture of them which is called 'SABR' (stochastic-$\alpha\beta\rho$) model. This model is popular among practitioners because an accurate asymptotic expansion formula of implied volatilities is known and is well-fit to the volatility smile. This model was first

*The opinions expressed are those of the author and not necessarily those of Mitsubishi UFJ Securities. Please address correspondence to: Mitsubishi UFJ Securities Co., Ltd. Research and Development Division, Marunouchi Building, 2-4-1, Marunouchi, Chiyoda-ku, Tokyo 100-6317, Japan, E-mail: osajima-yasufumi@sc.mufg.jp

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introduced by Hagan-Kumar-Lesniewski-Woodward [7]. They gave the formula in [6], [7] using singular perturbation techniques.

We will give a mathematically rigorous proof for the formula. Our approach is small volatility asymptotic expansion based on infinite dimensional analysis called the Watanabe-Yoshida theory on the Malliavin calculus (cf. Watanabe [17], Yoshida [18] and Kunitomo-Takahashi [10]).

Furthermore, we introduce a new model 'FX hybrid SABR model' which is a foreign exchange model where interest rates and FX volatilities are stochastic. We apply our approach to this model and give an asymptotic expansion formula for implied volatilities of foreign exchange options.

1.1 Dynamic SABR model

Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{0\leqslant t\leqslant T})$ be a complete probability space satisfying the usual hypotheses and $T \in (0,\infty)$ denotes some fixed time horizon of economy. Let $(W_1(t), W_2(t)),\ 0 \le t \le T$, be a 2-dimensional correlated Brownian motion with correlation given by $\rho : [0,T] \to [-1,1]$ such that

$$d\langle W_1, W_2\rangle_t = \rho(t)dt,\ d\langle W_1\rangle_t = d\langle W_2\rangle_t = dt,$$

Let $C : \mathbb{R} \to \mathbb{R}_+$ be a smooth function whose derivatives of any order are bounded. Let σ, ν be continuous $\mathbb{R}_+$-valued functions defined on $[0,T]$. We consider the following stochastic differential equation for X and α;

$$\begin{aligned} &dX^\varepsilon(t) = \varepsilon\alpha^\varepsilon(t)\sigma(t)b(X^\varepsilon(t))dW_1(t),\\ &d\alpha^\varepsilon(t) = \varepsilon\nu(t)\alpha^\varepsilon(t)dW_2(t),\\ &X^\varepsilon(0) = X_0,\ \alpha^\varepsilon(0) = \alpha. \end{aligned} \tag{1.1}$$

Here X, α and ν are considered an underlying process, 'volatility-like' parameter, the volatility of volatility respectively. This model is known as 'dynamic SABR model' to practitioners.

Under this model, we want to calculate forward values of call options. Since no analytic formula is known, we will calculate the asymptotic expansion, and furthermore we will calculate the asymptotic expansion of implied volatilities.

First, we define implied normal volatility. We denote by $V(T,K)$ the forward value of a European call option with strike rate K and maturity T, i.e.,

$$V(T,K) = E[(X(T) - K)_+].$$

Since X is a martingale,

$$V(T,K) \ge (X_0 - K)_+.$$

Let $\phi : \mathbb{R} \to \mathbb{R}$ be given by

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2},\ x \in \mathbb{R}.$$

We define $G : \mathbb{R} \to \mathbb{R}_+$ by

$$G(x) = \int_x^\infty (y - x)\phi(y)dy,\ x \in \mathbb{R}.$$

When we consider the following normal model

$$d\tilde{X}(t) = \sigma d\tilde{W}(t),\ \tilde{X}(0) = X_0,$$

the forward value of a call option is given by

$$V_N(T,K,\sigma) = \sigma\sqrt{T}G\Big(\frac{K - X_0}{\sigma\sqrt{T}}\Big).$$

Since V_N is strictly increasing in σ and

$$\lim_{\sigma\downarrow 0} V_N(T,K,\sigma) = (X_0 - K)_+,$$
$$\lim_{\sigma\uparrow\infty} V_N(T,K,\sigma) = +\infty,$$

there is an unique $\sigma_N(K) > 0$ satisfying

$$V(T,K) = V_N(T,K,\sigma_N(K)),$$

which is called *implied normal volatility.*

In the same way, under the following log-normal model

$$\frac{d\tilde{X}(t)}{\tilde{X}(t)} = \sigma d\tilde{W}(t),\ \ \tilde{X}(0) = X_0,$$

the forward value of a call option is given by the famous Black-Scholes formula

$$V_{BS}(T,K,\sigma_L) = X_0\Phi(d_1) - K\Phi(d_2)$$

where Φ is the normal distribution function and

$$d_{1,2} = \frac{\log(X_0/K) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}.$$

Since V_{BS} is also strictly increasing in σ and

$$\lim_{\sigma\downarrow 0} V_{BS}(T,K,\sigma) = (X_0 - K)_+,$$
$$\lim_{\sigma\uparrow\infty} V_{BS}(T,K,\sigma) = +\infty,$$

there is an unique $\sigma_{BS}(T,K) > 0$ satisfying

$$V(T,K) = V_{BS}(T,K,\sigma_{BS}(T,K)),$$

which is called *implied volatility.*

The asymptotic expansion of the implied volatilities for dynamic SABR model is the following.

THEOREM 1.1. *For each $y \in \mathbb{R}$, let $K_\varepsilon = K_\varepsilon(y) = X_0 + \varepsilon\Sigma_n y = X_0(1+\varepsilon\Sigma_l y),\ \varepsilon \in (0,1]$. For any $r \in [0,\infty)$, there is a constant $R > 0$ such that*

$$(1)\qquad \Big|\frac{\sigma_N(T,K_\varepsilon)}{\varepsilon} - \frac{\Sigma_n}{\sqrt{T}}\Big\{1 + \varepsilon\Big(\frac{\gamma_1}{2} + C_1\Big)\Sigma_n y + \varepsilon^2\Big(\frac{2\gamma_2 - \gamma_1^2}{12} + C_2\Big)(\Sigma_n y)^2$$
$$+\varepsilon^2\Big(\frac{2\gamma_2 - \gamma_1^2}{24} + \frac{\gamma_1}{2}C_1 + C_3\Big)\Sigma_n^2\Big\}\Big| \le \varepsilon^3 R,\ \ \varepsilon \in (0,1],\ \ y \in [-r,r].$$

$$(2)\qquad \Big|\frac{\sigma_{BS}(T,K_\varepsilon)}{\varepsilon} - \frac{\Sigma_l}{\sqrt{T}}\Big\{1 + \varepsilon\Big(\frac{\tilde{\gamma}_1}{2} + \tilde{C}_1\Big)\Sigma_l y + \varepsilon^2\Big(\frac{2\tilde{\gamma}_2 - \tilde{\gamma}_1^2 - \tilde{\gamma}_1}{12} - \frac{\tilde{C}_1}{2} + \tilde{C}_2\Big)(\Sigma_l y)^2$$
$$+\varepsilon^2\Big(\frac{2\tilde{\gamma}_2 - \tilde{\gamma}_1^2 + 2\tilde{\gamma}_1}{24} + \frac{1+\tilde{\gamma}_1}{2}\tilde{C}_1 + \tilde{C}_3\Big)\Sigma_l^2\Big\}\Big| \le \varepsilon^3 R,\ \ \varepsilon \in (0,1],\ \ y \in [-r,r].$$

Here,

$$\xi = \Big(\int_0^T \sigma^2(t)dt\Big)^{1/2},\ C(x) = b(x)/x,\ \Sigma_n = b(X_0)\alpha\xi,\ \Sigma_l = C(X_0)\alpha\xi,$$

$$\gamma_1 = \frac{b'(X_0)}{b(X_0)},\ \gamma_2 = \frac{b''(X_0)}{b(X_0)},\ \tilde{\gamma}_1 = \frac{C'(X_0)X_0}{C(X_0)},\ \tilde{\gamma}_2 = \frac{C''(X_0)X_0}{C(X_0)},$$

$$\eta_1 = \frac{2}{\xi^4}\int_0^T \Big(\int_0^t \sigma(s)\nu(s)\rho(s)ds\Big)\sigma^2(t)dt = \frac{2}{\xi^4}\int_0^T \Big(\int_t^T \sigma^2(s)ds\Big)\sigma(t)\nu(t)\rho(t)dt, \tag{1.2}$$

$$\eta_2^2 = \frac{12}{\xi^8}\int_0^T \Big(\int_0^t \Big(\int_s^t \sigma(u)\nu(u)\rho(u)du\Big)^2 \sigma^2(s)ds\Big)\sigma^2(t)dt, \tag{1.3}$$

$$v_1^2 = \frac{3}{\xi^6}\int_0^T \Big(\int_t^T \sigma^2(s)ds\Big)^2 \nu^2(t)dt = \frac{6}{\xi^6}\int_0^T \Big(\int_0^t \Big(\int_0^s \nu^2(u)du\Big)\sigma^2(s)ds\Big)\sigma^2(t)dt, \tag{1.4}$$

$$v_2^2 = \frac{6}{\xi^6}\int_0^T \Big(\int_t^T \sigma^2(s)ds\Big)\Big(\int_0^t \sigma^2(s)ds\Big)\nu^2(t)dt, \tag{1.5}$$

$$C_1 = \frac{\eta_1}{2b(X_0)\alpha},\ C_2 = \frac{4v_1^2 + 3(\eta_2^2 - 3\eta_1^2)}{24b(X_0)^2\alpha^2},\ C_3 = \frac{2v_2^2 - 3\eta_2^2}{24b(X_0)^2\alpha^2},$$

$$\tilde{C}_1 = C_1X_0,\ \tilde{C}_2 = C_2X_0^2,\ \tilde{C}_3 = C_3X_0^2,$$

In the case $\sigma(t) \equiv 1,\ \rho(t) \equiv \rho,\ \nu(t) \equiv \nu$, these parameters are $\eta_1 = \eta_2 = \nu\rho,\ v_1 = v_2 = \nu$, and

$$C_1 = \frac{\rho}{2}\Big(\frac{\nu}{b(X_0)\alpha}\Big),\ C_2 = \frac{2-3\rho^2}{12}\Big(\frac{\nu}{b(X_0)\alpha}\Big)^2,\ C_3 = \frac{2-3\rho^2}{24}\Big(\frac{\nu}{b(X_0)\alpha}\Big)^2.$$

In the case $\nu \equiv 0$, i.e. deterministic volatility case, this formula is almost the same as 'Equivalent Black volatilities' given in Hagan-Woodward [6].

Furthermore we obtain an another asymptotic expansion using large deviation approach. Berestycki-Busca-Florent [3] has also investigated this problem in more general settings.

THEOREM 1.2. *For any $K_0 > X_0$, the implied normal volatility for SABR model satisfies*

$$\lim_{\varepsilon\downarrow 0}\sup_{X_0\le K\le K_0}\Big|\frac{\sigma_N(T,K)}{\varepsilon} - \frac{\alpha(K-X_0)}{\int_{X_0}^K \frac{dx}{b(x)}}\cdot\frac{\zeta}{\hat{x}(\zeta)}\Big| = 0,$$

and the implied volatility for SABR model satisfies

$$\lim_{\varepsilon\downarrow 0}\sup_{X_0\le K\le K_0}\Big|\frac{\sigma_{BS}(T,K)}{\varepsilon} - \frac{\alpha\log(K/X_0)}{\int_{X_0}^K \frac{dx}{b(x)}}\cdot\frac{\zeta}{\hat{x}(\zeta)}\Big| = 0,$$

where

$$\zeta = -\frac{\nu}{\alpha}\int_{X_0}^K \frac{dz}{b(z)},\ \hat{x}(\zeta) = \log\Big(\frac{\sqrt{1-2\rho\zeta+\zeta^2} - \rho + \zeta}{1-\rho}\Big).$$

In the case $K < X_0$, we can apply our approach to the put option and obtain the same formula.

In Remark 3.2, we will show the relation with Hagan et al. [7].

1.2 FX Hybrid SABR model

We introduce a new model 'FX hybrid SABR model'. We apply our approach to foreign exchange model where interest rates and volatilities are stochastic. Let $(\Omega, \mathcal{F}, \tilde{P}, \{\mathcal{F}_t\}_{0\leqslant t\leqslant T})$ be a complete probability space satisfying the usual hypotheses. Let $\rho_{ij} : [0,T] \to [-1,1]$ $(0 \le i,j \le 3)$ be continuous functions such that $\rho_{ii} = 1$ and the matirix $(\rho_{ij})_{0\le i,j\le 3}$ is non-negative definite symmetric matrix. We assume $\rho_{13} = \rho_{23} = 0$. Let $\tilde{W}_i(t),\ 0 \le t \le T,\ 0 \le i \le 3$, be a 4-dimensional correlated Brownian motion with correlation such that

$$d\langle \tilde{W}_i, \tilde{W}_j\rangle_t = \rho_{ij}(t)dt,\ d\langle \tilde{W}_i\rangle_t = dt.$$

Let $C : \mathbb{R} \to \mathbb{R}_+$ be a smooth function whose derivatives of any order are bounded. Let κ_i and $\theta_i,\ i = 1,2$, be continuous $\mathbb{R}$-valued functions defined on $[0,T]$. Let $\sigma_i,\ i = 1,2,\ \sigma,\ \nu$ and L be continuous $\mathbb{R}_+$-valued functions defined on $[0,T]$. We consider the following stochastic differential equation for S, r, q and α;

$$\begin{aligned}
\frac{dS^{\varepsilon,\delta}(t)}{S^{\varepsilon,\delta}(t)} &= (r^\delta(t) - q^\delta(t))dt + \varepsilon\alpha^\varepsilon(t)\sigma(t)C(S^{\varepsilon,\delta}(t)/L(t))d\tilde{W}_0(t),\\
dr^\delta(t) &= (\theta_1^\delta(t) - \kappa_1(t)r^\delta(t))dt + \delta\sigma_1(t)d\tilde{W}_1(t),\\
dq^\delta(t) &= (\theta_2^\delta(t) - \kappa_2(t)q^\delta(t))dt + \delta\sigma_2(t)d\tilde{W}_2(t),\\
d\alpha^\varepsilon(t) &= \varepsilon\nu(t)\alpha(t)d\tilde{W}_3(t).
\end{aligned}$$

We consider $\tilde{P}$ a risk neutral measure, $S = S^{\varepsilon,\delta}$ a foreign exchange rate process, $r = r^\delta$ and $q = q^\delta$ processes for short rate of domestic and foreign economy respectively, and $\alpha = \alpha^\varepsilon$ a stochastic volatility process of foreign exchange. L is a time-dependent scaling constant. We assume $L(t)$ as a forward foreign exchange rate at time t as explained below. Piterbarg [15] considered the case $\nu \equiv 0$, i.e. deterministic volatility case. In real financial markets, volatilities of short rate processes are much smaller than that of foreign exchange rates, and so it is reasonable to assume $\delta = \varepsilon^2$.

Let $P_i(t,T),\ i = 1,2$, be the price at time t of zero-coupon bonds maturing at time T of domestic and foreign currency respectively. These are given by

$$P_1(t,T) = E^{\tilde{P}}\big[\exp\big(-\int_t^T r(s)ds\big)\big|\mathcal{F}_t\big],$$

and

$$P_2(t,T) = E^{\tilde{P}}\big[\frac{S(T)}{S(t)}\exp\big(-\int_t^T r(s)ds\big)\big|\mathcal{F}_t\big],$$

respectively. The drifts of short rate $\theta_1^\delta,\ \theta_2^\delta$ are chosen so that

$$P_1^\delta(0,t) = E^{\tilde{P}}\big[\exp\big(-\int_0^t r^\delta(s)ds\big)\big],\ 0 \le t \le T,$$

and

$$P_2^{\varepsilon,\delta}(0,t) = E^{\tilde{P}}\big[\frac{S^{\varepsilon,\delta}(t)}{S_0}\exp\big(-\int_0^t r^\delta(s)\big)\big],\ 0 \le t \le T,$$

are coincides the discount factors calculated from the initial yield curve. In particular,

$$P_1^0(t,T) = \exp(-\int_t^T r^0(s)ds),\ P_2^{0,0}(t,T) = \exp(-\int_t^T q^0(s)ds),\ S^{0,0}(t) = F(0,t).$$

We denote by P the measure associated with the numeraire $P_1(t,T),\ 0 \le t \le T$, and by E the corresponding expectation. This measure P is called T-forward measure. To calculate the value of a European call option with maturity T, it is useful to calculate under T-forward measure P. By Girsanov's theorem, a 4-dimensional P-Brownian motion $W_i(t),\ 0 \le t \le T,\ 0 \le i \le 3$, is given by

$$\begin{aligned}
dW_0(t) &= d\tilde{W}_0(t) + \varepsilon\rho_1(t)\alpha^\varepsilon(t)\sigma(t)C(S^{\varepsilon,\delta}(t)/L(t))dt,\\
dW_1(t) &= d\tilde{W}_1(t) + \delta\sigma_1(t,T)dt,\\
dW_2(t) &= d\tilde{W}_2(t) + \delta\sigma_2(t,T)\rho_{12}(t)dt - \varepsilon\alpha^\varepsilon(t)\sigma(t)C(S^{\varepsilon,\delta}(t)/L(t))\rho_2(t)dt,\\
dW_3(t) &= d\tilde{W}_3(t),
\end{aligned}$$

with correlation such that

$$d\langle W_i, W_j\rangle_t = \rho_{ij}(t)dt,\ d\langle W_i\rangle_t = dt.$$

The foreign exchange rate S and zero-coupon bonds of each currency $P_i(t,T),\ 0\le t\le T,\ i=1,2$ satisfies the following stochastic differential equation;

$$\begin{aligned}
&\frac{dS^{\varepsilon,\delta}(t)}{S^{\varepsilon,\delta}(t)} = \mu^{\varepsilon,\delta}(t)dt + \varepsilon\alpha^\varepsilon(t)\sigma(t)C(S^{\varepsilon,\delta}(t)/L(t))dW_0(t),\\
&\frac{dP_1^{\delta}(t,T)}{P_1^{\delta}(t,T)} = (r^\delta(t) + \delta^2\sigma_1(t,T)^2)dt + \delta\sigma_1(t,T)dW_1(t),\\
&\frac{dP_2^{\varepsilon,\delta}(t,T)}{P_2^{\varepsilon,\delta}(t,T)} = \Big(q^\delta(t) - \varepsilon\delta\rho_{02}(t)\alpha^\varepsilon(t)\sigma(t)C(S^{\varepsilon,\delta}(t)/L(t))\sigma_2(t,T) + \delta^2\rho_{12}(t)\sigma_1(t,T)\sigma_2(t,T)\Big)dt\\
&+\delta\sigma_2(t,T)dW_2(t),
\end{aligned}$$

where

$$\begin{aligned}
\mu^{\varepsilon,\delta}(t) &= r^\delta(t) - q^\delta(t) + \varepsilon\delta\rho_{01}(t)\alpha^\varepsilon(t)\sigma(t)C(S^{\varepsilon,\delta}(t)/L(t))\sigma_1(t,T),\\
\varphi_i(t) &= \exp(\int_0^t \kappa_i(s)ds),\ \psi_i(t) = \int_0^t \varphi_i(s)^{-1}ds,\\
\sigma_i(t,T) &= -\sigma_i(t)\varphi_i(t)\{\psi_i(T)-\psi_i(t)\},\ i=1,2.
\end{aligned}$$

Let $F(t,T) = F^{\varepsilon,\delta}(t,T)$ be the forward exchange rate at time t for maturity T;

$$F(t,T) = \frac{S(t)P_2(t,T)}{P_1(t,T)}.$$

Then the forward exchange rate $F^{\varepsilon,\delta}(t,T),\ 0\le t\le T$, is a martingale and satisfies the following stochastic differential equation;

$$\frac{dF^{\varepsilon,\delta}(t,T)}{F^{\varepsilon,\delta}(t,T)} = \varepsilon\alpha^\varepsilon(t)\sigma(t)C(S^{\varepsilon,\delta}(t)/L(t))dW_0(t) - \delta\sigma_1(t,T)dW_1(t) + \delta\sigma_2(t,T)dW_2(t).$$

In the present paper, we assume $L(t) = F(0,t)$. In this model, we are interested in the forward value of a foreign exchange call option;

$$V(T,K) = E[(S(T)-K)_+] = E[(F(T,T)-K)_+].$$

The basic hybrid model is the case $C\equiv 1$ and $\alpha\equiv 1$ (cf. Dempster-Hutton [4]). In this case, the implied volatility of the call option is given by

$$\sigma_{BS}(T,K) = \Big(\frac{1}{T}\int_0^T (\delta^2 a(t,T) + 2\varepsilon\delta b(t,T)\sigma(t) + \varepsilon^2\sigma^2(t))dt\Big)^{1/2},$$

where

$$\begin{aligned}
a(t,T) &= \sigma_1^2(t,T) + \sigma_2^2(t,T) - 2\rho_{12}(t)\sigma_1(t,T)\sigma_2(t,T),\\
b(t,T) &= \rho_{02}(t)\sigma_2(t,T)\sigma(t) - \rho_{01}(t)\sigma_1(t,T)\sigma(t).
\end{aligned}$$

In general case, we give the asymptotic expansion formula of the implied volatility as follows.

Theorem 1.3. *For each $y\in\mathbb{R}$, let $K_\varepsilon = K_\varepsilon(y) = F(0,T)(1+\varepsilon\Sigma_l y),\ \varepsilon\in(0,1]$. For any $r\in[0,\infty)$ there is a constant $R>0$ such that*

$$\begin{aligned}
\Big|\frac{\sigma_{BS}(T,K_\varepsilon)}{\varepsilon} - \frac{\Sigma_l}{\sqrt{T}}\Big\{\frac{\Sigma_{fwd}}{\Sigma_l} + \varepsilon\Big(\frac{\tilde\gamma_1}{2} + C_1 + \varepsilon\big(D_1\tilde\gamma_1 + D_2\big)\Big)\Sigma_l y + \varepsilon^2\Big(\frac{2\tilde\gamma_2 - \tilde\gamma_1^2 - \tilde\gamma_1}{12} - \frac{C_1}{2} + C_2\Big)(\Sigma_l y)^2\\
+\varepsilon^2\Big(\frac{2\tilde\gamma_2 - \tilde\gamma_1^2 + 2\tilde\gamma_1}{24} + \frac{1+\tilde\gamma_1}{2}C_1 + C_3\Big)\Sigma_l^2\Big\}\Big| \le \varepsilon^3 R,\ \varepsilon\in(0,1],\ y\in[-r,r].
\end{aligned}$$

Here, $\eta_1,\ \eta_2,\ v_1, v_2,$ are (1.2), (1.3), (1.4), (1.5) respectively and

$$\Sigma_{fwd} = \Big(\int_0^T (\sigma^2(t) + 2\varepsilon b(t,T)\sigma(t) + \varepsilon^2 a(t,T))dt\Big)^{1/2},$$

$$\xi = \Big(\int_0^T \sigma^2(t)dt\Big)^{1/2},\ \ \Sigma_l = C(1)\alpha\xi,\ \tilde{\gamma}_1 = \frac{C'(1)}{C(1)},\ \tilde{\gamma}_2 = \frac{C''(1)}{C(1)},$$

$$C_1 = \frac{\eta_1}{2C(1)\alpha},\ C_2 = \frac{4v_1^2 + 3(\eta_2^2 - 3\eta_1^2)}{24C(1)^2\alpha^2},\ C_3 = \frac{2v_2^2 - 3\eta_2^2}{24C(1)^2\alpha^2},$$

$$D_1 = -\frac{1}{2\xi^2}\int_0^T b(t,T)dt - \frac{1}{\xi^4}\int_0^T \Big(\int_0^t b(s,t)ds\Big)\sigma^2(t)dt,$$

$$D_2 = \frac{1}{\xi^4}\int_0^T \Sigma_{\sigma\nu}(t)b(t,T)dt - \Big(\frac{3}{\xi^6}\int_0^T \Sigma_{\sigma\nu}(t)\sigma^2(t)dt\Big)\Big(\int_0^T b(t,T)dt\Big).$$

2 Dynamic SABR model

In this section, we will prove Theorem 1.1. Since the stochastic differential equation for X can be written as

$$dX(t) = \big(\frac{\alpha(t)}{\alpha}\big)\big(\alpha b(X_0)\sigma(t)\big)\big(\frac{b(X_t)}{b(X_0)}\big)dW_1(t),$$

it is enough to prove in the case, $\alpha = 1$ and $b(X_0) = 1$. Then $\Sigma_n = \xi$. First we calculate the asymptotic expansion of the forward value of call option.

Proposition 2.1. *For each $y \in \mathbb{R}$, let $K_\varepsilon = K_\varepsilon(y) = X_0 + \varepsilon\xi y,\ \varepsilon \in (0,1]$. For any $r \in [0,\infty)$, there is a constant $R > 0$ such that*

$$\sup_{y\in[-r,r]}\Big|V(T,K_\varepsilon) - \varepsilon\xi\Big[G(y) + \varepsilon c_1 y\phi(y) + \varepsilon^2\Big(c_2(y^2-1) + c_3\Big)\phi(y)$$
$$+ \frac{\varepsilon^2}{2}\Big(c_1^2(y^4 - 6y^2 + 3) + c_4(y^2-1) + c_5\Big)\phi(y)\Big]\Big| \le \varepsilon^4 R.$$

Here, we define $\Sigma_\sigma : [0,T] \to \mathbb{R}$, $\Sigma_\nu : [0,T] \to \mathbb{R}$ and $\Sigma_{\sigma\nu} : [0,T] \to \mathbb{R}$ by

$$\Sigma_\sigma(t) = \int_0^t \sigma^2(s)ds,\ \Sigma_\nu(t) = \int_0^t \nu^2(s)ds,\ \Sigma_{\sigma\nu}(t) = \int_0^t \rho(s)\sigma(s)\nu(s)ds,\ t \in [0,T],$$

and

$$c_1 = \frac{\gamma_1\xi}{2} + \frac{1}{\xi^3}\int_0^T \Sigma_{\sigma\nu}(t)\sigma^2(t)dt,$$

$$c_2 = \frac{(\gamma_2 + \gamma_1^2)\xi^2}{6} + \frac{\gamma_1}{\xi^2}\int_0^T \Sigma_{\sigma\nu}(t)\sigma^2(t)dt + \frac{1}{2\xi^4}\int_0^T \Sigma_{\sigma\nu}^2(t)\sigma^2(t)dt,$$

$$c_3 = \frac{\gamma_2\xi^2}{4} + \frac{\gamma_1}{\xi^2}\int_0^T \Sigma_{\sigma\nu}(t)\sigma^2(t)dt,$$

$$c_4 = \gamma_1^2\xi^2 + \frac{4\gamma_1}{\xi^2}\int_0^T \Sigma_{\sigma\nu}(t)\sigma^2(t)dt + \frac{2}{\xi^4}\int_0^T\Big(\int_0^t \Sigma_\nu(s)\sigma^2(s)ds\Big)\sigma^2(t)dt + \frac{2}{\xi^4}\int_0^T \Sigma_{\sigma\nu}^2(t)\sigma^2(t)dt,$$

$$c_5 = \frac{\gamma_1^2\xi^2}{2} + \frac{2\gamma_1}{\xi^2}\int_0^T \Sigma_{\sigma\nu}(t)\sigma^2(t)dt + \frac{1}{\xi^2}\int_0^T \Sigma_\nu(t)\sigma^2(t)dt.$$

Proof. Let $H \equiv L^2([0,T],\mathbb{R}^2)$ be the Cameron-Martin space. Let $D^{k,p}$ $(k,p>1)$ be the Sobolev spaces of k-times Malliavin differentiable random variables which norms are defined by;

$$\|F\|_{k,p} \equiv \big[E|F|^p + \sum_{j=1}^k E[\|D^jF\|^p_{H^{\otimes j}}]\big]^{1/p}.$$

We define

$$D^{\infty} = \bigcap_{k>0} \bigcap_{1<p<\infty} D^{k,p},\ D^{-\infty} = \bigcup_{k>0} \bigcup_{1<p<\infty} D^{k,-p}.$$

The small volatility asymptotic expansion of processes X and α are given by

$$\begin{aligned} X^{\varepsilon}(t) &= X_0 + \xi(\varepsilon g_1(t) + \varepsilon^2 g_2(t) + \varepsilon^3 g_3(t) + \varepsilon^4 r(t,\varepsilon)), \\ \alpha^{\varepsilon}(t) &= 1 + \varepsilon \int_0^t \nu(s)dW_2(s) + \varepsilon^2 \int_0^t \Big(\int_0^s \nu(u)dW_2(u)\Big)\nu(s)dW_2(s) + \varepsilon^3 a(t,\varepsilon), \end{aligned}$$

as Watanabe [17], Yoshida [18] and

$$\begin{aligned} g_1(t) &= \frac{1}{\xi}\frac{\partial X^{\varepsilon}(t)}{\partial \varepsilon}\Big|_{\varepsilon=0} = \frac{1}{\xi}\int_0^t \sigma(s)dW_1(s), \\ g_2(t) &= \frac{1}{2\xi}\frac{\partial^2 X^{\varepsilon}(t)}{\partial \varepsilon^2}\Big|_{\varepsilon=0} \\ &= \frac{\gamma_1}{\xi}\int_0^t \Big(\int_0^s \sigma(u)dW_1(u)\Big)\sigma(s)dW_1(s) + \frac{1}{\xi}\int_0^t \Big(\int_0^s \nu(u)dW_2(u)\Big)\sigma(s)dW_1(s), \\ g_3(t) &= \frac{1}{6\xi}\frac{\partial^3 X^{\varepsilon}(t)}{\partial \varepsilon^3}\Big|_{\varepsilon=0} \\ &= \frac{\gamma_1^2}{\xi}\int_0^t \Big(\int_0^s \Big(\int_0^u \sigma(v)dW_1(v)\Big)\sigma(u)dW_1(u)\Big)\sigma(s)dW_1(s) \\ &+ \frac{\gamma_2}{2\xi}\int_0^t \Big(\int_0^s \sigma(u)dW_1(u)\Big)^2 \sigma(s)dW_1(s) \\ &+ \frac{\gamma_1}{\xi}\int_0^t \Big(\int_0^s \Big(\int_0^u \nu(v)dW_2(v)\Big)\sigma(u)dW_1(u)\Big)\sigma(s)dW_1(s) \\ &+ \frac{\gamma_1}{\xi}\int_0^t \Big(\int_0^s \sigma(u)dW_1(u)\Big)\Big(\int_0^s \nu(u)dW_2(u)\Big)\sigma(s)dW_1(s) \\ &+ \frac{1}{\xi}\int_0^t \Big(\int_0^s \Big(\int_0^u \nu(v)dW_2(v)\Big)\nu(u)dW_2(u)\Big)\sigma(s)dW_1(s). \end{aligned} \tag{2.1}$$

Then, $g_1 \sim N(0,1)$ and using formulas in Appendix A, the conditional expectations are given as follows;

$$\begin{aligned} E[g_2|g_1 = x] &= c_1(x^2-1), \\ E[g_3|g_1 = x] &= c_2(x^3-3x) + c_3 x, \\ E[g_2^2|g_1 = x] &= c_1^2(x^4 - 6x^2 + 3) + c_4(x^2-1) + c_5. \end{aligned}$$

We define

$$Y^{\varepsilon} = \frac{X^{\varepsilon}(T) - X_0}{\varepsilon\xi} = g_1 + \varepsilon g_2 + \varepsilon^2 g_3 + \varepsilon^3 r(T,\varepsilon).$$

We define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x_+$. Then the forward value of a call option is given by

$$\begin{aligned} V(T, K_\varepsilon) &= E[f(X^{\varepsilon}(T) - K_\varepsilon)] \\ &= \varepsilon\xi E[f(Y^{\varepsilon} - y)]. \end{aligned}$$

For every $y \in [-r, r]$, we define $T_y \in \mathcal{S}'(\mathbb{R})$ as $T_y(x) = f(x-y)$. Since the non-degeneracy of Malliavin covariance (See Appendix B), we can apply Theorem 2.3 in Watanabe [17]. So there exist $s>0$ such that for every $y\in[-r,r]$, $T_y(Y^\varepsilon)$ has the asymptotic expansion in $\bigcap_{1<p<\infty} D_p^{-s}$ and

$$\sup_{y\in[-r,r]}\left\|T_y(Y^\varepsilon)-\Phi_0(y)-\varepsilon\Phi_1(y)-\varepsilon^2\Phi_2(y)\right\|_{D_p^{-s}}=O(\varepsilon^3),\ 1<p<\infty, \tag{2.2}$$

where

$$\begin{aligned}\Phi_0(y)&=T_y(g_1),\ \Phi_1(y)=\frac{\partial}{\partial x}T_y(g_1)g_2,\\ \Phi_2(y)&=\frac{\partial}{\partial x}T_y(g_1)g_3+\frac{1}{2!}\frac{\partial^2}{\partial x^2}T_y(g_1)g_2^2.\end{aligned}$$

Here, we will remark the uniformness in (2.2). In the proof of Theorem 2.3 in Watanabe [17], we can find a positive integer m and a bounded function $\phi_y(x)$ on $\mathbb{R}$ which is 3-times continuously differentiable in (x,y) with bounded derivatives up to 3rd order such that $T_y=(1+x^2-\Delta)^m\phi_y$. Then we can prove (2.2) in the same way as the proof of Theorem 2.3 in Watanabe [17].

Therefore there is a constant $R>0$ such that

$$\sup_{y\in[-r,r]}\left|E[T_y(Y^\varepsilon)]-E[\Phi_0(y)]-\varepsilon E[\Phi_1(y)]-\varepsilon^2E[\Phi_2(y)]\right|\le\varepsilon^3R.$$

Since

$$\frac{\partial}{\partial x}T_y(x)=1_{(0,\infty)}(x-y),\ \frac{\partial^2}{\partial x^2}T_y(x)=\delta(y),$$

we can calculate each terms explicitly as follows;

$$\begin{aligned}E[\Phi_0(y)]&=G(y),\\ E[\Phi_1(y)]&=\int_y^\infty E[g_2|g_1=x]\phi(x)dx=c_1y\phi(y),\\ E[\Phi_2(y)]&=\int_y^\infty E[g_3|g_1=x]\phi(x)dx+\frac{\varepsilon^2}{2}\int_{-\infty}^\infty E[g_2^2|g_1=x]\delta(x-y)\phi(x)dx\\ &=\Big(c_2(y^2-1)+c_3\Big)\phi(y)+\frac{\varepsilon^2}{2}\Big(c_1^2(y^4-6y^2+3)+c_4(y^2-1)+c_5\Big)\phi(y).\end{aligned}$$

We define

$$\begin{aligned}J(\varepsilon,y)=G(y)&+\varepsilon c_1y\phi(y)+\varepsilon^2\Big(c_2(y^2-1)+c_3\Big)\phi(y)\\ &+\frac{\varepsilon^2}{2}\Big(c_1^2(y^4-6y^2+3)+c_4(y^2-1)+c_5\Big)\phi(y).\end{aligned}$$

Then from above asymptotic expansion, we have

$$\sup_{y\in[-r,r]}\Big|E[f(Y^\varepsilon-y)]-J(\varepsilon,y)\Big|\le\varepsilon^3R.$$

Therefore asymptotic expansion of call option price is given as follows;

$$\begin{aligned}\sup_{y\in[-r,r]}\Big|V(T,K_\varepsilon)-\varepsilon\xi\Big[G(y)&+\varepsilon c_1y\phi(y)+\varepsilon^2\Big(c_2(y^2-1)+c_3\Big)\phi(y)\\ &+\frac{\varepsilon^2}{2}\Big(c_1^2(y^4-6y^2+3)+c_4(y^2-1)+c_5\Big)\phi(y)\Big]\Big|\le\varepsilon^4R.\end{aligned}$$

Thus we have our assertion. □

To calculate the implied normal volatility, we need inverse function of V_N.

LEMMA 2.2. *Let $H(z;y) \equiv zG(y/z)$, then, there is a smooth function $L(x;y) : \mathbb{R}_+ \to \mathbb{R}$ such that*

$$H(L(x;y);y) = x$$

and $H(z_0;y) = x_0$ then

$$\frac{d}{dx}L(x;y)|_{x=x_0} = \frac{1}{\phi(y/z_0)},$$
$$\frac{d^2}{dx^2}L(x;y)|_{x=x_0} = -\frac{y^2}{z_0^3}\frac{1}{\phi(y/z_0)^2}.$$

Proof. Since $\frac{d}{dz}H(z;y) = \phi(y/z) > 0$, H is strongly increasing in z. Therefore, from implicit function theorem, H has a smooth inverse function $L(x;y) : \mathbb{R} \to \mathbb{R}$ such that

$$H(L(x;y);y) = x.$$

Let $L(x_0;y) = z_0$. Then we have

$$\frac{d^2}{dz^2}H(z;y) = \frac{y^2}{z^3}\phi(y/z)$$

We can calculate L' and L'' easily. □

Finally we will prove Theorem 1.1.

Proof of Theorem 1.1. Since

$$V_N(T, X_0 + \varepsilon\xi y, \sigma) = \sigma\sqrt{T} \cdot G(\frac{\varepsilon\xi y}{\sigma\sqrt{T}}) = H(\sigma\sqrt{T}; \varepsilon\xi y) = \varepsilon\xi H(\frac{\sigma\sqrt{T}}{\varepsilon\xi}; y),$$

the implied normal volatility can be written using L as follows;

$$\sigma_N(K) = \frac{\varepsilon\xi}{\sqrt{T}}L(\frac{V(T;K_\varepsilon)}{\varepsilon\xi}; y) = \frac{\varepsilon\xi}{\sqrt{T}}L(J(\varepsilon,y);y).$$

Since $J(0,y) = G(y) = H(1;y)$, we have

$$L(J(0,y);y) = 1.$$

Using Proposition 2.1 and Lemma 2.2, the Taylor expansion of $L(J(\varepsilon,y))$ around $\varepsilon = 0$ is given by

$$L(J(\varepsilon,y);y) = 1 + \varepsilon\Big(\frac{\gamma_1}{2} + C_1\Big)\xi y + \varepsilon^2\Big(\frac{2\gamma_2 - \gamma_1^2}{12} + C_2\Big)\xi^2 y^2 + \varepsilon^2\Big(\frac{2\gamma_2 - \gamma_1^2}{24} + \frac{\gamma_1 C_1}{2} + C_3\Big)\xi^2,$$
$$\varepsilon \in (0,1],\ y \in [-r, r],$$

where

$$C_1 = -\frac{\gamma_1}{2} + \frac{c_1}{\xi},$$
$$C_2 = -\frac{2\gamma_2^2 - \gamma_1^2}{12} + \frac{1}{\xi^2}(c_2 - 3c_1^2 + \frac{c_4}{2}),$$
$$C_3 = -\frac{2\gamma_2 - \gamma_1^2}{24} - \frac{\gamma_1 C_1}{2} + \frac{1}{\xi^2}(-c_2 + c_3 + \frac{3}{2}c_1^2 - \frac{c_4}{2} + \frac{c_5}{2}).$$

This implies our assertion (1).

Next we will prove the asymptotic expansion for implied volatility. Apply the above formula to the log-normal case i.e. $C(x) = x$, we obtain

$$\begin{aligned}\sigma_{BS}(T, K_\varepsilon) &= \Big(1 - \frac{\varepsilon\xi y}{2X_0} + \frac{1}{3}\big(\frac{\varepsilon\xi y}{X_0}\big)^2\Big)\frac{\sigma_N(T, K_\varepsilon)}{X_0}\big(1 + \frac{\varepsilon^2\xi^2}{24X_0^2} + O(\varepsilon^3)\big)\\ &= \frac{\xi}{X_0\sqrt{T}}\Big\{1 + \varepsilon\big(\frac{\gamma_1}{2} + C_1 - \frac{1}{2X_0}\big)\xi y + \varepsilon^2\big(\frac{2\gamma_2 - \gamma_1^2}{12} + C_2 - \frac{\gamma_1 + 2C_1}{4X_0}\big)\xi^2 y^2\\ &\quad + \varepsilon^2\big(\frac{2\gamma_2 - \gamma_1^2}{24} + \frac{\gamma_1 C_1}{2} + C_3 + \frac{1}{24X_0^2}\big)\xi^2 + O(\varepsilon^3)\Big\}.\end{aligned}$$

Thus we have our assertion (2). This completes the proof of Theorem 1.1. □

3 Large deviation approach

In this section, we will prove Theorem 1.2. We consider the original SABR model (i.e. ρ and ν are constant). Let $(W(t), Z(t))$, $0 \le t \le T$, be a 2-dimensional standard Brownian motion. We consider the following stochastic differential equation for X and α;

$$\begin{aligned}&dX^\varepsilon(t) = \varepsilon\alpha^\varepsilon(t)b(X^\varepsilon(t))\big(\sqrt{1-\rho^2}dW(t) + \rho dZ(t)\big),\\ &d\alpha^\varepsilon(t) = \varepsilon\nu\alpha^\varepsilon(t)dW(t),\\ &X^\varepsilon(0) = X_0,\ \alpha^\varepsilon(0) = \alpha.\end{aligned}$$

We calculate the asymptotic expansion of density function of X under original SABR model. Since $X \in D^\infty$ and non-degeneracy of Malliavin covariance, using Watanabe [17] we can define

$$p^\varepsilon(t; y) = E[\delta(X^\varepsilon(t) - y)].$$

We consider the following associated ordinary differential equation;

$$\begin{aligned}&\frac{df(t;h)}{dt} = a(t;h)b(f(t;h))(\sqrt{1-\rho^2}\dot{h}_1(t) + \rho\dot{h}_2(t)),\\ &\frac{da(t;h)}{dt} = \nu a(t;h)\dot{h}_2(t),\\ &f(0,h) = X_0,\ a(0,h) = \alpha.\end{aligned}$$

We define

$$e(y) = \inf\{\frac{1}{2}\int_0^T |\dot{h}(s)|^2 ds; f(T,h) = y\},\ y \in \mathbb{R}.$$

The function e is called 'energy' and $e(0) = 0$. Let

$$a_\varepsilon(y) = (2\pi\varepsilon^2)^{1/2}p^\varepsilon(T, y)\exp\big(\frac{e(y)}{\varepsilon^2}\big),\ y \in \mathbb{R}.$$

Then from Watanabe [17] and Kusuoka-Stroock [12], there exists a bounded function $a : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{\varepsilon\downarrow 0} a_\varepsilon(y) = a(y) > 0.$$

In particular, we have

$$e(y) = \lim_{\varepsilon\downarrow 0}\varepsilon^2 \log p^\varepsilon(t, y).$$

In this model, we can calculate the energy term explicitly.

THEOREM 3.1. *In SABR model, the energy term is*

$$e(K) = \frac{1}{2\nu^2 T}\log\Big(\frac{\sqrt{1-2\rho\zeta+\zeta^2}-\rho+\zeta}{1-\rho}\Big)^2 = \frac{\hat{x}(\zeta(K))^2}{2\nu^2 T},$$

where

$$\zeta(K) = -\frac{\nu}{\alpha}\int_{X_0}^{K}\frac{dz}{b(z)}.$$

See Appendix C for the proof.

Using this theorem, we can calculate the asymptotic expansion of the implied volatility to the first order.

Proof of Theorem 1.2. We will calculate the forward value of a call option

$$\begin{aligned} V^{\varepsilon}(T,K) &= \int_{-\infty}^{\infty}(y-K)_+ p^{\varepsilon}(T,y)dy \\ &= \tilde{V}^{\varepsilon}(T,K) + R_{\varepsilon}(K_0), \end{aligned}$$

where

$$\begin{aligned} \tilde{V}^{\varepsilon}(T,K) &= \int_{K}^{2K_0}(y-K)p^{\varepsilon}(T,y)dy = \int_{K}^{2K_0}(y-K)\big(\frac{1}{2\pi\varepsilon^2}\big)^{1/2}e^{-e(y)/\varepsilon^2}a_{\varepsilon}(y)dy, \\ R^{\varepsilon}(K_0) &= \int_{2K_0}^{\infty}(y-K)p^{\varepsilon}(T,y)dy = E[X^{\varepsilon}(T)-K,\ X^{\varepsilon}(T)>2K_0]. \end{aligned}$$

We define $g:\mathbb{R}\to\mathbb{R}$ as

$$g(x) = \frac{\hat{x}(\zeta(x))}{\nu},$$

then g satisfies

$$e(g(x)) = \frac{x^2}{2}.$$

Then

$$\tilde{V}^{\varepsilon}(T,K) = \int_{g^{-1}(K)}^{g^{-1}(2K_0)}(g(x)-K)_+\big(\frac{1}{2\pi\varepsilon^2}\big)^{1/2}e^{-x^2/2\varepsilon^2}a_{\varepsilon}(g(x))g'(x)dx.$$

Let $A_{\varepsilon}(x) \equiv a_{\varepsilon}(g(x))g'(x)$ and $\tilde{K_0} = g^{-1}(2K_0) - g^{-1}(K)$. Putting $x = z + g^{-1}(K)$, we have

$$\begin{aligned} &\exp\Big(\frac{g^{-1}(K)^2}{2\varepsilon^2}\Big)\tilde{V}^{\varepsilon}(T,K) \\ &= \int_0^{\tilde{K_0}}\big(g(z+g^{-1}(K))-K\big)\big(\frac{1}{2\pi\varepsilon^2}\big)^{1/2}\exp\Big(-\frac{z^2}{2\varepsilon^2}-\frac{zg^{-1}(K)}{2\varepsilon^2}\Big)A_{\varepsilon}(z+g^{-1}(K))dz \\ &\le g^{-1}(K_0)K_0\big(\frac{1}{2\pi\varepsilon^2}\big)^{1/2}\sup_{z\in[0,g^{-1}(2K_0)]}\big|A_{\varepsilon}(z)\big| \end{aligned}$$

We will estimate the l.h.s. from below. Since there is a $0\le\theta\le 1$,

$$\inf_{z\in[0,K_0]}\frac{g(z+g^{-1}(K))-K}{z} = g'(\theta_0 z + g^{-1}(K)) \ge \inf_{z\in[0,g^{-1}(2K_0)]}g'(z) \equiv C,$$

therefore

$$\begin{aligned}
&\exp(\frac{g^{-1}(K)^2}{2\varepsilon^2})\tilde{V}(T,K)\\
&\geq \int_0^{\varepsilon^2} Cz\big(\frac{1}{2\pi\varepsilon^2}\big)^{1/2}\exp(-\frac{z^2}{2\varepsilon^2}-\frac{zg^{-1}(K)}{2\varepsilon^2})\inf_{z\in[0,\tilde{K}_0]} A_\varepsilon(z+g^{-1}(K))dz\\
&\geq \exp(-\frac{1}{2}-\frac{g^{-1}(K)}{2})\frac{C\varepsilon^3}{2}\big(\frac{1}{2\pi}\big)^{1/2}
\end{aligned}$$

Since $e(K)=g^{-1}(K)^2/2$ and A_ε is bounded, we have

$$\big|\varepsilon^2\log\tilde{V}(T,K)+e(K)\big|\leq\varepsilon A,\ \ X_0\leq K\leq K_0.$$

For any $\delta>0$

$$\begin{aligned}
R_\varepsilon(K_0)&\leq E[X^\varepsilon(T);X^\varepsilon(T)>2K_0]\\
&\leq E[X^\varepsilon(T)^{1/\delta}]^\delta P(X^\varepsilon(T)>2K_0)^{1-\delta},
\end{aligned}$$

and so

$$\lim_{\varepsilon\downarrow 0}\varepsilon^2\log R_\varepsilon(K_0)\leq\lim_{\varepsilon\downarrow 0}\varepsilon^2(1-\delta)\log P(X^\varepsilon(T)>2K_0)=-(1-\delta)e(2K_0).$$

Since $e(2K_0)>e(K_0)$,

$$\lim_{\varepsilon\downarrow 0}\varepsilon^2\log R^\varepsilon(K_0)<-e(K_0).$$

Therefore

$$\lim_{\varepsilon\downarrow 0}\sup_{K\in[X_0,K_0]}\big|\varepsilon^2\log V^\varepsilon(T,K)+e(K)\big|=0.$$

On the other hand, the forward value of call option under the normal model is given by

$$V_N^\varepsilon(T,K,\varepsilon\sigma)=\exp\big(-\frac{(K-X_0)^2}{2\varepsilon^2\sigma^2T}\big)\int_0^\infty\frac{x}{\sqrt{2\pi\varepsilon^2\sigma^2T}}\exp\big(-\frac{x^2}{2\varepsilon^2\sigma^2T}\big)\exp\big(-\frac{(K-X_0)x}{\varepsilon^2\sigma^2T}\big)dx.$$

Therefore, for any $0<\sigma_0<\sigma_1$ we have

$$\lim_{\varepsilon\downarrow 0}\sup_{\substack{K\in[X_0,K_0]\\ \sigma\in[\sigma_0,\sigma_1]}}\big|\varepsilon^2\log V_N(T,K,\varepsilon\sigma)+\frac{(K-X_0)^2}{2\sigma^2T}\big|=0.$$

In the log-normal model, we can calculate in the same way;

$$\lim_{\varepsilon\downarrow 0}\sup_{\substack{K\in[X_0,K_0]\\ \sigma\in[\sigma_0,\sigma_1]}}\big|\varepsilon^2\log V_{BS}(T,K,\varepsilon\sigma)+\frac{\log(K/X_0)^2}{2\sigma^2T}\big|=0.$$

Since $K\in[X_0,K_0]$, there exist $\sigma_0,\ \sigma_1\in\mathbb{R}_+$ such that $\sigma_0\leq\sigma_N(T,K)\leq\sigma_1$ and $\sigma_0\leq\sigma_{BS}(T,K)\leq\sigma_1$. Noting

$$V_N(T,K,\varepsilon\frac{\sigma_N^\varepsilon(T,K)}{\varepsilon})=V^\varepsilon(T,K),$$

and

$$V_{BS}(T,K,\varepsilon\frac{\sigma_{BS}^\varepsilon(T,K)}{\varepsilon})=V^\varepsilon(T,K),$$

we have

$$\lim_{\varepsilon\downarrow 0}\sup_{K\in[X_0,K_0]}\Big|e(K)-\frac{(K-X_0)^2}{2\big(\frac{\sigma_N(K)}{\varepsilon}\big)^2T}\Big|=\lim_{\varepsilon\downarrow 0}\sup_{K\in[X_0,K_0]}\Big|e(K)-\frac{\log(K/X_0)^2}{2\big(\frac{\sigma_{BS}(K)}{\varepsilon}\big)^2T}\Big|=0.$$

This implies our Theorem 1.2. □

Finally, we show the relation with SABR formula in Hagan et al. [7]. We will investigate more in the forthcoming paper [14].

Remark 3.2 (SABR formula). In the case $\sigma(t) \equiv \sigma$ and $\nu(t) \equiv \nu$, the implied normal volatility for SABR model is given as follows using Theorem 1.1;

$$\begin{aligned}\sigma_N(T,K) = b(X_0)\alpha\Big\{1 &+ \varepsilon\Big(\frac{\gamma_1}{2} + \frac{\rho}{2}\big(\frac{\nu}{b(X_0)\alpha}\big)\Big)(K-X_0)\\ &+ \varepsilon^2\Big(\frac{2\gamma_2-\gamma_1^2}{12} + \frac{2-3\rho^2}{12}\big(\frac{\nu}{b(X_0)\alpha}\big)^2\Big)(K-X_0)^2\\ &+ \varepsilon^2\Big(\frac{2\gamma_2-\gamma_1^2}{24}b(X_0)^2\alpha^2 + \frac{\rho\nu\gamma_1 b(X_0)\alpha}{4} + \frac{2-3\rho^2}{24}\nu^2\Big)T + O(\varepsilon^3)\Big\}.\end{aligned}$$

On the other hand, if $K - X_0 = O(\varepsilon)$, then the Taylor expansion of $\sigma_N(K)$ around X_0 in Theorem 1.2 coincides the first two terms, i.e.

$$\begin{aligned}\frac{\varepsilon\alpha(K-X_0)}{\int_{X_0}^K \frac{dz}{b(z)}}\Big(\frac{\zeta}{\hat{x}(\zeta)}\Big) \sim b(X_0)\alpha\Big\{1 &+ \varepsilon\Big(\frac{\gamma_1}{2} + \frac{\rho}{2}\big(\frac{\nu}{b(X_0)\alpha}\big)\Big)(K-X_0)\\ &+ \varepsilon^2\Big(\frac{2\gamma_2-\gamma_1^2}{12} + \frac{2-3\rho^2}{12}\big(\frac{\nu}{b(X_0)\alpha}\big)^2\Big)(K-X_0)^2\Big\}.\end{aligned}$$

Therefore, as $\varepsilon \downarrow 0$, the implied normal volatility for SABR model is given as follows;

$$\begin{aligned}\sigma_N(T,K) &= \frac{\varepsilon\alpha(K-X_0)}{\int_{X_0}^K \frac{dx}{b(x)}}\left(\frac{\zeta}{\hat{x}(\zeta)}\right)\\ &\left\{1 + \left[\frac{2\gamma_2-\gamma_1^2}{24}\alpha^2 b^2(X_0) + \frac{1}{4}\rho\nu\alpha\gamma_1 b(X_0) + \frac{2-3\rho^2}{24}\nu^2\right]\varepsilon^2 T + O(\varepsilon^3)\right\}.\end{aligned}$$

Similarly, the implied volatility for SABR model is given as follows;

$$\begin{aligned}\sigma_{BS}(T,K) &= \frac{\varepsilon\alpha\log(K/X_0)}{\int_{X_0}^K \frac{dx}{b(x)}}\left(\frac{\zeta}{\hat{x}(\zeta)}\right)\\ &\left\{1 + \left[\frac{2\gamma_2-\gamma_1^2+1/X_0^2}{24}\alpha^2 b^2(X_0) + \frac{1}{4}\rho\nu\alpha\gamma_1 b(X_0) + \frac{2-3\rho^2}{24}\nu^2\right]\varepsilon^2 T + O(\varepsilon^3)\right\}.\end{aligned}$$

These are the SABR formulas.

4 FX Hybrid SABR Model

In this section, we will prove Theorem 1.3. Since the stochastic differential equation for S can be written as

$$\frac{dS(t)}{S(t)} = \mu(t)dt + \varepsilon\big(\frac{\alpha(t)}{\alpha}\big)\big(\alpha C(1)\sigma(t)\big)\big(\frac{C(S(t)/L(t))}{C(1)}\big)dW_0(t),$$

it is enough to prove in the case, $\alpha = 1$ and $C(1) = 1$. Then $\Sigma_n = \xi$. First we calculate the asymptotic expansion of the forward value of a call option.

Proposition 4.1. *For each $y \in \mathbb{R}$, let $K_\varepsilon = K_\varepsilon(y) = F(0,T)(1+\varepsilon\xi y)$, $\varepsilon \in (0,1]$. For any $r \in [0,\infty)$, there is a constant $R > 0$ such that,*

$$\begin{aligned}&\Big|V(T,K_\varepsilon) - \varepsilon\xi F(0,T)\Big\{G(y) + \varepsilon(c_1 y + d_1)\phi(y) + \varepsilon^2(c_2(y^2-1) + c_3 + d_2 y)\phi(y)\\ &+ \frac{\varepsilon^2}{2}\Big(c_1^2(y^4 - 6y^2 + 3) + c_4(y^2-1) + c_5\\ &+ d_1^2(y^2-1) + d_3 + 2c_1 d_1(y^3 - 3y) + 2d_4 y\Big)\phi(y)\Big\}\Big| \le \varepsilon^3 R,\ \varepsilon \in (0,1],\ y \in [-r,r],\end{aligned}$$

where

$$
\begin{aligned}
c_1 &= \frac{\gamma_1 \xi}{2}, + \frac{1}{\xi^3}\int_0^T \Sigma_{\sigma\nu}(t)\sigma^2(t)dt, \\
c_2 &= \frac{(\gamma_2+\gamma_1^2)\xi^2}{6} + \frac{\gamma_1}{\xi^2}\int_0^T \Sigma_{\sigma\nu}(t)\sigma^2(t)dt + \frac{1}{2\xi^4}\int_0^T \Sigma_{\sigma\nu}^2(t)\sigma^2(t)dt, \\
c_3 &= \frac{\gamma_2\xi^2}{4} + \frac{\gamma_1}{\xi^2}\int_0^T \Sigma_{\sigma\nu}(t)\sigma^2(t)dt, \\
c_4 &= \gamma_1^2\xi^2 + \frac{4\gamma_1}{\xi^2}\int_0^T \Sigma_{\sigma\nu}(t)\sigma^2(t)dt + \frac{2}{\xi^4}\int_0^T \Big(\int_0^t \Sigma_{\nu}(s)\sigma^2(s)ds\Big)\sigma^2(t)dt + \frac{2}{\xi^4}\int_0^T \Sigma_{\sigma\nu}^2(t)\sigma^2(t)dt, \\
c_5 &= \frac{\gamma_1^2\xi^2}{2} + \frac{2\gamma_1}{\xi^2}\int_0^T \Sigma_{\sigma\nu}(t)\sigma^2(t)dt + \frac{1}{\xi^2}\int_0^T \Sigma_{\nu}(t)\sigma^2(t)dt, \\
d_1 &= \frac{1}{\xi^2}\int_0^T b(t,T)dt, \\
d_2 &= \frac{1}{\xi}\int_0^T b(t,T)dt - \frac{\gamma_1 - 1}{\xi^3}\int_0^T \Big(\int_0^t b(s,t)ds\Big)\sigma^2(t)dt, \\
d_3 &= \frac{1}{\xi^2}\int_0^T a(t,T)dt, \\
d_4 &= \frac{\gamma_1}{\xi}\int_0^T b(t,T)dt + \frac{1}{\xi^3}\int_0^T \Sigma_{\sigma\nu}(t)b(t,T)dt, \\
\gamma_1 &= 1 + C'(1),\ \gamma_2 = C''(1) + 2C'(1),\ \xi = \int_0^T \sigma^2(t)dt.
\end{aligned}
$$

Proof. We can calculate the asymptotic expansion in the same way as Theorem 1.1. We define

$$
\begin{aligned}
X^{\varepsilon,\delta}(t) &= \frac{F^{\varepsilon,\delta}(t,T)}{F(0,T)}, \\
R(t) &= \int_0^t \sigma_1(s,T)dW_1(s) - \int_0^t \sigma_2(s,T)dW_2(s), \\
l(t) &= \int_0^t \varphi_1(s)^{-1}\Big(\int_0^s \varphi_1(u)\sigma_1(u)dW_1(u)\Big)ds - \int_0^t \varphi_2(s)^{-1}\Big(\int_0^s \varphi_2(u)\sigma_2(u)dW_2(u)\Big)ds.
\end{aligned}
$$

Since we assume $\delta = \varepsilon^2$, the small volatility asymptotic expansion of $X^{\varepsilon,\delta}$ up to $O(\varepsilon^3)$ is

$$
X^{\varepsilon,\delta}(t) = 1 + \xi(\varepsilon g_1(t) + \varepsilon^2 g_2(t) + \varepsilon^3 g_3(t) + \delta h_1(t) + \varepsilon\delta h_2(t) + r(\varepsilon,\delta)),
$$

where

$$
\begin{aligned}
g_1(t) &= \frac{1}{\xi}\frac{\partial X^{\varepsilon,\delta}(t)}{\partial\varepsilon}\Big|_{\varepsilon,\delta=0},\ g_2(t) = \frac{1}{2\xi}\frac{\partial^2 X^{\varepsilon,\delta}(t)}{\partial\varepsilon^2}\Big|_{\varepsilon,\delta=0},\ g_3(t) = \frac{1}{6\xi}\frac{\partial^3 X^{\varepsilon,\delta}(t)}{\partial\varepsilon^3}\Big|_{\varepsilon,\delta=0}, \\
h_1(t) &= \frac{1}{\xi}\frac{\partial X^{\varepsilon,\delta}(t)}{\partial\delta}\Big|_{\varepsilon,\delta=0} = -\frac{1}{\xi}R(t), \\
h_2(t) &= \frac{1}{\xi}\frac{\partial^2 X^{\varepsilon,\delta}(t)}{\partial\varepsilon\partial\delta}\Big|_{\varepsilon,\delta=0} \\
&= -\int_0^t g_1(s)dR(s) - \frac{1}{\xi}\int_0^t \sigma(s)R(s)dW(s) + \frac{\gamma_1 - 1}{\xi}\int_0^t \sigma(s)l(s)dW(s), \\
h_3(t) &= \frac{1}{2\xi}\frac{\partial^2 X^{\varepsilon,\delta}(t)}{\partial\delta^2}\Big|_{\varepsilon,\delta=0} = \frac{1}{2\xi}\int_0^t R(s)dR(s).
\end{aligned}
$$

Since $\delta = 0$ means deterministic interest rate, g_1, g_2, g_3 is (2.1) replace γ_1 by $1 + C'(1)$, γ_2 by $2C'(1) + C''(1)$. Then, $g_1 \sim N(0,1)$ and conditional expectations are

$$\begin{aligned}
E[g_2|g_1 = x] &= c_1(x^2 - 1),\\
E[g_3|g_1 = x] &= c_2(x^3 - 3x) + c_3 x,\\
E[g_2^2|g_1 = x] &= c_1^2(x^4 - 6x^2 + 3) + c_4(x^2 - 1) + c_5.
\end{aligned}$$

We need to calculate

$$E[h_1|g_1 = x],\ E[h_2|g_1 = x],\ E[h_1 g_2|g_1 = x],\ E[h_1^2|g_1 = x].$$

These conditional expectations are given by

$$\begin{aligned}
E[h_1|g_1 = x] &= d_1 x,\\
E[h_2|g_1 = x] &= d_2(x^2 - 1),\\
E[h_1^2|g_1 = x] &= d_1^2(x^2 - 1) + d_3,\\
E[g_2 h_1|g_1 = x] &= c_1 d_1(x^3 - 3x) + d_4 x.
\end{aligned}$$

Putting $\delta = \varepsilon^2$, we define

$$Y^\varepsilon = \frac{X^{\varepsilon,\varepsilon^2}(T) - 1}{\varepsilon\xi} = g_1 + \varepsilon(g_2 + h_1) + \varepsilon^2(g_3 + h_2) + \varepsilon^3 r(T,\varepsilon),$$

and $y = \frac{k-1}{\varepsilon\xi}$. Then the forward value of the call option is given by

$$V(T, K_\varepsilon) = \varepsilon F(0,T)\xi E[f(Y^\varepsilon - y)].$$

Then as in (2.2), there exists $R > 0$ such that

$$\sup_{y\in[-r,r]} \left| E[T_y(Y^\varepsilon)] - E[\Phi_0(y)] - \varepsilon E[\Phi_1(y)] - \varepsilon^2 E[\Phi_2(y)] \right| \le \varepsilon^3 R,$$

where

$$\begin{aligned}
\Phi_0(y) &= T_y(g_1),\ \Phi_1(y) = \frac{\partial}{\partial x} T_y(g_1)(g_2 + h_1),\\
\Phi_2(y) &= \frac{\partial}{\partial x} T_y(g_1)(g_3 + h_2) + \frac{1}{2!}\frac{\partial^2}{\partial x^2} T_y(g_1)(g_2 + h_1)^2.
\end{aligned}$$

We can calculate each term explicitly as follows;

$$\begin{aligned}
E[\Phi_0(y)] &= G(y),\\
E[\Phi_1(y)] &= \int_y^\infty E[g_2 + h_1|g_1 = x]\phi(x)dx = (c_1 y + d_1)\phi(y),\\
E[\Phi_2(y)] &= \int_y^\infty E[g_3 + h_2|g_1 = x]\phi(x)dx + \frac{\varepsilon^2}{2}\int_{-\infty}^\infty E[(g_2 + h_1)^2|g_1 = x]\delta(x - y)\phi(x)dx\\
&= \Big(c_2(y^2 - 1) + c_3 + d_2 y\Big)\phi(y) + \frac{\varepsilon^2}{2}\Big(c_1^2(y^4 - 6y^2 + 3) + c_4(y^2 - 1) + c_5\\
&\quad + d_1^2(y^2 - 1) + d_3 + 2c_1 d_1(y^3 - 3y) + 2d_4 y\Big)\phi(y).
\end{aligned}$$

This implies our assertion. □

Proof of Theorem 1.3. As in the proof of Theorem 1.1, the implied normal volatility is given by

$$\sigma_N(T,K_\varepsilon)=\frac{\varepsilon\xi F(0,T)}{\sqrt{T}}L\Big(\frac{V(T;K_\varepsilon)}{\varepsilon\xi F(0,T)};y\Big).$$

From Proposition 4.1, there is a constant $R'>0$ such that

$$\Big|\frac{\sigma_N(T,K_\varepsilon)}{\varepsilon}-\frac{F(0,T)\xi}{\sqrt{T}}L(J(\varepsilon;y);y)\Big|\le\varepsilon^3R'.$$

Using the Taylor expansion given in Lemma 1.2, the implied normal volatility satisfies

$$\begin{aligned}\Big|\frac{\sigma_N(T,K_\varepsilon)}{\varepsilon}&-\frac{F(0,T)\xi}{\sqrt{T}}\Big\{\Big(1+\varepsilon d_1-\frac{\varepsilon^2d_1^2}{2}+\frac{\varepsilon^2d_3}{2}\Big)+\varepsilon\Big(\frac{\gamma_1}{2}+C_1+\varepsilon\frac{d_2-3c_1d_1+d_4}{\xi}\Big)\xi y\\&+\varepsilon^2\Big(\frac{2\gamma_2-\gamma_1^2}{12}+C_2\Big)\xi^2y^2+\varepsilon^2\Big(\frac{2\gamma_2-\gamma_1^2}{24}+\frac{\gamma_1C_1}{2}+C_3\Big)\xi^2\Big\}\Big|\le\varepsilon^3R'.\end{aligned}$$

Therefore, there is a constant $R>0$ such that the implied volatility satisfies

$$\begin{aligned}\Big|\frac{\sigma_{BS}(T,K_\varepsilon)}{\varepsilon}&-\frac{\xi}{\sqrt{T}}\Big\{\Big(1+\varepsilon d_1-\frac{\varepsilon^2d_1^2}{2}+\frac{\varepsilon^2d_3}{2}\Big)+\varepsilon\Big(\frac{\gamma_1-1}{2}+C_1\Big)\xi y+\varepsilon^2(D_1(\gamma_1-1)+D_2)\xi y\\&+\varepsilon^2\Big(\frac{2\gamma_2-\gamma_1^2-3\gamma_1+4}{12}-\frac{C_1}{2}+C_2\Big)\xi^2y^2+\varepsilon^2\Big(\frac{2\gamma_2-\gamma_1^2+1}{24}+\frac{\gamma_1C_1}{2}+C_3\Big)\xi^2\Big\}\Big|\le\varepsilon^3R,\end{aligned}$$

where

$$\begin{aligned}D_1&=-\frac{1}{2\xi^2}\int_0^T b(t,T)dt-\frac{1}{\xi^4}\int_0^T\Big(\int_0^t b(s,t)ds\Big)\sigma^2(t)dt,\\D_2&=\frac{1}{\xi^4}\int_0^T\Sigma_{\sigma\nu}(t)b(t,T)dt-\Big(\frac{3}{\xi^6}\int_0^T\Sigma_{\sigma\nu}(t)\sigma^2(t)dt\Big)\int_0^T b(t,T)dt.\end{aligned}$$

Since Σ_{fwd} is given by

$$\Sigma_{fwd}=\frac{\xi}{\sqrt{T}}\Big(1+\varepsilon d_1-\frac{\varepsilon^2d_1^2}{2}+\frac{\varepsilon^2d_3}{2}+O(\varepsilon^3)\Big),$$

and $\gamma_1=1+\tilde\gamma_1,\ \gamma_2=\tilde\gamma_2+2\tilde\gamma_1$, we have

$$\begin{aligned}\Big|\frac{\sigma_{BS}(T,K_\varepsilon)}{\varepsilon}&-\frac{\xi}{\sqrt{T}}\Big\{\frac{\Sigma_{fwd}}{\xi}+\varepsilon\Big(\frac{\tilde\gamma_1}{2}+C_1\Big)\xi y+\varepsilon^2\Big(D_1\tilde\gamma_1+D_2\Big)\xi y\\&+\varepsilon^2\Big(\frac{2\tilde\gamma_2-\tilde\gamma_1^2-\tilde\gamma_1}{12}-\frac{C_1}{2}+C_2\Big)\xi^2y^2+\varepsilon^2\Big(\frac{2\tilde\gamma_2-\tilde\gamma_1^2+2\tilde\gamma_1}{24}+\frac{(1+\tilde\gamma_1)C_1}{2}+C_3\Big)\xi^2\Big\}\Big|\le\varepsilon^3R.\end{aligned}$$

This completes the proof of Theorem 1.3. □

A Conditional expectations of Wiener chaos

Lemma A.1. *Let $\{(W_0, W_1, W_2, W_3), 0 \le t \le T\}$ be a 4-dimensional correlated Brownian motion with correlation given by $\rho_{ij} : [0,T] \to [-1,1]$ such that*

$$d\langle W_i, W_j\rangle_t = \rho_{ij}(t)dt,\ d\langle W_i\rangle_t = dt.$$

Let $q,\ q_1,\ q_2$ and $q_3 : [0,T] \to \mathbb{R}$ be deterministic functions and we assume

$$\int_0^T q^2(t)dt = 1.$$

Then conditional expectations of Wiener chaos are

$$\begin{aligned}
&E\Big[\int_0^t \Big(\int_0^s q_2(u)dW_2(u)\Big) q_1(s)dW_1(s)\Big|\int_0^t q(s)dW_0(s) = x\Big] = c_1(x^2-1),\\
&E\Big[\Big(\int_0^t q_1(s)dW_1(s)\Big)\Big(\int_0^t q_2(s)dW_2(s)\Big)\Big|\int_0^t q(s)dW_0(s) = x\Big] = c_2(x^2-1)+c_3,\\
&E\Big[\Big(\int_0^t \Big(\int_0^s q_2(u)dW_2(u)\Big) q_1(s)dW_1(s)\Big)\Big(\int_0^t q_3(s)dW_3(s)\Big)\Big|\int_0^t q(s)dW_0(s) = x\Big]\\
&= d_1(x^3-3x)+d_2x,\\
&E\Big[\Big(\int_0^t \Big(\int_0^s q_2(u)dW_2(u)\Big) q_1(s)dW_1(s)\Big)^2\Big|\int_0^t q(s)dW_0(s) = x\Big]\\
&= d_3(x^4-6x^2+3)+d_4(x^2-1)+d_5,
\end{aligned}$$

where

$$\begin{aligned}
c_1 &= \int_0^t \Big(\int_0^s q_2(u)q(u)\rho_{02}(u)du\Big) q_1(s)q(s)\rho_{01}(s)ds,\\
c_2 &= \Big(\int_0^t q_1(s)q(s)\rho_{01}(s)ds\Big)\Big(\int_0^t q_2(s)q(s)\rho_{02}(s)ds\Big),\\
c_3 &= \int_0^t q_1(s)q_2(s)\rho_{12}(s)ds,\\
d_1 &= \Big(\int_0^t \Big(\int_0^s q_2(u)q(u)\rho_{02}(u)du\Big) q_1(s)q(s)\rho_{01}(s)ds\Big)\Big(\int_0^t q_3(s)q(s)\rho_{03}(s)ds\Big),\\
d_2 &= \int_0^t \Big(\int_0^s q_2(u)q(u)\rho_{02}(u)du\Big) q_1(s)q_3(s)\rho_{13}(s)ds + \int_0^t \Big(\int_0^s q_2(u)q_3(u)\rho_{23}(u)du\Big) q_1(s)q(s)\rho_{01}(s)ds,\\
d_3 &= \Big(\int_0^t \Big(\int_0^s q_2(u)q(u)\rho_{02}(u)du\Big) q_1(s)q(s)\rho_{01}(s)ds\Big)^2,\\
d_4 &= 2\int_0^t \Big(\int_0^s \Big(\int_0^u q_2(v)^2dv\Big) q_1(u)q(u)\rho_{01}(u)du\Big) q_1(s)q(s)\rho_{01}(s)ds\\
&+ 2\int_0^t \Big(\int_0^s \Big(\int_0^u q_2(v)q(v)\rho_{02}(v)dv\Big) q_1(u)q_2(u)\rho_{12}(u)du\Big) q_1(s)q(s)\rho_{01}(s)ds\\
&+ 2\int_0^t \Big(\int_0^s \Big(\int_0^u q_2(v)q(v)\rho_{02}(v)dv\Big) q_2(u)q(u)\rho_{02}(u)du\Big) q_1(s)^2ds,\\
d_5 &= \int_0^t \Big(\int_0^s q_2(u)^2\Big) q_1(s)^2ds.
\end{aligned}$$

See Nualart-Ustunel-Zakai [13] for the proof.

B Non degeneracy of Malliavin covariance

In this section, we will show the non-degeneracy of X in the sense of Malliavin. Let $\{(\tilde{W}_1(t), \tilde{W}_2(t)), 0 \leq t \leq T\}$ be a 2-dimensional standard Brownian motion. Let $C : \mathbb{R} \to \mathbb{R}_+$ be a smooth function whose derivatives of any order are bounded. Let σ, ν_1 and ν_2 be $\mathbb{R}_+$-valued continuous functions defined on $[0, T]$. We consider the following stochastic differential equation for X_1 and X_2;

$$
\begin{aligned}
&dX_1(t) = X_2(t)\sigma(t)C(X_1(t))d\tilde{W}_1(t),\\
&dX_2(t) = \nu_1(t)X_2(t)dW_1(t) + \nu_2(t)X_2(t)d\tilde{W}_2(t),\\
&X_1(0) = x_1 > 0,\ X_2(0) = x_2 > 0.
\end{aligned}
$$

When we assume $\nu_1(t) = \rho(t)\nu(t),\ \nu_2(t) = \sqrt{1-\rho(t)^2}\nu(t)$, the distribution of $(X_1(T), X_2(T))$ is same as $(X(T), \alpha(T))$. We will calculate the Malliavin covariance $\gamma_t^{ij} = \langle DX^i(t,x), DX^j(t,x)\rangle$. We define $V_i : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ $(i = 1, 2)$ as follows

$$
V_1(t,x) = \begin{pmatrix} x_2 C(x_1)\sigma(t) \\ x_2\nu_1(t) \end{pmatrix},\ V_2(t,x) = \begin{pmatrix} 0 \\ x_2\nu_2(t) \end{pmatrix}.
$$

Then, $V_1(0, X_0)$ and $V_2(0, X_0)$ are linearly independent. Let $Y(t)$ be defined by $Y_{ij}(t,x) = (\partial X^i/\partial x^j)(t,x)$. Then the Malliavin covariance $\Gamma_t = (\gamma_t^{ij})$ is given by

$$
\gamma_t^{ij} = \sum_{r=1}^{2} \int_0^t (Y_t Y_s^{-1} V_r(X_s))^i (Y_t Y_s^{-1} V_r(X_s))^j ds.
$$

We denote by $\lambda(t)$ the minimum eigenvalue of Γ_t. Then, from Kusuoka-Stroock [11] and Shigekawa [16] Theorem 6.7, $\lambda(t)$ satisfies

$$
E[\lambda(t)^{-p}] < \infty,\ p > 0.
$$

Since

$$
\gamma_t^{11} \geq {}^t\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Gamma_t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \geq \lambda(t),
$$

we have

$$
E[(\gamma_t^{11})^{-p}] < \infty.
$$

Then we can show that $X_1(T)$ is non-degenerate in the sense of Malliavin.

C Solution of Hamilton equation

We use the relations between Euler-Lagrange equation and Hamilton equation, please see Abraham-Marsden [1] for the details. We define $B : \mathbb{R} \to \mathbb{R}$ as

$$B(x) = \int_{X_0}^{x} \frac{dz}{b(z)}.$$

And define

$$q_1(t) = B(f(t;h)),\ q_2(t) = a(t;h),\ k = B(K).$$

Then q_1, q_2 satisfies the following ordinary differential equation;

$$\begin{aligned}
&\frac{dq_1(t)}{dt} = q_2(t)(\sqrt{1-\rho^2}\dot{h}_1(t) + \rho\dot{h}_2(t)),\\
&\frac{dq_2(t)}{dt} = \nu q_2(t)\dot{h}_2(t).\\
&q_1(0) = 0,\ q_1(T) = k,\ q_2(0) = \alpha.
\end{aligned}$$

We define Riemanian metric on $\mathbb{R}^2$ as

$$ds^2 = \sum_{i,j=1}^{2} g_{ij}(q) dq_i dq_j,$$

where (g_{ij}) is inverse matrix of (g^{ij}) and

$$\begin{pmatrix} g^{11}(q) & g^{12}(q) \\ g^{21}(q) & g^{22}(q) \end{pmatrix} = \begin{pmatrix} q_2^2 & \rho\nu q_2^2 \\ \rho\nu q_2^2 & \nu^2 q_2^2 \end{pmatrix}.$$

Then, We can interpret $E(y)$ as the square of minimum geodesic distance between the point $\{(q_1, q_2) = (0, \alpha)\}$ and the line $\{q_1 = k\}$. We define an action functional $\mathcal{S}$ as

$$\mathcal{S}(q, \dot{q}) = \int_0^T L(q(t), \dot{q}(t))dt,$$

where L is a Lagrangian given by

$$L(q(t), \dot{q}(t)) = \frac{1}{2}\sum_{i,j=1}^{2} g_{ij}(q(t))\dot{q}_i(t)\dot{q}_j(t).$$

The first variation of $\mathcal{S}$ to the direction ξ is given by

$$\nabla_\xi \mathcal{S} = \int_0^T \nabla_q L(q, \dot{q})\xi_t + \nabla_{\dot{q}} L(q, \dot{q})\dot{\xi}_t dt.$$

Then function q minimizing $S(q, \dot{q})$ satisfies the following Euler-Lagrange equation;

$$\begin{aligned}
&\nabla_q L(q, \dot{q}) - \frac{d}{dt}\nabla_{\dot{q}} L(q, \dot{q}) = 0,\\
&\nabla_{\dot{q}} L(q, \dot{q})\xi_1 = 0.
\end{aligned}$$

Let $p(t) = \nabla_{\dot{q}} L(q(t), \dot{q}(t))$. Since $\xi_1^1 = 0$, we have $p^2(T) = 0$. Let us consider the following Hamiltonian

$$H(p, q) = \frac{1}{2}\sum g^{ij}(q)p_i p_j = \frac{1}{2}q_2(t)^2(p_1(t)^2 + 2\nu\rho p_1(t)p_2(t) + \nu^2 p_2(t)^2),$$

and the associated Hamilton equation

$$\begin{aligned}
\frac{dq_1(t)}{dt} &= q_2(t)^2(p_1(t) + \nu\rho p_2(t)), \\
\frac{dq_2(t)}{dt} &= q_2(t)^2(\nu\rho p_1(t) + \nu^2 p_2(t)), \\
\frac{dp_1(t)}{dt} &= 0, \\
\frac{dp_2(t)}{dt} &= -q_2(t)(p_1(t)^2 + 2\nu\rho p_1(t)p_2(t) + \nu^2 p_2(t)^2),
\end{aligned}$$

with boundary conditions

$$q_1(0) = 0,\ q_1(T) = k,\ q_2(0) = \alpha,\ p_2(T) = 0. \tag{C.1}$$

We can easily check that H, p_1 and $p_1q_1 + p_2q_2$ are the first integrals of this Hamiltonian system i.e.

$$\frac{d}{dt}H(p(t), q(t)) = 0,\ \frac{d}{dt}p_1(t) = 0,\ \frac{d}{dt}(p_1(t)q_1(t) + p_2(t)q_2(t)) = 0.$$

We denote $H_0 = H(p(t), q(t))$, $P_1 = p_1(t)$ and $I = p_1q_1 + p_2q_2$. First, we solve p_2 and calculate $p_2(0)$.

$$\begin{aligned}
&\frac{dp_2(t)}{dt} = -(p_1(t)^2 + 2\nu\rho p_1(t)p_2(t) + \nu^2 p_2(t)^2)^{1/2}(2H_0)^{1/2}, \\
&p_2(T) = 0.
\end{aligned} \tag{C.2}$$

Using the following indefinite integral

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{a}} \log|2ax + b + 2\sqrt{a(ax^2 + bx + c)}|,$$

we can solve (C.2) as follows;

$$\nu^2 p_2(t) + \nu\rho P_1 + \nu\sqrt{P_1^2 + 2\nu\rho P_1 p_2(t) + \nu^2 p_2(t)^2} = \frac{C}{2}e^{-\sqrt{2H_0}\nu t}, \tag{C.3}$$

where C is an integral constant. By Equation C.1, we have

$$e^{-\sqrt{2H}\nu T} = \begin{cases} \frac{2(1-\rho)P_1\nu}{C} \text{ if } \frac{P_1}{C} \geq 0 \\ -\frac{2(1+\rho)P_1\nu}{C} \text{ if } \frac{P_1}{C} < 0 \end{cases}.$$

From (C.3), we can calculate $p_2(0)$ as

$$p_2(0) = \frac{C}{4\nu^2} - \frac{\rho P_1}{\nu} - \frac{(1-\rho^2)P_1^2}{C}. \tag{C.4}$$

On the other hand, since $I \equiv p_1(t)q_1(t) + p_2(t)q_2(t)$, we have

$$I = p_2(0)\alpha = P_1 k. \tag{C.5}$$

Let $X = P_1/C$. Then from (C.4) and (C.5), we have the following quadratic equation,

$$(1-\rho^2)X^2 + (\frac{\rho}{\nu} + \frac{k}{\alpha})X - \frac{1}{4\nu^2} = 0. \tag{C.6}$$

Put $\zeta = -\nu k/\alpha$. Then we have the solutions of (C.6)

$$X = \frac{-\rho + \zeta \pm \sqrt{1 - 2\rho\zeta + \zeta^2}}{1-\rho} \frac{1}{2(1+\rho)\nu}.$$

Therefore

$$e^{-\sqrt{2H_0}\nu T} = \frac{\sqrt{1-2\rho\zeta+\zeta^2} \pm \rho \mp \zeta}{1 \mp \rho}.$$

Since $e^{-\sqrt{2H_0}\nu T} < 1$,

$$e^{-\sqrt{2H_0}\nu T} = \frac{\sqrt{1-2\rho\zeta+\zeta^2} + \rho - \zeta}{1 - \rho}.$$

We see that

$$H_0 = \frac{1}{2\nu^2 T^2}\Big\{\log(\frac{\sqrt{1-2\rho\zeta+\zeta^2} - \rho + \zeta}{1 - \rho})\Big\}^2.$$

Finally the action functional $\mathcal{S}$ is given by

$$\mathcal{S}(q,\dot{q}) = \int_0^T H_0 dt = \frac{1}{2\nu^2 T}\Big\{\log(\frac{\sqrt{1-2\rho\zeta+\zeta^2} - \rho + \zeta}{1 - \rho})\Big\}^2.$$

This completes the proof of Theorem 3.1.

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