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NEW REALIZATION OF THE PSEUDOCONVEXITY AND ITS APPLICATION TO AN INVERSE PROBLEM

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Abstract. We consider a hyperbolic differential operator $P = a_0(x)^2\partial_t^2 - \Delta$ with variable principal term. We first give a sufficient condition for the pseudoconvexity which yields a Carleman estimate and a necessary condition. The former condition implies that level sets generated by the weight function in the Carleman estimate, is convex with respect to the set of rays given by $a_0(x)$, and gives a more general explicit condition of a_0 for the pseudoconvexity. Second we apply the Carleman estimate to an inverse problem of determining a_0 by Cauchy data on a lateral boundary with relaxed constraints on a_0.

§1. Introduction.

We consider a hyperbolic differential operator

(1.1)

$$(Pu)(x,t) = (P(x,D)u)(x,t) = a_0(x)^2\partial_t^2 u(x,t) - \Delta u(x,t), \qquad x \in \mathbb{R}^n,\ t \in \mathbb{R},$$

where $a_0 > 0$ is a function of C^2-class, $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $\partial_t = \frac{\partial}{\partial t}$, $\partial_j = \frac{\partial}{\partial x_j}$, $1 \le j \le n$, $\Delta = \sum_{j=1}^n \partial_j^2$.

One of the fundamental problems is the uniqueness in the initial value problem for the equation $Pu = 0$ or the unique continuation. For these purposes, a basic

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tool is a Carleman estimate, and for general theories, we refer to Hörmander [5], Isakov [13] - [16], for example. Especially for hyperbolic operators, see further Imanuvilov [6], Triggiani and Yao [26]. Here, according to Hörmander [5], we will state a necessary and sufficient condition for a relevant Carleman estimate. Let us define the principal symbol $P_m(x,\zeta)$ by

$$P_m(x,\zeta) = -a_0(x)^2\zeta_{n+1}^2 + \sum_{j=1}^{n}\zeta_j^2,$$

$$x \in \mathbb{R}^n,\ t \in \mathbb{R},\ \zeta = (\zeta_1, ..., \zeta_{n+1}) \in \mathbb{C}^{n+1}. \tag{1.2}$$

We set

$$t = x_{n+1}, \quad \partial_{n+1} = \partial_t,\ \nabla' = (\partial_1, ..., \partial_n),\ \nabla = (\partial_1, ..., \partial_n, \partial_t),$$

$$\xi = (\xi_1, ..., \xi_n, \xi_{n+1}) = (\xi', \xi_{n+1}), \quad \xi' = (\xi_1, ..., \xi_n)$$

and

$$\varphi(x,t) = e^{\lambda\psi(x,t)} \tag{1.3}$$

where $\lambda > 0$ is a parameter.

Then we directly see that P is principally normal (see Definition 8.5.1 in [5]), and we notice that the results of Chapter VIII in [5] are applicable to P. Throughout this paper, we assume that $Q \subset \mathbb{R}^n \times \mathbb{R}$ is a bounded domain. We set

$$\Omega \equiv \{x;\ (x,t) \in Q \text{ for some } t \in \mathbb{R}\}.$$

Moreover let $\psi, \varphi \in C^2(\overline{Q})$ satisfy $\nabla\psi \ne 0$ and $\nabla\varphi \ne 0$ on $\overline{Q}$. Then

Theorem A. *(Theorems 8.4.1, 8.5.2 and 8.6.3 in [5]).*

(i) (Sufficiency). *We assume that*

$$\sum_{j,k=1}^{n+1}(\partial_j\partial_k\psi)(x,t)\frac{\partial P_m}{\partial\xi_j}(x,\xi)\frac{\partial P_m}{\partial\xi_k}(x,\xi)$$

$$+\sum_{j,k=1}^{n+1}\left(\frac{\partial^2 P_m}{\partial\xi_j\partial x_k}(x,\xi)\frac{\partial P_m}{\partial\xi_k}(x,\xi) - \frac{\partial P_m}{\partial x_k}(x,\xi)\frac{\partial^2 P_m}{\partial\xi_j\partial\xi_k}(x,\xi)\right)(\partial_j\psi)(x,t) > 0 \tag{1.4}$$

if $(x,t) \in Q$ *and* $\xi \in \mathbb{R}^{n+1} \setminus \{0\}$ *satisfy*

$$P_m(x,\xi) = \sum_{j=1}^{n+1} \frac{\partial P_m}{\partial \xi_j}(x,\xi)(\partial_j \psi)(x,t) = 0 \tag{1.5}$$

and

$$\begin{aligned}&\sum_{j,k=1}^{n+1} (\partial_j \partial_k \psi)(x,t) \frac{\partial P_m}{\partial \zeta_j}(x,\zeta)\overline{\frac{\partial P_m}{\partial \zeta_k}(x,\zeta)}\\ &+s^{-1}\sum_{k=1}^{n+1} Im\left(\partial_k P_m(x,\zeta)\overline{\frac{\partial P_m}{\partial \zeta_k}(x,\zeta)}\right) > 0\end{aligned} \tag{1.6}$$

if $(x,t) \in Q$ *and* $\zeta = \xi + \sqrt{-1}s\nabla\psi$, $\xi \in \mathbb{R}^{n+1}$, $s \neq 0$, *satisfy*

$$P_m(x,\zeta) = \sum_{j=1}^{n+1} \frac{\partial P_m}{\partial \zeta_j}(x,\zeta)(\partial_j \psi)(x,t) = 0. \tag{1.7}$$

For sufficiently large $\lambda > 0$, *we define* $\varphi(x,t)$ *by (1.3). Then*

there exist constants $s_0 > 0$ *and* $C_1 > 0$ *such that*

$$s\int_Q |\nabla u|^2 e^{2s\varphi} dxdt + s^3 \int_Q u^2 e^{2s\varphi} dxdt \le C_1 \int_Q |Pu|^2 e^{2s\varphi} dxdt \tag{1.8}$$

for $s > s_0$ *and* $u \in H_0^2(Q)$. *Here the constant* $C_1 > 0$ *depends on* ψ.

(ii) (Necessity). *We assume that (1.8) holds. Then*

$$\begin{aligned}\frac{1}{2C_1}|\xi|^2 \le &\sum_{j,k=1}^{n+1} (\partial_j \partial_k \varphi)(x,t) \frac{\partial P_m}{\partial \xi_j}(x,\xi)\frac{\partial P_m}{\partial \xi_k}(x,\xi)\\ &+\sum_{j,k=1}^{n+1}\left(\frac{\partial^2 P_m}{\partial \xi_j \partial x_k}(x,\xi)\frac{\partial P_m}{\partial \xi_k}(x,\xi) - \frac{\partial P_m}{\partial x_k}(x,\xi)\frac{\partial^2 P_m}{\partial \xi_j \partial \xi_k}(x,\xi)\right)(\partial_j\varphi)(x,t)\end{aligned} \tag{1.9}$$

if $(x,t) \in Q$ *and* $\xi \in \mathbb{R}^{n+1} \setminus \{0\}$ *satisfy*

$$P_m(x,\xi) = \sum_{j=1}^{n+1} \frac{\partial P_m}{\partial \xi_j}(x,\xi)(\partial_j \varphi)(x,t) = 0. \tag{1.10}$$

Here and henceforth, for $z \in \mathbb{C}$, Im α and Re z denote the imaginary part and the real part respectively, and $\overline{z}$ is the complex conjugate.

An estimate of form (1.8) is called a Carleman estimate with the weight function φ, by which we can establish the unique continuation or stability in the Cauchy problem (e.g., Isakov [13] - [16]), observability inequalities (e.g., Cheng, Isakov, Yamamoto and Zhou [4], Kazemi and Klibanov [18], Klibanov and Timonov [21]) and inverse problems (e.g., Bukhgeim [2], Bukhgeim and Klibanov [3], Imanuvilov, Isakov and Yamamoto [7], Imanuvilov and Yamamoto [8] - [11], Isakov [12], [13], [15], [16], Isakov and Yamamoto [17], Khaĭdarov [19], Klibanov [20], Klibanov and Timonov [21], Yamamoto [27]). By the inverse problem, we mean the determination of $a_0(x)$ by overdetermining data of u on some boundary of Q. Thus it is critically important to find a weight function ψ satisfying (1.4) and (1.6) under conditions (1.5) and (1.7) respectively. However the existing searches for ψ are restricted and one mainly takes $\psi(x,t) = |x-x^0|^2 - \beta t^2$ where $x^0 \in \mathbb{R}^n$ and $\beta > 0$ is a parameter, and after such a fixed choice of ψ, we have to assume conditions on a_0 in order that conditions (1.4) and (1.6) are satisfied. That is, the following is known:

Proposition B. *We assume that there exists $x^0 \in \mathbb{R}^n \setminus \overline{\Omega}$ such that*

$$(\nabla' \log a_0(x) \cdot (x - x^0)) > -1 \tag{1.11}$$

for any $x \in \overline{\Omega}$. If we set

$$\psi(x,t) = |x-x^0|^2 - \beta t^2, \tag{1.12}$$

then with sufficiently small $\beta > 0$, (1.4) and (1.6) hold respectively under (1.5) and (1.7). In particular, Carleman estimate (1.8) holds.

Here and henceforth $(\zeta \cdot \widetilde{\zeta})$ denotes the scalar product in $\mathbb{R}^n$ or $\mathbb{R}^{n+1}$. For the proof, it suffices to verity that (1.12) satisfies the conditions in Theorem A (i), and see Imanuvilov and Yamamoto [10] for example. Condition (1.11) is quite restrictive, and we have to limit unknown coefficients to a class meeting (1.11) when we consider the inverse problem of determining a_0. We note that condition (1.11) is merely one sufficient condition for (1.8). In other words, even though a_0 does not satisfy (1.11), other choice of ψ may be able to satisfy (1.4) and (1.6).

The main purpose of this paper is to establish a sufficient condition of ψ for Carleman estimate (1.8) which is more directly related with a_0 and then to propose more flexible choices of ψ in Theorem A to relax constraint (1.11) for the principal term. Moreover we will also show a necessary condition for (1.8) which is similar but weaker than the sufficient condition. Next we will apply such a Carleman estimate to the inverse problem of determining a principal term within a more general class.

Now we will state our first main result which shows a sufficient condition and a necessary condition for Carleman estimate (1.8).

Theorem 1.

(i) (Sufficiency). *The following statement (a) implies (b):*

(a). *There exists $\beta_0 > 0$, depending on Q and a_0, such that a function $d \in C^2(\overline{\Omega})$ satisfies*

$$|\nabla' d(x)| \neq 0 \qquad on\ \overline{\Omega}. \tag{1.13}$$

and

$$\inf\left\{ \sum_{j,k=1}^{n} (\partial_j\partial_k d)(x)\xi_j\xi_k + (\nabla' d(x) \cdot \nabla' \log a_0(x))\,; \right. \\ (x,t) \in Q,\ \xi' \in \mathbb{R}^n \ \text{with}\ |\xi'| = 1 \ \text{and} \\ \left. (\xi' \cdot \nabla' d(x)) = \pm 2\beta t a_0(x), \quad 0 < \beta < \beta_0 \right\} > 0. \tag{1.14}$$

(b). *There exist $\lambda_0 > 0$ and $\beta_1 > 0$, depending on Q and a_0, such that if $\lambda > \lambda_0$ and $0 < \beta < \beta_1$, then Carleman estimate (1.8) holds with $\varphi(x,t) = e^{\lambda(d(x)-\beta t^2)}$.*

(ii) (Necessity). *The following statement (b) in (i) implies (a'):*

(a'). *For arbitrarily given $\varepsilon > 0$, there exists $\beta_0 > 0$, depending on Q and a_0, such that a function $d \in C^2(\overline{\Omega})$ satisfies (1.13) and*

$$\inf\left\{ \sum_{j,k=1}^{n} (\partial_j\partial_k d)(x)\xi_j\xi_k + (\nabla' d(x) \cdot \nabla' \log a_0(x)) \,; \right.$$
$$(x,t) \in Q,\ \xi' \in \mathbb{R}^n \ \textit{with}\ |\xi'| = 1 \ \textit{and}$$
$$\left. (\xi' \cdot \nabla' d(x)) = \pm 2\beta t a_0(x), \quad 0 < \beta < \beta_0 \right\} \ge -\varepsilon. \tag{1.14'}$$

In our main theorem, we establish a sufficient condition and a necessary condition in the statements where x, t, ξ' are more decoupled than in Theorem A. Thus especially (i) gives more flexible choices of ψ. However, at the cost of such statements, we do not have a sufficient and necessary condition. In particular, we do not know whether $\varepsilon = 0$ can be taken in (a').

Remark. We set

$$T = \sup\{|t|;\ (x,t) \in Q \quad \text{for some } x \in \Omega\}.$$

Sine Q is bounded, we see that $T < \infty$. In Theorem 1, more precisely, $\beta > 0$ should be sufficiently small such that βT is sufficienly small, but $T\sqrt{\beta}$ is not necessarily small. When we choose $d(x) = |x - x^0|^2$ with some $x^0 \in \mathbb{R}^n \setminus \overline{\Omega}$ under condition (1.11) and consider an inverse problem of determining a_0 over Ω, it is necessary that $T\sqrt{\beta}$ should be proportional to

$$\left(\sup_{x\in\Omega} |x - x^0|^2 - \inf_{x\in\Omega} |x - x^0|^2 \right)^{\frac{1}{2}}$$

(e.g., [7], [10]) and $T\sqrt{\beta}$ cannot be small. The same remark is valid for the following Theorem 2. Also see condition (3.3) for our inverse problem.

Our main result is involved with t-variable (see (1.14)). On the other hand, since our equation (1.1) is autonomous, it is reasonable for us to expect characterization of d for Carleman estimate (1.8) which is described only by x-variable and should relax (1.11). As such a sufficient condition, we show

Theorem 2. *For some costant $\varepsilon_0 > 0$, we suppose that $d \in C^2(\overline{\Omega})$ satisfies (1.13) and*

$$\sum_{j,k=1}^{n} (\partial_j \partial_k d(x))\xi_j \xi_k + (\nabla' d(x) \cdot \nabla' \log a_0(x)) \ge \varepsilon_0$$

$$\text{for } x \in \overline{\Omega},\ \xi' \in \mathbb{R}^n \text{ with } |\xi'| = 1. \tag{1.15}$$

We define φ by (1.3) and $\psi(x,t) = d(x) - \beta t^2$. Then there exist constants $\beta_0 > 0$ and $\lambda_0 > 0$ such that if $0 < \beta < \beta_0$ and $\lambda > \lambda_0$, then Carleman estimate (1.8) holds with φ.

By Theorem 2, we directly derive

Corollary. *Let an $n \times n$ matrix $(\partial_j \partial_k a_0(x))_{1 \le j,k \le n}$ be non-negative definite for $x \in \overline{\Omega}$ and let $|\nabla' a_0(x')| \ne 0$ on $\overline{\Omega}$ be true. Then Carleman estimate (1.8) holds true with the weight function $\varphi = e^{\lambda(a_0(x) - \beta t^2)}$, where $\lambda > 0$ and $\beta > 0$ are sufficiently large and small respectively.*

In fact, we can choose $a_0(x)$ as $d(x)$ in Theorem 2. Theorem 2 follows immediately from Theorem 1 and we will prove Theorem 1 in Section 2, whose proof is based on Theorem A.

We note that if in (1.15), we set $d(x) = |x - x^0|^2$, then (1.15) is rewritten as

$$(\nabla' \log a_0(x) \cdot (x - x^0)) > -1 + \frac{\varepsilon_0}{2} > -1$$

which implies (1.11). Therefore condition (1.11) is a special case of (1.15) with a fixed choice $d(x) = |x - x^0|^2$.

Remark. Theorem 2 really generalizes condition (1.11) in Proposition B. Let us set

$$\Omega = \left\{ x \in \mathbb{R}^n; \sqrt{\frac{9}{10}} < |x| < 1 \right\}, \qquad a_0(x) = 1 - \frac{2}{3}|x|^2, \quad x \in \overline{\Omega}.$$

Then (1.11) can not be satisfied for any $x^0 \in \mathbb{R}^n$. In fact, (1.11) is equivalent to

$$\frac{4|x|^2 - 4(x \cdot x^0)}{3 - 2|x|^2} < 1 \quad \text{if} \ \sqrt{\frac{9}{10}} \le |x| \le 1. \tag{1.16}$$

For any $x^0 \in \mathbb{R}^n$, we can choose $x^1 \in \Omega$ such that $(x^1 \cdot x^0) = 0$ and $|x^1| = \sqrt{\frac{10}{11}}$ for example, which breaks condition (1.16). However if we take $d(x) = -|x|^2$ for $x \in \overline{\Omega}$, then (1.15) holds trues: [the left hand side of (1.15)] $= -2 + \frac{8|x|^2}{3-2|x|^2} \ge 4$ if $\sqrt{\frac{9}{10}} \le |x| \le 1$.

Our condition (1.15) for the Carleman estimate can be interpreted in terms of the ray. For the interpretation of (1.15), we define the ray (e.g., Chapter 3 in Romanov [24]). Let us consider the three dimensional case and let $L(x, x^0)$ denote an arbitrary smooth curve connecting x, $x^0 \in \mathbb{R}^3$ and ds be an element of the arc length of $L(x, x^0)$. Then a ray $\Gamma(x, x^0)$ is defined as L attaining an extremal of the functional of L:

$$\int_{L(x,x^0)} a_0 dx.$$

Note that a_0^{-1} corresponds to the wave speed and that the ray is not necessarily determined uniquely for given x and x^0. Then (1.15) is interpreted that each surface $d(x) = C$ for any constant C is convex with respect to the set of rays, and, under some conditions on a smooth real-valued function d, we know the following fact:

Let us assume that (1.15) holds for any $\xi' = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ satisfying $|\xi'| = 1$ and $\xi' \cdot \nabla' d = 0$. Then any ray touching the surface $\{x; d(x) = C\}$, belongs to the domain $\{x; d(x) > C\}$ at any other point.

As for the details, see Chapter 3 in Romanov [24]. Intuitively we can understand that rays remaining on a surface prevent us from detecting interior information of solutions inside the domain, so that if such remaining rays exist, then the property of unique continuation may be very complicated. Since the Carleman estimate implies the unique continuation (e.g., Hörmander [5], Isakov [15]), the above fact suggests that our condition (1.15) is reasonable for proving the Carleman estimate. However we do not know whether (1.15) is a necessary condition for Carleman estimates.

As related papers, for a more general hyperbolic operator $\partial_t^2 - \sum_{j,k=1}^n \partial_j(a_{jk}(x)\partial_k)$, Lasiecka, Triggiani and Yao [22] and Yao [28] introduce the weight function of the form $\varphi(x,t) = d(x) - \beta t^2$, where d is strictly convex with respect to the Riemann metric derived by the elliptic part, and establish an inequality of Carleman's type. In our case of $a_{jk}(x) = \delta_{jk} a_0(x)^{-2}$ where $\delta_{jk} = 1$ if $j = k$ and $= 0$ if $j \ne k$, we can verify that d is strictly convex with respect to the Riemann metric if and only if the following $n \times n$ matrix $(m_{jk})_{1 \le j,k \le n}$ is positive definite in the domain under consideration:

$$m_{jk}(x) = \begin{cases} \partial_j^2 d - 2a_0^{-1}(\partial_j a_0)(\partial_j d) + a_0^{-1}(\nabla' a_0 \cdot \nabla' d), & \text{if } j = k, \\ \partial_j \partial_k d - a_0^{-1}(\partial_j a_0)(\partial_k d) - a_0^{-1}(\partial_k a_0)(\partial_j d), & \text{if } j \ne k. \end{cases}$$

In [22], the second large parameter $\lambda > 0$ is not considered unlike in our paper and such a parameter is generally useful for guaranteeing the relevant convexity (e.g., [5], [15]). In [22] and [28], the inequality of Carleman's type yields observability inequalities with a generous condition on the principal term, but their inequality includes some extra lower order terms, so that it is not directly applicable to our inverse problem and in [26] the authors proved a Carleman estimate without lower order terms. As for weight functions with factor $d(x) - \beta t^2$, see further Isakov and Yamamoto [17], Lasiecka, Triggiani and Zhang [23]. Bellassoued [1] proved a sufficient condition of the principal part for Carleman estimate for an anisotropic hyperbolic operator $a_0(x)^2\partial_t^2 - \sum_{j,k=1}^n \partial_j(a_{jk}(x)\partial_k)$ and discussed an inverse problem of determining one coefficient in the principal term. His method is based on the Riemannian geometry like in [22] and [28]. As a recent paper, see Romanov [25] where $d(x)$ is chosen by means of the Riemannian distance.

§2. Proof of Theorem 1.

The proof will be done by Theorem A.

Proof of (a) $\Longrightarrow$ (b). We assume (1.4) and (1.6) under conditions (1.5) and (1.7) respectively. First, for sufficiently small $\beta > 0$, we prove that any $\zeta = \xi + \sqrt{-1}s\nabla\psi$, $\xi \in \mathbb{R}^{n+1}$, $s \ne 0$, cannot satisfy (1.7). In fact, (1.7) is equivalent to

$$(\xi' \cdot \nabla'\psi(x,t)) = -2\beta t a_0^2(x)\xi_{n+1} \tag{2.1}$$

$$|\xi'|^2 - s^2|\nabla'\psi(x,t)|^2 = a_0(x)^2\xi_{n+1}^2 - 4s^2a_0(x)^2\beta^2t^2 \tag{2.2}$$

and

$$|\nabla'\psi(x,t)|^2 = 4\beta^2t^2a_0(x)^2. \tag{2.3}$$

Since Q is bounded, we recall that

$$T \equiv \sup\{|t|;\ (x,t) \in Q \quad \text{for some } x \in \Omega\} < \infty. \tag{2.4}$$

In view of (1.13), we set

$$c_1 = \inf_{x\in\Omega} |\nabla'\psi(x,t)| = \inf_{x\in\Omega} |\nabla' d(x)| > 0$$

and $M = \|a_0\|_{C(\overline{\Omega})}$. Therefore (2.3) implies $\beta T \geq \frac{c_1}{2M}$. Hence for sufficiently small $\beta > 0$ there are no solutions (x, t) to equation (2.3). More precisely, it suffices to choose $\beta_0 > 0$ such that

$$\beta_0 < 1 \text{ and } \beta_0 T \text{ are sufficienly small} \tag{2.5}$$

and we assume that $0 < \beta < \beta_0$.

Thus we need not consider (1.6) for proving that (a) $\Longrightarrow$ (b), and we have only to verify (1.4) under condition (1.5). We directly see that condition (1.5) is equivalent to (2.1) and

$$|\xi'|^2 = a_0(x)^2 \xi_{n+1}^2. \tag{2.6}$$

We denote the left hand side of (1.4) by $H(x, t, \xi)$. Henceforth we set $\xi' = (\xi_1, ..., \xi_n)$. Since $\frac{\partial P_m}{\partial \xi_j} = 2\xi_j$ for $1 \leq j \leq n$, $\frac{\partial P_m}{\partial \xi_{n+1}} = -2a_0^2 \xi_{n+1}$, $\partial_k P_m = -2a_0(\partial_k a_0)\xi_{n+1}^2$, $1 \leq k \leq n$ and $\partial_{n+1} P_m = 0$, and noting that $\partial_{n+1}\partial_j \psi = 0$ for $1 \leq j \leq n$, we can directly calculate to obtain

$$\begin{aligned} H(x,t,\xi) = &\sum_{j,k=1}^{n} 4\xi_j \xi_k (\partial_j \partial_k \psi) + 4a_0^4 (\partial_{n+1}^2 \psi)\xi_{n+1}^2 \\ &+ 4a_0 \xi_{n+1}^2 (\nabla' a_0 \cdot \nabla' \psi) + 16\beta t \xi_{n+1} a_0 (\nabla' a_0 \cdot \xi'). \end{aligned}$$

Using (2.6), we have

$$\begin{aligned} H(x,t,\xi) =& 4|\xi'|^2 \left(\sum_{j,k=1}^{n} (\partial_k \partial_k d) \frac{\xi_j}{|\xi'|} \frac{\xi_k}{|\xi'|} + (\nabla' \log a_0 \cdot \nabla' \psi) \right) \\ &-8\{\beta a_0^2 |\xi'|^2 - 2\beta t \xi_{n+1} a_0 (\nabla' a_0 \cdot \xi')\}. \end{aligned} \tag{2.7}$$

Here by (2.6) and $\xi \neq 0$, we see that $|\xi'| \neq 0$. Henceforth by the homogeneity of H in ξ, we may assume that $|\xi'| = 1$. Then, in terms of (2.1) and (2.7), the variable (x, t, ξ) has to satisfy

$$(\xi' \cdot \nabla' \psi(x)) = \pm 2\beta t a_0(x). \tag{2.8}$$

By ε_1, we denote the infimum in (1.14), and we see that $\varepsilon_1 > 0$. Therefore by (2.7) and (2.8), we have

$$H(x,t,\xi) \ge 4\varepsilon_1 - C(\beta + \beta T).$$

Here and henceforth, $C > 0$ denotes generic constants depending only on a_0 and Q. Hence, if $\beta \in (0, \beta_0)$ is sufficiently small such that (2.5) and $4\varepsilon_1 - C(\beta + \beta T) > 0$ are true, then $H(x,t,\xi) > 0$ for this β. Thus, in terms of Theorem A (i), the proof of (a) $\Longrightarrow$ (b) is complete.

Remark. In proving Carleman estimate, we need not verify (1.6) if we can take small $\beta > 0$. This fact is stated also in Isakov [16]. According to the terminology in [5], (1.4) corresponds to the pseudoconvexity, while (1.4) with (1.6) correspond to the strong pseudoconvexity.

Proof of (b) $\Longrightarrow$ (a'). By From (ii) of Theorem A, it follows that for $\varphi = e^{\lambda(d(x)-\beta t^2)}$ with $\lambda > \lambda_0$ and $0 < \beta < \beta_1$, we have (1.9) under (1.10). We directly verify that (1.10) is equivalent to (2.1) and (2.6). Let us denote the right hand side of (1.9) by $H_1(x,t,\xi)$. Similarly to (2.7), in terms of (2.1), we can calculate H_1:

$$\begin{aligned}
H_1(x,t,\xi) =&4\sum_{j,k=1}^{n} \xi_j\xi_k(\lambda^2(\partial_j\psi)(\partial_k\psi)\varphi + \lambda(\partial_j\partial_k\psi)\varphi) \\
&-8a_0^2\xi_{n+1}\sum_{j=1}^{n}\xi_j\lambda^2(\partial_j\psi)(\partial_{n+1}\psi)\varphi + 4a_0^4\xi_{n+1}^2\{\lambda^2(\partial_{n+1}\psi)^2\varphi + \lambda(\partial_{n+1}^2\psi)\varphi\} \\
&+4a_0\lambda\varphi(\nabla' a_0\cdot\nabla'\psi)\xi_{n+1}^2 - 8a_0\lambda\varphi(\nabla' a_0\cdot\xi')(\partial_{n+1}\psi)\xi_{n+1} \\
=&\left(4\lambda\varphi\sum_{j,k=1}^{n}(\partial_j\partial_k\psi)\xi_j\xi_k + 4a_0\lambda\varphi(\nabla' a_0\cdot\nabla'\psi)\xi_{n+1}^2\right) \\
&-8a_0^4\beta\lambda\varphi\xi_{n+1}^2 + 16a_0\lambda\varphi\beta t\xi_{n+1}(\nabla' a_0\cdot\xi').
\end{aligned}$$

By $\xi \ne 0$, (2.6) and the homogeneity of H_1 in ξ, we can set $|\xi'| = 1$. Therefore

$$H_1(x,t,\xi) \le 4\lambda\varphi\left\{\sum_{j,k=1}^{n}(\partial_j\partial_k d)\xi_j\xi_k + (\nabla'\log a_0\cdot\nabla' d)\right\} + C\lambda\varphi\beta T. \tag{2.9}$$

By (1.9) and (2.9), as long as $0 < \beta < \beta_1$, we see: if $|\xi'| = 1$ and (2.8) hold, then

$$\sum_{j,k=1}^{n} (\partial_j \partial_k d)\xi_j \xi_k + (\nabla' \log a_0 \cdot \nabla' d) + C\beta T > 0.$$

That is, for any $\beta \in (0, \beta_1)$, we have

$$\inf \left\{ \begin{array}{l} \displaystyle\sum_{j,k=1}^{n} (\partial_j \partial_k d)(x)\xi_j \xi_k + (\nabla' d(x) \cdot \nabla' \log a_0(x))\,; \\ (x,t) \in Q,\ \xi' \in \mathbb{R}^n \text{ with } |\xi'| = 1 \text{ and} \\ (\xi' \cdot \nabla' d(x)) = \pm 2\beta t a_0(x) \end{array} \right\} \ge -C\beta T. \tag{2.10}$$

Let $\varepsilon > 0$ be arbitrarily given. We set $\beta_0 = \min\left\{\frac{\varepsilon}{CT}, \beta_1\right\}$. Then

$$\begin{aligned} &\inf_{0<\beta<\beta_0} \left\{ \begin{array}{l} \displaystyle\sum_{j,k=1}^{n} (\partial_j \partial_k d)(x)\xi_j \xi_k + (\nabla' d(x) \cdot \nabla' \log a_0(x))\,; \\ (x,t) \in Q,\ \xi' \in \mathbb{R}^n \text{ with } |\xi'| = 1 \text{ and} \\ (\xi' \cdot \nabla' d(x)) = \pm 2\beta t a_0(x) \end{array} \right\} \\ \ge &\inf_{0<\beta<\beta_0} -C\beta T \ge -\varepsilon. \end{aligned}$$

Thus the proof of (b) $\Longrightarrow$ (a') is complete. ■

§3. Application to an inverse problem of determining principal terms.

Let $\Omega \in \mathbb{R}^n$ be a bounded domain with C^2- boundary $\partial\Omega$ and let us consider

$$(P_k u)(x,t) = (a_k(x)^2 \partial_t^2 - \Delta)u(x,t), \quad k = 0, 1,\ x \in \Omega,\ t \in \mathbb{R}, \tag{3.1}$$

with given initial values $u(\cdot, 0)$ and $\partial_t u(\cdot, 0)$. Here $a_k > 0$ on $\overline{\Omega}$ and $a_k \in C^2(\overline{\Omega})$. We discuss

Uniqueness in Inverse Problem. Let u_k satisfy $P_k u_k = 0$ in $\Omega \times (-T, T)$, $k = 0, 1$. Then, with some positivity condition on $u(\cdot, 0)$, can we conclude that $a_0 = a_1$ in Ω by

$$\begin{cases} u_0(x,0) = u_1(x,0), \quad \partial_t u_0(x,0) = \partial_t u_1(x,0), \qquad x \in \Omega, \\ u_0 = u_1, \quad \dfrac{\partial u_0}{\partial \nu} = \dfrac{\partial u_1}{\partial \nu} \quad \text{on } \partial\Omega \times (-T, T)? \end{cases} \tag{3.2}$$

Here and henceforth, $\nu = \nu(x)$ is the unit outward normal vector to $\partial\Omega$ at x and $\frac{\partial}{\partial\nu}$ denotes the normal derivative: $\frac{\partial u}{\partial \nu} = \nabla' u \cdot \nu$.

In this kind of inverse problems, unknown coefficients appear in principal terms and for the Carleman estimate which is the key, we have to assume conditions of type (1.11) in Imanuvilov, Isakov and Yamamoto [7], Imanuvilov and Yamamoto [10], [11], Isakov [12], [13]. Condition (1.11) definitely restricts an admissible set of unknown coefficients and the relaxation of the condition for the Carleman estimate is very desirable.

In this section, for simplicity, we mainly discuss the uniqueness in determining $a_1(x)$ around a given $a_0(x)$. For known a_0, we assume that there exists $d \in C^2(\overline{\Omega})$ satisfying (1.13) and (1.14) (or (1.13) and (1.15)). We set $\psi(x,t) = d(x) - \beta t^2$ and $\varphi(x,t) = e^{\lambda\psi(x,t)}$, where $\beta > 0$ and $\lambda > 0$ are defined in Theorem 1.

Now we are ready to state the main result on the uniqueness.

Theorem 3. *We assume that for a_0, there exists $d \in C^2(\mathbb{R}^n)$ satisfying (1.13) and (1.14). Let*

$$T > \frac{\sqrt{\sup_{x\in\Omega} d(x) - \inf_{x\in\Omega} d(x)}}{\sqrt{\beta}}, \tag{3.3}$$

and let $u_k \in C^2(\overline{\Omega} \times [-T,T])$, $k = 0,1$, satisfy $\partial_t u_k \in C^2(\overline{\Omega} \times [-T,T])$,

$$P_k u_k = 0 \qquad in\ \Omega \times (-T,T), \tag{3.4}$$

$$u_0(x,0) = u_1(x,0), \quad \partial_t u_0(x,0) = \partial_t u_1(x,0), \quad x \in \Omega, \tag{3.5}$$

and

$$u_0 = u_1, \quad \frac{\partial u_0}{\partial \nu} = \frac{\partial u_1}{\partial \nu} \quad on\ \partial\Omega \times (-T,T). \tag{3.6}$$

Moreover let

$$\Delta u_0(x,0) > 0, \qquad x \in \overline{\Omega}. \tag{3.7}$$

Then

$$a_0(x) = a_1(x), \qquad x \in \overline{\Omega}. \tag{3.8}$$

This theorem asserts the uniqueness, provided that strict positivity (3.7) of an initial value is satisfied. Such positivity is not very practical but for applications of Carleman estimates, we have to assume like in [1] - [3], [7] - [13], [17], [19], [20], [27]. Moreover within a suitable admissible set of a_k's, we can prove the conditional stability which estimates $a_0 - a_1$ by means of $u_0 - u_1$ and $\frac{\partial u_0}{\partial \nu} - \frac{\partial u_1}{\partial \nu}$ on $\partial\Omega \times (-T,T)$, but we will not discuss here, and for simplicity, we will consider the case where the boundary observation is taken over the whole boundary $\partial\Omega$ (see (3.6)).

Proof. Now that we have established a Carleman estimate in Theorem 2, the proof is done along the line of Imanuvilov and Yamamoto [8]. The difference $y = u_1 - u_0$ satisfies

$$P_0 y = R(x,t) f(x) \qquad \text{in } \Omega \times (-T,T), \tag{3.9}$$

$$y(x,0) = \partial_t y(x,0) = 0, \qquad x \in \Omega, \tag{3.10}$$

and

$$y = \frac{\partial y}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (-T,T). \tag{3.11}$$

Here we set

$$f(x) = a_0^2(x) - a_1^2(x), \quad R(x,t) = \partial_t^2 u_1(x,t) = \frac{1}{a_1^2(x)} \Delta u_1(x,t), \tag{3.12}$$

for $x \in \Omega$ and $t \in (-T, T)$.

For application of the Carleman estimate, we have to introduce a suitable cutoff function. For this, we will define relevant level sets. We set

$$Q(\delta) = \{(x,t) \in \mathbb{R}^n \times \mathbb{R};\ \varphi(x,t) > \delta\}$$

for $\delta > 0$ and

$$\rho_0 = \inf_{x \in \mathbb{R}^n} d(x). \tag{3.13}$$

By (3.3), we can choose $\varepsilon_2 > 0$ such that

$$T = \frac{\sqrt{\sup_{x\in\Omega} d(x) - (\inf_{x\in\Omega} d(x) - \varepsilon_2)}}{\sqrt{\beta}}. \tag{3.14}$$

Without loss of generality, we may assume that $\varepsilon_2 > 0$ is sufficiently small.

Moreover, by (1.13), the function $d(x)$ cannot attain the minimum on $\overline{\Omega}$. Hence $\inf_{x\in\Omega} d(x) > \rho_0$. Setting $\rho_1 = \inf_{x\in\Omega} d(x) - \varepsilon_2$ and noting that $\varepsilon_2 > 0$ is assumed to be sufficiently small, we see that

$$\{x \in \mathbb{R}^n; d(x) > \rho_1\} \supset \overline{\Omega}, \quad \rho_1 > \rho_0,$$

$$\{x \in \mathbb{R}^n; d(x) > \rho_1\} \supsetneqq \overline{\{x \in \mathbb{R}^n; d(x) > \rho\}}, \quad \text{if } \rho > \rho_1. \tag{3.15}$$

Now we set

$$\delta_1 = e^{\lambda \rho_1}. \tag{3.16}$$

Therefore, by (3.14), we can easily verify that

$$Q(\delta_1) \cap (\Omega \times \mathbb{R}) \subset \Omega \times (-T, T). \tag{3.17}$$

In fact, let $(x,t) \in Q(\delta_1) \cap (\Omega \times \mathbb{R})$, that is, $d(x) - \beta t^2 > \rho_1$ and $x \in \Omega$. Then

$$|t| \le \frac{\sqrt{d(x) - \rho_1}}{\sqrt{\beta}} \le \frac{\sqrt{\sup_{x\in\Omega} d(x) - (\inf_{x\in\Omega} d(x) - \varepsilon_2)}}{\sqrt{\beta}} = T$$

by (3.14). Thus (3.17) follows.

Taking $\rho_2, \rho_3 > 0$ such that $0 < \rho_1 < \rho_2 < \rho_3$ and $|\rho_1 - \rho_2| + |\rho_1 - \rho_3|$ is sufficiently small, so that

$$\{x \in \mathbb{R}^n; d(x) > \rho_3\} \supset \overline{\Omega},$$
$$\{x \in \mathbb{R}^n; d(x) > \rho_j\} \supsetneqq \overline{\{x \in \mathbb{R}^n; d(x) > \rho_{j+1}\}} \quad j = 1, 2, \tag{3.18}$$

in terms of (3.15). We set

$$\delta_2 = e^{\lambda\rho_2}, \qquad \delta_3 = e^{\lambda\rho_3}. \tag{3.19}$$

For $j = 1, 2, 3$, we note that

$$(x, t) \in \partial Q(\delta_j) \text{ if and only if } t = \pm\frac{\sqrt{d(x) - \rho_j}}{\sqrt{\beta}}. \tag{3.20}$$

By (3.18) and (3.20), we see that $\overline{Q(\delta_{j+1})} \subsetneqq Q(\delta_j)$ for $j = 1, 2$. Therefore we can define $\chi \in C_0^\infty(Q(\delta_1))$ such that $0 \le \chi \le 1$ and

$$\chi(x, t) = \begin{cases} 1, & (x, t) \in Q(\delta_3), \\ 0, & (x, t) \in Q(\delta_1) \setminus \overline{Q(\delta_2)}. \end{cases} \tag{3.21}$$

We set

$$z = (\partial_t y)e^{s\varphi}\chi \in C^2(\overline{\Omega} \times [-T, T]). \tag{3.22}$$

Then, by (3.9), we have

$$\begin{aligned} P_0 z = {} & f(\partial_t R)e^{s\varphi}\chi + s\{-2(\nabla'\varphi \cdot \nabla' z) + 2a_0^2(\partial_t\varphi)\partial_t z + (P_0\varphi)z\} \\ & -s^2(a_0^2|\partial_t\varphi|^2 - |\nabla'\varphi|^2)z \\ & +2e^{s\varphi}\{a_0^2(\partial_t^2 y)\partial_t\chi - (\nabla'(\partial_t y) \cdot \nabla'\chi)\} + (\partial_t y)e^{s\varphi}P_0\chi \quad \text{in } Q(\delta_1) \cap (\Omega \times \mathbb{R}). \end{aligned} \tag{3.23}$$

In fact,

$$\partial_j z = (\partial_j\partial_t y)e^{s\varphi}\chi + s(\partial_j\varphi)z + (\partial_t y)e^{s\varphi}\partial_j\chi,$$

and

$$(\partial_j\partial_t y)e^{s\varphi}\chi = \partial_j z - s(\partial_j\varphi)z - (\partial_t y)e^{s\varphi}\partial_j\chi. \tag{3.24}$$

Hence, by (3.24), we see

$$\partial_j^2 z = (\partial_j^2\partial_t y)e^{s\varphi}\chi + (\partial_j\partial_t y)s(\partial_j\varphi)e^{s\varphi}\chi + 2(\partial_j\partial_t y)e^{s\varphi}(\partial_j\chi) + s(\partial_j^2\varphi)z + s(\partial_j\varphi)\partial_j z$$

$$+(\partial_t y)s(\partial_j\varphi)e^{s\varphi}\partial_j\chi + (\partial_t y)e^{s\varphi}\partial_j^2\chi$$

$$=(\partial_j^2\partial_t y)e^{s\varphi}\chi + s(\partial_j\varphi)\{\partial_j z - s(\partial_j\varphi)z - (\partial_t y)e^{s\varphi}\partial_j\chi\}$$

$$+s(\partial_j^2\varphi)z + s(\partial_j\varphi)\partial_j z + 2(\partial_j\partial_t y)e^{s\varphi}(\partial_j\chi) + (\partial_t y)s(\partial_j\varphi)e^{s\varphi}\partial_j\chi + (\partial_t y)e^{s\varphi}\partial_j^2\chi$$

$$=(\partial_j^2\partial_t y)e^{s\varphi}\chi + 2s(\partial_j\varphi)\partial_j z + s(\partial_j^2\varphi)z - s^2(\partial_j\varphi)^2 z$$

$$+2(\partial_j\partial_t y)e^{s\varphi}\partial_j\chi + (\partial_t y)e^{s\varphi}\partial_j^2\chi.$$

Substitution into $(a_0^2\partial_{n+1}^2 - \Delta)z$ yields (3.23).

Moreover, setting $w = (\partial_t y)\chi$, we obtain

$$P_0 w = f(\partial_t R)\chi + 2a_0^2(\partial_t^2 y)\partial_t\chi - 2(\nabla'(\partial_t y)\cdot\nabla'\chi)$$

$$+(\partial_t y)P_0\chi \quad \text{in } Q(\delta_1)\cap(\Omega\times\mathbb{R}). \tag{3.25}$$

By (3.11), (3.17) and (3.21), it follows that $w \in H_0^2(Q(\delta_1)\cap(\Omega\times\mathbb{R}))$, so that we can apply Theorem 1 to w in $Q \equiv Q(\delta_1)\cap(\Omega\times\mathbb{R})$:

$$\begin{aligned}\int_Q (s^3w^2 + s|\nabla w|^2)e^{2s\varphi}dxdt \le C\int_Q f^2|\partial_t R|^2\chi^2 e^{2s\varphi}dxdt\\ +C\int_Q |2a_0^2(\partial_t^2 y)\partial_t\chi - 2(\nabla'(\partial_t y)\cdot\nabla'\chi) + (\partial_t y)P_0\chi|^2 e^{2s\varphi}dxdt\\ \le C\int_Q f^2\chi^2 e^{2s\varphi}dxdt + Ce^{2s\delta_3}\end{aligned} \tag{3.26}$$

for all sufficiently large $s > 0$. At the last inequality, we have used

$$\partial_t\chi = |\nabla'\chi| = P_0\chi = 0 \quad \text{in } \overline{Q(\delta_3)}\cup\overline{(Q(\delta_1)\setminus Q(\delta_2))}, \tag{3.27}$$

which follows from (3.21), and $e^{2s\varphi} \le e^{2s\delta_3}$ in $Q(\delta_2)\setminus Q(\delta_3)$. Noting that $z = we^{s\varphi}$, we can rewrite (3.26) in terms of z:

$$\int_Q (s^3|z|^2 + s|\nabla z|^2)dxdt \le C\int_Q f^2\chi^2 e^{2s\varphi}dxdt + Ce^{2s\delta_3} \tag{3.28}$$

for sufficiently large $s > 0$.

We set $Q_- = \{(x,t) \in Q; t < 0\}$. We multiply (3.23) by $\partial_t z$ and integrate over Q_-:

$$\begin{aligned} I_1 \equiv \int_{Q_-} (P_0 z)\partial_t z dxdt = \int_{Q_-} f(\partial_t R)e^{s\varphi}\chi\partial_t z dxdt \\ + \int_{Q_-} s\{-2(\nabla'\varphi\cdot\nabla' z) + 2a_0^2(\partial_t\varphi)\partial_t z + (P_0\varphi)z\}\partial_t z dxdt \\ -s^2\int_{Q_-} (a_0^2|\partial_t\varphi|^2 - |\nabla'\varphi|^2)z\partial_t z dxdt \\ + \int_{Q_-} \{2a_0^2(\partial_t^2 y)\partial_t\chi - 2(\nabla'(\partial_t y)\cdot\nabla'\chi) + (\partial_t y)P_0\chi\}e^{s\varphi}\partial_t z dxdt \equiv I_2. \end{aligned} \tag{3.29}$$

By (3.11) and (3.21), we integrate I_1 by parts:

$$\begin{aligned} I_1 = \int_{Q_-} (a_0^2(\partial_t^2 z)\partial_t z - (\Delta z)\partial_t z)dxdt = \int_{Q_-} \left\{\frac{1}{2}\partial_t(|\partial_t z|^2 a_0^2) + \frac{1}{2}\partial_t(|\nabla' z|^2)\right\}dxdt \\ = \int_\Omega \frac{1}{2}(|\partial_t z|^2 a_0^2 + |\nabla' z|^2)\nu_{n+1}dx \end{aligned}$$

where ν_{n+1} is the $(n+1)$-component of the unit outward normal vector to ∂Q_-. Hence (3.7), (3.9), (3.10) and (3.22) imply

$$\begin{aligned} I_1 = \frac{1}{2}\int_\Omega |(\partial_t z)(x,0)|^2 a_0^2(x)dx \\ = \frac{1}{2}\int_\Omega a_0^2(x)|(\partial_t^2 y)(x,0)|^2\chi^2(x,0)e^{2s\varphi(x,0)}dx \\ = \frac{1}{2}\int_\Omega f^2(x)\frac{|\Delta u_1(x,0)|^2}{a_0^2(x)a_1^4(x)}\chi^2(x,0)e^{2s\varphi(x,0)}dx \\ \ge C_1\int_\Omega f^2(x)\chi^2(x,0)e^{2s\varphi(x,0)}dx. \end{aligned} \tag{3.30}$$

For I_2, we use Schwarz's inequality and (3.27), (3.28) to obtain

$$I_2 \le C\int_Q f^2\chi^2 e^{2s\varphi}dxdt + Ce^{2s\delta_3} \tag{3.31}$$

for all large $s > 0$. Consequently (3.30) and (3.31) yield

$$\int_\Omega f^2(x)\chi^2(x,0)e^{2s\varphi(x,0)}dx \le C\int_Q f^2\chi^2 e^{2s\varphi}dxdt + Ce^{2s\delta_3}$$

for all large $s > 0$. By (3.18), we see that $x \in \Omega$ implies $\varphi(x,0) > e^{\lambda\rho_3} = \delta_3$, which yields $\chi(x,0) = 1$ by (3.21). Consequently

$$\int_\Omega f^2(x)e^{2s\varphi(x,0)}dx \le C\int_Q f^2 e^{2s\varphi}dxdt + Ce^{2s\delta_3} \tag{3.32}$$

for all large $s > 0$.

On the other hand, by (3.17), we see that $Q \subset \Omega \times (-T,T)$. Hence,

$$\begin{aligned}\int_Q f^2 e^{2s\varphi}dxdt &\le \int_\Omega \left(\int_{-T}^T e^{2s\varphi(x,t)}dt\right) f^2(x)dx\\ &= \int_\Omega f^2(x)e^{2s\varphi(x,0)}\left(\int_{-T}^T e^{2s(\varphi(x,t)-\varphi(x,0))}dt\right)dx.\end{aligned}$$

Recalling the definition of φ and applying the Lebesgue theorem, we have

$$\begin{aligned}&\sup_{x\in\Omega}\left|\int_{-T}^T e^{2s(\varphi(x,t)-\varphi(x,0))}dt\right| = \sup_{x\in\Omega}\left|\int_{-T}^T \exp(2se^{\lambda d(x)}(e^{-\lambda\beta t^2}-1))dt\right|\\ &\le \int_{-T}^T \exp(2se^{\lambda d_0}(e^{-\lambda\beta t^2}-1))dt = o(1),\end{aligned}$$

where $d_0 = \inf_{x\in\Omega} d(x)$, as $s \longrightarrow \infty$. Therefore

$$\int_Q f^2 e^{2s\varphi}dxdt = o(1)\int_\Omega f^2(x)e^{2s\varphi(x,0)}dx,$$

with which inequality (3.32) yields

$$(1-o(1))\int_\Omega f^2(x)e^{2s\varphi(x,0)}dx \le Ce^{2s\delta_3}$$

as $s \longrightarrow \infty$. By (3.18), there exists $\rho_4 > \rho_3$ such that $\{x \in \mathbb{R}^n; d(x) > \rho_4\} \supset \overline{\Omega}$, that is, $x \in \overline{\Omega}$ implies that $\varphi(x,0) > e^{\lambda\rho_4} \equiv \delta_4 > \delta_3$. Hence

$$(1 - o(1))e^{2s\delta_4} \int_\Omega f^2(x)dx \le Ce^{2s\delta_3},$$

so that

$$\int_\Omega f^2(x)dx \le Ce^{-2s(\delta_4-\delta_3)}$$

as $s \longrightarrow \infty$. Consequently, by letting $s \longrightarrow \infty$, we see by $\delta_4 - \delta_3 > 0$ that $f(x) = 0$, $x \in \Omega$. Thus the proof of Theorem 3 is complete. ■

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