An inverse problem for a dynamical Lamé system with residual stress

by

Victor ISAKOV, Jenn-Nan WANG
and Masahiro YAMAMOTO

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

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Victor Isakov[*]   Jenn-Nan Wang[†]   Masahiro Yamamoto[‡]

Abstract

In this paper we prove a Hölder and Lipschitz stability estimates of determining all coefficients of a dynamical Lamé system with residual stress, including the density, Lamé parameters, and the residual stress, by three pairs of observations from the whole boundary or from a part of it. These estimates imply first uniqueness results for determination of all parameters in the residual stress systems from few boundary measurements. Our essential assumptions are that the Lamé system possesses a suitable pseudoconvex function, residual stress is small, and three sets of the initial data satisfy some independency condition.

1   Introduction

We consider an elasticity system with residual stress. This system is anisotropic, i.e. it describes elastic properties of materials different in various directions. The assumption about isotropy is too restrictive in most important applications, although it allows deeper mathematical analysis of direct and especially of inverse problems. While theory of unique solvability of direct problems in quite general anisotropic case is relatively well developed [3], almost nothing is known about determination of anisotropic elastic parameters from additional boundary value data (i.e. about inverse problems). We handle simplest anisotropy known as Lamé system with residual stress, which is a small perturbation of classical isotropic Lamé system by a scalar anisotropic second order operator. Smallness of perturbation is motivated by applications to material science [13]. Assuming that speeds of propagation of shear and

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[*]Department of Mathematics and Statistics, Wichita State University, Wichita, KS 67206, USA. Email:victor.isakov@wichita.edu

[†]Department of Mathematics, National Taiwan University, Taipei 106, Taiwan. Email:jnwang@math.ntu.edu.tw

[‡]Department of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo 153-8914, Japan. Email:myama@ms.u-tokyo.ac.jp

compression waves in unperturbed system satisfy some pseudoconvexity type conditions (which exclude trapped elastic rays) and that three sets of initial conditions are in certain sense independent, we obtain first uniqueness and stability results about identification of all 9 elastic parameters of isotropic medium with residual stress from lateral boundary observations. When observation time and observed part of the boundary are arbitrary we explicitly describe a domain where coefficients are guaranteed to be unique and we give a Hölder stability estimate. When observation time is sufficiently large and observation is from the whole lateral boundary we derive Lipschitz stability estimates. These estimates indicate the possibility of numerical solution of inverse problem with high resolution and therefore a substantial applied potential. While our assumptions exclude zero initial data (most natural in many applications), recent progress in generating wave fields by interior sources in geophysics, material sciences, and medicine, and also substantial amount of historical seismic data from earthquakes (which are interior sources) make our assumptions more realistic.

Let Ω be an open bounded domain in $\mathbb{R}^3$ with boundary $\partial\Omega \in C^8$. The residual stress is modelled by a symmetric second-rank tensor $R(x) = (r_{jk}(x))_{j,k=1}^3 \in C^7(\overline{\Omega})$ which is divergence free

$$\mathrm{div}R = 0 \quad \text{in} \quad \Omega \tag{1.1}$$

and satisfies the boundary condition

$$R\nu = 0 \quad \text{on} \quad \partial\Omega, \tag{1.2}$$

where $\mathrm{div}R$ is a vector-valued function with components given by

$$(\mathrm{div}R)_j = \sum_{k=1}^3 \partial_k r_{jk}, \quad 1 \le j \le 3.$$

In this paper $x = (x_1, x_2, x_3) \in \mathbf{R}^3$ and $\nu = (\nu_1, \nu_2, \nu_3)^\top$ is the unit outer normal vector to $\partial\Omega$. Here and below, differential operators ∇ and Δ without subscript are with respect to x variables. Let $Q = \Omega \times (-T, T)$ and $\mathbf{u} = (u_1, u_2, u_3)^\top : Q \to \mathbb{R}^3$ be the displacement vector in Q. We remind that $\epsilon(\mathbf{u}) = (\nabla\mathbf{u} + \nabla\mathbf{u}^\top)/2$ is the strain tensor. We consider the initial boundary value problem:

$$\mathbf{A}_E\mathbf{u} := \rho\partial_t^2\mathbf{u} - \mu\Delta\mathbf{u} - (\lambda + \mu)\nabla(\mathrm{div}\mathbf{u}) - (\nabla\lambda)\mathrm{div}\mathbf{u} - 2\epsilon(\mathbf{u})\nabla\mu - \mathrm{div}((\nabla\mathbf{u})R) = 0 \quad \text{in} \quad Q, \tag{1.3}$$

$$\mathbf{u} = \mathbf{u}_0, \quad \partial_t\mathbf{u} = \mathbf{u}_1 \quad \text{on} \quad \Omega \times \{0\}, \tag{1.4}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on} \quad \partial\Omega \times (-T, T), \tag{1.5}$$

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where ρ is density and λ and μ are Lamé parameters depending only on x and satisfying inequalities

$$\varepsilon_1 < \mu, \ \varepsilon_1 < \rho, \ \varepsilon_1 < \lambda + \mu \quad \text{on} \quad \Omega \qquad (1.6)$$

for some positive constant ε_1. Hereafter, we use E to represent the set of elastic coefficients in (1.3), i.e. $E = (\rho, \lambda, \mu, R)$. We will assume that $\rho \in C^6(\overline{\Omega})$ and $\lambda, \mu \in C^7(\overline{\Omega})$. The system (1.3) can be written as

$$\rho \, \partial_t^2 \mathbf{u} - \text{div}\sigma(\mathbf{u}) = 0,$$

where $\sigma(\mathbf{u}) = \lambda(\text{tr}\epsilon)I + 2\mu\epsilon + R + (\nabla\mathbf{u})R$ is stress tensor. Note that the term $\text{div}R$ does not appear in (1.3) due to (1.1). Also, due to the same condition, we can see that

$$(\text{div}((\nabla\mathbf{u})R))_i = \sum_{j,k=1}^{3} r_{jk}\partial_j\partial_k u_i, \quad 1 \le i \le 3.$$

To make sure that the problem (1.3) with (1.4), (1.5) is well-posed, it suffices to assume that

$$\|R\|_{C^1(\overline{\Omega})} < \varepsilon_0 \qquad (1.7)$$

for some (small) constant $\varepsilon_0 > 0$. The assumption (1.7) is also motivated by material science applications [13]. Indeed, residual stress of interest for engineers is due to past thermal changes in steel production which do not significantly change elastic properties of steel. It is not hard to see that if ε_0 is sufficiently small, then the boundary value problem (1.3), (1.4), (1.5) is hyperbolic, and hence for any initial data $(\mathbf{u}_0, \mathbf{u}_1) \in H^1(\Omega) \times L^2(\Omega)$ and lateral Dirichlet data $\mathbf{g} \in C^1([-T, T]; H^1(\Omega))$, $\mathbf{u}_0 = \mathbf{g}$ on $\partial\Omega \times \{0\}$, there exists a unique solution $\mathbf{u}(\cdot; E; \mathbf{u}_0, \mathbf{u}_1, \mathbf{g}) \in C([-T, T]; H^1(\Omega))$ to (1.3)-(1.5).

In this paper we are interested in the following inverse problem:

Let Γ be an open subset of $\partial\Omega$ with $\partial\Gamma \in C^1$. Determine density ρ, Lamé parameters λ, μ, and the residual stress R (total 9 functions) from Cauchy type data $(\mathbf{u}, \sigma(\mathbf{u})\nu)$ on $\Gamma \times (-T, T)$, where $\mathbf{u} = \mathbf{u}(\cdot; E; \mathbf{u}_0, \mathbf{u}_1, \mathbf{g})$, given for a finite number of pairs of initial data $(\mathbf{u}_0, \mathbf{u}_1)$.

We will address uniqueness and stability issues. The focus is on the stability, since stability implies uniqueness. This work is a sequel of our recent paper [10] where we demonstrated uniqueness of only R assuming known constant ρ, λ, μ. Our method is based on Carleman estimates techniques initiated by Bukhgeim and Klibanov [2]. For works on Carleman estimates and related inverse problems for scalar equations, we refer to books [1] and [11] for further details and references. The method of [2] was modified for scalar equations in the paper of Imanuvilov and Yamamoto [5]. It was found by Imanuvilov, Isakov, and Yamamoto [7] that this modification allows to obtain uniqueness and stability for coefficients of systems of equations, in particular in

[7] there is a first uniqueness result for all three elastic parameters ρ, λ, μ of isotropic elasticity. For further results on identification of the isotropic Lame system we refer to [6]. For Carleman estimates and uniqueness of the continuation for the residual stress system (1.3) and for identification of source term and R with given constant ρ, λ, μ, we refer to [9], [10], [12]. In case of many boundary measurements and zero initial data there only partial results on identification of residual stress [4], [14]. In the present work we will show that we can determine all 9 parameters in (1.3)-(1.5) by three pairs of Cauchy data. We will derive a Hölder stability estimate in the convex hull of the observation surface Γ and a Lipschitz stability estimate for (ρ, λ, μ, R) in Ω when $\Gamma = \partial\Omega$ and observation time T is large.

We are now ready to state main results of the paper. Let $d = \inf |x|$ and $D = \sup |x|$ over $x \in \Omega$. We will assume that

$$0 < d. \tag{1.8}$$

For a function $c \in C^1(\overline{\Omega})$ we introduce the following condition

$$\theta^2 < c \quad \text{and} \quad \theta^2 c + d\theta\sqrt{c}|\nabla c| + \frac{1}{2}c\, x \cdot \nabla c < c^2 \quad \text{on} \quad \overline{\Omega}. \tag{1.9}$$

Let $\mathcal{E}_{\varepsilon_0,M}$ be the class of parameters defined by

$$\begin{aligned}
\mathcal{E}_{\varepsilon_0,M} = \quad & \{\ \|\rho\|_{C^6(\overline{\Omega})} + \|\lambda\|_{C^7(\overline{\Omega})} + \|\mu\|_{C^7(\overline{\Omega})} + \|R\|_{C^6(\overline{\Omega})} < M : \\
& \rho, \lambda, \mu \text{ satisfy } (1.6) \text{ and } c = \tfrac{\mu}{\rho},\ c = \tfrac{\lambda+2\mu}{\rho} \text{ satisfy } (1.9), \\
& R \text{ is symmetric and satisfies } (1.1), (1.2), \text{ and } (1.7)\ \}.
\end{aligned}$$

To study the inverse problem, we need not only the well-posedness of (1.3)-(1.5) but also some extra regularity of the solution $\mathbf{u}$. To achieve the latter property, the initial and Dirichlet data $(\mathbf{u}_0, \mathbf{u}_1, \mathbf{g})$ are required to satisfy some smoothness and compatibility conditions. More precisely, we will assume that $\mathbf{u}_0 \in H^9(\Omega), \mathbf{u}_1 \in H^8(\Omega)$ and $\mathbf{g} \in C^8([-T,T]; H^1(\partial\Omega)) \cap C^5([-T,T]; H^4(\partial\Omega))$ and they satisfy standard compatibility conditions of order 8 at $\partial\Omega \times \{0\}$. By using energy estimates [3] and Sobolev embedding theorems as in [7] one can show that

$$\|\partial_x^\alpha \partial_t^\beta \mathbf{u}\|_{C^0(\overline{Q})} \leq C \tag{1.10}$$

for $|\alpha| \leq 2$ and $0 \leq \beta \leq 5$. We will use three sets of initial data $(\mathbf{u}_0(\cdot; j), \mathbf{u}_1(\cdot; j))$, $j = 1, 2, 3$. To guarantee uniqueness in the inverse problem, we impose some non-degeneracy condition on the initial data. Namely, let $\mathbf{M}$ denote the 18×13 matrix

$$\begin{pmatrix}
\mu_1\Delta\mathbf{u}_0(\cdot;1) + (\lambda_1+\mu_1)\nabla(\mathrm{div}\mathbf{u}_0(\cdot;1)) & \mathrm{div}\mathbf{u}_0(\cdot;1)I_3 & 2\epsilon(\mathbf{u}_0(\cdot;1)) & \mathbf{R}(\mathbf{u}_0(\cdot;1)) \\
\mu_1\Delta\mathbf{u}_1(\cdot;1) + (\lambda_1+\mu_1)\nabla(\mathrm{div}\mathbf{u}_1(\cdot;1)) & \mathrm{div}\mathbf{u}_1(\cdot;1)I_3 & 2\epsilon(\mathbf{u}_1(\cdot;1)) & \mathbf{R}(\mathbf{u}_1(\cdot;1)) \\
\mu_1\Delta\mathbf{u}_0(\cdot;2) + (\lambda_1+\mu_1)\nabla(\mathrm{div}\mathbf{u}_0(\cdot;2)) & \mathrm{div}\mathbf{u}_0(\cdot;2)I_3 & 2\epsilon(\mathbf{u}_0(\cdot;2)) & \mathbf{R}(\mathbf{u}_0(\cdot;2)) \\
\mu_1\Delta\mathbf{u}_1(\cdot;2) + (\lambda_1+\mu_1)\nabla(\mathrm{div}\mathbf{u}_1(\cdot;2)) & \mathrm{div}\mathbf{u}_1(\cdot;2)I_3 & 2\epsilon(\mathbf{u}_1(\cdot;2)) & \mathbf{R}(\mathbf{u}_1(\cdot;2)) \\
\mu_1\Delta\mathbf{u}_0(\cdot;3) + (\lambda_1+\mu_1)\nabla(\mathrm{div}\mathbf{u}_0(\cdot;3)) & \mathrm{div}\mathbf{u}_0(\cdot;3)I_3 & 2\epsilon(\mathbf{u}_0(\cdot;3)) & \mathbf{R}(\mathbf{u}_0(\cdot;3)) \\
\mu_1\Delta\mathbf{u}_1(\cdot;3) + (\lambda_1+\mu_1)\nabla(\mathrm{div}\mathbf{u}_1(\cdot;3)) & \mathrm{div}\mathbf{u}_1(\cdot;3)I_3 & 2\epsilon(\mathbf{u}_1(\cdot;3)) & \mathbf{R}(\mathbf{u}_1(\cdot;3))
\end{pmatrix}$$

where I_3 is the 3×3 identity matrix, $\mathbf{R}(\mathbf{v})$ a 3×6 matrix defined by

$$\mathbf{R}(\mathbf{v}) = \begin{pmatrix} \partial_1^2 \mathbf{v} & 2\partial_1 \partial_2 \mathbf{v} & 2\partial_1 \partial_3 \mathbf{v} & \partial_2^2 \mathbf{v} & 2\partial_2 \partial_3 \mathbf{v} & \partial_3^2 \mathbf{v} \end{pmatrix}. \tag{1.11}$$

We will assume that

> there exists a 13×13 minor of $\mathbf{M}$ such that the absolute value of its determinant is greater than ε_0 on $\overline{\Omega}$. $\hspace{1cm}$ (1.12)

One can check that $\mathbf{u}_0(\cdot; 1) = (x_1 x_2, 0, 0)^\top$, $\mathbf{u}_1(\cdot; 1) = (0, 0, 0)^\top$, $\mathbf{u}_0(\cdot; 2) = (x_1, x_2, x_3)^\top$, $\mathbf{u}_1(\cdot; 2) = (0, x_2, x_3)$, $\mathbf{u}_0(\cdot; 3) = (x_1^2, x_2^2, x_3^2)^\top$ and $\mathbf{u}_1(\cdot; 3) = (x_2 x_3, x_1 x_3, x_1 x_2)^\top$ satisfy (1.12). Here 13 row vectors from row 2,7-18 are linearly independent on $\overline{\Omega}$.

We will use the following notation:

C, γ are generic constants depending only on $\Omega, T, \delta, \varepsilon_0, M, \mathbf{u}_0(\cdot; j), \mathbf{u}_1(\cdot; j)$, $j = 1, 2, 3$, any other dependence will be indicated, $\| \cdot \|_{(k)}(Q)$ is the norm in the Sobolev space $H^k(Q)$. $Q(\varepsilon) = Q \cap \{\varepsilon < |x|^2 - \theta^2 t^2 - d_1^2\}$ and $\Omega(\varepsilon) = \Omega \cap \{\varepsilon < |x|^2 - d_1^2\}$, where $d_1 \geq d$. Let $\mathbf{u}(; 1; j)$ and $\mathbf{u}(; 2; j)$ be solutions of (1.3), (1.4) with initial data $(\mathbf{u}_0(; j), \mathbf{u}_1(; j))$, for $j = 1, 2, 3$, corresponding to coefficients E_1 and E_2, respectively. We will consider the Dirichlet data (displacements) as measurements (observations). We introduce the norm of the differences of the data

$$\begin{aligned} F = \sum_{j=1}^3 \sum_{\beta=2}^4 (\|\partial_t^\beta (\mathbf{u}(; 2; j) - \mathbf{u}(; 1; j))\|_{(\frac{5}{2})}(\Gamma \times (-T, T)) \\ + \|\partial_t^\beta \sigma(\mathbf{u}(; 2; j) - \mathbf{u}(; 1; j))\nu\|_{(\frac{3}{2})}(\Gamma \times (-T, T))). \end{aligned}$$

We first state the Hölder type estimate of determining coefficients in $\Omega(\varepsilon)$.

Theorem 1.1. *Assume that the domain Ω satisfies (1.8), θ satisfies*

$$\theta^2 < \frac{d^2}{T^2}, \tag{1.13}$$

and for some d_1,

$$|x|^2 - d_1^2 < 0 \text{ when } x \in (\partial\Omega \setminus \Gamma), \text{ and } D^2 - \theta^2 T^2 - d_1^2 < 0. \tag{1.14}$$

Let the initial data $(\mathbf{u}_0(; j), \mathbf{u}_1(; j))$, $j = 1, 2, 3$, satisfy (1.12).

Then there exist ε_0 and constants $C, \gamma \in (0, 1)$ such that for $E_1, E_2 \in \mathcal{E}_{\varepsilon_0, M}$ with

$$\lambda_1 = \lambda_2 \quad and \quad \mu_1 = \mu_2 \quad on \quad \Gamma, \tag{1.15}$$

one has

$$\|\rho_1 - \rho_2\|_{(0)}(\Omega(\varepsilon)) + \|\lambda_1 - \lambda_2\|_{(0)}(\Omega(\varepsilon)) + \|\mu_1 - \mu_2\|_{(0)}(\Omega(\varepsilon)) + \|R_1 - R_2\|_{(0)}(\Omega(\varepsilon)) \leq CF^\gamma. \tag{1.16}$$

Remark 1.2. *If $d_1 < D$, then the second condition of (1.14) and (1.13) imply that*

$$\frac{D^2 - d_1^2}{\theta^2} < T^2 < \frac{d^2}{\theta^2}.$$

In other words, the observation time T needs not be too large. In this case, we can determine elastic parameters in the domain $\Omega(\varepsilon)$. The domain $\Omega(\varepsilon)$ is discussed in [8], section 3.4.

If Γ is the whole lateral boundary and T is sufficiently large, then a much stronger (and in a certain sense best possible) Lipschitz stability estimate holds.

Theorem 1.3. *Let $d_1 = d$. Assume that*

$$D^2 < 2d^2, \tag{1.17}$$

and

$$\frac{D^2 - d^2}{\theta^2} < T^2 < \frac{d^2}{\theta^2}. \tag{1.18}$$

Let the initial data $(\mathbf{u}_0(; j), \mathbf{u}_1(; j))$, $j = 1, 2, 3$, satisfy (1.12) and $\Gamma = \partial\Omega$.
Then there exist an ε_0 in (1.7) and C such that for $E_1, E_2 \in \mathcal{E}_{\varepsilon_0, M}$ satisfying the conditions

$$\rho_1 = \rho_2, \ R_1 = R_2, \ \partial^\alpha \lambda_1 = \partial^\alpha \lambda_2 \ and \ \partial^\alpha \mu_1 = \partial^\alpha \mu_2 \ , \quad on \quad \Gamma \quad when \quad |\alpha| \leq 1, \tag{1.19}$$

one has

$$\|\rho_1 - \rho_2\|_{(0)}(\Omega) + \|\lambda_1 - \lambda_2\|_{(0)}(\Omega) + \|\mu_1 - \mu_2\|_{(0)}(\Omega) + \|R_1 - R_2\|_{(0)}(\Omega) \leq CF \tag{1.20}$$

Let us show compatibility of conditions (1.18). From conditions (1.17) we have

$$\frac{D^2 - d^2}{\theta^2} < \frac{d^2}{\theta^2}$$

and hence we can find T^2 between these two numbers.

As mentioned previously, the proofs of these theorems rely on Carleman estimates. We briefly described needed Carleman estimates in Section 2. Using this estimate we will prove in Section 3 the Hölder stability estimate (1.16). In Section 4, we derive the Lipschitz stability estimate for our inverse problem.

2 Carleman estimate

In this section we will describe Carleman estimates needed to solve our inverse problem. Their proofs can be found in [9] and [10]. Let $\psi(x,t) = |x|^2 - \theta^2 t^2 - d_1^2$ and $\varphi(x,t) = \exp(\frac{\eta}{2}\psi(x,t))$, where θ is chosen in (1.13). Due to condition (1.9) and known sufficient conditions of pseudoconvexity [8], Theorem 3.4.1, we can fix (large) $\eta > 0$ so that the phase function φ is strongly pseudoconvex on $\overline{Q}$ with respect to

$$\frac{\rho}{\mu}\partial_t^2 - \Delta, \ \frac{\rho}{\lambda + 2\mu}\partial_t^2 - \Delta.$$

Theorem 2.1. *There are constants ε_0 and C such that for $E \in \mathcal{E}_{\varepsilon_0,M}$*

$$\int_Q(\tau|\nabla_{x,t}\mathbf{u}|^2 + \tau|\nabla_{x,t}\mathrm{div}\mathbf{u}|^2 + \tau|\nabla_{x,t}\mathrm{curl}\mathbf{u}|^2 + \tau^3|\mathbf{u}|^2 + \tau^3|\mathrm{div}\mathbf{u}|^2 + \tau^3|\mathrm{curl}\mathbf{u}|^2)e^{2\tau\varphi}$$
$$\leq C\int_Q(|\mathbf{A}_E\mathbf{u}|^2 + |\nabla(\mathbf{A}_E\mathbf{u})|^2)e^{2\tau\varphi}$$

$$(2.1)$$

for all $\mathbf{u} \in H_0^3(Q)$ and

$$\int_Q(\tau^2|\mathbf{u}|^2 + |\mathrm{div}\mathbf{u}|^2 + |\mathrm{curl}\mathbf{u}|^2 + \tau^{-1}|\nabla\mathbf{u}|)e^{2\tau\varphi} \leq C\int_Q|\mathbf{A}_E\mathbf{u}|^2 e^{2\tau\varphi} \qquad (2.2)$$

for all $\mathbf{u} \in H_0^2(Q)$.

Carleman estimates of Theorem 2.1 is our basic tool for treating the inverse problem. The basic idea in proving Theorem 2.1 is to reduce (1.3) to an extended system of dimension 7 for $(\mathbf{u}, \mathrm{div}\mathbf{u}, \mathrm{curl}\mathbf{u})$. The resulting new system is not principally diagonalized. However, when the residual stress R is small, the second derivatives of $\mathbf{u}$ can be bounded by first derivatives of $\mathrm{div}\mathbf{u}$ and $\mathrm{curl}\mathbf{u}$. We refer to [9] and [10] for detailed computations. For the case considered here, we only need to verify the strong pseudoconvexity of φ on $\overline{Q}$. Under conditions (1.9) and (1.13), one can check that φ satisfies the required property when ε_0 is small and η is large (see [8] or [9]). An estimate similar to (2.2) was also derived in [7].

In order to use (2.1), it is required that the Cauchy data of the solution and the source term vanish on the lateral boundary. To handle non-vanishing Cauchy data, the following lemma is useful.

Lemma 2.2. *For any pair of $(\mathbf{g}_0, \mathbf{g}_1) \in H^{\frac{5}{2}}(\partial\Omega \times (-T,T)) \times H^{\frac{3}{2}}(\partial\Omega \times (-T,T))$, we can find a vector-valued function $\mathbf{u}^* \in H^3(Q)$ such that*

$$\mathbf{u}^* = \mathbf{g}_0, \sigma(\mathbf{u}^*)\nu = \mathbf{g}_1, \mathbf{A}_E\mathbf{u}^* = 0 \quad on \quad \partial\Omega \times (-T,T),$$

and

$$\|\mathbf{u}^*\|_{(3)}(Q) \leq C(\|\mathbf{g}_0\|_{(\frac{5}{2})}(\partial\Omega \times (-T,T)) + \|\mathbf{g}_1\|_{(\frac{3}{2})}(\partial\Omega \times (-T,T))) \qquad (2.3)$$

for some $C > 0$ provided ε_0 in (1.7) is sufficiently small.

Proof. By standard extensions theorems for any $\mathbf{g}_2 \in H_{(\frac{1}{2})}(\partial\Omega \times (-T,T))$ we can find $\mathbf{u}^{**} \in H^3(Q)$ so that

$$\mathbf{u}^{**} = \mathbf{g}_0, \sigma(\mathbf{u}^{**})\nu = \mathbf{g}_1, \partial_\nu^2 \mathbf{u}^{**} = \mathbf{g}_2 \quad \text{on} \quad \partial\Omega \times (-T,T)$$

and

$$\|\mathbf{u}^{**}\|_{(3)}(Q) \leq C(\|\mathbf{g}_2\|_{(\frac{1}{2})}(\partial\Omega\times(-T,T))+\|\mathbf{g}_1\|_{(\frac{3}{2})}(\partial\Omega\times(-T,T))+\|\mathbf{g}_0\|_{(\frac{5}{2})}(\partial\Omega\times(-T,T))).$$

Since $\partial\Omega \times (-T,T)$ is non-characteristic with respect to $\mathbf{A}_E$ provided (1.6) holds and ε_0 is small, the condition $\mathbf{A}_E\mathbf{u}^{**} = 0$ on $\partial\Omega\times(-T,T)$ is equivalent to the fact that $\mathbf{g}_2$ can be written as a linear combination (with C^1 coefficients) of $\partial_t^2\mathbf{g}_0$ and tangential derivatives of $\mathbf{g}_0$ (up to second order) and of $\mathbf{g}_1$ (up to first order) along $\partial\Omega$. In particular,

$$\|\mathbf{g}_2\|_{(\frac{1}{2})}(\partial\Omega \times (-T,T)) \leq C(\|\mathbf{g}_1\|_{(\frac{3}{2})}(\partial\Omega \times (-T,T) + \|\mathbf{g}_0\|_{(\frac{5}{2})}(\partial\Omega \times (-T,T)).$$

Choosing $\mathbf{g}_2$ as this linear combination we obtain (2.3). $\qquad\square$

To handle $\nabla\lambda$ and $\nabla\mu$ in (1.3), we need other Carleman estimates. We first derive the estimate needed in the proof of Theorem 1.1. Let d_1 be given as in Theorem 1.1. Then we can see that $\partial\Omega(\varepsilon) = (\Gamma \cup \{|x|^2 = d_1^2 + \varepsilon\}) \cap \bar{\Omega}$.

Lemma 2.3. *Let $f \in C^1(\overline{\Omega})$ satisfy $f|_\Gamma = 0$. Then*

$$\tau \int_{\Omega(\varepsilon)} |f(x)|^2 e^{2\tau\varphi(x,0)}dx \leq C \int_{\Omega(\varepsilon)} |\nabla f(x)|^2 e^{2\tau\varphi(x,0)}dx. \tag{2.4}$$

Proof. Denote $\varphi_0(x) = \varphi(x,0)$. Let $g = e^{\tau\varphi_0}f$, then $e^{\tau\varphi_0}\nabla f = \nabla g - \tau\nabla\varphi_0 g$. Note that $g|_\Gamma = 0$. We observe that $\nabla\varphi_0(x) = \eta x\varphi_0(x)$ and thus on $\partial\Omega(\varepsilon) \setminus \Gamma$ with the unit outer normal $\nu(= -x/|x|)$

$$\partial_\nu\varphi_0(x) = \nabla\varphi_0 \cdot \nu = -\eta|x|\varphi_0(x). \tag{2.5}$$

Using integration by parts and (2.5), we have that

$$\int_{\Omega(\varepsilon)} |\nabla g - \tau\nabla\varphi_0 g|^2$$
$$= \int_{\Omega(\varepsilon)} |\nabla g|^2 + \tau^2 \int_{\Omega(\varepsilon)} |\nabla\varphi_0 g|^2 - 2\tau \int_{\Omega(\varepsilon)} \nabla g \cdot \nabla\varphi_0 g$$
$$\geq -\tau \int_{\Omega(\varepsilon)} \nabla\varphi_0 \cdot \nabla(g^2)$$
$$= -\tau \int_{\partial\Omega(\varepsilon)\setminus\Gamma} \partial_\nu\varphi_0 g^2 + \tau \int_{\Omega(\varepsilon)} \Delta\varphi_0 g^2$$
$$= \tau \int_{\partial\Omega(\varepsilon)\setminus\Gamma} \eta|x|\varphi_0(x)g^2(x)d\Gamma(x) + \tau \int_{\Omega(\varepsilon)} (3\eta + \eta^2|x|^2)\varphi_0 g^2(x)dx$$
$$\geq C \int_{\Omega(\varepsilon)} g^2,$$

which implies (2.4). $\qquad\square$

The following estimate is useful in proving Theorem 1.3 (see also [7, Lemma 3.6]).

Corollary 2.4. *Let $f \in C^1(\overline{\Omega})$ and $f = 0$ on $\partial\Omega$, Then we have*

$$\tau \int_\Omega |f(x)|^2 e^{2\tau\varphi(x,0)}dx \leq C \int_\Omega |\nabla f(x)|^2 e^{2\tau\varphi(x,0)}.$$

3 Hölder stability for the determination of coefficients

In this section we prove the first main result of the paper, Theorem 1.1. Let us denote $\mathbf{u}(;j) = \mathbf{u}(;2;j) - \mathbf{u}(;1;j)$ for $j = 1, 2, 3$, and $\mathbf{F} = (f_1, f_2, \cdots, f_9, R)^\top$, where $f_1 = \rho_1 - \rho_2$, $f_2 = \lambda_1 - \lambda_2$, $f_3 = \mu_1 - \mu_2$, $(f_4, f_5, f_6)^\top = \nabla f_2$, $(f_7, f_8, f_9)^\top = \nabla f_3$, and

$$R^\top = \begin{pmatrix} r_{11} \\ r_{12} \\ r_{13} \\ r_{22} \\ r_{23} \\ r_{33} \end{pmatrix} = \begin{pmatrix} r_{1,11} - r_{2,11} \\ r_{1,12} - r_{2,12} \\ r_{1,13} - r_{2,13} \\ r_{1,22} - r_{2,22} \\ r_{1,23} - r_{2,23} \\ r_{1,33} - r_{2,33} \end{pmatrix}.$$

By subtracting equations (1.3) for $\mathbf{u}(;1;j)$ from the equations for $\mathbf{u}(;2;j)$ we yield

$$\mathbf{A}_{E_2}\mathbf{u}(;j) = \mathcal{A}(\mathbf{u}(;1;j))\mathbf{F} \quad \text{on} \quad Q, \tag{3.1}$$

where

$$\begin{aligned}
\mathcal{A}(\mathbf{v})\mathbf{F} = \ & -f_1\partial_t^2\mathbf{v} + (f_2 + f_3)\nabla(\mathrm{div}\mathbf{v}) + f_3\Delta\mathbf{v} + \mathrm{div}\mathbf{v}(f_4, f_5, f_6)^\top \\
& + 2\epsilon(\mathbf{v})(f_7, f_8, f_9)^\top + \sum_{j,k=1}^3 r_{jk}\partial_j\partial_k\mathbf{v},
\end{aligned}$$

and

$$\mathbf{u}(;j) = \partial_t\mathbf{u}(;j) = 0 \quad \text{on} \quad \Omega \times \{0\}. \tag{3.2}$$

Differentiating (3.1) in t and using time-independence of the coefficients of the system, we get

$$\mathbf{A}_{E_2}\mathbf{U}(;j) = \mathcal{A}(\mathbf{U}(;1;j))\mathbf{F} \quad \text{on} \quad Q, \tag{3.3}$$

where

$$\mathbf{U}(;j) = \begin{pmatrix} \partial_t^2\mathbf{u}(;j) \\ \partial_t^3\mathbf{u}(;j) \\ \partial_t^4\mathbf{u}(;j) \end{pmatrix} \quad \text{and} \quad \mathbf{U}(;1;j) = \begin{pmatrix} \partial_t^2\mathbf{u}(;1;j) \\ \partial_t^3\mathbf{u}(;1;j) \\ \partial_t^4\mathbf{u}(;1;j) \end{pmatrix}.$$

By extension theorems for Sobolev spaces there exists $\mathbf{U}^*(;j) \in H^2(Q)$ such that

$$\mathbf{U}^*(;j) = \mathbf{U}(;j), \ \sigma(\mathbf{U}^*(;j))\nu = \sigma(\mathbf{U}(;j))\nu \quad \text{on} \quad \Gamma \times (-T, T), \tag{3.4}$$

and

$$\|\mathbf{U}^*(;j)\|_{(2)}(Q) \le C(\|\mathbf{U}(;j)\|_{(\frac{3}{2})}(\Gamma \times (-T, T)) + \|\sigma(\mathbf{U})(;j)\nu\|_{(\frac{1}{2})}(\Gamma \times (-T, T))) \le CF \tag{3.5}$$

for all $j = 1, 2, 3$, due to the definitions of $\mathbf{u}(;j), \mathbf{U}(;j)$, and F. We now introduce $\mathbf{V}(;j) = \mathbf{U}(;j) - \mathbf{U}^*(;j)$. Then

$$\mathbf{A}_{E_2}\mathbf{V}(;j) = \mathcal{A}(\mathbf{U}(;1;j))\mathbf{F} - \mathbf{A}_{E_2}\mathbf{U}^*(;j) \quad \text{on} \quad Q \tag{3.6}$$

and

$$\mathbf{V}(;j) = \sigma(\mathbf{V})(;j)\nu = 0 \quad \text{on} \quad \Gamma \times (-T, T). \tag{3.7}$$

To use the Carleman estimate (2.2), we introduce a cut-off function $\chi \in C^2(\mathbb{R}^4)$ such that $0 \le \chi \le 1$, $\chi = 1$ on $Q(\frac{\varepsilon}{2})$ and $\chi = 0$ on $Q \setminus Q(0)$. By the Leibniz' formula

$$\mathbf{A}_{E_2}(\chi\mathbf{V}(;j)) = \chi\mathbf{A}_{E_2}(\mathbf{V}(;j)) + \mathbf{A}_1\mathbf{V}(;j) = \chi\mathcal{A}\mathbf{F} - \chi\mathbf{A}_{E_2}\mathbf{U}^*(;j) + \mathbf{A}_1\mathbf{V}(;j)$$

due to (3.6). Here (and below) $\mathbf{A}_1$ denotes a first order matrix differential operator with coefficients uniformly bounded by $C(\varepsilon)$. By the choice of χ, $\mathbf{A}_1\mathbf{V}(;j) = 0$ on $Q(\frac{\varepsilon}{2})$. Because of (3.7) and of the first condition of (1.14), the function $\chi\mathbf{V}(;j) \in H_0^2(Q)$, so we can apply to it the Carleman estimate (2.2) to get

$$\int_Q \tau|\chi\mathbf{V}(;j)|^2 e^{2\tau\varphi} \le$$

$$C \int_Q (|\mathbf{F}|^2 + |\mathbf{A}_{E_2}(\mathbf{U}^*(;j))|^2) e^{2\tau\varphi} + C \int_{Q \setminus Q(\frac{\varepsilon}{2})} |\mathbf{A}_1\mathbf{V}(;j)|^2 e^{2\tau\varphi} \le$$

$$C(\int_Q |\mathbf{F}|^2 e^{2\tau\varphi} + F^2 e^{2\tau\Phi} + C(\varepsilon)e^{2\tau\varepsilon_1}) \tag{3.8}$$

where $\Phi = \sup\varphi$ over Q and $\varepsilon_1 = e^{\frac{\eta\varepsilon}{4}}$. To get the last inequality we used the bounds (3.5) and (1.10).

On the other hand, from (1.3), (3.1), (3.2) we have

$$\rho_2\partial_t^2\mathbf{u}(;j) = \mathcal{A}(\mathbf{u}(;1;j))\mathbf{F},$$

$$\rho_2\partial_t^3\mathbf{u}(;j) = \mathcal{A}(\partial_t\mathbf{u}(;1;j))\mathbf{F}$$

on $\Omega \times \{0\}$. We now want to rearrange the formulas above. Let $\mathbf{a}_{kj} = -\partial_t^{2+k}\mathbf{u}(,0;1;j)$, $\mathbf{b}_{kj} = \nabla(\text{div}\mathbf{u}_k(;j))$, $\mathbf{c}_{kj} = \Delta\mathbf{u}_k(;j) + \nabla\text{div}\mathbf{u}_k(;j)$, $\mathbf{B}_{kj} = \text{div}\mathbf{u}_k(;j)$, $\mathbf{C}_{kj} = 2\epsilon(\mathbf{u}_k(;j))$, and $\mathbf{R}_{kj} = \mathbf{R}(\mathbf{u}_k(;j))$ (see the definition of $\mathbf{R}$ in (1.11)), where $k = 0, 1$ and $j = 1, 2, 3$. Using that $\mathbf{u}(;1;j) = \mathbf{u}_0(;j), \partial_t\mathbf{u}(;1;j) = \mathbf{u}_1(;j)$ on $\Omega \times \{0\}$ we have

$$\begin{pmatrix} \mathbf{a}_{01} & \mathbf{B}_{01}I_3 & \mathbf{C}_{01} & \mathbf{R}_{01} \\ \mathbf{a}_{11} & \mathbf{B}_{11}I_3 & \mathbf{C}_{11} & \mathbf{R}_{11} \\ \mathbf{a}_{02} & \mathbf{B}_{02}I_3 & \mathbf{C}_{02} & \mathbf{R}_{02} \\ \mathbf{a}_{12} & \mathbf{B}_{12}I_3 & \mathbf{C}_{12} & \mathbf{R}_{12} \\ \mathbf{a}_{03} & \mathbf{B}_{03}I_3 & \mathbf{C}_{03} & \mathbf{R}_{03} \\ \mathbf{a}_{13} & \mathbf{B}_{13}I_3 & \mathbf{C}_{13} & \mathbf{R}_{13} \end{pmatrix} \begin{pmatrix} f_1 \\ f_4 \\ \vdots \\ f_9 \\ r_{11} \\ \vdots \\ r_{33} \end{pmatrix} = \rho_2 \begin{pmatrix} \partial_t^2\mathbf{u}(,0;1) \\ \partial_t^3\mathbf{u}(,0;1) \\ \partial_t^2\mathbf{u}(,0;2) \\ \partial_t^3\mathbf{u}(,0;2) \\ \partial_t^2\mathbf{u}(,0;3) \\ \partial_t^3\mathbf{u}(,0;3) \end{pmatrix} - \begin{pmatrix} \mathbf{b}_{01} & \mathbf{c}_{01} \\ \mathbf{b}_{11} & \mathbf{c}_{11} \\ \mathbf{b}_{02} & \mathbf{c}_{02} \\ \mathbf{b}_{12} & \mathbf{c}_{12} \\ \mathbf{b}_{03} & \mathbf{c}_{03} \\ \mathbf{b}_{13} & \mathbf{c}_{13} \end{pmatrix} \begin{pmatrix} f_2 \\ f_3 \end{pmatrix} \tag{3.9}$$

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on Ω. From the system (1.3) at $t = 0$ and from this system differentiated in t and taken at $t = 0$, we obtain

$$
\begin{aligned}
\mathbf{a}_{kj} &= -\tfrac{\mu_1}{\rho_1}\Delta\mathbf{u}_k(;j) - \tfrac{\lambda_1+\mu_1}{\rho_1}\nabla(\mathrm{div}\mathbf{u}_k(;j)) - \mathrm{div}\mathbf{u}_k(;j)\tfrac{\nabla\lambda_1}{\rho_1} \\
&\quad -2\epsilon(\mathbf{u}_k(;j))\tfrac{\nabla\mu_1}{\rho_1} - \sum_{\ell,m=1}^{3} r_{1,\ell m}\partial_\ell\partial_m\mathbf{u}_k(;j) \\
&= -\tfrac{\mu_1}{\rho_1}\Delta\mathbf{u}_k(;j) - \tfrac{\lambda_1+\mu_1}{\rho_1}\nabla(\mathrm{div}\mathbf{u}_k(;j)) - \mathbf{B}_{kj}\tfrac{\nabla\lambda_1}{\rho_1} \\
&\quad -\mathbf{C}_{kj}\tfrac{\nabla\mu_1}{\rho_1} - \sum_{\ell,m=1}^{3} r_{1,\ell m}\partial_\ell\partial_m\mathbf{u}_k(;j)
\end{aligned}
\tag{3.10}
$$

when $k = 0, 1$ and $j = 1, 2, 3$.

We now consider the matrix on the left hand side of (3.9). Using (3.10), one can add to the first column the remaining columns multiplied by suitable factors such that $-\mathrm{div}\mathbf{u}_k(;j)\tfrac{\nabla\lambda_1}{\rho_1}$, $-2\epsilon(\mathbf{u}_k(;j))\tfrac{\nabla\mu_1}{\rho_1}$, and $-\sum_{\ell,m=1}^{3} r_{1,\ell m}\partial_\ell\partial_m\mathbf{u}_k(;j)$ are eliminated from the first column of this matrix. Then we multiply the first column of the new matrix by the minus. We end up with the matrix $\mathbf{M}$ defined in Section 1. Obviously, determinants of corresponding minors of the matrix on the left side of (3.10) and of the matrix $\mathbf{M}$ are the same. It follows from the condition (1.12) and bounds (1.10) that

$$
|\mathbf{F}|^2 \le C\Big(\sum_{j=1}^{3}\sum_{\beta=2}^{3}|\partial_t^\beta\mathbf{u}(0;j)|^2 + |f_2|^2 + |f_3|^2\Big) \quad \text{on} \quad \overline{\Omega}.
\tag{3.11}
$$

Since $\chi(\cdot, T) = 0$,

$$
\int_\Omega |\chi\partial_t^\beta\mathbf{u}(0;j)|^2 e^{2\tau\varphi(x,0)}dx = -\int_0^T \partial_t\Big(\int_\Omega |\chi\partial_t^\beta\mathbf{u}(;j)|^2 e^{2\tau\varphi(x,t)}\Big)dx\Big)dt \le
$$

$$
\int_Q 2\chi^2(|\partial_t^{\beta+1}\mathbf{u}(;j)||\partial_t^\beta\mathbf{u}(;j)| + \tau|\partial_t\varphi||\partial_t^\beta\mathbf{u}(;j)|^2)e^{2\tau\varphi} + 2\int_{Q\setminus Q(\frac{\varepsilon}{2})} |\partial_t^\beta\mathbf{u}(;j)|^2\chi|\partial_t\chi|e^{2\tau\varphi}
$$

where $\beta = 2, 3$. The right side does not exceed

$$
C\Big(\int_Q \tau|\chi\mathbf{U}(;j)|^2 e^{2\tau\varphi} + C(\varepsilon)\int_{Q\setminus Q(\frac{\varepsilon}{2})} |\mathbf{U}(;j)|^2 e^{2\tau\varphi}\Big) \le
$$

$$
C\Big(\int_Q \tau|\chi\mathbf{V}(;j)|^2 e^{2\tau\varphi} + C(\varepsilon)\int_{Q\setminus Q(\frac{\varepsilon}{2})} |\mathbf{U}(;j)|^2 e^{2\tau\varphi} + \tau\int_Q |\mathbf{U}^*(;j)|^2 e^{2\tau\varphi}\Big)
$$

because $\mathbf{U}(;j) = \mathbf{V}(;j) + \mathbf{U}^*(;j)$. Using that $\chi = 1$ on $\Omega(\frac{\varepsilon}{2})$, $\varphi < \varepsilon_1$ on $Q \setminus Q(\frac{\varepsilon}{2})$ and $\varphi < \Phi$ on Q from these inequalities, from (3.8), (3.5), and (1.10) we yield

$$
\int_{\Omega(\frac{\varepsilon}{2})} |\partial_t^\beta\mathbf{u}(0;j)|^2 e^{2\tau\varphi(,0)} \le C\Big(\int_Q |\mathbf{F}|^2 e^{2\tau\varphi} + C(\varepsilon)e^{2\tau\varepsilon_1} + \tau e^{2\tau\Phi}F^2\Big)
\tag{3.12}
$$

for $\beta = 2, 3$ and $j = 1, 2, 3$. Using that $\chi = 1$ on $\Omega(\frac{\varepsilon}{2})$, from (3.11) and (3.12) we obtain

$$
\int_{\Omega(\frac{\varepsilon}{2})} |\mathbf{F}|^2 e^{2\tau\varphi(,0)} \le C\Big(\int_{Q(\frac{\varepsilon}{2})} |\mathbf{F}|^2 e^{2\tau\varphi} + \tau e^{2\tau\Phi}F^2 + C(\varepsilon)e^{2\tau\varepsilon_1} + \int_{\Omega(\frac{\varepsilon}{2})} (|f_2|^2 + |f_3|^2)e^{2\tau\varphi(,0)}\Big)
\tag{3.13}
$$

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where we also split Q in the right side of (3.12) into $Q(\frac{\varepsilon}{2})$ and its complement, and used that $|\mathbf{F}| \leq C$ and $\varphi < \varepsilon_1$ on the complement.

To eliminate the first integral in the right side of (3.13) we observe that

$$\int_{Q(\frac{\varepsilon}{2})} |\mathbf{F}|^2(x) e^{2\tau\varphi(x,t)} dx dt \leq \int_{\Omega(\frac{\varepsilon}{2})} |\mathbf{F}|^2(x) e^{2\tau\varphi(x,0)} \Big(\int_{-T}^{T} e^{2\tau(\varphi(x,t)-\varphi(x,0))} dt\Big) dx.$$

Due to our choice of function φ we have $\varphi(x,t) - \varphi(x,0) < 0$ when $t \neq 0$. Hence by the Lebesgue Theorem the inner integral (with respect to t) converges to 0 as τ goes to infinity. By reasons of continuity of φ, this convergence is uniform with respect to $x \in \Omega$. Choosing $\tau > C$ we therefore can absorb the integral over $Q(\frac{\varepsilon}{2})$ in the right side of (3.13) by the left side and arrive at the inequality

$$\int_{\Omega(\frac{\varepsilon}{2})} |\mathbf{F}|^2 e^{2\tau\varphi(,0)} \leq C(\tau e^{2\tau\Phi} F^2 + C(\varepsilon) e^{2\tau\varepsilon_1} + \int_{\Omega(\frac{\varepsilon}{2})} (|f_2|^2 + |f_3|^2) e^{2\tau\varphi(,0)}). \qquad (3.14)$$

On the other hand, to eliminate the last integral on the right side of (3.14), we use Lemma 2.3 with the condition (1.15) to get

$$\int_{\Omega(\frac{\varepsilon}{2})} (|f_2|^2 + |f_3|^2) e^{2\tau\varphi(,0)} \leq \frac{C}{\tau} \int_{\Omega(\frac{\varepsilon}{2})} (|\nabla f_2|^2 + |\nabla f_3|^2) e^{2\tau\varphi(,0)}. \qquad (3.15)$$

Using (3.15) with large τ and the inequality $\tau \leq e^\tau$, we absorb the last integral in the right side of (3.14) into the left side and obtain

$$\int_{\Omega(\frac{\varepsilon}{2})} |\mathbf{F}|^2 e^{2\tau\varphi(,0)} \leq C(e^{2\tau(\Phi_1+1)} F^2 + C(\varepsilon) e^{2\tau\varepsilon_1}).$$

Letting $\varepsilon_2 = e^{\frac{\eta\varepsilon}{2}} \leq \varphi$ on $\Omega(\varepsilon)$ and dividing the both parts by $e^{2\tau\varepsilon_2}$ we yield

$$\int_{\Omega(\varepsilon)} |\mathbf{F}|^2 \leq C(\tau e^{2\tau(\Phi+1-\varepsilon_2)} F^2 + e^{-2\tau(\varepsilon_2-\varepsilon_1)}) \leq C(\varepsilon)(e^{2\tau(\Phi+1)} F^2 + e^{-2\tau(\varepsilon_2-\varepsilon_1)}) \qquad (3.16)$$

since $\tau e^{-2\tau\varepsilon_2} < C(\varepsilon)$. To prove (1.16) it suffices to assume that $F < \frac{1}{C}$. Then $\tau = \frac{-logF}{\Phi+1+\varepsilon_2-\varepsilon_1} > C$ and we can use this τ in (3.16). Due to the choice of τ,

$$e^{-2\tau(\varepsilon_2-\varepsilon_1)} = e^{2\tau(\Phi+1)} F^2 = F^{2\frac{\varepsilon_2-\varepsilon_1}{\Phi+1+\varepsilon_2-\varepsilon_1}}$$

and from (3.16) we obtain (1.16) with $\gamma = \frac{\varepsilon_2-\varepsilon_1}{\Phi+1+\varepsilon_2-\varepsilon_1}$. The proof of Theorem 1.1 is now complete. $\qquad \square$

4 Lipschitz stability for the determination of coefficients

In this section we will prove Theorem 1.3. The key ingredient is the following Lipschitz stability estimate for the Cauchy problem for the system $\mathbf{A}_E \mathbf{u} = \mathbf{f}$.

Theorem 4.1. *Suppose that Ω and T satisfy the assumptions of Theorem 1.3. Let $\mathbf{u} \in (H^3(Q))^3$ solve the Cauchy problem*

$$\begin{cases} \mathbf{A}_E \mathbf{u} = \mathbf{f} & in \quad Q \\ \mathbf{u} = \sigma_\nu(\mathbf{u}) = 0 \ on \ \partial\Omega \times (-T, T) \end{cases} \tag{4.1}$$

with $\mathbf{f} \in L^2((-T, T); H^1(\Omega))$ and $\mathbf{f} = 0$ on $\partial\Omega \times (-T, T)$. Furthermore, assume that (1.7) holds for sufficiently small ε_0.

Then there exists a constant $C > 0$ such that

$$\|\mathbf{u}\|^2_{H^1(Q)} + \|\mathrm{div}\mathbf{u}\|^2_{H^1(Q)} + \|\mathrm{curl}\mathbf{u}\|^2_{H^1(Q)} \leq C\|\mathbf{f}\|^2_{L^2((-T,T);H^1(\Omega))}. \tag{4.2}$$

This estimate was proved in [10].

By virtue of (4.2) and an equivalence of the norms $\|\mathbf{u}\|_{(1)}(\Omega)$ and of

$$\|\mathrm{div}\mathbf{u}\|_{(0)}(\Omega) + \|\mathrm{curl}\mathbf{u}\|_{(0)}(\Omega) + \|\mathbf{u}\|_{(0)}(\Omega)$$

in $H_0^1(\Omega)$ (e.g., [3], pp.358-359), it is not hard to derive the following

Corollary 4.2. *Under conditions of Theorem 4.1*

$$\|\mathbf{u}\|_{(0)}(Q) + \|\nabla_{x,t}\mathbf{u}\|_{(0)}(Q) + \|\partial_t\nabla\mathbf{u}\|_{(0)}(Q) \leq C\|\mathbf{f}\|_{L^2((-T,T);H^1(\Omega))}. \tag{4.3}$$

Now we are ready to prove Theorem 1.3. We will use the notations in Section 3. Recall that

$$\mathbf{A}_{E_2}\mathbf{U}(;1;j) = \mathcal{A}(\mathbf{U}(;1;j))\mathbf{F}$$

where

$$\begin{aligned} \mathcal{A}(\mathbf{U}(;1;j))\mathbf{F} = \ & -f_1\partial_t^2\mathbf{U}(;1;j) + (f_2 + f_3)\nabla(\mathrm{div}\mathbf{U}(;1;j)) + f_3\Delta\mathbf{U}(;1;j) \\ & +\mathrm{div}\mathbf{U}(;1;j)(f_4, f_5, f_6)^\top + 2\epsilon(\mathbf{U}(;1;j))(f_7, f_8, f_9)^\top \\ & + \sum_{j,k=1}^3 r_{jk}\partial_j\partial_k\mathbf{U}(;1;j). \end{aligned}$$

So, from (1.19) we have

$$\mathbf{A}_{E_2}\mathbf{U}(;j) = 0 \quad \text{on} \quad \partial\Omega \times (-T, T). \tag{4.4}$$

Furthermore, in view of Lemma 2.2, there exists $\mathbf{U}^*(;j) \in H^3(Q)$ such that

$$\mathbf{U}^*(;j) = \mathbf{U}(;j), \ \sigma(\mathbf{U}^*(;j))\nu = \sigma(\mathbf{U}(;j))\nu, \ \mathbf{A}_{E_2}\mathbf{U}^*(;j) = 0 \quad \text{on} \quad \partial\Omega \times (-T, T), \tag{4.5}$$

and

$$\|\mathbf{U}^*(;j)\|_{(3)}(Q) \leq C(\|\mathbf{U}(;j)\|_{(\frac{5}{2})}(\partial\Omega \times (-T, T)) + \|\sigma(\mathbf{U})(;j)\nu\|_{(\frac{3}{2})}(\partial\Omega \times (-T, T))) \leq CF \tag{4.6}$$

due to the definition of F. As before, we set $\mathbf{V}(;j) = \mathbf{U}(;j) - \mathbf{U}^*(;j)$. Due to (4.4) and (4.5), we get

$$\mathbf{V}(;j) = \sigma(\mathbf{V})(;j)\nu = 0, \ \mathbf{A}_{E_2}\mathbf{V}(;j) = 0 \text{ on } \partial\Omega \times (-T, T). \tag{4.7}$$

With (4.7), applying Corollary 4.2 to (3.6), (3.7) and using (4.6) gives

$$\|\mathbf{V}(;j)\|_{(0)}^2(Q) + \|\nabla_{x,t}\mathbf{V}(;j)\|_{(0)}^2(Q) + \|\partial_t\nabla\mathbf{V}(;j)\|_{(0)}^2(Q) \le C(\|\mathbf{F}\|_{(1)}(\Omega)^2 + F^2) \tag{4.8}$$

for $j = 1, 2, 3$.

On the other hand, as in the proof of Theorem 1.1 we will bound the right side of (4.8) by $\mathbf{V}$. To use the Carleman estimate (2.1) we need to cut off $\mathbf{V}(;j)$ near $t = T$ and $t = -T$. We first observe that from the definition

$$1 \le \varphi(x, 0), x \in \Omega,$$

and from the condition (1.18)

$$\varphi(x, T) = \varphi(x, -T) < 1 \quad \text{when} \quad x \in \overline{\Omega}.$$

So there exists a $\delta > \frac{1}{C}$ such that

$$1 - \delta < \varphi \quad \text{on} \quad \Omega \times (0, \delta), \quad \varphi < 1 - 2\delta \quad \text{on} \quad \Omega \times (T - 2\delta, T). \tag{4.9}$$

We now choose a smooth cut-off function $0 \le \chi_0(t) \le 1$ such that $\chi_0(t) = 1$ for $-T + 2\delta < t < T - 2\delta$ and $\chi(t) = 0$ for $|t| > T - \delta$. Using (4.7) and according to [12], Lemma A.1, $\chi_0\mathbf{V}(;j) \in H_0^3(Q)$. Using the Leibniz' formula

$$\mathbf{A}_{E_2}(\chi_0\mathbf{V}(;j)) = \chi_0\mathcal{A}(\mathbf{U}(;1;j))\mathbf{F} - \chi_0\mathbf{A}_{E_2}\mathbf{U}^*(;j) + 2\rho_2(\partial_t\chi_0)\partial_t\mathbf{V}(;j) + \rho_2(\partial_t^2\chi_0)\mathbf{V}(;j)$$

and Carleman estimate (2.1), we yield

$$\int_Q \chi_0^2(\tau^3|\mathbf{V}(;j)|^2 + \tau|\nabla\mathbf{V}(;j)|^2)e^{2\tau\varphi} \le$$

$$C(\int_Q (|\mathbf{F}|^2 + |\nabla\mathbf{F}|^2 + |\mathbf{A}_{E_2}\mathbf{U}^*(;j)|^2 + |\nabla(\mathbf{A}_{E_2}\mathbf{U}^*)(;j)|^2)e^{2\tau\varphi} +$$

$$\int_{\Omega \times \{T - 2\delta < |t| < T\}} (|\mathbf{V}(;j)|^2 + |\nabla_{x,t}\mathbf{V}(;j)|^2 + |\partial_t\nabla\mathbf{V}(;j)|^2)e^{2\tau\varphi}) \le$$

$$C(\int_Q (|\mathbf{F}|^2 + |\nabla\mathbf{F}|^2)e^{2\tau\varphi} + e^{2\tau\Phi}F^2 + e^{2\tau(1-2\delta)}\int_\Omega (|\mathbf{F}|^2 + |\nabla\mathbf{F}|^2)),$$

where we let $\Phi = \sup_Q \varphi$ and used (4.6), (4.8), (4.9). Since $\mathbf{U}(;j) = \mathbf{V}(;j) + \mathbf{U}^*(;j)$ from (4.6) we obtain

$$\int_Q \chi_0^2(\tau^3|\mathbf{U}(;j)|^2 + \tau|\nabla\mathbf{U}(;j)|^2)e^{2\tau\varphi} \le$$

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$$C(\tau^3 e^{2\tau\Phi}F^2 + \int_\Omega (\int_{-T}^T e^{2\tau\varphi(x,t)}dt + e^{2\tau(1-2\delta)})(|\mathbf{F}|^2 + |\nabla\mathbf{F}|^2)(x))dx. \qquad (4.10)$$

Utilizing (3.2) and (1.12), similarly to deriving (3.11), we get from (3.9) that

$$|\mathbf{F}|^2 + |\nabla\mathbf{F}|^2 \le C(\sum_{j=1}^3 \sum_{\beta=2,3;k=0,1} |\partial_t^\beta \nabla^k \mathbf{u}(0;j)|^2 + \sum_{k=0,1}(|\nabla^k f_2|^2 + |\nabla^k f_3|^2)). \quad (4.11)$$

Therefore, by (4.11) and using Corollary 2.4 (with conditions (1.19) for Lamé coefficients), we have

$$\int_\Omega (|\mathbf{F}|^2 + |\nabla\mathbf{F}|^2)e^{2\tau\varphi(,0)}$$

$$\le C(\int_\Omega \sum_{j=1}^3 \sum_{\beta=2,3;k=0,1} |\partial_t^\beta \nabla^k \mathbf{u}(0;j)|^2 e^{2\tau\varphi(,0)} + \int_\Omega \sum_{k=0,1}(|\nabla^k f_2|^2 + |\nabla^k f_3|^2)e^{2\tau\varphi(,0)}) \le$$

$$-C\int_0^T \partial_t(\int_\Omega \sum_{j=1}^3 \sum_{\beta=2,3;k=0,1} \chi_0^2 |\partial_t^\beta \nabla^k \mathbf{u}(;j)|^2 e^{2\tau\varphi}dx)dt + \frac{C}{\tau}\int_\Omega(|\mathbf{F}|^2 + |\nabla\mathbf{F}|^2)e^{2\tau\varphi(;0)}.$$

Choosing τ large we eliminate the last term and obtain

$$\int_\Omega(|\mathbf{F}|^2 + |\nabla\mathbf{F}|^2)e^{2\tau\varphi(,0)} \le$$

$$C\int_Q \chi_0^2 \sum_{j=1}^3 \sum_{\beta=2,3;k=0,1}(|\partial_t^\beta \nabla^k \mathbf{u}(;j)||\partial_t^{\beta+1}\nabla^k \mathbf{u}(;j)| + \tau|\partial_t\varphi||\partial_t^\beta \nabla^k \mathbf{u}(;j)|^2)e^{2\tau\varphi}+$$

$$C\int_{\Omega\times(T-2\delta,T)} \chi_0|\partial_t\chi_0|\sum_{j=1}^3 \sum_{\beta=2,3;k=0,1} |\partial_t^\beta \nabla^k \mathbf{u}(;j)|^2 e^{2\tau\varphi}.$$

Now as in the proofs of section 3 the right side is less than

$$C(\int_Q \tau\chi_0^2(|\mathbf{U}(;j)|^2 + |\nabla\mathbf{U}(;j)|^2)e^{2\tau\varphi} + \int_{\Omega\times(T-2\delta,T)}(|\mathbf{U}(;j)|^2 + |\nabla\mathbf{U}(;j)|^2)e^{2\tau\varphi}) \le$$

$$C(\int_Q \tau\chi_0^2(|\mathbf{U}(;j)|^2 + |\nabla\mathbf{U}(;j)|^2)e^{2\tau\varphi} + e^{2\tau(1-2\delta)}(\|\mathbf{F}\|_{(1)}^2(\Omega) + F^2)).$$

where we used equality $\mathbf{U}(;j) = \mathbf{U}^*(;j) + \mathbf{V}(;j)$ and (4.6), (4.8). From the two previous bounds and (4.10) we conclude that

$$\int_\Omega(|\mathbf{F}|^2 + |\nabla\mathbf{F}|^2)e^{2\tau\varphi(,0)} \le C(\tau^3 e^{2\tau\Phi}F^2 + \int_\Omega(\int_{-T}^T e^{2\tau\varphi(,t)}dt + e^{2\tau(1-2\delta)})(|\mathbf{F}|^2 + |\nabla\mathbf{F}|^2)).$$

$$(4.12)$$

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Due to our choice of φ, $1 \leq \varphi(,0), \varphi(,t) - \varphi(,0) < 0$ when $t \neq 0$. Thus by the Lebesgue Theorem as in the proofs of section 3, we have

$$2C(\int_{-T}^{T} e^{2\tau\varphi(,t)}dt + e^{2\tau(1-\delta)}) \leq e^{2\tau\varphi(,0)}$$

uniformly on Ω when $\tau > C$. Hence choosing and fixing such large τ we eliminate the second term on the right side of (4.12). The proof of Theorem 1.3 is now complete. $\square$

5 Conclusion

While natural in some applications, assumption about smallness of residual stress is restrictive. In our opinion it can be relaxed by using methods of papers [7], [10], and of this paper. More restrictive and much more difficult to remove is the condition that the initial data are not zero. At present, even for scalar isotropic hyperbolic equations global uniqueness of speed of propagation or of potential from few lateral boundary measurement is an open outstanding research problem (see, for example, [8]). Also of substantial interest is uniqueness in inverse problems for more general anisotropic systems, for example, for dynamical elasticity system with transversal isotropy. For such systems there are no Carleman estimates or uniqueness of the continuation results. On the other hand, they are quite important for applications to geophysics, material science, and medicine, and they are notorious mathematical challenges.

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ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001   FAX +81-3-5465-7012