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# WHITTAKER FUNCTIONS FOR $P_J$-PRINCIPAL SERIES REPRESENTATIONS OF $Sp(3, \mathbf{R})$

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**ABSTRACT:** In this paper, we give explicit formulas for the secondary and the primary Whittaker functions for $P_J$-principal series representations of $Sp(3, \mathbf{R})$.

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**Key words:** Whittaker functions, Whittaker models.

## 1. Introduction

For the real symplectic group $Sp(2, \mathbf{R})$ of degree two, explicit formulas of the Whittaker functions for various representations have been considerably developed in these fifteen years ([26], [21], [22], [24], [12], [8]) as well as the generalized Whittaker functions (e.g. [20], [5], [6], [7]). These results are applied to obtain global results such as the entireness of the spinor $L$-functions for generic cusp forms on $GSp(2)$ ([23], [13]). To present the first step toward the extension of these studies to the higher degree cases, we discuss the Whittaker functions for $P_J$-principal series representations of $Sp(3, \mathbf{R})$ as a continuation to the previous paper ([10]).

The $P_J$-principal series representation we treat here is the induced representation from the discrete series representation of the Levi part $GL^+(2, \mathbf{R})$ of the parabolic subgroup $P_J$ (see section 2.3 for the precise). We have the following two reasons to investigate such representations. Firstly, it is easy to handle these representations because they have the convenient scalar $K$-types and thus the corresponding Whittaker functions are scalar valued. Secondly, the invariants such as the Gelfand-Kirillov dimension and the Bernstein degree ([35]) of these representations are the same as those of the large discrete series representations of $Sp(3, \mathbf{R})$, which are sole cohomological representations having Whittaker models. In view of the role of the discrete series representations in the cohomological theory of discrete subgroups in $Sp(n, \mathbf{R})$ ([25, sections 3,4]), we may expect that the automorphic $L$-functions associated with automorphic forms generating discrete series representations have "geometric meaning." This is the reason why we stick to the discrete series. As shown in [26], when the group is $Sp(2, \mathbf{R})$, the Whittaker functions for $P_J$-principal series are resemble to that for discrete series. Therefore we can heuristically expect that the same analogy can be seen in the higher degree cases.

In the previous paper ([10]) we gave explicit formulas of the *secondary* Whittaker functions (i.e., the power series solutions around the regular singularity of the holonomic system characterizing the Whittaker functions) by using the generalized hypergeometric series ${}_4F_3(1)$. Here we obtain not only another expressions for the secondary Whittaker functions but also the integral representations of Euler type and of Mellin-Barnes type for the *primary* Whittaker functions (i.e., the

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Whittaker functions having the moderate growth property). Our new observation which is not given in [10] is that our Whittaker functions can be written in terms of the Whittaker functions for the class one principal series representation of the split orthogonal group $SO(5,\mathbf{R})$ (Theorems 6.2, 7.3 and 8.2). This relation also holds between $Sp(2,\mathbf{R})$ and $SO(3,\mathbf{R})$ and we might expect a kind of bootstrap procedure from $SO(2n-1,\mathbf{R})$ to $Sp(n,\mathbf{R})$. We also prove the linear relation between the primary and the secondary Whittaker functions analogous to the results of Harish-Chandra ([3]) for the spherical functions and of Hashizume ([4]) for the class one Whittaker functions.

Although we believe our result itself is interesting as a new example of (confluent type) special functions on $Sp(3,\mathbf{R})$, we mention a possible application to number theory. As is indicated in some previous works ([32], [33], [23], [13]), integral representations of Mellin-Barnes type for the primary Whittaker functions are very powerful tool to compute the gamma factors of automorphic $L$-functions. Then our formula might be enable us to compute the archimedean parts of the zeta integrals for the spinor $L$-functions for $GSp(3)$ and for $GSp(3)\times GL(2)$ constructed by Bump and Ginzburg ([1]) (see also Vo [34]) and to show the global functional equations. On the other hand the secondary Whittaker functions play a fundamental role in constructing the Poincaré series (*cf.* [19], [27]).

Here is the outline of this paper. In sections 2 and 3 we review the basic notions such as $P_J$-principal series representations and Whittaker functions. Section 4 is devoted to deduce the differential equations for Whittaker functions, which is not precisely explained in [10]. After the review of the main result in [10] about the explicit formula of the secondary Whittaker functions in section 5, we give another expression for the secondary Whittaker functions by using the secondary Whittaker functions for the class one principal series representations on $SO(5,\mathbf{R})$. Analogous results for the primary Whittaker functions and the relation to the secondary Whittaker functions are given in sections 7 and 8.

## 2. Preliminaries

2.1. **Groups and algebras.** We denote by $\mathbf{Z}$, $\mathbf{R}$, and $\mathbf{C}$ the ring of rational integers, the real number field and the complex number field, respectively, and by $\mathbf{Z}_{\geq m}$ the set of integers $n$ such that $n\geq m$. Let $M_n(\mathbf{R})$ be the space of real matrices of size $n$ and $1_n$ (resp. $O_n$) be the unit (resp. the zero) matrix in $M_n(\mathbf{R})$. Moreover for $1\leq i\leq 3$, let $\mathbf{e}_i$ be the unit vector of degree 3 with its $i$-th component 1 and the remaining component 0.

The real symplectic group $G=Sp(3,\mathbf{R})$ of degree three is defined by

$$G=Sp(3,\mathbf{R})=\left\{g\in M_6(\mathbf{R})\,|\,{}^tgJ_3=J_3g^{-1},\det g=1\right\},\quad J_3=\begin{pmatrix} O_3 & 1_3\\ -1_3 & O_3\end{pmatrix},$$

which is connected, semisimple, and split over $\mathbf{R}$. Here ${}^tg$ and $g^{-1}$ mean the transpose and the inverse of $g$, respectively. Let $\theta(g)={}^tg^{-1}$, $g\in G$, be a Cartan involution of $G$. Then $K=\{g\in G\,|\,\theta(g)=g\}$ is a maximal compact subgroup of $G$ which is isomorphic to the unitary group $U(3)$ of degree three.

Let

$$\mathfrak{g} = \mathfrak{sp}(3, \mathbf{R}) = \left\{X \in M_6(\mathbf{R}) \mid J_3X + {}^tXJ_3 = 0\right\},$$

be the Lie algebra of $G$. If we denote the differential of $\theta$ again by $\theta$, then we have $\theta(X) = -{}^tX$ for $X \in \mathfrak{g}$. Let $\mathfrak{k}$ and $\mathfrak{p}$ be the $+1$ and the $-1$ eigenspaces of $\theta$ in $\mathfrak{g}$, respectively, that is,

$$\begin{aligned}\mathfrak{k} &= \left\{X \in \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \middle| \; A, B \in M_3(\mathbf{R}), {}^tA = -A, {}^tB = B\right\}, \\ \mathfrak{p} &= \left\{X \in \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \middle| \; A, B \in M_3(\mathbf{R}), {}^tA = A, {}^tB = B\right\}.\end{aligned}$$

Then $\mathfrak{k}$ is the Lie algebra of $K$ which is isomorphic to the unitary algebra

$$\mathfrak{u}(3) = \{X \in M_3(\mathbf{C}) \mid X + {}^t\bar{X} = 0\},$$

of degree three, and $\mathfrak{g}$ has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We fix an isomorphism $\kappa$ between $\mathfrak{u}(3)$ and $\mathfrak{k}$ given by

$$\kappa : \mathfrak{u}(3) \ni X \mapsto \frac{1}{2}\begin{pmatrix} X + \bar{X} & \sqrt{-1}(\bar{X} - X) \\ \sqrt{-1}(X - \bar{X}) & X + \bar{X} \end{pmatrix} \in \mathfrak{k}.$$

For a Lie algebra $\mathfrak{l}$, we denote by $\mathfrak{l}_{\mathbb{C}} = \mathfrak{l} \otimes_{\mathbf{R}} \mathbf{C}$ the complexification of $\mathfrak{l}$. Take a compact Cartan subalgebra $\mathfrak{h} = \oplus_{i=1}^3 \mathbf{R}T_i$ of $\mathfrak{g}$, where $T_i = \kappa(\sqrt{-1}E_{ii}) \in \mathfrak{k}$ with the matrix unit $E_{ij}$ in $M_3(\mathbf{R})$ of $(i, j)$ entry. For each $1 \leq i \leq 3$, define a linear form $\beta_i$ on $\mathfrak{h}_{\mathbb{C}}$ by $\beta_i(T_j) = \sqrt{-1}\delta_{ij}$, $1 \leq j \leq 3$. Here $\delta_{ij}$ is the Kronecker's delta. Then the set $\Delta$ of roots of $(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$ is given by

$$\Delta = \Delta(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}) = \left\{\pm 2\beta_i \, (1 \leq i \leq 3), \, \pm\beta_j \pm \beta_k \, (1 \leq j < k \leq 3)\right\},$$

and the subset $\Delta^+ = \left\{2\beta_i \, (1 \leq i \leq 3), \, \beta_j \pm \beta_k \, (1 \leq j < k \leq 3)\right\}$ forms a positive root system. Let

$$\begin{aligned}\Delta_c^+ &= \left\{\beta_j - \beta_k \, (1 \leq j < k \leq 3)\right\}, \\ \Delta_n^+ &= \left\{2\beta_i \, (1 \leq i \leq 3), \, \beta_j + \beta_k \, (1 \leq j < k \leq 3)\right\},\end{aligned}$$

be the set of compact and non-compact positive roots, respectively. If we denote the root space for $\beta \in \Delta$ by $\mathfrak{g}_\beta$, then $\mathfrak{k}_{\mathbb{C}} \simeq \mathfrak{gl}(3, \mathbf{C})$ and $\mathfrak{p}_{\mathbb{C}}$ have the decompositions

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \left(\oplus_{\beta \in \Delta_c^+} \mathfrak{g}_{\pm\beta}\right), \quad \mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-, \quad \mathfrak{p}_\pm = \oplus_{\beta \in \Delta_n^+} \mathfrak{g}_{\pm\beta}.$$

Now we take a basis of $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_\pm$ consisting of root vectors. If we denote the extension of the isomorphism $\kappa$ to their complexifications again by $\kappa$, then we have $\kappa(E_{ij}) \in \mathfrak{g}_{\beta_i - \beta_j}$ for each $1 \leq i, j \leq 3$ satisfying $i \neq j$ and thus the set $\left\{\kappa(E_{ij}) \mid 1 \leq i, j \leq 3\right\}$ forms a basis of $\mathfrak{k}_{\mathbb{C}}$. On the other hand, if we define a map

$$p_\pm : \left\{X \in M_3(\mathbf{C}) \mid X = {}^tX\right\} \ni X \mapsto \begin{pmatrix} X & \pm\sqrt{-1}X \\ \pm\sqrt{-1}X & -X \end{pmatrix} \in \mathfrak{p}_\pm,$$

then the element $X_{\pm ij} = p_\pm\left(\frac{1}{2}(E_{ij} + E_{ji})\right)$ is a root vector in $\mathfrak{g}_{\pm(\beta_i + \beta_j)}$ for each $1 \leq i \leq j \leq 3$ and the set $\{X_{\pm ij} \mid 1 \leq i \leq j \leq 3\}$ gives a basis of $\mathfrak{p}_\pm$.

Put $\mathfrak{a}_{\mathfrak{p}} = \oplus_{i=1}^3 \mathbf{R}H_i$ with $H_1 = \mathrm{diag}\,(1, 0, 0, -1, 0, 0)$, $H_2 = \mathrm{diag}\,(0, 1, 0, 0, -1, 0)$, and $H_3 = \mathrm{diag}\,(0, 0, 1, 0, 0, -1)$. Then $\mathfrak{a}_{\mathfrak{p}}$ is a maximal abelian subalgebra of $\mathfrak{p}$. For each $1 \leq i \leq 3$, we define $e_i \in \mathfrak{a}_{\mathfrak{p}}^*$ by $e_i(H_j) = \delta_{ij}$ for $1 \leq j \leq 3$. The set $\Sigma$ of the restricted roots of $(\mathfrak{a}_{\mathfrak{p}}, \mathfrak{g})$ is given by

$$\Sigma = \Sigma(\mathfrak{a}_{\mathfrak{p}}, \mathfrak{g}) = \big\{\pm 2e_i\,(1 \leq i \leq 3),\ \pm e_j \pm e_k\,(1 \leq j < k \leq 3)\big\},$$

and the subset $\Sigma^+ = \big\{2e_i\,(1 \leq i \leq 3),\ e_j \pm e_k\,(1 \leq j < k \leq 3)\big\}$ forms a positive root system. For each $\alpha \in \Sigma$, we denote the restricted root space by $\mathfrak{g}_\alpha$ and choose a restricted root vector $E_\alpha$ in $\mathfrak{g}_\alpha$ as follows.

$$E_{2e_i} = \left(\begin{array}{c|c} O_3 & E_{ii} \\ \hline O_3 & O_3 \end{array}\right),\ 1 \leq i \leq 3,$$

$$E_{e_i+e_j} = \left(\begin{array}{c|c} O_3 & E_{ij} + E_{ji} \\ \hline O_3 & O_3 \end{array}\right),\ E_{e_i-e_j} = \left(\begin{array}{c|c} E_{ij} & O_3 \\ \hline O_3 & -E_{ji} \end{array}\right),\ 1 \leq i < j \leq 3,$$

and $E_{-\alpha} = \theta E_\alpha$ for $\alpha \in \Sigma^+$. If we put $\mathfrak{n}_{\mathfrak{p}} = \oplus_{\alpha\in\Sigma^+}\mathfrak{g}_\alpha$, then $\mathfrak{g}$ has an Iwasawa decomposition $\mathfrak{g} = \mathfrak{n}_{\mathfrak{p}} \oplus \mathfrak{a}_{\mathfrak{p}} \oplus \mathfrak{k}$. Also we have $G = NAK$, where $A$ (resp. $N$) is the analytic subgroup with Lie algebra $\mathfrak{a}_{\mathfrak{p}}$ (resp. $\mathfrak{n}_{\mathfrak{p}}$).

Set

$$\mathfrak{a}_J = \oplus_{i=1}^2 \mathbf{R}H_i,\ \ \mathfrak{n}_J = \oplus_{\alpha\in\Sigma^+\setminus\{2e_3\}}\mathfrak{g}_\alpha,\ \ \mathfrak{m}_J = \mathbf{R}H_3 \oplus \mathfrak{g}_{2e_3} \oplus \mathfrak{g}_{-2e_3} \simeq \mathfrak{sl}(2,\mathbf{R}).$$

Moreover let $A_J$, $N_J$, and $M_{J,0} \simeq SL(2,\mathbf{R})$ be the analytic subgroups with Lie algebras $\mathfrak{a}_J$, $\mathfrak{n}_J$, and $\mathfrak{m}_J$, respectively. Then $P_J = M_J A_J N_J$ with $M_J = Z_K(\mathfrak{a}_J)M_{J,0}$ is a parabolic subgroup of $G$ corresponding to the root $2e_3$ and the right-hand side gives its Langlands decomposition. Here $Z_K(\mathfrak{a}_J) = \{1_6, \mu_1\} \times \{1_6, \mu_2\}$ with $\mu_i = \exp \pi T_i$ is the centralizer of $\mathfrak{a}_J$ in $K$. We call $P_J$ *the second Jacobi parabolic subgroup* of $G$.

2.2. **Representation of $K$.** The equivalence classes of irreducible representations of $K \simeq U(3)$ can be parameterized by the set $\Lambda = \big\{\lambda = (\lambda_1, \lambda_2, \lambda_3) \,|\, \lambda_i \in \mathbf{Z}, \lambda_1 \geq \lambda_2 \geq \lambda_3\big\}$ from the highest weight theory. We denote the representation of $K$ associated to $\lambda \in \Lambda$ by $(\tau_\lambda, V_\lambda)$.

The representation space $V_\lambda$ of a representation $\tau_\lambda$ has the Gelfand-Zelevinsky (or the canonical) basis $\{f(M)\}_{M\in G(\lambda)}$ parameterized by the set $G(\lambda)$ of all G-patterns of type $\lambda$. Here a G-pattern $M \in G(\lambda)$ of type $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \Lambda$ is a triangular array

$$M = \begin{pmatrix} \lambda_1\ \lambda_2\ \lambda_3 \\ \alpha_1\ \alpha_2 \\ \beta \end{pmatrix}$$

of integers satisfying the conditions $\lambda_1 \geq \alpha_1 \geq \lambda_2 \geq \alpha_2 \geq \lambda_3$ and $\alpha_1 \geq \beta \geq \alpha_2$. For the definition of the Gelfand-Zelevinsky basis and the explicit action of $\mathfrak{k}_{\mathbb{C}} = \mathrm{Lie}\,(K)_{\mathbb{C}} = \mathfrak{u}(3)_{\mathbb{C}}$ on this basis, we refer to the papers [2] and [9]. In particular, when $\lambda = (m, m, m) \in \Lambda$, the associated representation $(\tau_\lambda, V_\lambda)$ is one dimensional and the action of $\mathfrak{k}_{\mathbb{C}} = \mathfrak{gl}(3,\mathbf{C})$ on $v \in V_\lambda$ is given by

$$\tau_\lambda\big(\kappa(E_{ij})\big)v = \delta_{ij} m v,\ \ 1 \leq i, j \leq 3.$$

It is known that both of $\mathfrak{p}_\pm$ become $K$-modules via the adjoint action of $K$. Concerning this, we have the following lemma.

**Lemma 2.1.** *We have isomorphisms* $\mathfrak{p}_+ \simeq V_{2\mathbf{e}_1}$ *and* $\mathfrak{p}_- \simeq V_{-2\mathbf{e}_3}$ *by the correspondences between their basis*

$$\big(X_{+11}, X_{+22}, X_{+33}, X_{+12}, X_{+13}, X_{+23}\big)$$
$$\leftrightarrow \left( f\begin{pmatrix} 2\mathbf{e}_1 \\ 2\,0 \\ 2 \end{pmatrix}, f\begin{pmatrix} 2\mathbf{e}_1 \\ 2\,0 \\ 0 \end{pmatrix}, f\begin{pmatrix} 2\mathbf{e}_1 \\ 0\,0 \\ 0 \end{pmatrix}, f\begin{pmatrix} 2\mathbf{e}_1 \\ 2\,0 \\ 1 \end{pmatrix}, f\begin{pmatrix} 2\mathbf{e}_1 \\ 1\,0 \\ 1 \end{pmatrix}, f\begin{pmatrix} 2\mathbf{e}_1 \\ 1\,0 \\ 0 \end{pmatrix} \right),$$
$$\big(X_{-11}, X_{-22}, X_{-33}, -X_{-12}, X_{-13}, -X_{-23}\big)$$
$$\leftrightarrow \left( f\begin{pmatrix} -2\mathbf{e}_3 \\ 0\,-2 \\ -2 \end{pmatrix}, f\begin{pmatrix} -2\mathbf{e}_3 \\ 0\,-2 \\ 0 \end{pmatrix}, f\begin{pmatrix} -2\mathbf{e}_3 \\ 0\,0 \\ 0 \end{pmatrix}, f\begin{pmatrix} -2\mathbf{e}_3 \\ 0\,-2 \\ -1 \end{pmatrix}, f\begin{pmatrix} -2\mathbf{e}_3 \\ 0\,-1 \\ -1 \end{pmatrix}, f\begin{pmatrix} -2\mathbf{e}_3 \\ 0\,-1 \\ 0 \end{pmatrix} \right).$$

*Proof.* Let $\lambda_+ = 2\mathbf{e}_1$ and $\lambda_- = -2\mathbf{e}_3$ be two elements in $\Lambda$. Then we can find the action $\tau_{\lambda_\pm}$ of the generators of $\mathfrak{gl}(3,\mathbf{C})$ on the Gelfand-Zelevinsky basis $\{f(M)\}_{M\in G(\lambda_\pm)}$ of $V_{\lambda_\pm}$ in [2], Theorem 4 (*cf.* [9], Proposition 1.4). On the other hand, we have the following tables of the adjoint actions of $\mathfrak{k}_{\mathbb{C}}$ on the basis $\{X_{\pm ij}\}$ of $\mathfrak{p}_\pm$, which are obtained by direct computation. Comparing these two actions, we have the assertion. $\square$

| | $\kappa(E_{11})$ | $\kappa(E_{22})$ | $\kappa(E_{33})$ | $\kappa(E_{12})$ | $\kappa(E_{21})$ | $\kappa(E_{23})$ | $\kappa(E_{32})$ | $\kappa(E_{13})$ | $\kappa(E_{31})$ |
|---|---|---|---|---|---|---|---|---|---|
| $X_{+11}$ | $2X_{+11}$ | 0 | 0 | 0 | $2X_{+12}$ | 0 | 0 | 0 | $2X_{+13}$ |
| $X_{+12}$ | $X_{+12}$ | $X_{+12}$ | 0 | $X_{+11}$ | $X_{+22}$ | 0 | $X_{+13}$ | 0 | $X_{+23}$ |
| $X_{+22}$ | 0 | $2X_{+22}$ | 0 | $2X_{+12}$ | 0 | 0 | $2X_{+23}$ | 0 | 0 |
| $X_{+13}$ | $X_{+13}$ | 0 | $X_{+13}$ | 0 | $X_{+23}$ | $X_{+12}$ | 0 | $X_{+11}$ | $X_{+33}$ |
| $X_{+23}$ | 0 | $X_{+23}$ | $X_{+23}$ | $X_{+13}$ | 0 | $X_{+22}$ | $X_{+33}$ | $X_{+12}$ | 0 |
| $X_{+33}$ | 0 | 0 | $2X_{+33}$ | 0 | 0 | $2X_{+23}$ | 0 | $2X_{+13}$ | 0 |

TABLE 1. The adjoint actions of $\kappa(E_{ij})$ on $\{X_{+ij}\}$.

| | $\kappa(E_{11})$ | $\kappa(E_{22})$ | $\kappa(E_{33})$ | $\kappa(E_{12})$ | $\kappa(E_{21})$ | $\kappa(E_{23})$ | $\kappa(E_{32})$ | $\kappa(E_{13})$ | $\kappa(E_{31})$ |
|---|---|---|---|---|---|---|---|---|---|
| $-X_{-11}$ | $2X_{-11}$ | 0 | 0 | $2X_{-12}$ | 0 | 0 | 0 | $2X_{-13}$ | 0 |
| $-X_{-12}$ | $X_{-12}$ | $X_{-12}$ | 0 | $X_{-22}$ | $X_{-11}$ | $X_{-13}$ | 0 | $X_{-23}$ | 0 |
| $-X_{-22}$ | 0 | $2X_{-22}$ | 0 | 0 | $2X_{-12}$ | $2X_{-23}$ | 0 | 0 | 0 |
| $-X_{-13}$ | $X_{-13}$ | 0 | $X_{-13}$ | $X_{-23}$ | 0 | 0 | $X_{-12}$ | $X_{-33}$ | $X_{-11}$ |
| $-X_{-23}$ | 0 | $X_{-23}$ | $X_{-23}$ | 0 | $X_{-13}$ | $X_{-33}$ | $X_{-22}$ | 0 | $X_{-12}$ |
| $-X_{-33}$ | 0 | 0 | $2X_{-33}$ | 0 | 0 | 0 | $2X_{-23}$ | 0 | $2X_{-13}$ |

TABLE 2. The adjoint actions of $\kappa(E_{ij})$ on $\{-X_{-ij}\}$.

2.3. **$P_J$-principal series representation of $G$.** Let $\sigma = (\varepsilon_1, \varepsilon_2, D)$ be a representation of $M_J = \{1_6, \mu_1\} \times \{1_6, \mu_2\} \times M_{J,0}$ with characters $\varepsilon_i : \{1_6, \mu_i\} \to \mathbf{C}^\times$, $i = 1, 2$, and a (limit of) discrete series representation $D = \mathcal{D}_k^\pm$ of $M_{J,0} \simeq SL(2,\mathbf{R})$ with the Blattner parameter $\pm k$ ($k \in \mathbf{Z}_{\geq 1}$). Moreover take a quasi-character $\nu$ of $A_J$ such that

$$\nu\big(\operatorname{diag}(a_1, a_2, 1, a_1^{-1}, a_2^{-1}, 1)\big) = a_1^{\nu_1} a_2^{\nu_2}, \quad (\nu_1, \nu_2) \in \mathbf{C}^2.$$

Then we can construct an induced representation $\operatorname{Ind}_{P_J}^G(\sigma \otimes \nu \otimes 1_{N_J})$ of $G$ from the second Jacobi parabolic subgroup $P_J$ in the usual manner, which we call *a $P_J$-principal series representation* of $G$. The multiplicity theorem for the $K$-types can be computed by the Frobenius reciprocity for induced representations.

**Proposition 2.2.** *Let $\pi = \mathrm{Ind}_{P_J}^G(\sigma\otimes\nu\otimes 1_{N_J})$ be a $P_J$-principal series representation of $G$ with data $\sigma = (\varepsilon_1, \varepsilon_2, D)$ and $\nu$. Then each irreducible $K$-module $(\tau_\lambda, V_\lambda)$ associated with $\lambda \in \Lambda$ occurs in the restriction $\pi|_K$ of $\pi$ to $K$ with the following multiplicity $m_\lambda$.*

$$m_\lambda = \#\left\{ M \in G(\lambda) \;\middle|\; \begin{array}{c} \varepsilon_i(\mu_i) = (-1)^{w_i},\ i = 1, 2 \\ k \equiv w_3 \,(\mathrm{mod}\, 2),\ k \leq \mathrm{sgn}\,(D) w_3 \end{array} \right\}.$$

*Here $w = (w_1, w_2, w_3)$ is the weight for $M = \begin{pmatrix} \lambda_1, \lambda_2, \lambda_3 \\ \alpha_1, \alpha_2 \\ \beta \end{pmatrix} \in G(\lambda)$ defined by the formula*

$$w_1 = \beta,\ w_2 = \alpha_1 + \alpha_2 - \beta,\ w_3 = \lambda_1 + \lambda_2 + \lambda_3 - \alpha_1 - \alpha_2,$$

*and* $\mathrm{sgn}\,(D) = 1$ *(resp. $-1$) for $D = \mathcal{D}_k^+$ (resp. $\mathcal{D}_k^-$).*

*Proof.* First, we observe that

$$K \cap M_J = \{1_6, \mu_1\} \times \{1_6, \mu_2\} \times \{\exp\theta T_3 \,|\, 0 \leq \theta < 2\pi\} \simeq \{\pm 1\}^2 \times SO(2).$$

Therefore, if we define a character $\delta_i$ of $\{1_6, \mu_i\}$ by $\delta_i(\mu_i) = -1$ and a character $\chi_m$ of $\{\exp\theta T_3 \,|\, 0 \leq \theta < 2\pi\}$ by $\chi_m(\exp\theta T_3) = e^{\sqrt{-1}m\theta}$, then

$$(\widehat{K \cap M_J}) = \big\{(\delta_1^{n_1}, \delta_2^{n_2}, \chi_m) \,\big|\, n_i \in \{0, 1\}, m \in \mathbf{Z}\big\}.$$

The Frobenius reciprocity for induced representations (*cf.* Knapp[14]) tells us that for each $\lambda \in \Lambda$ the multiplicity $m_\lambda$ is given by

$$m_\lambda = \sum_{w \in (\widehat{K \cap M_J})} \big[\sigma|_{K\cap M_J} : w\big]\big[\tau_\lambda|_{K\cap M_J} : w\big].$$

Since the action of $K \cap M_J$ on the Gelfand-Zelevinsky basis $\{f(M)\}_{M\in G(\lambda)}$ of $V_\lambda$ is given by

$$\begin{array}{l} \tau_\lambda(\mu_i) f(M) = (-1)^{w_i} f(M), \quad i = 1, 2, \\ \tau_\lambda(\exp\theta T_3) f(M) = e^{\sqrt{-1}\theta w_3} f(M), \end{array}$$

we have

$$\tau_\lambda|_{K\cap M_J} = \bigoplus_{M\in G(\lambda)} \big(\delta_1^{w_1}, \delta_2^{w_2}, \chi_{w_3}\big).$$

On the other hand, it follows from the decomposition of $D$ into its $K$-type that the restriction of $\sigma$ to $K \cap M_J$ has the decomposition

$$\sigma|_{K\cap M_J} = \bigoplus_{m\in \mathbf{Z}_{\geq 0}} \big(\varepsilon_1, \varepsilon_2, \chi_{\mathrm{sgn}\,(D)(k+2m)}\big).$$

Hence we have

$$m_\lambda = \sum_{m\in\mathbf{Z}_{\geq 0}} \#\left\{ M \in G(\lambda) \;\middle|\; \begin{array}{l} \varepsilon_i = \delta_i^{w_i},\ i = 1, 2 \\ \mathrm{sgn}\,(D)(k + 2m) = w_3 \end{array} \right\},$$

and thus, the assertion of Proposition follows. $\square$

Let $\pi = \mathrm{Ind}_{P_J}^G(\sigma \otimes \nu \otimes 1_{N_J})$ be a $P_J$-principal series representation with $\sigma = (\varepsilon_1, \varepsilon_2, \mathcal{D}_k^+)$ such that $\varepsilon_i(\mu_i) = (-1)^k$. According to the above proposition, in the restriction $\pi|_K$ to $K$ of $\pi$, the multiplicity $m_{(k,k,k)}$ of the $K$-module $\tau_{(k,k,k)}$ is one whereas $m_{(k-2,k-2,k-2)} = 0$. The $K$-type $\tau_{(k,k,k)}$ of $\pi$ is called *corner*.

2.4. **Unitary characters of $N$.** Let $\eta$ be a unitary character of $N$ and denote the derivative of $\eta$ by the same letter. Since

$$\mathfrak{n}_{\mathfrak{p}}^{\mathrm{ab}} = \mathfrak{n}_{\mathfrak{p}}/[\mathfrak{n}_{\mathfrak{p}}, \mathfrak{n}_{\mathfrak{p}}] \simeq \mathfrak{g}_{e_1-e_2} \oplus \mathfrak{g}_{e_2-e_3} \oplus \mathfrak{g}_{2e_3},$$

$\eta$ is specified by three real numbers $c_{12}$, $c_{23}$, and $c_3$ such that

$$\eta(E_{e_1-e_2}) = 2\pi\sqrt{-1}c_{12},\ \eta(E_{e_2-e_3}) = 2\pi\sqrt{-1}c_{23},\ \text{and}\ \eta(E_{2e_3}) = 2\pi\sqrt{-1}c_3.$$

When $c_{12}c_{23}c_3 \neq 0$, a unitary character $\eta$ of $N$ is called *non-degenerate.*

## 3. Whittaker functions

For a finite dimensional representation $(\tau, V_\tau)$ of $K$ and a non-degenerate unitary character $\eta$ of $N$, we consider the space $C^\infty_{\eta,\tau}(N\backslash G/K)$ of smooth functions $\varphi : G \to V_\tau$ with the property

$$\varphi(ngk) = \eta(n)\tau(k)^{-1}\varphi(g), \quad (n, g, k) \in N \times G \times K.$$

Here we remark that any function $f \in C^\infty_{\eta,\tau}(N\backslash G/K)$ is determined by its restriction $f|_A$ to $A$ from the Iwasawa decomposition $G = NAK$ of $G$. Also let $(\tau^*, V_{\tau^*})$ be the contragredient representation of $(\tau, V_\tau)$ and $C^\infty\mathrm{Ind}_N^G(\eta)$ be the $C^\infty$-induced representation from $\eta$ with the representation space

$$C^\infty_\eta(N\backslash G) = \{\varphi \in C^\infty(G) \,|\, \varphi(ng) = \eta(n)\varphi(g), (n, g) \in N \times G\},$$

on which $G$ acts by right translation. Then the space $C^\infty_{\eta,\tau}(N\backslash G/K)$ is isomorphic to $\mathrm{Hom}_K(\tau^*, C^\infty\mathrm{Ind}_N^G(\eta))$ via the correspondence between $\iota \in \mathrm{Hom}_K(\tau^*, C^\infty\mathrm{Ind}_N^G(\eta))$ and $F^{[\iota]} \in C^\infty_{\eta,\tau}(N\backslash G/K)$ given by the relation $\iota(v^*)(g) = \langle v^*, F^{[\iota]}(g)\rangle$ for $v^* \in V_{\tau^*}$ and $g \in G$ with the canonical bilinear form $\langle\cdot,\cdot\rangle$ on $V_{\tau^*} \times V_\tau$.

Let $(\pi, H_\pi)$ be an irreducible admissible representation of $G$, and take a multiplicity one $K$-type $(\tau^*, V_{\tau^*})$ of $\pi$ with an injection $i : \tau^* \to \pi$. Then, for each element $T$ in the intertwining space $\mathcal{I}_{\eta,\pi} = \mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(\pi, C^\infty\mathrm{Ind}_N^G(\eta))$ between $(\mathfrak{g}_\mathbb{C}, K)$-modules consisting of all $K$-finite vectors, the relation $T(i(v^*))(g) = \langle v^*, T_i(g)\rangle$ for $v^* \in V_{\tau^*}$ and $g \in G$ determines an element $T_i \in C^\infty_{\eta,\tau}(N\backslash G/K)$. Now we put

$$\mathrm{Wh}(\pi, \eta, \tau) = \bigcup_{i\in\mathrm{Hom}_K(\tau^*,\pi)} \{T_i \in C^\infty_{\eta,\tau}(N\backslash G/K) \,|\, T \in \mathcal{I}_{\eta,\pi}\},$$

and call $\mathrm{Wh}(\pi, \eta, \tau)$ *the space of Whittaker functions* for $(\pi, \eta, \tau)$. Moreover, we denote by $\mathcal{I}^\circ_{\eta,\pi}$ the subspace of $\mathcal{I}_{\eta,\pi}$ consisting of the intertwining operators whose images in $C^\infty_\eta(N\backslash G)$ are moderate growth functions ([36] §8.1) and define

$$\mathrm{Wh}(\pi, \eta, \tau)^{\mathrm{mod}} = \bigcup_{i\in\mathrm{Hom}_K(\tau^*,\pi)} \{T_i \in C^\infty_{\eta,\tau}(N\backslash G/K) \,|\, T \in \mathcal{I}^\circ_{\eta,\pi}\}.$$

An element in $\mathrm{Wh}(\pi, \eta, \tau)^{\mathrm{mod}}$ is called *a Whittaker function of moderate growth.*

## 4. Differential Equations

Let $\sigma = (\varepsilon_1, \varepsilon_2, \mathcal{D}_k^+)$ be a representation of $M_J$ such that $\varepsilon_i(\mu_i) = (-1)^k$ and $\nu$ a quasi-character of $A_J$ defined by

$$\nu\big(\mathrm{diag}\,(a_1, a_2, 1, a_1^{-1}, a_2^{-1}, 1)\big) = a_1^{\nu_1} a_2^{\nu_2}, \quad (\nu_1, \nu_2) \in \mathbf{C}^2.$$

In the rest of this paper, we consider the Whittaker functions for the irreducible $P_J$-principal series representation $\pi = \mathrm{Ind}_{P_J}^G(\sigma \otimes \nu \otimes 1_{N_J})$, a non-degenerate unitary character $\eta$ of $N$ specified by three real numbers $c_{12}$, $c_{23}$ and $c_3$, and the $K$-module $\tau = \tau_{(-k,-k,-k)}$ whose contragredient representation gives the corner $K$-type of $\pi$.

First of all, we define the $\pm$-chirality matrices as follows.

**Definition 4.1.** *The $\pm$-chirality matrices $m_i(C_\pm)$ for $1 \le i \le 3$ are defined by*

$$m_1(C_\pm) = \begin{bmatrix} X_{\pm 11} & X_{\pm 12} & X_{\pm 13} \\ X_{\pm 12} & X_{\pm 22} & X_{\pm 23} \\ X_{\pm 13} & X_{\pm 23} & X_{\pm 33} \end{bmatrix}, \quad m_2(C_\pm) = \begin{bmatrix} M_{\pm 11} & -M_{\pm 12} & M_{\pm 13} \\ -M_{\pm 12} & M_{\pm 22} & -M_{\pm 23} \\ M_{\pm 13} & -M_{\pm 23} & M_{\pm 33} \end{bmatrix},$$

*and $m_3(C_\pm) = \det\big(m_1(C_\pm)\big)$. Here $M_{\pm ij}$ is the $(i,j)$-minor of the matrix $m_1(C_\pm)$ for each $1 \le i \le j \le 3$, that is,*

$$M_{\pm 11} = \begin{vmatrix} X_{\pm 22} & X_{\pm 23} \\ X_{\pm 23} & X_{\pm 33} \end{vmatrix}, M_{\pm 22} = \begin{vmatrix} X_{\pm 11} & X_{\pm 13} \\ X_{\pm 13} & X_{\pm 33} \end{vmatrix}, M_{\pm 33} = \begin{vmatrix} X_{\pm 11} & X_{\pm 12} \\ X_{\pm 12} & X_{\pm 22} \end{vmatrix},$$

$$M_{\pm 12} = \begin{vmatrix} X_{\pm 12} & X_{\pm 23} \\ X_{\pm 13} & X_{\pm 33} \end{vmatrix}, M_{\pm 13} = \begin{vmatrix} X_{\pm 12} & X_{\pm 22} \\ X_{\pm 13} & X_{\pm 23} \end{vmatrix}, M_{\pm 23} = \begin{vmatrix} X_{\pm 11} & X_{\pm 12} \\ X_{\pm 13} & X_{\pm 23} \end{vmatrix}.$$

Then we can find the following lemma immediately from the definition of the chirality matrices.

**Lemma 4.2.** *For each $1 \le i \le 3$, the element $C_{2i} = \mathrm{Tr}\,\big(m_i(C_+) m_i(C_-)\big)$ in $U(\mathfrak{g}_{\mathbb{C}})$ is invariant under the adjoint action of $K$, that is,*

$$C_{2i} \in U(\mathfrak{g}_{\mathbb{C}})^K = \{X \in U(\mathfrak{g}_{\mathbb{C}}) \mid \mathrm{Ad}\,(k)X = X, k \in K\}.$$

**Remark 4.3.** *In the case of $Sp(n, \mathbf{R})$, we can define $C_{2i}$ for each $1 \le i \le n$ belonging to $U(\mathfrak{g}_{\mathbb{C}})^K$ similarly. The operator $C_{2n}$ is essentially the same as the so-called Maass shift operator in the classical literature* [17]. *Also, the chirality matrices are used to construct the Capelli elements for a symmetric pair in* [16]*, recently.*

Now we consider a system of differential equations which are satisfied by the $A$-radial part of each element in $\mathrm{Wh}(\pi, \eta, \tau)$. The elements $C_2$, $C_4$, and $C_6$ in $U(\mathfrak{g}_{\mathbb{C}})^K$ defined in Lemma 4.2 are acting on the space $C_\eta^\infty(N\backslash G)$ as differential operators. In particular, since $K$-type $\tau^*$ occurs with multiplicity one in $\pi|_K$, it follows that these operators are acting on the space $\mathrm{Wh}(\pi, \eta, \tau)$ as scalar operators. Here we examine the scalar action of the operator $C_6 = m_3(C_+) m_3(C_-)$ in more detail. By definition, the operator $m_3(C_-)$ maps the $K$-type $\tau^* = \tau_{(k,k,k)}$ into $\tau_{(k-2,k-2,k-2)}$ in the Harish-Chandra module of $\pi$. Therefore each element in $\mathrm{Wh}(\pi, \eta, \tau)$ vanishes by the action of $m_3(C_-)$, because the $K$-module $\tau_{(k-2,k-2,k-2)}$ does not occur in $\pi|_K$. Therefore each element in $\mathrm{Wh}(\pi, \eta, \tau)$ satisfies the following system of differential equations

$$C_2 \phi = \chi_{2,k,\nu} \phi, \quad C_4 \phi = \chi_{4,k,\nu} \phi, \quad m_3(C_-)\phi = 0,$$

if we denote the scalar value for the action of the operator $C_{2i}$ by $\chi_{2i,k,\nu}$.

To obtain the explicit actions of the operators $C_2$, $C_4$, and $m_3(C_-)$ and their eigenvalues $\chi_{2i,k,\nu}$, we may express these operators in the normal order modulo $[\mathfrak{n}_\mathfrak{p}, \mathfrak{n}_\mathfrak{p}]$ with respect to the Iwasawa decomposition of $\mathfrak{g}$, according to the following lemma.

**Lemma 4.4.** *Let $f \in C^\infty_{\eta,\tau}(N\backslash G/K)$. For $X \in U(\mathfrak{k}_\mathbb{C})$, $Y \in U(\mathfrak{n}_{\mathfrak{p}\mathbb{C}})$, $Z \in U(\mathfrak{a}_\mathbb{C})$ and $a \in A$, we have $\big(\mathrm{Ad}\,(a^{-1})Y\big)ZXf(a) = \eta(Y)\tau(-X)(Zf)(a)$. In particular, for $a = \mathrm{diag}\,(a_1, a_2, a_3, a_1^{-1}, a_2^{-1}, a_3^{-1}) \in A$, we have $H_i f(a) = a_i \dfrac{\partial}{\partial a_i} f(a)$ and*

$$E_{e_1-e_2} f(a) = 2\pi\sqrt{-1}c_{12}\frac{a_1}{a_2}f(a), \quad E_{e_2-e_3} f(a) = 2\pi\sqrt{-1}c_{23}\frac{a_2}{a_3}f(a),$$
$$E_{2e_3} f(a) = 2\pi\sqrt{-1}c_3 a_3^2 f(a),$$

*and $E_\alpha f(a) = 0$ for $\forall\alpha \in \Sigma^+\backslash\{e_1-e_2, e_2-e_3, 2e_3\}$.*

The proof is omitted (*cf.* Knapp[14], Chapter VIII).

Moreover, we have the following fundamental lemma which is required to get the expressions of the elements in $U(\mathfrak{g}_\mathbb{C})$ in normal order. In the following, we denote $X \equiv Y$ for two elements $X$ and $Y$ in $U(\mathfrak{g}_\mathbb{C})$ when $X - Y \in [\mathfrak{n}_\mathfrak{p}, \mathfrak{n}_\mathfrak{p}]U(\mathfrak{g}_\mathbb{C})$.

**Lemma 4.5.** *The root vectors $X_{\pm ij}$ in $\mathfrak{p}_\pm$ have the following expressions according to the Iwasawa decomposition of $\mathfrak{g}$.*

$$X_{+ij} = \begin{cases} 2\sqrt{-1}E_{2e_i} + H_i + \kappa(E_{ii}), & i = j, \\ \big(E_{e_i-e_j} + \sqrt{-1}E_{e_i+e_j}\big) + \kappa(E_{ji}), & i < j, \end{cases}$$
$$X_{-ij} = \begin{cases} -2\sqrt{-1}E_{2e_i} + H_i - \kappa(E_{ii}), & i = j, \\ \big(E_{e_i-e_j} - \sqrt{-1}E_{e_i+e_j}\big) - \kappa(E_{ij}), & i < j. \end{cases}$$

*Therefore, we have*

$$X_{+ij} \equiv \begin{cases} H_i + \kappa(E_{ii}), & i = j = 1, 2, \\ E_{e_i-e_j} + \kappa(E_{ji}), & (i,j) = (1,2),\ (2,3), \end{cases}$$
$$X_{-ij} \equiv \begin{cases} H_i - \kappa(E_{ii}), & i = j = 1, 2, \\ E_{e_i-e_j} - \kappa(E_{ij}), & (i,j) = (1,2),\ (2,3), \end{cases}$$

*and*

$$X_{\pm 33} \equiv \pm 2\sqrt{-1}E_{2e_3} + H_3 \pm \kappa(E_{33}), \;\; X_{+13} \equiv \kappa(E_{31}), \;\; X_{-13} \equiv -\kappa(E_{13}).$$

*Proof.* These are obtained by direct computation. □

Let us compute the normal order of the operators $C_2$, $C_4$, and $m_3(C_-)$. First we treat the operator

$$C_2 = \mathrm{Tr}\,\big(m_1(C_+)m_1(C_-)\big) = \sum_{i=1}^{3} X_{+ii}X_{-ii} + 2\sum_{1\le i<j\le 3} X_{+ij}X_{-ij},$$

of degree two. By using the expressions of $X_{+ij}$ in Lemma 4.5 and Table 2 for the action of $\mathfrak{k}_\mathbb{C}$ on $X_{-ij}$ in the proof of Lemma 2.1, each term in the right hand side of the above expression of $C_2$ can be computed as

$$X_{+ii}X_{-ii} \equiv \begin{cases} H_iX_{-ii} + X_{-ii}\big(\kappa(E_{ii}) - 2\big), & i = 1, 2, \\ \big(2\sqrt{-1}E_{2e_3} + H_3\big)X_{-33} + X_{-33}\big(\kappa(E_{33}) - 2\big), & i = 3, \end{cases}$$

and

$$X_{+ij}X_{-ij} \equiv \begin{cases} E_{e_i-e_j}X_{-ij} + X_{-ij}\kappa(E_{ji}) - X_{-ii}, & (i,j) = (1,2),\ (2,3), \\ X_{-13}\kappa(E_{31}) - X_{-11}, & (i,j) = (1,3). \end{cases}$$

Thus we have

$$\begin{aligned} C_2 \ \equiv\ & (H_1 - 6)X_{-11} + X_{-11}\kappa(E_{11}) + (H_2 - 4)X_{-22} + X_{-22}\kappa(E_{22}) \\ & + \big(H_3 + 2\sqrt{-1}E_{2e_3} - 2\big)X_{-33} + X_{-33}\kappa(E_{33}) + 2E_{e_1-e_2}X_{-12} \\ & + 2X_{-12}\kappa(E_{21}) + 2E_{e_2-e_3}X_{-23} + 2X_{-23}\kappa(E_{32}) + 2X_{-13}\kappa(E_{31}). \end{aligned}$$

Next we consider the operator $C_4$ of degree four. By definition, $C_4$ can be expressed as

$$C_4 = \sum_{i=1}^{3} M_{+ii}M_{-ii} + 2\sum_{1\le i<j\le 3} M_{+ij}M_{-ij}.$$

We compute the right hand side of this expression by using the following lemmas.

**Lemma 4.6.** 1. *Each $(i,j)$-minor $M_{+ij}$ in the matrix $m_2(C_+)$ has the following expression.*

$$\begin{aligned} M_{+11} &\equiv (H_2 - 1)X_{+33} + X_{+33}\kappa(E_{22}) - E_{e_2-e_3}X_{+23} - X_{+23}\kappa(E_{32}), \\ M_{+22} &\equiv (H_1 - 1)X_{+33} + X_{+33}\kappa(E_{11}) - X_{+13}\kappa(E_{31}), \\ M_{+33} &\equiv (H_1 - 1)X_{+22} + X_{+22}\kappa(E_{11}) - E_{e_1-e_2}X_{+12} - X_{+12}\kappa(E_{21}), \\ M_{+12} &\equiv E_{e_1-e_2}X_{+33} + X_{+33}\kappa(E_{21}) - X_{+23}\kappa(E_{31}), \\ M_{+23} &\equiv (H_1 - 1)X_{+23} + X_{+23}\kappa(E_{11}) - X_{+12}\kappa(E_{31}), \\ M_{+13} &\equiv E_{e_1-e_2}X_{+23} + X_{+23}\kappa(E_{21}) - X_{+22}\kappa(E_{31}). \end{aligned}$$

2. *Each $(i,j)$-minor $M_{-ij}$ in the matrix $m_2(C_-)$ has the following expression.*

$$\begin{aligned} M_{-11} &\equiv (H_2 - 1)X_{-33} - X_{-33}\kappa(E_{22}) - E_{e_2-e_3}X_{-23} + X_{-23}\kappa(E_{23}), \\ M_{-22} &\equiv (H_1 - 1)X_{-33} - X_{-33}\kappa(E_{11}) + X_{-13}\kappa(E_{13}), \\ M_{-33} &\equiv (H_1 - 1)X_{-22} - X_{-22}\kappa(E_{11}) - E_{e_1-e_2}X_{-12} + X_{-12}\kappa(E_{12}), \\ M_{-12} &\equiv E_{e_1-e_2}X_{-33} - X_{-33}\kappa(E_{12}) + X_{-23}\kappa(E_{13}), \\ M_{-23} &\equiv (H_1 - 1)X_{-23} - X_{-23}\kappa(E_{11}) + X_{-12}\kappa(E_{13}), \\ M_{-13} &\equiv E_{e_1-e_2}X_{-23} - X_{-23}\kappa(E_{12}) + X_{-22}\kappa(E_{13}). \end{aligned}$$

*Proof.* By using the expressions in Lemma 4.5, the minor $M_{+11} = X_{+22}X_{+33} - X_{+23}X_{+23}$ can be expressed as

$$M_{+11} \equiv \big(H_2 + \kappa(E_{22})\big)X_{+33} - \big(E_{e_2-e_3} + \kappa(E_{32})\big)X_{+23}.$$

From this and the relations

$$\kappa(E_{22})X_{+33} = X_{+33}\kappa(E_{22}), \quad \kappa(E_{32})X_{+23} = X_{+23}\kappa(E_{32}) + X_{+33},$$

which can be seen from Table 1 for the action of $\mathfrak{k}_{\mathbb{C}}$ on $\mathfrak{p}_+$ in the proof of Lemma 2.1, we have the expression of $M_{+11}$ in the assertion of lemma. The expressions for the other minors can be obtained similarly. □

**Lemma 4.7.** *We have the following commutation relations.*

1. *For each* $1 \le i, j \le 3$, $\kappa(E_{ii})M_{-jj} = M_{-jj}\big\{\kappa(E_{ii}) - 2(1-\delta_{ij})\big\}$.
2. *For each permutation* $\{i,j,k\}$ *of* $\{1,2,3\}$, $\kappa(E_{ij})M_{-kk} = M_{-kk}\kappa(E_{ij})$.
3. $\kappa(E_{31})M_{-12} = M_{-12}\kappa(E_{31}) + M_{-23}$, $\kappa(E_{32})M_{-12} = M_{-12}\kappa(E_{32}) - M_{-13}$, $\kappa(E_{21})M_{-12} = M_{-12}\kappa(E_{21}) - M_{-22}$, $\kappa(E_{12})M_{-12} = M_{-12}\kappa(E_{12}) - M_{-11}$, $\kappa(E_{33})M_{-12} = M_{-12}\big(\kappa(E_{33}) - 2\big)$.
4. $\kappa(E_{21})M_{-13} = M_{-13}\kappa(E_{21}) - M_{-23}$, $\kappa(E_{31})M_{-13} = M_{-13}\kappa(E_{31}) + M_{-33}$, $\kappa(E_{32})M_{-13} = M_{-13}\kappa(E_{32})$, $\kappa(E_{13})M_{-13} = M_{-13}\kappa(E_{13}) + M_{-11}$, $\kappa(E_{22})M_{-13} = M_{-13}\big(\kappa(E_{22}) - 2\big)$.
5. $\kappa(E_{21})M_{-23} = M_{-23}\kappa(E_{21})$, $\kappa(E_{31})M_{-23} = M_{-23}\kappa(E_{31})$, $\kappa(E_{32})M_{-23} = M_{-23}\kappa(E_{32}) - M_{-33}$, $\kappa(E_{11})M_{-23} = M_{-23}\big(\kappa(E_{11}) - 2\big)$.

*Proof.* This lemma follows from the definition of the minors $M_{-ij}$ and Table 2 for the action of $\mathfrak{k}_{\mathbb{C}}$ on $\mathfrak{p}_-$ in the proof of Lemma 2.1 by direct computation. □

Now we proceed our computation for $C_4$. From Lemma 4.6 and Lemma 4.7, we have

$$\begin{aligned} M_{+11}M_{-11} &\equiv (H_2-1)X_{+33}M_{-11} + X_{+33}M_{-11}\big(\kappa(E_{22})-2\big) \\ &\quad -E_{e_2-e_3}X_{+23}M_{-11} - X_{+23}M_{-11}\kappa(E_{32}). \end{aligned}$$

Also, by using the relations

$$\begin{aligned} X_{+33}M_{-11} &\equiv \big(2\sqrt{-1}E_{2e_3} + H_3\big)M_{-11} + M_{-11}\big(\kappa(E_{33})-2\big), \\ X_{+23}M_{-11} &\equiv E_{e_2-e_3}M_{-11} + M_{-11}\kappa(E_{32}). \end{aligned}$$

which can be seen from Lemma 4.5 and Lemma 4.7, we can compute $M_{+11}M_{-11}$ further as

$$\begin{aligned} M_{+11}M_{-11} &\equiv (H_2-1)\Big\{\big(2\sqrt{-1}E_{2e_3} + H_3\big)M_{-11} + M_{-11}\big(\kappa(E_{33})-2\big)\Big\} \\ &\quad +\Big\{\big(2\sqrt{-1}E_{2e_3} + H_3\big)M_{-11} + M_{-11}\big(\kappa(E_{33})-2\big)\Big\}\big(\kappa(E_{22})-2\big) \\ &\quad -E_{e_2-e_3}^2M_{-11} - 2E_{e_2-e_3}M_{-11}\kappa(E_{32}) - M_{-11}\kappa(E_{32})^2. \end{aligned}$$

Similar calculation shows the following expressions for the other required products of two minors.

$$\begin{aligned} M_{+22}M_{-22} &\equiv (H_1-1)\Big\{\big(2\sqrt{-1}E_{2e_3} + H_3\big)M_{-22} + M_{-22}\big(\kappa(E_{33})-2\big)\Big\} \\ &\quad +\Big\{\big(2\sqrt{-1}E_{2e_3} + H_3\big)M_{-22} + M_{-22}\big(\kappa(E_{33})-2\big)\Big\}\big(\kappa(E_{11})-2\big) \\ &\quad -M_{-22}\kappa(E_{31})^2, \\ M_{+33}M_{-33} &\equiv (H_1-1)\Big\{H_2M_{-33} + M_{-33}\big(\kappa(E_{22})-2\big)\Big\} \\ &\quad +\Big\{H_2M_{-33} + M_{-33}\big(\kappa(E_{22})-2\big)\Big\}\big(\kappa(E_{11})-2\big) \\ &\quad -E_{e_1-e_2}^2M_{-33} - 2E_{e_1-e_2}M_{-33}\kappa(E_{21}) - M_{-33}\kappa(E_{21})^2, \end{aligned}$$

$$\begin{aligned}
M_{+12}M_{-12} &\equiv E_{e_1-e_2}\Big\{\big(2\sqrt{-1}E_{2e_3}+H_3\big)M_{-12}+M_{-12}\big(\kappa(E_{33})-2\big)\Big\}\\
&\quad -\big(2\sqrt{-1}E_{2e_3}+H_3\big)M_{-22}-M_{-22}\big(\kappa(E_{33})-2\big)-E_{e_2-e_3}M_{-23}\\
&\quad +M_{-33}+X_{+33}M_{-12}\kappa(E_{21})-X_{+23}M_{-12}\kappa(E_{31})-M_{-23}\kappa(E_{32}),\\
M_{+23}M_{-23} &\equiv (H_1-1)\big(E_{e_2-e_3}M_{-23}+M_{-23}\kappa(E_{32})-M_{-33}\big)\\
&\quad +\big(E_{e_2-e_3}M_{-23}+M_{-23}\kappa(E_{32})-M_{-33}\big)\big(\kappa(E_{11})-2\big)\\
&\quad -X_{+12}M_{-23}\kappa(E_{31}),\\
M_{+13}M_{-13} &\equiv E_{e_1-e_2}\big(E_{e_2-e_3}M_{-13}+M_{-13}\kappa(E_{32})\big)-E_{e_2-e_3}M_{-23}\\
&\quad +M_{-33}-H_2M_{-33}-M_{-33}\big(\kappa(E_{22})-2\big)\\
&\quad +X_{+23}M_{-13}\kappa(E_{21})-M_{-23}\kappa(E_{32})-X_{+22}M_{-13}\kappa(E_{31}).
\end{aligned}$$

By adding up them, we can obtain a tractable expression of the operator $C_4$.

Finally we discuss the operator

$$m_3(C_-)=\det C_-=X_{-11}M_{-11}-X_{-12}M_{-12}+X_{-13}M_{-13}.$$

Its expression in the normal order can be computed by combining the expressions in Lemma 4.5 and the commutation relations in Lemma 4.7. The resulted expression is given as follows.

$$\begin{aligned}
m_3(C_-) &\equiv (H_1-2)M_{-11}-M_{-11}\kappa(E_{11})-E_{e_1-e_2}M_{-12}\\
&\quad +M_{-12}\kappa(E_{12})-M_{-13}\kappa(E_{13}).
\end{aligned}$$

From the above computation and the action of $\mathfrak{k}_{\mathbb{C}}$ on the representation space $V_\tau$ of $\tau=\tau_{(-k,-k,-k)}$ given in §2.2, we can summarize the explicit actions of the operators $C_2$, $C_4$, and $m_3(C_-)$ on the space $C^\infty_{\eta,\tau}(N\backslash G/K)$.

**Proposition 4.8.** *The operators $C_2$, $C_4$ and $m_3(C_-)$ acting on $C^\infty_{\eta,\tau}(N\backslash G/K)$ are given as follows.*

$$\begin{aligned}
C_2 &\equiv (H_1-6+k)(H_1-k)+(H_2-4+k)(H_2-k)\\
&\quad +\big(H_3+2\sqrt{-1}E_{2e_3}-2+k\big)\big(H_3-2\sqrt{-1}E_{2e_3}-k\big)\\
&\quad +2E^2_{e_1-e_2}+2E^2_{e_2-e_3},\\
C_4 &\equiv \Big\{(H_2+k-3)\big(2\sqrt{-1}E_{2e_3}+H_3+k-2\big)-E^2_{e_2-e_3}\Big\}M_{-11}\\
&\quad +(H_1+k-5)\big(2\sqrt{-1}E_{2e_3}+H_3+k-2\big)M_{-22}\\
&\quad +\big\{(H_1+k-5)(H_2+k-4)-E^2_{e_1-e_2}\big\}M_{-33}\\
&\quad +2E_{e_1-e_2}\big(2\sqrt{-1}E_{2e_3}+H_3+k-2\big)M_{-12}+2E_{e_1-e_2}E_{e_2-e_3}M_{-13}\\
&\quad +2(H_1+k-5)E_{e_2-e_3}M_{-23},\\
m_3(C_-) &\equiv (H_1-k-2)M_{-11}-E_{e_1-e_2}M_{-12},
\end{aligned}$$

*and*

$$M_{-11} \equiv \big(H_3-2\sqrt{-1}E_{2e_3}-k\big)(H_2-k-1)-E^2_{e_2-e_3},$$

$$\begin{aligned}
M_{-22} &\equiv \big(H_3 - 2\sqrt{-1}E_{2e_3} - k\big)(H_1 - k - 1),\\
M_{-33} &\equiv (H_1 - k - 1)(H_2 - k) - E_{e_1-e_2}^2,\\
M_{-12} &\equiv E_{e_1-e_2}\big(H_3 - 2\sqrt{-1}E_{2e_3} - k\big),\\
M_{-23} &\equiv E_{e_2-e_3}(H_1 - k - 1),\\
M_{-13} &\equiv E_{e_1-e_2}E_{e_2-e_3}.
\end{aligned}$$

On the other hand, the eigenvalues $\chi_{2,k,\nu}$ and $\chi_{4,k,\nu}$ for the actions of $C_2$ and $C_4$ can be evaluated in the usual manner. The result is given as follows.

**Lemma 4.9.** *The scalar values $\chi_{2,k,\nu}$ and $\chi_{4,k,\nu}$ for the action of $C_2$ and $C_4$ on* $\mathrm{Wh}(\pi,\eta,\tau)$ *are given by*

$$\chi_{2,k,\nu} = \big\{\nu_1^2 - (k-3)^2\big\} + \big\{\nu_2^2 - (k-2)^2\big\},\ \ \chi_{4,k,\nu} = \big\{\nu_1^2 - (k-2)^2\big\}\big\{\nu_2^2 - (k-2)^2\big\}.$$

To state an explicit form of a holonomic system of partial differential equations satisfied by the $A$-radial part of each element in $\mathrm{Wh}(\pi,\eta,\tau)$, we introduce the coordinate $x = (x_1, x_2, x_3)$ on $A$ defined by

$$x_1 = \left(\pi c_{12}\frac{a_1}{a_2}\right)^2,\ \ x_2 = \left(\pi c_{23}\frac{a_2}{a_3}\right)^2,\ \ x_3 = 4\pi c_3 a_3^2,$$

for $\mathrm{diag}\,(a_1, a_2, a_3, a_1^{-1}, a_2^{-1}, a_3^{-1}) \in A$. Then we have the following theorem.

**Theorem 4.10.** *Each element $\varphi$ in the space* $\mathrm{Wh}(\pi,\eta,\tau)|_A$ *of the restriction of Whittaker functions to $A$ satisfies the following holonomic system of partial differential equations of rank* 24.

$$(1)\qquad \begin{cases} \mathcal{D}_2\varphi(x) = 0,\\ \mathcal{D}_3\varphi(x) = 0,\\ \mathcal{D}_4\varphi(x) = 0.\end{cases}$$

*Here*

$$\begin{aligned}
\mathcal{D}_2 &= (2\partial_1 + k - 6)\,(2\partial_1 - k) + (-2\partial_1 + 2\partial_2 + k - 4)\,(-2\partial_1 + 2\partial_2 - k)\\
&\quad + (-2\partial_2 + 2\partial_3 - x_3 + k - 2)\,(-2\partial_2 + 2\partial_3 + x_3 - k) - 8x_1 - 8x_2 - \chi_{2,k,\nu},\\
\mathcal{D}_3 &= (2\partial_1 - k - 2)\big\{(-2\partial_1 + 2\partial_2 - k - 1)\,(-2\partial_2 + 2\partial_3 + x_3 - k) + 4x_2\big\}\\
&\quad + 4x_1\,(-2\partial_2 + 2\partial_3 + x_3 - k)\,,\\
\mathcal{D}_4 &= \big\{(-2\partial_1 + 2\partial_2 + k - 3)\,(-2\partial_2 + 2\partial_3 - x_3 + k - 2) + 4x_2\big\}\\
&\qquad \cdot\big\{(-2\partial_1 + 2\partial_2 - k - 1)\,(-2\partial_2 + 2\partial_3 + x_3 - k) + 4x_2\big\}\\
&\quad + (2\partial_1 + k - 5)\,(-2\partial_2 + 2\partial_3 - x_3 + k - 2)\\
&\qquad \cdot(2\partial_1 - k - 1)\,(-2\partial_2 + 2\partial_3 + x_3 - k)\\
&\quad + \big\{(2\partial_1 + k - 5)\,(-2\partial_1 + 2\partial_2 + k - 4) + 4x_1\big\}\\
&\qquad \cdot\big\{(2\partial_1 - k - 1)\,(-2\partial_1 + 2\partial_2 - k) + 4x_1\big\}\\
&\quad - 8x_1\,(-2\partial_2 + 2\partial_3 - x_3 + k - 2)\,(-2\partial_2 + 2\partial_3 + x_3 - k)\\
&\quad + 32x_1x_2 - 8x_2\,(2\partial_1 + k - 5)\,(2\partial_1 - k - 1) - \chi_{4,k,\nu},
\end{aligned}$$

*with $\chi_{2,k,\nu}$ and $\chi_{4,k,\nu}$ given in Lemma 4.9. Moreover, $\partial_i = x_i \dfrac{\partial}{\partial x_i}$ is the Euler operator with respect to the variable $x_i$.*

For later computation, the following form of the above holonomic system (1) is useful.

**Corollary 4.11.** *For a function $\varphi \in C^\infty(A)$, put*

$$\varphi(x) = x_1^{\frac{3}{2}} x_2^{\frac{5}{2}} x_3^3 \exp\left(-\frac{x_3}{2}\right) \tilde{\varphi}(x).$$

*Then the holonomic system (1) for $\varphi$ is equivalent to the following holonomic system for $\tilde{\varphi}$.*

$$\left\{ \begin{array}{l} \widetilde{\mathcal{D}}_2 \tilde{\varphi}(x) = 0, \\ \widetilde{\mathcal{D}}_3 \tilde{\varphi}(x) = 0, \\ \widetilde{\mathcal{D}}_4 \tilde{\varphi}(x) = 0. \end{array} \right. \tag{2}$$

*Here*

$$\begin{aligned}
\widetilde{\mathcal{D}}_2 \;=\;& (2\partial_1 + k - 3)(2\partial_1 - k + 3) + (-2\partial_1 + 2\partial_2 + k - 2)(-2\partial_1 + 2\partial_2 - k + 2) \\
& + (-2\partial_2 + 2\partial_3 - 2x_3 + k - 1)(-2\partial_2 + 2\partial_3 - k + 1) - 8x_1 - 8x_2 - \chi_{2,k,\nu}, \\
\widetilde{\mathcal{D}}_3 \;=\;& (2\partial_1 - k + 1)\big\{(-2\partial_1 + 2\partial_2 - k + 1)(-2\partial_2 + 2\partial_3 - k + 1) + 4x_2\big\} \\
& + 4x_1(-2\partial_2 + 2\partial_3 - k + 1), \\
\widetilde{\mathcal{D}}_4 \;=\;& \big\{(-2\partial_1 + 2\partial_2 + k - 1)(-2\partial_2 + 2\partial_3 - 2x_3 + k - 1) + 4x_2\big\} \\
& \quad \cdot \big\{(-2\partial_1 + 2\partial_2 - k + 1)(-2\partial_2 + 2\partial_3 - k + 1) + 4x_2\big\} \\
& + (2\partial_1 + k - 2)(-2\partial_2 + 2\partial_3 - 2x_3 + k - 1) \\
& \quad \cdot (2\partial_1 - k + 2)(-2\partial_2 + 2\partial_3 - k + 1) \\
& + \big\{(2\partial_1 + k - 2)(-2\partial_1 + 2\partial_2 + k - 2) + 4x_1\big\} \\
& \quad \cdot \big\{(2\partial_1 - k + 2)(-2\partial_1 + 2\partial_2 - k + 2) + 4x_1\big\} \\
& - 8x_1(-2\partial_2 + 2\partial_3 - 2x_3 + k - 1)(-2\partial_2 + 2\partial_3 - k + 1) \\
& + 32x_1x_2 - 8x_2(2\partial_1 + k - 2)(2\partial_1 - k + 2) - \chi_{4,k,\nu}.
\end{aligned}$$

From the results of Kostant ([15] Theorem 6.8.1) and Matumoto ([18] Corollary 2.2.2, Theorem 6.2.1), it follows that the dimension of the intertwining space $\mathcal{I}_{\eta,\pi}$, and thus, of the space $\mathrm{Wh}(\pi, \eta, \tau)$ of the Whittaker functions is 24. Therefore every solution of the holonomic system in Theorem 4.10 gives an element in $\mathrm{Wh}(\pi, \eta, \tau)|_A$.

## 5. Secondary Whittaker functions; the first formula

The holonomic system (1) has regular singularities along 3 divisors $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$ with normal crossing at $x = (0, 0, 0)$, in the sense of [28]. The power series solutions of the system (1) around the point $x = (0, 0, 0)$ are called *the secondary Whittaker functions*. To give an explicit formula for the secondary Whittaker functions, we treat the holonomic system (2) instead of (1).

Let us consider a formal power series solution

$$x_1^{\gamma_1}x_2^{\gamma_2}x_3^{\gamma_3}\sum_{n_1,n_2,n_3\geq 0} c^{\gamma}_{n_1,n_2,n_3}x_1^{n_1}x_2^{n_2}x_3^{n_3},\quad \gamma=(\gamma_1,\gamma_2,\gamma_3)\in\mathbf{C}^3, \tag{3}$$

of the holonomic system (2) around $x=(0,0,0)$ associated with a characteristic index $\gamma$. Then we can translate the holonomic system (2) into a system of difference equations for the coefficients $c^{\gamma}_{n_1,n_2,n_3}$ of the power series (3). In the following, we put $\delta=(\delta_1,\delta_2,\delta_3)=(\gamma_1,-\gamma_1+\gamma_2,-\gamma_2+\gamma_3)$ for the characteristic index $\gamma=(\gamma_1,\gamma_2,\gamma_3)$, and often use the symbol $\delta$ instead of $\gamma$, such as $c^{\delta}_{n_1,n_2,n_3}$.

**Lemma 5.1.** *The power series (3) satisfies the holonomic system (2) if and only if the coefficients $\{c^{\delta}_{n_1,n_2,n_3}\}$ satisfy the following system of difference equations:*

$$\begin{aligned}
&\Big[4(\delta_1+n_1)^2+4(\delta_2-n_1+n_2)^2+4(\delta_3-n_2+n_3)^2-\big\{\nu_1^2+\nu_2^2+(k-1)^2\big\}\Big]c^{\delta}_{n_1,n_2,n_3}\\
&-8c^{\delta}_{n_1-1,n_2,n_3}-8c^{\delta}_{n_1,n_2-1,n_3}-2\left(2\delta_3-2n_2+2n_3-k-1\right)c^{\delta}_{n_1,n_2,n_3-1}=0,
\end{aligned}$$

$$\begin{aligned}
&(2\delta_1+2n_1-k+1)\left(2\delta_2-2n_1+2n_2-k+1\right)\left(2\delta_3-2n_2+2n_3-k+1\right)c^{\delta}_{n_1,n_2,n_3}\\
&+4\left(2\delta_3-2n_2+2n_3-k+1\right)c^{\delta}_{n_1-1,n_2,n_3}+4\left(2\delta_1+2n_1-k+1\right)c^{\delta}_{n_1,n_2-1,n_3}=0,
\end{aligned}$$

$$\begin{aligned}
&\Big[\big\{4\left(\delta_2-n_1+n_2\right)^2-(k-1)^2\big\}\big\{4\left(\delta_3-n_2+n_3\right)^2-(k-1)^2\big\}\\
&\qquad+\big\{4\left(\delta_1+n_1\right)^2-(k-2)^2\big\}\big\{4\left(\delta_3-n_2+n_3\right)^2-(k-1)^2\big\}\\
&\qquad+\big\{4\left(\delta_1+n_1\right)^2-(k-2)^2\big\}\big\{4\left(\delta_2-n_1+n_2\right)^2-(k-2)^2\big\}\\
&\qquad-\big\{\nu_1^2-(k-2)^2\big\}\big\{\nu_2^2-(k-2)^2\big\}\Big]c^{\delta}_{n_1,n_2,n_3}\\
&+\Big[4\left(2\delta_1+2n_1-k\right)\left(2\delta_2-2n_1+2n_2-k+4\right)\\
&\qquad+4\left(2\delta_1+2n_1+k-2\right)\left(2\delta_2-2n_1+2n_2+k-2\right)\\
&\qquad-8\big\{4\left(\delta_3-n_2+n_3\right)^2-(k-1)^2\big\}\Big]c^{\delta}_{n_1-1,n_2,n_3}\\
&+\Big[4\left(2\delta_2-2n_1+2n_2+k-1\right)\left(2\delta_3-2n_2+2n_3+k-1\right)\\
&\qquad+4\left(2\delta_2-2n_1+2n_2-k-1\right)\left(2\delta_3-2n_2+2n_3-k+3\right)\\
&\qquad-8\big\{4\left(\delta_1+n_1\right)^2-(k-2)^2\big\}\Big]c^{\delta}_{n_1,n_2-1,n_3}\\
&+\Big[-2\big\{4\left(\delta_2-n_1+n_2\right)^2-(k-1)^2\big\}\left(2\delta_3-2n_2+2n_3-k-1\right)\\
&\qquad-2\big\{4\left(\delta_1+n_1\right)^2-(k-2)^2\big\}\left(2\delta_3-2n_2+2n_3-k-1\right)\Big]c^{\delta}_{n_1,n_2,n_3-1}\\
&+16c^{\delta}_{n_1-2,n_2,n_3}+16c^{\delta}_{n_1,n_2-2,n_3}+32c^{\delta}_{n_1-1,n_2-1,n_3}\\
&-8\left(2\delta_2-2n_1+2n_2+k-1\right)c^{\delta}_{n_1,n_2-1,n_3-1}\\
&+16\left(2\delta_3-2n_2+2n_3-k-1\right)c^{\delta}_{n_1-1,n_2,n_3-1}=0.
\end{aligned}$$

*Here we understand $c^{\delta}_{n_1,n_2,n_3}=0$ if $n_1$, $n_2$, or $n_3<0$.*

Observe that for a fixed characteristic index $\gamma$ all coefficients $c^{\delta}_{n_1,n_2,n_3}$ are determined inductively from an initial non-zero coefficient $c^{\delta}_{0,0,0}$ by the first difference equation in Lemma 5.1, which is obtained from the differential operator $\mathcal{D}_2$. We can find the 24 characteristic indices $\gamma$ as

$$\delta = \sigma\left(\frac{\epsilon_1\nu_1}{2}, \frac{\epsilon_2\nu_2}{2}, \kappa\right), \quad \kappa = \frac{k-1}{2}, \tag{4}$$

with $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ and $\sigma \in \mathfrak{S}_3$, by putting $n_1 = n_2 = n_3 = 0$ in the system of difference equations in Lemma 5.1. Here $\mathfrak{S}_3$ means the symmetric group of degree 3.

Before giving an explicit formula for the secondary Whittaker functions, we shall discuss the convergence of the power series whose coefficients are given by a solution of the first difference equation in Lemma 5.1. The followings are based on the idea of Harish-Chandra (cf. [3, Lemma 4.5]). For complex numbers $a, b, c, d$, put

$$\Delta_{n_1,n_2,n_3} \equiv \Delta_{n_1,n_2,n_3}(a,b,c,d) = n_1^2 + n_2^2 + \frac{1}{2}n_3^2 - n_1n_2 - n_2n_3 + an_1 + bn_2 + cn_3 + d.$$

We can define complex numbers $A_{n_1,n_2,n_3} \equiv A_{n_1,n_2,n_3}(a,b,c,d,p)$ inductively by the recurrence relation

$$\begin{cases} A_{0,0,0} = 1, \\ \Delta_{n_1,n_2,n_3}A_{n_1,n_2,n_3} = A_{n_1-1,n_2,n_3} + A_{n_1,n_2-1,n_3} + \frac{1}{2}(-n_2+n_3+p)A_{n_1,n_2,n_3-1}, \end{cases} \tag{5}$$

if $\Delta_{n_1,n_2,n_3}$ does not vanish for all $(n_1,n_2,n_3) \neq (0,0,0)$.

**Lemma 5.2.** *Set*

$X := \{(a,b,c,d,p) \in \mathbf{C}^5 \mid \Delta_{n_1,n_2,n_3}(a,b,c,d) \neq 0$ *for all* $(n_1,n_2,n_3) \in \mathbf{N}^3 \backslash \{(0,0,0)\}\}$.

*Let $U$ be any compact subset in $X$. There exists a positive constant $c_U$ depending only on $U$ such that*

$$|A_{n_1,n_2,n_3}(a,b,c,d,p)| \leq c_U^{n_1+n_2+n_3}/(n_1+n_2+n_3)! \tag{6}$$

*for all $(n_1,n_2,n_3) \in \mathbf{N}^3$ and $(a,b,c,d,p) \in U$. Thus the power series*

$$\sum_{n_1,n_2,n_3 \geq 0} A_{n_1,n_2,n_3} x_1^{n_1} x_2^{n_2} x_3^{n_3}$$

*converges absolutely and uniformly on $(x_1,x_2,x_3) \in \mathbf{R}^3_+$ and $(a,b,c,d,p) \in X$.*

*Proof.* We prove (6) by induction on $n_1+n_2+n_3$. The case of $n_1+n_2+n_3 = 0$ is obvious. We first estimate $\Delta_{n_1,n_2,n_3}$. In the following, $c_i$ and $d_i$ mean constants depending only on $U$. For $(n_1,n_2,n_3) \in \mathbf{N}^3$ we have

$$\begin{aligned} |\Delta_{n_1,n_2,n_3}(a,b,c,d)| &= \Bigg|\frac{1}{2}(n_1+a+b+c)^2 + \frac{1}{2}(n_1-n_2-b-c)^2 + \frac{1}{2}(n_2-n_3-c)^2 \\ &\qquad + d - \frac{1}{2}(a+b+c)^2 - \frac{1}{2}(b+c)^2 - \frac{1}{2}c^2\Bigg| \\ &\geq \frac{1}{2}\left|(n_1+a+b+c)^2 + (n_1-n_2-b-c)^2 + (n_2-n_3-c)^2\right| + d_1 \\ &\geq c_1\left(|n_1+a+b+c| + |n_1-n_2-b-c| + |n_2-n_3-c|\right)^2 + d_1 \end{aligned}$$

$$\geq c_2(n_1+n_2+n_3+d_2)^2+d_1$$

with $c_1, c_2 > 0$ and $d_1, d_2 \in \mathbf{R}$. Then there exists a positive integer $N$ depending on $U$ such that

$$|\Delta_{n_1,n_2,n_3}(a,b,c,d)| \geq c_3(n_1+n_2+n_3)^2$$

for all $n_1+n_2+n_3 > N$ and $(a,b,c,d,p) \in U$. Put

$$c_4 = \min_{\substack{n_1+n_2+n_3 \leq N \\ (a,b,c,d,p)\in U}} \frac{|\Delta_{n_1,n_2,n_3}(a,b,c,d)|}{(n_1+n_2+n_3)^2}.$$

Then we can see $c_4 > 0$ and have

$$|\Delta_{n_1,n_2,n_3}(a,b,c,d)| \geq c_5(n_1+n_2+n_3)^2$$

for all $(n_1,n_2,n_3) \in \mathbf{N}^3$ with $c_5 = \min(c_3,c_4) > 0$. We also take a positive constant $c_6$ such that

$$|-n_2+n_3+p| \leq c_6(n_2+n_3+1)$$

for $(n_1,n_2,n_3) \in \mathbf{N}^3$.

Then the recurrence relation (5) and the induction hypothesis imply that

$$\begin{aligned}
|A_{n_1,n_2,n_3}| &\leq \frac{1}{|\Delta_{n_1,n_2,n_3}|}\left(|A_{n_1-1,n_2,n_3}|+|A_{n_1,n_2-1,n_3}|+\frac{1}{2}|-n_2+n_3+p|\cdot|A_{n_1,n_2,n_3-1}|\right) \\
&\leq \frac{c_5^{-1}}{(n_1+n_2+n_3)^2}\left\{\frac{2c_U^{n_1+n_2+n_3-1}}{(n_1+n_2+n_3-1)!}+\frac{c_U^{n_1+n_2+n_3-1}c_6(n_2+n_3+1)}{(n_1+n_2+n_3-1)!}\right\} \\
&= \frac{c_5^{-1}c_U^{n_1+n_2+n_3-1}}{(n_1+n_2+n_3)!}\cdot\frac{2+c_6(n_2+n_3+1)}{n_1+n_2+n_3}.
\end{aligned}$$

Therefore we take a positive constant $c_7$ such that

$$2+c_6(n_2+n_3+1) \leq c_7(n_1+n_2+n_3)$$

for all $(n_1,n_2,n_3) \neq (0,0,0)$ and replace $c_U$ by $\max(c_U, c_5^{-1}c_7)$, to obtain the assertion. □

Now we give the following explicit formula of the secondary Whittaker functions which is the main result of our previous paper [10].

**Theorem 5.3.** *For each characteristic index $\gamma$ of the holonomic system (2) given in (4), put*

$$M_\gamma(x) = x_1^{\frac{3}{2}+\gamma_1}x_2^{\frac{5}{2}+\gamma_2}x_3^{3+\gamma_3}\exp\left(-\frac{x_3}{2}\right)\sum_{n_1,n_2,n_3\geq 0} C^\delta_{n_1,n_2,n_3}x_1^{n_1}x_2^{n_2}x_3^{n_3},$$

*where the coefficients $\{C^\delta_{n_1,n_2,n_3}\}$ are defined as follows: For $l,m,n \in \mathbf{Z}_{\geq 0}$ and constants $a,b,c,a',b',c'$, put*

$$\begin{aligned}
k_{l,m,n} &= k_{l,m,n}(a,b,c,a',b',c') \\
&= \frac{1}{n!}\cdot\frac{(m+a)_n(-l+b)_n}{(c)_n}\,{}_4F_3\left(\left.\begin{matrix} -n,\ 1-n-c,\ -m+a',\ l+b' \\ 1-n-m-a,\ 1-n+l-b,\ c'\end{matrix}\right|1\right),
\end{aligned}$$

*where* $(a)_n = \dfrac{\Gamma(a+n)}{\Gamma(a)}$ *is Pochhammer's symbol and* ${}_pF_q$ *is the generalized hypergeometric function (cf.* [29]*). If either* $\delta_1$ *or* $\delta_2$ *is equal to* $\kappa$*, then*

$$\begin{aligned} C^{\delta}_{n_1,n_2,n_3} &= \frac{1}{n_2!}\cdot\frac{(\alpha_2+n_1)_{n_2-n_3}}{(\alpha_1)_{n_2-n_3}(\alpha_3)_{n_2}(\alpha_4)_{n_3}(\alpha_5)_{n_1}(\alpha_6)_{n_1}} \\ &\qquad \times k_{n_1,n_2,n_3}(\alpha_4,-\alpha_2+1,-\alpha_3+\alpha_4+1,0,\alpha_2+\alpha_4-1,\alpha_3+\alpha_4-1), \end{aligned}$$

*with the parameters*

$$\begin{array}{lll} \alpha_1 = -\delta_3+\kappa+1, & \alpha_2 = \delta_1-\delta_3+1, & \alpha_3 = \delta_*-\delta_3+1, \\ \alpha_4 = \delta_*+\delta_3+1, & \alpha_5 = \delta_1-\kappa+1, & \alpha_6 = -\delta_2+\kappa+1, \end{array}$$

*where* $\delta_* = \delta_1+\delta_2-\kappa$*. If* $\delta_3 = \kappa$*, then*

$$\begin{aligned} C^{\delta}_{n_1,n_2,n_3} &= \frac{1}{n_1!(n_2-n_3)!}\cdot\frac{(\beta_1+n_1)_{n_2-n_3}}{(\beta_1)_{n_2}(\beta_2)_{n_1}(\beta_3)_{n_3}(\beta_4)_{n_2}} \\ &\qquad \times k_{n_2,n_1,n_3}(\beta_3,\ -\beta_1+1,\ -\beta_2+\beta_3+1,\ 0,\ \beta_1+\beta_3-1,\ \beta_2+\beta_3-1), \end{aligned}$$

*for* $n_2 \geq n_3$ *and* $C^{\delta}_{n_1,n_2,n_3} = 0$ *for* $n_2 < n_3$*, where*

$$\beta_1 = \delta_1-\kappa+1,\ \ \beta_2 = \delta_1-\delta_2+1,\ \ \beta_3 = \delta_1+\delta_2+1,\ \ \beta_4 = \delta_2-\kappa+1.$$

*Then, the set* $\{M_\gamma(x)\}$ *gives a system of linearly independent solutions of the holonomic system (1) at* $x = (0,0,0)$*.*

**Remark 5.4.** *If the constants* $a, b, c, a', b', c'$ *satisfy the relation*

$$a+b-c = -(a'+b'-c'), \tag{7}$$

*then Euler's transformation formula*

$${}_2F_1\left(\begin{array}{c} a, b \\ c \end{array}\middle|\, z\right) = (1-z)^{c-a-b}{}_2F_1\left(\begin{array}{c} c-a, c-b \\ c \end{array}\middle|\, z\right)$$

*leads the equation*

$$k_{l,m,n}(a,b,c,a',b',c') = k_{m,l,n}(c-b,c-a,c,c'-b',c'-a',c').$$

*The parameters of the function* $k$ *appeared in the formulas of* $C^{\delta}_{n_1,n_2,n_3}$ *in Theorem 5.3 satisfy this condition (7). In particular, the relation* $k_{n_1,n_2,n_3} = k_{n_2,n_1,n_3}$ *holds in the formula of* $C^{\delta}_{n_1,n_2,n_3}$ *when* $\delta_2 = \kappa$*.*

## 6. Secondary Whittaker functions; the second formula

In this section, we present different expressions of the secondary Whittaker functions from Theorem 5.3. Since our new formula involves the secondary Whittaker functions for the class one principal series representations on the split orthogonal group $SO(5,\mathbf{R})$, or equivalently on $Sp(2,\mathbf{R})$, we first recall the explicit formula of them ([11], cf. [12]).

Let $y = (y_1, y_2) \in \mathbf{R}_+^2$ be the coordinate of the maximal torus of $SO(5,\mathbf{R})$ and $\pi_{(\nu_1,\nu_2)}$ be the class one principal series representation of $SO(5,\mathbf{R})$ as in [11, §§2-3].

Then the radial part of the secondary Whittaker function $M^{o}_{(\nu_1,\nu_2)}(y)$ on $SO(5,\mathbf{R})$ for the characteristic index $(\nu_1,\nu_1+\nu_2)$ is of the form

$$M^{o}_{(\nu_1,\nu_2)}(y) = y_1^{\frac{3}{2}} y_2^{2} \sum_{k_1,k_2\geq 0} C^{o,(\nu_1,\nu_2)}_{k_1,k_2} y_1^{2k_1+\nu_1} y_2^{2k_2+\nu_1+\nu_2},$$

where the coefficients $\{C^{o,(\nu_1,\nu_2)}_{k_1,k_2}\}$ are uniquely determined by the initial condition $C^{o,(\nu_1,\nu_2)}_{0,0} = 1$ and the recurrence relation ([11, (4.1)]):

$$\left(k_1^2 + \frac{1}{2}k_2^2 - k_1k_2 + \frac{\nu_1-\nu_2}{2}k_1 + \frac{\nu_2}{2}k_2\right) C^{o,(\nu_1,\nu_2)}_{k_1,k_2} = C^{o,(\nu_1,\nu_2)}_{k_1-1,k_2} + \frac{1}{2}C^{o,(\nu_1,\nu_2)}_{k_1,k_2-1}. \tag{8}$$

In [11, Theorem 4.1] we expressed $C^{o,(\nu_1,\nu_2)}_{k_1,k_2}$ in terms of a generalized hypergeometric series ${}_3F_2$. Let us rewrite it in the following way.

**Proposition 6.1.** *The solution of the recurrence relation (8) can be written as*

$$\begin{aligned} C^{o,(\nu_1,\nu_2)}_{k_1,k_2} = & \sum_{0\leq i_1\leq k_1} \sum_{0\leq i_2\leq \min(i_1,k_2)} \frac{1}{(i_1-i_2)!i_2!(k_1-i_1)!(k_2-i_2)!} \\ & \times \frac{1}{(\frac{\nu_1-\nu_2}{2}+1)_{i_1}(\nu_1+1)_{i_2}(\frac{\nu_1+\nu_2}{2}+1)_{k_1}(\nu_2+1)_{k_2-i_1}}. \end{aligned}$$

*Proof.* By [11, Theorem 4.1],

$$C^{o,(\nu_1,\nu_2)}_{k_1,k_2} = \frac{1}{k_1!k_2!(\nu_1+1)_{k_1}(\nu_2+1)_{k_2}} \, {}_3F_2\left( \begin{matrix} -k_1,\ -k_2-\frac{\nu_1+\nu_2}{2},\ k_2+\frac{\nu_1+\nu_2}{2}+1 \\ \frac{\nu_1+\nu_2}{2}+1,\ \frac{\nu_1-\nu_2}{2}+1 \end{matrix} \,\middle|\, 1 \right).$$

By using the formula

$${}_3F_2\left( \begin{matrix} -k_1,\ a,\ b \\ c,\ d \end{matrix} \,\middle|\, 1 \right) = \frac{(c+d-a-b)_{k_1}}{(c)_{k_1}} \, {}_3F_2\left( \begin{matrix} -k_1,\ d-a,\ d-b \\ c+d-a-b,\ d \end{matrix} \,\middle|\, 1 \right)$$

([29, 7.4.4.83]), we have

$$\begin{aligned} C^{o,(\nu_1,\nu_2)}_{k_1,k_2} &= \frac{1}{k_1!k_2!(\frac{\nu_1+\nu_2}{2}+1)_{k_1}(\nu_2+1)_{k_2}} \, {}_3F_2\left( \begin{matrix} -k_1,\ k_2+\nu_1+1,\ -k_2-\nu_2 \\ \nu_1+1,\ \frac{\nu_1-\nu_2}{2}+1 \end{matrix} \,\middle|\, 1 \right) \\ &= \frac{1}{k_1!k_2!(\frac{\nu_1+\nu_2}{2}+1)_{k_1}(\nu_2+1)_{k_2}} \sum_{0\leq i_1\leq k_1} \frac{(-k_1)_{i_1}(k_2+\nu_1+1)_{i_1}(-k_2-\nu_2)_{i_1}}{i_1!(\nu_1+1)_{i_1}(\frac{\nu_1-\nu_2}{2}+1)_{i_1}}. \end{aligned}$$

In view of

$$(-k_1)_{i_1} = (-1)^{i_1}\frac{k_1!}{(k_1-i_1)!}, \qquad \frac{(-k_2-\nu_2)_{i_1}}{(\nu_2+1)_{k_2}} = \frac{(-1)^{i_1}}{(\nu_2+1)_{k_2-i_1}}$$

and

$$\begin{aligned} \frac{(k_2+\nu_1+1)_{i_1}}{(\nu_1+1)_{i_1}} &= \sum_{0\leq i_2\leq \min(i_1,k_2)} \frac{(-k_1)_{i_2}(-i_1)_{i_2}}{i_2!(\nu_1+1)_{i_2}} \\ &= \sum_{0\leq i_2\leq \min(i_1,k_2)} \frac{k_1!i_1!}{(k_1-i_2)!(i_1-i_2)!i_2!(\nu_1+1)_{i_2}}, \end{aligned}$$

we can reach the desired formula. □

We now state another representation for $C^{\delta}_{n_1,n_2,n_3}$.

**Theorem 6.2.** *Under the same notation as in Theorem 5.3, $C^{\delta}_{n_1,n_2,n_3}$ can be written as follows:*
*(i) if $\delta_1=\kappa$, then*

$$C^{\delta}_{n_1,n_2,n_3}=\sum_{0\le k_1\le n_2}\frac{(-1)^{n_3}C^{o,(2\delta_2,2\delta_3)}_{k_1,n_3}}{n_1!(n_2-k_1)!(-\delta_2+\kappa+1)_{n_1-k_1}(-\delta_3+\kappa+1)_{n_2-n_3}};$$

*(ii) if $\delta_2=\kappa$, then*

$$C^{\delta}_{n_1,n_2,n_3}=\sum_{0\le k_1\le \min(n_1,n_2)}\frac{(-1)^{n_3}C^{o,(2\delta_1,2\delta_3)}_{k_1,n_3}}{(n_1-k_1)!(n_2-k_1)!(\delta_1-\kappa+1)_{n_1}(-\delta_3+\kappa+1)_{n_2-n_3}};$$

*(iii) if $\delta_3=\kappa$, then*

$$C^{\delta}_{n_1,n_2,n_3}=\sum_{0\le k_1\le n_1}\frac{(-1)^{n_3}C^{o,(2\delta_1,2\delta_2)}_{k_1,n_3}}{(n_1-k_1)!(n_2-n_3)!(\delta_1-\kappa+1)_{n_1}(\delta_2-\kappa+1)_{n_2-k_1}}$$

*for $n_2\ge n_3$ and $C^{\delta}_{n_1,n_2,n_3}=0$ for $n_2<n_3$.*

*Proof.* From the first difference equation in Lemma 5.1, we have only to check the following recurrence relation:

$$\begin{aligned}&\Delta_{n_1,n_2,n_3}(\delta_1-\delta_2,\delta_2-\delta_3,\delta_3,0)C^{\delta}_{n_1,n_2,n_3}\\&=C^{\delta}_{n_1-1,n_2,n_3}+C^{\delta}_{n_1,n_2-1,n_3}+\frac{1}{2}(-n_2+n_3+\delta_3-\kappa-1)C^{\delta}_{n_1,n_2,n_3-1}.\end{aligned}\tag{9}$$

Here $\Delta_{n_1,n_2,n_3}(a,b,c,d)$ is the symbol introduced in the previous section.

Let us consider the case of $\delta_1=\kappa$. Put

$$P_{n_1,n_2,n_3,k_1}=\frac{(-1)^{n_3}}{n_1!(n_2-k_1)!(-\delta_2+\kappa+1)_{n_1-k_1}(-\delta_3+\kappa+1)_{n_2-n_3}}.$$

Then the right hand side of (9) is written as

$$\begin{aligned}&\sum_{0\le k_1\le n_2}\frac{P_{n_1-1,n_2,n_3,k_1}}{P_{n_1,n_2,n_3,k_1}}\cdot P_{n_1,n_2,n_3,k_1}C^{o,(2\delta_2,2\delta_3)}_{k_1,n_3}\\&+\sum_{0\le k_1\le n_2-1}\frac{P_{n_1,n_2-1,n_3,k_1}}{P_{n_1,n_2,n_3,k_1}}\cdot P_{n_1,n_2,n_3,k_1}C^{o,(2\delta_2,2\delta_3)}_{k_1,n_3}\\&+\frac{1}{2}(-n_2+n_3+\delta_3-\kappa-1)\sum_{0\le k_1\le n_2}\frac{P_{n_1,n_2,n_3-1,k_1}}{P_{n_1,n_2,n_3,k_1}}\cdot P_{n_1,n_2,n_3,k_1}C^{o,(2\delta_2,2\delta_3)}_{k_1,n_3-1}.\end{aligned}\tag{10}$$

In view of

$$\begin{aligned}&\frac{P_{n_1-1,n_2,n_3,k_1}}{P_{n_1,n_2,n_3,k_1}}+\frac{P_{n_1,n_2-1,n_3,k_1}}{P_{n_1,n_2,n_3,k_1}}-\frac{P_{n_1,n_2,n_3,k_1+1}}{P_{n_1,n_2,n_3,k_1}}\\&=n_1(n_1-k_1-\delta_2+\kappa)+(n_2-k_1)(n_2-n_3-\delta_3+\kappa)-(n_2-k_1)(n_1-k_1-\delta_2+\kappa)\\&=\Delta_{n_1,n_2,n_3}(\delta_1-\delta_2,\delta_2-\delta_3,\delta_3,0)-\left\{k_1^2+\frac{1}{2}n_3^2-k_1n+(\delta_2-\delta_3)k_1+\delta_3n_3\right\}\end{aligned}$$

and

$$\frac{P_{n_1,n_2,n_3-1,k_1}}{P_{n_1,n_2,n_3,k_1}} = -\frac{1}{m_2 - n_3 - \delta_3 + \kappa + 1},$$

(10) becomes a sum of the following four terms:

$$\Delta_{n_1,n_2,n_3}(\delta_1 - \delta_2, \delta_2 - \delta_3, \delta_3, 0) \sum_{0 \le k_1 \le n_2} P_{n_1,n_2,n_3,k_1} C^{o,(2\delta_2,2\delta_3)}_{k_1,n_3},$$

$$\sum_{0 \le k_1 \le n_2} \frac{P_{n_1,n_2,n_3,k_1+1}}{P_{n_1,n_2,n_3,k_1}} \cdot P_{n_1,n_2,n_3,k_1} C^{o,(2\delta_2,2\delta_3)}_{k_1,n_3} = \sum_{0 \le k_1 \le n_2} P_{n_1,n_2,n_3,k_1} C^{o,(2\delta_2,2\delta_3)}_{k_1-1,n_3},$$

$$\frac{1}{2} \sum_{0 \le k_1 \le n_2} P_{n_1,n_2,n_3,k_1} C^{o,(2\delta_2,2\delta_3)}_{k_1,n_3-1},$$

and

$$-\sum_{0 \le k_1 \le n_2} \left\{ k_1^2 + \frac{1}{2} n_3^2 - k_1 n + (\delta_2 - \delta_3) k_1 + \delta_3 n_3 \right\} P_{n_1,n_2,n_3,k_1} C^{o,(2\delta_2,2\delta_3)}_{k_1,n_3}.$$

Now by means of the recurrence relation (8) for $C^{o,(2\delta_2,2\delta_3)}_{k_1,n_3}$, we can see that the sum of last three terms equals to zero, and thus we finish the proof of (9). The cases of $\delta_2 = \kappa$ and $\delta_3 = \kappa$ can be similarly done. □

**Remark 6.3.** *We can directly derive the second expressions for $C^{\delta}_{n_1,n_2,n_3}$ from Theorem 5.3 by applying the identity*

$$\frac{\Gamma(p+q+m+1)}{\Gamma(p+1)\Gamma(q+m+1)\Gamma(p+q+1)} = \sum_{0 \le i \le m} \frac{m!}{i!(m-i)!\Gamma(p-i+1)\Gamma(q+i+1)},$$

*which is equivalent to Gauss' summation formula (*[37]*).*

## 7. Primary Whittaker functions

In this section we consider the space $\mathrm{Wh}(\pi, \eta, \tau)^{\mathrm{mod}}$. According to [30] the dimension this space is at most one. We call the unique (up to a constant) element in this space *primary Whittaker function* and give an explicit integral formula of (a radial part of) it. As in the previous section our formula comes from the class one Whittaker function on $SO(5, \mathbf{R})$, that is, unique moderate growth Whittaker function for the class one principal series representation $\pi_{(\nu_1,\nu_2)}$ on $SO(5, \mathbf{R})$. Let us recall the integral representation of it.

**Proposition 7.1.** *(*[11, Theorem 4.2]*) Let $W^o(y) = y_1^{\frac{3}{2}} y_2^2 \widetilde{W}^o(y)$ with*

$$\begin{aligned}\widetilde{W}^o(y) &= 4 y_1^{-\frac{\nu_1+\nu_2}{2}} y_2^{\frac{\nu_1+\nu_2}{2}} \\ &\times \int_0^\infty \int_0^\infty K_{\frac{\nu_1-\nu_2}{2}}\left(2y_1\sqrt{(1+1/u_1)(1+1/u_2)}\right) K_{\frac{\nu_1+\nu_2}{2}}\left(2y_2\sqrt{1+u_1+u_2}\right) \\ &\quad \times \left(\frac{u_1^2 u_2^2}{1+u_1+u_2}\right)^{\frac{\nu_1+\nu_2}{4}} \left(\frac{u_1(1+u_1)}{u_2(1+u_2)}\right)^{\frac{\nu_1-\nu_2}{4}} \frac{du_1}{u_1}\frac{du_2}{u_2}.\end{aligned}$$

*Here $K_\nu(z)$ is the $K$-Bessel function ([37]). Then, up to a constant multiple, $W^o(y)$ is the radial part of the class one Whittaker function on $SO(5,\mathbf{R})$.*

Hashizume ([4]) obtained explicit linear relations between the class one Whittaker functions and the secondary Whittaker functions for arbitrary semisimple Lie groups. See also [11, §4] and [12, §4] for an elementary proof of Hashizume's result in the case of $SO(5,\mathbf{R})$.

**Proposition 7.2.** *([4, Theorem 7.8], [11, Theorem 4.2])*

$$W^o(y) = \sum_{w\in\mathcal{W}} w\left[\Gamma(-\nu_1)\Gamma(-\nu_2)\Gamma\left(-\frac{\nu_1+\nu_2}{2}\right)\Gamma\left(-\frac{\nu_1-\nu_2}{2}\right)M^o_{(\nu_1,\nu_2)}(y)\right].$$

*Here $\mathcal{W} = \langle w_1, w_2\rangle \cong \mathfrak{S}_2 \ltimes (\mathbf{Z}/2\mathbf{Z})^2$ is the Weyl group of $SO(5,\mathbf{R})$, and the action of the generators on the parameter of the class one principal series $\pi_{(\nu_1,\nu_2)}$ is given by $w_1(\nu_1,\nu_2) = (\nu_2,\nu_1)$ and $w_2(\nu_1,\nu_2) = (-\nu_1,\nu_2)$.*

Our main result in this section is as follows.

**Theorem 7.3.** *Let $W(x) = x_1^{\frac{3}{2}}x_2^{\frac{5}{2}}x_3^3\exp(-\frac{x_3}{2})\widetilde{W}(x)$ with*

$$\begin{aligned}\widetilde{W}(x) = &\int_0^\infty\int_0^\infty \exp\left(-t_1 - t_2 - \frac{x_1}{t_1} - \frac{x_2}{t_2}\right)\\ &\times\left(\frac{x_1x_2x_3}{t_1t_2}\right)^\kappa \widetilde{W^o}\left(\sqrt{x_2\frac{t_1}{t_2}}, \sqrt{x_3t_2}\right)\frac{dt_1}{t_1}\frac{dt_2}{t_2}.\end{aligned} \tag{11}$$

*Then, up to a constant multiple, $W(x)$ is the radial part of the primary Whittaker function and can be written as a linear combination of the secondary Whittaker functions $M_\delta(x)$ in Theorem 5.3 as follows:*

$$\begin{aligned}W(x) = \sum_{w\in\mathcal{W}} w\Bigg[&\Gamma(-\nu_1)\Gamma(-\nu_2)\Gamma\left(-\frac{\nu_1+\nu_2}{2}\right)\Gamma\left(-\frac{\nu_1-\nu_2}{2}\right)\\ &\times\left\{\Gamma\left(\frac{\nu_1}{2}-\kappa\right)\Gamma\left(\frac{\nu_2}{2}-\kappa\right)M_{(\kappa,\frac{\nu_1}{2},\frac{\nu_2}{2})}(x)\right.\\ &+\Gamma\left(-\frac{\nu_1}{2}+\kappa\right)\Gamma\left(\frac{\nu_2}{2}-\kappa\right)M_{(\frac{\nu_1}{2},\kappa,\frac{\nu_2}{2})}(x)\\ &\left.+\Gamma\left(-\frac{\nu_1}{2}+\kappa\right)\Gamma\left(-\frac{\nu_2}{2}+\kappa\right)M_{(\frac{\nu_1}{2},\frac{\nu_2}{2},\kappa)}(x)\right\}\Bigg].\end{aligned} \tag{12}$$

*Proof.* The moderate growth property follows from the rapid decay of the class one Whittaker function $W^o$ on $SO(5,\mathbf{R})$. Since the dimension of the space $\mathrm{Wh}(\pi,\eta,\tau)^{\mathrm{mod}}$ is at most one ([30]), we need to show $W(x)$ is contained in the space $\mathrm{Wh}(\pi,\eta,\tau)^{\mathrm{mod}}$. Hence it is enough to prove the expansion formula (12).

We substitute the expansion formula for $W^o$ into (11) to find

$$\begin{aligned}\widetilde{W}(x) = \sum_w w\Bigg[&\Gamma(-\nu_1)\Gamma(-\nu_2)\Gamma\left(-\frac{\nu_1+\nu_2}{2}\right)\Gamma\left(-\frac{\nu_1-\nu_2}{2}\right)\\ &\times \int_0^\infty\int_0^\infty \exp\left(-t_1-t_2-\frac{x_1}{t_1}-\frac{x_2}{t_2}\right)\cdot\left(\frac{x_1x_2x_3}{t_1t_2}\right)^\kappa\\ &\times \sum_{k_1,k_2\geq 0} C^{o,(\nu_1,\nu_2)}_{k_1,k_2}\left(x_2\frac{t_1}{t_2}\right)^{k_1+\frac{\nu_1}{2}}(x_3t_2)^{k_2+\frac{\nu_1+\nu_2}{2}}\frac{dt_1}{t_1}\frac{dt_2}{t_2}\Bigg].\end{aligned} \tag{13}$$

If we allow the change of the order of integration and infinite sum in (13) (this interchange is justified later), it can be written as

$$\begin{aligned}\widetilde{W}(x) = \sum_w w\Bigg[&\Gamma(-\nu_1)\Gamma(-\nu_2)\Gamma\left(-\frac{\nu_1+\nu_2}{2}\right)\Gamma\left(-\frac{\nu_1-\nu_2}{2}\right)\\ &\times \sum_{k_1,k_2\geq 0} C^{o,(\nu_1,\nu_2)}_{k_1,k_2} x_3^{k_2+\frac{\nu_1+\nu_2}{2}+\kappa} J_1(x_1)J_2(x_2)\Bigg],\end{aligned} \tag{14}$$

with

$$\begin{aligned}J_1(x_1) &= x_1^\kappa\int_0^\infty \exp\left(-t_1-\frac{x_1}{t_1}\right)t_1^{k_1+\frac{\nu_1}{2}-\kappa}\frac{dt_1}{t_1},\\ J_2(x_2) &= x_2^{k_1+\frac{\nu_1}{2}+\kappa}\int_0^\infty \exp\left(-t_2-\frac{x_2}{t_2}\right)t_2^{-k_1+k_2+\frac{\nu_2}{2}-\kappa}\frac{dt_2}{t_2}.\end{aligned}$$

By using

$$\begin{aligned}&\int_0^\infty \exp\left(-t-\frac{x}{t}\right)t^{s+k}\frac{dt}{t}\\ &= 2x^{\frac{s+k}{2}}K_{s+k}(2\sqrt{x})\\ &= \frac{\pi}{\sin(s+k)\pi}\left(\sum_{i\geq 0}\frac{x^i}{i!\Gamma(i-s-k+1)}-\sum_{i\geq 0}\frac{x^{i+s}}{i!\Gamma(i+s+k+1)}\right)\\ &= (-1)^k\left(\sum_{i\geq 0}\frac{\Gamma(s)\cdot x^i}{i!(-s+1)_{i-k}}+\sum_{i\geq 0}\frac{\Gamma(-s)\cdot x^{i+s+k}}{i!(s+1)_{i+k}}\right)\end{aligned}$$

for $k\in\mathbf{Z}$ and $s\notin\mathbf{Z}$ ([37]), we get

$$J_1(x_1) = (-1)^{k_1}\left\{\sum_{n_1\geq 0}\frac{\Gamma(\frac{\nu_1}{2}-\kappa)\cdot x_1^{n_1+\kappa}}{n_1!(-\frac{\nu_1}{2}+\kappa+1)_{n_1-k_1}}+\sum_{n_1\geq 0}\frac{\Gamma(-\frac{\nu_1}{2}+\kappa)\cdot x_1^{n_1+k_1+\frac{\nu_1}{2}}}{n_1!(\frac{\nu_1}{2}-\kappa+1)_{n_1+k_1}}\right\},$$

$$J_2(x_2) = (-1)^{k_1-k_2}\left\{\sum_{n_2\geq 0}\frac{\Gamma(\frac{\nu_2}{2}-\kappa)\cdot x_2^{n_2+k_1+\frac{\nu_1}{2}+\kappa}}{n_2!(-\frac{\nu_2}{2}+\kappa+1)_{n_2+k_1-k_2}}+\sum_{n_2\geq 0}\frac{\Gamma(-\frac{\nu_2}{2}+\kappa)\cdot x_2^{n_2+k_2+\frac{\nu_1+\nu_2}{2}}}{n_2!(\frac{\nu_2}{2}-\kappa+1)_{n_2-k_1+k_2}}\right\}.$$

We substitute the above into (14) and arrange the order of summation to find

$$\begin{aligned}W^o(x) = \sum_w w\Big[&\Gamma(-\nu_1)\Gamma(-\nu_2)\Gamma\left(-\frac{\nu_1+\nu_2}{2}\right)\Gamma\left(-\frac{\nu_1-\nu_2}{2}\right)\\
&\times\Big(\sum_{n_1,n_2,n_3\geq 0} C^1_{n_1,n_2,n_3} x_1^{n_1+\kappa} x_2^{n_2+\frac{\nu_1}{2}+\kappa} x_3^{n_3+\frac{\nu_1+\nu_2}{2}+\kappa}\\
&+\sum_{n_1,n_2,n_3\geq 0} C^2_{n_1,n_2,n_3} x_1^{n_1+\kappa} x_2^{n_2+\frac{\nu_1+\nu_2}{2}} x_3^{n_3+\frac{\nu_1+\nu_2}{2}+\kappa}\\
&+\sum_{n_1,n_2,n_3\geq 0} C^3_{n_1,n_2,n_3} x_1^{n_1+\frac{\nu_1}{2}} x_2^{n_2+\frac{\nu_1}{2}+\kappa} x_3^{n_3+\frac{\nu_1+\nu_2}{2}+\kappa}\\
&+\sum_{n_1,n_2,n_3\geq 0} C^4_{n_1,n_2,n_3} x_1^{n_1+\frac{\nu_1}{2}} x_2^{n_2+\frac{\nu_1+\nu_2}{2}} x_3^{n_3+\frac{\nu_1+\nu_2}{2}+\kappa}\Big)\Big],\end{aligned}$$

where

$$C^1_{n_1,n_2,n_3} = \sum_{0\leq k_1\leq n_2} \frac{\Gamma(\frac{\nu_1}{2}-\kappa)\Gamma(\frac{\nu_2}{2}-\kappa)\cdot(-1)^{n_3}C^{o,(\nu_1,\nu_2)}_{k_1,n_3}}{n_1!(n_2-k_1)!(-\frac{\nu_1}{2}+\kappa+1)_{n_1-k_1}(-\frac{\nu_2}{2}+\kappa+1)_{n_2-n_3}},$$

$$C^2_{n_1,n_2,n_3} = \sum_{0\leq k_1} \frac{\Gamma(\frac{\nu_1}{2}-\kappa)\Gamma(-\frac{\nu_2}{2}+\kappa)\cdot(-1)^{n_3}C^{o,(\nu_1,\nu_2)}_{k_1,n_3}}{n_1!(n_2-n_3)!(-\frac{\nu_1}{2}+\kappa+1)_{n_1-k_1}(\frac{\nu_2}{2}-\kappa+1)_{n_2-k_1}},$$

$$C^3_{n_1,n_2,n_3} = \sum_{0\leq k_1\leq \min(n_1,n_2)} \frac{\Gamma(-\frac{\nu_1}{2}+\kappa)\Gamma(\frac{\nu_2}{2}-\kappa)\cdot(-1)^{n_3}C^{o,(\nu_1,\nu_2)}_{k_1,n_3}}{(n_1-k_1)!(n_2-k_1)!(\frac{\nu_1}{2}-\kappa+1)_{n_1}(-\frac{\nu_2}{2}+\kappa+1)_{n_2-n_3}},$$

$$C^4_{n_1,n_2,n_3} = \sum_{0\leq k_1\leq n_1} \frac{\Gamma(-\frac{\nu_1}{2}+\kappa)\Gamma(-\frac{\nu_2}{2}+\kappa)\cdot(-1)^{n}C^{o,(\nu_1,\nu_2)}_{k_1,n_3}}{(n_1-k_1)!(n_2-n_3)!(\frac{\nu_1}{2}-\kappa+1)_{n_1}(\frac{\nu_2}{2}-\kappa+1)_{n_2-k_1}}.$$

Here $C^2_{n_1,n_2,n_3} = C^4_{n_1,n_2,n_3} = 0$ for $n_2 < n_3$. The following lemma ensures the change of the order of integration and infinite sum in (13).

**Lemma 7.4.** *The coefficient $C^i_{n_1,n_2,n_3}$ satisfies the recurrence relation*

$$\begin{aligned}&\Delta_{n_1,n_2,n_3}(a_i,b_i,c_i,d_i)C^2_{n_1,n_2,n_3}\\
&= C^i_{n_1-1,n_2,n_3} + C^i_{n_1,n_2-1,n_3} + \frac{1}{2}(-n_2+n_3+c_i-\kappa-1)C^i_{n_1,n_2,n_3-1},\end{aligned} \tag{15}$$

*with*

$$(a_i,b_i,c_i,d_i) = \begin{cases} \left(\kappa-\frac{\nu_1}{2}, \frac{\nu_1-\nu_2}{2}, \frac{\nu_2}{2}, 0\right) & i=1,\\ \left(-\frac{\nu_1+\nu_2}{2}+2\kappa, \frac{\nu_1+\nu_2}{2}-2\kappa, \kappa, (\frac{\nu_1}{2}-\kappa)(\frac{\nu_2}{2}-\kappa)\right) & i=2,\\ \left(\frac{\nu_1}{2}-\kappa, \kappa-\frac{\nu_2}{2}, \frac{\nu_2}{2}, 0\right) & i=3,\\ \left(\frac{\nu_1-\nu_2}{2}, \frac{\nu_2}{2}-\kappa, \kappa, 0\right) & i=4. \end{cases}$$

*Moreover the power series $\sum_{n_1,n_2,n_3\geq 0} C^i_{n_1,n_2,n_3} x_1^{n_1}x_2^{n_2}x_3^{n_3}$ converges absolutely and uniformly for $(\nu_1,\nu_2) \in \{(\nu_1,\nu_2)\in \mathbf{C}^2 \mid \Delta_{n_1,n_2,n_3}(a_i,b_i,c_i,d_i)\neq 0,\ \forall(n_1,n_2,n_3)\in \mathbf{N}^3\backslash\{(0,0,0)\}\}$ and $(x_1,x_2,x_3)\in\mathbf{R}^3_+$.*

*Proof of lemma.* The claim for $i = 1, 3, 4$ is immediate from Theorem 6.2. The case of $C^2_{n_1,n_2,n_3}$ is similarly done. Actually (15) follows from the identity:

$$\begin{aligned} & n_1\left(n_1 - k_1 - \frac{\nu_1}{2} + \kappa\right) + (n_2 - n_3)\left(n_2 - k_1 + \frac{\nu_2}{2} - \kappa\right) \\ & - \left(n_1 - k_1 - \frac{\nu_1}{2} + \kappa\right)\left(n_2 - k_1 + \frac{\nu_2}{2} - \kappa\right) \\ & = \Delta_{n_1,n_2,n_3}(a_2, b_2, c_2, d_2) - \left(k_1^2 + \frac{1}{2}n_3^2 - k_1 n_3 + \frac{\nu_1 - \nu_2}{2}k_1 + \frac{\nu_2}{2}n_3\right). \end{aligned}$$

Therefore, in view of Lemma 5.2, we can finish the proof of lemma. □

Let us return to the proof of Theorem 7.3. From Lemma 7.4 and Theorem 6.2, it suffices to show

$$\begin{aligned} \sum_w w \Bigg[ & \Gamma(-\nu_1)\Gamma(-\nu_2)\Gamma\left(-\frac{\nu_1+\nu_2}{2}\right)\Gamma\left(-\frac{\nu_1-\nu_2}{2}\right) \\ & \times \sum_{n_1,n_2,n_3 \geq 0} C^2_{n_1,n_2,n_3} x_1^{n_1+\kappa} x_2^{n_2+\frac{\nu_1+\nu_2}{2}} x_3^{n_3+\frac{\nu_1+\nu_2}{2}+\kappa} \Bigg] = 0 \end{aligned}$$

to prove the expansion formula (12). Hence the following lemma concludes Theorem 7.3. □

**Lemma 7.5.** *Put*

$$B^{(\nu_1,\nu_2)}_{n_1,n_2,n_3} = \Gamma(-\nu_1)\Gamma(-\nu_2)\Gamma\left(-\frac{\nu_1+\nu_2}{2}\right)\Gamma\left(-\frac{\nu_1-\nu_2}{2}\right) \cdot C^2_{n_1,n_2,n_3},$$

*and*

$$\widetilde{B}^{(\nu_1,\nu_2)}_{n_1,n_2,n_3} = B^{(\nu_1,\nu_2)}_{n_1,n_2,n_3} + B^{w_1(\nu_1,\nu_2)}_{n_1,n_2,n_3} = B^{(\nu_1,\nu_2)}_{n_1,n_2,n_3} + B^{(\nu_2,\nu_1)}_{n_1,n_2,n_3}.$$

*Then we have* $\widetilde{B}^{(\nu_1,\nu_2)}_{n_1,n_2,n_3} = 0$ *for all* $(n_1, n_2, n_3) \in \mathbf{N}^3$.

*Proof of lemma.* We can see that the recurrence relation (15) is invariant under the action of $w_1$ on $(\nu_1, \nu_2)$, then $B^{w_1(\nu_1,\nu_2)}_{n_1,n_2,n_3}$ satisfies (15), therefore $\widetilde{B}^{(\nu_1,\nu_2)}_{n_1,n_2,n_3}$ also satisfies it. Then if we can verify $\widetilde{B}^{(\nu_1,\nu_2)}_{0,0,0} = 0$, we may conclude that $\widetilde{B}^{(\nu_1,\nu_2)}_{n_1,n_2,n_3} = 0$ inductively. Since the recurrence relation (8) leads

$$C^{o,(\nu_1,\nu_2)}_{k_1,0} = \frac{\Gamma(\frac{\nu_1-\nu_2}{2}+1)}{k_1!\Gamma(k_1 + \frac{\nu_1-\nu_2}{2}+1)},$$

we have

$$\begin{aligned} B^{(\nu_1,\nu_2)}_{0,0,0} = {} & \Gamma(-\nu_1)\Gamma(-\nu_2)\Gamma\left(-\frac{\nu_1+\nu_2}{2}\right)\Gamma\left(-\frac{\nu_1-\nu_2}{2}\right)\Gamma\left(\frac{\nu_1}{2}-\kappa\right)\Gamma\left(-\frac{\nu_2}{2}+\kappa\right) \\ & \times \sum_{k_1 \geq 0} \frac{\Gamma(-\frac{\nu_1}{2}+\kappa+1)\Gamma(\frac{\nu_2}{2}-\kappa+1)}{\Gamma(-k_1-\frac{\nu_1}{2}+\kappa+1)\Gamma(-k_1+\frac{\nu_2}{2}-\kappa+1)} \cdot \frac{\Gamma(\frac{\nu_1-\nu_2}{2}+1)}{k_1!\Gamma(k_1+\frac{\nu_1-\nu_2}{2}+1)}. \end{aligned}$$

By means of the identity:

$$\sum_{k_1 \geq 0} \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{k_1!\Gamma(a-k_1)\Gamma(b-k_1)\Gamma(c+k_1)} = \frac{\Gamma(c)\Gamma(a+b+c-2)}{\Gamma(a+c-1)\Gamma(b+c-1)},$$

which follows from Gauss' summation formula ([37]), we get

$$\begin{aligned}B^{(\nu_1,\nu_2)}_{0,0,0} &= \Gamma(-\nu_1)\Gamma(-\nu_2)\Gamma\left(-\frac{\nu_1+\nu_2}{2}\right)\Gamma\left(-\frac{\nu_1-\nu_2}{2}\right)\Gamma\left(\frac{\nu_1}{2}-\kappa\right)\Gamma\left(-\frac{\nu_2}{2}+\kappa\right)\\ &\quad\times\frac{\Gamma(\frac{\nu_1-\nu_2}{2}+1)}{\Gamma(-\frac{\nu_2}{2}+\kappa+1)\Gamma(\frac{\nu_1}{2}-\kappa+1)}\\ &= \Gamma(-\nu_1)\Gamma(-\nu_2)\Gamma\left(-\frac{\nu_1+\nu_2}{2}\right)\cdot\frac{\pi}{\sin\,(-\frac{\nu_1-\nu_2}{2})\pi}\cdot\frac{1}{(\frac{\nu_1}{2}-\kappa)(-\frac{\nu_2}{2}+\kappa)}.\end{aligned}$$

Then we can see that $\widetilde{B}^{(\nu_1,\nu_2)}_{0,0,0} = B^{(\nu_1,\nu_2)}_{0,0,0} + B^{w_1(\nu_1,\nu_2)}_{0,0,0} = 0$. □

**Remark 7.6.** *Miyazaki and Oda ([22]) obtained the explicit formula of the $P_J$-principal series Whittaker functions on $Sp(2,\mathbf{R})$. The radial part of the primary Whittaker function is $W^2(x_1,x_2) = x_1x_2^{\frac{3}{2}}\exp(-\frac{x_2}{2})\widetilde{W}^2(x_1,x_2)$ with*

$$\widetilde{W}^2(x_1,x_2) = x_1^{\frac{k-1}{2}}x_2^{k-1}\int_0^\infty t^{-k+\frac{1}{2}}W_{0,\nu}(t)\exp\left(-\frac{t^2}{16x_2}-\frac{16x_1x_2}{t^2}\right)\frac{dt}{t}.$$

*Here $W_{\kappa,\mu}$ is the classical Whittaker function ([37]). By using $W_{0,\nu}(t) = \sqrt{t/\pi}K_\nu(t/2)$ and substituting $t \to 4\sqrt{tx_2}$, we arrive at*

$$\widetilde{W}^2(x_1,x_2) = 2^{1-2k}\pi^{-\frac{1}{2}}\int_0^\infty\left(\frac{x_1x_2}{t}\right)^{\frac{k-1}{2}}K_\nu\left(2\sqrt{tx_2}\right)\exp\left(-t-\frac{x_1}{t}\right)\frac{dt}{t}.$$

*We note that $\sqrt{x}K_\nu(x)$ is a radial part of the class one Whittaker function on $SO(3,\mathbf{R})$. Hence we may expect the primary $P_J$-principal series Whittaker functions on $Sp(n,\mathbf{R})$ are written in terms of the class one Whittaker functions $SO(2n+1,\mathbf{R})$.*

## 8. Mellin-Barnes integral representations

In this section we give a Mellin-Barnes integral representation of the primary Whittaker function $W(x)$. A (single) Mellin-Barnes integral is the contour integral of the form

$$\int_z F(z)z^s\,dz,$$

where the path of integration is a line parallel to the imaginary axis in the complex plane, of sufficiently large real part to ensure that all the poles of $F(z)$ on its left.

A Mellin-Barnes integral representation for the class one Whittaker function $W^o(y) = y_1^{\frac{3}{2}}y_2^2\widetilde{W}^o(y)$ on $SO(5,\mathbf{R})$ is given in [11] (cf. [12]). We first rewrite it.

**Proposition 8.1.** *We have*

$$\widetilde{W}^o(y) = \frac{2^2}{(2\pi\sqrt{-1})^2}\int_{\sigma_1}\int_{\sigma_2}V^o(\sigma_1,\sigma_2)\,y_1^{-2\sigma_1}y_2^{-2\sigma_2}d\sigma_2d\sigma_1,$$

*where the 2-chain of integration is a product of paths given by*

$$(\rho_1-\sqrt{-1}\infty,\ \rho_1+\sqrt{-1}\infty)\times(\rho_2-\sqrt{-1}\infty,\ \rho_2+\sqrt{-1}\infty)$$

*with the real numbers $\rho_1$ and $\rho_2$ fixed as*

$$\rho_1 > \sup\{|\mathrm{Re}(\nu_1)|, |\mathrm{Re}(\nu_2)|\};\ \rho_2 > \sup\{|\mathrm{Re}(\nu_1+\nu_2)|, |\mathrm{Re}(\nu_1-\nu_2)|\},$$

*and*

$$\begin{aligned} V^o(\sigma_1,\sigma_2) &= \Gamma\left(\sigma_1+\frac{\nu_2}{2}\right)\Gamma\left(\sigma_1-\frac{\nu_2}{2}\right)\Gamma(\sigma_2) \\ &\times \frac{1}{2\pi\sqrt{-1}}\int_\tau \frac{\Gamma(\sigma_1+\tau)\Gamma(\sigma_2+\tau+\frac{\nu_2}{2})\Gamma(\sigma_2+\tau-\frac{\nu_2}{2})\Gamma(-\tau+\frac{\nu_1}{2})\Gamma(-\tau-\frac{\nu_1}{2})}{\Gamma(\sigma_1+\sigma_2+\tau)}d\tau. \end{aligned}$$

*Proof.* By [11, p.532 (4.5)],

$$\begin{aligned} V^o(\sigma_1,\sigma_2) &= \Gamma\left(\sigma_1+\frac{\nu_1}{2}\right)\Gamma\left(\sigma_1-\frac{\nu_1}{2}\right)\Gamma\left(\sigma_1+\frac{\nu_2}{2}\right)\Gamma\left(\sigma_1-\frac{\nu_2}{2}\right) \\ &\times \frac{\Gamma(\sigma_2)\Gamma(\sigma_2+\frac{\nu_1+\nu_2}{2})\Gamma(\sigma_2+\frac{\nu_1-\nu_2}{2})\Gamma(\sigma_2-\frac{\nu_1-\nu_2}{2})}{\Gamma(\sigma_1+\sigma_2+\frac{\nu_1}{2})\Gamma(\sigma_1+\sigma_2+\frac{\nu_2}{2})} \\ &\times {}_3F_2\left(\begin{matrix}\sigma_1+\frac{\nu_1}{2},\ \sigma_1+\frac{\nu_2}{2},\ \sigma_2+\frac{\nu_1+\nu_2}{2} \\ \sigma_1+\sigma_2+\frac{\nu_1}{2},\ \sigma_1+\sigma_2+\frac{\nu_2}{2}\end{matrix}\ \middle|\ 1\right). \end{aligned}$$

If we use the formula

$$\begin{aligned} &\frac{\Gamma(a)\Gamma(b)\Gamma(d-a)\Gamma(d-b)\Gamma(e-c)}{\Gamma(d)\Gamma(e)}\,{}_3F_2\left(\begin{matrix}a,\ b,\ c \\ d,\ e\end{matrix}\ \middle|\ 1\right) \\ &= \frac{1}{2\pi\sqrt{-1}}\int_\tau \frac{\Gamma(a+\tau)\Gamma(b+\tau)\Gamma(e-c+\tau)\Gamma(d-a-b-\tau)\Gamma(-\tau)}{\Gamma(e+\tau)}\,d\tau \end{aligned}$$

([31, 4.2.2]) with $a=\sigma_1+\frac{\nu_1}{2}$, $b=\sigma_2+\frac{\nu_1+\nu_2}{2}$, $c=\sigma_1+\frac{\nu_2}{2}$, $d=\sigma_1+\sigma_2+\frac{\nu_2}{2}$, $e=\sigma_1+\sigma_2+\frac{\nu_1}{2}$ and substitute $\tau\to\tau-\frac{\nu_1}{2}$, we find the desired expression. □

**Theorem 8.2.** *We have*

$$\begin{aligned} \widetilde{W}(x) &= \frac{4}{(2\pi\sqrt{-1})^4}\int_{\sigma_1}\int_{\sigma_2}\int_{s_1}\int_{s_2} V^o(\sigma_1,\sigma_2)\Gamma(s_1)\Gamma(s_1-\sigma_1-\kappa) \\ &\times \Gamma(s_2)\Gamma(s_2+\sigma_1-\sigma_2-\kappa)\,x_1^{-s_1+\kappa}x_2^{-s_2-\sigma_1+\kappa}x_3^{-\sigma_2+\kappa}ds_2ds_1d\sigma_2d\sigma_1. \end{aligned}$$

*Here the 4-chain of multiple Mellin-Barnes integrals is taken such that*

$$\mathrm{Re}(\sigma_1+\kappa) > \mathrm{Re}(s_1) > 0;\ \mathrm{Re}(-\sigma_1+\sigma_2+\kappa) > \mathrm{Re}(s_2) > 0;$$
$$\mathrm{Re}(\sigma_1) > \sup\{|\mathrm{Re}(\nu_1)|, |\mathrm{Re}(\nu_2)|\};\ \mathrm{Re}(\sigma_2) > \sup\{|\mathrm{Re}(\nu_1+\nu_2)|, |\mathrm{Re}(\nu_1-\nu_2)|\}.$$

*Proof.* We substitute the Mellin-Barnes integral representation of $\widetilde{W}^o(y)$ into (11) to find

$$\begin{aligned} \widetilde{W}(x) &= \frac{4}{(2\pi\sqrt{-1})^2}\int_{\sigma_1}\int_{\sigma_2} V^o(\sigma_1,\sigma_2)\int_0^\infty\!\!\int_0^\infty \exp\left(-t_1-t_2-\frac{x_1}{t_1}-\frac{x_2}{t_2}\right) \\ &\times \left(\frac{x_1x_2x_3}{t_1t_2}\right)^\kappa\left(x_2\frac{t_1}{t_2}\right)^{-\sigma_1}(x_3t_2)^{-\sigma_2}\frac{dt_1}{t_1}\frac{dt_2}{t_2}d\sigma_2d\sigma_1. \end{aligned}$$

In view of

$$\int_0^\infty \exp\left(-t-\frac{x}{t}\right)t^\alpha\frac{dt}{t} = 2x^{\frac{\alpha}{2}}K_\alpha\left(2\sqrt{x}\right)$$

$$= \frac{1}{2\pi\sqrt{-1}} \int_s \Gamma(s)\Gamma(s+\alpha)x^{-s}\, ds,$$

for $\alpha \in \mathbf{C}$ ([37]) we have the assertion. □

**Correction** There is an obvious mistake in our previous paper [8]. In the definition of $I(P_J; \sigma, \nu)$ in the end of section 1, $\exp(\nu+1)$ should be $\exp(\nu+2)$. We thank Mr. Tadashi Miyazaki for pointing out this error.

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