UTMS 2006–27 October 13, 2006

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by

Ganghua Yuan and Masahiro Yamamoto

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

LIPSCHITZ STABILITY IN THE DETERMINATION OF PRINCIPAL PARTS OF A PARABOLIC EQUATION BY BOUNDARY OBSERVATIONS

Ganghua Yuan[1] and Masahiro Yamamoto[2]

[1], [2] Department of Mathematical Sciences, The University of Tokyo
Komaba Meguro Tokyo 153-8914 Japan
e-mail:[1] ghyuan@ms.u-tokyo.ac.jp, [2] myama@ms.u-tokyo.ac.jp
[1] School of Mathematics & Statistics, Northeast Normal University
Changchun, Jilin, 130024 P.R. China

Abstract. Let $y(h)(t,x)$ be one solution to

$$\partial_t y(t,x) - \sum_{i,j=1}^n \partial_j(a_{ij}(x)\partial_i y(t,x)) = h(t,x),\ 0<t<T,\ x\in\Omega$$

with a nonhomogeneous term h, and $y|_{(0,T)\times\partial\Omega} = 0$, where $\Omega\subset\mathbb{R}^n$ is a bounded domain. We discuss an inverse problem of determining $n(n+1)/2$ unknown functions a_{ij} by $\{\partial_\nu y(h_\ell)|_{(0,T)\times\Gamma_0}, y(h_\ell)(\theta,\cdot)\}_{1\le\ell\le\ell_0}$ after selecting inputs $h_1,...,h_{\ell_0}$ suitably, where Γ_0 is an arbitrary subboundary, ∂_ν denotes the normal derivative, $0<\theta<T$ and $\ell_0\in\mathbb{N}$. In the case of $\ell_0 = (n+1)^2n/2$, we prove the Lipschitz stability in the inverse problem if we choose $(h_1,...,h_{\ell_0})$ from a set $\mathcal{H}\subset\{C_0^\infty((0,T)\times\omega)\}^{\ell_0}$ with an arbitrarily fixed subdomain $\omega\subset\Omega$. Moreover we can take $\ell_0 = (n+3)n/2$ with special choice h_ℓ. The proof is based on a Carleman estimate.

§1. Introduction and main results.

In this paper we consider the following parabolic equation:

$$\partial_t y(t,x) - \sum_{i,j=1}^n \partial_j(a_{ij}(x)\partial_i y(t,x)) = h(t,x), \quad (t,x)\in Q\equiv(0,T)\times\Omega \tag{1.1}$$

$$y(t,x) = 0, \quad (t,x)\in\Sigma\equiv(0,T)\times\partial\Omega, \quad y(0,\cdot)\in L^2(\Omega). \tag{1.2}$$

Here $n\le 5$, $\Omega\subset\mathbb{R}^n$ is a bounded domain whose boundary $\partial\Omega$ is sufficiently smooth, and $x=(x_1,...,x_n)\in\mathbb{R}^n$, $\partial_t = \frac{\partial}{\partial t}$, $\partial_j = \frac{\partial}{\partial x_j}$, $\nabla = (\partial_1,\cdots,\partial_n)$, $h\in C_0^\infty((0,T)\times\omega)$,

Key words and phrases. inverse parabolic problem, Carleman estimate, Lipschitz stability.
2000 Mathematics Subject Classification: 35R30, 35K20.

ω is an arbitrarily fixed subdomain of Ω. Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ be a multi-index with $\alpha_j \in \mathbb{N} \cup \{0\}$. We set $\partial_x^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, and $\nu = \nu(x) = (\nu_1(x), \cdots, \nu_n(x))$ is the external unit normal vector to $\partial\Omega$ at x. Let $\partial_\nu = \nu \cdot \nabla$.

Assume that

$$a_{ij} \in C^5(\overline{\Omega}), \quad a_{ij} = a_{ji}, \quad 1 \le i, j \le n, \tag{1.3}$$

and that the coefficients $\{a_{ij}\} \equiv \{a_{ij}\}_{1\le i,j\le n}$ satisfy the uniform ellipticity: there exists a constant $r > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\zeta_i\zeta_j \ge r|\zeta|^2, \qquad \zeta \in \mathbb{R}^n, \quad x \in \overline{\Omega}. \tag{1.4}$$

For $y(0,\cdot) \in L^2(\Omega)$, we can prove (e.g., Pazy [32]) that $y(\{a_{ij}\}, h) \in C([0,T]; L^2(\Omega)) \cap C((0,T); H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1((0,T); L^2(\Omega))$ and see also (1.6) below. By $y\left(\{a_{ij}\}, h\right)(t,x)$ we denote one function satisfying (1.1) - (1.2). We note that $y(\{a_{ij}\}, h)$ is not uniquely determined, because we do not specify an initial value.

We consider the following inverse problem:

Inverse problem. Let $\theta \in (0,T)$ be arbitrarily fixed and $\Gamma_0 \ne \emptyset$ be an arbitrary relatively open subset of $\partial\Omega$. Select $\ell_0 \in \mathbb{N}$, $h_\ell \in C_0^\infty((0,T) \times \omega)$, $1 \le \ell \le \ell_0$ suitably and determine $a_{ij}(x)$, $x \in \Omega$, $1 \le i,j \le n$ by observation data $\partial_\nu y(\{a_{ij}\}, h_\ell)|_{(0,T)\times\Gamma_0}$ and $y(\{a_{ij}\}, h_\ell)(\theta, x)$, $x \in \Omega$, $1 \le \ell \le \ell_0$.

In the formulation of the inverse problem, the initial values are also unknown. The nonhomogeneous terms h_ℓ, $1 \le \ell \le \ell_0$, are considered as inputs to system (1.1) - (1.2) and are spatially restricted to a small subdomain $\omega \subset \Omega$. Then we determine $a_{ij}(x)$, $x \in \Omega$ by observation data $\partial_\nu y(\{a_{ij}\}, h_\ell)|_{(0,T)\times\Gamma_0}$ and $y(\{a_{ij}\}, h_\ell)(\theta, \cdot)$, $1 \le \ell \le \ell_0$, which are regarded as outputs.

More precisely, around known $a_{ij}^{(2)}$, we will determine $a_{ij}^{(1)}$, which means that we can know solutions corresponding to the coefficients $\{a_{ij}^{(2)}\}$. In our formualtion, in order to determine $\frac{n(n+1)}{2}$ coefficients $a_{ij}^{(1)}$, we are assumed to be able to operate the heat processes by suitably changing inputs h_ℓ. We note that we need not know initial data in repeating the processes. Our main concern is the stability estimate for the inverse problem: Estimate $\sum_{i,j=1}^n \|a_{ij}^{(1)} - a_{ij}^{(2)}\|_{H^1(\Omega)}$ by suitable norms of $\partial_\nu y\left(\{a_{ij}^{(1)}\}, h_\ell\right) - \partial_\nu y\left(\{a_{ij}^{(2)}\}, h_\ell\right)$ and $y\left(\{a_{ij}^{(1)}\}, h_\ell\right)(\theta,\cdot) - y\left(\{a_{ij}^{(2)}\}, h_\ell\right)(\theta,\cdot)$, $1 \le \ell \le \ell_0$. The stability is a fundamental mathematical subject in the inverse problem and immediately yields the uniqueness.

Here we assume that initial data are also unknown. After determination of the coefficients, the determination of initial values is the parabolic equation backward in time, where we are requested to determine $y(0,\cdot)$ by $y(\theta,\cdot)$. As for the backward heat equation, see the monographs Ames and Straughan [2], Payne [31] and Klibanov [26] as a recent paper. Our main concern is the determination of coefficients and so we will omit the determination of initial values.

We can consider an inverse problem for a usual initial value/boundary value problem by setting $\theta = 0$. In the case where $\theta = 0$ and Γ_0 is an arbitrary subboundary of Ω, the corresponding inverse problem is open (e.g., Chapter 9, Section 2 in Isakov [20]) even for the inverse problem of derermining a single coefficient in a parabolic equation. In the case of $\theta = 0$, if $\Gamma_0 \subset \partial\Omega$ is a sufficienly large portion and unknown coefficients a_{ij} satisfy some extra conditions which cannot be interpreted by the parabolicity of equation (1.1), we may be able to prove the stability provided that initial values satisfy some nondegeneracy condition similar to (1.7) below. Due to the extra conditions on Γ_0 and a_{ij}, in the case of $\theta = 0$, the available results for the inverse parabolic problem are still incomplete. On the other hand, since the finiteness of the propagation speed does not hold in the parabolic equation unlike a hyperbolic equation, it is practically difficult to select initial values satisfying the nondegeneracy condition exactly at the initial time $t = 0$ and initiate the corresponding heat process. In our formulation with $\theta > 0$, as inputs initiating the heat processes, we have to select exterior heat sources h_ℓ restricted to any small part of the domain Ω, and we need not select spatially varying function at $t = \theta$ which are observation data as outputs. Therefore we can assert that our formulation is more realizable.

Our inverse problem is related to determination of thermal conductivity of an anisotropic medium by heat conduction process. To the authors' best knowledge, there are no papers on the determination of multiple coefficients in the principal part of a parabolic equation, although we have an available methodology which was initiated by Bukhgeim and Klibanov [8]. The determination of multiple coefficients requires repeat of observations, and the application of the method in [8] needs independent consideration. Moreover, since we aim at the global stability in the whole domain Ω by means of lateral Cauchy data on an arbitrary small subboundary $\Gamma_0 \subset \partial\Omega$, we have to establish a relevant Carleman estimate (Theorem 2.1 below).

For statement of our main results, we need to introduce some notations. For a sequence $\{\rho_\ell(x)\} := \{\rho_\ell(x)\}_{1\le\ell\le\frac{(n+1)^2n}{2}}$ of C^2-functions and $1 \le k \le \frac{n(n+1)}{2}$, we set

$$
\begin{aligned}
D_{ij}^k &= D_{ij}^k(\{\rho_\ell\})(x) \\
=&\det \begin{pmatrix}
\partial_i\partial_j\rho_{(k-1)(n+1)+1}(x) & \partial_1\rho_{(k-1)(n+1)+1}(x) & \cdots & \partial_n\rho_{(k-1)(n+1)+1}(x) \\
\partial_i\partial_j\rho_{(k-1)(n+1)+2}(x) & \partial_1\rho_{(k-1)(n+1)+2}(x) & \cdots & \partial_n\rho_{(k-1)(n+1)+2}(x) \\
\vdots & \vdots & \ddots & \vdots \\
\partial_i\partial_j\rho_{(k-1)(n+1)+n+1}(x) & \partial_1\rho_{(k-1)(n+1)+n+1}(x) & \dots & \partial_n\rho_{(k-1)(n+1)+n+1}(x)
\end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
&D(\{\rho_\ell\})(x) \\
=&\det \begin{pmatrix}
D_{11}^1 & D_{12}^1 & \dots & D_{1n}^1 & D_{22}^1 & \dots & D_{2n}^1 & \dots & D_{nn}^1 \\
D_{11}^2 & D_{12}^2 & \dots & D_{1n}^2 & D_{22}^2 & \dots & D_{2n}^2 & \dots & D_{nn}^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
D_{11}^{\frac12(n+1)n} & D_{12}^{\frac12(n+1)n} & \dots & D_{1n}^{\frac12(n+1)n} & D_{22}^{\frac12(n+1)n} & \dots & D_{2n}^{\frac12(n+1)n} & \dots & D_{nn}^{\frac12(n+1)n}
\end{pmatrix}.
\end{aligned}
$$

Next we introduce an admissible set of unknown coefficients $\{a_{ij}\}$. Let us fix constants $M_0 > 0$, $r > 0$ and smooth functions $\eta_{ij} = \eta_{ij}(x)$, $1 \le i,j \le n$, a subdomain $\omega_1 \subset \Omega$ such that $\partial\omega_1 \supset \partial\Omega$. We set

$$
\mathcal{U} = \{\{a_{ij}\};\ \|a_{ij}\|_{C^5(\overline{\Omega})} \le M_0,\ a_{ij} = \eta_{ij} \text{ in } \omega_1 \text{ and (1.4) is satisfied.}\}. \tag{1.5}
$$

For $m \in \mathbb{N} \cup \{0\}$ and $0 < \tau_1 < \tau_2 < T$, we can prove

$$
\|y(\{a_{ij}\},h)\|_{C^m([\tau_1,\tau_2];H^6(\Omega))} \le C_0(\|y(\{a_{ij}\},h)(0,\cdot)\|_{L^2(\Omega)} + \|h\|_{W^{m,1}(0,T;H^6(\omega))}). \tag{1.6}
$$

Here $C_0 > 0$ depends only m, τ_1, τ_2 and $\mathcal{U}$, and $\|\eta\|_{W^{m,1}(0,T;H^6(\omega))} = \sum_{j=0}^m \|\partial_t^j\eta\|_{L^1(0,T;H^6(\omega))}$. The proof is done by the semigroup theory (e.g., [32]) and is given in Appendix B. Henceforth, for arbitrarily fixed $M > 0$, we assume that

$$
\|y(\{a_{ij}\},h)(0,\cdot)\|_{L^2(\Omega)} \le M,
$$

which means that unknown initial values are bounded with a priori bound $M > 0$.

Now we are ready to state our main results.

Theorem 1.1. *Let $n \le 5$, $0 < \tau_1 < \theta < \tau_2 < T$, $\Gamma_0 \ne \emptyset$ be an arbitrary relatively open subset of $\partial\Omega$, and let $\{a_{ij}^{(2)}\} \in \mathcal{U}$ be arbitrarily fixed. We assume that $h_\ell \in C_0^\infty((0,T)\times\omega)$, $1 \le \ell \le \frac{(n+1)^2 n}{2}$, satisfy*

$$D(y(\{a_{ij}^{(2)}\}, h_\ell))(\theta, x) \ne 0, \quad x \in \overline{\Omega \setminus \omega_1}. \tag{1.7}$$

Then there exists a constant $C_1 = C_1(\mathcal{U}, M, \{h_\ell\}) > 0$ such that

$$\sum_{i,j=1}^{n} \|a_{ij}^{(1)} - a_{ij}^{(2)}\|_{H^1(\Omega)} \le C_1 \sum_{\ell=1}^{\frac{(n+1)^2 n}{2}} \|\partial_\nu y(\{a_{ij}^{(1)}\}, h_\ell) - \partial_\nu y(\{a_{ij}^{(2)}\}, h_\ell)\|_{H^2(\tau_1,\tau_2;L^2(\Gamma_0))}$$
$$+C_1 \sum_{\ell=1}^{\frac{(n+1)^2 n}{2}} \|y(\{a_{ij}^{(1)}\}, h_\ell)(\theta,\cdot) - y(\{a_{ij}^{(2)}\}, h_\ell)(\theta,\cdot)\|_{H^3(\Omega)} \tag{1.8}$$

for all $\{a_{ij}^{(1)}\} \in \mathcal{U}$.

In order to estimate $\{a_{ij}^{(1)}\}$ around given $\{a_{ij}^{(2)}\}$, we have to choose h_ℓ, $1 \le \ell \le \frac{(n+1)^2 n}{2}$ whose supports are restricted to a small set $(0,T)\times\omega$, so that the systems are steered to satisfy (1.7) at the time θ. The choice is related with the approximate controllability (e.g., [34]).

In fact, we can prove

Proposition 1.1. *Let $\{a_{ij}\}$ satisfy (1.3) and (1.4). For each $\theta > 0$ and $\mu \in L^2(\Omega)$, the set*

$$\{y(\{a_{ij}\}, h, \mu)(\theta,\cdot);\ h \in C_0^\infty((0,T)\times\omega)\}$$

is dense in $\mathcal{D}(A^3) = \{y \in H^6(\Omega);\ y|_{\partial\Omega} = Ay|_{\partial\Omega} = A^2 y|_{\partial\Omega} = 0\}$.

Here and henceforth we define an operator A in $L^2(\Omega)$ by

$$\begin{cases} (Ay)(x) = -\sum_{i,j=1}^{n} \partial_j(a_{ij}(x)\partial_i y(x)), & x \in \Omega, \\ \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega), \end{cases} \tag{1.9}$$

where $\mathcal{D}(A)$ denotes the domain of the operator A, and let $y(\{a_{ij}\}, h, \mu)$ denote the solution to (1.1) and (1.2) with $y(0, x) = \mu(x)$, $x \in \Omega$.

Therefore one can prove that there exist $h_\ell \in C_0^\infty((0,T) \times \omega)$, $1 \le \ell \le \frac{(n+1)^2 n}{2}$ such that (1.7) holds, which guarantees the Lipschitz stability in determining $\{a_{ij}^{(1)}\}$.

Now we discuss the set of such h_ℓ, $1 \le \ell \le \frac{(n+1)^2 n}{2}$. For simplicity, for system with known $a_{ij}^{(2)}$, we mainly consider the zero initial value. That is, let $y(\{a_{ij}\}, h, 0)$ be a unique solution to (1.1) and (1.2) with $y(x, 0) = 0$, $x \in \Omega$. We set $\ell_0 = \frac{(n+1)^2 n}{2}$ and

$$\mathcal{H} = \{(h_1, ..., h_{\ell_0}) \in \{C_0^\infty((0,T)\times\omega)\}^{\ell_0};\, D(y(\{a_{ij}^{(2)}\}, h_\ell, 0))(\theta, x) \ne 0 \quad \text{for } x \in \overline{\Omega \setminus \omega_1}.\}$$

By the elliptic regularity (e.g., Theorem 8.13 in [13]) and the semigroup theory (e.g., [32]), we can prove that

$$\|y(\{a_{ij}\}, h, 0)\|_{C([0,T];C^2(\overline{\Omega}))} \le C_2 \|h\|_{L^1(0,T;H^5(\Omega))} \tag{1.10}$$

(see Appenix B for the proof).

Therefore we can prove that for $(h_1, ..., h_{\ell_0}) \in \mathcal{H}$, there exists $\varepsilon = \varepsilon(h_1, ..., h_{\ell_0}) > 0$ such that if $(\widetilde{h}_1, ..., \widetilde{h}_{\ell_0}) \in \{C_0^\infty((0,T)\times\omega)\}^{\ell_0}$ and $\max_{1\le\ell\le\ell_0} \|h_\ell - \widetilde{h}_\ell\|_{L^1(0,T;H^5(\Omega))} < \varepsilon$, then $(\widetilde{h}_1, ..., \widetilde{h}_{\ell_0}) \in \mathcal{H}$. This means the stability of inputs $(h_1, ..., h_{\ell_0})$ realizing the Lipschitz stability.

Since $C_0^\infty((0,T) \times \omega)$ is dense in $C_0^m((0,T) \times \omega)$ with $m \in \mathbb{N}$, we can take $C_0^m((0,T) \times \omega)$ as a class of interior inputs, using the regularity property of the parabolic equation (e.g., [32])).

Furthermore we can prove an even better result with smaller ℓ_0 in Theorem 1.1. That is, whatever initial values to system (1.1) with $a_{ij}^{(2)}$ are, we can choose h_ℓ, $1 \le \ell \le \frac{(n+3)n}{2}$ to establish the Lipschitz stability around $a_{ij}^{(2)}$ by means of $\frac{(n+3)n}{2}$ data.

Theorem 1.2. *Let $n \le 5$, $0 < \tau_1 < \theta < \tau_2 < T$, $\Gamma_0 \ne \emptyset$ be an arbitrary relatively open subset of $\partial\Omega$ and let us fix $\{a_{ij}^{(2)}\} \in \mathcal{U}$. Then we can choose suitable $h_\ell \in C_0^\infty((0,T)\times\omega)$, $1 \le \ell \le \frac{n(n+3)}{2}$ such that there exists a constant $C_2 = C_2(\mathcal{U}, M, \{h_\ell\}) > 0$ such that*

$$\sum_{i,j=1}^{n} \|a_{ij}^{(1)} - a_{ij}^{(2)}\|_{H^1(\Omega)} \le C_2 \sum_{\ell=1}^{\frac{n(n+3)}{2}} \|\partial_\nu y(\{a_{ij}^{(1)}\}, h_\ell) - \partial_\nu y(\{a_{ij}^{(2)}\}, h_\ell)\|_{H^2(\tau_1,\tau_2;L^2(\Gamma_0))}$$
$$+C_2 \sum_{\ell=1}^{\frac{n(n+3)}{2}} \|y(\{a_{ij}^{(1)}\}, h_\ell)(\theta,\cdot) - y(\{a_{ij}^{(2)}\}, h_\ell)(\theta,\cdot)\|_{H^3(\Omega)} \tag{1.11}$$

for all $\{a_{ij}^{(1)}\} \in \mathcal{U}$.

As for inverse problems of determining coefficients in parabolic equations, we refer to Danilaev [9], Elayyan and Isakov [10], Imanuvilov and Yamamoto [16], [18], Isakov [20], Isakov and Kindermann [21], Ivanchov [22], Klibanov [25], Klibanov and Timonov [27], Yamamoto and Zou [36]. In those existing papers, the determination of a single coefficient is discussed, while we here consider an inverse problem of determining multiple coefficients of the principal part by a finite set of observations.

Our formulation is with a finite number of observations and this kind of inverse problems was firstly solved by Bukhgeim and Klibanov [8], whose methodology is based on Carleman estimates. For similar inverse problems for other equations, we refer to Baudouin and Puel [3], Bellssoued [4], Bellassoued and Yamamoto [5], Bukhgeim [7], Imanuvilov and Yamamoto [17], [19], Isakov [20], Khaĭdarov [23], Klibanov [24], [25], Klibanov and Timonov [27], Klibanov and Yamamoto [28], Yamamoto [35].

For proving Theorems 1.1 and 1.2, we establish a Carleman estimate (Theorem 2.1) for functions without compact supports, and we apply a modification of arguments in [8], [19].

This paper is composed of four sections and two appendices. In Section 2 we present Carleman estimates and the proof is given in Appendix A. In Section 3, we prove Theorems 1.1 and 1.2. In Section 4, we prove Proposition 1.1. In Appendix B, we prove estimates (1.6) and (1.10).

§2. Carleman estimates.

In this section we will prove Carleman estimates for the parabolic equation. The results in this section may have independent interests.

Lemma 2.1. *Let $\Gamma_0 \ne \emptyset \subset \partial\Omega$ be an arbitrary relatively open subset. Then there exists a function $d \in C^2(\overline{\Omega})$ such that*

$$d(x) > 0 \quad \textit{for } x \in \Omega, \qquad |\nabla d(x)| > 0 \quad \textit{for } x \in \overline{\Omega} \tag{2.1}$$

and

$$\sum_{i,j=1}^{n} a_{ij}(x)\partial_i d(x)\nu_j(x) \le 0, \quad x \in \partial\Omega \setminus \Gamma_0 \tag{2.2}$$

for all $a_{ij} \in C^1(\overline{\Omega})$, $a_{ij} = a_{ji}$, $1 \le i,j \le n$ satisfying (1.4).

Lemma 2.1 can be easily seen from the proof of Lemma 2.1 in [16], and so we omit the proof.

Example. Let us consider a special case where $a_{ij} = 0$ if $i \ne j$ and $a_{ii} = 1$ and

$$\Omega = \{x \in \mathbb{R}^n;\, |x| < R\}, \qquad \Gamma_0 = \{x \in \partial\Omega;\, (x - x_0, \nu(x)) \ge 0\} \tag{2.3}$$

with an arbitrarily fixed $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$. Here $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^n$. Then we can take $d(x) = |x - x_0|^2$.

We present Carleman estimates for operator L:

$$(Ly)(t,x) = \partial_t y(t,x) - \sum_{i,j=1}^{n} \partial_j(a_{ij}(x)\partial_i y(t,x)).$$

Theorem 2.1. *Assume that (1.4) holds and that* $a_{ij} \in C^1(\overline{\Omega})$, $a_{ij} = a_{ji}$, $1 \le i,j \le n$. *Let* $d \in C^2(\overline{\Omega})$ *be a function satisfying (2.1) and (2.2), and let* $0 \le \tau_1 < \theta < \tau_2$ *be fixed.*

(1) Let $\varphi(t,x) = e^{\lambda(d(x)-\beta|t-\theta|^2)}$, *where* $\beta > 0$ *is a constant. Then there exists a number* $\lambda_0 > 0$ *such that for an arbitrary* $\lambda \ge \lambda_0$, *we can choose a constant* $s_0(\lambda) \ge 0$ *satisfying: there exists a constant* $C_1 = C_1(s_0, \lambda) > 0$ *such that*

$$\begin{aligned}&\int_{(\tau_1,\tau_2)\times\Omega} \left\{ \frac{1}{s}\left(|\partial_t v|^2 + \sum_{i,j=1}^n |\partial_i\partial_j v|^2 \right) + s|\nabla v|^2 + s^3|v|^2 \right\} e^{2s\varphi} dxdt \\ \le& C_1 \int_{(\tau_1,\tau_2)\times\Omega} |Lv|^2 e^{2s\varphi} dxdt + C_1 s \int_{\tau_1}^{\tau_2}\int_{\Gamma_0} |\partial_\nu v|^2 e^{2s\varphi} d\Sigma \end{aligned} \tag{2.4}$$

for all $s > s_0$ *and all* v *satisfying*

$$\begin{cases} Lv \in L^2((\tau_1,\tau_2)\times\Omega), \quad v \in L^2(\tau_1,\tau_2; H^2(\Omega)\cap H_0^1(\Omega)), \\ \partial_\nu v \in L^2(\tau_1,\tau_2; L^2(\partial\Omega)), \qquad v(\tau_1,\cdot) = v(\tau_2,\cdot) = 0. \end{cases} \tag{2.5}$$

Moreover the constants s_0 *and* C_1 *continuously depend on* λ *and* $\sum_{i,j=1}^n \|a_{ij}\|_{C^1(\overline{\Omega})}$, *while* λ_0 *continuously depends on* $\sum_{i,j=1}^n \|a_{ij}\|_{C^1(\overline{\Omega})}$.

(2) Let $\varphi(t,x) = e^{\lambda(d(x)-\beta|t-\theta|^2+M_1)}$, *where* $M_1 > \sup_{t\in(\tau_1,\tau_2)} \beta(t-\theta)^2$. *Then there exist positive constants* λ_0, s_0 *and* $C_2 = C_2(\lambda_0, s_0)$ *such that*

$$\begin{aligned}&\int_{(\tau_1,\tau_2)\times\Omega} \left\{ \frac{1}{s\varphi}\left(|\partial_t v|^2 + \sum_{i,j=1}^n |\partial_i\partial_j v|^2 \right) + s\lambda^2\varphi|\nabla v|^2 + s^3\lambda^4\varphi^3|v|^2 \right\} e^{2s\varphi} dxdt \\ \le& C_2 \int_{(\tau_1,\tau_2)\times\Omega} |Lv|^2 e^{2s\varphi} dxdt + C_2 s\lambda \int_{\tau_1}^{\tau_2}\int_{\Gamma_0} \varphi|\partial_\nu v|^2 e^{2s\varphi} d\Sigma \end{aligned} \tag{2.6}$$

for all $s > s_0$, $\lambda > \lambda_0$ *and all* v *satisfying (2.5). The constants* λ_0, s_0 *and* C_2 *continuously depend on* $\sum_{i,j=1}^n \|a_{ij}\|_{C^1(\overline{\Omega})}$.

We prove the theorem in Appendix A.

As for Carleman estimates with $e^{\lambda\psi(x,t)}$ with regular $\psi(x,t)$, see Eller and Isakov [11], Hörmander [14], Isakov [20], Khaĭdarov [23], Klibanov and Timonov [27], Lavrent'ev, Romanov and Shishat·skiĭ[29]. For those Carleman estimates for the parabolic equation, we often have to change independent variables in order to apply them to the case of an arbitrary $\Gamma_0 \subset \partial\Omega$, so that it is complicated to derive the Lipschitz stability over the whole domain Ω. As for Carleman estimates for parabolic equations with singular weight functions, we can refer to Fursikov and Imanuvilov [12], Imanuvilov [15], Imanuvilov and Yamamoto [18].

Inequality (2.6) is a Carleman estimate with two large parameters λ and s for the functions without compact support. Our Carleman estimate can be applied to inverse problems for a coupling system of parabolic and hyperbolic equations and thermoelastic plate equations in case (2.3) for example. However the Carleman estimate in [12], [15] and [18] are not applicable to such parabolic-hyperbolic systems. For such applicability, we prove (2.6) in Theorem 2.1. As for Carleman estimates with two large parameters for functions with compact support. we can refer to [11].

§3. Proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. By $0 < \tau_1 < \tau_2 < T$, we choose and fix $\tau_3, \tau_4 > 0$ such that

$$0 < \tau_3 < \tau_1 < \tau_2 < \tau_4 < T.$$

It is sufficient to prove (1.8) with the norm in $H^2(\tau_3, \tau_4; L^2(\Gamma_0))$ of the first term on the righ-hand side. Let $d \in C^2(\overline{\Omega})$ satisfy (2.1) and (2.2). We choose $\beta > 0$ such that $\sup_{x\in\Omega} d(x) < \beta \min\{|\tau_1 - \theta|^2, |\tau_2 - \theta|^2\}$. We set

$$\varphi(t,x) = \exp\{\lambda(d(x) - \beta|t-\theta|^2)\}.$$

Let $d_0 = \inf_{x\in\Omega} \exp\{\lambda d(x)\} \ge 1$. Then, by the choice of $\beta > 0$, we have

$$\varphi(\theta, x) \ge d_0, \quad \varphi(\tau_1, x) = \varphi(\tau_2, x) < 1 \le d_0, \quad x \in \overline{\Omega}.$$

Thus for a sufficiently small $\varepsilon > 0$, we can choose a small $\delta = \delta(\varepsilon) > 0$ such that

$\tau_1 < \tau_1 + 2\delta < \theta - \delta < \theta + \delta < \tau_2 - 2\delta < \tau_2$,

$$\varphi(t,x) \ge d_0 - \varepsilon, \quad (t,x) \in [\theta - \delta, \theta + \delta] \times \overline{\Omega}$$

and

$$\varphi(t,x) \le d_0 - 2\varepsilon, \quad (t,x) \in ([\tau_1, \tau_1 + 2\delta] \cup [\tau_2 - 2\delta, \tau_2]) \times \overline{\Omega}.$$

We introduce a cut-off function χ satisfying $0 \le \chi \le 1$, $\chi \in C_0^\infty(0,T)$ and

$$\chi(t) = \begin{cases} 0, & t \in [\tau_1, \tau_1 + \delta] \cup [\tau_2 - \delta, \tau_2], \\ 1, & t \in [\tau_1 + 2\delta, \tau_2 - 2\delta]. \end{cases} \tag{3.1}$$

Let us set

$$f_{ij}(x) = a_{ij}^{(1)}(x) - a_{ij}^{(2)}(x), \qquad R_\ell(t,x) = y(\{a_{ij}^{(2)}\}, h_\ell)(t,x), \tag{3.2}$$

$$(L^{(1)}y)(t,x) \equiv \partial_t y - \sum_{i,j=1}^n \partial_j (a_{ij}^{(1)}(x)\partial_i y).$$

By (1.1) and (1.2), we can see that the differences $\widetilde{y}_\ell(t,x) = y(\{a_{ij}^{(1)}\}, h_\ell)(t,x) - y(\{a_{ij}^{(2)}\}, h_\ell)(t,x)$ satisfy

$$L^{(1)}\widetilde{y}_\ell(t,x) = \sum_{i,j=1}^n \partial_j \left(f_{ij}(x)\partial_i R_\ell(t,x) \right), \quad (t,x) \in (0,T) \times \Omega, \tag{3.3}$$

$$\widetilde{y}_\ell(t,x) = 0, \quad (t,x) \in (0,T) \times \partial\Omega, \quad 1 \le \ell \le \frac{(n+1)^2 n}{2}. \tag{3.4}$$

We set

$$z_\ell(t,x) = \partial_t \widetilde{y}_\ell(t,x), \qquad \Phi = \sup_{(t,x)\in(\tau_1,\tau_2)\times\Omega} \varphi(t,x) \tag{3.5}$$

and

$$U = \Bigg(\sum_{\ell=1}^{\frac{(n+1)^2 n}{2}} (\|z_\ell\|^2_{L^2(\tau_1,\tau_2;L^2(\Omega))} + \|\nabla z_\ell\|^2_{L^2(\tau_1,\tau_2;L^2(\Omega))}$$
$$+ \|\partial_t z_\ell\|^2_{L^2(\tau_1,\tau_2;L^2(\Omega))} + \|\nabla \partial_t z_\ell\|^2_{L^2(\tau_1,\tau_2;L^2(\Omega))}) \Bigg)^{\frac{1}{2}},$$

$$V = \left(\sum_{\ell=1}^{\frac{(n+1)^2 n}{2}} \|\partial_\nu \widetilde{y}_\ell\|^2_{H^2(\tau_1,\tau_2,T;L^2(\Gamma_0))} \right)^{\frac{1}{2}}.$$

Then by (3.1), (3.3) and (3.5), we have

$$L^{(1)}(z_\ell \chi) = \sum_{i,j=1}^{n} \partial_j \left(\chi f_{ij}(x) \partial_i \partial_t R_\ell(t,x) \right) + z_\ell \partial_t \chi \tag{3.6}$$

and

$$L^{(1)}(\chi \partial_t z_\ell) = \sum_{i,j=1}^{n} \partial_j \left(\chi f_{ij}(x) \partial_i \partial_t^2 R_\ell(t,x) \right) + (\partial_t \chi) \partial_t z_\ell. \tag{3.7}$$

We set

$$Q_1 = (\tau_1, \tau_2) \times \Omega.$$

Let $1 \le \ell \le \frac{(n+1)^2 n}{2}$. By (1.6), we see that $y(\{a_{ij}^{(k)}\}, h_\ell) \in C^3([\tau_1, \tau_2]; H^6(\Omega))$, $k = 1, 2$, so that the right-hand sides of (3.6) and (3.7) are in $L^2(Q_1)$. Moreover from (3.4) it follows that $\chi \partial_t z_\ell, \chi z_\ell \in C([\tau_1, \tau_2]; H^2(\Omega) \cap H_0^1(\Omega))$. Furthermore, by (3.1), we have $(\chi \partial_t z_\ell)(\tau_1, \cdot) = (\chi z_\ell)(\tau_1, \cdot) = (\chi \partial_t z_\ell)(\tau_2, \cdot) = (\chi z_\ell)(\tau_2, \cdot) = 0$.

Henceforth C_j denote generic constants which are dependent on Ω, T, λ, M, $\mathcal{U}$, $\{h_\ell\}$, but independent of s. Thus we can apply Theorem 2.1 (1) to (3.6) and (3.7) in Q_1. Then

$$\int_{Q_1} \{s|\nabla(\chi z_\ell)|^2 + s^3 |\chi z_\ell|^2\} e^{2s\varphi} dxdt \le C_1 \sum_{i,j=1}^{n} \sum_{|\alpha| \le 1} \int_{Q_1} \chi^2 |\partial_x^\alpha f_{ij}|^2 e^{2s\varphi} dxdt$$
$$+ C_1 U^2 e^{2s(d_0 - 2\varepsilon)} + C_1 s \int_{\tau_1}^{\tau_2} \int_{\Gamma_0} |\partial_\nu (\chi z_\ell)|^2 e^{2s\varphi} d\Sigma, \quad s \ge s_0 \tag{3.8}$$

and

$$\int_{Q_1} \{s|\nabla(\chi \partial_t z_\ell)|^2 + s^3 |\chi \partial_t z_\ell|^2] e^{2s\varphi} dxdt \le C_1 \sum_{i,j=1}^{n} \sum_{|\alpha| \le 1} \int_{Q_1} \chi^2 |\partial_x^\alpha f_{ij}(x)|^2 e^{2s\varphi} dxdt$$
$$+ C_1 U^2 e^{2s(d_0 - 2\varepsilon)} + C_1 s \int_{\tau_1}^{\tau_2} \int_{\Gamma_0} |\partial_\nu (\chi \partial_t z_\ell)|^2 e^{2s\varphi} d\Sigma, \quad s \ge s_0. \tag{3.9}$$

Here we note that $\partial_t\chi \neq 0$ only if $\varphi(t,x) \leq d_0 - 2\varepsilon$. On the other hand, we have

$$\begin{aligned}
&\int_\Omega |\partial_t\widetilde{y}_\ell(\theta,x)|^2 e^{2s\varphi(\theta,x)}dx = \int_\Omega |\chi(\theta)\partial_t\widetilde{y}_\ell(\theta,x)|^2 e^{2s\varphi(\theta,x)}dx \\
=&\int_{\tau_1}^{\theta} \partial_t\left(\int_\Omega |\chi(t)\partial_t\widetilde{y}_\ell(t,x)|^2 e^{2s\varphi(t,x)}dx\right)dt \\
\leq&\int_{Q_1} 2(|\partial_t^2\widetilde{y}_\ell||\partial_t\widetilde{y}_\ell|\chi^2 + s|\partial_t\varphi||\chi\partial_t\widetilde{y}_\ell|^2)e^{2s\varphi}dxdt + \int_{Q_1} 2|\partial_t\widetilde{y}_\ell|^2\chi|\partial_t\chi|e^{2s\varphi}dxdt \\
\leq& C_2\int_{Q_1} |\chi\partial_t z_\ell|^2 e^{2s\varphi(t,x)}dxdt + C_2(s+1)\int_{Q_1} |\chi z_\ell|^2 e^{2s\varphi(t,x)}dxdt + C_2U^2e^{2s(d_0-2\varepsilon)}.
\end{aligned} \tag{3.10}$$

By (3.8) - (3.10), we obtain

$$\begin{aligned}
&\int_\Omega |\partial_t\widetilde{y}_\ell(\theta,x)|^2 e^{2s\varphi(\theta,x)}dx \\
\leq& C_3\left\{\sum_{i,j=1}^n \sum_{|\alpha|\leq 1}\int_{Q_1} \chi^2|\partial_x^\alpha f_{ij}(x)|^2 e^{2s\varphi}dxdt + U^2e^{2s(d_0-2\varepsilon)} + se^{2s\Phi}V^2\right\}
\end{aligned} \tag{3.11}$$

for sufficiently large $s > 0$. Similarly we have

$$\begin{aligned}
&\int_\Omega |\nabla\partial_t\widetilde{y}_\ell(\theta,x)|^2 e^{2s\varphi(\theta,x)}dx \\
\leq& C_4\left\{\sum_{i,j=1}^n \sum_{|\alpha|\leq 1}\int_{Q_1} \chi^2|\partial_x^\alpha f_{ij}(x)|^2 e^{2s\varphi}dxdt + U^2e^{2s(d_0-2\varepsilon)} + se^{2s\Phi}V^2\right\}
\end{aligned} \tag{3.12}$$

for sufficiently large $s > 0$. By (3.3), we have

$$L^{(1)}\widetilde{y}_\ell(\theta,x) = \sum_{i,j=1}^n (\partial_j f_{ij}(x))\partial_i R_\ell(\theta,x) + \sum_{i,j=1}^n f_{ij}(x)\partial_i\partial_j R_\ell(\theta,x), \quad x\in\Omega \tag{3.13}$$

for $1 \leq \ell \leq \frac{(n+1)^2 n}{2}$. Let us consider the above equations for $1 \leq \ell \leq n+1$. Then we have

$$\begin{pmatrix} \partial_1R_1(\theta,x) & \partial_2R_1(\theta,x) & \dots & \partial_nR_1(\theta,x) \\ \partial_1R_2(\theta,x) & \partial_2R_2(\theta,x) & \dots & \partial_nR_2(\theta,x) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1R_{n+1}(\theta,x) & \partial_2R_{n+1}(\theta,x) & \dots & \partial_nR_{n+1}(\theta,x) \end{pmatrix} \begin{pmatrix} \sum_{j=1}^n \partial_j f_{1j}(x) \\ \sum_{j=1}^n \partial_j f_{2j}(x) \\ \vdots \\ \sum_{j=1}^n \partial_j f_{nj}(x) \end{pmatrix} \tag{3.14}$$

$$= \begin{pmatrix} L^{(1)}\widetilde{y}_1(\theta,x) - \sum_{i,j=1}^n f_{ij}\partial_i\partial_j R_1(\theta,x) \\ L^{(1)}\widetilde{y}_2(\theta,x) - \sum_{i,j=1}^n f_{ij}\partial_i\partial_j R_2(\theta,x) \\ \vdots \\ L^{(1)}\widetilde{y}_{n+1}(\theta,x) - \sum_{i,j=1}^n f_{ij}\partial_i\partial_j R_{n+1}(\theta,x) \end{pmatrix}.$$

Because linear system (3.14) is composed of $(n+1)$ equations with respect to n unknowns, and possesses a solution $\left(\sum_{j=1}^n \partial_j f_{1j}(x), \sum_{j=1}^n \partial_j f_{2j}(x), \ldots, \sum_{j=1}^n \partial_j f_{nj}(x)\right)$, the coefficients matrix must satisfy

$$\det\begin{pmatrix} L^{(1)}\widetilde{y}_1(\theta,x) - \sum_{i,j=1}^n f_{ij}\partial_i\partial_j R_1(\theta,x) & \partial_1 R_1(\theta,x) & \ldots & \partial_n R_1(\theta,x) \\ L^{(1)}\widetilde{y}_2(\theta,x) - \sum_{i,j=1}^n f_{ij}\partial_i\partial_j R_2(\theta,x) & \partial_1 R_2(\theta,x) & \ldots & \partial_n R_2(\theta,x) \\ \vdots & \vdots & \ddots & \vdots \\ L^{(1)}\widetilde{y}_{n+1}(\theta,x) - \sum_{i,j=1}^n f_{ij}\partial_i\partial_j R_{n+1}(\theta,x) & \partial_1 R_{n+1}(\theta,x) & \ldots & \partial_n R_{n+1}(\theta,x) \end{pmatrix} = 0.$$

Let us set $D_{ij}^k(x) \equiv D_{ij}^k(y(\{a_{ij}^{(2)}\}, h_\ell))(\theta,x)$. Then we have

$$\sum_{j=1}^n D_{jj}^1(x) f_{jj}(x) + 2\sum_{i<j} D_{ij}^1(x) f_{ij}(x) = Y_1(x), \quad x \in \overline{\Omega\setminus\omega_1}, \tag{3.15}$$

where

$$Y_1(x) = \det\begin{pmatrix} L^{(1)}\widetilde{y}_1(\theta,x) & \partial_1 y(\{a_{ij}^{(2)}\}, h_1)(\theta,x) & \ldots & \partial_n y(\{a_{ij}^{(2)}\}, h_1)(\theta,x) \\ L^{(1)}\widetilde{y}_2(\theta,x) & \partial_1 y(\{a_{ij}^{(2)}\}, h_2)(\theta,x) & \ldots & \partial_n y(\{a_{ij}^{(2)}\}, h_2)(\theta,x) \\ \vdots & \vdots & \ddots & \vdots \\ L^{(1)}\widetilde{y}_{n+1}(\theta,x) & \partial_1 y(\{a_{ij}^{(2)}\}, h_{n+1})(\theta,x) & \ldots & \partial_n y(\{a_{ij}^{(2)}\}, h_{n+1})(\theta,x) \end{pmatrix}.$$

We set

$$\begin{aligned} &Y_2(x) \\ =&\det\begin{pmatrix} L^{(1)}\widetilde{y}_{n+2}(\theta,x) & \partial_1 y(\{a_{ij}^{(2)}\}, h_{n+2})(\theta,x) & \ldots & \partial_n y(\{a_{ij}^{(2)}\}, h_{n+2})(\theta,x) \\ L^{(1)}\widetilde{y}_{n+3}(\theta,x) & \partial_1 y(\{a_{ij}^{(2)}\}, h_{n+3})(\theta,x) & \ldots & \partial_n y(\{a_{ij}^{(2)}\}, h_{n+3})(\theta,x) \\ \vdots & \vdots & \ddots & \vdots \\ L^{(1)}\widetilde{y}_{2n+2}(\theta,x) & \partial_1 y(\{a_{ij}^{(2)}\}, h_{2n+2})(\theta,x) & \ldots & \partial_n y(\{a_{ij}^{(2)}\}, h_{2n+2})(\theta,x) \end{pmatrix}, \end{aligned}$$

$$\vdots \quad \vdots$$

$$\begin{aligned} &Y_{\frac{(n+1)n}{2}}(x) \\ =&\det\left(\begin{matrix} L^{(1)}\widetilde{y}_{\frac{1}{2}(n+1)^2n-n}(\theta,x) & \partial_1 y(\{a_{ij}^{(2)}\}, h_{\frac{1}{2}(n+1)^2n-n})(\theta,x) & \ldots \\ L^{(1)}\widetilde{y}_{\frac{1}{2}(n+1)^2n-n+1}(\theta,x) & \partial_1 y(\{a_{ij}^{(2)}\}; h_{\frac{1}{2}(n+1)^2n-n+1})(\theta,x) & \ldots \\ \vdots & \vdots & \ddots \\ L^{(1)}\widetilde{y}_{\frac{1}{2}(n+1)^2n}(\theta,x) & \partial_1 y(\{a_{ij}^{(2)}\}; h_{\frac{1}{2}(n+1)^2n})(\theta,x) & \ldots \end{matrix}\right. \\ &\left.\begin{matrix} \partial_n y(\{a_{ij}^{(2)}\}, h_{\frac{1}{2}(n+1)^2n-n})(\theta,x) \\ \partial_n y(\{a_{ij}^{(2)}\}, h_{\frac{1}{2}(n+1)^2n-n+1})(\theta,x) \\ \vdots \\ \partial_n y(\{a_{ij}^{(2)}\}, h_{\frac{1}{2}(n+1)^2n})(\theta,x) \end{matrix}\right). \end{aligned}$$

Similarly to (3.15), we can obtain

$$\sum_{j=1}^{n} D_{jj}^{k}(x)f_{jj}(x) + 2\sum_{i<j} D_{ij}^{k}(x)f_{ij}(x) = Y_k(x), \quad x \in \overline{\Omega \setminus \omega_1} \tag{3.16}$$

for $1 \le k \le \frac{n(n+1)}{2}$. By (1.7) we can solve (3.16) uniquely with respect to $\frac{n(n+1)}{2}$ unknowns f_{ij}. By $n \le 5$ and the Sobolev embedding theorem (e.g., [1], [30]), we see that $H^6(\Omega) \subset C^3(\overline{\Omega})$. Hence $y(\{a_{ij}^{(2)}\}, h_\ell) \in C^3([\tau_1, \tau_2]; H^6(\Omega)) \subset C^3([\tau_1, \tau_2]; C^3(\overline{\Omega}))$, and so there exist $c_{ij}^{\ell} \in C^1(\overline{\Omega \setminus \omega_1})$, $1 \le i, j \le n$, $1 \le \ell \le \frac{(n+1)^2 n}{2}$, such that

$$f_{ij}(x) = \sum_{\ell=1}^{\frac{(n+1)^2 n}{2}} c_{ij}^{\ell}(x) L^{(1)} \widetilde{y}_\ell(\theta, x), \quad x \in \overline{\Omega \setminus \omega_1}, \quad 1 \le i, j \le n. \tag{3.17}$$

By noting also that $f_{ij}(x) = 0$, $x \in \omega_1$, $1 \le i, j \le n$, by means of (3.17) and $c_{ij}^{\ell} \in C^1(\overline{\Omega \setminus \omega_1})$, we have

$$\begin{aligned} &\int_\Omega \sum_{|\alpha| \le 1} |\partial_x^\alpha f_{ij}(x)|^2 e^{2s\varphi(\theta,x)} dx \le C_5 \sum_{\ell=1}^{\frac{(n+1)^2 n}{2}} \sum_{|\alpha| \le 1} \int_\Omega |\partial_x^\alpha \partial_t \widetilde{y}_\ell(\theta, x)|^2 e^{2s\varphi(\theta,x)} dx \\ &+ C_5 \sum_{\ell=1}^{\frac{(n+1)^2 n}{2}} \sum_{|\alpha| \le 3} \int_\Omega |\partial_x^\alpha \widetilde{y}_\ell(\theta, x)|^2 e^{2s\varphi(\theta,x)} dx, \quad 1 \le i, j \le n, \quad 1 \le \ell \le \frac{(n+1)^2 n}{2}. \end{aligned} \tag{3.18}$$

By (3.11) and (3.12), we have

$$\begin{aligned} &\sum_{\ell=1}^{\frac{(n+1)^2 n}{2}} \sum_{|\alpha| \le 1} \int_\Omega |\partial_x^\alpha \partial_t \widetilde{y}_\ell(\theta, x)|^2 e^{2s\varphi(\theta,x)} dx \\ \le & C_6 \sum_{i,j=1}^{n} \sum_{|\alpha| \le 1} \int_\Omega |\partial_x^\alpha f_{ij}(x)|^2 e^{2s\varphi(\theta,x)} \left(\int_{\tau_1}^{\tau_2} e^{2s(\varphi(t,x) - \varphi(\theta,x))} dt \right) dx \\ &+ C_6 U^2 e^{2s(d_0 - 2\varepsilon)} + C_6 s e^{2s\Phi} V^2 \end{aligned} \tag{3.19}$$

for all large $s > 0$. By (3.18) and (3.19), we obtain

$$\begin{aligned} &\sum_{i,j=1}^{n} \sum_{|\alpha| \le 1} \int_\Omega |\partial_x^\alpha f_{ij}(x)|^2 e^{2s\varphi(\theta,x)} dx \\ \le & C_7 \sum_{i,j=1}^{n} \sum_{|\alpha| \le 1} \int_\Omega |\partial_x^\alpha f_{ij}(x)|^2 e^{2s\varphi(\theta,x)} \left(\int_{\tau_1}^{\tau_2} e^{2s(\varphi(t,x) - \varphi(\theta,x))} dt \right) dx \\ &+ C_7 \sum_{\ell=1}^{\frac{(n+1)^2 n}{2}} \sum_{|\alpha| \le 3} \int_\Omega |\partial_x^\alpha \widetilde{y}_\ell(\theta, x)|^2 e^{2s\varphi(\theta,x)} dx + C_7 U^2 e^{2s(d_0 - 2\varepsilon)} + C_7 s e^{2s\Phi} V^2 \end{aligned} \tag{3.20}$$

for large $s > 0$. Applying the Lebesgue theorem, we have

$$\sup_{x\in\Omega}\left|\int_{\tau_1}^{\tau_2} e^{2s(\varphi(t,x)-\varphi(\theta,x))}dt\right| = \sup_{x\in\Omega}\left|\int_{\tau_1}^{\tau_2} \exp\left(2se^{\lambda d(x)}(e^{-\lambda\beta|t-\theta|^2}-1)\right)dt\right|$$
$$\le \int_{\tau_1}^{\tau_2} \exp\left(2se^{\lambda d_1}(e^{-\lambda\beta|t-\theta|^2}-1)\right)dt = o(1) \quad \text{as } s \longrightarrow \infty,$$

where $d_1 = \inf_{x\in\Omega} d(x)$. Then

$$\sum_{i,j=1}^{n}\sum_{|\alpha|\le 1}\int_\Omega |\partial_x^\alpha f_{ij}(x)|^2 e^{2s\varphi(\theta,x)}\left(\int_{\tau_1}^{\tau_2} e^{2s(\varphi(t,x)-\varphi(\theta,x))}dt\right)dx$$
$$= o(1)\sum_{i,j=1}^{n}\sum_{|\alpha|\le 1}\int_\Omega |\partial_x^\alpha f_{ij}(x)|^2 e^{2s\varphi(\theta,x)}dx \quad \text{as } s \longrightarrow \infty.$$

Hence, from (3.20) we have

$$(1-o(1))\sum_{i,j=1}^{n}\sum_{|\alpha|\le 1}\int_\Omega |\partial_x^\alpha f_{ij}(x)|^2 e^{2s\varphi(\theta,x)}dx \le C_8U^2e^{2s(d_0-2\varepsilon)} + C_8se^{2s\Phi}V^2$$
$$+C_8\sum_{\ell=1}^{\frac{(n+1)^2n}{2}}\sum_{|\alpha|\le 3}\int_\Omega |\partial_x^\alpha \widetilde{y}_\ell(\theta,x)|^2 e^{2s\varphi(\theta,x)}dx \quad \text{as } s \longrightarrow \infty.$$

By $\varphi(\theta,x) \ge d_0$ for $x \in \overline{\Omega}$, we obtain

$$(1-o(1))\sum_{i,j=1}^{n}\sum_{|\alpha|\le 1} e^{2sd_0}\int_\Omega |\partial_x^\alpha f_{ij}(x)|^2 dx \le C_9U^2e^{2s(d_0-2\varepsilon)} + C_9se^{2s\Phi}V^2$$
$$+C_9\sum_{\ell=1}^{\frac{(n+1)^2n}{2}}\sum_{|\alpha|\le 3}\int_\Omega |\partial_x^\alpha \widetilde{y}_\ell(\theta,x)|^2 e^{2s\varphi(\theta,x)}dx \quad \text{as } s \longrightarrow \infty,$$

that is,

$$(1-o(1))\sum_{i,j=1}^{n}\sum_{|\alpha|\le 1}\int_\Omega |\partial_x^\alpha f_{ij}(x)|^2 dx \le C_9U^2e^{-4s\varepsilon} + C_9se^{2s(\Phi-d_0)}V^2$$
$$+C_9\sum_{\ell=1}^{\frac{(n+1)^2n}{2}}\sum_{|\alpha|\le 3}\int_\Omega |\partial_x^\alpha \widetilde{y}_\ell(\theta,x)|^2 e^{2s(\Phi-d_0)}dx \quad \text{as } s \longrightarrow \infty. \tag{3.21}$$

On the other hand, we can prove the following estimate:

$$U^2 \le C_{10}V^2 + C_{10}\sum_{i,j=1}^{n}\sum_{|\alpha|\le 1}\int_\Omega |\partial_x^\alpha f_{ij}(x)|^2 dx. \tag{3.22}$$

In fact, by (3.3) and (3.4) we have

$$\begin{cases} L^{(1)}\partial_t\widetilde{y}_\ell(t,x) = \sum_{i,j=1}^{n} \partial_j(f_{ij}(x)\partial_i\partial_t R_\ell(t,x)), \quad (t,x)\in(0,T)\times\Omega, \\ \partial_t\widetilde{y}_\ell(t,x)=0, \quad (t,x)\in(0,T)\times\partial\Omega,\ 1\le\ell\le \dfrac{(n+1)^2 n}{2}. \end{cases} \tag{3.23}$$

Apply Lemma 2.4 in [16] to (3.23). Then we can see that there exist $\psi\in C^2(\overline{\Omega})$ and a constant $\sigma_0>0$ such that for a constant $\sigma\ge\sigma_0$ we can choose $\eta_0(\sigma)>0$ such that for each $\eta\ge\eta_0(\sigma)$, we have

$$\begin{aligned}
&\int_{(\tau_3,\tau_4)\times\Omega}\left(\frac{\eta e^{\sigma\psi}}{(t-\tau_3)(\tau_4-t)}|\nabla\partial_t\widetilde{y}_\ell|^2+\frac{\eta^3e^{3\sigma\psi}}{(t-\tau_3)^3(\tau_4-t)^3}|\partial_t\widetilde{y}_\ell|^2\right)\\
&\exp\left\{\frac{2\eta\left(e^{\sigma\psi}-e^{2\sigma\|\psi\|_{C(\overline{\Omega})}}\right)}{(t-\tau_3)(\tau_4-t)}\right\}dxdt\\
\le& C_{11}\sum_{i,j=1}^{n}\sum_{|\alpha|=1}\int_{(\tau_3,\tau_4)\times\Omega}|\partial_x^\alpha f_{ij}(x)|^2\exp\left\{\frac{2\eta\left(e^{\sigma\psi}-e^{2\sigma\|\psi\|_{C(\overline{\Omega})}}\right)}{(t-\tau_3)(\tau_4-t)}\right\}dxdt\\
&+C_{11}\eta\int_{\tau_3}^{\tau_4}\int_{\Gamma_0}|\partial_\nu\partial_t\widetilde{y}_\ell|^2\frac{\eta e^{\sigma\psi}}{(t-\tau_3)(\tau_4-t)}\exp\left\{\frac{2\eta\left(e^{\sigma\psi}-e^{2\sigma\|\psi\|_{C(\overline{\Omega})}}\right)}{(t-\tau_3)(\tau_4-t)}\right\}dxdt,
\end{aligned}$$

where the constant $C_{11}>0$ depends on $\mathcal{U}$, σ, but independent of η, and the constant σ_0 depends on $\mathcal{U}$. We fix $\sigma>\sigma_0$ and $\eta>\eta_0(\sigma)$. Then

$$0<C_{11}\le\exp\left\{\frac{2\eta\left(e^{\sigma\psi}-e^{2\sigma\|\psi\|_{C(\overline{\Omega})}}\right)}{(t-\tau_3)(\tau_4-t)}\right\}$$

for $x\in\overline{\Omega}$ and $\tau_1<t<\tau_2$ and

$$\exp\left\{\frac{2\eta\left(e^{\sigma\psi}-e^{2\sigma\|\psi\|_{C(\overline{\Omega})}}\right)}{(t-\tau_3)(\tau_4-t)}\right\},\quad \frac{\eta e^{\sigma\psi}}{(t-\tau_3)(\tau_4-t)}\exp\left\{\frac{2\eta\left(e^{\sigma\psi}-e^{2\sigma\|\psi\|_{C(\overline{\Omega})}}\right)}{(t-\tau_3)(\tau_4-t)}\right\}\le C_{12}$$

for $x\in\overline{\Omega}$ and $\tau_3<t<\tau_4$. Hence we have

$$\begin{aligned}
&\int_{Q_1}\left(|\nabla\partial_t\widetilde{y}_\ell|^2+|\partial_t\widetilde{y}_\ell|^2\right)dxdt\\
\le& C_{13}\sum_{i,j=1}^{n}\sum_{|\alpha|\le1}\int_{(\tau_3,\tau_4)\times\Omega}|\partial_x^\alpha f_{ij}(x)|^2dxdt+C_{13}\int_{\tau_3}^{\tau_4}\int_{\Gamma_0}|\partial_\nu\partial_t\widetilde{y}_\ell|^2dxdt.
\end{aligned}\tag{3.24}$$

Similarly, we can obtain

$$\begin{aligned}&\int_{Q_1}(|\nabla\partial_t^2\widetilde{y}_\ell|^2+|\partial_t^2\widetilde{y}_\ell|^2)dxdt\\ \le&C_{13}\sum_{i,j=1}^n\sum_{|\alpha|\le1}\int_{(\tau_3,\tau_4)\times\Omega}|\partial_x^\alpha f_{ij}(x)|^2dxdt+C_{13}\int_{\tau_3}^{\tau_4}\int_{\Gamma_0}|\partial_\nu\partial_t^2\widetilde{y}_\ell|^2dxdt.\end{aligned}\tag{3.25}$$

By (3.24) and (3.25), we complete the proof of (3.22).

We can obtain (1.6) by substituting (3.22) into (3.21) and taking s large enough. Thus the proof of Theorem 1.1 is complete.

Proof of Theorem 1.2. Let $B=(b_{ij})_{1\le i,j\le n}$ be an $n\times n$ matrix such that $b_{ij}\in\mathbb{R}$ and $\det B>0$. We set

$$\widetilde{g}_i(x)=\sum_{j=1}^n b_{ij}x_j,\qquad 1\le i\le n$$

and

$$\begin{aligned}&\widehat{g}_1(x)=x_1^2,\ \widehat{g}_2(x)=2x_1x_2,\ \widehat{g}_3(x)=2x_1x_3,\ \cdots,\ \widehat{g}_n(x)=2x_1x_n,\\ &\widehat{g}_{n+1}(x)=x_2^2,\ \widehat{g}_{n+2}(x)=2x_2x_3,\ \cdots,\ \widehat{g}_{2n-1}(x)=2x_2x_n,\\ &\vdots\quad\vdots\\ &\widehat{g}_{\frac{(n+1)n}{2}-2}(x)=x_{n-1}^2,\ \widehat{g}_{\frac{(n+1)n}{2}-1}(x)=2x_{n-1}x_n,\\ &\widehat{g}_{\frac{(n+1)n}{2}}(x)=x_n^2.\end{aligned}$$

Let us define an $\frac{(n+1)^2n}{2}-$ dimensional vector by

$$\begin{aligned}&\Big(g_1(x),g_2(x),\cdots,g_{n+1}(x),\quad g_{n+2}(x),g_{n+3}(x),\cdots,g_{2n+2}(x),\quad\cdots\\ &g_{\frac{n^3+2n^2-n}{2}}(x),g_{\frac{n^3+2n^2-n}{2}+1}(x),\cdots,g_{\frac{n^3+2n^2-n}{2}+n}(x)\Big)\\ =&\Big(\widehat{g}_1(x),\widetilde{g}_1(x),\quad\cdots,\widetilde{g}_n(x),\quad\widehat{g}_2(x),\widetilde{g}_1(x),\cdots,\widetilde{g}_n(x),\quad\cdots\\ &\widehat{g}_{\frac{n(n+1)}{2}}(x),\widetilde{g}_1(x),\cdots,\widetilde{g}_n(x)\Big).\end{aligned}\tag{3.26}$$

Therefore, noting that $\partial_i\partial_j\widetilde{g_k}=0$, we obtain

$$D^k_{ij}(\{g_\ell\})(x)=\det\begin{pmatrix}\partial_i\partial_j\widehat{g}_k & * \\ 0 & B\end{pmatrix}$$
$$=(\partial_i\partial_j\widehat{g}_k)\det B, \qquad 1\le k\le \frac{n(n+1)}{2},\ 1\le i,j\le n.$$

Hence

$$D(\{g_\ell\})(x)=\begin{pmatrix} 2\det B & 0 & 0 & \cdots & 0 & 0\\ 0 & 2\det B & 0 & \cdots & 0 & 0\\ 0 & 0 & 2\det B & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 2\det B & 0\\ 0 & 0 & 0 & \cdots & 0 & 2\det B\end{pmatrix}.$$

Consequently we have $\det D(\{g_\ell\})(x)=(2\det B)^{\frac{(n+1)n}{2}}>0$. We introduce a cut-off function $\chi_1\in C_0^\infty(\Omega)$ such that $\chi_1=1$ on $\overline{\Omega\setminus\omega_1}$. Then we have

$$\chi_1 g_\ell\in\mathcal{D}(A^3)\quad\text{and}\quad D(\{\chi_1 g_\ell\})(x)>0,\quad x\in\overline{\Omega\setminus\omega_1}. \tag{3.27}$$

Here we recall that A is defined by (1.9).

By Proposition 1.1, for arbitrarily $\mu_\ell\in L^2(\Omega)$, we can choose $h_\ell\in C_0^\infty((0,T)\times\omega)$, $1\le\ell\le\frac{(n+3)n}{2}$, so that for a sufficiently small $\varepsilon>0$ we have

$$\|y(\{a^{(2)}_{ij}\},h_\ell,\mu_\ell)(\theta,\cdot)-\chi_1\widehat{g}_\ell\|_{H^6(\Omega)}\le\varepsilon,\quad 1\le\ell\le\frac{(n+1)n}{2}$$

and

$$\left\|y\left(\{a^{(2)}_{ij}\},h_{\frac{(n+1)n}{2}+k},\mu_{\frac{(n+1)n}{2}+k}\right)(\theta,\cdot)-\chi_1\widetilde{g}_k\right\|_{H^6(\Omega)}\le\varepsilon,\quad 1\le k\le n.$$

Here we note that $y(\{a^{(2)}_{ij}\},h,\mu)$ denotes the solution to (1.1) and (1.2) with $y(0,\cdot)=\mu$. Since $n\le 5$, we have $H^6(\Omega)\subset C^2(\overline{\Omega})$. Then we can obtain

$$\|y(\{a^{(2)}_{ij}\},h_\ell,\mu_\ell)(\theta,\cdot)-\chi_1\widehat{g}_\ell\|_{C^2(\overline{\Omega\setminus\omega_1})}\le\varepsilon,\quad 1\le\ell\le\frac{(n+1)n}{2} \tag{3.28}$$

and

$$\left\| y\left(\{a_{ij}^{(2)}\}, h_{\frac{(n+1)n}{2}+k}, \mu_{\frac{(n+1)n}{2}+k}\right)(\theta,\cdot) - \chi_1 \widetilde{g}_k \right\|_{C^2(\overline{\Omega\setminus\omega_1})} \le \varepsilon, \quad 1 \le k \le n. \quad (3.29)$$

Let

$$\begin{aligned}
&(\widehat{h}_m)_{1\le m\le \frac{(n+1)^2 n}{2}} \\
=&\Big(\widehat{h}_1, \widehat{h}_2, \cdots, \widehat{h}_{n+1}, \quad \widehat{h}_{n+2}, \widehat{h}_{n+3}, \cdots, \widehat{h}_{2n+2}, \quad \cdots \\
&\widehat{h}_{\frac{n^3+2n^2-n}{2}}, \widehat{h}_{\frac{n^3+2n^2-n}{2}+1}, \cdots, \widehat{h}_{\frac{n^3+2n^2-n}{2}+n}\Big) \\
\equiv&\Big(h_1, h_{\frac{n(n+1)}{2}+1}, \cdots, h_{\frac{n(n+1)}{2}+n}, \quad h_2, h_{\frac{n(n+1)}{2}+1}, \cdots, h_{\frac{n(n+1)}{2}+n}, \quad \cdots \\
&h_{\frac{n(n+1)}{2}}, h_{\frac{n(n+1)}{2}+1}, \cdots, h_{\frac{n(n+1)}{2}+n}\Big)
\end{aligned}$$

and

$$\begin{aligned}
&(\widehat{\mu}_m)_{1\le m\le \frac{(n+1)^2 n}{2}} \equiv \Big(\widehat{\mu}_1, \widehat{\mu}_2, \cdots, \widehat{\mu}_{n+1}, \quad \widehat{\mu}_{n+2}, \widehat{\mu}_{n+3}, \cdots, \widehat{\mu}_{2n+2}, \quad \cdots \\
&\widehat{\mu}_{\frac{n^3+2n^2-n}{2}}, \widehat{\mu}_{\frac{n^3+2n^2-n}{2}+1}, \cdots, \widehat{\mu}_{\frac{n^3+2n^2-n}{2}+n}\Big) \\
=&\Big(\mu_1, \mu_{\frac{n(n+1)}{2}+1}, \cdots, \mu_{\frac{n(n+1)}{2}+n}, \quad \mu_2, \mu_{\frac{n(n+1)}{2}+1}, \cdots, \mu_{\frac{n(n+1)}{2}+n}, \quad \cdots \\
&\mu_{\frac{n(n+1)}{2}}, \mu_{\frac{n(n+1)}{2}+1}, \cdots, \mu_{\frac{n(n+1)}{2}+n}\Big). \qquad (3.31)
\end{aligned}$$

By (3.27) - (3.31), we can obtain

$$D(y(\{a_{ij}^{(2)}\}, \widehat{h}_m, \widehat{\mu}_m))(\theta, x) > 0, \quad x \in \overline{\Omega\setminus\omega_1}$$

by taking ε small enough. Thus, by applying Theorem 1.1 to $\widehat{h}_m$, $1 \le m \le \frac{(n+1)^2 n}{2}$, the proof of Theorem 1.2 is complete.

§4. Proof of Proposition 1.1.

By $\|\cdot\|$ and $(\cdot,\cdot)$ we denote the norm and the scalar product in $L^2(\Omega)$, respectively. We recall that the operator A in $L^2(\Omega)$ is defined by (1.9). By $a_{ij} \in C^5(\overline{\Omega})$, we apply the elliptic regularity (e.g., Theorem 8.13 (p.187) in [13]) and we obtain

$$C_1^{-1}\|A^3u\| \le \|u\|_{H^6(\Omega)} \le C_1\|A^3u\|, \quad u \in \mathcal{D}(A^3).$$

Moreover it is known that there exists a sequence of eigenvalues $\{\kappa_m\}_{m\in\mathbb{N}}$ of A:

$$0 < \kappa_1 \le \kappa_2 \le \cdots \longrightarrow \infty,$$

where κ_m appears the same time as its multiplicity. Then we can form an orthonormal basis $\{e_m\}_{m\in\mathbb{N}}$ in $L^2(\Omega)$ such that $Ae_m = \kappa_m e_m$. We have

$$\|A^\ell u\| = \left(\sum_{m=1}^{\infty} \kappa_m^{2\ell}(u, e_m)^2\right)^{\frac{1}{2}}$$

and $\mathcal{D}(A^\ell)$, $\ell \in \mathbb{N}\cup\{0\}$, is a Hilbert space with the scalar product

$$(u,v)_{\mathcal{D}(A^\ell)} = \sum_{m=1}^{\infty} \kappa_m^{2\ell}(u,e_m)(v,e_m).$$

In particular, $\mathcal{D}(A^0) = L^2(\Omega)$, and $\mathcal{D}(A^3)$ is dense in $L^2(\Omega)$, and the embedding is continuous. Identifying the dual space $(L^2(\Omega))'$ with itself, we have $\mathcal{D}(A^3) \subset L^2(\Omega) \subset (\mathcal{D}(A^3))'$ topologically (e.g., [6]). Henceforth we set $(\mathcal{D}(A^3))' = \mathcal{D}(A^{-3})$ and ${}_{\mathcal{D}(A^3)}< u, \xi >_{\mathcal{D}(A^{-3})}$ denotes the value of a linear functional $\xi \in (D(A^3))'$ at u. We note that

$${}_{\mathcal{D}(A^3)}< u, \xi >_{\mathcal{D}(A^{-3})} = (u, \xi)$$

if $u \in \mathcal{D}(A^3)$ and $\xi \in L^2(\Omega)$ (e.g., V.2 in [6]).

Then we note that $L^2(\Omega)$ is dense in $\mathcal{D}(A^{-3})$, A^{-3} is extended uniquely to a bounded operator in $\mathcal{D}(A^{-3})$ and $\|u\|_{\mathcal{D}(A^{-3})} = \|A^{-3}u\|$. By the density of $C_0^\infty(\Omega)$ in $L^2(\Omega)$, we see also that $C_0^\infty(\Omega)$ is dense in $\mathcal{D}(A^{-3})$. Furthermore it is seen that e^{-tA} is an analytic semigroup in $\mathcal{D}(A^{-3})$ and $A^{-3}e^{-tA} = e^{-tA}A^{-3}$.

Now we proceed to the proof of the proposition. Without loss of generality, we can suppose that $\mu = 0$, because the parabolic equation (1.1) and (1.2) with $y(0) = \mu$ is linear. We will use the duality argument. First we consider

$$\begin{cases} -\dfrac{\partial z}{\partial t} + Az(t,x) = 0, \qquad (t,x) \in Q, \\ z = 0, \qquad (t,x) \in \Sigma, \\ z(T,x) = \xi(x), \quad x \in \Omega, \end{cases} \tag{4.1}$$

where $\xi \in \mathcal{D}(A^{-3})$. We can verify (e.g., [32]) that for every $\xi \in \mathcal{D}(A^{-3})$, there exists a unique solution $z \in C([0,T];\mathcal{D}(A^{-3}))$ such that

$$\|z\|_{C([0,T];\mathcal{D}(A^{-3}))} \le C\|\xi\|_{\mathcal{D}(A^{-3})}.$$

Recall that $y(\{a_{ij}\},h,0)(t,x)$ is the solution to (1.1) and (1.2) with $y(0) = 0$ where $h \in C_0^\infty((0,T)\times\omega)$. We will prove

$${}_{\mathcal{D}(A^3)}\left\langle y(\{a_{ij}\},h,0)(T,\cdot),\xi\right\rangle_{\mathcal{D}(A^{-3})} = {}_{L^2(0,T;\mathcal{D}(A^3))}\left\langle h,z\right\rangle_{L^2(0,T;\mathcal{D}(A^{-3}))}. \tag{4.2}$$

In fact, by the density of $C_0^\infty(\Omega)$ in $\mathcal{D}(A^{-3})$, there exists a sequence $\xi_k \in C_0^\infty(\Omega)$, $k \in \mathbb{N}$ such that $\xi_k \longrightarrow \xi$ in $\mathcal{D}(A^{-3})$. By z_k we denote the solution to (4.1) with the final value ξ_k at $t = T$. Then $z_k, y(\{a_{ij}\},h,0) \in C^1([0,T];L^2(\Omega)) \cap C([0,T];\mathcal{D}(A))$ (e.g., Theorem 3.5 (p.114) in [32]). Therefore we can multiply (1.1) with $z_k(t,x)$, so that by integrating by parts, we have

$$(y(\{a_{ij}\},h,0)(T,\cdot),\xi_k)_{L^2(\Omega)} = (h,z_k)_{L^2((0,T)\times\Omega)}.$$

Noting that $h \in C_0^\infty((0,T)\times\omega)$, we can further rewrite it as

$${}_{\mathcal{D}(A^3)}\left\langle y(\{a_{ij}\},h,0)(T,\cdot),\xi_k\right\rangle_{\mathcal{D}(A^{-3})} = {}_{L^2(0,T;\mathcal{D}(A^3))}\left\langle h,z_k\right\rangle_{L^2(0,T;\mathcal{D}(A^{-3}))}.$$

Since $y(\{a_{ij}\},h,0)(t,\cdot) = \int_0^t e^{-(t-s)A}h(s,\cdot)ds$ for $t > 0$ (e.g., [32]) and $h \in C_0^\infty((0,T)\times\omega)$, we directly see that $y(\{a_{ij}\},h,0)(T,\cdot) \in \mathcal{D}(A^3)$. Hence, as $k \longrightarrow \infty$, we have

$${}_{\mathcal{D}(A^3)}\left\langle y(\{a_{ij}\},h,0)(T,\cdot),\xi\right\rangle_{\mathcal{D}(A^{-3})} = {}_{L^2(0,T;\mathcal{D}(A^3))}\left\langle h,z\right\rangle_{L^2(0,T;\mathcal{D}(A^{-3}))}.$$

Thus we proved (4.2).

For the proof of the proposition, it is sufficient to verify that if

$$_{\mathcal{D}(A^3)}\langle y(\{a_{ij}\}, h, 0)(T, \cdot), \xi\rangle_{\mathcal{D}(A^{-3})} = 0 \tag{4.3}$$

for all $h \in C_0^\infty((0,T)\times\omega)$, then $\xi = 0$. Let us assume (4.3). Then for any $\delta \in (0,T)$, by (4.2) we have

$$_{L^2(0,T-\delta;\mathcal{D}(A^3))}\langle h, z\rangle_{L^2(0,T-\delta;\mathcal{D}(A^{-3}))} = 0 \quad \text{for all } h \in C_0^\infty((0,T-\delta)\times\omega).$$

By the smoothing property for the parabolic equation (e.g., [32]), we know that $z \in L^2(0,T-\delta;\mathcal{D}(A)) \subset L^2(0,T-\delta;H^2(\Omega)\cap H_0^1(\Omega))$. Therefore

$$\begin{aligned} &_{L^2(0,T-\delta;\mathcal{D}(A^3))}\langle h, z\rangle_{L^2(0,T-\delta;\mathcal{D}(A^{-3}))} = (h, z)_{L^2(0,T-\delta;L^2(\Omega))} \\ =&(h, z)_{L^2(0,T-\delta;L^2(\omega))} \end{aligned}$$

for all $h \in C_0^\infty((0,T-\delta)\times\omega)$. Hence we have $z = 0$ in $(0,T-\delta)\times\omega$. By the unique continuation for the parablic equation (e.g., Saut and Scheurer [33]), we can see that $z = 0$ in $(0,T-\delta)\times\Omega$. Since δ is arbitrary and $z \in C([0,T];\mathcal{D}(A^{-3}))$, we can obtain $\xi = 0$. Thus the proof of Proposition 1.1 is complete.

Acknowledgements. The first named author is supported partially by the Japanese Government Scholarship. The second named author is supported partially by Grant 15340027 from the Japan Society for the Promotion of Science and Grant 17654019 from the Ministry of Education, Cultures, Sports and Technology.

Appendix A. Proof of Theorem 2.1.

The proof is done by modifying the proofs in [12] and [15], where the authors treat the case when the weight function contains a singular function.

We first prove (2.6). It suffices to prove (2.6) for the operator

$$\widetilde{L}v = \partial_t v - \sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j v.$$

Henceforth we set

$$a(x,\zeta,\xi) \equiv \sum_{i,j=1}^{n} a_{ij}(x)\zeta_i\xi_j, \quad \zeta = (\zeta_1,...,\zeta_n),\ \xi = (\xi_1,...,\xi_n) \in \mathbb{R}^n, \quad (t,x) \in (\tau_1,\tau_2)\times\Omega.$$

Henceforth we set

$$Q_1 = (\tau_1,\tau_2)\times\Omega, \qquad \Sigma_1 = (\tau_1,\tau_2)\times\partial\Omega.$$

Let $w(t,x) = e^{s\varphi}v(t,x)$. By (2.5) we have

$$w(\tau_1,\cdot) = w(\tau_2,\cdot) = 0 \quad \text{in } \Omega. \tag{A.1}$$

Let

$$Pw \equiv e^{s\varphi}\widetilde{L}e^{-s\varphi}w = e^{s\varphi}\widetilde{L}v \quad \text{in } Q_1.$$

It is easy to see that the operator P has the form

$$\begin{aligned} Pw = \partial_t w - \sum_{i,j=1}^{n} a_{ij}\partial_i\partial_j w + 2s\lambda\varphi\sum_{i,j=1}^{n} a_{ij}(\partial_i d)\partial_j w \\ +s\lambda^2\varphi w\sum_{i,j=1}^{n} a_{ij}(\partial_i d)\partial_j d - s^2\lambda^2\varphi^2 w\sum_{i,j=1}^{n} a_{ij}(\partial_i d)\partial_j d \\ +s\lambda\varphi w\sum_{i,j=1}^{n} a_{ij}\partial_i\partial_j d - sw\partial_t\varphi. \end{aligned} \tag{A.2}$$

We set

$$\begin{aligned} P_1 w + P_2 w = Pw - s\lambda^2\varphi w\sum_{i,j=1}^{n} a_{ij}(\partial_i d)\partial_j d \\ -s\lambda\varphi w\sum_{i,j=1}^{n} a_{ij}\partial_i\partial_j d + sw\partial_t\varphi \equiv f_s \quad \text{in } Q_1, \end{aligned} \tag{A.3}$$

where

$$P_1 w = -\sum_{i,j=1}^{n} a_{ij}\partial_i\partial_j w - s^2\lambda^2\varphi^2 a(x,\nabla d,\nabla d)w, \tag{A.4}$$

$$P_2 w = \partial_t w + 2s\lambda\varphi\sum_{i,j=1}^{n} a_{ij}(\partial_i d)\partial_j w. \tag{A.5}$$

Equation (A.3) implies

$$\|f_s\|^2_{L^2(Q_1)} = \|P_1 w\|^2_{L^2(Q_1)} + \|P_2 w\|^2_{L^2(Q_1)} + 2(P_1 w, P_2 w)_{L^2(Q_1)}. \tag{A.6}$$

By virtue of (A.4) and (A.5) we have

$$\begin{aligned}(P_1 w, P_2 w)_{L^2(Q_1)} = &\left(-\sum_{i,j=1}^{n} a_{ij}\partial_i\partial_j w - s^2\lambda^2\varphi^2 w a(x,\nabla d,\nabla d), \partial_t w\right)_{L^2(Q_1)}\\
&-\int_{Q_1} 2s^3\lambda^3 w\varphi^3 a(x,\nabla d,\nabla d)a(x,\nabla d,\nabla w)dxdt - \int_{Q_1} 2s\lambda\varphi\sum_{i,j=1}^{n} a_{ij}\partial_i\partial_j w\sum_{k,\ell=1}^{n} a_{k\ell}(\partial_k d)\partial_\ell w dxdt\\
&\equiv I_1 + I_2 + I_3. \end{aligned}\tag{A.7}$$

We note

$$\nabla w = (\partial_\nu w)\nu \qquad \text{on } \Sigma_1, \tag{A.8}$$

because $v \in L^2(\tau_1,\tau_2; H^2(\Omega)\cap H^1_0(\Omega))$ implies $w|_{\Sigma_1} = 0$.

Noting also that $a_{ij} = a_{ji}$ and $w(\tau_1,\cdot) = w(\tau_2,\cdot) = 0$, we transform I_1, I_2 and I_3 by integrating by parts respectively:

$$\begin{aligned}I_1 = &\int_{Q_1}\left[\partial_t w\sum_{i,j=1}^{n}(\partial_i a_{ij})\partial_j w + \sum_{i,j=1}^{n} a_{ij}(\partial_j w)\partial_i\partial_t w - \frac{s^2\lambda^2}{2}\varphi^2 a(x,\nabla d,\nabla d)\partial_t(w^2)\right]dxdt\\
= &\int_{Q_1}\left[\partial_t w\sum_{i,j=1}^{n}(\partial_i a_{ij})\partial_j w + w^2\frac{s^2\lambda^2}{2}\partial_t(\varphi^2 a(x,\nabla d,\nabla d))\right]dxdt,\end{aligned}\tag{A.9}$$

$$\begin{aligned}I_2 = &-\int_{Q_1} s^3\lambda^3\varphi^3\sum_{i,j=1}^{n} a_{ij}a(x,\nabla d,\nabla d)(\partial_i d)\partial_j(w^2)dxdt\\
= &\int_{Q_1}\left[3s^3\lambda^4 w^2\varphi^3 a(x,\nabla d,\nabla d)^2 + s^3\lambda^3 w^2\varphi^3\sum_{i,j=1}^{n}\partial_j(a_{ij}a(x,\nabla d,\nabla d)\partial_i d)\right]dxdt\end{aligned}\tag{A.10}$$

and

$$
\begin{aligned}
I_3 &= \int_{Q_1} -\left(\sum_{i,j=1}^n a_{ij}\partial_i\partial_j w\right)\left(2s\lambda\varphi\sum_{k,\ell=1}^n a_{k\ell}(\partial_k d)\partial_\ell w\right) dxdt \\
&= \int_{Q_1}\left\{\sum_{i,j=1}^n 2s\lambda\varphi(\partial_j a_{ij})(\partial_i w)\sum_{k,\ell=1}^n a_{k\ell}(\partial_k d)\partial_\ell w + 2s\lambda^2\varphi\sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j d)\sum_{k,\ell=1}^n a_{k\ell}(\partial_k d)\partial_\ell w\right. \\
&\left. +2s\lambda\varphi\sum_{i,j=1}^n\left[a_{ij}\partial_i w\sum_{k,\ell=1}^n \partial_j(a_{k\ell}\partial_k d)\partial_\ell w\right] + 2s\lambda\varphi\sum_{i,j=1}^n\left(a_{ij}\partial_i w\sum_{k,\ell=1}^n a_{k\ell}(\partial_k d)\partial_j\partial_\ell w\right)\right\} dxdt \\
&-2\int_{\Sigma_1}\left(\sum_{i,j=1}^n a_{ij}\nu_j\partial_i w\right)\left(s\lambda\varphi\sum_{k,\ell=1}^n a_{k\ell}(\partial_k d)\partial_\ell w\right) d\Sigma.
\end{aligned}
$$

By using (A.8) and $a_{ij} = a_{ji}$, we can obtain

$$
\begin{aligned}
I_3 &= \int_{Q_1}\left\{\sum_{i,j=1}^n 2s\lambda\varphi(\partial_j a_{ij})\partial_i w\sum_{k,\ell=1}^n a_{k\ell}(\partial_k d)\partial_\ell w + 2s\lambda^2\varphi\sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j d)\sum_{k,\ell=1}^n a_{k\ell}(\partial_k d)\partial_\ell w\right. \\
&\left. +2s\lambda\varphi\sum_{i,j=1}^n\left[a_{ij}(\partial_i w)\sum_{k,\ell=1}^n \partial_j(a_{k\ell}\partial_k d)\partial_\ell w\right] + s\lambda\varphi\sum_{k,\ell=1}^n\left[a_{k\ell}\partial_k d\sum_{i,j=1}^n a_{ij}\partial_\ell\left((\partial_i w)\partial_j w\right)\right]\right\} dxdt \\
&-2s\lambda\int_{\Sigma_1}\varphi|\partial_\nu w|^2 a(x,\nu,\nu)a(x,\nabla d,\nu)d\Sigma.
\end{aligned}
$$

Integrating by parts, we have

$$
\begin{aligned}
I_3 &= \int_{Q_1}\left\{\sum_{i,j=1}^n 2s\lambda\varphi(\partial_j a_{ij})(\partial_i w)\sum_{k,\ell=1}^n a_{k\ell}(\partial_k d)\partial_\ell w\right. \\
&+2s\lambda^2\varphi a(x,\nabla d,\nabla w)^2 + 2s\lambda\varphi\sum_{i,j=1}^n\left[a_{ij}\partial_i w\sum_{k,\ell=1}^n \partial_j(a_{k\ell}\partial_k d)\partial_\ell w\right] \\
&-s\lambda^2\varphi a(x,\nabla d,\nabla d)a(x,\nabla w,\nabla w) - s\lambda\varphi\sum_{k,\ell=1}^n \partial_\ell(a_{k\ell}\partial_k d)a(x,\nabla w,\nabla w) \\
&\left. -s\lambda\varphi\sum_{k,\ell=1}^n\left[a_{k\ell}\partial_k d\sum_{i,j=1}^n(\partial_\ell a_{ij})(\partial_i w)\partial_j w\right]\right\} dxdt \\
&-s\lambda\int_{\Sigma_1}\varphi|\partial_\nu w|^2 a(x,\nu,\nu)a(x,\nabla d,\nu)d\Sigma. \qquad \text{(A.11)}
\end{aligned}
$$

By (A.9) - (A.11), we have

$$(P_1 w, P_2 w)_{L^2(Q_1)} = \int_{Q_1} \Bigg[3s^3\lambda^4\varphi^3 w^2 a(x,\nabla d,\nabla d)^2 + \left(\sum_{i,j=1}^n (\partial_j a_{ij})\partial_i w \right) P_2 w$$
$$+ 2s\lambda^2\varphi a(x,\nabla d,\nabla w)^2 - s\lambda^2\varphi a(x,\nabla d,\nabla d)a(x,\nabla w,\nabla w) \Bigg] dxdt$$
$$- s\lambda \int_{\Sigma_1} \varphi |\partial_\nu w|^2 a(x,\nu,\nu) a(x,\nabla d,\nu) d\Sigma + X_1, \tag{A.12}$$

where

$$X_1 = \int_{Q_1} \Bigg\{ w^2 \frac{s^2\lambda^2}{2} \partial_t(\varphi^2 a(x,\nabla d,\nabla d)) + s^3\lambda^3 w^2 \varphi^3 \sum_{i,j=1}^n \partial_j(a_{ij} a(x,\nabla d,\nabla d)\partial_i d)$$
$$+ 2s\lambda\varphi \sum_{i,j=1}^n \left[a_{ij}(\partial_i w) \sum_{k,\ell=1}^n \partial_j(a_{k\ell}\partial_k d)\partial_\ell w \right] - s\lambda\varphi \sum_{k,\ell=1}^n \partial_\ell(a_{k\ell}\partial_k d) a(x,\nabla w,\nabla w)$$
$$- s\lambda\varphi \sum_{k,\ell=1}^n \left[a_{k\ell}\partial_k d \sum_{i,j=1}^n (\partial_\ell a_{ij})(\partial_i w)\partial_j w \right] \Bigg\} dxdt.$$

Henceforth we take $\lambda > 1$ and by C_j we denote generic constants which do not depend on s and λ, and continuously depends on $\sum_{i,j=1}^n \|a_{ij}\|_{C^1(\overline{\Omega})}$. Then by $a_{ij} \in C^1(\overline{\Omega})$, we obtain

$$|X_1| \le C_1 \int_{Q_1} \left[(s\lambda\varphi + 1)|\nabla w|^2 + (s^3\lambda^3\varphi^3 + s^2\lambda^3\varphi^2) w^2 \right] dxdt. \tag{A.13}$$

Multiply (A.3) by $s\lambda^2\varphi w a(x,\nabla d,\nabla d)$ and integrate by parts in Q_1, and we obtain

$$\int_{Q_1} s\lambda^2\varphi f_s a(x,\nabla d,\nabla d) w dxdt$$
$$= \int_{Q_1} \Bigg\{ s\lambda^2\varphi a(x,\nabla d,\nabla d) w P_2 w - s^3\lambda^4\varphi^3 a(x,\nabla d,\nabla d)^2 w^2$$
$$+ s\lambda^2\varphi a(x,\nabla w,\nabla w) a(x,\nabla d,\nabla d) + s\lambda^3\varphi a(x,\nabla d,\nabla d) a(x,\nabla d,\nabla w) w$$
$$+ s\lambda^2\varphi w \sum_{i,j=1}^n \partial_j(a_{ij} a(x,\nabla d,\nabla d))\partial_i w \Bigg\} dxdt.$$

Consequently

$$2s^3\lambda^4 \int_{Q_1} \varphi^3 a(x,\nabla d,\nabla d)^2 w^2 dxdt$$
$$= 2\int_{Q_1} s\lambda^2\varphi a(x,\nabla w,\nabla w) a(x,\nabla d,\nabla d) dxdt + 2X_2, \tag{A.14}$$

where

$$X_2 = \int_{Q_1} \left\{ s\lambda^2\varphi w \sum_{i,j=1}^{n} \partial_j(a_{ij}a(x,\nabla d,\nabla d))\partial_i w + s\lambda^3\varphi a(x,\nabla d,\nabla d)a(x,\nabla d,\nabla w)w \right.$$

$$\left. + s\lambda^2\varphi a(x,\nabla d,\nabla d)wP_2 w - s\lambda^2 f_s\varphi a(x,\nabla d,\nabla d)w \right\} dxdt.$$

By $a_{ij} \in C^1(\overline{\Omega})$ and the Schwarz inequality, we obtain

$$|X_2| \le \frac{1}{16}\|P_2 w\|_{L_2(Q_1)} + C_2 \int_{Q_1} \left[(s^2\lambda^4\varphi^2 + s^2\lambda^4\varphi)w^2 + \lambda^2\varphi|\nabla w|^2 \right] dxdt$$
$$+\frac{1}{2}\|f_s\|^2_{L^2(Q_1)}. \tag{A.15}$$

Using $3s^3\lambda^4\varphi^3w^2a(x,\nabla d,\nabla d)^2 = s^3\lambda^4\varphi^3w^2a(x,\nabla d,\nabla d)^2 + 2s^3\lambda^4\varphi^3w^2a(x,\nabla d,\nabla d)^2$ in (A.12) and substituting (A.14) into the above second term, we have

$$(P_1 w, P_2 w)_{L^2(Q_1)} = \int_{Q_1} \left[s^3\lambda^4\varphi^3w^2a(x,\nabla d,\nabla d)^2 + \left(\sum_{i,j=1}^{n} (\partial_j a_{ij})\partial_i w \right) P_2 w \right.$$
$$\left. + 2s\lambda^2\varphi a(x,\nabla d,\nabla w)^2 + s\lambda^2\varphi a(x,\nabla d,\nabla d)a(x,\nabla w,\nabla w) \right] dxdt$$
$$-s\lambda \int_{\Sigma_1} \varphi|\partial_\nu w|^2 a(x,\nu,\nu)a(x,\nabla d,\nu)d\Sigma + X_1 + 2X_2.$$

Therefore we see that

$$2(P_1 w, P_2 w)_{L^2(Q_1)} \ge \int_{Q_1} 2[s^3\lambda^4\varphi^3w^2a(x,\nabla d,\nabla d)^2 + s\lambda^2\varphi a(x,\nabla d,\nabla d)a(x,\nabla w,\nabla w)]dxdt$$
$$+ \int_{Q_1} 2\left(\frac{1}{2}P_2 w\right)\left(2\sum_{i,j=1}^{n} (\partial_j a_{ij})\partial_i w\right) dxdt$$
$$-2s\lambda \int_{\Sigma_1} \varphi|\partial_\nu w|^2 a(x,\nu,\nu)a(x,\nabla d,\nu)d\Sigma + 2X_1 + 4X_2.$$

Applying

$$2\left|\frac{1}{2}(P_2 w)\left(2\sum_{i,j=1}^{n} (\partial_j a_{ij})\partial_i w\right)\right| \le \frac{1}{4}|P_2 w|^2 + 4\left|\sum_{i,j=1}^{n} (\partial_j a_{ij})\partial_i w\right|^2,$$

by virtue of $\lambda > 1$, (A.6), (A.13) and (A.15), we obtain

$$\begin{aligned}
&\|f_s\|^2_{L^2(Q_1)} = \|P_1 w\|^2_{L^2(Q_1)} + \|P_2 w\|^2_{L^2(Q_1)} + 2(P_1 w, P_2 w)_{L^2(Q_1)} \\
\geq &\|P_1 w\|^2_{L^2(Q_1)} + \frac{1}{2}\|P_2 w\|^2_{L^2(Q_1)} \\
+ &\int_{Q_1} 2[s^3\lambda^4\varphi^3 w^2 a(x,\nabla d,\nabla d)^2 + s\lambda^2\varphi a(x,\nabla d,\nabla d)a(x,\nabla w,\nabla w)]dxdt \\
- &C_3\int_{Q_1}[(\lambda^2\varphi + s\lambda\varphi + 1)|\nabla w|^2 + (s^3\lambda^3\varphi^3 + s^2\lambda^4\varphi^2 + s^2\lambda^4\varphi)w^2]dxdt \\
- &2\|f_s\|^2_{L^2(Q_1)} - 2s\lambda\int_{\Sigma_1}\varphi|\partial_\nu w|^2 a(x,\nu,\nu)a(x,\nabla d,\nu)d\Sigma.
\end{aligned}$$

Since $d \in C^2(\overline{\Omega})$ satisfies $|\nabla d(x)| > 0$, $x \in \overline{\Omega}$, by (1.4) we can obtain

$$\begin{aligned}
&\|f_s\|^2_{L^2(Q_1)} \geq \frac{1}{3}\|P_1 w\|^2_{L^2(Q_1)} + \frac{1}{6}\|P_2 w\|^2_{L^2(Q_1)} \\
+ &C_4\int_{Q_1}(s^3\lambda^4\varphi^3 w^2 + s\lambda^2\varphi|\nabla w|^2)dxdt \\
- &C_5\int_{Q_1}[(\lambda^2\varphi + s\lambda\varphi + 1)|\nabla w|^2 + (s^3\lambda^3\varphi^3 + s^2\lambda^4\varphi^2 + s^2\lambda^4\varphi)w^2]dxdt \\
- &\frac{2}{3}s\lambda\int_{\Sigma_1}\varphi|\partial_\nu w|^2 a(x,\nu,\nu)a(x,\nabla d,\nu)d\Sigma.
\end{aligned}$$

In terms of the definition of f_s in (A.3), we have

$$\|f_s\|^2_{L^2(Q_1)} \leq C_6\int_{Q_1}(s^2\lambda^4\varphi^2 w^2 + |Pw|^2)dxdt.$$

Therefore, using also (2.2), we obtain

$$\begin{aligned}
&C_7\|Pw\|^2_{L^2(Q_1)} + C_7 s\lambda\int_{\tau_1}^{\tau_2}\int_{\Gamma_0}\varphi|\partial_\nu w|^2 d\Sigma \\
\geq &\frac{1}{3}\|P_1 w\|^2_{L^2(Q_1)} + \frac{1}{6}\|P_2 w\|^2_{L^2(Q_1)} + \int_{Q_1}(C_4 s^3\lambda^4\varphi^3 - C_7 s^3\lambda^3\varphi^3 - C_7 s^2\lambda^4\varphi^2 - C_7 s^2\lambda^4\varphi)w^2 dxdt \\
+ &\int_{Q_1}(C_4 s\lambda^2\varphi - C_7 s\lambda\varphi - C_7\lambda^2\varphi - C_7)|\nabla w|^2 dxdt.
\end{aligned}$$

Noting that $\varphi \geq 1$ on $\overline{Q_1}$, we can find constants $\lambda_0 > 0$ and $s_0 > 0$ which continuously depend on $\sum_{i,j=1}^n \|a_{ij}\|_{C^1(\overline{\Omega})}$ such that

$$\begin{aligned}
&C_8 s\lambda\int_{\tau_1}^{\tau_2}\int_{\Gamma_0}\varphi|\partial_\nu w|^2 d\Sigma + C_8\|Pw\|^2_{L^2(Q_1)} \geq \|P_1 w\|^2_{L^2(Q_1)} + \|P_2 w\|^2_{L^2(Q_1)} \\
+ &\int_{Q_1}(s^3\lambda^4\varphi^3 w^2 + s\lambda^2\varphi|\nabla w|^2)dxdt
\end{aligned}$$

for all $s > s_0$ and $\lambda > \lambda_0$. By (A.4) and (A.5), we have

$$|(P_1 w)(x,t)|^2 \ge C_9 \left| \sum_{i,j=1}^n a_{ij}\partial_i\partial_j w \right|^2 - C_{10} s^4\lambda^4\varphi^4 w^2$$

and

$$|(P_2 w)(x,t)|^2 \ge C_9 |\partial_t w|^2 - C_{10} s^2\lambda^2\varphi^2 |\nabla w|^2,$$

so that

$$\begin{aligned}&\int_{Q_1} \left\{ \frac{1}{s\varphi}\left(|\partial_t w|^2 + \left| \sum_{i,j=1}^n a_{ij}\partial_i\partial_j w \right|^2 \right) + s\lambda^2\varphi|\nabla w|^2 + s^3\lambda^4\varphi^3 w^2 \right\} dxdt \\ \le & C_{11}\int_{Q_1} |Pw|^2 dxdt + C_{11} s\lambda \int_{\tau_1}^{\tau_2}\int_{\Gamma_0} \varphi |\partial_\nu w|^2 d\Sigma \end{aligned} \tag{A.16}$$

for all $s > s_0$ and $\lambda > \lambda_0$.

Moreover we have

$$\begin{aligned}\partial_i\partial_j\left(\frac{w}{\sqrt{\varphi}}\right) =& \frac{\partial_i\partial_j w}{\sqrt{\varphi}} - \frac{\partial_i\partial_j\varphi}{2\varphi^{\frac{3}{2}}} w \\ -&\frac{1}{2\varphi^{\frac{3}{2}}}\{(\partial_j w)(\partial_i\varphi) + (\partial_i w)(\partial_j\varphi)\} + \frac{3}{4\varphi^{\frac{5}{2}}}(\partial_i\varphi)(\partial_j\varphi)w, \quad 1 \le i,j \le n, \end{aligned} \tag{A.17}$$

and

$$\begin{aligned}&\sum_{i,j=1}^n a_{ij}\partial_i\partial_j\left(\frac{w}{\sqrt{\varphi}}\right) \\ =&\frac{g}{\sqrt{\varphi}} - \frac{\sum_{i,j=1}^n a_{ij}\partial_i\partial_j\varphi}{2\varphi^{\frac{3}{2}}} w + \frac{3}{4\varphi^{\frac{5}{2}}} w \sum_{i,j=1}^n a_{ij}(\partial_i\varphi)(\partial_j\varphi) \\ -&\frac{1}{\varphi^{\frac{3}{2}}} \sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j\varphi)\end{aligned}$$

where we set $g = \sum_{i,j=1}^n a_{ij}\partial_i\partial_j w$. Since $w(t,\cdot) \in H_0^1(\Omega)$ for almost all $t \in [\tau_1, \tau_2]$, we apply a usual a priori estimate for the Dirichlet problem for the elliptic equation (e.g., [13]), so that

$$\int_\Omega \sum_{i,j=1}^n \left|\partial_i\partial_j\left(\frac{w}{\sqrt{\varphi}}\right)\right|^2(t,x)dx \le C_{12}\int_\Omega \frac{g(t,x)^2}{\varphi}dx + C_{12}\int_\Omega \frac{\left|\sum_{i,j=1}^n a_{ij}\partial_i\partial_j\varphi\right|^2}{\varphi^3}|w(t,x)|^2dx$$

$$+C_{12}\int_\Omega \frac{w(t,x)^2}{\varphi^5}\left|\sum_{i,j=1}^n a_{ij}(\partial_i\varphi)(\partial_j\varphi)\right|^2 dx$$

$$+C_{12}\int_\Omega \frac{1}{\varphi^3}\left|\sum_{i,j=1}^n a_{ij}(\partial_i w)\partial_j\varphi\right|^2 dx. \tag{A.18}$$

On the other hand, (A.17) yields

$$\begin{aligned}&\int_\Omega \frac{1}{\varphi}|\partial_i\partial_j w(t,x)|^2dx\\ \le& C_{13}\int_\Omega\left\{\left|\partial_i\partial_j\left(\frac{w}{\sqrt{\varphi}}\right)\right|^2 + \frac{|\partial_i\partial_j\varphi|^2}{\varphi^3}w^2 + \frac{1}{\varphi^3}(|\partial_j w|^2|\partial_i\varphi|^2 + |\partial_i w|^2|\partial_j\varphi|^2)\right.\\ &\left.+\frac{1}{\varphi^5}|\partial_i\varphi|^2|\partial_j\varphi|^2w^2\right\}(t,x)dx.\end{aligned} \tag{A.19}$$

Since $\partial_i\varphi = \lambda(\partial_i d)\varphi$ and $\partial_i\partial_j\varphi = \lambda(\partial_i\partial_j d)\varphi + \lambda^2(\partial_i d)(\partial_j d)\varphi$, we see by $\lambda > 1$ that

$$\begin{aligned}&|\partial_i\varphi(t,x)| \le C_{14}\lambda\varphi(t,x),\\ &|\partial_i\partial_j\varphi(t,x)| \le C_{14}\lambda^2\varphi(t,x), \quad 1\le i,j\le n,,\ (t,x)\in\overline{Q_1}.\end{aligned} \tag{A.20}$$

Hence, $\varphi \ge 1$, (A.18) and (A.19) yield

$$\sum_{i,j=1}^n\int_\Omega \frac{1}{\varphi(t,x)}|\partial_i\partial_j w(t,x)|^2dx \le C_{15}\int_\Omega\frac{g^2(t,x)}{\varphi(t,x)} + C_{15}\int_\Omega(\lambda^4w^2+\lambda^2|\nabla w|^2)(t,x)dx.$$

With (A.16), we obtain

$$\begin{aligned}&\int_{Q_1}\left\{\frac{1}{s\varphi}\left(|\partial_t w|^2 + \sum_{i,j=1}^n|\partial_i\partial_j w|^2\right) + s\lambda^2\varphi|\nabla w|^2 + s^3\lambda^4\varphi^3w^2\right\}dxdt\\ \le& C_{16}\int_{Q_1}|Pw|^2dxdt + C_{16}s\lambda\int_{\tau_1}^{\tau_2}\int_{\Gamma_0}\varphi|\partial_\nu w|^2d\Sigma\end{aligned}$$

for all $s > s_0$ and $\lambda > \lambda_0$. Substituting $w = e^{s\varphi}v$ and noting $v|_{\Sigma_1} = 0$ and (A.20), we can complete the proof of (2.6).

In (2.6), fixing $\lambda > \lambda_0$ and replacing $e^{\lambda M_1}s$ by s, we can derive (2.4) from (2.6). Thus the proof of Theorem 2.1 is complete.

Appendix B. Proof of (1.6) and (1.10).

For $\{a_{ij}\}$ satisfying (1.3) and (1.4), we recall that the operator A in $L^2(\Omega)$ is defined by (1.9). By $a_{ij} \in C^5(\overline{\Omega})$, we apply the elliptic regularity (e.g., Theorem 8.13 (p.187) in [13]), and we see that

$$C_1^{-1}\|A^3u\| \le \|u\|_{H^6(\Omega)} \le C_1\|A^3u\|, \quad u \in \mathcal{D}(A^3). \tag{B.1}$$

Here the constant $C_1 > 0$ is independent of $u \in \mathcal{D}(A^3)$, and $\|\cdot\|_{H^6(\Omega)}$, $\|\cdot\|$ denote the norms in $H^6(\Omega)$ and $L^2(\Omega)$ respectively. Moreover the fractional power A^γ, $\gamma \in \mathbb{R}$ is defined (e.g., [32]), and by the interpolation theorem (e.g., [30]) we see that

$$C_1^{-1}\|A^{\frac{5}{2}}u\| \le \|u\|_{H^5(\Omega)} \le C_1\|A^{\frac{5}{2}}u\|, \quad u \in \mathcal{D}(A^{\frac{5}{2}}). \tag{B.2}$$

On the other hand, $-A$ generates an analytic semigroup in $L^2(\Omega)$ (e.g., [32]) and we have

$$y(t) \equiv y(\{a_{ij}\}, h, \mu)(t, \cdot) = e^{-tA}\mu + \int_0^t e^{-sA}h(t-s)ds, \qquad 0 < t < T.$$

Here and henceforth we regard $h(t) = h(t, \cdot)$ as an element in $L^2(0, T; L^2(\Omega))$. Therefore by $h \in C_0^\infty((0, T) \times \omega)$, we have

$$\partial_t^m y(t) = (-A)^m e^{-tA}\mu + \int_0^t e^{-sA}\partial_t^m h(t-s)ds.$$

Furthermore $h \in C_0^\infty((0, T) \times \omega)$ yields $\|A^3\partial_t^m h(t)\| \le C_2\|\partial_t^m h(t)\|_{H^6(\omega)}$ by (B.1). Hence, since $\|A^m e^{-tA}\| \le \frac{C_3}{t^m}$ for $t > 0$ (e.g., §2.6 in [32]), we obtain

$$\begin{aligned} \|A^3\partial_t^m y(t)\| &\le \frac{C_3}{\tau_1^{m+3}}\|\mu\| + C_3\int_0^t \|A^3\partial_t^m h(t-s)\|ds \\ &\le \left(\frac{C_3}{\tau_1^{m+3}} + C_3\right)(\|\mu\| + \|h\|_{W^{m,1}(0,T;H^6(\omega))}), \quad \tau_1 \le t \le \tau_2. \end{aligned} \tag{B.3}$$

Thus, in terms of (B.1), the proof of (1.6) is complete.

Next we prove (1.10). By $n \le 5$ and the Sobolev embedding (e.g., [1], [30]), we see that $H^5(\Omega) \subset C^2(\overline{\Omega})$. Similarly to (B.3), in terms of (B.2) we have

$$\|y(\{a_{ij}\}, h, 0)\|_{C[0,T];C^2(\overline{\Omega}))} \le C_4 \|y(\{a_{ij}\}, h, 0)\|_{C[0,T];H^5(\Omega))}$$
$$\le C_5 \int_0^t \|A^{\frac{5}{2}} e^{-sA} h(t-s)\| ds = C_5 \int_0^t \|e^{-sA} A^{\frac{5}{2}} h(t-s)\| ds \le C_6 \int_0^t \|h(t-s)\|_{H^5(\Omega)} ds.$$

Thus the proof of (1.10) is complete.

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ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012