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A quarkonial decomposition of Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces

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Abstract

The aim of this paper is to define the Besov-Morrey spaces and the Triebel-Lizorkin-Morrey spaces and to present a decomposition of functions belonging to these spaces. Our results contain an answer to the conjecture proposed by Mazzucato.

Keywords Besov, Triebel-Lizorkin space, Morrey space, decomposition of functions

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Contents

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1 Introduction

The aim of this paper is to define the Besov-Morrey spaces and the Triebel-Lizorkin-Morrey spaces and to present a decomposition of these function spaces. Our results contain an answer to the conjecture proposed by Mazzucato in [6].

Let us recall briefly the definition of the Besov spaces and the Triebel-Lizorkin spaces. Following the notation in [12], we write $\eta(D)f = \mathcal{F}^{-1}(\eta \cdot \mathcal{F}f)$ for $\eta \in \mathcal{S}$ and $f \in \mathcal{S}'$. Here and below we set

$$\mathcal{F}f(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\,dx,\ \mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(\xi)e^{ix\cdot\xi}\,d\xi$$

for definiteness. Let $0 < p, q \le \infty$. Given a sequence of measurable functions $\{f_j\}_{j=0}^{\infty}$, we define

$$\|f_j \,:\, l_q(L_p)\| := \left(\sum_{j=0}^{\infty} \|f_j \,:\, L_p\|^q\right)^{\frac{1}{q}},\ \|f_j \,:\, L_p(l_q)\| := \left\|\left(\sum_{j=0}^{\infty} |f_j|^q\right)^{\frac{1}{q}} \,:\, L_p\right\|. \quad (1)$$

In order to deal with the sum with the parameter running through all the integers $0, 1, 2, \ldots$, we define $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$. Let $\phi_0, \phi_1 \in \mathcal{S}$ satisfy

$$\chi_{B(2)} \le \phi_0 \le \chi_{B(4)},\ \chi_{B(4)\setminus B(2)} \le \phi_1 \le \chi_{B(8)\setminus B(1)}, \quad (2)$$

where χ_A denotes the indicator function of the set A and $B(r)$ is the ball centered at the origin of radius r : Below we denote by $B(x, r)$ the ball centered at x of radius r. We set $\phi_j(x) = \phi_1(2^{-j+1}\cdot)$ for $j \ge 2$. Let the parameters p, q, s satisfy

$$0 < p < \infty,\ 0 < q \le \infty,\ s \in \mathbb{R}.$$

Then the Besov norm and the Triebel-Lizorkin norm are defined by

$$\|f \,:\, B^s_{pq}\| := \|2^{js}\phi_j(D)f \,:\, l_q(L_p)\|$$

and

$$\|f \,:\, F^s_{pq}\| := \|2^{js}\phi_j(D)f \,:\, L_p(l_q)\|$$

respectively. Here the condition on $\{\phi_j\}_{j\in\mathbb{N}_0}$ is loosened or transformed as long as it forms the Littlewood-Paley patch. The Besov-Morrey spaces and the Triebel-Lizorkin-Morrey spaces are function spaces whose norms are obtained by replacing the L_p-norms with the Morrey norms. Let $0 < u \le p < \infty$. The Morrey norm is given by

$$\|f \,:\, \mathcal{M}^p_u\| = \sup_{x\in\mathbb{R}^n,\ r>0} r^{\frac{n}{p}-\frac{n}{u}} \left(\int_{B(x,r)} |f(y)|^u \, dy \right)^{\frac{1}{u}},$$

where $B(x,r)$ is a ball centered at x of radius $r > 0$. Motivated by the Lebesgue differential theorem, we define $\mathcal{M}^\infty_u = L^\infty$ if $0 < u \le \infty$. The Morrey spaces are functions spaces which are larger than the L_p spaces in the sense that $\mathcal{M}^p_u \supset L_p$ for $0 < u \le p < \infty$. In [8] the Morrey spaces are originally used to investigate the smoothness of functions in terms of the regularity of their gradient.

Motivated by (1) we define

$$\|f_j \,:\, l_q(\mathcal{M}^p_u)\| := \left(\sum_{j=0}^{\infty} \|f_j \,:\, \mathcal{M}^p_u\|^q \right)^{\frac{1}{q}}, \ \|f_j \,:\, \mathcal{M}^p_u(l_q)\| := \left\| \left(\sum_{j=0}^{\infty} |f_j|^q \right)^{\frac{1}{q}} \,:\, \mathcal{M}^p_u \right\|$$

for $0 < u \le p \le \infty$ and $0 < q \le \infty$.

Let $0 < u \le p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. In [10] T. Lin and J. Xu defined function spaces normed by

$$\|f \,:\, \mathcal{N}^s_{pqu}\| := \|2^{js}\phi_j(D)f \,:\, l_q(\mathcal{M}^p_u)\|, \ \|f \,:\, \mathcal{E}^s_{pqu}\| := \|2^{js}\phi_j(D)f \,:\, \mathcal{M}^p_u(l_q)\|$$

for $f \in \mathcal{S}'$. We note that the spaces $\mathcal{N}^s_{pqu}$ with $1 < u \le p < \infty$, $1 < q \le \infty$ and $s \in \mathbb{R}$ was defined originally by H. Kozono and M. Yamazaki in [5]. In [10] Lin and Xu extended the range of parameters p, q, u to $0 < u \le p < \infty$ and $0 < q \le \infty$. Note that $\mathcal{N}^s_{pqu}$ and $\mathcal{E}^s_{pqu}$ are not Banach spaces, if p, q or u is less than 1. However, the attempt has been made to extend naturally the function spaces with underlying parameters less than 1. For example the modulation space $M^{p,q}$ with $0 < p, q \le \infty$ was considered recently in [4].

Before we go into the detail we fix some more notations. By "cube " we mean a cube whose edges are parallel to the coordinate axis and denote by $\mathcal{Q}$ the totality of the cubes in $\mathbb{R}^n$. $Q(r)$ means a cube given by

$$Q(r) := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \,:\, \max(|x_1|, |x_2|, \ldots, |x_n|) \le r\},$$

while $B(r)$ is a ball centered at the origin of radius r. Let $r > 0$ and $x \in \mathbb{R}^n$. We also set

$$Q(x,r) := \{y \in \mathbb{R}^n \,:\, x - y \in Q(r)\}.$$

The spaces $F^s_{pq}, B^s_{pq}, \mathcal{N}^s_{pqu}$ and $\mathcal{E}^s_{pqu}$ are called non-homogeneous, while the homogeneous spaces are defined as follows: Let $\phi \in \mathcal{S}$ be taken so that

$$\chi_{B(4)\setminus B(2)} \le \phi \le \chi_{B(8)\setminus B(1)}.$$

For $j \in \mathbb{Z}$ we set $\phi_j = \phi(2^{-j}\cdot)$. Denote by $\mathcal{P}$ the set of all polynomials, which form a linear subspace of $\mathcal{S}'$. For $f \in \mathcal{S}'/\mathcal{P}$, the space modulo $\mathcal{P}$, the definition

$$\phi_j(D)f = \mathcal{F}^{-1}(\phi_j \cdot \mathcal{F}f)$$

makes sense: The definition is independent of the choice of the representative $f \in \mathcal{S}'/\mathcal{P}$. Thus, we can define the norms by

$$\begin{aligned}
\|f : \dot{B}^s_{pq}\| &:= \|\{2^{js}\phi_j(D)f\}_{j\in\mathbb{Z}} : l_q(L_p)\| \\
\|f : \dot{\mathcal{N}}^s_{pqu}\| &:= \|\{2^{js}\phi_j(D)f\}_{j\in\mathbb{Z}} : l_q(\mathcal{M}^p_u)\| \\
\|f : \dot{F}^s_{pq}\| &:= \|\{2^{js}\phi_j(D)f\}_{j\in\mathbb{Z}} : L_p(l_q)\| \\
\|f : \dot{\mathcal{E}}^s_{pqu}\| &:= \|\{2^{js}\phi_j(D)f\}_{j\in\mathbb{Z}} : \mathcal{M}^p_u(l_q)\|.
\end{aligned}$$

for $f \in \mathcal{S}'/\mathcal{P}$. Here the parameters p, q, u, s are the same as the corresponding function space of non-homogeneous type. Both homogeneous function spaces and non-homogeneous function spaces are widely applied to the partial-differential equations. For example, Kozono and Yamazaki applied the non-homogeneous Besov-Morrey spaces to the Navier-Stokes equation [5]. The homogeneous Besov space is applied to the wave equation in [1]. Pseudo differential operators on the Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces are investigated in [7, 10].

In this paper we present the decomposition of the elements in these function spaces. In [6] Mazzucato conjectured that any element in $\mathcal{M}^p_u$ admits a decomposition into atom functions. To describe her conjecture, we recall the definitions of molecules and atoms.

Definition 1.1 (Molecule). A C^K-function m is called an (s,p)-molecule, $s \in \mathbb{R}$, $1 \le p \le \infty$, if it satisfies the following oscillation and decay conditions hold for some point $x_0 \in \mathbb{R}^n$ and $\nu \in \mathbb{Z}$, where M is a constant sufficiently large:

1. $\displaystyle\int_{\mathbb{R}^n} x^\alpha a(x)\,dx = 0$ for $|\alpha| \le L$.
2. $|\partial^\alpha m(x)| \le 2^{\nu(s-n/p+|\alpha|)}\langle 2^\nu(x-x_0)\rangle^{-M-|\alpha|}$ if $|\alpha| \le K$.

Compared with the molecules, the atom functions are more localized than the molecules.

Definition 1.2 (Atom). A C^K-function a is called an (s,p)-atom, if the following support, smoothness and the cancellation conditions are satisfied for some cube $Q \in \mathcal{Q}$:

1. $\mathrm{supp}(a) \subset 3Q$.
2. $\displaystyle\int_{\mathbb{R}^n} x^\alpha a(x)\,dx = 0$ for $|\alpha| \le L$.

3. $\|\partial^\alpha a : L_\infty\| \le \ell(Q)^{s-\frac{n}{p}-|\alpha|}$ if $|\alpha| \le K$.

In [6] she conjectured that the Morrey functions can be decomposed into atoms and molecules.

Conjecture 1.3. [6, p292] *Suppose that* $1 < u \le p < \infty$, $K \in \mathbb{N}$ *and* $L \in \mathbb{N}_0$.

(A) $f \in \mathcal{M}^p_u$ *admits the following decompositions.*

(a) $f = \sum_{Q:dyadic} s_Q m_Q$, *where* m_Q *are* $(0,p)$*-molecules.*

(b) $f = \sum_{Q:dyadic} s_Q a_Q$, *where* a_Q *are* $(0,p)$*-atoms.*

The coefficients s_Q *can be taken so that*

$$\|f : \mathcal{M}^p_u\| \simeq \left\| \left(\sum_Q |s_Q|^2 \chi_Q^{(p)2} \right)^{\frac{1}{2}} : \mathcal{M}^p_u \right\|. \tag{3}$$

Here we set $\chi_Q^{(p)} = |Q|^{-\frac{1}{p}} \chi_Q$, *the p-normalized indicator.*

(B) *Conversely if* (3) *is satisfied, then*

$$\sum_{Q:dyadic} s_Q a_Q, \quad \sum_{Q:dyadic} s_Q m_Q \in \mathcal{M}^p_u,$$

where m_Q *are* $(0,p)$*-molecules and* a_Q *are* $(0,p)$*-atoms.*

This type of decomposition dates back to the work of Uchiyama for BMO in [15]. In [2, 3] the decomposition was made for the functions in the Besov spaces and the Triebel-Lizorkin spaces. In [6] she gave this type of decomposition of $\mathcal{N}^s_{pqu}$ with $1 \le u \le p < \infty$, $1 \le q \le \infty$ and $s \in \mathbb{R}$. She proposed Conjecture 1.3 in analogy with her result.

One of the aim of this paper is to answer her conjecture. Furthermore we will present another nice decomposition of functions. In [14] Triebel presented a new decomposition method called quarkonial decomposition. Compared with the results in [2, 3, 15], which was her motivation to Conjecture 1.3, the quarkonial decomposition enjoys a good property. While the coefficients do not depend linearly in the method in [2, 3], the quarkonial decomposition gives us the linear dependency of the coefficients. In [6] the function spaces $\mathcal{M}^p_u$ are covered only with $1 \le u \le p < \infty$. In this present paper we will form the decompositions for the parameters $0 < u \le p \le \infty$.

Throughout this paper we will denote by M the Hardy-Littlewood maximal operator of uncentered type, which is given by

$$Mf(x) := \sup_{x \in Q \in \mathcal{Q}} \frac{1}{|Q|} \int_Q |f(y)| \, dy$$

for a measurable function f. Here and below we denote by $m_A(f) = \dfrac{1}{|A|}\displaystyle\int_A f(x)\,dx$, the average of $f \in L_{1\,loc}$ over the measurable set A. It is also convenient to introduce its powered version. For $\eta > 0$ we define

$$M^{(\eta)} f(x) := \sup_{x \in Q \in \mathcal{Q}} m_Q(|f|^\eta)^{\frac{1}{\eta}}$$

for a measurable function f. It is trivial that $\|Mf \,:\, L_\infty\| \le c\,\|f \,:\, L_\infty\|$. The following maximal inequality, which is so-called the Fefferman-Stein vector-valued maximal inequality, will be a key to our later considerations.

Theorem 1.4. [9, Theorem 2.2], [10, Lemma 2.5] *Suppose that the parameters p, q, u satisfy*

$$1 < u \le p < \infty, \ 1 < q \le \infty.$$

Then there is a constant $C > 0$ such that

$$\|Mf_j \,:\, \mathcal{M}^p_u(l_q)\| \le C\,\|f_j \,:\, \mathcal{M}^p_u(l_q)\| \tag{4}$$

for every sequence of measurable functions $\{f_j\}_{j=0}^\infty$.

It is convenient to transform Theorem 1.4 to the following powered version.

Corollary 1.5. *Suppose that the parameters p, q, u, η satisfy*

$$0 < \eta < u \le p < \infty, \ \eta < q \le \infty.$$

Then there is a constant $C > 0$ such that

$$\left\|M^{(\eta)} f_j \,:\, \mathcal{M}^p_u(l_q)\right\| \le C\,\|f_j \,:\, \mathcal{M}^p_u(l_q)\| \tag{5}$$

for every sequence of measurable functions $\{f_j\}_{j=0}^\infty$.

Finally we describe the organization of this paper. In Section 2 we make a brief look at the theory of band-limited Schwartz distributions. f is said to be band-limited, if its Fourier transform is supported on a compact set. In Section 3 we present the definition and investigate its validness. We also collect some elementary properties that is needed for later sections. We follow the line in [10] in Section 2 and Section 3 and supply some proofs if necessary. In Section 4 we deal with the atomic decompositions. First we obtain an equivalent norm and then by using this equivalent norm we obtain an atomic decomposition. In Section 5 we form the quarkonial decomposition for the elements in $\mathcal{N}^s_{pqu}$ and $\mathcal{E}^s_{pqu}$. Although a similar proof to the Besov spaces and the Triebel-Lizorkin spaces works for many propositions, we supply complete proofs in order to verify that our theory works well for the Morrey spaces. Finally in Section 6 we answer Conjecture 1.3. What is needed for solving the conjecture will have been completed by the end of this section. In Section 6 we make a brief sketch of the homogeneous spaces and then we answer Conjecture 1.3.

2 Entire analytic functions

In this section we recall the properties of band-limited distributions. In particular we are concerned with distributions f whose Fourier transform is contained in a compact set. Here and below c denotes the constants that may change from one occurrence to another.

Definition 2.1. Given a bounded subset $A \subset \mathbb{R}^n$, we define

$$\mathcal{S}'_A := \left\{ f \in \mathcal{S}' \,:\, \mathrm{supp}(\mathcal{F}f) \subset \bar{A} \right\},$$

where $\bar{A}$ denotes the closure of A. We also define $(\mathcal{M}^p_u)_A := \mathcal{S}'_A \cap \mathcal{M}^p_u$ for $0 < u \le p < \infty$.

Observe that if A is a bounded set, then, as it will turn out below, $\mathcal{S}'_A$ consists of regular distributions whose Fourier transform are supported in $\bar{A}$. Thus, $(\mathcal{M}^p_u)_A$ is made up of the regular distributions f such that $\mathrm{supp}(\mathcal{F}f) \subset \bar{A}$ and $f \in \mathcal{M}^p_u$.

Theorem 2.2. [11, Theorem 1.3.1, Section 1.4.1] *Let $f \in (\mathcal{M}^p_u)_{B(1)}$. Then for all $\eta > 0$, there exists $c > 0$ such that there holds*

$$\sup_{y \in \mathbb{R}^n} \frac{|f(x-y)|}{1 + |y|^{\frac{n}{\eta}}} \le c\, M^{(\eta)} f(x)$$

for every $x \in \mathbb{R}^n$.

From this theorem, we obtain that $(\mathcal{M}^p_u)_{B(1)}$ is embedded into L_∞.

Corollary 2.3. *$(\mathcal{M}^p_u)_{B(1)}$ is continuously embedded into L_∞ whenever $0 < u \le p < \infty$. That is, there exists $c > 0$ such that*

$$\|f \,:\, L_\infty\| \le c\, \|f \,:\, \mathcal{M}^p_u\| \tag{6}$$

for every $f \in (\mathcal{M}^p_u)_{B(1)}$.

Proof. Let B be an arbitrary ball of radius 1. Take an auxiliary η so that it is slightly less than $\min(1, u)$. Then

$$\sup_{x \in B} |f(x)| \le c \inf_{z \in B} M^{(\eta)} f(z) \le c\, \|M^{(\eta)} f \,:\, \mathcal{M}^p_u\| \le c\, \|f \,:\, \mathcal{M}^p_u\|.$$

This is the desired result. ∎

Later we use (6) after scaling. Let $R > 0$. Then

$$\|f \,:\, L_\infty\| \le c\, R^{\frac{n}{p}} \|f \,:\, \mathcal{M}^p_u\| \tag{7}$$

for $f \in (\mathcal{M}^p_u)_{B(R)}$. Later this inequality will turn out to be useful in obtaining smoothness information.

Next we turn to the multiplier theorem used frequently in this paper. Let $s > 0$. Recall that H_2^s is a function space consisting of $f \in L_2$ with

$$\|f \,:\, H_2^s\| := \|\langle\cdot\rangle^s \mathcal{F} f \,:\, L_2\| < \infty.$$

Theorem 2.4. [10, Theorem 2.7] *Suppose that the parameters p, q, u, σ satisfy*

$$0 < u \le p \le \infty,\ 0 < q \le \infty,\ \sigma > 0$$

and that

$$h, h_1, h_2, \ldots \in H_2^\sigma,\ R, R_1, R_2, \ldots > 0.$$

1. *Let $\sigma > \dfrac{1}{\eta} + \dfrac{n}{2}$ with $\eta > 0$. For all $f \in (\mathcal{M}^p_u)_{B(R)}$ the integral*

$$h(D)f(x) := (2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n} \mathcal{F}^{-1} h(x-y) f(y)\, dy$$

   converges for almost every $x \in \mathbb{R}^n$ and it satisfies the estimate

$$\int_{\mathbb{R}^n} |\mathcal{F}^{-1} h(x-y) f(y)|\, dy \le c\, \|h(R\cdot) \,:\, H_2^\sigma\| \cdot M^{(\eta)} f(x). \tag{8}$$

2. *Suppose $\sigma > \dfrac{n}{\min(u,1)} + \dfrac{n}{2}$. Let $f \in (\mathcal{M}^p_u)_{B(R)}$. Then we have*

$$\|h(D)f \,:\, \mathcal{M}^p_u\| \le c\, \|h(R\cdot) \,:\, H_2^\sigma\| \cdot \|f \,:\, \mathcal{M}^p_u\|. \tag{9}$$

3. *Suppose that $0 < p < \infty$ and $\sigma > \dfrac{n}{\min(1,q,u)} + \dfrac{n}{2}$. Let $\{f_j\}_{j=0}^\infty \subset \mathcal{M}^p_u$ with $f_j \in (\mathcal{M}^p_u)_{B(R_j)}$ for each $j \in \mathbb{N}_0$. Then we have*

$$\|h_j(D) f_j \,:\, \mathcal{M}^p_u(l^q)\| \le c \left(\sup_{k \in \mathbb{N}_0} \|h_k(R_k\cdot) \,:\, H_2^\sigma\| \right) \cdot \|f_j \,:\, \mathcal{M}^p_u(l^q)\|. \tag{10}$$

Here the constant $c > 0$ does not depend on $R, R_1, R_2, \ldots$, $h, h_1, h_2, \ldots$, nor $f, f_1, f_2, \ldots$ appearing in (8), (9) *and* (10).

Outline of the proof. (8) can be obtained from Theorem 2.2 and the Plancherel theorem. Once (8) is established, *2* and *3* are proved with (5) and (8). ∎

3 Besov-Morrey and Triebel-Lizorkin-Morrey spaces

In this section we define the function spaces $\mathcal{N}^s_{pqu}, \mathcal{E}^s_{pqu}$ with $0 < u \le p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$ and then investigate their properties.

Recall that we have assumed in Section 1 that $\phi_0, \phi_1 \in \mathcal{S}$ satisfy (2). We set $\phi_j(x) = \phi_1(2^{-j+1}\cdot)$ for $j \ge 2$.

Definition 3.1. [10, Definition 2.3] Let $0 < u \le p < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Then for $f \in \mathcal{S}'$ we define

$$\begin{aligned} \|f \,:\, \mathcal{N}^s_{pqu}\| &:= \|2^{js}\phi_j(D)f \,:\, l_q(\mathcal{M}^p_u)\| \\ \|f \,:\, \mathcal{E}^s_{pqu}\| &:= \|2^{js}\phi_j(D)f \,:\, \mathcal{M}^p_u(l_q)\|. \end{aligned}$$

In order to unify the statement that follows, we denote by $\mathcal{A}^s_{pqu}$ either $\mathcal{N}^s_{pqu}$ or $\mathcal{E}^s_{pqu}$. The case when $p = \infty$ is admissible only for $\mathcal{A} = \mathcal{N}$. Below we always pose this condition on $\mathcal{A}^s_{pqu}$.

First we establish the validness of this definition.

Theorem 3.2. *Let $0 < u \le p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$.*

1. *The definition of the function space $\mathcal{A}^s_{pqu}$ does not depend on the choice of ϕ_0, ϕ_1 satisfying* (2). [10, Theorem 2.8]
2. *$\mathcal{S} \subset \mathcal{A}^s_{pqu} \subset \mathcal{S}'$ in the sense of continuous embedding.*
3. *$\mathcal{A}^s_{pqu}$ is complete in the following sense: Let $\{f_j\}_{j \in \mathbb{N}_0}$ be a sequence that satisfies*

$$\lim_{K \to \infty} \left(\sup_{l,m \ge K} \|f_l - f_m \,:\, \mathcal{A}^s_{pqu}\| \right) = 0.$$

Then there exists $f \in \mathcal{A}^s_{pqu}$ such that

$$\lim_{l \to \infty} \|f - f_l \,:\, \mathcal{A}^s_{pqu}\| = 0.$$

For the sake of convenience for readers we will supply the proof of *2* and *3*. We also note that *1* can be obtained by using Theorem 2.4. The proof of $\mathcal{S} \subset \mathcal{A}^s_{pqu}$ is not so hard, since $\mathcal{S} \subset B^s_{pq}$.

Thus, what remains to be proved is $\mathcal{A}^s_{pqu} \subset \mathcal{S}'$ and *3*. We postpone their proofs.

For the proof of Theorem 3.2 it is convenient to use smoothness information. The lift operator and the differential properties are investigated in [10, Theorem 2.15]. The assertion *2* and *3* in the next theorem will be helpful when we decompose functions into the sum of atoms.

Theorem 3.3. *Let $\sigma \in \mathbb{R}$ and $m \in \mathbb{N}$. Then*

$$\partial_j \,:\, \mathcal{A}^s_{pqu} \to \mathcal{A}^{s-1}_{pqu} \tag{11}$$

is a continuous mapping. Furthermore the following mappings are all isomorphisms.

1. *$(1 - \Delta)^\sigma \,:\, \mathcal{A}^s_{pqu} \to \mathcal{A}^{s-2\sigma}_{pqu}$.*
2. *$(1 + (-\Delta)^m) \,:\, \mathcal{A}^s_{pqu} \to \mathcal{A}^{s-2m}_{pqu}$.*

3. $(1+\partial_1{}^{4m}+\ldots+\partial_n{}^{4m}) : \mathcal{A}^s_{pqu} \to \mathcal{A}^{s-4m}_{pqu}$.

Proof. All assertions are proved easily by using Theorem 2.4. ∎

An immediate corollary of this theorem is the following.

Corollary 3.4. [10, Theorem 2.15] *Let $m \in \mathbb{N}$. Then there exists a constant $c > 0$ such that*

$$c^{-1} \|f : \mathcal{A}^{s+m}_{pqu}\| \le \|f : \mathcal{A}^s_{pqu}\| + \sum_{j=1}^n \|\partial_j{}^m f : \mathcal{A}^s_{pqu}\| \le c \|f : \mathcal{A}^{s+m}_{pqu}\|.$$

Proof. Both sides of inequalities are derived from Theorem 3.3. In particular the right inequality is easy.

For convenience for readers we will give a short proof of the left inequality. By using (11) and *3* in Theorem 3.3 we obtain

$$\begin{aligned} \|f : \mathcal{A}^{s+m}_{pqu}\| &\le c \|(1+\partial_1{}^{4m}+\ldots+\partial_n{}^{4m})f : \mathcal{A}^{s-3m}_{pqu}\| \\ &\le c \left(\|f : \mathcal{A}^{s-3m}_{pqu}\| + \sum_{j=1}^n \|\partial_j{}^{4m} f : \mathcal{A}^{s-3m}_{pqu}\| \right) \\ &\le c \left(\|f : \mathcal{A}^s_{pqu}\| + \sum_{j=1}^n \|\partial_j{}^m f : \mathcal{A}^s_{pqu}\| \right). \end{aligned}$$

Thus, the proof is now complete. ∎

We collect one more property of these function spaces.

Lemma 3.5. *Suppose that $\{f_k\}_{k \in \mathbb{N}_0}$ is a bounded sequence in $\mathcal{A}^s_{pqu}$. If the limit*

$$f = \lim_{k \to \infty} f_k$$

exists in $\mathcal{S}'$, then $f \in \mathcal{A}^s_{pqu}$ with

$$\|f : \mathcal{A}^s_{pqu}\| \le c \sup_{k \in \mathbb{N}_0} \|f_k : \mathcal{A}^s_{pqu}\|.$$

Proof. The proof is obtained directly from the Fatou property of $\mathcal{M}^p_u$. ∎

Next we collect some elementary inclusions.

Proposition 3.6. *Let the parameters $p, q, q_1, q_2, u, s, \epsilon$ satisfy*

$$0 < u \le p \le \infty,\ 0 < q, q_1, q_2 \le \infty,\ s \in \mathbb{R},\ \epsilon > 0.$$

Then we have

1. $\mathcal{N}^{s+\epsilon}_{pq_1u} \subset \mathcal{E}^s_{pq_2u}$ *and* $\mathcal{E}^{s+\epsilon}_{pq_1u} \subset \mathcal{N}^s_{pq_2u}$.
2. $\mathcal{A}^s_{pq_1u} \subset \mathcal{A}^s_{pq_2u}$, *if* $q_1 \le q_2$.
3. $\mathcal{E}^s_{pqu} \subset \mathcal{N}^s_{pq\max(p,q)}$.
4. $\mathcal{N}^s_{pqp} = B^s_{pq}$ *and* $\mathcal{E}^s_{pqp} = F^s_{pq}$.

Proof. The proofs are standard or can be obtained from the definitions. So we omit them. ∎

The proof of Theorem 3.2 *2* **and** *3* We return to the point left open in Theorem 3.2 *2* and *3*. The proof of *2* can be reduced to showing the following proposition, which will serve to measure the regularity of the distributions.

Proposition 3.7. *If* $s > \dfrac{n}{p}$, *then* $\mathcal{A}^s_{pqu} \subset L_\infty$ *in the sense of continuous embedding.*

Proof. To prove this proposition, by using the embedding $\mathcal{A}^s_{pqu} \subset \mathcal{N}^{s-\delta}_{pqu}$ with δ sufficiently small, we have only to prove $\mathcal{N}^s_{pqu} \subset L_\infty$. Since Theorem 3.2 *1* is proved, we can assume $\sum_{j=0}^{\infty} \phi_j(\cdot) \equiv 1$. By (7) we have

$$\|\phi_j(D)f \,:\, L_\infty\| \le c\, 2^{\frac{n}{p}} \|\phi_j(D)f \,:\, \mathcal{M}^p_u\|.$$

Consequently we obtain

$$\|f \,:\, L_\infty\| \le \sum_{j=0}^{\infty} \|\phi_j(D)f \,:\, L_\infty\| \le c\, \|f \,:\, \mathcal{N}^s_{pqu}\|.$$

This is the desired result. ∎

Remark 3.8. From the proof of this proposition we can even say that $\mathcal{A}^s_{pqu}$ is embedded into the space of bounded and uniformly continuous functions. Since $\sum_{j=0}^{J} \phi_j(D)f$ converges to f in L_∞ and $\sum_{j=0}^{J} \phi_j(D)f$ is bounded and uniformly continuous, so is f.

To prove $\mathcal{A}^s_{pqu} \subset \mathcal{S}'$, by Theorem 3.3 we may assume s is large enough. In this case Proposition 3.7 says more: $\mathcal{A}^s_{pqu}$ is continuously embedded into L_∞. Finally we shall prove *3*. Assume s is larger than n/p. Then $\{f_j\}_{j\in\mathbb{N}_0}$ converges to $f \in L_\infty$ by Proposition 3.7. By Lemma 3.5 we see the convergence also takes place in $\mathcal{A}^s_{pqu}$.

To conclude this section let us see the approach of Mazzucato with which to propose Conjecture 1.3. She tried to attack Conjecture 1.3 by using the following theorem.

Theorem 3.9. [6, Proposition 4.1] *Let $1 < u \le p < \infty$. Then $\mathcal{E}^0_{p2u} = \mathcal{M}^p_u$ with norm equivalence.*

Outline of the proof. Her observation is that the integral operator whose kernel is

$$K(x,y) = (\mathcal{F}^{-1}\phi_j(x-y))_{j \in \mathbb{N}_0}$$

is a Calderón-Zygmund operator from L_2 to $L_2(l_2)$. The kernel satisfies

$$\|K(x,y) \,:\, l_2\| \le c\,|x-y|^{-n}, \; \|\nabla_x K(x,y) \,:\, l_2\| + \|\nabla_y K(x,y) \,:\, l_2\| \le c\,|x-y|^{-n-1}.$$

As is well-known, the kernel satisfying this condition can be extended to the bounded operator from $\mathcal{M}^p_u$ to $\mathcal{M}^p_u(l_2)$ with $1 < u \le p < \infty$. She proved the theorem by using this singular integral operator. ∎

4 Atomic decomposition

A main goal of this section is to consider the atomic decomposition.

4.1 Local mean

Here we will obtain an equivalent norm. Take $\phi \in \mathcal{S}$ so that $\chi_{B(1)} \le \phi \le \chi_{B(2)}$. Then we define $k_0 = \phi$ and $k = \Delta^{2N}\phi$, where $N > 0$ is a large integer depending on the parameters p, q, u, s. Let $k_j(x) = 2^{jn}k(2^j x)$ for $j \in \mathbb{N}$.

This section is devoted to establishing the following norm equivalence.

Theorem 4.1. *Suppose that the parameters p, q, u, s satisfy*

$$0 < u \le p \le \infty, \; 0 < q \le \infty, \; s \in \mathbb{R}.$$

1. *There exists a constant $c > 0$ such that for every $f \in \mathcal{N}^s_{pqu}$*

$$c^{-1}\,\|f \,:\, \mathcal{N}^s_{pqu}\| \le \|2^{js}k_j * f \,:\, l_q(\mathcal{M}^p_u)\| \le c\,\|f \,:\, \mathcal{N}^s_{pqu}\|.$$

2. *Assume in addition that $p < \infty$. Then there exists a constant $c > 0$ such that for every $f \in \mathcal{E}^s_{pqu}$*

$$c^{-1}\,\|f \,:\, \mathcal{E}^s_{pqu}\| \le \|2^{js}k_j * f \,:\, \mathcal{M}^p_u(l_q)\| \le c\,\|f \,:\, \mathcal{E}^s_{pqu}\|.$$

Proof. We will prove *2* only (*1* being proved similarly). We take $r_0 > 0$ so that $B(r_0) \subset \{\mathcal{F}\phi \ne 0\}$. Let $\psi \in \mathcal{S}$ be $\chi_{B(r_0/4)} \le \psi \le \chi_{B(r_0/2)}$. We may assume $\phi_0, \phi_1, \ldots$ satisfy

$$\phi_0(x) = \psi(x), \; \phi_j(x) = \psi(2^{-j}x) - \psi(2^{-j+1}x), \; j \in \mathbb{N} \tag{12}$$

by virtue of Theorem 3.2.

Let $f \in \mathcal{E}^s_{pqu}$. Then f can be decomposed as $f = \sum_{j=0}^{\infty} \phi_j(D)f$ by virtue of (12). We insert this relation to $k_j * f$:

$$2^{js} k_j * f(x) = \sum_{l=0}^{\infty} 2^{js} k_j * \phi_l(D)f(x) = \sum_{l=0}^{\infty} 2^{j(n+s)} \int_{\mathbb{R}^n} k(2^j y)\phi_l(D)f(x-y)\,dy$$

for $j \in \mathbb{N}$ and

$$k_0 * f(x) = \sum_{l=0}^{\infty} \int_{\mathbb{R}^n} k_0(y)\phi_l(D)f(x-y)\,dy.$$

From this we are led to consider

$$2^{j(n+s)} \int_{\mathbb{R}^n} k(2^j y)\phi_l(D)f(x-y)\,dy$$

for $j \in \mathbb{N}$ and the formula corresponding to $j = 0$.

Let $j \geq l$ and $j \neq 0$. Recall that $k = \Delta^{2N}\phi$. Thus carrying out integration by parts, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} k(2^j y)\phi_l(D)f(x-y)\,dy &= 2^{-4jN} \int_{\mathbb{R}^n} \Delta_y^{2N}(\phi(2^j y))\phi_l(D)f(x-y)\,dy \\ &= 2^{-4jN} \int_{\mathbb{R}^n} (\phi(2^j y))\Delta_y^{2N}[\phi_l(D)f(x-y)]\,dy. \quad (13) \end{aligned}$$

We set $\tau_l(x) := |2^{-l}x|^{4N}\phi_l(x)$ for $l \in \mathbb{N}_0$. Then (13) can be rephrased as

$$\int_{\mathbb{R}^n} k(2^j y)\phi_l(D)f(x-y)\,dy = 2^{-4jN+4lN} \int_{\mathbb{R}^n} \phi(2^j y)\tau_l(D)f(x-y)\,dy.$$

By virtue of Theorem 2.2 we have

$$|\tau_l(D)f(x-y)| \leq c\,(1+|2^l y|)^{\frac{n}{\eta}} M^{(\eta)}[\tau_l(D)f](x)$$

with η slightly less than $\min(1,q,u)$. From this we obtain

$$|2^{js}k_j * \phi_l(D)f(x)| \leq c\,2^{j(s-4N+n)+4lN} M^{(\eta)}[\tau_l(D)f](x) \cdot \int_{\mathbb{R}^n} |\phi(2^j y)|(1+|2^l y|)^{\frac{n}{\eta}}\,dy. \quad (14)$$

Finally we estimate the above integral. A straight calculation gives

$$\int_{\mathbb{R}^n} |\phi(2^j y)|(1+|2^l y|)^{\frac{n}{\eta}}\,dy = 2^{-jn} \int_{\mathbb{R}^n} |\phi(y)|(1+|2^{l-j} y|)^{\frac{n}{\eta}}\,dy \leq c\,2^{-jn}. \quad (15)$$

Inserting (15) to (14), we obtain

$$|2^{js}k_j * \phi_l(D)f(x)| \leq c\,2^{-2\delta(j-l)} M^{(\eta)}[2^{ls}\tau_l(D)f](x), \quad (16)$$

provided N is large enough. Here δ is a constant larger than $\dfrac{n}{\min(1,q,u)} + \dfrac{n}{2}$.

Next, let $l \geq j$ with $l \neq 0$. Take a large integer $M \in \mathbb{N}$ fixed later. We define

$$\zeta_l(x) := \frac{1}{|2^{-l}x|^{4M}}\phi_l(x),\ l \geq 1.$$

From this definition we have $\phi_l(D)f(x) = 2^{-4lM}\Delta^{2M}\zeta_l(D)f(x)$.

Insert this formula to $k_j * \phi_l(D)f$ and carry out the integration by parts. Then we obtain

$$\begin{aligned} 2^{js}k_j * \phi_l(D)f(x) &= 2^{j(n+s)}\int_{\mathbb{R}^n} k(2^j y)\phi_l(D)f(x-y)\,dy \\ &= 2^{j(n+s)-4lM}\int_{\mathbb{R}^n} k(2^j y)\Delta^{2M}\zeta_l(D)f(x-y)\,dy \\ &= 2^{j(n+s+4M)-4lM}\int_{\mathbb{R}^n} \Delta^{2M}k(2^j y)\zeta_l(D)f(x-y)\,dy. \end{aligned}$$

Proceeding in the same way as (14) and (15) as before we obtain

$$|2^{js}k_j * \phi_l(D)f(x)| \leq c\,2^{(j-l)\left(s+4M-\frac{n}{\eta}\right)}M^{(\eta)}[2^{ls}\zeta_l(D)f](x).$$

If we choose M large enough, then for $l \geq 1$ there holds

$$|2^{js}k_j * \phi_l(D)f(x)| \leq c\,2^{-2\delta(l-j)}M^{(\eta)}[2^{ls}\zeta_l(D)f](x), \tag{17}$$

where $\delta > \dfrac{n}{\min(1,q,u)} + \dfrac{n}{2}$. In order to include the case when $j \geq l = 0$ we put $\zeta_0 := \phi_0$. Reexamine the argument we just gave to reduce (17). Then (17) is still available for $j \geq l = 0$: The rough estimate without using the integration by parts works.

Therefore (16) and (17) give us a key estimate:

$$|2^{js}k_j * \phi_l(D)f(x)| \leq c\,2^{-2\delta|j-l|}\left(M^{(\eta)}[2^{ls}\tau_l(D)f](x) + M^{(\eta)}[2^{ls}\zeta_l(D)f](x)\right) \tag{18}$$

with $\delta > \dfrac{n}{\min(1,q,u)} + \dfrac{n}{2}$ for every $j, l \in \mathbb{N}_0$.

By using (18) we obtain

$$\begin{aligned} &\left(\sum_{j=0}^{\infty} 2^{jsq}|k_j * f(x)|^q\right)^{\frac{1}{q}} \\ &\leq c\left(\sum_{j=0}^{\infty}\left(\sum_{l=0}^{\infty} 2^{-2\delta|j-l|}\left(M^{(\eta)}[2^{ls}\tau_l(D)f](x) + M^{(\eta)}[2^{ls}\zeta_l(D)f](x)\right)\right)^q\right)^{\frac{1}{q}}. \end{aligned}$$

If $q \leq 1$, then we use $\left(\sum_{l=0}^{\infty} a_l\right)^q \leq \sum_{l=0}^{\infty} {a_l}^q$ for any positive sequence $\{a_l\}_{l=0}^{\infty}$, which gives us

$$\left(\sum_{j=0}^{\infty}\left(\sum_{l=0}^{\infty}2^{-2\delta|j-l|}\left(M^{(\eta)}[2^{ls}\tau_l(D)f](x)+M^{(\eta)}[2^{ls}\zeta_l(D)f](x)\right)\right)^q\right)^{\frac{1}{q}}$$

$$\leq \left(\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}2^{-2\delta q|j-l|}\left(M^{(\eta)}[2^{ls}\tau_l(D)f](x)+M^{(\eta)}[2^{ls}\zeta_l(D)f](x)\right)^q\right)^{\frac{1}{q}}$$

$$\leq c\left(\sum_{l=0}^{\infty}M^{(\eta)}[2^{ls}\tau_l(D)f](x)^q+M^{(\eta)}[2^{ls}\zeta_l(D)f](x)^q\right)^{\frac{1}{q}}.$$

If $q>1$, then we use the Hölder inequality instead.

$$\left(\sum_{j=0}^{\infty}\left(\sum_{l=0}^{\infty}2^{-2\delta|j-l|}\left(M^{(\eta)}[2^{ls}\tau_l(D)f](x)+M^{(\eta)}[2^{ls}\zeta_l(D)f](x)\right)\right)^q\right)^{\frac{1}{q}}$$

$$\leq c\left(\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}2^{-\delta q|j-l|}\left(M^{(\eta)}[2^{ls}\tau_l(D)f](x)+M^{(\eta)}[2^{ls}\zeta_l(D)f](x)\right)^q\right)^{\frac{1}{q}}$$

$$\leq c\left(\sum_{l=0}^{\infty}M^{(\eta)}[2^{ls}\tau_l(D)f](x)^q+M^{(\eta)}[2^{ls}\zeta_l(D)f](x)^q\right)^{\frac{1}{q}}.$$

Thus, we obtain, whether q is less than 1 or not,

$$\left(\sum_{j=0}^{\infty}2^{jsq}|k_j*f(x)|^q\right)^{\frac{1}{q}}\leq c\left(\sum_{l=0}^{\infty}M^{(\eta)}[2^{ls}\tau_l(D)f](x)^q+M^{(\eta)}[2^{ls}\zeta_l(D)f](x)^q\right)^{\frac{1}{q}}. \quad (19)$$

Taking η slightly less than $\min(1,q,u)$ and using Theorem 2.4, we obtain

$$\|2^{js}k_j*f\ :\ \mathcal{M}^p_u(l_q)\|\leq c\,\|f\ :\ \mathcal{E}^s_{pqu}\|. \quad (20)$$

We shall prove the left inequality. To this end we recall $\{\phi_j\}_{j\in\mathbb{N}_0}$ satisfies (12). By this fact, we define

$$\psi_j:=\frac{\phi_j}{\mathcal{F}k_j},\ j\geq 0. \quad (21)$$

Note that the support of the numerators are contained in the interior of the support of the denominators. As a consequence the definition above makes sense. Then we have $\phi_j(D)f=\psi_j(D)k_j*f$ for $j\geq 0$. Let $K\in\mathbb{N}$ taken large enough. We factorize

$$\phi_j(D)f=\phi_0(2^{-K-j}D)\psi_j(D)k_j*f.$$

Let

$$\frac{n}{\min(p,q,1)}+\frac{n}{2}<\sigma<\delta$$

be an auxiliary constant. Using this factorization and Theorem 2.4, we obtain

$$\|2^{js}\phi_j(D)f \,:\, L_p\| \le c_K \,\|2^{js}\phi_0(2^{-K-j}D)k_j * f \,:\, L_p(l_q)\|,$$

where c_K is given by

$$c_K = c\left(\sup_{k\in\mathbb{N}} \|\psi_k(2^{k+K}\cdot) \,:\, H_2^\sigma\| + \|\psi_0(2^K\cdot) \,:\, H_2^\sigma\|\right) \le c\,2^{\sigma K}. \qquad (22)$$

Here and below c_K denotes a constant dependent on K satisfying $c_K \le c\,2^{\sigma K}$ as in (22). Furthermore we decompose

$$\begin{aligned}
&\|2^{js}\phi_j(D)f \,:\, L_p\| \\
&\le\ c_K\,\|2^{js}\phi_0(2^{-K-j}D)k_j * f \,:\, L_p(l_q)\| \\
&\le\ c_K\left(\|2^{js}k_j * f \,:\, L_p(l_q)\| + \|2^{js}(1-\phi_0(2^{-K-j}D))k_j * f \,:\, L_p(l_q)\|\right).
\end{aligned}$$

Observe that

$$(1-\phi_0(2^{-K-j}D))k_j * f = k_j * (1-\phi_0(2^{-K-j}D))f.$$

Now we use (16) with f replaced by $(1-\phi_0(2^{-K-j}D))f$. Then we obtain

$$\begin{aligned}
&|2^{js}\phi_l(D)(k_j * (1-\phi_0(2^{-K-j}D)f))(x)| \\
&\le\ c\,2^{-2\delta(l-j)}M^{(\eta)}[2^{ls}\zeta_l(D)(1-\phi_0(2^{-K-j}D))f](x)
\end{aligned} \qquad (23)$$

for $\delta > \dfrac{n}{\min(1,q,u)} + \dfrac{n}{2}$. From (23) we can deduce

$$\begin{aligned}
&\left(\sum_{j=0}^\infty \left|2^{js}(1-\phi_0(2^{-K-j}D))k_j * f(x)\right|^q\right)^{\frac{1}{q}} \\
&\le\ c\left(\sum_{j=0}^\infty\left(\sum_{l=0}^\infty 2^{-2\delta(l-j)}M^{(\eta)}[2^{ls}(1-\phi_0(2^{-K-j}D))\zeta_l(D)f](x)\right)^q\right)^{\frac{1}{q}} \\
&\le\ c\left(\sum_{j=0}^\infty\sum_{l=0}^\infty 2^{-\delta q(l-j)}M^{(\eta)}[2^{ls}(1-\phi_0(2^{-K-j}D))\zeta_l(D)f](x)^q\right)^{\frac{1}{q}}
\end{aligned}$$

in the same way as (19). The Fefferman-Stein vector-valued maximal inequality (5) then gives

$$\begin{aligned}
&\left\|\left(\sum_{j=0}^\infty \left|2^{js}(1-\phi_0(2^{-K-j}D))k_j * f\right|^q\right)^{\frac{1}{q}} \,:\, \mathcal{M}_u^p\right\| \\
&\le\ c\left\|\left(\sum_{j=0}^\infty\sum_{l=0}^\infty 2^{-\delta q(l-j)}M^{(\eta)}[2^{ls}(1-\phi_0(2^{-K-j}D))\zeta_l(D)f]^q\right)^{\frac{1}{q}} \,:\, \mathcal{M}_u^p\right\| \\
&\le\ c\left\|\left(\sum_{l=0}^\infty\sum_{j=0}^\infty 2^{-\delta q(l-j)}\left|2^{ls}(1-\phi_0(2^{-K-j}D))\zeta_l(D)f\right|^q\right)^{\frac{1}{q}} \,:\, \mathcal{M}_u^p\right\|.
\end{aligned}$$

Note that from the support condition the term with $j \geq l - K + 5$ is zero. We divide the last term into two parts:

$$\left\| \left(\sum_{l=0}^{\infty} \sum_{j=0}^{\infty} 2^{-\delta q(l-j)} \left| 2^{ls}(1 - \phi_0(2^{-K-j}D))\zeta_l(D)f \right|^q \right)^{\frac{1}{q}} : \mathcal{M}_u^p \right\|$$

$$\leq \quad c \left\| \left(\sum_{\substack{j,l \in \mathbb{N}_0 \\ |j-l+K| \leq 4}} 2^{-\delta q(l-j)} \left| 2^{ls}(1 - \phi_0(2^{-K-j}D))\zeta_l(D)f \right|^q \right)^{\frac{1}{q}} : \mathcal{M}_u^p \right\|$$

$$+ \quad c \left\| \left(\sum_{\substack{j,l \in \mathbb{N}_0 \\ j-l+K < -4}} 2^{-\delta q(l-j)} \left| 2^{ls}(1 - \phi_0(2^{-K-j}D))\zeta_l(D)f \right|^q \right)^{\frac{1}{q}} : \mathcal{M}_u^p \right\|.$$

Now we invoke the fact that $(1-\phi_0(2^{-K-j}D))\zeta_l(D)f = \zeta_l(D)f$, whenever $j-k+l < -4$:

$$\left\| \left(\sum_{l=0}^{\infty} \sum_{j=0}^{\infty} 2^{-\delta q(l-j)} \left| 2^{ls}(1 - \phi_0(2^{-K-j}D))\zeta_l(D)f \right|^q \right)^{\frac{1}{q}} : \mathcal{M}_u^p \right\|$$

$$\leq \quad c \left\| \left(\sum_{\substack{j,l \in \mathbb{N}_0 \\ |j-l+K| \leq 4}} 2^{-\delta q(l-j)} \left| 2^{ls}(1 - \phi_0(2^{-K-j}D))\zeta_l(D)f \right|^q \right)^{\frac{1}{q}} : \mathcal{M}_u^p \right\|$$

$$+ \quad c \left\| \left(\sum_{\substack{j,l \in \mathbb{N}_0 \\ j-l+K < -4}} 2^{-\delta q(l-j)} \left| 2^{ls}\zeta_l(D)f \right|^q \right)^{\frac{1}{q}} : \mathcal{M}_u^p \right\|.$$

Furthermore Theorem 2.4 allows us to delate $(1 - \phi_0(2^{-K-j}D))$ in the first term of the most right side. The result is

$$\left\| \left(\sum_{j=0}^{\infty} \left| \sum_{l=0}^{\infty} 2^{js}(1 - \phi_0(2^{-K-j}D))k_j * f \right|^q \right)^{\frac{1}{q}} : \mathcal{M}_u^p \right\|$$

$$\leq \quad c \left\| \left(\sum_{l=0}^{\infty} 2^{-\delta K} \left| 2^{ls}\zeta_l(D)f(x) \right|^q \right)^{\frac{1}{q}} + \left(\sum_{\substack{j,l \in \mathbb{N}_0 \\ j-l+K < -4}} 2^{-\delta q(l-j)} \left| 2^{ls}\zeta_l(D)f \right|^q \right)^{\frac{1}{q}} : \mathcal{M}_u^p \right\|$$

$$\leq \quad c\, 2^{-\delta K} \left\| \left(\sum_{l=0}^{\infty} \left| 2^{ls}\zeta_l(D)f \right|^q \right)^{\frac{1}{q}} : \mathcal{M}_u^p \right\| \leq c\, 2^{-\delta K} \| f : \mathcal{E}_{pqu}^s \|.$$

Inserting this formula, we obtain

$$\|2^{js}\phi_j(D)f \,:\, \mathcal{M}^p_u(l_q)\| \le c\,\|2^{js}k_j * f \,:\, \mathcal{M}^p_u(l_q)\| + c_K\, 2^{-\delta K}\|2^{js}\phi_j(D)f \,:\, \mathcal{M}^p_u(l_q)\|.$$

Recall that

$$c_K \le c\, 2^{\sigma K} \text{ with } \sigma < \delta$$

and $K > 0$ is still at our disposal. Furthermore we are assuming that $f \in \mathcal{E}^s_{pqu}$. So we are in the position of bringing the second term of the right side to the left side by taking K large enough. Thus, we finally obtain

$$\|2^{js}\phi_j(D)f \,:\, \mathcal{M}^p_u(l_q)\| \le c\,\|2^{js}k_j * f \,:\, \mathcal{M}^p_u(l_q)\|.$$

This, together with (20), is the result we wish to prove. ∎

4.2 Atomic decomposition

Now we will deal with the atomic decomposition.

Definition 4.2. 1. Let $\nu \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$. Then we define

$$Q_{\nu m} := \prod_{j=1}^{n} \left[\frac{m_j}{2^\nu}, \frac{m_j+1}{2^\nu}\right).$$

2. Let $0 < p \le \infty$, $\nu \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$. Then we define the p-normalized indicator $\chi^{(p)}_{\nu m}$ by

$$\chi^{(p)}_{\nu m} := 2^{n\nu/p}\chi_{Q_{\nu m}}.$$

3. Let $0 < u \le p \le \infty$, $0 < q \le \infty$. Then, given a doubly indexed complex sequence $\lambda = \{\lambda_{\nu m}\}_{\nu \in \mathbb{N}_0,\, m \in \mathbb{Z}^n}$, we define

$$\begin{aligned} \|\lambda \,:\, \mathbf{n}_{pqu}\| &:= \|\Lambda_\nu \,:\, l_q(\mathcal{M}^p_u)\| = \left\| \left\{ \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\chi^{(p)}_{\nu m} \right\}_{\nu\in\mathbb{N}_0} \,:\, l_q(\mathcal{M}^p_u) \right\| \\ \|\lambda \,:\, \mathbf{e}_{pqu}\| &:= \|\Lambda_\nu \,:\, \mathcal{M}^p_u(l_q)\| = \left\| \left\{ \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\chi^{(p)}_{\nu m} \right\}_{\nu\in\mathbb{N}_0} \,:\, \mathcal{M}^p_u(l_q) \right\|, \end{aligned}$$

where

$$\Lambda_\nu = \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\chi^{(p)}_{\nu m}.$$

As before to unify our results, we use $\mathbf{a}_{pqu}$ to denote $\mathbf{n}_{pqu}$ and $\mathbf{e}_{pqu}$. In denoting $\mathbf{e}_{pqu}$, we tacitly exclude the case when $p = \infty$.

Definition 4.3. Let $d > 1$ be a fixed number and $K, L \in \mathbb{Z}$ with $K \ge 0$ and $L \ge -1$.

1. Let $m \in \mathbb{Z}^n$. We say that $a \in C^K$ is an atom centered at Q_{0m}, if

$$\begin{array}{ll} (1) & \mathrm{supp}(a) \subset d\,Q_{0m}, \\ (2) & \|\partial^\alpha a\|_\infty \le 1 \text{ for all } \alpha \in {\mathbb{N}_0}^n \text{ with } |\alpha| \le K. \end{array}$$

2. Let $\nu \in \mathbb{N}$ and $m \in \mathbb{Z}^n$. We say that $a \in C^K$ is an atom centered at $Q_{\nu m}$, if

$$\begin{array}{ll} (1) & \mathrm{supp}(a) \subset d\,Q_{\nu m}, \\ (2) & \|\partial^\alpha a\|_\infty \le 2^{-\nu\left(s-\frac{n}{p}\right)+\nu|\alpha|} \text{ for all } \alpha \in {\mathbb{N}_0}^n \text{ with } |\alpha| \le K, \\ (3) & \displaystyle\int_{\mathbb{R}^n} x^\beta a(x)\,dx = 0 \text{ for all } \beta \in {\mathbb{N}_0}^n \text{ with } |\beta| \le L. \end{array}$$

If $L = -1$, then the moment condition (3) in the definition above is vacuous automatically.

First, let us see that the atom belongs to the function space $\mathcal{A}^s_{pqu}$ without obtaining the norm estimate. This fact will be needed because we are going to apply Theorem 4.1.

Lemma 4.4. *Let $a \in C^K$ be a compactly supported function with $K \ge (1+[s])_+$. Then $a \in \mathcal{N}^s_{pqu} \cap \mathcal{E}^s_{pqu}$.*

Proof. To prove this lemma, we have only to prove that $a \in B^s_{pq}$ by virtue of Proposition 3.6. It is easy to show that, for every $P \in \mathbb{N}$, there exists $c > 0$ (depending on a) with

$$|\phi_0(D)a(x)| \le c\,\langle x\rangle^{-P}.$$

Now we turn our attention to $\phi_j(D)a(x)$ with $j \ge 1$. We decompose $|\xi|^{4K}$:

$$|\xi|^{4K} = \sum_{|\alpha|=K} P_\alpha(\xi)\xi^\alpha,$$

where $P_\alpha(\xi)$ is a homogeneous polynomial of degree $3K$. We set

$$\tau^\alpha_j(\xi) := P_\alpha(2^{-j}\xi)|2^{-j}\xi|^{-4K}\phi_j(\xi).$$

With this decomposition we obtain

$$\phi_j(D)a(x) = 2^{-j|\alpha|}\partial^\alpha[\tau^\alpha_j(D)a](x) = 2^{-jK}\tau^\alpha_j(D)[\partial^\alpha a](x).$$

From this expression we see that, for every $P \in \mathbb{N}$, there exists a constant $c > 0$ such that the estimate

$$|\phi_j(D)a(x)| \le c\,2^{-jK}\langle x\rangle^{-P}$$

holds for every $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$. From this estimate we obtain

$$2^{js}\|\phi_j(D)a \,:\, L_p\| \le c\,2^{j(s-K)},\ j \in \mathbb{N}_0.$$

Since $s < K$, this inequality gives us that $a \in B^s_{pq}$. ∎

Finally we define

$$\sigma_p := n\left(\frac{1}{\min(1,p)} - 1\right),\ \sigma_{pq} := n\left(\frac{1}{\min(1,p,q)} - 1\right). \tag{24}$$

Lemma 4.5. *Suppose that the parameters p, q, u, s satisfy*

$$0 < u \le p \le \infty,\ 0 < q \le \infty,\ s \in \mathbb{R}.$$

Let $K, L \in \mathbb{Z}$ be an integer satisfying

$$K \ge (1 + [s])_+,\ L \ge \min(-1, [\sigma_p - s]).$$

For $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ we are given an atom $a_{\nu m}$ centered at $Q_{\nu m}$. Then for $\lambda = \{\lambda_{\nu m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathbf{a}_{pqu}$ the series

$$\lim_{P \to \infty} \sum_{\nu=0}^{P} \left(\sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m} \right)$$

converges in $\mathcal{S}'$.

Proof. For the proof of this lemma, we have only to prove

$$\lim_{P \to \infty} \sum_{\nu=1}^{P} \left(\sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m} \right)$$

converges in $\mathcal{S}'$, since $\sum_{m \in \mathbb{Z}^n} \lambda_{0m} a_{0m} \in L_\infty$. Let $\phi \in \mathcal{S}$. Then by virtue of the moment condition we have

$$\left\langle \sum_{\nu=1}^{P} \left(\sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m} \right), \phi \right\rangle = \sum_{\nu=1}^{P} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \int_{\mathbb{R}^n} a_{\nu m}(x) \cdot \phi_{\nu m}(x)\, dx,$$

where $\phi_{\nu m}$ is given by

$$\phi_{\nu m}(x) = \phi(x) - \left(\sum_{|\beta| \le L} \frac{\partial^\beta \phi(2^{-\nu} m)}{\beta!} (x - 2^{-\nu} m)^\beta \right).$$

By the mean value theorem we have

$$|\phi_{\nu m}(x)| \le c\, 2^{-\nu(L+1)} \left(\sup_{\substack{|\gamma| = L+1 \\ y \in d\, Q_{\nu m}}} |\partial^\gamma \phi(y)| \right)$$

for $x \in d\, Q_{\nu m}$. Thus, the pointwise estimate $|a_{\nu m}(x)| \le 2^{-\nu(s - n/p)},\ x \in \mathbb{R}^n$ yields

$$\langle x \rangle^N |a_{\nu m}(x) \phi_{\nu m}(x)| \le c\, 2^{-\nu\left(s - \frac{n}{p} + L + 1\right)} \left(\sup_{\substack{|\gamma| = L+1 \\ y \in d\, Q_{\nu m}}} \langle y \rangle^N |\partial^\gamma \phi(y)| \right) \chi_{d\, Q_{\nu m}}(x).$$

For the sake of simplicity we write

$$p(\phi) := \sup_{\substack{|\gamma|=L+1\\ y\in\mathbb{R}^n}} \langle y\rangle^N |\partial^\gamma \phi(y)|.$$

p is a continuous seminorm of $\mathcal{S}$. Adding the above estimate over $m \in \mathbb{Z}^n$, we obtain

$$\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m} a_{\nu m}(x)\phi_{\nu m}(x)| \le c\, 2^{-\nu\left(s-\frac{n}{p}+L+1\right)} p(\phi)\cdot \langle x\rangle^{-N} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|\chi_{d\,Q_{\nu m}}(x).$$

As a consequence we obtain

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} |\lambda_{\nu m} a_{\nu m}(x)\phi_{\nu m}(x)|\, dx\\
\le\;& c\, 2^{-\nu\left(s-\frac{n}{p}+L+1\right)} p(\phi) \int_{\mathbb{R}^n} \langle x\rangle^{-N} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|\chi_{d\,Q_{\nu m}}(x)\, dx\\
\le\;& c\, 2^{-\nu\left(s-\frac{n}{p}+L+1\right)} p(\phi) \sum_{k=0}^{\infty} 2^{-kN} \int_{Q(2^k)} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|\chi_{d\,Q_{\nu m}}(x)\, dx\\
=\;& c\, 2^{-\nu\left(s-\frac{n}{p}+n+L+1\right)} p(\phi) \sum_{k=0}^{\infty} 2^{-kN} \int_{Q(2^k)} \left|\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\chi^{(1)}_{Q_{\nu m}}(x)\right| dx.
\end{aligned}$$

Notice that

$$\begin{aligned}
&\int_{Q(2^k)} \left|\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\chi^{(1)}_{Q_{\nu m}}(x)\right| dx\\
\le\;& \left(\int_{Q(2^k)} \left|\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\chi^{(\min(1,u))}_{Q_{\nu m}}(x)\right|^{\min(1,u)} dx\right)^{\frac{1}{\min(1,u)}}\\
\le\;& c\, 2^{\frac{kn}{\min(1,u)}} \left(\frac{1}{2^{kn}}\int_{Q(2^k)} \left|\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\chi^{(u)}_{Q_{\nu m}}(x)\right|^{u} dx\right)^{\frac{1}{u}}\\
\le\;& c\, 2^{kn\left(\frac{1}{\min(1,u)}-\frac{1}{p}\right)} \left\|\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m} \,:\, \mathcal{M}^p_u\right\|.
\end{aligned}$$

Recall that N is at our disposal. Thus, we finally obtain

$$\sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} |\lambda_{\nu m} a_{\nu m}(x)\phi_{\nu m}(x)|\, dx \le c\, 2^{-\nu\left(s-\frac{n}{p}+n+L+1\right)} p(\phi) \left\|\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m} \,:\, \mathcal{M}^p_u\right\|.$$

Now by assumption L is sufficiently large:

$$s-\frac{n}{p}+n+L+1 > s-\frac{n}{p}+n+\sigma_p - s \ge 0.$$

Thus, we are in the position of adding these inequalities over $\nu \in \mathbb{N}$:

$$\sum_{\nu=1}^{\infty} \left|\int_{\mathbb{R}^n} \left(\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}(x)\right)\phi(x)\, dx\right| \le c\, p(\phi)\, \|\lambda \,:\, \mathbf{a}_{pqu}\|. \tag{25}$$

This proves

$$\lim_{P\to\infty}\sum_{\nu=0}^{P}\left(\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}a_{\nu m}\right)$$

exists in $\mathcal{S}'$. ∎

4.3 The norm estimate of the sum of atoms

In this section we shall take up the norm estimate of the atomic decomposition.

Theorem 4.6. *Assume that the parameters p, q, u, s satisfy*

$$0 < u \le p \le \infty,\ 0 < q \le \infty,\ s \in \mathbb{R}.$$

Let K, L be integers with

$$K \ge (1+[s])_+,\ L \ge \max(-1, [\sigma_q - s])$$

for $\mathcal{N}$-scale and

$$K \ge (1+[s])_+,\ L \ge \max(-1, [\sigma_{qu} - s])$$

for $\mathcal{E}$-scale. Let $\lambda = \{\lambda_{\nu m}\}_{\nu\in\mathbb{N}_0,\ m\in\mathbb{Z}^n} \in \mathbf{a}_{pqu}$. Suppose that we are given an atom $a_{\nu m}$ centered at $Q_{\nu m}$.

$$\left\|\sum_{\nu\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}a_{\nu m}\ :\ \mathcal{A}^s_{pqu}\right\| \le c\,\|\lambda\ :\ \mathbf{a}_{pqu}\|.$$

Here c is a constant independent of λ.

Proof. First we remark that by Lemma 4.5 we may assume that the coefficients are zero with finite exception. In this case we have that $f \in \mathcal{A}^s_{pqu}$ by Lemma 4.4. Thus, to measure its norm we are in the position of using Theorem 4.1. Let $j \ge \nu \ge 0$ with $j \ne 0$. Then

$$k_j * a_{\nu m}(x) = 2^{jn}\int_{\mathbb{R}^n} k(2^j y)a_{\nu m}(x-y)\,dy.$$

We first calculate the size of the support of $k_j * a_{\nu m}$:

$$\operatorname{supp}(k_j * a_{\nu m}) \subset B(2^{-j+1}) + d\,Q_{\nu m} \subset c\,Q_{\nu m}.$$

We take a homogeneous polynomial $P_\alpha(\xi)$ of degree $3K$ with $|\xi|^{4N} = \sum_{|\alpha|=K}\xi^\alpha P_\alpha(\xi)$ as before. Then $k(2^j y) = [\Delta^{2N}\phi](2^j y) = 2^{-Kj}\sum_{|\alpha|=K}\partial_y^\alpha\left([P_\alpha(\partial)\phi](2^j y)\right)$. As a result we obtain

$$k_j * a_{\nu m}(x) = 2^{j(n-K)}\sum_{|\alpha|=K}(-1)^K\int_{\mathbb{R}^n} P_\alpha(\partial)\phi(2^j y)\partial^\alpha a_{\nu m}(x-y)\,dy.$$

Note each term of the above formula can be estimated by $c\,2^{-\nu\left(s-\frac{n}{p}\right)+\nu K}2^{-jn}$. As a consequence we obtain

$$|2^{js}k_j * a_{\nu m}(x)| \le c\,2^{-\nu s+\nu K-jK+js}\chi^{(p)}_{c\,Q_{\nu m}}(x) = c\,2^{-(j-\nu)(K-s)}\chi^{(p)}_{c\,Q_{\nu m}}(x). \tag{26}$$

Let $\nu > j \ge 0$. Then $\mathrm{supp}(k_j * a_{\nu m}) \subset B(2^{-j+1}) + d\,Q_{\nu m} \subset B(2^{-\nu}m, c\,2^{-j})$. Now we use the moment condition of $a_{\nu m}$. Then if $j \ne 0$, we have

$$k_j * a_{\nu m}(x) = 2^{jn}\int_{\mathbb{R}^n}\left(k(2^j y) - \sum_{|\beta|\le L}\frac{2^{j|\beta|}\partial^\beta k(2^j x)}{\beta!}(y-x)^\beta\right)a_{\nu m}(x-y)\,dy.$$

If $y \in Q(x-2^{-\nu}m, d\,2^{-\nu})$, then

$$\left|k(2^j y) - \sum_{|\beta|\le L}\frac{2^{j|\beta|}\partial^\beta k(2^j x)}{\beta!}(y-x)^\beta\right| \le c\,2^{-(\nu-j)(L+1)}.$$

If we insert this formula, then we obtain

$$|2^{js}k_j * a_{\nu m}(x)| \le c\,2^{-(\nu-j)(L+1+s+n)+j\nu/p}\chi_{c\,Q(2^{-\nu}m,2^{-j})} \tag{27}$$

for $\nu > j > 0$. If $\nu > j = 0$, the same argument for $j > 1$ works again without carrying out the integration by parts. In (26) and (27) we have treated the cases when $j, \nu \in \mathbb{N}_0$ and $j + \nu > 0$. However, we can easily incorporate the case when $\nu = j = 0$ to (27). We need only reexamine the argument above and mimic it. Thus, we have

$$\sum_{m\in\mathbb{Z}^n}|2^{js}\lambda_{\nu m}k_j * a_{\nu m}(x)| \le c\,2^{-(\nu-j)(L+1+s+n)+j\nu/p}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\chi_{c\,Q(2^{-\nu}m,2^{-j})}$$

for $j \in \mathbb{N}_0$.

Let η be a constant slightly smaller than $\min(1, q, u)$. For some constant $c > 0$ we have

$$\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\chi_{c\,Q(2^{-\nu}m,2^{-j})}(x) \le \sum_{\substack{m\in\mathbb{Z}^n\\ x\in B(2^{-\nu}m,c\,2^{-j})}}|\lambda_{\nu m}| \le \left(\sum_{\substack{m\in\mathbb{Z}^n\\ x\in B(2^{-\nu}m,c\,2^{-j})}}|\lambda_{\nu m}|^\eta\right)^{\frac{1}{\eta}}.$$

By using the maximal operator we obtain

$$\begin{aligned}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\chi_{c\,Q(2^{-\nu}m,2^{-j})} &\le c\,2^{\frac{\nu n}{\eta}}\left(\int_{B(x,c\,2^{-j})}\left|\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi_{Q_{\nu m}}(y)\right|^\eta dy\right)^{\frac{1}{\eta}}\\ &\le c\,2^{\frac{n(\nu-j)}{\eta}}m_{B(x,c\,2^{-j})}\left(\left|\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi_{Q_{\nu m}}\right|^\eta\right)^{\frac{1}{\eta}}\\ &\le c\,2^{\frac{n(\nu-j)}{\eta}}M^{(\eta)}\left(\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi_{Q_{\nu m}}\right)(x).\end{aligned}$$

Inserting this formula to (27) gives

$$\sum_{m\in\mathbb{Z}^n}|2^{js}\lambda_{\nu m}k_j * a_{\nu m}(x)| \le c\,2^{-(\nu-j)(L+1+s+n-n/\eta)}M^{(\eta)}\left(\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi^{(p)}_{Q_{\nu m}}\right)(x). \quad (28)$$

The same but simpler argument using the maximal operator works for (26) and the result is

$$\sum_{m\in\mathbb{Z}^n}|2^{js}\lambda_{\nu m}k_j * a_{\nu m}(x)| \le c\,2^{-(\nu-j)(K-s)}M^{(\eta)}\left(\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi^{(p)}_{Q_{\nu m}}\right)(x). \quad (29)$$

Since η is slightly smaller than $\min(1,q,u)$, we have $L+1+s+n-n/\eta, K-s>0$ by assumption. Thus, (28) and (29) can be unified and we obtain

$$\sum_{m\in\mathbb{Z}^n}|2^{js}\lambda_{\nu m}k_j * a_{\nu m}(x)| \le c\,2^{-2\tilde{\delta}|\nu-j|}M^{(\eta)}\left(\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi^{(p)}_{Q_{\nu m}}\right)(x), \quad (30)$$

where $\tilde{\delta}$ is a positive number. With (30) achieved, we have only to prove

$$\left\|\left\{\sum_{\nu\in\mathbb{N}_0}2^{-2\tilde{\delta}|\nu-j|}M^{(\eta)}\left(\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi^{(p)}_{Q_{\nu m}}\right)\right\}_{j\in\mathbb{N}_0} : \mathcal{M}^p_u(l_q)\right\| \le c\,\|\lambda : \mathbf{e}_{pqu}\|. \quad (31)$$

As before by the Hölder inequality we obtain

$$\begin{aligned}
&\left\{\sum_{j\in\mathbb{N}_0}\left[\sum_{\nu\in\mathbb{N}_0}2^{-2\tilde{\delta}|\nu-j|}M^{(\eta)}\left(\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi^{(p)}_{Q_{\nu m}}\right)\right]^q\right\}^{\frac{1}{q}}\\
&\le c\left\{\sum_{j\in\mathbb{N}_0}\sum_{\nu\in\mathbb{N}_0}2^{-\tilde{\delta}q|\nu-j|}M^{(\eta)}\left(\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi^{(p)}_{Q_{\nu m}}\right)(x)^q\right\}^{\frac{1}{q}}\\
&\le c\left\{\sum_{\nu\in\mathbb{N}_0}M^{(\eta)}\left(\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi^{(p)}_{Q_{\nu m}}\right)(x)^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

This inequality and the Fefferman-Stein vector-valued maximal inequality (5) yield (31). Thus, we have the desired result. ∎

4.4 Decomposition of distributions into the sum of atoms

Finally in this section we decompose distributions into the sum of atoms with suitable norm estimates. The next lemma gives a quantitative information on the coefficients.

Lemma 4.7. *Let $\kappa_0, \kappa_1 \in \mathcal{S}$ supported on $B(4)$ and $B(8) \setminus B(1)$ respectively. Set $\kappa_j(x) = \kappa_1(2^{-j+1}x)$ for $j \geq 2$. Then we have*

$$\left\| \left\{ 2^{k\left(s-\frac{n}{p}\right)} \sup_{y \in Q_{km}} |\kappa_k(D)f(y)| \right\}_{k \in \mathbb{N}_0,\ m \in \mathbb{Z}^n} : \mathbf{a}_{pqu} \right\| \leq c \, \|f \,:\, \mathcal{A}^s_{pqu}\|.$$

Proof. Note that from Theorem 2.2 we obtain

$$\begin{aligned} &\sum_{m \in \mathbb{Z}^n} 2^{k\left(s-\frac{n}{p}\right)} \left(\sup_{y \in Q_{km}} |\kappa_k(D)f(y)| \right) \chi^{(p)}_{km}(x) \\ &\quad \leq c \, 2^{ks} \sup_{y \in \mathbb{R}^n} \frac{|\kappa_k(D)f(x-y)|}{1+|2^k y|^{\frac{n}{\eta}}} \leq c \, M^{(\eta)}[\kappa_k(D)f](x), \end{aligned}$$

where $\eta > 0$ can be taken as small as we wish, say, less than $\min(1, q, u)$. Thus, the Fefferman-Stein vector-valued maximal inequality (5) and Theorem 2.4 prove Lemma 4.7. ∎

With these preparations in mind, we turn to prove the atomic decomposition.

Theorem 4.8. *Assume that the parameters p, q, u, s satisfy*

$$0 < u \leq p \leq \infty, \ 0 < q \leq \infty, \ s \in \mathbb{R}.$$

Let K, L be integers with

$$K \geq (1+[s])_+, \ L \geq \max(-1, [\sigma_q - s])$$

for $\mathcal{N}$-scale and

$$K \geq (1+[s])_+, \ L \geq \max(-1, [\sigma_{qu} - s])$$

for $\mathcal{E}$-scale. Let $f \in \mathcal{A}^s_{pqu}$. Then f can be decomposed as

$$f = \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m},$$

where $a_{\nu m}$ is an atom centered at $Q_{\nu m}$ and the coefficient $\lambda = \{\lambda_{\nu m}\}_{\nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n}$ satisfy

$$\|\lambda \,:\, \mathbf{a}_{pqu}\| \leq c \, \|f \,:\, \mathcal{A}^s_{pqu}\|.$$

Proof of Theorem 4.8 with $L = -1$. Let $f \in \mathcal{A}^s_{pqu}$. Assume $\{\phi_j\}_{j \in \mathbb{N}_0}$ is a family of functions such that

$$\phi_j(x) = \phi_0(2^{-j}x) - \phi_0(2^{-j+1}x) \text{ for } j \geq 1, \ \chi_{Q(1)} \leq \phi_0 \leq \chi_{Q(3/2)}.$$

We also take a smooth function κ such that $\mathrm{supp}(\kappa) \subset Q(1)$, $Q(2) \subset \{\mathcal{F}\kappa \neq 0\}$. For $j \in \mathbb{N}_0$ set $\kappa_j(x) = 2^{jn}\kappa(2^j x)$ and we define $\psi_j \in \mathcal{S}$ uniquely so that $\phi_j = (2\pi)^{-\frac{n}{2}} \mathcal{F}\psi_j \cdot \mathcal{F}\kappa_j$. Then

$$f = \sum_{j=0}^{\infty} \phi_j(D)f = \sum_{j=0}^{\infty} \psi_j * \kappa_j * f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \int_{Q_{jm}} \psi_j(x-y) \kappa_j * f(y) \, dy.$$

Set $\lambda_{jm} := 2^{j\left(s-\frac{n}{p}\right)} \sup_{y \in Q_{jm}} |\kappa_j * f(y)|$ and

$$a_{jm} := \begin{cases} 0 & \text{if } \lambda_{\nu m} = 0 \\ \frac{1}{\lambda_{\nu m}} \int_{Q_{jm}} \psi_j(x-y)\kappa_j * f(y)\, dy & \text{otherwise.} \end{cases}$$

Then $f = \sum_{j \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm}$ and a_{jm} is an atom centered at Q_{jm} modulo multiplicative constants. Lemma 4.7 yields

$$\begin{aligned} &\|\{\lambda_{jm}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \,:\, \mathbf{a}_{pqu}\| \\ &= \left\| \left\{ 2^{j\left(s-\frac{n}{p}\right)} \sup_{y \in Q_{jm}} |\kappa_j * f(y)| \right\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} \,:\, \mathbf{a}_{pqu} \right\| \le \|f \,:\, \mathcal{A}^s_{pqu}\|. \end{aligned}$$

This is the desired estimate of the coefficients. As a consequence, any $f \in \mathcal{A}^s_{pqu}$ is decomposed into the sum of atoms, unless we require the moment condition. ∎

Proof of Theorem 4.8 with $L \ge 0$. Let $M \in \mathbb{N}$ be a constant larger than $K+1$. Then $f \in \mathcal{A}^s_{pqu}$ can be decomposed as

$$f = g + (-\Delta)^M g, \ \|f \,:\, \mathcal{A}^s_{pqu}\| \simeq \|g \,:\, \mathcal{A}^{s+2M}_{pqu}\|$$

where $g \in \mathcal{A}^{s+2M}_{pqu}$. Observe that $g \in C^K$ by virtue of Theorem 3.3 and Proposition 3.6, provided M is large enough. Let $\psi \in \mathcal{S}$ be a compactly supported function with $\sum_{m \in \mathbb{Z}^n} \psi(x-m) \equiv 1$. Suppose that $\operatorname{supp}(\psi) \subset B(2^r)$. We decompose

$$g = \sum_{m \in \mathbb{Z}^n} \psi(\cdot - m) g.$$

We define coefficients λ_{0m} and functions a_{0m} by

$$\begin{aligned} \lambda_{0m} &:= \sup_{|\alpha| \le K} \|\partial^\alpha(\psi(\cdot - m)g) \,:\, L_\infty\| \\ a_{0m} &:= \begin{cases} 0 & \text{if } \psi(\cdot - m)g \equiv 0 \\ {\lambda_{0m}}^{-1}\psi(\cdot - m)g & \text{otherwise} \end{cases} \end{aligned}$$

for $m \in \mathbb{Z}^n$. If c_0 is a constant large enough, then $\dfrac{a_{0m}}{c_0}$ is an atom centered at $Q_{\nu m}$ and

$$g = \sum_{m \in \mathbb{Z}^n} \lambda_{0m} a_{0m}. \tag{32}$$

Next, we note that

$$\left\| \sum_{m \in \mathbb{Z}^n} \lambda_{0m} \chi^{(p)}_{0m} \,:\, \mathcal{M}^p_u \right\| \le c \sum_{|\alpha| \le K} \left\| \sup_{|\cdot - y| \le c} |D^\alpha g(y)| \,:\, \mathcal{M}^p_u \right\|. \tag{33}$$

We decompose

$$D^\alpha g(y) = \sum_{j \in \mathbb{N}_0} \phi_j(D)(D^\alpha g)(y)$$

and we use Theorem 2.2 to obtain a pointwise estimate; for all $x \in \mathbb{R}^n$

$$\begin{aligned} &\sup_{y \in B(x,c)} |D^\alpha g(y)| \\ &\quad \le \sum_{j \in \mathbb{N}_0} \sup_{y \in B(x,c)} |\phi_j(D)(D^\alpha g)(y)| \le c \sum_{j \in \mathbb{N}_0} 2^{\frac{jn}{\eta}} M^{(\eta)} \phi_j(D)(D^\alpha g)(x), \end{aligned}$$

where η is slightly less than $\min(1, q, u)$. Taking into account the triangle inequality

$$\left\| \sum_{j \in \mathbb{N}} |h_j| \,:\, \mathcal{M}^p_u \right\|^\eta \le \sum_{j \in \mathbb{N}} \|h_j \,:\, \mathcal{M}^p_u\|^\eta,$$

we obtain, with the help of Theorem 3.3 and (5),

$$\left\| \sup_{|\cdot - y| \le c} |D^\alpha g(y)| \,:\, \mathcal{M}^p_u \right\|^\eta \le c \sum_{j \in \mathbb{N}} 2^{jn} \|\phi_j(D)(D^\alpha g) \,:\, \mathcal{M}^p_u\|^\eta \le c \, \|g \,:\, \mathcal{A}^{2n/\eta + |\alpha|}_{pqu}\|. \quad (34)$$

As a result if M is large enough, another application of Theorem 3.3 and combination of (33) and (34) give us

$$\left\| \sum_{m \in \mathbb{Z}^n} \lambda_{0m} \chi^{(p)}_{0m} \,:\, \mathcal{M}^p_u \right\| \le c \, \|g \,:\, \mathcal{A}^{2n/\eta + |\alpha|}_{pqu}\| \le c \, \|f \,:\, \mathcal{A}^s_{pqu}\|.$$

Next, since $g \in \mathcal{A}^{s+2M}_{pqu}$ with $s + 2M > \sigma_{qu}$, we are in the position of applying the case when $L = -1$. Using the case when $L = -1$, we obtain

$$g = \sum_{\nu=1}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} b_{\nu m}. \quad (35)$$

Note the parameter ν here runs through $\mathbb{N}$, which is just a matter of scaling. Here the coefficients $\lambda = \{\lambda_{\nu m}\}_{\nu \in \mathbb{N},\, m \in \mathbb{Z}^n}$ and the function $b_{\nu m}$ satisfy the following conditions.

1. $\|\lambda \,:\, \mathbf{a}_{pqu}\| \le c \, \|g \,:\, \mathcal{A}^{s+2M}_{pqu}\| \le c \, \|f \,:\, \mathcal{A}^s_{pqu}\|$.
2. $\mathrm{supp}(b_{\nu m}) \subset d \, Q_{\nu m}$.
3. $\|\partial^\alpha b_{\nu m}\| \le 2^{-\nu\left(s + 2M - \frac{n}{p}\right) + |\alpha|\nu}$ for $|\alpha| \le K + 2M$.

Thus, if we set $a_{\nu m} := (-\Delta)^M b_{\nu m}$ and take $M \ge L$, then we see that $a_{\nu m}$ is an atom centered at $Q_{\nu m}$ with the desired cancellation condition and the expansion

$$f = \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m},$$

the convergence taking place in $\mathcal{A}^s_{pqu}$. ∎

5 Quarkonial decomposition

Finally in this paper we form the quarkonial decomposition of functions to give some kind of answer to Conjecture 1.3. To describe the quarkonial decomposition, we fix some notations.

Definition 5.1. Throughout this section $\psi \in \mathcal{S}$ is a fixed function satisfying

$$\sum_{m \in \mathbb{Z}^n} \psi(x-m) \equiv 1$$

for all $x \in \mathbb{R}^n$. Accordingly the number $r > 0$ is fixed so that

$$\mathrm{supp}(\psi) \subset B(2^r). \tag{36}$$

With ψ specified, we define the quark.

Definition 5.2. Let $\beta \in {\mathbb{N}_0}^n,\ \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n$ and $\rho > r$, where r is given by (36).

1. $\psi^\beta(x) := x^\beta \psi(x)$.
2. $(\beta qu)_{\nu m}(x) = 2^{-\nu\left(s-\frac{n}{p}\right)}\psi^\beta(2^\nu x - m)$, where p and s are parameters of the function space $\mathcal{A}^s_{pqu}$ under consideration.
3. Let the parameters p, q, u satisfy

   $$0 < u \le p < \infty,\ 0 < q \le \infty.$$

   Given a triply parameterized sequence $\lambda = \{\lambda^\beta_{\nu m}\}_{\beta \in {\mathbb{N}_0}^n,\ \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n} = \{\lambda^\beta\}_{\beta \in {\mathbb{N}_0}^n}$, we define

   $$\|\lambda \,:\, \mathbf{a}_{pqu}\|_\rho := \sup_{\beta \in {\mathbb{N}_0}^n} 2^{\rho|\beta|}\|\lambda^\beta \,:\, \mathbf{a}_{pqu}\|.$$

   Here we tacitly exclude the case when $p = \infty$ if we consider $\mathbf{e}_{pqu}$.

5.1 Quarkonial decomposition for regular case

In this section we consider the quarkonial decomposition for regular case. We assume

$$0 < u \le p \le \infty,\ 0 < q \le \infty,\ s > \sigma_q \tag{37}$$

for $\mathcal{N}$-scale and

$$0 < u \le p < \infty,\ 0 < q \le \infty,\ s > \sigma_{qu} \tag{38}$$

for $\mathcal{E}$-scale.

With this preparation in mind, we present our main theorem.

Theorem 5.3. *Suppose that the parameters p, q, u, s satisfy* (37) *for $\mathcal{N}$-scale and* (38) *for $\mathcal{E}$-scale. Let $f \in \mathcal{S}'$. Then $f \in \mathcal{A}^s_{pqu}$ if and only if there exists a triply indexed sequence $\lambda = \{\lambda^\beta_{\nu m}\}_{\beta \in \mathbb{N}_0{}^n,\ \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n}$ such that f can be expressed as*

$$f = \sum_{\beta \in \mathbb{N}_0{}^n} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{\nu m} (\beta q u)_{\nu m}$$

with

$$\|\lambda \,:\, \mathbf{a}_{pqu}\| < \infty. \tag{39}$$

If this is the case, then λ can be taken so that

$$\|\lambda \,:\, \mathbf{a}_{pqu}\| \simeq \|f \,:\, \mathcal{A}^s_{pqu}\|. \tag{40}$$

Before we come to the proof, we observe that the "if " part is proved easily. Indeed, let $0 < \epsilon < \rho - r$. Then $2^{-(r+\epsilon)|\beta|}(\beta q u)_{\nu m}$ is an atom centered at $Q_{\nu m}$ modulo multiplicative constants. Thus, letting

$$f^\beta := \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{\nu m} (\beta q u)_{\nu m},$$

we obtain from Theorem 4.6 that

$$\|f^\beta \,:\, \mathcal{A}^s_{pqu}\| \le c\, 2^{-(\rho - r - \epsilon)|\beta|} \|\lambda \,:\, \mathbf{a}_{pqu}\|_\rho.$$

Since we have

$$\|f_1 + f_2 \,:\, \mathcal{A}^s_{pqu}\|^{\min(q,u)} \le \|f_1 \,:\, \mathcal{A}^s_{pqu}\|^{\min(q,u)} + \|f_2 \,:\, \mathcal{A}^s_{pqu}\|^{\min(q,u)}$$

for $f_1, f_2 \in \mathcal{A}^s_{pqu}$, we obtain $f \in \mathcal{A}^s_{pqu}$ with the norm estimate (39).

Thus, we devote ourselves to the proof of "only if" part, the possibility of the decomposition of given $f \in \mathcal{A}^s_{pqu}$.

First, we prove a lemma to decompose the functions.

Lemma 5.4. *Let $f \in \mathcal{S}'$ with* $\operatorname{supp}(\mathcal{F}f) \subset Q(3R), R > 0$. *Then*

$$f = \sum_{m \in \mathbb{Z}^n} f\left(\frac{m}{R}\right) \mathcal{F}^{-1}\kappa(R \cdot - m),$$

where κ is an auxiliary bump function with $\chi_{Q(3)} \le \kappa \le \chi_{Q(3+1/100)}$.

Proof. Set $\kappa_R(\cdot) = \kappa(R^{-1}\cdot)$. First we take $\tau \in \mathcal{S}$ arbitrarily. Then the support condition on f gives us

$$\langle \mathcal{F}f, \tau \rangle = \left\langle \mathcal{F}f, \kappa_{\frac{101R}{100}} \cdot \kappa_R \cdot \tau \right\rangle. \tag{41}$$

We consider

$$\tau^*(x) := \sum_{l \in \mathbb{Z}^n} \kappa_R(x - 2\pi R l)\tau(x - 2\pi R l),$$

which is $2\pi R$-periodic. Expand τ^* to the Fourier series. Then we obtain

$$\tau^*(x) = \sum_{m \in \mathbb{Z}^n} a_m \exp\left(\frac{(* \cdot m)i}{R}\right), \tag{42}$$

where the coefficient is given by

$$\begin{aligned} a_m &= \frac{1}{(2\pi R)^n} \int_{Q(\pi R)} \tau^*(x) \exp\left(-\frac{(x \cdot m)i}{R}\right) dx \\ &= \frac{1}{(2\pi R)^n} \int_{Q(\pi R)} \left(\sum_{l \in \mathbb{Z}^n} \kappa_R(x - 2\pi R l)\tau(x - 2\pi R l)\right) \exp\left(-\frac{(x \cdot m)i}{R}\right) dx \\ &= \frac{1}{(2\pi R)^n} \int_{\mathbb{R}^n} \kappa_R(x)\tau(x) \exp\left(-\frac{(x \cdot m)i}{R}\right) dx. \end{aligned}$$

Taking into account the support condition of the functions, we obtain

$$\kappa_{\frac{101R}{100}}(x)\kappa_R(x)\tau(x) = \kappa_{\frac{101R}{100}}(x)\kappa_R(x)\tau^*(x) = \sum_{m \in \mathbb{Z}^n} a_m \, \kappa_{\frac{101R}{100}}(x) \exp\left(\frac{(* \cdot m)i}{R}\right). \tag{43}$$

We write out (41) in full by using (42) and (43).

$$\begin{aligned} \langle \mathcal{F}f, \tau \rangle &= \sum_{m \in \mathbb{Z}^n} a_m \left\langle \mathcal{F}f, \kappa_{\frac{101R}{100}} \exp\left(\frac{(* \cdot m)i}{R}\right)\right\rangle \\ &= \sum_{m \in \mathbb{Z}^n} \frac{1}{(2\pi R)^n} \left\langle \kappa_R \exp\left(-\frac{(* \cdot m)i}{R}\right), \tau \right\rangle \cdot \left\langle \mathcal{F}f, \kappa_R \exp\left(\frac{(* \cdot m)i}{R}\right)\right\rangle \\ &= \left\langle \left\{ \sum_{m \in \mathbb{Z}^n} \frac{1}{(2\pi R)^n} \left\langle \mathcal{F}f, \kappa_R \exp\left(\frac{(* \cdot m)i}{R}\right)\right\rangle \cdot \kappa_R \exp\left(-\frac{(* \cdot m)i}{R}\right)\right\}, \tau \right\rangle. \end{aligned}$$

Finally observe that

$$\left\langle \mathcal{F}f, \kappa_R \cdot \exp\left(\frac{(* \cdot m)i}{R}\right)\right\rangle = (2\pi)^{\frac{n}{2}} f\left(\frac{m}{R}\right).$$

Since τ is arbitrary, we finally obtain

$$\mathcal{F}f = (2\pi)^{\frac{n}{2}} \sum_{m \in \mathbb{Z}^n} \frac{1}{(2\pi R)^n} f\left(\frac{m}{R}\right) \cdot \kappa_R \exp\left(-\frac{(* \cdot m)i}{R}\right).$$

By taking the inverse Fourier transform to both sides, we have the desired result. ∎

Here and below we assume that $\{\phi_j\}_{j \in \mathbb{N}_0} \subset \mathcal{S}$ is a special family of functions satisfying

$$\chi_{B(2)} \le \phi_0 \le \chi_{B(3)}, \ \phi_j(x) = \phi_0(2^{-j}x) - \phi_0(2^{-j+1}x), \ j \in \mathbb{N}.$$

Corollary 5.5. *Let $f \in \mathcal{S}'$. Then f has the following expansion.*

$$f = \sum_{k \in \mathbb{N}_0} \left(\sum_{m \in \mathbb{Z}^n} \phi_k(D) f\left(\frac{m}{2^k}\right) \mathcal{F}^{-1}\kappa(2^k \cdot - m) \right).$$

The lemma below enables us to control the index-shifting

$$\{\lambda_{\nu m}\}_{\nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n} \mapsto \{\lambda_{\nu m+l}\}_{\nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n}.$$

in terms of $l \in \mathbb{Z}^n$.

Lemma 5.6. *Given $l \in \mathbb{Z}^n$ and $\lambda = \{\lambda_{\nu m}\}_{\nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n} \in \mathbf{a}_{pqu}$, we write*

$$\lambda^l := \{\lambda_{\nu\, m+l}\}_{\nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n}.$$

Then we have

$$\|\lambda^l \,:\, \mathbf{a}_{pqu}\| \le c\,(1+|l|)^a \|\lambda \,:\, \mathbf{a}_{pqu}\|$$

for some $a, c > 0$.

Proof. For the proof of this lemma, if suffices to show that, for each ν

$$\left| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu\, l+m} \chi_{Q_{\nu m}} \right| \le c\,(1+|l|)^a M^{(\eta)} \left(\sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{Q_{\nu m}} \right)$$

with $0 < \eta < \min(1, q, u)$ by virtue of (5).

Let $x \in Q_{\nu m}$. Then

$$\begin{aligned} |\lambda_{\nu\, m+l}| &\le c\,(1+|l|)^{n\eta} m_{B(x, c\, 2^{-\nu}(1+|l|))} \left(\left| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{Q_{\nu m}} \right|^\eta \right)^{\frac{1}{\eta}} \\ &\le c\,(1+|l|)^{n\eta} M^{(\eta)} \left(\sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \chi_{Q_{\nu m}} \right)(x), \end{aligned}$$

which proves the lemma. ∎

Lemma 5.7. *Under the assumption* (39) *in Theorem 5.3 the sum*

$$f = \sum_{\beta \in \mathbb{N}_0{}^n} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{\nu m} (\beta q u)_{\nu m}$$

converges absolutely in the topology of $\mathcal{S}'$.

Proof. We write $f^\beta := \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{\nu m} (\beta q u)_{\nu m}$ as before. Let $0 < \epsilon < \rho - r$. Note that $2^{-(r+\epsilon)|\beta|}(\beta q u)_{\nu m}$ is an atom centered at $Q_{\nu m}$ modulo multiplicative constant depending only on ψ. Thus, from (25) we obtain

$$|\langle f^\beta, \phi \rangle| \le c\, 2^{-(\rho - r - \epsilon)|\beta|} p(\phi) \|\lambda \,:\, \mathbf{a}_{pqu}\|_\rho. \tag{44}$$

We remark that in (25) the term corresponding to $\nu = 0$ is excluded in the proof of (25). However, it can be easily incorporated and hence (44) holds. Summing (44) gives

$$|\langle f, \phi \rangle| \leq c\, p(\phi) \|\lambda : \mathbf{a}_{pqu}\|_\rho.$$

Thus, the series converges in $\mathcal{S}'$. ∎

Having set down the preliminary facts, we return to the point left open above, namely, the "only if" part of Theorem 5.3.

Conclusion of the proof of Theorem 5.3. Let $f \in \mathcal{S}$ be given. Then we have

$$f = \sum_{k \in \mathbb{N}_0} \phi_k(D) f = \sum_{k \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \phi_k(D) f\left(\frac{m}{2^k}\right) \mathcal{F}^{-1}\kappa(2^k x - m) \tag{45}$$

by virtue of Corollary 5.5. We set

$$\Lambda_{km} = 2^{k\left(s - \frac{n}{p}\right)} \phi_k(D) f\left(\frac{m}{2^k}\right)$$

and we rewrite (45) as

$$f = c \sum_{k \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} 2^{-k\left(s - \frac{n}{p}\right)} \Lambda_{km} \mathcal{F}^{-1}\kappa(2^k x - m).$$

Now we turn our attention to $\mathcal{F}^{-1}\kappa(2^k x - m)$. We shall obtain a quantitative information of $|\mathcal{F}^{-1}\kappa(x)|$ as $|x| \to \infty$. The repeated integration by parts yields

$$\begin{aligned}
\partial^\alpha \mathcal{F}^{-1}\kappa(y) &= \int_{\mathbb{R}^n} (iz)^\alpha \kappa(z) \exp(iz \cdot y)\, dz \\
&= \frac{1}{(1+|y|^2)^N} \int_{\mathbb{R}^n} (iz)^\alpha \kappa(z) \left((1 - \Delta_z)^N \exp(iz \cdot y)\right) dz \\
&= \frac{1}{(1+|y|^2)^N} \int_{\mathbb{R}^n} \left((1 - \Delta_z)^N (iz)^\alpha \kappa(z)\right) \exp(iz \cdot y)\, dz.
\end{aligned}$$

As a result, for a fixed large N at our disposal, we obtain

$$|\partial^\alpha \mathcal{F}^{-1}\kappa(y)| \leq c\,(1 + |\alpha|^{2N})(1+|y|^2)^{-N}. \tag{46}$$

To prove Theorem 5.3 we may assume that ρ is a large integer by replacing ρ with a larger integer if necessary. By the Taylor expansion we have

$$\begin{aligned}
&\psi(2^{k+\rho}x - l)\mathcal{F}^{-1}\kappa(2^k x - m) \\
&= \sum_{\beta \in \mathbb{N}_0{}^n} \frac{D^\beta \mathcal{F}^{-1}\kappa(2^{-\rho}l - m)(2^k x - 2^{-\rho}l)^\beta \psi(2^{k+\rho}x - l)}{\beta!} \\
&= \sum_{\beta \in \mathbb{N}_0{}^n} \frac{2^{-\rho|\beta|} D^\beta \mathcal{F}^{-1}\kappa(2^{-\rho}l - m)\psi^\beta(2^{k+\rho}x - l)}{\beta!}.
\end{aligned}$$

Since $\sum_{m\in\mathbb{Z}^n} \psi(x-m) \equiv 1$, $\phi_k(D)f$ has the following expansion.

$$\phi_k(D)f = c\,2^{-k\left(s-\frac{n}{p}\right)} \sum_{m\in\mathbb{Z}^n} \sum_{l\in\mathbb{Z}^n} \sum_{\beta\in\mathbb{N}_0{}^n} \frac{2^{-\rho|\beta|}}{\beta!} \Lambda_{km} D^\beta \mathcal{F}^{-1}\kappa(2^{-\rho}l-m)\psi^\beta(2^{k+\rho}x-l).$$

Note that this sum converges in L_∞ topology and hence we can interchange the order of the sums. In terms of the quark functions this formula can be rewritten as

$$\phi_k(D)f(x) = c \sum_{l\in\mathbb{Z}^n} \sum_{\beta\in\mathbb{N}_0{}^n} \sum_{m\in\mathbb{Z}^n} \frac{2^{-\rho|\beta|}}{\beta!} \Lambda_{km}\, D^\beta \mathcal{F}^{-1}\kappa(2^{-\rho}l-m)(\beta qu)_{k+\rho\,l}(x).$$

Let $\lambda^\beta_{k+\rho\,l} := \frac{2^{-\rho|\beta|}}{\beta!} \sum_{m\in\mathbb{Z}^n} \Lambda_{km}\, D^\beta \mathcal{F}^{-1}\kappa(2^{-\rho}l-m)$. Inserting this expression gives

$$f = \sum_{k\in\mathbb{N}_0} \phi_k(D)f(x) = c \sum_{k\in\mathbb{N}_0} \sum_{l\in\mathbb{Z}^n} \sum_{\beta\in\mathbb{N}_0{}^n} \lambda^\beta_{k+\rho\,l}(\beta qu)_{k+\rho\,l}(x).$$

Next, we investigate how large the coefficient is. Let $x \in Q_{k+\rho\,2^\rho l+l_0}$ where $l \in \mathbb{Z}^n$ and each component of l_0 is an integer in $[0, 2^\rho)$. By (46), we have

$$|\lambda^\beta_{k+\rho\,2^\rho l+l_0}| \le c\,2^{-\rho|\beta|} \sum_{m\in\mathbb{Z}^n} \frac{|\Lambda_{k\,m}|}{1+|l-m|^N} = c\,2^{-\rho|\beta|} \sum_{m\in\mathbb{Z}^n} \frac{|\Lambda_{k\,m+l}|}{1+|m|^N}.$$

Thus, if we write $\Lambda^m = \{|\Lambda_{k\,m+l}|\}$, then we obtain with $\eta_0 = \min(1, q, u)$

$$\begin{aligned} \|\lambda \,:\, \mathbf{a}_{pqu}\|_\rho{}^{\eta_0} &\le c \left(\left\| 2^{-\rho|\beta|} \sum_{m\in\mathbb{Z}^n} \frac{\Lambda^m}{1+|m|^N} \,:\, \mathbf{a}_{pqu} \right\|_\rho \right)^{\eta_0} \\ &\le c \sum_{m\in\mathbb{Z}^n} \frac{\|\Lambda^m \,:\, \mathbf{a}_{pqu}\|^{\eta_0}}{1+|m|^{N\eta_0}} \le c \sum_{m\in\mathbb{Z}^n} \frac{(1+|m|)^{a\eta_0}}{1+|m|^{N\eta_0}} \|\Lambda \,:\, \mathbf{a}_{pqu}\|^{\eta_0}, \end{aligned}$$

where $a > 1$ is a constant from Lemma 5.6. Since N is still at our disposal, by taking N sufficiently large we obtain

$$\|\lambda \,:\, \mathbf{a}_{pqu}\|_\rho \le c\,\|\Lambda \,:\, \mathbf{a}_{pqu}\|.$$

Thus, we have the desired result. ∎

5.2 Quarkonial decomposition for general case

Now we cover the case when s is arbitrary.

Definition 5.8. Let $L \in \{-1, 1, 3, 5, \dots\}$. Then we define

$$(\beta qu)^{(L)}_{\nu m}(x) := 2^{-\nu\left(s-\frac{n}{p}\right)} \left((-\Delta)^{\frac{L+1}{2}}\psi^\beta\right)(2^\nu x-m),$$

where p and s are parameters of the function space $\mathcal{A}^s_{pqu}$ under consideration.

We state our main theorem.

Theorem 5.9. *Let*

$$0 < u \le p \le \infty,\ 0 < q \le \infty,\ s \in \mathbb{R}.$$

Suppose that the parameters ρ, σ, L satisfy

$$\rho > r,\ \sigma > 0,\ L \in \{-1, 1, 3, 5, \ldots\},\ \sigma > \max(\sigma_u, s),\ L \ge \max(-1, [\sigma_u - s])$$

for $\mathcal{N}$-scale and

$$\rho > r,\ \sigma > 0,\ L \in \{-1, 1, 3, 5, \ldots\},\ \sigma > \max(\sigma_{qu}, s),\ L \ge \max(-1, [\sigma_{qu} - s])$$

for $\mathcal{E}$-scale. Set

$$(\beta qu)_{\nu m}(x) = 2^{-\nu\left(\sigma - \frac{n}{p}\right)} \psi^\beta(2^\nu x - m)$$

and

$$(\beta qu)^{(L)}_{\nu m}(x) = 2^{-\nu\left(s - \frac{n}{p}\right)} \left((-\Delta)^{\frac{L+1}{2}} \psi^\beta\right)(2^\nu x - m).$$

Then $f \in \mathcal{A}^s_{pqu}$ if and only if there exists a triply indexed sequence

$$\eta = \{\eta^\beta_{\nu m}\}_{\beta \in \mathbb{N}_0{}^n,\ \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n} \text{ and } \lambda = \{\lambda^\beta_{\nu m}\}_{\beta \in \mathbb{N}_0{}^n,\ \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n}$$

such that f can be expressed as

$$f = \sum_{\beta \in \mathbb{N}_0{}^n} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \eta^\beta_{\nu m} (\beta qu)_{\nu m} + \sum_{\beta \in \mathbb{N}_0{}^n} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{\nu m} (\beta qu)^{(L)}_{\nu m}$$

with

$$\|\eta \,:\, \mathbf{a}_{pqu}\|_\rho + \|\lambda \,:\, \mathbf{a}_{pqu}\|_\rho < \infty. \tag{47}$$

If this is the case, then λ can be taken so that

$$\|\eta \,:\, \mathbf{a}_{pqu}\|_\rho + \|\lambda \,:\, \mathbf{a}_{pqu}\|_\rho \simeq \|f \,:\, \mathcal{A}^s_{pqu}\|. \tag{48}$$

Proof. First, "if" part is still obvious by Theorem 4.6. As for the "only if" part, we use Theorem 3.3. Let M be an odd integer large enough, say, $M > \max(L, \sigma - s)$. Let

$$\begin{aligned} g_1 &:= (1 + (-\Delta)^{\frac{M+1}{2}})^{-1} f \in \mathcal{A}^{s+M+1}_{pqu} \subset \mathcal{A}^\sigma_{pqu} \\ g_2 &:= (-\Delta)^{\frac{M-L}{2}} (1 + (-\Delta)^{\frac{M+1}{2}})^{-1} f \in \mathcal{A}^{s+L+1}_{pqu}. \end{aligned}$$

With the help of Theorem 3.3 we have

$$f = g_1 + (-\Delta)^{\frac{L+1}{2}} g_2 \text{ with } g_1 \in \mathcal{A}^\sigma_{pqu},\ g_2 \in \mathcal{A}^{s+L+1}_{pqu}.$$

We are in the position of applying the result in the previous section to g_1 and g_2.

$$\begin{aligned} g_1 &= \sum_{\beta \in \mathbb{N}_0{}^n} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \eta^\beta_{\nu m} (\beta qu)_{\nu m} \\ g_2 &= \sum_{\beta \in \mathbb{N}_0{}^n} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{\nu m} \left\{ 2^{-\nu\left(s+L+1-\frac{n}{p}\right)} \psi^\beta(2^\nu * - m) \right\}. \end{aligned}$$

Here the coefficients satisfy

$$\|\eta : \mathbf{a}_{pqu}\|_\rho + \|\lambda : \mathbf{a}_{pqu}\|_\rho \le c\,(\|g_2 : \mathcal{A}^{s+L+1}_{pqu}\| + \|g_1 : \mathcal{A}^{\sigma}_{pqu}\|) \le c\,\|f : \mathcal{A}^{s}_{pqu}\|.$$

Since $(-\Delta)^{\frac{L+1}{2}} g_2 = \sum_{\beta \in \mathbb{N}_0{}^n} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda^{\beta}_{\nu m} (\beta q u)^{(L)}_{\nu m}$, we have

$$f = \sum_{\beta \in \mathbb{N}_0{}^n} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \eta^{\beta}_{\nu m} (\beta q u)_{\nu m} + \sum_{\beta \in \mathbb{N}_0{}^n} \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda^{\beta}_{\nu m} (\beta q u)^{(L)}_{\nu m}$$

with the desired estimate of coefficients. ∎

6 Another atomic decomposition

In this section we answer Conjecture 1.3.

What we have been considering in this paper is so-called the non-homogeneous type. To answer Conjecture 1.3 we need to deal with the homogeneous space. However, the construction and the properties are completely analogous. Thus, we do not supply the proofs. To begin with, we take $\psi \in \mathcal{S}$ so that

$$\chi_{B(4)\setminus B(2)} \le \psi \le \chi_{B(8)\setminus B(1)}.$$

Then we define

$$\begin{aligned} \|f : \dot{\mathcal{N}}^s_{pqu}\| &= \|2^{js}\phi_j(D) f : l_q(\mathcal{M}^p_u)\| \quad \text{for } 0 < u \le p \le \infty,\ 0 < q \le \infty, s \in \mathbb{R} \\ \|f : \dot{\mathcal{E}}^s_{pqu}\| &= \|2^{js}\phi_j(D) f : \mathcal{M}^p_u(l_q)\| \quad \text{for } 0 < u \le p < \infty,\ 0 < q \le \infty, s \in \mathbb{R}. \end{aligned}$$

It will be understood again that $\mathcal{M}^\infty_u = L^\infty$ for $0 < u \le \infty$. The function spaces $\dot{\mathcal{N}}^s_{pqu}$ and $\dot{\mathcal{E}}^s_{pqu}$ are subsets of $\mathcal{S}'/\mathcal{P}$ whose $\dot{\mathcal{N}}^s_{pqu}$ and $\dot{\mathcal{E}}^s_{pqu}$ are finite respectively. The properties of function spaces and the statement, especially Theorems 3.9, 4.6 and 4.8, remain true if we replace the range of j and ν from $\mathbb{N}_0$ to $\mathbb{Z}$. Since the homogeneous analogue of Theorem 4.6 is valid, we have only to consider the possibility of decomposition into atoms, especially the spaces $\mathcal{E}^s_{pqu}$.

Finally in this paper we prove the following theorem, which is a formulation of Conjecture 1.3 based on our notation.

Theorem 6.1. *Suppose that $1 < u \le p < \infty$, $K \in \mathbb{N}$ and $L \in \mathbb{N}_0$.*

(A) *$f \in \mathcal{M}^p_u$ admits the following decompositions.*

(a) $f = \sum_{\nu \in \mathbb{Z}} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} m_{\nu m}$, where $m_{\nu m}$ are $(0,p)$-molecules.

(b) $f = \sum_{\nu \in \mathbb{Z}} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}$, where each $a_{\nu m}$ is an atom centered at $Q_{\nu m}$.

The coefficients $\lambda_{\nu m}$ can be taken so that

$$\|f : \mathcal{M}^p_u\| \simeq \left\| \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^2 \chi^{(p)}_{\nu m} : \mathcal{M}^p_u(l_2)\right\|.$$

(B) *Conversely if* (3) *is satisfied, then*

$$\sum_{\nu\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} m_{\nu m}, \ \sum_{\nu\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m} \in \mathcal{M}^p_u,$$

where $m_{\nu m}$ are $(0,p)$-molecules and $a_{\nu m}$ are $(0,p)$-atoms.

Proof. We have only to consider the assertion on atoms. The construction is almost the same as Theorem 4.6. Suppose $\psi \in \mathcal{S}$ is a compactly supported function such that $B(2) \subset \{\mathcal{F}\psi \neq 0\}$. We set $\psi_j = 2^{jn}\psi(2^j\cdot)$. $\kappa_j \in \mathcal{S}$ is defined by $\mathcal{F}\kappa_j = \dfrac{\phi_j}{\mathcal{F}\psi_j}$. Indeed, for $f \in \dot{E}^s_{pqu}$, we have

$$f = \sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^n} \int_{Q_{jm}} \psi_j(x-y)\kappa_j * f(y)\,dy.$$

We construct and estimate the coefficients in the same way as the non-homogeneous case. What remains to be done is to arrange the atoms satisfy the moment condition. To achieve this, we have only to replace ψ with $(-\Delta)^{L+1}\psi$. Correspondingly we replace κ_j so that $\mathcal{F}\kappa_j = \dfrac{\phi_j}{\mathcal{F}\left[(-\Delta)^{L+1}\psi)_j\right]}$ Then a_{jm}, given by $c\displaystyle\int_{Q_{jm}} [(-\Delta)^{L+1}\psi]_j(x-y)\kappa_j *$ $f(y)\,dy$ with some constant c, inherits the moment condition from that of ψ. ∎

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