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by

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Chapter 1

Introduction

Historically, statistical mechanics and Gibbs measure were derived from thermodynamics and classical mechanics independently. Precisely, one can get statistical mechanics from physical fundemental equation, Newtonian mechanics, or quantum mechanics, under Boltzman's ergodic hypothesis. After the ergodic hypothesis was introduced by Boltzman, in order the justify it, von Neumann and Birkhoff developed ergodic theorems.

However, we have to emphasis that, although ergodic hypothesis can justify Gibbs' measure, it is not enough to explain all of the phenomena.

It would be a very interesting and important question to derive the results (facts) of statistical mechanics from classical mechanics or quantum mechanics directly. However, this is almost not done.

The simpliest example would be Brownian motion. This problem was discussed by, *e.g.*, Holley [7], Dürr-Goldstein-Lebowitz [3], [4], [5], Calderoni-Dürr-Kusuoka [2], *etc.*. Brownion motion was first observed, without knowing the reason, by Brown in 1827, as the irregular motion of a rather big particle which is put into water. This was later explained by Einstein in the following way: since a big number of water atoms collide with the big particle randomly, the motion of the big particle could be considered as a sum of many *i.i.d.* random variables, so after taking limit, this will give us a Brownian motion. This is also the explaination given by many physical textbooks.

However, we have to notice that, in real problems, there exists the possibility of recollision, moreover, when considering the problem of interaction caused by potentials, the state of each small particle is not independent to the history. Therefore, the actual motion is not a sum of *i.i.d.* random variables, and we have to construct some new model, which includes the mentioned re-interaction. This is the aim of this research.

Let us describe our problem and results in detail now. Let $m > 0$, $N \geq 1$, $d \geq 1$, and $M_1, \cdots, M_N > 0$. Here N stands for the number of big particles (atoms), $M_1, \cdots, M_N$ for the masses of each atom, m for the mass of the small particles, and d is the dimension of the space. Let $U_i \in C_0^\infty(\mathbf{R}^d)$, $i = 1, \cdots, N$. Also, let

$X_{i,0}, V_{i,0} \in \mathbf{R}^d$, $i = 1, \cdots, N$, which stands for the initial positions and initial speeds of the big particles. For any $\omega \in Conf(\mathbf{R}^d \times \mathbf{R}^d)$, we consider the infinite system given by the following ODE (so we are considering the case when there is no direct interaction between big particles or between small particles):

$$\begin{cases} \dfrac{d}{dt} X_i^{(m)}(t,\omega) = V_i^{(m)}(t,\omega), \\ M_i \dfrac{d}{dt} V_i^{(m)}(t,\omega) = -\displaystyle\int_{\mathbf{R}^d \times \mathbf{R}^d} \nabla U_i(X_i^{(m)}(t,\omega) - x^{(m)}(t,x,v,\omega))\mu_\omega(dx,dv), \\ (X_i^{(m)}(0,\omega), V_i^{(m)}(0,\omega)) = (X_{i,0}, V_{i,0}), \qquad i = 1, \cdots, N, \\ \dfrac{d}{dt} x^{(m)}(t,x,v,\omega) = v^{(m)}(t,x,v,\omega), \\ m \dfrac{d}{dt} v^{(m)}(t,x,v,\omega) = -\displaystyle\sum_{i=1}^N \nabla U_i(x^{(m)}(t,x,v,\omega) - X_i^{(m)}(t,\omega)), \\ (x^{(m)}(0,x,v,\omega), v^{(m)}(0,x,v,\omega)) = (x,v). \end{cases} \tag{1.0.1}$$

Here $Conf(\mathbf{R}^d \times \mathbf{R}^d)$ means the set of all non-empty closed subsets of $\mathbf{R}^d \times \mathbf{R}^d$ which have no cluster point. (See Chapter 2 for more discussion about the structure of closed sets). We will omit the superscript (m) when there is no risk of confusion.

Let $\rho : \mathbf{R} \to [0, \infty)$ be a continuous function such that $\rho(s) \to 0$ rapidly as $s \to \infty$. Let λ_m be the non-atomic Radon measure on $\mathbf{R}^d \times \mathbf{R}^d$ given by

$$\lambda_m(dx, dv) = m^{\frac{d-1}{2}} \rho\Big(\frac{m}{2}|v|^2 + \sum_{i=1}^N U_i(x - X_{i,0})\Big) dx dv,$$

and let $P_m(d\omega)$ be the Poisson point process determined by λ_m. So P_m is a probability measure on $Conf(\mathbf{R}^d \times \mathbf{R}^d)$. (See Chapters 2 and 3 for the definition and properties of Poisson point process).

We are mostly interested in the following two problems:

1. **Existence of the solution**.

2. **The limit behavior of the distribution of**

$$\begin{aligned} &(\vec{X}^{(m)}(t,\omega), \vec{V}^{(m)}(t,\omega)) \\ &= ((X_1^{(m)}(t,\omega), \cdots, X_N^{(m)}(t,\omega)), (V_1^{(m)}(t,\omega), \cdots, V_N^{(m)}(t,\omega))) \end{aligned}$$

under $P_m(d\omega)$ **as** $m \to 0$**.**

For the first problem, we have the following result (see Section 4.3 for details). Assume that $d \geq 2$ and $\int_{-\infty}^{\infty} (1 + |s|)^d \rho(s) ds < \infty$, then there exists a unique solution to (1.0.1) for P_{λ_m}-*a.s.* ω.

In order to answer the second question, we need to modify our assumptions a little bit. Assume that $U_i \in C_0^\infty(\mathbf{R}^d)$ satisfy $U_i(-x) = U_i(x)$, $x \in \mathbf{R}^d$, $i = 1, \cdots, N$, and $U_i(x) = 0$ if $|x| \geq R_i$. Define constants $C_0 = \left(2\sum_{i=1}^N R_i \|\nabla U_i\|_\infty\right)^{1/2}$, $e_0 = \frac{1}{2}(2C_0+1)^2 + \sum_{i=1}^N \|U_i\|_\infty$, and assume that $\rho : \mathbf{R} \to [0,\infty)$ is a measurable function satisfying the following.

1. $\rho(s) = 0$ if $s \leq e_0$,
2. for any $c > 0$, there exists a $\widetilde{\rho_c} : \mathbf{R} \to [0,\infty)$ such that

$$\sup_{|a|\leq c} \rho(s+a) \leq \widetilde{\rho_c}(s), \qquad \text{for any } s \in \mathbf{R},$$

   and

$$\int_{\mathbf{R}^d} (1+|v|^3)\widetilde{\rho_c}(\frac{1}{2}|v|^2)dv < \infty.$$

Also, assume that the initial position $(X_{1,0}, \cdots, X_{N,0})$ satifties $|X_{i,0} - X_{j,0}| > R_i + R_j$ for any $i \neq j$.

It is easy to check that under this setting, the existence and the uniqueness of the solution of the considered ODE still holds, *i.e.*, there exists a unique solution to (1.0.1) for P_{λ_m}-*a.s.* ω. (See Theorem 5.0.2 for details). Moreover, we have the convergence result.

To describe the limit process, let us first define some notations. For any $\vec{X} \in \mathbf{R}^{dN}$, let us consider the following ODE:

$$\begin{cases} \dfrac{d}{dt}\tilde{x}(t,x,v;\vec{X}) = \tilde{v}(t,x,v;\vec{X}) \\ m\dfrac{d}{dt}\tilde{v}(t,x,v;\vec{X}) = -\sum\limits_{i=1}^N \nabla U_i(\tilde{x}(t,x,v;\vec{X}) - X_i) \\ (\tilde{x}(0,x,v;\vec{X}), \tilde{v}(0,x,v;\vec{X})) = (x,v). \end{cases} \tag{1.0.2}$$

Let

$$\begin{aligned} E &= \{(x,v) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus \{0\});\ x \cdot v = 0\}, \\ E_v &= \{x \in \mathbf{R}^d; x \cdot v = 0\}, \qquad v \in \mathbf{R}^d \setminus \{0\}, \end{aligned}$$

and let $\tilde{\nu}(dx;v)$ be the Lebesgue measure on E_v, let $\nu(dx,dv) = |v|\tilde{\nu}(dx;v)dv$.

Define

$$\Psi : \mathbf{R} \times E \to \mathbf{R}^d \times (\mathbf{R}^d \setminus \{0\}), (s,(x,v)) \mapsto (x - sv, v),$$

and let

$$\begin{aligned} \psi^0(t,x,v;\vec{X}) &= \lim_{s\to\infty} \tilde{x}(t+s, \Psi(s,x,v);\vec{X}), \\ \psi^1(t,x,v;\vec{X}) &= \lim_{s\to\infty} \tilde{v}(t+s, \Psi(s,x,v);\vec{X}), \end{aligned}$$

which is well-defined for any $t \in \mathbf{R}$ and any $(x, v) \in E$. where $(\widetilde{x}, \widetilde{v})$ stands for the solution of (1.0.2).

Let

$$\begin{aligned} a_{ik;jl}(\vec{X}) &= \frac{1}{M_i M_j} \int_E \Big(\int_{-\infty}^{\infty} \nabla_k U_i(\psi^0(t, x, v; \vec{X}) - X_i) dt \Big) \\ &\qquad \Big(\int_{-\infty}^{\infty} \nabla_l U_j(\psi^0(t, x, v; \vec{X}) - X_j) dt \Big) \rho(\frac{1}{2}|v|^2) \nu(dx, dv), \end{aligned}$$

and let $b_{ik;jl} : \mathbf{R}^{Nd} \to \mathbf{R}$ be the C^∞-functions given in the following way: Let $z(t, x, v, \vec{X}, \vec{V}, a)$ denote the solution of

$$\begin{cases} \dfrac{d^2}{dt^2} z(t) = -\displaystyle\sum_{i=1}^{N} \nabla^2 U_i(\psi^0(t, x, v, \vec{X}) - X_i)(z(t) - (t + a)V_i), \\ z(-\infty) = \dfrac{d}{dt} z(-\infty) = 0. \end{cases}$$

Then

$$z^1(x, v, \vec{X}, \vec{V}, a) := \lim_{t \to \infty} \frac{d}{dt} z(t, x, v, \vec{X}, \vec{V}, a)$$

is a linear function on $\vec{V}$. $b_{ik;jl}$ are determined by

$$\begin{aligned} B_i(\vec{X}, \vec{V}) &:= -\frac{1}{2} \frac{1}{M_i} \int_E \Big(\int_{-\infty}^{\infty} dt \nabla^2 U_i(\psi^0(t, x, v, \vec{X}) - X_i) z(t, x, v, \vec{X}, \vec{V}, -t) \Big) \\ &\qquad \rho(\frac{1}{2}|v|^2) \nu(dx, dv) \\ &= \frac{1}{M_i} \sum_{k=1}^{d} \sum_{j=1}^{N} b_{i\cdot;jk}(\vec{X}) V_j^k. \end{aligned}$$

Let L be the 2nd order differential operator on $\mathbf{R}^{2Nd}$ given by

$$L = \sum_{i,j=1}^{N} \sum_{k,l=1}^{d} a_{ik,jl}(\vec{X}) \frac{\partial^2}{\partial V_i^k \partial V_j^l} + \sum_{i,j=1}^{N} \sum_{k,l=1}^{d} b_{ik,jl}(\vec{X}) V_j^k \frac{\partial}{\partial V_i^k} + \sum_{i=1}^{N} \sum_{k=1}^{d} V_i^k \frac{\partial}{\partial X_i^k}.$$

Our convergence results are the following.

RESULT 1 *Assume $N = 1$. Then $\{(X_1(t), V_1(t)), t \geq 0\}$ converges to the diffusion in $C([0, \infty); \mathbf{R}^{2d})$ with generator L as $m \to 0$.*

RESULT 2 *Assume $N \geq 2$. Let*

$$\sigma_0(\omega) = \inf \{t > 0; \min_{i \neq j} \{|X_i(t) - X_j(t)| - (R_i + R_j)\} \leq 0\}.$$

Then $\{(\vec{X}(t \wedge \sigma_0), \vec{V}(t \wedge \sigma_0)), t \geq 0\}$ converges to the diffusion with generator L stopped at σ_0 in $C([0, \infty); \mathbf{R}^{2dN})$ as $m \to 0$.

RESULT 3 *Let $N = 2$ and $d \geq 3$. Assume that there exist functions h_1, h_2 such that*

$$U_i(x) = h_i(|x|), \qquad i = 1, 2,$$

and there exists a constant $\varepsilon_0 > 0$ such that

$$(-1)^{i-1} h_i(s) > 0, \quad (-1)^{i-1} h_i''(s) > 0, \qquad s \in (R_i - \varepsilon_0, R_i), i = 1, 2.$$

Then $\{(\vec{X}(t), \vec{V}(t)), t \geq 0\}$ converges to a Markov process as $m \to 0$.

This article is prepared as a lecture note. We included also some elementary well-known results, to make it as self-complacency as possible.

In Chapter 2, we review some basic facts about closed sets and Poisson point processes. In Chapter 3, we prepare some basic facts about classical mechanics, especially about Hamilton's equation, Newton's equation, ray representations, and classical scattering. (See Reed-Simon [11] for more details about classical scattering theory). In Chapter 4, we use random fields to prove the P_m-*a.s.* existence of the solution for every $m > 0$. (See also Evstigneev [6] for random fields). Chapter 5 contains some preparation for the proof of convergence, especially, it gives the decomposition of $V_i(t)$ (Lemma 5.3.1) and some properties followed, which are used in Chapter 7 to prove convergence results 1 and 2. See Chapter 6 for the proof of these lemmas. Finally, in Chapter 8, we prove Result 3, *i.e.*, we discusss the example of two big particles with ball symmetric potentials for dimension $d \geq 3$, and show that under certain conditions, as $m \to 0$, the phase process $\{(\vec{X}(t), \vec{V}(t)), t \geq 0\}$ converges to the Markov process given as the "reflecting diffusion process with generator L".

We emphasis again that as mentioned at the beginning of this chapter, in our present problem, the forces at any fixed time are not independent to the history. Therefore, since both the big particles and the small "environment" particles are changing, the system is very complicated and difficult to be handled. Our basic idea for the proof of convergence is that, although all of the particles are moving all the time, since the mass of big particles are very big compared with the small particles, when considering the scattering of the small particles, we can use the approximation that the big particles are fixed, with the caused error small enough. With the help of this approximation, the ODE for the motion of small particles could be approximated by the one in which the big particles are "fixed". This will certainly make our life easier. (See Chapters 5 $\sim$ 7 for more details).

Also, we want to remark that, for any fixed $m > 0$, although $V_i(t)$ is continuous with respect to t (since it is described by the ODE (1.0.1), our martingale part $M_i(t)$ in the decomposition of $V_i(t)$ (see Lemma 5.3.1) does not need to be continuous. The only thing we can say is that the jumps of it are dominated by some constant times $m^{1/2}$, (see Lemma 5.3.1). This is also one of our ideas: use the martingale theorem only to the part for which it is usable, for the remaining term, instead of trying to deal with it in detail, we show that the whole term is negligible as $m \to 0$ from the beginning.

Chapter 2

Closed Sets in Polish Spaces

2.1 Structure of measurable

Let M be a Polish space, i.e., a separable complete metric space. Also, let $\mathcal{O}(M)$ denote the family of all non-empty closed subsets of M.

DEFINITION 2.1.1 (1) *Let $\mathcal{E}_0$ denote the σ-algebra on $\mathcal{O}(M)$ generated by $\{\{C \in \mathcal{O}(M); C \cap G = \emptyset\}; G$ is open in $M\}$.*

(2) *Let $\mathcal{E}_1$ denote the σ-algebra on $\mathcal{O}(M)$ generated by $\{\{C \in \mathcal{O}(M); C \subset G\}; G$ is open in $M\}$.*

PROPOSITION 2.1.2 $\mathcal{E}_0 \subset \mathcal{E}_1$.

Before giving the proof of Proposition 2.1.2, let us prepare some notations. For any subset $A \subset M$ and $r > 0$, we define

$$(A)_r = \{x \in M; dist(x, A) < r\}.$$

Also, we write

$$B(x, r) = (\{x\})_r = \{y \in M; dist(x, y) < r\},$$

which is an open set.

Proof of Proposition 2.1.2. Let G be any open set. Then for any $C \in \mathcal{O}(M)$,

$$\begin{aligned} C \cap G = \emptyset \quad &\Longleftrightarrow \quad C \subset G^C \\ &\Longleftrightarrow \quad C \subset (G^C)_{1/n} \text{ for any } n \geq 1, \end{aligned}$$

so

$$\begin{aligned} & \{C \in \mathcal{O}(M); C \cap G = \emptyset\} \\ = \ & \bigcap_{n=1}^{\infty} \{C \in \mathcal{O}(M); C \subset (G^C)_{1/n}\} \\ \in \ & \mathcal{E}_1, \end{aligned}$$

which implies our assertion. ∎

We define several more notations. Let

$$\begin{aligned} Comp(M) &= \{C \in \mathcal{O}(M); C \text{ is compact}\}, \\ Conf(M) &= \{C \in \mathcal{O}(M); C \text{ has no cluster point}\}, \\ Fin(M) &= \{C \in \mathcal{O}(M); \sharp(C) < \infty\}. \end{aligned}$$

PROPOSITION 2.1.3 *For any closed set $K \subset M$, we have that*

$$\{C \in \mathcal{O}(M); C \subset K\} \in \mathcal{E}_0.$$

Proof. This is easy since $C \subset K \Longleftrightarrow C \cap K^C = \emptyset$. ∎

PROPOSITION 2.1.4 $Comp(M) \in \mathcal{E}_0$.

Proof. First note that

$$C \in \mathcal{O}(M) \text{ is compact} \Longleftrightarrow C \text{ is closed and totally bounded.}$$

Now choose and fix a sequence $\{x_n\}_{n=1}^\infty$ such that it is dense in M. (This is possible since M is separable). Then we have

$$\begin{aligned} & C \in Comp(M) \\ \Longleftrightarrow \quad & C \in \mathcal{O}(M) \text{ and } "\forall n \geq 1, \exists m \geq 1, s.t. C \subset \cup_{k=1}^m \overline{B(x_k, 1/n)}". \end{aligned}$$

Therefore, by Proposition 2.1.3,

$$\begin{aligned} Comp(M) &= \bigcap_{n=1}^\infty \Big(\bigcup_{m=1}^\infty \left\{ C \in \mathcal{O}(M); C \subset \cup_{k=1}^m \overline{B(x_k, 1/n)} \right\} \Big) \\ &\in \mathcal{E}_0. \end{aligned}$$

∎

PROPOSITION 2.1.5 *We have $\{C \in \mathcal{O}(M); \sharp(C) \leq n\} \in \mathcal{E}_0$ for any $n \in \mathbf{N}$.*

Proof. Choose and fix a sequence $\{x_n\}_{n=1}^\infty$ such that it is dense in M. (As before, this is possible since M is separable).

Claim.

$$\begin{aligned} & \{C \in \mathcal{O}(M); \sharp(C) \leq n\} \\ = & \bigcap_{\ell=1}^\infty \bigcup_{1 \leq k_1 < \cdots < k_n} \left\{ C \in \mathcal{O}(M); C \subset \bigcup_{i=1}^n \overline{B(x_{k_i}, 1/\ell)} \right\}. \end{aligned}$$

Proof of Claim. The "$\subset$" side is easy. We show the " $\supset$ " side in the following. Suppose $\sharp(C) \geq n+1$. Then there exist $y_1, \cdots, y_{n+1} \in C$ with $dist(y_i, y_j) > 0$ for any $i \neq j$. Choose $\ell \in \mathbf{N}$ such that

$$\frac{2}{\ell} < \min_{i \neq j} dist(y_i, y_j).$$

Then there do NOT exist $x_{k_1}, \cdots, x_{k_n}$ such that $C \subset \bigcup_{i=1}^{n} \overline{B(x_{k_i}, 1/\ell)}$. This completes the proof of the claim.

Now, Proposition 2.1.5 is easy by Claim and Proposition 2.1.4. ∎

COROLLARY 2.1.6 *$Fin(M) \in \mathcal{E}_0$.*

PROPOSITION 2.1.7 *Suppose that M is locally compact in addition. Then $Conf(M) \in \mathcal{E}_0$.*

Proof. Since M is locally compact, for any $x \in M$, there exists a $r_x > 0$ such that $\overline{B(x, r_x)}$ is compact. We take $r_x > 0$ as the largest possible number less than 1, *i.e.*, $r_x = \sup\{r \leq 1 : \overline{B(x, r_x)}$ is compact$\}$. Also, since M is separable, there exists a sequence $\{x_n\}_{n \in \mathbf{N}}$ such that it is dense in M.

We have that $M = \cup_{n \in \mathbf{N}} B(x_n, r_{x_n})$. Actually, if not, then there exists a $y \in M$ such that $y \notin B(x_n, r_{x_n})$ for any $n \in \mathbf{N}$. Since $\{x_n\}_{n \in \mathbf{N}}$ is dense in M, there exists a subsequence n_k such that $x_{n_k} \to y$ as $k \to \infty$. Therefore, there exists a $K \in \mathbf{N}$ such that $x_{n_k} \in B(y, r_y/2)$ for any $k \geq K$. So by the definition of $r_{x_{n_k}}$, we get that $r_{x_{n_k}} \geq r_y/2$. Therefore, $y \notin B(x_{n_k}, r_{x_{n_k}})$ implies $y \notin B(x_{n_k}, r_y/2)$, in other words, $x_{n_k} \notin B(y, r_y/2)$ for any $k \geq K$. This contradicts with the fact that $x_{n_k} \to y$.

It is easy to see that

$$\begin{aligned} &\mathcal{O}(M) \setminus Conf(M) \\ = &\bigcup_{n=1}^{\infty} \Big\{ C \in \mathcal{O}(M); B(x_n, r_{x_n}) \cap C \text{ has infinitely many elements} \Big\}. \end{aligned}$$

Actually, the "$\supset$" part is trivial. For the "$\subset$" part, choose any $C \in \mathcal{O}(M) \setminus Conf(M)$, we only need to show that there exists a $k \in \mathbf{N}$ such that $B(x_k, r_{x_k}) \cap C$ has infinitely many elements. Since $C \in \mathcal{O}(M) \setminus Conf(M)$, there exists (at least one) $a \in M$ such that a is a cluster point of C. So $B(a, r_a) \cap C$ has infinitely many elements. On the other hand, the fact that $\cup_n B(x_n, r_{x_n}) = M \supset \overline{B(a, r_a)}$ combined with the compactness of $\overline{B(a, r_a)}$ implies that there exists a $m \in \mathbf{N}$ and $n_1, \cdots, n_m \in \mathbf{N}$ such that $\cup_{l=1}^{m} B(x_{n_l}, r_{x_{n_l}}) \supset \overline{B(a, r_a)}$, hence $\cup_{l=1}^{m} \Big(B(x_{n_l}, r_{x_{n_l}}) \cap C \Big) \supset \overline{B(a, r_a)} \cap C$. Therefore, there exists at least one $l \in \{1, \cdots, m\}$ such that $B(x_{n_l}, r_{x_{n_l}}) \cap C$ has infinitely many elements.

Therefore, by the Claim in the proof of Proposition 2.1.5,

$$\mathcal{O}(M) \setminus Conf(M)$$

$$
\begin{aligned}
&= \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \left\{ C \in \mathcal{O}(M); \sharp(B(x_n, r_{x_n}) \cap C) > m \right\} \\
&= \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{\ell=1}^{\infty} \bigcap_{k_1 < \cdots < k_m} \\
&\qquad \left\{ C \in \mathcal{O}(M); C \cap B(x_n, r_{x_n}) \cap (\bigcup_{i=1}^{m} \overline{B(x_{k_i}, 1/\ell)})^C \neq \emptyset \right\} \\
&\in \mathcal{E}_0.
\end{aligned}
$$

∎

2.2 Castaing's axiom of choice

Let $(\Omega, \mathcal{F})$ be a measurable space through out this section.

DEFINITION 2.2.1 *A map $\Gamma : \Omega \to \mathcal{O}(M)$ is said to be measurable if it is $\mathcal{F}/\mathcal{E}_0$-measurable.*

Also, we use the following notation: for any $A \subset M$, let

$$\Gamma^{-w}(A) = \{\omega \in \Omega; \Gamma(\omega) \cap A \neq \emptyset\}.$$

PROPOSITION 2.2.2 *$\Gamma : \Omega \to \mathcal{O}(M)$ is measurable if and only if $\Gamma^{-w}(G) \in \mathcal{F}$ for all open G.*

Proof. This is easy since

$$
\begin{aligned}
& \Gamma^{-w}(G) \\
={}& \Gamma^{-1}(\{C \in \mathcal{O}(M); C \cap G \neq \emptyset\}) \\
={}& \Gamma^{-1}(\{C \in \mathcal{O}(M); C \cap G = \emptyset\})^C.
\end{aligned}
$$

∎

PROPOSITION 2.2.3 *Suppose that $\Gamma : \Omega \to \mathcal{O}(M)$ is measurable. Then there exists a measurable $\gamma : \Omega \to M$ such that*

$$\gamma(\omega) \in \Gamma(\omega), \qquad \textit{for all } \omega \in \Omega.$$

Proof. In general, if $\tilde{d}$ is a complete metric in M, then

$$d(x, y) := \tilde{d}(x, y) \wedge 1, \qquad x, y \in M,$$

is also a complete metric in M. Therefore, without loss of generality, we may and do assume that

$$\sup_{x,y \in M} d(x, y) \leq 1.$$

Also, since M is separable, there exists a sequence $\{x_n\}_{n=0}^{\infty}$ such that it is dense in M.

In the following, we construct a sequence of measurable maps $\gamma_n : \Omega \to M$, $n \geq 0$, that satisfies the following:

$$\begin{cases} d(\gamma_{k-1}(\omega), \gamma_k(\omega)) \leq 2^{-(k-2)}, \\ B(\gamma_k(\omega), 2^{-k}) \cap \Gamma(\omega) \neq \emptyset, \quad k \geq 0, \omega \in \Omega. \end{cases} \tag{2.2.1}$$

We construct it inductively. First let $\gamma_0(\omega) = x_0$. Next, when $\gamma_0, \cdots, \gamma_n$ are given with (2.2.1) hold for $k = 1, \cdots, n$, we define γ_{n+1} in the following way such that it also satisfies (2.2.1). For any $m \geq 0$, let

$$\begin{aligned} E_m &= \{\omega \in \Omega; B(x_m, 2^{-(n+1)}) \cap \Gamma(\omega) \neq \emptyset\}, \\ F_m &= \{\omega \in \Omega; \gamma_n(\omega) \in B(x_m, 2^{-(n-1)})\}, \end{aligned}$$

and let $G_m = E_m \cap F_m$. Then $E_m = \Gamma^{-w}(B(x_m, 2^{-(n+1)})) \in \mathcal{F}$ by Proposition 2.2.2, $F_n = \gamma_n^{-1}(B(x_m, 2^{-(n-1)})) \in \mathcal{F}$ since γ_n is measurable. Therefore, $G_m \in \mathcal{F}$. For any $\omega \in \Omega$, we have by inductive condition that there exists a $y \in \Gamma(\omega)$ such that

$$dist(\gamma_n(\omega), y) < 2^{-n}. \tag{2.2.2}$$

Also, since $\{x_n\}_{n=0}^{\infty}$ is dense in M, there exists a x_ℓ such that

$$dist(y, x_\ell) < 2^{-(n+1)}, \tag{2.2.3}$$

hence

$$B(x_\ell, 2^{-(n+1)}) \cap \Gamma(\omega) \neq \emptyset.$$

Moreover, by (2.2.2) and (2.2.3), we have that

$$\gamma_n(\omega) \in B(x_\ell, 2^{-(n-1)}).$$

These give us that $\omega \in G_\ell$. Therefore, $\bigcup_{m=0}^{\infty} G_m = \Omega$. Let

$$\begin{aligned} &G'_0 = G_0, \\ &G'_{m+1} = G_{m+1} \setminus (\cup_{k=0}^{m} G_k). \end{aligned}$$

Then $\{G'_m\}_{m\geq 0}$ are disjoint to each other, and

$$\bigcup_{m=0}^{\infty} G'_m = \Omega.$$

Define

$$\gamma_{n+1}(\omega) = x_m, \qquad \text{if } \omega \in G'_m, \quad m = 0, 1, 2, \cdots,$$

Then it is easy to check by the definition of G_m that γ_{n+1} also satisfies (2.2.1) with $k = n + 1$. This completes our definition of $\gamma_n : \Omega \to M$, $n \geq 0$, inductively.

For any ω, $\{\gamma_n(\omega)\}_{n\geq 0}$ is a Cauchy sequence by definition, so there exists a limit $\gamma(\omega) := \lim_{n\to\infty} \gamma_n(\omega)$, and

$$d(\gamma(\omega), \Gamma(\omega)) = \lim_{n\to\infty} d(\gamma_n(\omega), \Gamma(\omega)) = 0.$$

Since $\Gamma(\omega)$ is closed, this implies that $\gamma(\omega) \in \Gamma(\omega)$, $\omega \in \Omega$. ∎

THEOREM 2.2.4 (Castaing) *Suppose that $\Gamma : \Omega \to \mathcal{O}(M)$ is measurable. Then there exists a sequence of measureable maps $\gamma_n : \Omega \to M$, $n \in \mathbf{N}$, such that*

(1) $\gamma_n(\omega) \in \Gamma(\omega), \quad \omega \in \Omega,$

(2) $\overline{\{\gamma_n(\omega); n \in \mathbf{N}\}} = \Gamma(\omega), \quad \omega \in \Omega.$

Proof. Let $\{x_n\}_{n\in\mathbf{N}}$ be a dense set in M (This is possible since M is separable). Also, for any $n, m \in \mathbf{N}$, let

$$\Gamma_{n,m}(\omega) = \begin{cases} \Gamma(\omega) \cap B(x_n, 2^{-m}), & \text{if } \omega \in \Gamma^{-w}(B(x_n, 2^{-m})), \\ \Gamma(\omega), & \text{otherwise.} \end{cases}$$

Then for any open $G \subset M$, we have that

$$\begin{aligned} & (\overline{\Gamma_{n,m}})^{-w}(G) \\ = \ & \{\omega \in \Omega; \overline{\Gamma_{n,m}(\omega)} \cap G \neq \emptyset\} \\ = \ & \{\omega \in \Omega; \Gamma_{n,m}(\omega) \cap G \neq \emptyset\} \\ = \ & \Gamma^{-w}(B(x_n, 2^{-m}) \cap G) \cup \Big(\Gamma^{-w}(G) \setminus \Gamma^{-w}(B(x_n, 2^{-m}))\Big) \\ \in \ & \mathcal{F}. \end{aligned}$$

So by Proposition 2.2.2, $\overline{\Gamma_{n,m}} : \Omega \to \mathcal{O}(M)$ is measurable. Therefore, by Proposition 2.2.3, there exists a measurable map $\gamma_{n,m} : \Omega \to M$ such that

$$\gamma_{n,m}(\omega) \in \overline{\Gamma_{n,m}(\omega)} \subset \Gamma(\omega), \qquad \omega \in \Omega.$$

Next, we show that the second condition of the theorem is also satisfied. Actually, for any $\omega \in \Omega$, $x \in \Gamma(\omega)$ and $m \geq 1$, since $\{x_n\}_{n\in\mathbf{N}}$ is dense in M, there exists a x_n such that

$$d(x_n, x) < 2^{-m-1}. \tag{2.2.4}$$

This gives us that $\Gamma(\omega) \cap B(x_n, 2^{-m-1}) \neq \emptyset$, so by definition, $\omega \in \Gamma^{-w}(B(x_n, 2^{-m-1}))$. Hence by the definition of $\Gamma_{n,m+1}$, $\Gamma_{n,m+1}(\omega) = \Gamma(\omega) \cap B(x_n, 2^{-m-1})$. Therefore, $\gamma_{n,m+1}(\omega) \in \Gamma_{n,m+1}(\omega)$ implies that $\gamma_{n,m+1}(\omega) \in B(x_n, 2^{-m-1})$, i.e.,

$$d(x_n, \gamma_{n,m+1}(\omega)) < 2^{-m-1}.$$

This combined with (2.2.4) gives us that

$$d(x, \gamma_{n,m+1}(\omega)) < 2^{-m}.$$

Therefore,

$$\overline{\{\gamma_{n,m}(\omega); n, m \in \mathbf{N}\}} = \Gamma(\omega).$$

This completes the proof by re-ordering $\{\gamma_{n,m}(\omega)\}_{n,m\in\mathbf{N}}$. ∎

2.3 Measures determined by elements of $Conf(M)$

THEOREM 2.3.1 *Let M be a locally compact space. For any $\omega \in Conf(M)$, let μ_ω be the measure on M given by*

$$\mu_\omega(A) = \sharp(\omega \cap A), \qquad A \in \mathcal{B}(M).$$

Then for any measurable $f : M \to [0, \infty)$, we have that the map

$$Conf(M) \to [0, \infty), \quad \omega \mapsto \int_M f d\mu_\omega$$

is measurable.

Proof. Let

$$\Omega = Conf(M), \qquad \mathcal{F} = \mathcal{E}_0\Big|_{Conf(M)}.$$

Then $(\Omega, \mathcal{F})$ is a measurable space, and the map

$$\Omega \to \mathcal{O}(M), \omega \mapsto \omega$$

is measurable. Also, let Δ be a point that does not belong to M.

By Castaing's Theorem 2.2.4, there exists a sequence of measurable maps $\{\gamma_n : \Omega \to M\}_{n\in\mathbf{N}}$ such that

$$\overline{\{\gamma_n(\omega); n \in \mathbf{N}\}} = \omega$$

for any $\omega \in \Omega$. Since $\omega \in \Omega = Conf(M)$, we have by definition that ω has no cluster point. Therefore, the equation above implies that

$$\{\gamma_n(\omega); n \in \mathbf{N}\} = \omega, \qquad \omega \in \Omega.$$

Define $\widetilde{\gamma_n} : \Omega \to M \cup \{\Delta\}$ by

$$\widetilde{\gamma_n}(\omega) = \begin{cases} \gamma_n(\omega), & \text{if } \gamma_n(\omega) \neq \gamma_k(\omega) \text{ for } k = 1, \cdots, n-1, \\ \Delta, & \text{otherwise.} \end{cases}$$

Then $\widetilde{\gamma_n}$ is also measurable.

For any measurable $f : M \to [0, \infty)$, define

$$\widetilde{f} : M \cup \{\Delta\} \to [0, \infty), \quad x \mapsto \widetilde{f}(x) = \begin{cases} f(x), & x \in M, \\ 0, & x = \Delta. \end{cases}$$

Then $\widetilde{f}$ is also measurable. Therefore,

$$\int_M f d\mu_\omega = \sum_{n=1}^{\infty} \widetilde{f}(\widetilde{\gamma_n}(\omega)) = \sum_{x\in\omega} f(x)$$

is also measurable.

This completes the proof. ∎

2.4 Poisson point process

THEOREM 2.4.1 *Let M be a locally compact space, and let ν be a σ-finite non-atomic Radon measure on M. Then there exists a unique probability measure P_ν on $(Conf(M), \mathcal{E}_0\big|_{Conf(M)})$ such that*

(1) *The distribution of $\mu_\omega(A)$ under $P_\nu(d\omega)$ is the Poisson distribution with mean $\nu(A)$ for any $A \in \mathcal{B}(M)$,*

(2) *for any compact sets $K_1, \cdots, K_n \subset M$ that are disjoint to each other, we have that $\{\mu_\omega(K_i); i = 1, \cdots, n\}$ are independent under $P_\nu(d\omega)$.*

Proof. We consider the case with $\nu(M) = \infty$. The case of $\nu(M) < \infty$ can be done in the similar way and is easier.

Let ν_0 be the Lebesgue measure on $((0,\infty), \mathcal{B}((0,\infty)))$. Then there exist sets $A_0 \in \mathcal{B}((0,\infty))$, $A_1 \in \mathcal{B}(M)$ and a one-to-one onto bi-measurable $\varphi : A_0 \to A_1$ such that

$$\nu_0((0,\infty) \setminus A_0) = 0, \quad \nu(M \setminus A_1) = 0,$$
$$\nu_0 \circ \varphi^{-1}(B \cap A_1) = \nu(B \cap A_1), \qquad \text{for any } B \in \mathcal{B}(M).$$

Such a map exists. Actually, as well-known, a complete seperable metric space with a complete regular Borel probability measure on it is a Lebesgue space, *i.e.*, is a measurable space that is measurablely isomephic to the real line (or a bounded interval) with the usual Lebesgue measure plus countablely many atoms. (See, *e.g.*, Ikeda-Watanabe [8, p. 13] and Parthasarathy [10, section 1.2] for details). Since ν has no atom by assumption, we get that there exists a bijection $f : M \to \{0,1\}^{\mathbf{N}} \cong (0,\infty)$ such that f is $\mathcal{B}(M)/\mathcal{B}((0,\infty))$-measurable and f^{-1} is $\mathcal{B}((0,\infty))/\mathcal{B}(M)$-measurable. After converted the problem into the one on positive real numbers, the remaining is easy.

Let $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ be a new probability space, and let $X_i, i \in \mathbf{N}$, be *i.i.d.* random variables on it with distribution $e^{-x}dx$, $x > 0$. Also, let $S_n = X_1 + \cdots + X_n$. Then $S_n(\tilde{\omega}) \to \infty$, $\widetilde{P}$-*a.s.*. Let $C(\tilde{\omega}) = \{S_n(\tilde{\omega}); n \in \mathbf{N}\}$. Then $C(\tilde{\omega}) \in Conf((0,\infty))$, $\widetilde{P}$-*a.s.*. Also, we have that

1. $\mu_{C(\tilde{\omega})}(A)$ is the Poisson distribution with mean $\nu_0(A)$ for any $A \in \mathcal{B}((0,\infty))$,

2. if $A_1, \cdots, A_n \in \mathcal{B}((0,\infty))$ are disjoint to each other, then $\mu_{C(\tilde{\omega})}(A_i)$, $i = 1, \cdots, n$, are independent.

In particular, $C(\tilde{\omega}) \subset A_0$, $\widetilde{P}$-*a.s.*, and $\varphi(C(\tilde{\omega})) \in Conf(M)$, $\widetilde{P}$-*a.s.*. Let

$$P_\nu(B) = \widetilde{P}\Big(\{\tilde{\omega} \in Conf((0,\infty)); \varphi(C(\tilde{\omega})) \in B\}\Big).$$

We show that this P_ν satisfies the desired conditions. For any $A \in \mathcal{B}(M)$ and $k \in \mathbf{N}$, let $B = \{\omega \in Conf(M), \sharp(\omega \cap A) = k\}$. Then since φ is one-to-one, we have

$$\begin{aligned}
& P_\nu(\{\omega : \mu_\omega(A) = k\}) = P_\nu(B) \\
= \;& \widetilde{P}\Big(\{\widetilde{\omega} : \varphi(C(\widetilde{\omega})) \in B\}\Big) \\
= \;& \widetilde{P}\Big(\{\widetilde{\omega} : \sharp(\varphi(C(\widetilde{\omega})) \cap A) = k\}\Big) \\
= \;& \widetilde{P}\Big(\{\widetilde{\omega} : \sharp(C(\widetilde{\omega}) \cap \varphi^{-1}(A)) = k\}\Big) \\
= \;& \widetilde{P}\Big(\{\widetilde{\omega} : \mu_{C(\widetilde{\omega})}(\varphi^{-1}(A)) = k\}\Big).
\end{aligned}$$

Therefore, the distribution of $\mu_\omega(A)$ under P_ν is the same as the distribution of $\mu_{C(\widetilde{\omega})}(\varphi^{-1}(A))$ under $\widetilde{P}$, which by definition, is equal to the Poisson distribution with mean $\nu_0(\varphi^{-1}(A)) = \nu(A)$. This gives us that the defined P_ν satisfies the first desired condition. The fact that it also satisfies the second condition can by shown in exactly the same way, and we omit it here. This completes the proof of the existence. The uniqueness property is easy by definition. ∎

Chapter 3

Classical Mechanics

3.1 Hamilton's equation

Let $H : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$, $(q,p) \mapsto H(q,p)$ be a smooth function satisfying the following:

H1. The functions $\frac{\partial}{\partial q^i}H$, $\frac{\partial}{\partial p^i}H$, $i = 1, \cdots, N$, are globally Lipschitz continuous.

Consider the following equation:

$$\begin{cases} \frac{d}{dt}q^i(t) = \frac{\partial H}{\partial p^i}(q(t),p(t)), \\ \frac{d}{dt}p^i(t) = -\frac{\partial H}{\partial q^i}(q(t),p(t)), \qquad i = 1, \cdots, N, \\ (q(0),p(0)) = (q_0,p_0) \in \mathbf{R}^d \times \mathbf{R}^d, \end{cases}$$

and write the solution of it as $(q(t),p(t)) = \Phi(t,q_0,p_0)$. Then $\Phi : \mathbf{R} \times \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}^d \times \mathbf{R}^d$ is a smooth operator, $\Phi_t(\cdot) := \Phi(t,\cdot)$ is a diffeomorphism on $\mathbf{R}^d \times \mathbf{R}^d$, and by definition,

$$\begin{aligned} & \frac{d}{dt}H(\Phi(t,q_0,p_0)) \\ = \ & \sum_{i=1}^{N} \left\{ \frac{\partial H}{\partial q^i}(\Phi(t,q_0,p_0))\frac{d}{dt}q^i(t) + \frac{\partial H}{\partial p^i}(\Phi(t,q_0,p_0))\frac{d}{dt}p^i(t) \right\} \\ = \ & 0, \end{aligned}$$

i.e., $H(\Phi(t,q_0,p_0)) = H(q_0,p_0)$ for any $t \geq 0$.

Also, by the uniqueness of the solution, it is easy to see that the flow $\{\Phi_t\}_t$ satisfies

$$\Phi_t \circ \Phi_s = \Phi_{t+s}, \qquad \text{for any } t, s > 0. \tag{3.1.1}$$

Let $\omega = \sum_{i=1}^{N} dq^i \wedge dp^i$. Then

$$\frac{d}{dt}\Phi_t^*\omega$$

$$\begin{aligned}
&= \sum_{i=1}^{N} d\Big(\frac{d}{dt}q^i(t,q,p)\Big) \wedge dp^i(t,q,p) + \sum_{i=1}^{N} dq^i(t,q,p) \wedge d\Big(\frac{d}{dt}p^i(t,q,p)\Big) \\
&= \sum_{i=1}^{N} \Big\{ d\Big(\frac{\partial H}{\partial p^i}(\Phi_t(q,p))\Big) \wedge dp^i(t,q,p) - dq^i(t,q,p) \wedge d\Big(\frac{\partial H}{\partial q^i}(\Phi_t(q,p))\Big)\Big\}.
\end{aligned}$$

So

$$\begin{aligned}
& \frac{d}{dt}\Phi_t^*\omega\Big|_{t=0} \\
= \ & \sum_{i=1}^{N}\Big(\sum_{j=1}^{N}\frac{\partial^2 H}{\partial q^j \partial p^i}dq^j + \sum_{j=1}^{N}\frac{\partial^2 H}{\partial p^j \partial p^i}dp^j\Big) \wedge dp^i \\
& - \sum_{i=1}^{N} dq^i \wedge \Big(\sum_{j=1}^{N}\frac{\partial^2 H}{\partial q^j \partial q^i}dq^j + \sum_{j=1}^{N}\frac{\partial^2 H}{\partial p^j \partial q^i}dp^j\Big) \\
= \ & 0.
\end{aligned}$$

Therefore, by (3.1.1),

$$\frac{d}{dt}\Phi_t^*\omega = \Phi_t^*\Big(\frac{d}{ds}\Phi_s^*\omega\Big|_{s=0}\Big) = 0,$$

hence $\Phi_t^*\omega = \omega$. In the same way, we have that

$$\Phi_t^*\omega^d = \omega^d.$$

Therefore,

$$\begin{aligned}
& \Phi_t^*(f(q,p)dq^1 \wedge \cdots \wedge dq^d \wedge dp^1 \wedge \cdots \wedge dp^d) \\
= \ & f(\Phi_t(q,p))dq^1 \wedge \cdots \wedge dq^d \wedge dp^1 \wedge \cdots \wedge dp^d.
\end{aligned}$$

This gives us the following important formula:

$$\int_{\mathbf{R}^d \times \mathbf{R}^d} f(\Phi_t(q,p))dqdp = \int_{\mathbf{R}^d \times \mathbf{R}^d} f(q,p)dqdp.$$

3.2 Newton's equation

Let $N \geq 1$, $d \geq 1$, $U \in C_b^\infty(\mathbf{R}^{dN})$ and $M_i > 0$, $i = 1, \cdots, N$.

Let us consider the following Newton's equation:

$$\begin{cases} \frac{d}{dt}x_i(t) = v_i(t), \\ M_i \frac{d}{dt}v_i(t) = -\nabla_i U(\vec{x}(t)), \qquad i = 1, \cdots, N, \\ (\vec{x}(0), \vec{v}(0)) = (\vec{x_0}, \vec{v_0}) \in \mathbf{R}^{2dN}, \end{cases} \tag{3.2.1}$$

and write the solution as

$$\Phi(t, \vec{x_0}, \vec{v_0}) = (\vec{x}(t), \vec{v}(t)).$$

We first see the relation with Hamilton's equation. Let $H : \mathbf{R}^{dN} \times \mathbf{R}^{dN} \to \mathbf{R}$ be the function given by

$$H(\vec{q}, \vec{p}) = \sum_{i=1}^{N} \frac{1}{2M_i} |p_i|^2 + U(\vec{q}).$$

Hamilton's equation in Section 3.1 now becomes

$$\begin{cases} \frac{d}{dt} q^i(t) = \frac{1}{M_i} p_i(t), \\ \frac{d}{dt} p^i(t) = -\nabla_i U(\vec{q}(t)), \qquad i = 1, \cdots, N. \end{cases} \tag{3.2.2}$$

Let

$$v_i(t) = \frac{1}{M_i} p_i(t), \quad x_i(t) = q_i(t). \tag{3.2.3}$$

Then the fact $(\vec{q}, \vec{p})$ satisfies Hamilton's equation (3.2.2) implies that $(\vec{x}, \vec{v})$ given by (3.2.3) satisfies Newton's equation (3.2.1) (with initial condition $(\vec{q}_0, (\frac{1}{M_1} p_{01}, \cdots, \frac{1}{M_N} p_{0N}))$. Let $\widetilde{\Phi}(t, \vec{q}, \vec{p}) = (\vec{q}(t), \vec{p}(t))$, the solution of Hamilton equation (3.2.2). Also, let $\Theta : \mathbf{R}^{2dN} \to \mathbf{R}^{2dN}$ be the map given by

$$\Theta : \mathbf{R}^{2dN} \to \mathbf{R}^{2dN}, (\vec{y}, \vec{z}) = (\vec{y}, (z_1, \cdots, z_N)) \mapsto (\vec{y}, (M_1 z_1, \cdots, M_N z_N)).$$

Then we have the following relation

$$\Phi(t, \vec{x}, \vec{v}) = \Theta^{-1}(\widetilde{\Phi}(t, \Theta(\vec{x}, \vec{v}))).$$

This combined with the results of Section 3.1 gives us the following.

THEOREM 3.2.1 (1) *Let* $E(\vec{x}, \vec{v}) = \frac{1}{2} \sum_{i=1}^{N} M_i |v_i|^2 + U(\vec{x})$. *Then*

$$E(\Phi(t, \vec{x}, \vec{v})) = E(\vec{x}, \vec{v}), \qquad \textit{for any } t > 0.$$

(2) *For any measurable* $f : \mathbf{R}^{2dN} \to [0, \infty)$, *we have that*

$$\int_{\mathbf{R}^{2dN}} f(\Phi(t, \vec{x}, \vec{v})) d\vec{x} d\vec{v} = \int_{\mathbf{R}^{2dN}} f(\vec{x}, \vec{v}) d\vec{x} d\vec{v}.$$

As a result, we get the following.

THEOREM 3.2.2 *For any measurable* $f : \mathbf{R}^{2dN} \to [0, \infty)$ *and* $\rho : \mathbf{R} \to [0, \infty)$, *we have that*

$$\int_{\mathbf{R}^{2dN}} f(\Phi(t, \vec{x}, \vec{v})) \rho(E(\vec{x}, \vec{v})) d\vec{x} d\vec{v} = \int_{\mathbf{R}^{2dN}} f(\vec{x}, \vec{v}) \rho(E(\vec{x}, \vec{v})) d\vec{x} d\vec{v}.$$

3.3 Ray representation

Let $d \geq 1$. For any $v \in \mathbf{R}^d \setminus \{0\}$, let $E_v \in \mathbf{R}^d$ be the hyperplane given by

$$E_v = \{x \in \mathbf{R}^d; x \cdot v = 0\},$$

and let $\widetilde{\nu}(dx; v)$ be the Riemannian volume on E_v. Also, let

$$E = \{(x, v) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus \{0\}); x \cdot v = 0\},$$

and let

$$\Psi : \mathbf{R} \times E \to \mathbf{R}^d \times (\mathbf{R}^d \setminus \{0\}), (t, (x, v)) \mapsto (x - tv, v).$$

Then Ψ is one-to-one and onto.

Note that for any measurable $f : \mathbf{R}^d \to [0, \infty)$, we have that

$$\begin{aligned} & \int_{\mathbf{R}^d} f(x)dx \\ = & \int_{\mathbf{R}\times E_v} f(x - s|v|^{-1}v)ds\widetilde{\nu}(dx; v) \\ = & \int_{\mathbf{R}\times E_v} f(x - tv)|v|dt\widetilde{\nu}(dx; v), \end{aligned}$$

where in the last line, we used the variable change $s = t|v|$. Let

$$\nu(dx, dv) = |v|\widetilde{\nu}(dx; v)dv.$$

Then the above gives us the following.

THEOREM 3.3.1 *For any measurable $f : \mathbf{R}^{2d} \to [0, \infty)$, we have that*

$$\int_{\mathbf{R}^{2d}} f(x, v)dxdv = \int_{\mathbf{R}\times E} f(\Psi(t, x, v))dt\nu(dx, dv).$$

3.4 Classical scattering

For $i = 1, \cdots, N$, choose $U_i \in C_0^\infty(\mathbf{R}^d)$. Then there exist $R_i > 0$ such that $U_i(x) = 0$ if $|x| > R_i$, $i = 1, \cdots, N$.

Choose any $\vec{X} = (X_1, \cdots, X_N) \in (\mathbf{R}^d)^N$ and fix it for a while. Let

$$U(x) = U(x; \vec{X}) = \sum_{i=1}^{N} U_i(x - X_i).$$

Let $M_i = 1$, $i = 1, \cdots, N$, for a while. Then the function E in Theorem 3.2.1 is now given by

$$\widetilde{E}(x, v) = \widetilde{E}(x, v; \vec{X}) = U(x; \vec{X}) + \frac{1}{2}|v|^2.$$

Newton's equation now becomes

$$\begin{cases} \frac{d}{dt}x(t) = v(t), \\ \frac{d}{dt}v(t) = -\nabla U(x;\vec{X}), \\ (x(0), v(0)) = (x_0, v_0) \in \mathbf{R}^{2d}. \end{cases}$$

Write the solution as

$$\widetilde{\varphi}(t; x_0, v_0) = \widetilde{\varphi}(t; x_0, v_0; \vec{X}) = (x(t), v(t)) = (\widetilde{\varphi}^0(t; x_0, v_0), \widetilde{\varphi}^1(t; x_0, v_0)).$$

Let

$$R(\vec{X}) = \max\{R_i + |X_i|; i = 1, \cdots, N\},$$

and let $s_0 = \frac{R(\vec{X})}{|v|}$. Then for any $(x, v) \in E$, we have (notice that $x \cdot v = 0$ by definition of E)

$$\begin{aligned} \inf_{0\le r\le u} |x - (s_0 + u)v + rv| &= \inf_{0\le r\le u} |x - s_0 v - (u - r)v| \\ &\ge s_0|v| = R(\vec{X}), \qquad u \ge 0, \end{aligned}$$

hence by assumption,

$$\widetilde{\varphi}(u, \Psi(s_0 + u, x, v)) = (x - s_0 v, v) = \Psi(s_0, x, v), \qquad u \ge 0,$$

which implies that

$$\begin{aligned} & \widetilde{\varphi}(t + s_0 + u, \Psi(s_0 + u, x, v)) \\ = \; & \widetilde{\varphi}(t + s_0, \widetilde{\varphi}(u, \Psi(s_0 + u, x, v))) \\ = \; & \widetilde{\varphi}(t + s_0, \Psi(s_0, x, v)), \qquad t \in \mathbf{R}, u \ge 0. \end{aligned}$$

That is,

$$\widetilde{\varphi}(t + s, \Psi(s, x, v)) = \widetilde{\varphi}(t + s_0, \Psi(s_0, x, v)), \qquad \text{for any } s \ge s_0,$$

or equivalently, to say that $\widetilde{\varphi}(t+s, \Psi(s, x, v))$ is independent to s as long as $s \ge s_0$. Let

$$\widetilde{\psi}(t, x, v) = \lim_{s\to\infty} \widetilde{\varphi}(t + s; \Psi(s, x, v))(= \widetilde{\varphi}(t + s_0; \Psi(s_0, x, v))). \tag{3.4.1}$$

Also, let

$$C_0 = \Big\{2 \sum_{i=1}^{N} R_i \|\nabla U_i\|_\infty\Big\}^{1/2}.$$

PROPOSITION 3.4.1 *Suppose $|v| > 2C_0$. Then*

$$\widetilde{\varphi}^1(t, x, v) \cdot (|v|^{-1}v) > C_0, \qquad \text{for any } t \in \mathbf{R}, x \in E_v.$$

Proof. Notice that $\widetilde{\varphi}^1(0,x,v) = v$. Write $\eta = |v|^{-1}v$. Then by assumption, $v \cdot \eta = |v| > 2C_0$. Let

$$\tau_1 = \inf\{t \geq 0; \quad \widetilde{\varphi}^1(t,x,v)\cdot\eta \leq C_0\}.$$

We show that $\tau_1 = +\infty$.

Suppose $\tau_1 < +\infty$. Then $\widetilde{\varphi}^1(\tau_1,x,v)\cdot\eta = C_0$. By definition, we have

$$\begin{aligned}(\widetilde{\varphi}^0(t,x,v) - \widetilde{\varphi}^0(s,x,v))\cdot\eta &= \int_s^t \widetilde{\varphi}^1(u,x,v)\cdot\eta du \\ &> C_0|t-s|, \qquad \text{for any } 0 \leq s < t \leq \tau_1,\end{aligned}$$

which implies that

$$\frac{d}{dt}\left(\widetilde{\varphi}^0(t,x,v)\cdot\eta\right) \geq C_0, \qquad 0 \leq t \leq \tau_1. \tag{3.4.2}$$

In particular, $\frac{d}{dt}\left(\widetilde{\varphi}^0(t,x,v)\cdot\eta\right) > 0$ for $0 \leq t \leq \tau_1$. Also, since

$$\widetilde{\varphi}^1(\tau_1,x,v) - v = -\int_0^{\tau_1}\sum_{i=1}^N \nabla U_i(\widetilde{\varphi}^0(t,x,v) - X_i)dt,$$

we have by definition that

$$-\int_0^{\tau_1}\sum_{i=1}^N \nabla U_i(\widetilde{\varphi}^0(t,x,v) - X_i)\cdot\eta dt = \widetilde{\varphi}^1(\tau_1,x,v)\cdot\eta - v\cdot\eta < C_0 - 2C_0 = -C_0.$$

Therefore, with the help of (3.4.2), we have

$$\begin{aligned}C_0 &< \int_0^{\tau_1}\sum_{i=1}^N \nabla U_i(\widetilde{\varphi}^0(t,x,v) - X_i)\cdot\eta dt \\ &\leq \frac{1}{C_0}\sum_{i=1}^N\int_0^{\tau_1}\left|\nabla U_i(\widetilde{\varphi}^0(t,x,v) - X_i)\cdot\eta\right|\cdot\frac{d}{dt}\left(\widetilde{\varphi}^0(t,x,v)\cdot\eta\right)dt \\ &= \frac{1}{C_0}\sum_{i=1}^N\int_{t\in[0,\tau_1],|\widetilde{\varphi}^0(t,x,v)\cdot\eta - X_i\cdot\eta|<R_i}\left|\nabla U_i(\widetilde{\varphi}^0(t,x,v) - X_i)\cdot\eta\right|\cdot\frac{d}{dt}\left(\widetilde{\varphi}^0(t,x,v)\cdot\eta\right)dt \\ &\leq \frac{1}{C_0}\sum_{i=1}^N\|\nabla U_i\|_\infty\int_{t\in[0,\tau_1],|\widetilde{\varphi}^0(t,x,v)\cdot\eta - X_i\cdot\eta|<R_i}\frac{d}{dt}\left(\widetilde{\varphi}^0(t,x,v)\cdot\eta - X_i\cdot\eta\right)dt \\ &\leq \frac{1}{C_0}\sum_{i=1}^N\|\nabla U_i\|_\infty 2R_i \\ &= C_0,\end{aligned}$$

which makes a contradiction. Therefore, $\tau_1 = +\infty$, i.e., $\widetilde{\varphi}^1(t,x,v)\cdot(|v|^{-1}v) > C_0$ for any $t \geq 0$.

The assertion for $t < 0$ can be shown in the same way by considering

$$\tau_2 = \sup\{t < 0; \quad \widetilde{\varphi}^1(t,x,v)\cdot\eta \leq C_0\}.$$

∎

COROLLARY 3.4.2 *Let $(x,v) \in E$ with $|v| > 2C_0$. Then*

$$\left|\widetilde{\psi}^0(t,x,v) - X_i\right| > R_i, \qquad i = 1, \cdots, N,$$

if $t \geq 2C_0^{-1}R(\vec{X})$ or $t \leq -C_0^{-1}R(\vec{X})$.

Proof. Choose and fix any $(x,v) \in E$ with $|v| > 2C_0$, and let $\eta = |v|^{-1}v$. Then since $x \cdot v = 0$, we have that

$$\Psi^0(s,x,v) \cdot \eta = (x - sv) \cdot \eta = -sv \cdot \eta = -s|v|, \qquad \text{for any } s > 0.$$

Let $s_0 = \frac{R(\vec{X})}{|v|}$ as before. Then $s_0 < C_0^{-1}R(\vec{X})$, and

$$\widetilde{\varphi}^0(0, \Psi(s_0,x,v)) \cdot \eta = \Psi^0(s_0,x,v) \cdot \eta = -s_0|v| = -R(\vec{X}). \tag{3.4.3}$$

Moreover, by definition (3.4.1) of $\widetilde{\psi}$,

$$\widetilde{\psi}(t,x,v) = \widetilde{\varphi}(t + s_0, \Psi(s_0,x,v)). \tag{3.4.4}$$

Also, $|\Psi^1(s_0,x,v)| = |v| > 2C_0$ by assumption. Combining (3.4.4), Proposition 3.4.1 and (3.4.3), we get

$$\begin{aligned}
&\widetilde{\psi}^0(t,x,v) \cdot \eta = \widetilde{\varphi}^0(t+s_0, \Psi(s_0,x,v)) \cdot \eta \\
= \; & \int_0^{t+s_0} \widetilde{\varphi}^1(u, \Psi(s_0,x,v)) \cdot \eta du + \widetilde{\varphi}^0(0, \Psi(s_0,x,v)) \cdot \eta \\
> \; & (t+s_0)C_0 - R(\vec{X}), \qquad \text{for any } t > -s_0.
\end{aligned}$$

In particular, if $t > 2C_0^{-1}R(\vec{X})$, then

$$\widetilde{\psi}^0(t,x,v) \cdot \eta > (t+s_0)C_0 - R(\vec{X}) \geq R(\vec{X}).$$

In the same way, if $t < -C_0^{-1}R(\vec{X})$, then $t + s_0 < 0$, so

$$\begin{aligned}
&\widetilde{\psi}^0(t,x,v) \cdot \eta = \widetilde{\varphi}^0(t+s_0, \Psi(s_0,x,v)) \cdot \eta \\
= \; & -\int_{(t+s_0)}^0 \widetilde{\varphi}^1(u, \Psi(s_0,x,v)) \cdot \eta du + \widetilde{\varphi}^0(0, \Psi(s_0,x,v)) \cdot \eta \\
< \; & -C_0 \cdot (-(t+s_0)) - R(\vec{X}) \\
< \; & -R(\vec{X}).
\end{aligned}$$

This completes the proof of our assertion. ∎

Let

$$e_0 = \frac{1}{2}(2C_0+1)^2 + \sum_{i=1}^N \|U_i\|_\infty.$$

COROLLARY 3.4.3 *Suppose $\widetilde{E}(x,v) > e_0$. Then $|v| > 2C_0$. In particular,*

$$\widetilde{\varphi}^1(t, \Psi(s,x,v)) \cdot (|v|^{-1}v) > C_0, \qquad \text{for any } s,t \in \mathbf{R}, x \in E_v,$$

and

$$|\widetilde{\varphi}^0(t,x,v)| \to \infty, \qquad \text{as } t \to \infty.$$

Proof. Since $\widetilde{E}(x,v) > e_0$, we have by definition

$$\frac{1}{2}|v|^2 + \sum_{i=1}^{N} U_i(x - X_i) > e_0.$$

Therefore, by definition of e_0,

$$|v|^2 > 4C_0^2,$$

which implies that

$$|v| > 2C_0.$$

The second assertion follows then by Proposition 3.4.1. The last assertion is now easy since by the second assertion,

$$\frac{d}{dt}\left(\widetilde{\varphi}^0(t,x,v) \cdot \eta\right) = \widetilde{\varphi}^1(t,x,v) \cdot \eta \geq C_0 > 0,$$

where we used the notation $\eta = |v|^{-1}v$. ∎

PROPOSITION 3.4.4 *Let $\rho : \mathbf{R} \to [0,\infty)$ be a measurable function satisfying*

$$\rho(s) = 0, \qquad \text{for any } s < e_0.$$

Then for any measurable $f : \mathbf{R}^{2d} \to [0,\infty)$, we have

$$\begin{aligned} &\int_{\mathbf{R}^{2d}} f(x,v)\rho(\widetilde{E}(x,v))dxdv \\ = &\int_E \left(\int_{-\infty}^{\infty} f(\widetilde{\psi}(t,x,v))dt\right) \rho(\frac{1}{2}|v|^2)\nu(dx,dv). \end{aligned} \tag{3.4.5}$$

Proof. By using convergence theorem if necessary, we may and do assume, without loss of generality, that there exists a constant $\widetilde{R} > 0$ such that

$$\mathrm{supp}(f) \subset \{(x,v); |x| + |v| \leq \widetilde{R}\}.$$

Let

$$T = 2C_0^{-1}(\widetilde{R} + R(\vec{X})).$$

By Theorem 3.2.2 and Theorem 3.3.1, we have

$$\begin{aligned} &\int_{\mathbf{R}^{2d}} f(x,v)\rho(\widetilde{E}(x,v))dxdv \\ = &\int_{\mathbf{R}^{2d}} f(\widetilde{\varphi}(T,x,v))\rho(\widetilde{E}(x,v))dxdv \\ = &\int_{\mathbf{R}\times E} f(\widetilde{\varphi}(T,\Psi(t,x,v)))\rho(\widetilde{E}(\Psi(t,x,v)))dt\nu(dx,dv). \end{aligned} \tag{3.4.6}$$

Therefore, it suffices for us to show that the right hand side of (3.4.6) is equal to

$$\int_{\mathbf{R}\times E} f(\widetilde{\psi}(T-t,x,v))\rho(\frac{1}{2}|v|^2)dt\nu(dx,dv).$$

We only need to show that the integrands are equal, *i.e.*, it suffices to show that

$$f(\widetilde{\varphi}(T,\Psi(t,x,v)))\rho(\widetilde{E}(\Psi(t,x,v))) = f(\widetilde{\psi}(T-t,x,v))\rho(\frac{1}{2}|v|^2). \tag{3.4.7}$$

We show it from now on. We first show that if the left hand side above is not 0, then it is equal to the right hand side. Assume that $f(\widetilde{\varphi}(T,\Psi(t,x,v)))\rho(\widetilde{E}(\Psi(t,x,v))) \neq 0$. Then $\rho(\widetilde{E}(\Psi(t,x,v))) > 0$ implies by our assumption that $\widetilde{E}(\Psi(t,x,v)) > e_0$, so by Corollary 3.4.3, $|v| > 2C_0$ and

$$\widetilde{\varphi}^1(s,\Psi(t,x,v))\cdot\eta > C_0$$

for any $s \in \mathbf{R}$, where $\eta = |v|^{-1}v$. Therefore, since $(\widetilde{\varphi}^0,\widetilde{\varphi}^1)$ is the solution of the Newton's equation, we have by definition that

$$\begin{aligned}
& \Big(\widetilde{\varphi}^0(T,\Psi(t,x,v)) - \Psi^0(t,x,v)\Big)\cdot\eta \\
= & \int_0^T \frac{d}{ds}\widetilde{\varphi}^0(s,\Psi(t,x,v))ds\cdot\eta \\
= & \int_0^T \widetilde{\varphi}^1(s,\Psi(t,x,v))\cdot\eta ds \\
> & \ T\cdot C_0 = 2(\widetilde{R}+R(\vec{X})),
\end{aligned} \tag{3.4.8}$$

by the definition of T.

We also have $f(\widetilde{\varphi}(T,\Psi(t,x,v))) > 0$ in addtion, which gives us

$$|\widetilde{\varphi}^0(T,\Psi(t,x,v))\cdot\eta| \leq |\widetilde{\varphi}^0(T,\Psi(t,x,v))| + |\widetilde{\varphi}^1(T,\Psi(t,x,v))| \leq \widetilde{R}. \tag{3.4.9}$$

Combine (3.4.8) with (3.4.9), and notice that $x\cdot v = 0$ since $(x,v)\in E$, and we get by the definition of η that

$$\widetilde{R} + 2R(\vec{X}) < -\Psi^0(t,x,v)\cdot\eta = (x-tv)\cdot\eta = t|v|, \tag{3.4.10}$$

hence $t \geq \frac{\widetilde{R}+2R(\vec{X})}{|v|} \geq s_0$. So by the definition (3.4.1) of $\widetilde{\psi}$, we get

$$\widetilde{\psi}(T-t,x,v) = \widetilde{\varphi}(T-t+t,\Psi(t,x,v)) = \widetilde{\varphi}(T,\Psi(t,x,v)).$$

Also, (3.4.10) gives us that $|\Psi(t,x,v)| = |x-tv| \geq R(\vec{X})$, so by the definition of $\widetilde{E}$, we also get

$$\widetilde{E}(\Psi(t,x,v)) = \frac{1}{2}|v|^2.$$

This completes the proof of the fact that if the left hand side of (3.4.7) is not 0, then it is equal to the right hand side.

We next show the opposite, *i.e.*, we assume that the right hand side of (3.4.7), $f(\widetilde{\psi}(T-t,x,v))\rho(\frac{1}{2}|v|^2)$, is not 0, and show that it is equal to the left hand side, $f(\widetilde{\varphi}(T,\Psi(t,x,v)))\rho(\widetilde{E}(\Psi(t,x,v)))$. It is sufficient to show that $t \geq s_0(=\frac{R(\vec{X})}{|v|})$. (Actually, if $t \geq s_0$, then by using $x \cdot v = 0$, we get $|x - tv| \geq t|v| \geq R(\vec{X})$, hence $\widetilde{E}(\Psi(t,x,v)) = \frac{1}{2}|v|^2$ by definition. Also, since $t \geq s_0$, we have by (3.4.1) that $\widetilde{\psi}(T-t,x,v) = \widetilde{\varphi}(T-t+t,\Psi(t,x,v)) = \widetilde{\varphi}(T,\Psi(t,x,v))$, which will complete our proof). Since $\rho(\frac{1}{2}|v|^2) > 0$, we have $\frac{1}{2}|v|^2 > 2C_0^2$, hence $|v| > 2C_0$, which implies by Corollary 3.4.3 that

$$\widetilde{\varphi}^1(u,\Psi(s,x,v)) \cdot \eta > C_0 \tag{3.4.11}$$

for any $u, s \in \mathbf{R}$ and $x \in E_v$. If $t \geq T$, then by definition of T, since $|v| > 2C_0$, we have

$$t \geq T = \frac{2}{C_0}(\widetilde{R} + R(\vec{X})) > \frac{4}{|v|}(\widetilde{R} + R(\vec{X})) > \frac{R(\vec{X})}{|v|} = s_0.$$

If $t < T$, then we have by (3.4.11) and the definition of T that for any $r > 0$

$$\begin{aligned} & \Big(\widetilde{\varphi}^0(T-t+r,\Psi(r,x,v)) - \Psi^0(r,x,v)\Big) \cdot \eta \\ = \; & \int_0^{T-t+r} \widetilde{\varphi}^1(u,\Psi(r,x,v)) \cdot \eta du \\ > \; & (T-t+r) \cdot C_0 = 2\widetilde{R} + 2R(\vec{X}) + (r-t)C_0. \end{aligned}$$

Also, since $f(\widetilde{\psi}(T-t,x,v)) > 0$, we have

$$|\widetilde{\psi}^0(T-t,x,v)| + |\widetilde{\psi}^1(T-t,x,v)| \leq \widetilde{R}.$$

Therefore, we have for any $r \geq s_0$

$$|\widetilde{\varphi}^0(T-t+r,\Psi(r,x,v))| = |\widetilde{\psi}^0(T-t,x,v)| \leq \widetilde{R}.$$

Combining these two inequalities, we get

$$(r|v| =) - \Psi^0(r,x,v) \cdot \eta > \widetilde{R} + 2R(\vec{X}) + (r-t)C_0, \qquad \text{for any } r \geq s_0.$$

Applying the above to $r = s_0(=\frac{R(\vec{X})}{|v|})$, we get

$$R(\vec{X}) > \widetilde{R} + 2R(\vec{X}) + (s_0 - t)C_0.$$

Therefore, $t > s_0$. This completes our proof. ∎

Chapter 4

Random Field

4.1 Filtering

Let M be a Polish space and let $(\Omega, \mathcal{B}, P)$ be a complete probability space. Define

$$\aleph = \{A \in \mathcal{B}; P(A) = 0 \text{ or } 1\}.$$

DEFINITION 4.1.1 *$\mathcal{F} = \{\mathcal{F}_G; G$ is a open set in $M\}$ is called an increasing σ-algebre, if*

(1) *$\mathcal{F}_G$ is a sub-σ-algebra of $\mathcal{B}$ for any open set $G \subset M$,*

(2) *$\aleph \subset \mathcal{F}_G$ for any open set $G \subset M$,*

(3) *$\mathcal{F}_{G_1} \subset \mathcal{F}_{G_2}$ for any open sets $G_1 \subset G_2$.*

From now on, we choose and fix an increasing σ-algebra $\mathcal{F} = \{\mathcal{F}_G\}$.

DEFINITION 4.1.2 *A subset $A \subset M$ is called $\mathcal{F}$-regular if it satisfies the following condition: for any sequence of open sets G_n, $n \in \mathbf{N}$, satisfying $G_1 \supset G_2 \supset G_3 \supset \cdots$, if $G_n \supset A^o$ for any $n \in \mathbf{N}$, and $\overline{A} \supset \bigcap_{n=1}^{\infty} \overline{G_n}$, then $\mathcal{F}_{A^o} = \bigcap_{n=1}^{\infty} \mathcal{F}_{G_n}$.*

PROPOSITION 4.1.3 *If A is $\mathcal{F}$-regular, then for any A' satisfying $A^o \subset A' \subset \overline{A}$, we have that A' is also $\mathcal{F}$-regular.*

DEFINITION 4.1.4 *A map $T : \Omega \to \mathcal{O}(M)$ is called a $\mathcal{F}$-stopping time, if*

(1) *T is $\mathcal{B}/\mathcal{E}_0$-measurable,*

(2) *$\{\omega \in \Omega; T(\omega) \subset G\} \in \mathcal{F}_G$ for any $\mathcal{F}$-regular open set G.*

DEFINITION 4.1.5 *For any $\mathcal{F}$-stopping time $T : \Omega \to \mathcal{O}(M)$, we define*

$$\mathcal{F}_T = \{A \in \mathcal{B}; A \cap \{T \subset G\} \in \mathcal{F}_G \text{ for any open and } \mathcal{F}\text{-regular subset } G\}.$$

PROPOSITION 4.1.6 *Let S and T be two $\mathcal{F}_G$-stopping times satisfying*

$$S(\omega) \subset T(\omega), \qquad \text{for all } \omega \in \Omega.$$

Then

$$\mathcal{F}_S \subset \mathcal{F}_T.$$

Proof. This is easy since by condition,

$$A \cap \{T \subset G\} = A \cap \{S \subset G\} \cap \{T \subset G\}.$$

∎

DEFINITION 4.1.7 *$\mathcal{C}$ is said to be a $\mathcal{F}$-regular covering if*

(1) $\mathcal{C} \subset \mathcal{O}(M)$, *and* $\bigcup \mathcal{C} = M$,

(2) $\sharp(\mathcal{C}) < \infty$,

(3) $\bigcup_{k=1}^{n} C_k$ *and* $(\bigcup_{k=1}^{n} C_k)^C$ *are $\mathcal{F}$-regular for any $n \geq 1$ and $C_1, \cdots, C_n \in \mathcal{C}$.*

For any $\mathcal{F}$-regular covering $\mathcal{C}$ and any $K \in \mathcal{O}(M)$, we define

$$[K]_{\mathcal{C}} = \bigcup \{C \in \mathcal{C}; K \cap C \neq \emptyset\}.$$

PROPOSITION 4.1.8 *Let $\mathcal{C}$ be a $\mathcal{F}$-regular covering. Then*

(1) $[K]_{\mathcal{C}} \supset K$ *for any closed K,*

(2) $[K]_{\mathcal{C}} \subset G \Longleftrightarrow K \subset {[G^C]_{\mathcal{C}}}^C$ *for any open G and closed K.*

Proof. The first assertion is trivial. We show the second one. Notice that

$$\begin{aligned}
& [K]_{\mathcal{C}} \subset G \\
\Longleftrightarrow\ & \text{for any } C \in \mathcal{C}, K \cap C \neq \emptyset \text{ implies } C \subset G \\
\Longleftrightarrow\ & \text{for any } C \in \mathcal{C}, C \cap G^C \neq \emptyset \text{ implies } K \cap C = \emptyset \\
\Longleftrightarrow\ & [G^C]_{\mathcal{C}} \subset K^C.
\end{aligned}$$

This completes the proof. ∎

PROPOSITION 4.1.9 *Let $\mathcal{C}$ be a $\mathcal{F}$-regular covering and T be a $\mathcal{F}$-stopping time. Then $[T]_{\mathcal{C}}$ is also a $\mathcal{F}$-stopping time.*

Proof. By definition, we only need to show that

$$\{[T]_{\mathcal{C}} \subset G\} \in \mathcal{F}_G$$

for any open set G.

For any such G, we have that $[G^C]_{\mathcal{C}}^C$ is open, and by Proposition 4.1.8, since T is a F-stopping time,

$$\{[T]_{\mathcal{C}} \subset G\} = \{T \subset [G^C]_{\mathcal{C}}^C\} \in \mathcal{F}_{[G^C]_{\mathcal{C}}^C}.$$

Also, since $G^C \subset [G^C]_{\mathcal{C}}$ by Proposition 4.1.8, we have $G \supset [G^C]_{\mathcal{C}}^C$. Therefore, by the definition of increasing σ-algebra, we get $\mathcal{F}_{[G^C]_{\mathcal{C}}^C} \subset \mathcal{F}_G$. These give us our assertion. ∎

DEFINITION 4.1.10 *We call $\{\mathcal{C}_n\}_{n=1}^{\infty}$ a good sequence if*

(1) *$\mathcal{C}_n$ is a regular covering for any $n \geq 1$,*

(2) *for any $C \in \mathcal{C}_{n+1}$, there exists a $C' \in \mathcal{C}_n$ such that $C \subset C'$,*

(3) *$\bigcap_{n=1}^{\infty} [K]_{\mathcal{C}_n} = K$ for any $K \in \mathcal{O}(M)$.*

PROPOSITION 4.1.11 *Let $\mathcal{C}_n$ be a good sequence and T be a $\mathcal{F}$-stopping time. Then*

(1) *$\mathcal{F}_{[T]_{\mathcal{C}_n}}$ is monotone non-increasing with respect to n,*

(2) *$\mathcal{F}_T = \bigcap_{n=1}^{\infty} \mathcal{F}_{[T]_{\mathcal{C}_n}}$.*

Proof. First notice that for any $n \in \mathbf{N}$, we have by definition

$$[K]_{\mathcal{C}_{n+1}} \subset [K]_{\mathcal{C}_n}, \qquad \text{for any closed } K.$$

This combined with Proposition 4.1.6 gives us our first assertion.

Also, since $T(\omega) \subset [T(\omega)]_{\mathcal{C}_n}$ for any $\omega \in \Omega$, we have by Proposition 4.1.6

$$\mathcal{F}_T \subset \mathcal{F}_{[T]_{\mathcal{C}_n}}, \qquad n \geq 1.$$

So to show the second assertion, it suffices to show that $\bigcap_{n=1}^{\infty} \mathcal{F}_{[T]_{\mathcal{C}_n}} \subset \mathcal{F}_T$. Choose any $A \in \bigcap_{n=1}^{\infty} \mathcal{F}_{[T]_{\mathcal{C}_n}}$ and any regular open set G. Define

$$K_n = [\overline{G}]_{\mathcal{C}_n},$$
$$G_{n,m} = [\overline{(K_n)_{1/m}}]_{\mathcal{C}_m}^o,$$

where we used the notation $B_\varepsilon = \{x \in M; \mathrm{dist}(x, B) < \varepsilon\}$, $B \subset M$. Then $G_{n,m}$ is also open regular by definition and Proposition 4.1.3, since $\mathcal{C}$ is a regular covering. Also, by the definition of good sequence, we have for any $n, \ell \geq 1$,

$$\bigcap_{m=1}^{\infty} \overline{G_{n,m}} \subset \bigcap_{m \geq l} [\overline{(K_n)_{1/\ell}}]_{\mathcal{C}_m} = \overline{(K_n)_{1/\ell}}.$$

Since K_n is closed, this implies that $\bigcap_{m=1}^{\infty} \overline{G_{n,m}} \subset K_n$. On the other hand, for any $m \in \mathbf{N}$, since $[\overline{(K_n)_{1/m}}]_{\mathcal{C}_m} \supset \overline{(K_n)_{1/m}}$, we have $G_{n,m} = [\overline{(K_n)_{1/m}}]^o_{\mathcal{C}_m} \supset \overline{(K_n)_{1/m}}^o \supset K_n$. Combining the above, we get

$$\bigcap_{m=1}^{\infty} \overline{G_{n,m}} = K_n, \qquad n \in \mathbf{N},$$

hence

$$\bigcap_{n,m=1}^{\infty} \overline{G_{n,m}} = \bigcap_{n=1}^{\infty} K_n = \overline{G}. \tag{4.1.1}$$

Notice that since K_n is monotone non-increasing with respect to n, we have that $G_{n,m} \supset G_{m,m}$ for any $n, m \in \mathbf{N}$ with $m \geq n$. Also, for any $m \leq n$, we have by definition $G_{n,m} = [\overline{(K_n)_{1/m}}]^o_{\mathcal{C}_m} \supset [\overline{(K_n)_{1/n}}]^o_{\mathcal{C}_m} \supset [\overline{(K_n)_{1/n}}]^o_{\mathcal{C}_n} = G_{n,n}$. Therefore, (4.1.1) becomes

$$\bigcap_{n=1}^{\infty} \overline{G_{n,n}} = \overline{G}. \tag{4.1.2}$$

Also notice that $G_{n,n}$ is monotone non-increasing with respect to n, and $G_{n,n} \supset G$. Therefore, since G is regular, (4.1.2) implies

$$\bigcap_{n=1}^{\infty} \mathcal{F}_{G_{n,n}} = \mathcal{F}_G. \tag{4.1.3}$$

Now, for any $n \in \mathbf{N}$, since

$$T \subset G \Longrightarrow [T]_{\mathcal{C}_n} \subset [\overline{G}]_{\mathcal{C}_n} = K_n \subset G_{n,n},$$

we have

$$A \cap \{T \subset G\} = A \cap \{T \subset G\} \cap \{[T]_{\mathcal{C}_n} \subset G_{n,n}\}.$$

But $A \cap \{[T]_{\mathcal{C}_n} \subset G_{n,n}\} \subset \mathcal{F}_{G_{n,n}}$ since $A \in \mathcal{F}_{[T]_{\mathcal{C}_n}}$, and $\{T \subset G\} \in \mathcal{F}_G \subset \mathcal{F}_{G_{n,n}}$. Therefore,

$$A \cap \{T \subset G\} \in \mathcal{F}_{G_{n,n}}, \qquad \text{for any } n \geq 1.$$

So by (4.1.3),

$$A \cap \{T \subset G\} \in \bigcap_{n=1}^{\infty} \mathcal{F}_{G_{n,n}} = \mathcal{F}_G, \qquad \text{for any open regular } G,$$

hence $A \in \mathcal{F}_T$. This is true for any $A \in \bigcap_{n=1}^{\infty} \mathcal{F}_{[T]_{\mathcal{C}_n}}$. So $\bigcap_{n=1}^{\infty} \mathcal{F}_{[T]_{\mathcal{C}_n}} \subset \mathcal{F}_T$.

This completes the proof of the second assertion. ∎

4.2 Poisson point process

Let M be a locally compact Polish space, let ν be a σ-finite Radon measure on $(M, \mathcal{B}(M))$ with no atom, and let P_ν denote the Poisson point process on $Conf(M)$ with intensity measure ν. (See Section 2.4 for the definition of the notations). We also define $\Omega = Conf(M)$, $\mathcal{F} = \mathcal{E}_0\Big|_{Conf(M)}$, and $P = P_\nu$. Then $(\Omega, \mathcal{F}, P)$ is a complete probability space.

For any $A \in \mathcal{B}(M)$, define

$$X_A(\omega) = \mu_\omega(A), \qquad \omega \in \Omega,$$

and

$$\mathcal{F}_A = \sigma\{X_K; K \text{ is compact and } K \subset A\} \vee \aleph.$$

It is trivial that $\{\mathcal{F}_A | A \text{ is open}\}$ is an increasing σ-algebra.

PROPOSITION 4.2.1 (1) *Let $A_1, A_2 \in \mathcal{B}(M)$ with $A_1 \subset A_2$. Then X_{A_1} is $\mathcal{F}_{A_2}$-measurable, and the distribution of X_{A_1} is the Poisson distribution with mean $\nu(A_1)$.*

(2) *If $A_1, \cdots, A_n \in \mathcal{B}(M)$ are disjoint with each other, then $X_{A_1}, \cdots, X_{A_n}$ are independent.*

(3) *If $A_1, A_2 \in \mathcal{B}(M)$, then $\mathcal{F}_{A_1 \cup A_2} = \mathcal{F}_{A_1} \vee \mathcal{F}_{A_2}$.*

(4) *If $A_1, A_2 \in \mathcal{B}(M)$ and $A_1 \cap A_2 = \emptyset$, then $\mathcal{F}_{A_1}$ and $\mathcal{F}_{A_2}$ are independent.*

Proof. Just notice that

$$X_A(\omega) = \mu_\omega(A) = \sup\{X_K(\omega); K \text{ is compact and } K \subset A\},$$

which is easy to be checked by considering first the case $\nu(A) < \infty$ then the general case. Our assertion is now easy from Theorem 2.4.1. ∎

PROPOSITION 4.2.2 *If $A \subset M$ satisfies $\nu(\partial A) = 0$, then A is regular.*

Proof. First, since $\nu(\partial A) = 0$ by assumption, we have that X_K has mean 0 for any $K \subset \partial A$. So $\mathcal{F}_{\partial A} \subset \aleph$. Therefore, by Proposition 4.2.1,

$$\mathcal{F}_{\overline{A}} = \mathcal{F}_{A^o} \vee \mathcal{F}_{\partial A} = \mathcal{F}_{A^o},$$

hence

$$\mathcal{B} = \mathcal{F}_{\overline{A}} \vee \mathcal{F}_{(\overline{A})^C} = \mathcal{F}_{A^o} \vee \mathcal{F}_{(\overline{A})^C}.$$

Let $\{G_n\}_{n \in \mathbf{N}}$ be any sequence of monotone decreasing open sets satisfying

$$G_n \supset A, \qquad \bigcap_{n=1}^{\infty} \overline{G_n} \subset \overline{A}.$$

We show that $\mathcal{F}_{A^o} \supset \bigcap_{n=1}^{\infty} \mathcal{F}_{\overline{G_n}}$.

First,

$$\mathcal{B} = \mathcal{F}_{\overline{G_n}} \vee \mathcal{F}_{(\overline{G_n})^C}$$

for any $n \geq 1$. So

$$\begin{aligned}
&\left(\bigcap_{m=1}^{\infty} \mathcal{F}_{\overline{G_m}}\right) \vee \left(\bigvee_{n=1}^{\infty} \mathcal{F}_{(\overline{G_n})^C}\right) \\
= &\bigvee_{n=1}^{\infty} \left\{\left(\bigcap_{m=1}^{\infty} \mathcal{F}_{\overline{G_m}}\right) \vee \mathcal{F}_{(\overline{G_n})^C}\right\} \\
\subset &\bigvee_{n=1}^{\infty} \left\{\mathcal{F}_{\overline{G_n}} \vee \mathcal{F}_{(\overline{G_n})^C}\right\} = \bigvee_{n=1}^{\infty} \mathcal{B} = \mathcal{B} \\
= &\mathcal{F}_{A^o} \vee \mathcal{F}_{(\overline{A})^C}.
\end{aligned} \tag{4.2.1}$$

But since $\bigcup_{n=1}^{\infty} (\overline{G_n})^C \supset (\overline{A})^C$, we have

$$\bigvee_{n=1}^{\infty} \mathcal{F}_{(\overline{G_n})^C} \supset \mathcal{F}_{(\overline{A})^C}. \tag{4.2.2}$$

Also, it is easy to see that $\mathcal{F}_{\overline{G_n}} \supset \mathcal{F}_{A^o}$ for any $n \in \mathbf{N}$, hence

$$\bigcap_{n=1}^{\infty} \mathcal{F}_{\overline{G_n}} \supset \mathcal{F}_{A^o}. \tag{4.2.3}$$

On the other hand, it is easy to see that

$$\bigcap_{m=1}^{\infty} \mathcal{F}_{\overline{G_m}} \text{ and } \bigvee_{n=1}^{\infty} \mathcal{F}_{(\overline{G_n})^C} \text{ are independent.}$$

This combined with (4.2.1), (4.2.2) and (4.2.3) gives us that

$$\mathcal{F}_{A^o} \supset \bigcap_{n=1}^{\infty} \mathcal{F}_{\overline{G_n}},$$

which completes the proof. ∎

PROPOSITION 4.2.3 *There exists a good sequence.*

Proof. Since M is separable, there exists a sequence $\{x_n\}_{n \in \mathbf{N}}$ that is dense in M. Let

$$R_n = \sup\{r > 0; \nu(B(x_n, r)) < \infty\} \wedge 1.$$

Then $R_n > 0$, $n \geq 1$. For any $n \geq 1$, there are at most countablely many $r \in (0, R_n)$ such that $\nu(\partial B(x_n, r)) > 0$, so there exists a sequence $\{r_{n,m}\}_{m=1}^{\infty}$ such that it is dense in $(0, R_n)$ and $\nu(\partial B(x_n, r_{n,m})) = 0$. Let

$$\begin{aligned}
B_{n,m,0} &= \overline{B(x_n, r_{n,m})}, \\
B_{n,m,1} &= M \setminus B(x_n, r_{n,m}),
\end{aligned}$$

and let

$$\mathcal{C}_N = \left\{ \bigcap_{n,m=1}^{N} B_{n,m,i_{n,m}}; i_{n,m} = 0 \text{ or } 1 \right\}.$$

Then $\{\mathcal{C}_N\}_{N=1}^{\infty}$ is a good sequence. In order to say so, we only need to check that it satisfies all of the three conditions of Definition 4.1.10. We do it in the following. The fact that $\mathcal{C}_n$ is a regular covering is trivial by definition and Proposition 4.2.2. The second condition is also satisfied trivially. We show in the following that the third one is also satisfied, *i.e.*, for any $K \in \mathcal{O}(M)$, we show that $\bigcap_{n=1}^{\infty}[K]_{\mathcal{C}_n} \supset K$. Suppose not. Then there exists a $x \notin K$ and $x \in [K]_{\mathcal{C}_n} = \bigcup\{C \in \mathcal{C}_N : K \cap C \neq \emptyset\}$ for any $N \in \mathbf{N}$. Since K is closed, the fact $x \notin K$ gives us that there exists an $\varepsilon > 0$ such that $B(x, \varepsilon) \cap K = \emptyset$. Also, the latter condition gives us that for any $N \in \mathbf{N}$, there exists a family $\{i_{n,m}^N \in \{0,1\}\}_{n,m=1}^N$ such that $x \in \cap_{n,m=1}^N B_{n,m,i_{n,m}^N}$ and $\cap_{n,m=1}^N B_{n,m,i_{n,m}^N} \cap K \neq \emptyset$. Since $\{x_n\}_n$ is dense in M, there exists a sub-sequence n_k such that $x_{n_k} \to x$ as $k \to \infty$, hence there exists a $K \in \mathbf{N}$ such that for any $k \geq K$, we have $B(x_{n_k}, \frac{\varepsilon}{2}) \cap K = \emptyset$. Also, since x_{n_k} converges, it is easy to see that $\inf_k r_{n_k} > 0$. Without loss of generality, we assume that this positive number is greater than $\frac{\varepsilon}{4}$. So since $\{R_{n,m}\}_{m=1}^{\infty}$ is dense in $(0, R_n)$, there exists a sequence r_{n_k,m_k} such that $r_{n_k,m_k} \to \frac{\varepsilon}{4}$ as $k \to \infty$. Then for $k \in \mathbf{N}$ large enough, we have $\overline{B(x_{n_k}, r_{n_k,m_k})} \cap K = \emptyset$. Therefore, the condition $B_{n_k,m_k,i_{n_k,m_k}^N} \cap K \neq \emptyset$ implies that $i_{n_k,m_k}^N = 1$ for k large enough. So the condition $x \in B_{n_k,m_k,i_{n_k,m_k}^N}$ now becomes $x \in B_{n_k,m_k,1}$, hence $x \notin B(x_{n_k}, r_{n_k,m_k}) \supset B(x_{n_k}, \frac{\varepsilon}{8})$ for any k large enough, which contridicts with the fact that $x_{n_k} \to x$. This gives us that $\bigcap_{n=1}^{\infty}[K]_{\mathcal{C}_n} \supset K$, and completes our proof. ∎

In the second half of this section, let us discuss about strong Markov property.

Same as before, let M be a locally compact Polish space, let ν be a σ-finite Radon measure with no atom, and let P_ν denote the Poisson point process on $Conf(M)$ with intensity measure ν. Define

$$\mathcal{F}_G = \sigma\{\nu(A); A \in \mathcal{B}(M), A \subset G\} \vee \aleph.$$

THEOREM 4.2.4 *Let T be a $\{\mathcal{F}_G\}$-stopping time. Then*

(1) *$\mu_\omega(A \cap T(\omega))$ and $\nu(A \cap T(\omega))$ are $\mathcal{F}_T$-measurable for any $A \in \mathcal{B}(M)$,*

(2) *For any $A_i \in \mathcal{B}(M)$, $i = 1, \cdots, m$, disjoint with each other, we have that*

$$\begin{aligned} & E^{P_\nu}\Big[\exp(\sqrt{-1}\sum_{i=1}^m \xi_i \mu(A_i \setminus T))\Big|\mathcal{F}_T\Big] \\ = \ & \exp(\sum_{i=1}^m (e^{\sqrt{-1}\xi_i} - 1)\nu(A_i \setminus T)), \qquad P_\nu - a.s.. \end{aligned}$$

Proof. Let $\mathcal{C}_n$ be a good sequence, which exists by Proposition 4.2.3. Also, for any $n \in \mathbf{N}$, let $T_n = [T]_{\mathcal{C}_n}$. Then T_n, $n \in \mathbf{N}$, are also stopping times by Proposition 4.1.9, and satisfy $T_n \downarrow T$ as $n \to \infty$.

Fix any $n \in \mathbf{N}$ for a while. We have $\sharp(\mathcal{C}_n) < \infty$ by definition. Also, we have in general that there exists an $\varepsilon > 0$ such that

$$K, K' \in \mathcal{C}_n, K \not\subset K' \Longrightarrow K \not\subset (K')_\varepsilon. \tag{4.2.4}$$

Therefore, for any $K \in \mathcal{C}_n$ and G open with $K \subset G$, we have

$$\{T_n \subset K\} = \{T_n \subset (K)_\varepsilon \cap G\} \in \mathcal{F}_{(K)_\varepsilon \cap G} \subset \mathcal{F}_G.$$

The part "$\in$" in the equation above is easy since T_n is a stopping time and $(K)_\varepsilon \cap G$ is open. We show the "$=$" part. It is easy that $\{T_n \subset K\} \subset \{T_n \subset (K)_\varepsilon \cap G\}$. We show that the opposite one also holds. Actually, by the definition of T_n, $T_n = \bigcup\{C \in \mathcal{C}_n; T \cap C \neq \emptyset\}$, so with the help of (4.2.4),

$$\begin{aligned} & T_n \subset (K)_\varepsilon \cap G \\ \Longrightarrow\ & T_n \subset (K)_\varepsilon \\ \Longrightarrow\ & \text{for any } C \in \mathcal{C}_n \text{ satisfying } T \cap C \neq \emptyset, \text{ we have } C \subset (K)_\varepsilon \\ \Longrightarrow\ & \text{for any } C \in \mathcal{C}_n \text{ satisfying } T \cap C \neq \emptyset, \text{ we have } C \subset K \\ \Longrightarrow\ & T_n \subset K, \end{aligned}$$

i.e., $\{T_n \subset (K)_\varepsilon \cap G\} \subset \{T_n \subset K\}$. This gives us our assertion.

By using this fact, for any $B \in \mathcal{F}_{T_n}$, we have that

$$B \cap \{T_n \subset K\} = B \cap \{T_n \subset (K)_\varepsilon \cap G\} \in \mathcal{F}_G,$$

therefore,

$$B \cap \{T_n = K\} = \Big(B \cap \{T_n \subset K\}\Big) \setminus \bigcup_{K' \in \mathcal{C}_n, K' \subset K, K' \neq K} (B \cap \{T_n \subset K'\}) \in \mathcal{F}_G.$$

Claim.

$$\begin{aligned} & E^{P_\nu}\Big[\exp(\sqrt{-1}\sum_{j=1}^m \xi_j \mu_\omega(A_j \setminus T_n))\Big|\mathcal{F}_{T_n}\Big] \\ = \ & \exp(\sum_{j=1}^m (e^{\sqrt{-1}\xi_j} - 1)\nu(A_j \setminus T_n)), \qquad P_\nu - \text{a.s.}. \end{aligned}$$

Proof of Claim. For any $B \in \mathcal{F}_{T_n}$, we have by Proposition 4.2.1

$$\begin{aligned} & E^{P_\nu}\Big[1_B \exp(\sqrt{-1}\sum_{j=1}^m \xi_j \mu_\omega(A_j \setminus T_n))\Big] \\ = \ & \sum_{K \in \mathcal{C}_n} E^{P_\nu}\Big[\exp(\sqrt{-1}\sum_{i=j}^m \xi_j \mu_\omega(A_j \setminus T_n)), B \cap \{T_n = K\}\Big] \\ = \ & \sum_{K \in \mathcal{C}_n} E^{P_\nu}\Big[\exp(\sqrt{-1}\sum_{j=1}^m \xi_j \mu_\omega(A_j \setminus K)), B \cap \{T_n = K\}\Big] \end{aligned}$$

$$
\begin{aligned}
&= \lim_{\varepsilon\to 0}\sum_{K\in\mathcal{C}_n} E^{P_\nu}\Big[\exp(\sqrt{-1}\sum_{j=1}^m \xi_j\mu_\omega(A_j\setminus\overline{(K)_\varepsilon})), B\cap\{T_n=K\}\Big]\\
&= \lim_{\varepsilon\to 0}\sum_{K\in\mathcal{C}_n} \exp\Big(\sum_{j=1}^m (e^{\sqrt{-1}\xi_j}-1)\nu(A_j\setminus\overline{(K)_\varepsilon})\Big)P_\nu(B\cap\{T_n=K\})\\
&= \sum_{K\in\mathcal{C}_n} E^{P_\nu}\Big[1_B\exp(\sum_{j=1}^m (e^{\sqrt{-1}\xi_j}-1)\nu(A_j\setminus K), T_n=K\Big]\\
&= E^{P_\nu}\Big[1_B\exp(\sum_{j=1}^m (e^{\sqrt{-1}\xi_j}-1)\nu(A_j\setminus T_n)\Big].
\end{aligned}
$$

Also, in general, for any $A\in\mathcal{B}(M)$, we have

$$\nu(A\cap T_n)=\sum_{K\in\mathcal{C}_n}\nu(A\cap K)1_{\{T_n=K\}},$$

so

$$1_{\{T_n\subset G\}}\nu(A\cap T_n)=\sum_{K\in\mathcal{C}_n, K\subset G}\nu(A\cap K)1_{\{T_n=K\}},$$

which is $\mathcal{F}_G$-measurable. In the same way, $1_{\{T_n\subset G\}}\mu_\omega(A\cap T_n)$ is $\mathcal{F}_G$-measurable. Therefore,

$$\mu_\omega(A\cap T_n) \text{ and } \nu(A\cap T_n) \text{ are } \mathcal{F}_{T_n}\text{-measurable.} \tag{4.2.5}$$

As a result,

$$\nu(A_j\setminus T_n)=\nu(A_j)-\nu(A_j\cap T_n)$$

is also $\mathcal{F}_{T_n}$-measurable. This completes the proof of our Claim.

Let $n\to\infty$ in (4.2.5), and we get the first assertion of our Theorem by Proposition 4.1.11.

Also, for any $B\in\mathcal{F}_T$, we have by our Claim that

$$
\begin{aligned}
&E^{P_\nu}\Big[1_B\exp(\sqrt{-1}\sum_{j=1}^m \xi_j\mu_\omega(A_j\setminus T))\Big]\\
&= \lim_{n\to\infty} E^{P_\nu}\Big[1_B\exp(\sqrt{-1}\sum_{j=1}^m \xi_j\mu_\omega(A_j\setminus T_n))\Big]\\
&= E^{P_\nu}\Big[1_B\exp(\sum_{j=1}^m (e^{\sqrt{-1}\xi_j}-1)\nu(A_j\setminus T)\Big].
\end{aligned}
$$

This completes the proof of our Theorem. ∎

COROLLARY 4.2.5 (1) *Let $f: M\to[0,\infty)$ be measurable and let S be a stopping time. Then*

$$E\Big[\int_{S(\omega)} f\,d\mu_\omega\Big]=E\Big[\int_{S(\omega)} f\,d\nu\Big].$$

(2) *Let* $f : M \to [0, \infty)$ *be measurable and* S, T *be two stopping times satisfying*

(i) $T(\omega) \subset S(\omega)$ *for any* $\omega \in \Omega$,

(ii) $E\Big[\int_{S(\omega)} |f| d\nu\Big] < \infty$.

Then

$$E\Big[\int_{S(\omega)} f(d\mu_\omega - d\nu)\Big|\mathcal{F}_T\Big] = E\Big[\int_{T(\omega)} f(d\mu_\omega - d\nu)\Big].$$

Proof. First, for $A \in \mathcal{B}(M)$ with $\nu(A) < \infty$, we have by Theorem 4.2.4 that

$$E[\mu_\omega(A \setminus S)\big|\mathcal{F}_S] = \nu(A \setminus S)$$

So for any f that can be expressed as $f = \sum_{k=1}^n a_k 1_{A_k}$ with $a_k > 0$ and $\nu(A_k) < \infty$, we have

$$E\Big[\int_{M\setminus S} f d\mu_\omega\Big|\mathcal{F}_S\Big] = \int_{M\setminus S} f d\nu,$$

hence

$$E\Big[\int_M f d\mu_\omega\Big|\mathcal{F}_S\Big] = \int_{M\setminus S} f d\nu + \int_S f d\mu_\omega. \tag{4.2.6}$$

Also, it is easy by definition that

$$E\Big[\int_M f d\mu_\omega\Big] = \int_M f d\nu.$$

So

$$E\Big[\int_{S(\omega)} f d\mu_\omega\Big] = E\Big[\int_{S(\omega)} f d\nu\Big].$$

This gives us our first assertion by monotone convergence theorem.

Also, for any measureable $f : M \to [0, \infty)$ satisfying $\int_M f d\nu < \infty$, by monotone convergence theorem, we also get from (4.2.6) that

$$E\Big[\int_M f(d\mu_\omega - d\nu)\Big|\mathcal{F}_S\Big] = \int_{S(\omega)} f(d\mu_\omega - d\nu).$$

Therefore,

$$\begin{aligned}
& E\Big[\int_{S(\omega)} f(d\mu_\omega - d\nu)\Big|\mathcal{F}_T\Big] \\
= \;& E\Big[E\Big[\int_M f(d\mu_\omega - d\nu)\Big|\mathcal{F}_S\Big]\Big|\mathcal{F}_T\Big] \\
= \;& E\Big[\int_M f(d\mu_\omega - d\nu)\Big|\mathcal{F}_T\Big] \\
= \;& E\int_{T(\omega)} f(d\mu_\omega - d\nu).
\end{aligned}$$

This completes the proof of our corollary. ∎

4.3 Existence of solution for ODE of infinite particle system

In this section, we prove the *a.s.* existence of the solution of the considered ODE. (See Sinai [12], [13] for some related results).

As claimed in Chapter 1, let $N \geq 1$, $d \geq 2$, $m > 0$ and $M_i > 0$, $i = 1, \cdots, N$. Also, let $U_i \in C_0^\infty(\mathbf{R}^d)$, and let R_i be constants such that $U_i(x) \geq 0$ if $|x| \geq R_i$, $i = 1, \cdots, N$. For any $X_{i,0}, V_{i,0} \in \mathbf{R}^d$, $i = 1, \cdots, N$, and $\omega \in Conf(\mathbf{R}^{2d})$, we consider the following equation given in Chapter 1:

$$\begin{cases} \dfrac{d}{dt}X_i(t;\omega) = V_i(t;\omega) \\ M_i\dfrac{d}{dt}V_i(t;\omega) = -\displaystyle\int \nabla U_i(X_i(t;\omega) - x(t,x,v;\omega))\mu_\omega(dx,dv) \\ (X_i(0;\omega), V_i(0;\omega)) = (X_{i,0}, V_{i,0}) \\ \dfrac{d}{dt}x(t,x,v;\omega) = v(t,x,v;\omega) \\ m\dfrac{d}{dt}v(t,x,v;\omega) = -\displaystyle\sum_{i=1}^N \nabla U_i(x(t,x,v;\omega) - X_i(t;\omega)) \\ (x(0,x,v;\omega), v(0,x,v;\omega)) = (x,v) \end{cases} \tag{4.3.1}$$

It is easy to see that if $\omega \in Fin(\mathbf{R}^{2d})$, then (4.3.1) has a unique solution $(\vec{X}(t;\omega), \vec{V}(t;\omega)), (x(t,x,v;\omega), v(t,x,v;\omega))$.

Let $\rho : \mathbf{R} \to [0,\infty)$ be a measurable function satisfying the following:

Assumption. $\displaystyle\int_{-\infty}^{\infty}(1+|s|)^d\rho(s)ds < \infty.$

Notice that for any $c \in \mathbf{R}$ and $\alpha \geq 0$, since $d \geq 2$, we have $\alpha + \frac{d}{2} - 1 \geq 0$, so

$$\begin{aligned} & \int_{\mathbf{R}^d} |v|^{2\alpha}\rho(\frac{m}{2}|v|^2 + c)dv \\ = \ & C_d\int_0^\infty r^{2\alpha}\rho(\frac{m}{2}r^2 + c)r^{d-1}dr \\ = \ & C_{d,m}\int_0^\infty s^{\alpha+\frac{d}{2}-1}\rho(s+c)ds \\ \leq \ & C_{d,m}\int_{-\infty}^\infty |s-c|^{\alpha+\frac{d}{2}-1}\rho(s)ds \\ \leq \ & C_{d,m}\int_{-\infty}^\infty (|c|+|s|)^{\alpha+\frac{d}{2}-1}\rho(s)ds \end{aligned} \tag{4.3.2}$$

for some constants $C_d, C_{d,m} > 0$ independent to c, where when passing to the third line, we used the change of variable $s = \frac{m}{2}r^2$. So

$$\int_{\mathbf{R}^d} |v|^{2\alpha}\rho(\frac{m}{2}|v|^2 + c)dv < \infty, \qquad \text{if } 0 \leq \alpha \leq \frac{d}{2} + 1. \tag{4.3.3}$$

Let

$$\lambda_m(dx, dv) = m^{\frac{d-1}{2}} \rho\Big(\frac{m}{2}|v|^2 + \sum_{i=1}^N U_i(x - X_{i,0})\Big) dx dv,$$

and let P_{λ_m} be the Poisson point process on $Conf(\mathbf{R}^{2d})$ with intensity λ_m.

THEOREM 4.3.1 *There exists a unique solution to (4.3.1) for P_{λ_m}-a.s. ω.*

Proof. Fix any $T > 0$. We only need to show the existence and the uniqueness for $t \in [0, T]$. For any open subset G of $\mathbf{R}^{2d}$, we define

$$\theta_G : Conf(\mathbf{R}^{2d}) \to Conf(\mathbf{R}^{2d}); \quad \omega \mapsto \theta_G(\omega) = \omega \cap G.$$

Then θ_G is $\mathcal{E}_0/\mathcal{E}_0$-measurable. Let

$$R_0 = \max_{i=1,\cdots,N} (|X_{i,0}| + R_i + 1),$$

and let

$$G_n = \{(x, v) \in \mathbf{R}^{2d}; |x| < R_0 + nT + |v|T\}.$$

Let $C_1 = \sum_{i=1}^N \|U_i\|_\infty$. Then since

$$|\{x; (x, v) \in G_n\}| = 2^d(R_0 + nT + T|v|)^d \le 4^d(R_0 + nT)^d + 4^d T^d |v|^d,$$

we have by (4.3.2) and our Assumption

$$\begin{aligned}
& m^{-\frac{d-1}{2}} \lambda_m(G_n) \\
= & \int_{G_n} \rho\Big(\frac{m}{2}|v|^2 + \sum_{i=1}^N U_i(x - X_{i,0})\Big) dx dv \\
\le & \int_{|x| \le R_0} \rho\Big(\frac{m}{2}|v|^2 + \sum_{i=1}^N U_i(x - X_{i,0})\Big) dx dv + \int_{G_n \cap \{|x| > R_0\}} \rho\Big(\frac{m}{2}|v|^2\Big) dx dv \\
\le & (2R_0)^d C_{d,m} \int_{-\infty}^{\infty} (C_1 + |s|)^{\frac{d}{2}-1} \rho(s) ds \\
& \quad + 4^d (R_0 + nT)^d \int_{\mathbf{R}^d} \rho\Big(\frac{m}{2}|v|^2\Big) dv + 4^d T^d \int_{\mathbf{R}^d} |v|^d \rho\Big(\frac{m}{2}|v|^2\Big) dv \\
\le & (2R_0)^d C_{d,m} \int_{-\infty}^{\infty} (C_1 + |s|)^{\frac{d}{2}-1} \rho(s) ds \\
& \quad + 4^d (R_0 + nT)^d C_{d,m} \int_{-\infty}^{\infty} |s|^{\frac{d}{2}-1} \rho(s) ds + 4^d T^d C_{d,m} \int_{-\infty}^{\infty} |s|^{d-1} \rho(s) ds \\
< & \infty.
\end{aligned}$$

Let $\theta_n = \theta_{G_n}$. Since $\sharp(\theta_{G_n}\omega) = \mu_\omega(G_n)$ by definition, we have

$$E^{P_{\lambda_m}}[\sharp(\theta_{G_n}\omega)] = \lambda_m(G_n) < \infty,$$

hence

$$\theta_{G_n}\omega \in Fin(\mathbf{R}^{2d}), \qquad a.s. - \omega. \tag{4.3.4}$$

Therefore, $(\vec{X}(t, \theta_n\omega), \vec{V}(t, \theta_n\omega))$ is well-defined for $a.s.$-ω.

Next, for any $\omega \in Fin(\mathbf{R}^{2d})$ and $t \in [0, T]$, we define

$$S_t(\omega) = \Big\{(x, v) \in \mathbf{R}^{2d}; \exists i = 1, \cdots, N, s.t., \min_{0\le s\le t} |X_i(s, \omega) - (x + sv)| \le R_i + \frac{1}{2}\Big\}.$$

Claim. For any open set G and $\omega \in Fin(\mathbf{R}^{2d})$, we have

$$\{S_t(\omega) \subset G\} = \{S_t(\theta_G\omega) \subset G\}.$$

Proof of the Claim. Choose and fix any $\omega \in Fin(\mathbf{R}^{2d})$. We first show $\{S_t(\omega) \subset G\} \subset \{S_t(\theta_G\omega) \subset G\}$. Notice that by definition,

$$\begin{aligned} & (x, v) \notin S_t(\omega) \\ \Longrightarrow\ & |X_i(s, \omega) - (x + sv)| \ge R_i + \frac{1}{2}, \quad \forall s \in [0, t], i = 1, \cdots, N \\ \Longrightarrow\ & x(s, v, x; \omega) = x + sv, \quad v(s, v, x; \omega) = v, \quad \forall s \in [0, t]. \end{aligned}$$

So

$$\begin{aligned} & (x, v) \notin S_t(\omega) \\ \Longrightarrow\ & |X_i(s; \omega) - x(s, x, v; \omega)| \ge R_i + \frac{1}{2}, \quad \forall s \in [0, t], i = 1, \cdots, N \\ \Longrightarrow\ & \nabla U_i(X_i(s; \omega) - x(s, x, v; \omega)) = 0, \quad \forall s \in [0, t]. \end{aligned} \tag{4.3.5}$$

Moreover, it is trivial that

$$(x, v) \in G \Longrightarrow \mu_\omega(dx, dv) = \mu_{\theta_G\omega}(dx, dv). \tag{4.3.6}$$

(4.3.5) and (4.3.6) combined with the definition (4.3.1) imply

$$S_t(\omega) \subset G \Longrightarrow (\vec{X}(s, \omega), \vec{V}(s, \omega)) = (\vec{X}(s, \theta_G\omega), \vec{V}(s, \theta_G\omega)), \quad \forall s \in [0, t], \tag{4.3.7}$$

(as long as $\omega \in Fin(\mathbf{R}^{2d})$). Therefore,

$$S_t(\omega) \subset G \Longrightarrow S_t(\theta_G\omega) \subset G.$$

The opposite one can be seen in exactly the same way. This completes the proof of our Claim.

We next deal with general $\omega \in Conf(\mathbf{R}^d)$. Define

$$\mathcal{F}_G = \sigma\{\theta_G\omega\} \vee \aleph.$$

Then by (4.3.4) and the last Claim,

$$\{S_t(\theta_n\omega) \subset G\} = \{S_t(\theta_{G_n\cap G}\omega) \subset G\}, \qquad a.s.,$$

so by the definition of $\mathcal{F}.$, $\{S_t(\theta_n\omega) \subset G\} \in \mathcal{F}_{G_n\cap G} \subset \mathcal{F}_G$, *i.e.*,

$$S_t(\theta_n\omega) \text{ is a } \{\mathcal{F}_G\}\text{-stopping time.}$$

Let

$$\tau_n(\omega) = \inf\{t \geq 0; \max_{i=1,\cdots,N} |V_i(t,\theta_n\omega)| > n\} \wedge T.$$

We first show that the desired solution is well-defined if $\tau_n(\omega) = T$ for some $n \in \mathbf{N}$. Notice that

$$\tau_n(\omega) = T \Longrightarrow S_T(\theta_n\omega) \subset G_n.$$

Actually, if $\tau_n(\omega) = T$, then $|V_i(t,\theta_n\omega)| \leq n$ for any $0 \leq t \leq T$ and $i = 1, \cdots, N$, so by the first equation of (4.3.1), $|X_i(s,\theta_n G)| \leq nT + |X_{i,0}|$ for any $0 \leq s \leq T$ and $i = 1, \cdots, N$. Also, if $(x, v) \notin G_n$, then $|x| > R_0 + nT + |v|T$, so $|x + sv| > R_0 + nT$ for any $0 \leq s \leq T$. Therefore, $|X_i(s,\theta_n G) - (x + sv)| > R_i$, *i.e.*, $(x, v) \notin S_T(\theta_n G)$. This gives us that $S_T(\theta_n\omega) \subset G_n$ under the assumption $\tau_n(\omega) = T$.

Also, we have in the same way as in the proof of the Claim that

$$\{S_t(\theta_k\omega) \subset G_n\} = \{S_t(\theta_n\omega) \subset G_n\}, \qquad \text{for any } k > n.$$

Therefore, if $\tau_n(\omega) = T$, then we have by (4.3.7)

$$(\vec{X}(t,\theta_k\omega), \vec{V}(t,\theta_k\omega)) = (\vec{X}(t,\theta_n\omega), \vec{V}(t,\theta_n\omega)), \quad \forall t \in [0,T],$$

so we can define

$$\begin{aligned}
&(\vec{X}(t,\omega), \vec{V}(t,\omega)) = (\vec{X}(t,\theta_n\omega), \vec{V}(t,\theta_n\omega)),\\
&(x(t,x,v,\omega), v(t,x,v,\omega)) = (x(t,x,v,\theta_n\omega), v(t,x,v,\theta_n\omega)),
\end{aligned}$$

which exists for *a.s.*-ω by (4.3.4). Then $(\vec{X}(t,\omega), \vec{V}(t,\omega), x(t,x,v,\omega), v(t,x,v,\omega))$ satisfies (4.3.1).

Notice that $\tau_n(\omega) = T \Longrightarrow \tau_{n+1}(\omega) = T$. Therefore, to complete the proof of our theorem, it suffices to show that

$$P\Big(\bigcup_{n=1}^{\infty}\{\tau_n = T\}\Big) = 1.$$

We show it from now on.

For any $\theta_n\omega \in Fin(\mathbf{R}^{2d})$, we have by the invariance of energy

$$\begin{aligned}
&\sum_{i=1}^{N}\frac{1}{2}M_i|V_i(t,\theta_n\omega)|^2 + \frac{m}{2}\int_{\mathbf{R}^{2d}}|v(t,x,v,\theta_n\omega)|^2\mu_{\theta_n\omega}(dx,dv)\\
&\qquad + \sum_{i=1}^{N}\int_{\mathbf{R}^{2d}}U_i(X_i(t,\theta_n\omega) - x(t,x,v,\theta_n\omega))\mu_{\theta_n\omega}(dx,dv)\\
= \ &\sum_{i=1}^{N}\frac{1}{2}M_i|V_{i,0}|^2 + \frac{m}{2}\int_{\mathbf{R}^{2d}}|v|^2\mu_{\theta_n\omega}(dx,dv)\\
&\qquad + \sum_{i=1}^{N}\int_{\mathbf{R}^{2d}}U_i(X_{i,0} - x)\mu_{\theta_n\omega}(dx,dv).
\end{aligned}$$

If $(x,v) \notin S_t(\theta_n\omega)$, then $|X_i(s,\theta_n\omega)-(x+sv)| > R_i + \frac{1}{2}$ for any $s \in [0,t]$ and $i = 1,\cdots,N$, so by (4.3.1), $v(t,x,v,\theta_n\omega) = v$ and $U_i(X_i(t,\theta_n\omega)-x(t,x,v,\theta_n\omega)) = 0$. Therefore, the equation above implies

$$\begin{aligned}
& \sum_{i=1}^N \frac{1}{2} M_i |V_i(t,\theta_n\omega)|^2 + \frac{m}{2}\int_{S_t(\theta_n\omega)} |v(t,x,v,\theta_n\omega)|^2 \mu_{\theta_n\omega}(dx,dv) \\
& \quad + \sum_{i=1}^N \int_{S_t(\theta_n\omega)} U_i(X_i(t,\theta_n\omega)-x(t,x,v,\theta_n\omega))\mu_{\theta_n\omega}(dx,dv) \\
= & \sum_{i=1}^N \frac{1}{2} M_i |V_{i,0}|^2 + \frac{m}{2}\int_{S_t(\theta_n\omega)} |v|^2 \mu_{\theta_n\omega}(dx,dv) \\
& \quad + \sum_{i=1}^N \int_{S_t(\theta_n\omega)} U_i(X_{i,0}-x)\mu_{\theta_n\omega}(dx,dv).
\end{aligned}$$

So there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned}
& \sum_{i=1}^N \frac{1}{2} M_i |V_i(t,\theta_n\omega)|^2 \\
\le & \frac{1}{2}\sum_{i=1}^N M_i |V_{i,0}|^2 + 2\sum_{i=1}^N \|U_i\|_\infty \mu_{\theta_n\omega}(S_t(\theta_n\omega)) \\
& \quad + \frac{m}{2}\int_{S_t(\theta_n\omega)} |v|^2 \mu_{\theta_n\omega}(dx,dv) \\
\le & C_0 + C_1 \int_{S_t(\theta_n\omega)} (1+|v|^2)\mu_{\theta_n\omega}(dx,dv) \\
= & C_0 + C_1 \int_{S_t(\theta_n\omega)} 1_{G_n}(x,v)(1+|v|^2)\mu_\omega(dx,dv). \qquad (4.3.8)
\end{aligned}$$

Let $\mathcal{F}_t^{(n)} = \bigcap_{\varepsilon>0} \mathcal{F}_{S_{t+\varepsilon}(\theta_n\omega)}$, $0 \le t < T$. Then $\{\mathcal{F}_t^{(n)}\}_{t\in[0,T)}$ is a filtration, and τ_n is a $\{\mathcal{F}_t^{(n)}\}_{t\in[0,T)}$-stopping time. Let

$$M_t^{(n)} = \int_{S_t(\theta_n\omega)} 1_{G_n}(x,v)(1+|v|^2)(\mu_\omega(dx,dv) - \lambda_m(dx,dv)).$$

Then $\{M_t^{(n)}\}_{t\in[0,T)}$ is a $\{\mathcal{F}_t^{(n)}\}_{t\in[0,T)}$-martingale with mean 0. Actually, we have that $S_t(\theta_n\omega)$ is monotone non-decreasing with respect to t, also, since $|\{x;(x,v)\in G_n\}| = 2^d(R_0+nT+T|v|)^d$ and there exists a constant $C > 0$ (depending on R_0, n,T, d) such that $2^d(R_0+nT+T|v|)^d(1+|v|^2) \le C(1+|v|^{d+2})$, we get by (4.3.2) and our Assumption

$$\begin{aligned}
& m^{\frac{1-d}{2}} \int_{\mathbf{R}^{2d}} 1_{G_n}(x,v)(1+|v|^2)\lambda_m(dx,dv) \\
\le & \int_{|x|\le R_0} (1+|v|^2)\rho\Big(\frac{m}{2}|v|^2 + \sum_{i=1}^N U_i(x-X_{i,0})\Big)dxdv
\end{aligned}$$

$$
\begin{aligned}
& \quad + \int_{G_n \cap \{|x|>R_0\}} (1+|v|^2) \rho\Big(\frac{m}{2}|v|^2\Big) dx dv \\
\leq & \quad \int_{|x|\leq R_0} dx \int_{\mathbf{R}^d} (1+|v|^2)\rho\Big(\frac{m}{2}|v|^2 + \sum_{i=1}^N U_i(x - X_{i,0})\Big) dv \\
& \quad + \int_{\mathbf{R}^d} C(1+|v|^{d+2})\rho\Big(\frac{m}{2}|v|^2\Big) dv \\
\leq & \quad (2R_0)^d C_{d,m} \int_{-\infty}^{\infty} \Big[(C_1+|s|)^{\frac{d}{2}-1} + (C_1+|s|)^{\frac{d}{2}}\Big]\rho(s) ds \\
& \quad + C C_{d,m} \int_{-\infty}^{\infty} \Big[|s|^{\frac{d}{2}-1} + |s|^d\Big]\rho(s) ds \\
< & \quad \infty,
\end{aligned}
$$

i.e.,

$$
\int_{\mathbf{R}^{2d}} 1_{G_n}(x,v)(1+|v|^2)\lambda_m(dx,dv) < \infty.
$$

So Corollary 4.2.5 gives us that $\{M_t^{(n)}\}_{t\in[0,T)}$ is a $\{\mathcal{F}_t^{(n)}\}_{t\in[0,T)}$-martingale with mean 0.

Hence $E[M_{\tau_n}^{(n)}] = 0$. So by (4.3.8),

$$
\begin{aligned}
& \sum_{i=1}^N \frac{1}{2} M_i E[|V_i(\tau_n, \theta_n\omega)|^2] \\
\leq & \quad C_0 + C_1 E\Big[\int_{S_{\tau_n}(\theta_n\omega)} 1_{G_n}(x,v)(1+|v|^2)\lambda_m(dx,dv)\Big].
\end{aligned}
$$

Therefore, with $C_2 := (\min \frac{M_i}{2})^{-1}$, we have

$$
\begin{aligned}
& P[\tau_n < T] = P[\max_{i=1,\cdots,N} |V_i(\tau_n,\theta_n\omega)| \geq n] \qquad (4.3.9) \\
\leq & \quad \frac{C_2}{n^2} E\Big[\sum_{i=1}^N \frac{1}{2} M_i |V_i(\tau_n,\theta_n\omega)|^2\Big] \\
\leq & \quad \frac{1}{n^2} C_2 C_0 + \frac{1}{n^2} C_2 C_1 E\Big[\int_{S_{\tau_n}(\theta_n\omega)} 1_{G_n}(x,v)(1+|v|^2)\lambda_m(dx,dv)\Big]. \qquad (4.3.10)
\end{aligned}
$$

Notice that by definition, $\lambda_m(dx,dv) = \rho(\frac{m}{2}|v|^2)dxdv$ if $|x| > R_0$. Also, there exist constants $C_0', C_1' > 0$ such that

$$
\begin{aligned}
& \quad |\{x \in \mathbf{R}^d; (x,v) \in S_t(\theta_n\omega)\}| \\
= & \quad |\{x \in \mathbf{R}^d; \exists i = 1,\cdots,N, s.t., \min_{0\leq s\leq t} |x + sv - X_i(s,\theta_n\omega)| \leq R_i + \frac{1}{2}\}| \\
= & \quad |\{x \in \mathbf{R}^d; \exists i = 1,\cdots,N, s.t., \min_{0\leq s\leq t} |x + \int_0^s (v - V_i(r,\theta_n\omega))dr| \leq R_i + \frac{1}{2}\}| \\
\leq & \quad C_0' + C_1'(|v| + N \max_{0\leq s\leq t} |V_i(s,\theta_n\omega)|).
\end{aligned}
$$

Moreover, $|V_i(t,\theta_n\omega)| \le n$ if $t \in [0,\tau_n]$. Therefore, by Assumption and (4.3.3), there exist constants $C_0'', C_1'' > 0$ such that

$$\begin{aligned}
&\int_{S_{\tau_n}(\theta_n\omega)} 1_{G_n}(x,v)(1+|v|^2)\lambda_m(dx,dv) \\
\le\ & \int_{|x|\le R_0} (1+|v|^2)\lambda_m(dx,dv) \\
&\quad + m^{\frac{d-1}{2}} \int_{\mathbf{R}^d} \rho(\frac{m}{2}|v|^2)dv(1+|v|^2)|\{x\in\mathbf{R}^d; (x,v)\in S_{\tau_n}(\theta_n\omega)\}| \\
\le\ & m^{\frac{d-1}{2}} \int_{|x|\le R_0} dx \int_{\mathbf{R}^d} (1+|v|^2)\rho\Big(\frac{m}{2}|v|^2+\sum_{i=1}^N U_i(x-X_{i,0})\Big)dv \\
&\quad + m^{\frac{d-1}{2}} \int_{\mathbf{R}^d} (C_0'+C_1'(|v|+Nn))(1+|v|^2)\rho(\frac{m}{2}|v|^2)dv \\
\le\ & (2R_0)^d m^{\frac{d-1}{2}} C_{d,m} \int_{-\infty}^{\infty} \Big[(C_1+|s|)^{\frac{d}{2}-1} + (C_1+|s|)^{\frac{d}{2}}\Big]\rho(s)ds \\
&\quad + m^{\frac{d-1}{2}} (C_0'+C_1'nN) C_{d,m} \int_{-\infty}^{\infty} \Big[|s|^{\frac{d}{2}-1}+|s|^{\frac{d}{2}}\Big]\rho(s)ds \\
&\quad + m^{\frac{d-1}{2}} C_1' C_{d,m} \int_{-\infty}^{\infty} \Big[|s|^{\frac{d-1}{2}}+|s|^{\frac{d+1}{2}}\Big]\rho(s)ds \\
\le\ & C_0''+C_1''n.
\end{aligned}$$

This combined with (4.3.10) implies

$$P(\tau_n < T) \to 0, \qquad \text{as } n\to\infty,$$

which completes the proof. ∎

Chapter 5

Preparations for Limit Theorems

As announced in Chapter 1, from now on, we consider the problem of convergence under the following setting.

Let $d \geq 1$, $N \geq 1$, $M_i > 0$ and $X_{i,0}, V_{i,0} \in \mathbf{R}^d$ for $i = 1, \cdots, N$. Let $U_i \in C_0^\infty(\mathbf{R}^d)$ satisfying $U_i(-x) = U_i(x)$, and there exist constants $R_i > 0$ such that $U_i(x) = 0$ for $|x| \geq R_i$.

We define constants

$$C_0 = \left(2 \sum_{i=1}^N R_i \|\nabla U_i\|_\infty \right)^{1/2},$$

$$e_0 = \frac{1}{2}(2C_0 + 1)^2 + \sum_{i=1}^N \|U_i\|_\infty,$$

and let $\rho : \mathbf{R} \to [0, \infty)$ be a measurable function satisfying the following.

1. $\rho(s) = 0$ if $s \leq e_0$,
2. for any $c > 0$, there exists a $\widetilde{\rho_c} : \mathbf{R} \to [0, \infty)$ such that

$$\sup_{|a| \leq c} \rho(s + a) \leq \widetilde{\rho_c}(s), \qquad \text{for any } s \in \mathbf{R},$$

and

$$\int_{\mathbf{R}^d} (1 + |v|^3) \widetilde{\rho_c}(\frac{1}{2}|v|^2) dv < \infty.$$

Again, we consider the ODE (4.3.1), with $P_m(d\omega)$ the Poisson point process on $Conf(\mathbf{R}^{2d})$ with intensity $m^{\frac{d-1}{2}} \rho(\frac{m}{2}|v|^2 + \sum_{i=1}^N U_i(x - X_{i,0}))dxdv$.

We assume the following:

A1. $|X_{i,0} - X_{j,0}| > R_i + R_j$ for any $i \neq j$.

Under these assumptions, by using the same method as in Theorem 4.3.1, we first get the following existence.

THEOREM 5.0.2 *Under our assumptions, (4.3.1) has a unique solution for P_m-a.s. ω.*

Proof. The proof is just a combination of ray representation and the method in the proof of Theorem 4.3.1, so we give only a sketch. We use the same notations as in the proof of Theorem 4.3.1.

First use the ray representation as in Section 3.3 (see Section 5.2 below for details). (With a little abuse of notations, we write the corresponding $\widetilde{\omega}$ as ω). Write the intensity measure corresponding to $m^{\frac{d-1}{2}}\rho(\frac{m}{2}|v|^2+\sum_{i=1}^N U_i(x-X_{i,0}))dxdv$ as λ_m, *i.e.*, $\lambda_m(ds,dx,dv)=m^{-1}\rho\Big(\frac{1}{2}|v|^2+\sum_{i=1}^N U_i(x-m^{-1/2}sv-X_{i,0})\Big)ds\nu(dx,dv)$. Let

$$G_n=\{(t,x,v)\in \mathbf{R}\times E; |x|<R_0, |t|<T+C_0^{-1}R_0\},$$

(the G_n defined here actually does not depend on n, never the less, we use the subscript n to keep the notation same as in the proof of Theorem 4.3.1), and let $c=\sum_{i=1}^N\|U_i\|_\infty$. Then by the calculation in Section 5.2,

$$\begin{aligned}
& \lambda_m(G_n)\\
= & \int_{\mathbf{R}\times E} 1_{\{|x|<R_0,|t|<T+C_0^{-1}R_0\}}m^{-1}\rho\Big(\frac{1}{2}|v|^2+\sum_{i=1}^N(x-m^{-1/2}tv-X_{i,0})\Big)dt\nu(dx,dv)\\
\leq & (2R_0)^{d-1}2(T+C_0^{-1}R_0)m^{-1}\int_{\mathbf{R}^d}|v|\widetilde{\rho_c}(\frac{1}{2}|v|^2)dv,
\end{aligned}$$

which is finite by our assumption. Let $\theta_n=\theta_{G_n}$, then as in the proof of Theorem 4.3.1, since $E[\sharp(\theta_{G_n}\omega)]=\lambda_m(G_n)$, the above implies that $\theta_n\omega\in Fin(\mathbf{R}\times E)$ *a.s.*. Let

$$S_t(\omega)=\Big\{(u,x,v)\in\mathbf{R}\times E; \exists i=1,\cdots,N, s.t., \min_{0\leq s\leq t}|X_i(s,\omega)-(x-uv+sv)|\leq R_i+\frac{1}{2}\Big\}.$$

Then we have the following.

Claim. For any open set G and $\omega\in Fin(\mathbf{R}\times E)$, we have that $\{S_t(\omega)\subset G\}=\{S_t(\theta_G\omega)\subset G\}$.

Proof of the Claim. First, we have by definition

$$\begin{aligned}
& (u,x,v)\notin S_t(\omega)\\
\Longrightarrow\ & |X_i(s,\omega)-(x-uv+sv)|\geq R_i+\frac{1}{2}, \quad \forall s\in[0,t], i=1,\cdots,N\\
\Longrightarrow\ & x(s,v,x-uv;\omega)=x-uv+sv, \quad v(s,v,x-uv;\omega)=v, \quad \forall s\in[0,t].
\end{aligned}$$

So

$$\begin{aligned}
& (u,x,v)\notin S_t(\omega)\\
\Longrightarrow\ & |X_i(s;\omega)-x(s,x-uv,v;\omega)|\geq R_i+\frac{1}{2}, \quad \forall s\in[0,t], i=1,\cdots,N\\
\Longrightarrow\ & \nabla U_i(X_i(s;\omega)-x(s,x-uv,v;\omega))=0, \quad \forall s\in[0,t].
\end{aligned} \tag{5.0.1}$$

Moreover, it is trivial that

$$(u,x,v)\in G\Longrightarrow \mu_\omega(du,dx,dv)=\mu_{\theta_G\omega}(du,dx,dv). \tag{5.0.2}$$

(5.0.1) and (5.0.2) combined with the definition imply

$$S_t(\omega) \subset G \Longrightarrow (\vec{X}(s,\omega), \vec{V}(s,\omega)) = (\vec{X}(s,\theta_G\omega), \vec{V}(s,\theta_G\omega)), \quad \forall s \in [0,t], \quad (5.0.3)$$

(as long as $\omega \in Fin(\mathbf{R}\times E)$). Therefore,

$$S_t(\omega) \subset G \Longrightarrow S_t(\theta_G\omega) \subset G.$$

The opposite one can be seen in exactly the same way. This completes the proof of our Claim.

Let $\tau_n(\omega) = \inf\{t \geq 0; \max_{i=1,\cdots,N} |V_i(t,\theta_n\omega)| > n\} \wedge T$. Notice that since $\rho(\frac{m}{2}|v|^2 + \sum_{i=1}^N U_i(x-uv-X_{i,0})) \neq 0$ only if $|v| \geq 2C_0+1$, without loss of generality, we may and do assume that $|v| \geq 2C_0+1$. In order to ensure that the proof of Theorem 4.3.1 is also valid here, we only need to check that the following hold.

1. $S_T(\theta_n\omega) \subset G_n$ if $\tau_n(\omega) = T$,
2. $\displaystyle\int_{\mathbf{R}\times E} 1_{G_n}(t,x,v)(1+|v|^2)\lambda_m(dt,dx,dv) < \infty$,
3. there exist constants $C_0, C_1 > 0$ such that
$$\int_{S_{\tau_n}(\theta_n\omega)} 1_{G_n}(t,x,v)(1+|v|^2)\lambda_m(dt,dx,dv) < C_0 + C_1 n.$$

Actually, 1. ensures that the solution of the considered equation is well-defined *a.s.*'ly until σ_n for any $n \in \mathbf{N}$, 2. is used to show that the defined $\{M_t^{(n)}\}_{t\in[0,T]}$ is a martingale, and 3. is combined with the invariant of energy and Chebyshev's inequality to show that $P_m[\tau_n < T] \to 0$ as $n \to \infty$.

We show $1 \sim 3$ now. For the first one, first notice that if $\tau_n(\omega) = T$, then $|V_i(t,\theta_n\omega)| \leq n$ for any $t \in [0,T]$ and $i = 1,\cdots,N$, hence $|X_i(t,\theta_n\omega)| \leq nT + |X_{i,0}|$ for any $t \in [0,T]$ and $i = 1,\cdots,N$. Assume $(u,x,v) \notin G_n$. Then either $|x| \geq R_0$ or $|u| \geq C_0^{-1}R_0+T$. If $|x| \geq R_0$, then $|x+rv| \geq |x| \geq R_0$ for any $r \in \mathbf{R}$, so $|X_i(s,\theta_n\omega) - (x-uv+sv)| \geq R_i + \frac{1}{2}$ for any $s \in [0,T]$, which implies that $(u,x,v) \notin S_T(\theta_n\omega)$. If $|u| \geq C_0^{-1}R_0 + T$, then for any $s \in [0,T]$, we have $|x-uv+sv| \geq C_0^{-1}R_0|v| \geq R_0$, so in this case, we also have $|X_i(s,\theta_n\omega) - (x-uv+sv)| \geq R_i + \frac{1}{2}$ for any $s \in [0,T]$, which implies that $(u,x,v) \notin S_T(\theta_n\omega)$. In conclusion, we have in either case that $(u,x,v) \notin S_T(\theta_n\omega)$. This completes the proof of our first assertion.

The second one and the third one are easy since

$$\begin{aligned}&\int_{\mathbf{R}\times E} 1_{G_n}(t,x,v)(1+|v|^2)\lambda_m(dt,dx,dv)\\ \leq\ & (2R_0)^{d-1}2(T+C_0^{-1}R_0)m^{-1}\int_{\mathbf{R}^d}|v|(1+|v|^2)\widetilde{\rho_c}(\frac{1}{2}|v|^2)dv,\end{aligned}$$

which is finite by our assumption, and does not depend on $n \in \mathbf{N}$. By applying the same method as in the proof of Theorem 4.3.1, these complete the proof of our Theorem. ∎

By Theorem 5.0.2, the solution of (4.3.1) is well-defined for P_m-*a.s.* ω. Write it as $\Big(\vec{X}(t,\omega),\vec{V}(t,\omega)\Big) = \Big((X_1(t,\omega),\cdots,X_N(t,\omega)),(V_1(t,\omega),\cdots,V_N(t,\omega))\Big)$.

From now on, we prove the convergence results announced in Chaper 1.

5.1 Basic facts

First, in this section, we recall some basic facts without proof about the space $D = D^d$ given below and the tightness of the probability measures on it, which will be used later. (See Billingsley [1] for more details).

5.1.1 Space D

Let

$$\begin{aligned} D &= D^d = D^d[0,1] \\ &= \Big\{w : [0,1] \to \mathbf{R}^d; \quad w(t) = w(t+) := \lim_{s\downarrow t} w(s), t \in [0,1), \\ &\qquad \text{and } w(t-) := \lim_{s\uparrow t} w(s) \text{ exists}, t \in (0,1]\Big\}, \end{aligned}$$

and let

$$\Lambda = \Big\{\lambda : [0,1] \to [0,1]; \text{ continuous, non-decreasing}, \lambda(0) = 0, \lambda(1) = 1\Big\}.$$

For any $\lambda \in \Lambda$, we define

$$\|\lambda\|^0 = \sup_{0\le s<t\le 1} \Big|\log \frac{\lambda(t)-\lambda(s)}{t-s}\Big|.$$

Also, for any $w, \widetilde{w} \in D^d$, we define

$$d^0(w,\widetilde{w}) = \inf_{\lambda\in\Lambda} \Big\{\|\lambda\|^0 \vee \|w - \widetilde{w}\circ\lambda\|_\infty\Big\},$$

where $\|w\|_\infty = \sup_{0\le t\le 1} |w(t)|$.

THEOREM 5.1.1 *(D^d, d^0) is a complete metric space.*

We call the topology derived by this metric as Skorohod topology.

Let

$$C = \{w : [0,1] \to \mathbf{R}^d; \text{ continuous}\}.$$

PROPOSITION 5.1.2 (1) *C is closed in D^d.*

(2) *If $w_n \to w_\infty$ in D^d, $\widetilde{w_n} \to \widetilde{w_\infty}$ in D^d, and $w_n \in C$ for any $n = 1, 2, \cdots$, then $w_n + \widetilde{w_n} \to w_\infty + \widetilde{w_\infty}$.*

Remark 1 *Notice that (D^d, d^0) is NOT a topological vector space.*

For any $w \in D^d$, let

$$\begin{aligned}\Delta(w;\delta) &= \sup\{|w(t)-w(t_1)| \wedge |w(t_2)-w(t)|;\\ &\qquad 0 \le t_1 < t < t_2 \le 1, t_2 - t_1 \le \delta\}.\end{aligned}$$

Then we have the following.

THEOREM 5.1.3 *Let $A \subset D^d$. Then the following are equivalent to each other.*

(1) *A is relative compact,*

(2) *The following 3 conditions are satisfied:*

(i) $\sup_{w\in A} \|w\|_\infty < \infty$,

(ii) $\lim_{\delta\to 0} \sup_{w\in A} \Delta(w;\delta) = 0$,

(iii) $\lim_{\delta\to 0} \sup_{w\in A} \Big\{|w(0)-w(\delta)| + |w(1-\delta)-w(1-)|\Big\} = 0$.

5.1.2 Tightness

Let M be a Polish space, and let $\mathcal{P}(M)$ denote the set of probability measures on $(M, \mathcal{B}(M))$.

DEFINITION 5.1.4 *For $\mu_n, \mu_\infty \in \mathcal{P}(M)$, $n = 1, 2, \cdots$, we say that $\mu_n \to \mu_\infty$ weakly in $\mathcal{P}(M)$ if*

$$\int_M f d\mu_n \to \int_M f d\mu_\infty$$

for any $f : M \to \mathbf{R}$ that is bounded and continuous.

This gives the Prohorov metric on $\mathcal{P}(M)$.

DEFINITION 5.1.5 *For any $A \subset \mathcal{P}(M)$, we say that A is tight if for any $\varepsilon > 0$, there exists a compact set K in M such that*

$$\mu(K) > 1-\varepsilon, \qquad \text{for any } \mu \in A.$$

THEOREM 5.1.6 *Let $A \subset \mathcal{P}(M)$. Then*

$$A \text{ is relative compact} \iff A \text{ is tight.}$$

Now, let $(\Omega_n, \mathcal{F}_n, P_n)$, $n = 1, 2, \cdots$, be probability spaces, and let $X_n : \Omega_n \to D^d$, $n \in \mathbf{N}$, be measurable. Let $\mu_{X_n} = P_n \circ X_n^{-1}$. Then we have the following.

THEOREM 5.1.7 *Suppose that there exist constants $\varepsilon, \beta, \gamma, C > 0$ such that*

(1) $E^{P_n}[\|X_n(\,\cdot\,)\|_\infty^\varepsilon] \le C$,

(2) $E^{P_n}\Big[|X_n(r)-X_n(s)|^\beta |X_n(s)-X_n(t)|^\beta\Big] \le C|t-r|^{1+\varepsilon}$ *for any* $0 \le r \le s \le t \le 1$,

(3) $E^{P_n}\Big[|X_n(s)-X_n(t)|^\varepsilon\Big] \le C|t-s|^\gamma$ *for any* $0 \le s \le t \le 1$,

for any $n \in \mathbf{N}$. Then $\{\mu_{X_n}\}_{n=1}^\infty$ is tight in $\mathcal{P}(D^d)$.

5.2 Ray representation

As before, let $d \geq 1$, $N \geq 1$, $M_i > 0$ and $X_{i,0}, V_{i,0} \in \mathbf{R}^d$ for $i = 1, \cdots, N$. Let $U_i \in C_0^\infty(\mathbf{R}^d)$ satisfying $U_i(-x) = U_i(x)$, and let $R_i > 0$ be constants such that $U_i(x) = 0$ for $|x| \geq R_i$, $i = 1, \cdots, N$.

We consider the following ODE. Notice that we use $\widetilde{\omega}$ instead of ω here, because we will use ray representation to convert the problem into the one about $\mathbf{R} \times E$ in the second half of this section, and will consider the problem on the new space after that.

$$\begin{cases} \dfrac{d}{dt}X_i(t;\widetilde{\omega}) = V_i(t;\widetilde{\omega}) \\ M_i\dfrac{d}{dt}V_i(t;\widetilde{\omega}) = -\displaystyle\int \nabla U_i(X_i(t;\widetilde{\omega}) - x(t,x,v;\widetilde{\omega}))\mu_{\widetilde{\omega}}(dx,dv) \\ (X_i(0;\widetilde{\omega}), V_i(0;\widetilde{\omega})) = (X_{i,0}, V_{i,0}) \\ \dfrac{d}{dt}x(t,x,v;\widetilde{\omega}) = v(t,x,v;\widetilde{\omega}) \\ m\dfrac{d}{dt}v(t,x,v;\widetilde{\omega}) = -\displaystyle\sum_{i=1}^N \nabla U_i(x(t,x,v;\widetilde{\omega}) - X_i(t;\widetilde{\omega})) \\ (x(0,x,v;\widetilde{\omega}), v(0,x,v;\widetilde{\omega})) = (x,v) \end{cases}$$

As before, let $\widetilde{\Omega} = Conf(\mathbf{R}^{2d})$, let

$$\widetilde{\lambda}(dx,dv) = \widetilde{\lambda}_m(dx,dv) = m^{\frac{d-1}{2}}\rho\Big(\frac{m}{2}|v|^2 + \sum_{i=1}^N U_i(x - X_{i,0})\Big)dxdv,$$

and let $P_{\widetilde{\lambda}_m}$ be the Poisson point process with intensity $\widetilde{\lambda}_m$.

We consider the following "change of co-ordinates":

$$\Psi_m : \mathbf{R} \times E \to \mathbf{R}^d \times (\mathbf{R}^d \setminus \{0\}), \quad (s,x,v) \mapsto \Psi_m(s,x,v) = \Psi(s,x,m^{-\frac{1}{2}}v).$$

Let

$$\begin{aligned} f_m(x,v) &= f(x, m^{-\frac{1}{2}}v), \\ \rho_0(x,v) &= \rho\Big(\frac{1}{2}|v|^2 + \sum_{i=1}^N U_i(x - X_{i,0})\Big). \end{aligned}$$

Then we have

$$\begin{aligned} & \int_{\mathbf{R}^{2d}} f(x,v)\widetilde{\lambda}(dx,dv) \\ = \;& m^{\frac{d-1}{2}} \int_{\mathbf{R}^{2d}} f(x,v)\rho\Big(\frac{m}{2}|v|^2 + \sum_{i=1}^N U_i(x - X_{i,0})\Big)dxdv \\ = \;& m^{-\frac{1}{2}} \int_{\mathbf{R}^{2d}} f(x, m^{-\frac{1}{2}}v)\rho\Big(\frac{1}{2}|v|^2 + \sum_{i=1}^N U_i(x - X_{i,0})\Big)dxdv \end{aligned}$$

$$
\begin{aligned}
&= m^{-\frac{1}{2}} \int_{\mathbf{R}\times E} f_m(\Psi(s,x,v))\rho_0(\Psi(s,x,v))ds\nu(dx,dv) \\
&= m^{-1} \int_{\mathbf{R}\times E} f_m(\Psi(m^{-\frac{1}{2}}s,x,v))\rho_0(\Psi(m^{-\frac{1}{2}}s,x,v))ds\nu(dx,dv),
\end{aligned}
$$

where we used Theorem 3.3.1 when passing to the forth line. But

$$f_m(\Psi(m^{-\frac{1}{2}}s,x,v)) = f(x - m^{-\frac{1}{2}}sv, m^{-\frac{1}{2}}v) = f(\Psi_m(s,x,v)).$$

Therefore,

$$\int_{\mathbf{R}^{2d}} f(x,v)\widetilde{\lambda}(dx,dv) = \int_{\mathbf{R}\times E} f(\Psi_m(s,x,v))\lambda(ds,dx,dv),$$

where

$$
\begin{aligned}
& \lambda(ds,dx,dv) = \lambda_m(ds,dx,dv) \\
&= m^{-1}\rho_0(\Psi(m^{-\frac{1}{2}}s,x,v))ds\nu(dx,dv) \\
&= m^{-1}\rho\Big(\frac{1}{2}|v|^2 + \sum_{i=1}^{N} U_i(x - m^{-1/2}sv - X_{i,0})\Big)dx\nu(dx,dv).
\end{aligned}
$$

Let $\Omega = Conf(\mathbf{R}\times E)$. Then λ is a measure on Ω. Notice that when restrict Ψ_m on Ω, $\Psi_m : \Omega = Conf(\mathbf{R}\times E) \to Conf(\mathbf{R}^d \times (\mathbf{R}^d \setminus \{0\}))$. Let $P_m(d\omega) = P_{\lambda_m}(d\omega)$ be the Poisson point process on $Conf(\mathbf{R}\times E)$ with intensity function $\lambda_m(ds,dx,dv)$. Then since $\lambda_m(B) = \widetilde{\lambda}_m(\Psi_m(B))$ for any $B \in \mathcal{B}(\mathbf{R}\times E)$, we have that

$$P_{\lambda_m}(A) = P_{\widetilde{\lambda}_m}(\Psi_m(A)), \qquad \text{for all } A \in \mathcal{E}_0.$$

Therefore, we can convert our problem with respect to $\widetilde{\Omega}$ to the problem on Ω.

Our new equation is the following.

$$
\begin{cases}
\dfrac{d}{dt}X_i(t;\omega) = V_i(t;\omega) \\
M_i\dfrac{d}{dt}V_i(t;\omega) = -\displaystyle\int_{\mathbf{R}\times E} \nabla U_i(X_i(t;\omega) - x(t,\Psi(s,x,m^{-\frac{1}{2}}v)))\mu_\omega(ds,dx,dv) \\
(X_i(0;\omega),V_i(0;\omega)) = (X_{i,0},V_{i,0}) \\
\dfrac{d}{dt}x(t,x,v;\omega) = v(t,x,v;\omega) \\
m\dfrac{d}{dt}v(t,x,v;\omega) = -\displaystyle\sum_{i=1}^{N} \nabla U_i(x(t,x,v;\omega) - X_i(t;\omega)) \\
(x(0,x,v;\omega),v(0,x,v;\omega)) = (x,v)
\end{cases}
\tag{5.2.1}
$$

5.3 Basic lemmas

Choose any $T > 0$ and $n \geq 1$ and fix for a while. Let

$$\begin{aligned}
&\sigma(\omega) = \sigma_n(\omega) = \inf \Big\{ t \geq 0; \max_{i=1,\cdots,N} |V_i(t,\omega)| \geq n \Big\}, \\
&R_0 = R_0(n,T) = \max_{i=1,\cdots,N} (R_i + |X_{i,0}| + nT) + 1, \\
&C_0 = \Big\{ 2 \sum_{i=1}^{N} R_i \|\nabla U_i\|_\infty \Big\}^{1/2}, \\
&\tau = \tau(n,T) = C_0^{-1} R_0, \\
&\mathcal{F}_t = \mathcal{F}_t^{(T,n)} = \mathcal{F}_{(-\infty, t+2m^{1/2}\tau)\times E} \vee \aleph, \\
&\quad = \sigma\{\omega \cap (-\infty, t + 2m^{1/2}\tau) \times E\} \vee \aleph.
\end{aligned}$$

Notice that the definition of R_0 is different from before. We do so for the sake of simplicity, since we will use $\max_{i=1,\cdots,N} (R_i + |X_{i,0}| + nT) + 1$ as a whole thing only from now on. Then we will show that $(X_i(t \wedge \sigma), V_i(t \wedge \sigma)), i = 1, \cdots, N$, are $\mathcal{F}_t$-measurable.

Also, we define a new potential in the following way. Let

$$\widetilde{\rho}(t) = - \int_t^\infty \rho(s) ds, \qquad t \in \mathbf{R},$$

$$f(s) = \int_{\mathbf{R}^d} \widetilde{\rho}\Big(\frac{1}{2}|v|^2 + s\Big) dv,$$

and let

$$\widetilde{U}(\vec{X}) = \int_{\mathbf{R}^d} \Big(f\Big(\sum_{i=1}^{N} U_i(X_i - x) \Big) - f(0) \Big) dx.$$

Then we can show that the value of $\widetilde{U}$ for any $\vec{X}$ satisfying $|X_i - X_j| > R_i + R_j$ ($i \neq j$) is a constant (See (5.4.9) below). Write this constant as $\widetilde{U}_0$.

We are going to proof the limit theorems by showing the following lemmas.

Lemma 5.3.1 *For any $i = 1, \cdots, N$, there exist a $\mathbf{R}^d$-valued $(\mathcal{F}_t)_t$-adapted process $P_i^{*0}(t)$, a $\mathbf{R}^d$-valued $(\mathcal{F}_t)_t$-adapted C^1-class (in t) process $P_i^{*1}(t)$, a $\mathbf{R}^d$-valued $(\mathcal{F}_t)_t$-martingale $M_i(t)$ and a $\mathbf{R}^d$-valued $(\mathcal{F}_t)_t$-adapted process $\eta_i(t)$ such that*

(1)

$$M_i(V_i(t \wedge \sigma) - V_i(0)) = P_i^{*0}(t) + P_i^{*1}(t) - m^{-1/2} \int_0^{t\wedge\sigma} \nabla_i \widetilde{U}(X(s)) ds,$$

and $P_i^{*0}(t) = M_i(t) + \eta_i(t)$,

(2)

$$\sup_{m \in (0,1]} \sup_{t \in [0,T]} E^{P_m} \Big[|\frac{d}{dt} P_i^{*1}(t)|^2 \Big] < \infty,$$

(3) *there exists a constant C indenpendent of m such that*

$$\Big|d\langle M_i^k, M_j^\ell\rangle_t\Big| \le Cdt, \qquad P_m\text{-a.s.}$$

for any $k, \ell = 1, \cdots, d$ and $m \in (0,1]$, with $|\Delta M_i(t)| \le Cm^{1/2}$,

(4)

$$E^{P_m}[\sup_{t\in[0,T]} |\eta_i(t)|] \to 0, \qquad \text{as } m \to 0$$

for any $i = 1, \cdots, N$.

In particular, $P_i^{*0}(t)$ and $P_i^{*1}(t)$ are tight in $D([0,T]; \mathbf{R}^d)$, and the limits are continuous processes.

Lemma 5.3.2 *Let D be any open subset of $\mathbf{R}^{dN}$, and assume that for any $i = 1, \cdots, N$, there exists a C_b^1-class function $g_i : \overline{D} \to \mathbf{R}^d$ satisfying*

$$g_i(\vec{X}) \cdot \nabla_i \widetilde{U}(\vec{X}) = |\nabla_i \widetilde{U}(\vec{X})|, \qquad \text{for any } \vec{X} \in \overline{D}, i = 1, \cdots, N.$$

Let

$$\widetilde{\sigma_D} = \inf\{t \ge 0; \vec{X}(t) \in D^C\}.$$

Then

(1)

$$\sup_{m\in(0,1]} E^{P_m}\Big[\int_0^{T\wedge\sigma\wedge\widetilde{\sigma_D}} m^{-1/2} |\nabla_i \widetilde{U}(\vec{X}(t))| dt\Big] < \infty$$

for any $i = 1, \cdots, N$,

(2) $m^{-1/2}\Big(\widetilde{U}(\vec{X}(t\wedge\sigma\wedge\widetilde{\sigma_D})) - \widetilde{U}_0\Big) + \sum_{i=1}^N \frac{M_i}{2} |V_i(t\wedge\sigma\wedge\widetilde{\sigma_D})|^2$ *is tight in $C([0,T]; \mathbf{R})$.*

Lemma 5.3.3 *If $b_i : \mathbf{R}^{dN} \to \mathbf{R}^d$, $i = 1, \cdots, N$, are C_b^2-class, and*

$$\sum_{i=1}^N b_i(\vec{X}) \cdot \nabla_i \widetilde{U}(\vec{X}) = 0$$

for any $\vec{X} \in \mathbf{R}^{dN}$, then $\sum_{i=1}^N b_i(\vec{X}(t\wedge\sigma)) \cdot V_i(t\wedge\sigma)$ is tight in D,

Lemma 5.3.4 *Let $D_0 = (supp\widetilde{U})^C \subset \mathbf{R}^{dN}$, and assume that $f \in C_0^\infty(D_0 \times \mathbf{R}^{dN})$. Then $f(\vec{X}(t\wedge\sigma), \vec{V}(t\wedge\sigma))$ is tight in $C([0,T]; \mathbf{R})$. Also, the limit process $(\vec{X}_\infty(t), \vec{V}_\infty(t))$ is the solution of the L-martingale problem stopped at σ.*

5.4 Preparations

We prepare some estimates, which will be used later. Fix any $n \geq 1$. Since by definition

$$\sigma(\omega) = \sigma_n(\omega) = \inf\Big\{t \geq 0; \max_{i=1,\cdots,N} |V_i(t,\omega)| \geq n\Big\},$$

it is trivial by definition that

$$|X_i(t,\omega)| \leq |X_{i,0}| + nT, \qquad \text{for any } t \in [0, \sigma(\omega) \wedge T]. \tag{5.4.1}$$

PROPOSITION 5.4.1 *Suppose that* $|v| > (2C_0+1)m^{-1/2}$ *and* $n \leq m^{-1/2}$. *Then*

$$(|v|^{-1}v) \cdot v(t,x,v;\omega) \geq m^{-1/2}(C_0+1), \qquad \textit{for any } t \in [0, \sigma(\omega)].$$

Proof. Let $\eta = |v|^{-1}v$ and let

$$\xi = \inf\{t > 0; v(t,x,v;\omega) \cdot \eta < m^{-1/2}(C_0+1)\}.$$

We only need to show that $\xi \geq \sigma(\omega)$. Suppose not. Notice that by definition,

$$(v(\xi,x,v;\omega) - v) \cdot \eta = -m^{-1} \sum_{i=1}^{N} \int_0^{\xi} (\nabla U_i(x(t,x,v;\omega) - X_i(t;\omega)) \cdot \eta) dt.$$

Also, for any $t \in [0, \xi \wedge \sigma(\omega)]$, we have by assumption

$$\begin{aligned}
& \frac{d}{dt}(x(t,x,v;\omega) - X_i(t;\omega)) \cdot \eta \\
= \;& v(t,x,v;\omega) \cdot \eta - V_i(t;\omega) \cdot \eta \\
\geq \;& m^{-1/2}(C_0+1) - n \geq m^{-1/2}(C_0+1) - m^{-1/2} = m^{-1/2}C_0,
\end{aligned}$$

in particular, $(x(t,x,v;\omega) - X_i(t;\omega)) \cdot \eta$ is monotone increasing with respect to t. So since $v \cdot \eta = |v| > (2C_0+1)m^{-1/2}$ by assumption,

$$\begin{aligned}
m^{-1/2}C_0 &< -(v(\xi,x,v;\omega) - v) \cdot \eta \\
&= m^{-1} \sum_{i=1}^{N} \int_0^{\xi} \Big(\nabla U_i(x(t,x,v;\omega) - X_i(t;\omega)) \cdot \eta\Big) dt \\
&\leq m^{-1} \sum_{i=1}^{N} \int_0^{\xi} |\nabla U_i(x(t,x,v;\omega) - X_i(t;\omega)) \cdot \eta| \\
&\qquad \times (m^{-1/2}C_0)^{-1} d[(x(t,x,v;\omega) - X_i(t;\omega)) \cdot \eta] \\
&\leq m^{-1} \sum_{i=1}^{N} (m^{-1/2}C_0)^{-1} \|\nabla U_i\|_\infty \\
&\qquad \int_{|(x(t,x,v;\omega) - X_i(t;\omega)) \cdot \eta| \leq R_i} d[(x(t,x,v;\omega) - X_i(t;\omega)) \cdot \eta] \\
&\leq m^{-1} \sum_{i=1}^{N} (m^{-1/2}C_0)^{-1} \|\nabla U_i\|_\infty 2R_i \\
&= m^{-1/2}C_0,
\end{aligned}$$

which makes a contradiction. Therefore, $\xi \geq \sigma(\omega)$. ∎

We assume $n < m^{-1/2}$ from now on.

Remember that in Section 3.4, we wrote the solution of

$$\begin{cases} \dfrac{d}{dt}x(t) = v(t), \\ \dfrac{d}{dt}v(t) = -\displaystyle\sum_{i=1}^{N} \nabla U_i(x(t) - X_i), \\ (x(0), v(0)) = (x_0, v_0) \end{cases}$$

as

$$\widetilde{\varphi}(t, x_0, v_0; \vec{X}) = (x(t), v(t)),$$

and defined

$$\psi(t, x, v; \vec{X}) = \lim_{s\to\infty} \widetilde{\varphi}(t+s, \Psi(s, x, v); \vec{X})$$

for any $t \in \mathbf{R}$, $(x, v) \in E$.

Now, in our present setting, since

$$\begin{cases} \dfrac{d}{dt}x(t, \Psi(s, x, m^{-1/2}v)) = v(t, \Psi(s, x, m^{-1/2}v)), \\ m\dfrac{d}{dt}v(t, \Psi(s, x, m^{-1/2}v)) = -\displaystyle\sum_{i=1}^{N} \nabla U_i(x(t, \Psi(s, x, m^{-1/2}v)) - X_i(t, \omega)), \end{cases}$$

we have

$$\begin{aligned} & \frac{d^2}{dt^2}x(m^{1/2}t + s, \Psi(s, x, m^{-1/2}v)) \\ = \ & -\sum_{i=1}^{N} \nabla U_i(x(m^{1/2}t + s, \Psi(s, x, m^{-1/2}v)) - X_i(m^{1/2}t + s, \omega)). \end{aligned}$$

Also, for any $s > 0$ and $t \in [0, T \wedge \sigma(\omega)]$, we have by definition and (5.4.1) that

$$\Big(x(t, \Psi(s, x, m^{-1/2}v)), v(t, \Psi(s, x, m^{-1/2}v))\Big) = \Psi(s - t, x, m^{-1/2}v)$$

if $t < s - (m^{-1/2}C_0)^{-1}R_0$ and $|v| \geq 2C_0 + 1$.

Therefore,

$$\begin{aligned} & \Big(x(m^{1/2}t + s, \Psi(s, x, m^{-1/2}v)), \frac{d}{dt}x(m^{1/2}t + s, \Psi(s, x, m^{-1/2}v))\Big) \\ = \ & (\Psi^0(-m^{1/2}t, x, m^{-1/2}v), m^{1/2}\Psi^1(-m^{1/2}t, x, m^{-1/2}v)) = (x + tv, v) \\ = \ & \Psi(-t, x, v) \end{aligned} \tag{5.4.2}$$

if $t < -C_0^{-1}R_0$, $|v| \geq 2C_0 + 1$, and $0 \leq m^{1/2}t + s \leq T \wedge \sigma(\omega)$.

We recall the following famous Gronwall's Lemma, for later use.

Lemma 5.4.2 (Gronwall's Lemma) *Suppose that the continuous function* $g(t)$ *satisfies*

$$0 \le g(t) \le \alpha(t) + \beta \int_0^t g(s)ds, \qquad 0 \le t \le T,$$

with $\beta \ge 0$ *and* $\alpha : [0,T] \to \mathbf{R}$ *integrable. Then*

$$g(t) \le \alpha(t) + \beta \int_0^t \alpha(s)e^{\beta(t-s)}ds, \qquad 0 \le t \le T.$$

In particular, if $\alpha(t) = \alpha$ *is a constant, then*

$$g(t) \le \alpha e^{\beta t}, \qquad 0 \le t \le T.$$

PROPOSITION 5.4.3 *Fix any* $a \in \mathbf{R}$. *Suppose that* $0 \le s - am^{1/2} \le T \wedge \sigma(\omega)$ *and* $0 \le s - m^{1/2}\tau \le T \wedge \sigma(\omega)$. *Let*

$$y(t) = x(m^{1/2}t + s, \Psi(s, x, m^{-1/2}v)) - \psi^0(t, x, v, \vec{X}(s - am^{1/2}, \omega)).$$

Also, suppose that $|v| > 2C_0 + 1$. *Then*

(1) $y(t) = 0$ *if* $0 \le m^{1/2}t + s \le T \wedge \sigma(\omega)$ *and* $t \le -\tau$,

(2)

$$\begin{aligned} &\frac{d^2}{dt^2}y(t) \\ = \quad & -\sum_{i=1}^N \Big\{\nabla U_i\Big(y(t) + \psi^0(t, x, v; \vec{X}(s - am^{1/2}, \omega)) - X_i(m^{1/2}t + s, \omega)\Big) \\ & \qquad -\nabla U_i\Big(\psi^0(t, x, v; \vec{X}(s - am^{1/2}, \omega)) - X_i(s - am^{1/2}, \omega)\Big)\Big\}. \end{aligned}$$

(3) *there exists a constant* $\widetilde{C}$, *depending only on* n, τ *and* $\sum_{i=1}^N \|\nabla^2 U_i\|_\infty$, *such that*

$$|y(t)| + |\frac{d}{dt}y(t)| \le m^{1/2}\widetilde{C}(2\tau + |a|), \tag{5.4.3}$$

if $0 \le m^{1/2}t + s \le T \wedge \sigma(\omega)$ *and* $|t| \le 2\tau$.

Proof. We first show the first assertion. We have by (5.4.2) that $x(m^{1/2}t + s, \Psi(s, x, m^{-1/2}v)) = x + tv$ under our setting. Also, notice that $|X_i(s - am^{1/2}, \omega)| \le |X_{i,0}| + nT$ under our assumption, and since $t \le -\tau$ and $|v| \ge 2C_0 + 1$, we have for any $\widetilde{s}$ big enought

$$\inf_{u \in [0, t+\widetilde{s}]} |x - \widetilde{s}v + uv| \ge |t||v| \ge C_0^{-1}R_0(2C_0 + 1) \ge R_0,$$

therefore, $\psi^0(t, x, v, \vec{X}(s - am^{1/2}, \omega)) = \lim_{\widetilde{s} \to \infty} \varphi^0(t + \widetilde{s}, x - \widetilde{s}v, v, \vec{X}(s - am^{1/2}, \omega)) = x + tv$. This gives us our first assertion.

The second assertion is trivial by definition.

We show the last one. Notice that for any $|t| \le 2\tau$ satisfying $0 \le m^{1/2}t + s \le T \wedge \sigma(\omega)$, we have

$$\begin{aligned} & \Big|X_i(m^{1/2}t+s,\omega) - X_i(s-am^{1/2},\omega)\Big| \\ \le\ & n|(m^{1/2}t+s)-(s-am^{1/2})| \le nm^{1/2}(2\tau+|a|), \end{aligned}$$

so by (2),

$$\begin{aligned} & \Big|\frac{d^2}{dt^2}y(t)\Big| \\ \le\ & \sum_{i=1}^N \|\nabla^2 U_i\|_\infty \Big|y(t) - \Big[X_i(m^{1/2}t+s,\omega) - X_i(s-am^{1/2},\omega)\Big]\Big| \\ \le\ & \sum_{i=1}^N \|\nabla^2 U_i\|_\infty m^{1/2} n(2\tau+|a|) + \Big(\sum_{i=1}^N \|\nabla^2 U_i\|_\infty\Big)|y(t)|. \end{aligned}$$

Therefore,

$$\begin{aligned} & \Big|\frac{d}{dt}\Big|\Big(y(t),\frac{d}{dt}y(t)\Big)\Big|\Big| \le \Big|\frac{d}{dt}y(t)\Big| + \Big|\frac{d^2}{dt^2}y(t)\Big| \\ \le\ & m^{1/2}\Big(\sum_{i=1}^N \|\nabla^2 U_i\|_\infty n\Big)(2\tau+|a|) + \left(1+\sum_{i=1}^N \|\nabla^2 U_i\|_\infty\right)\Big|(y(t),\frac{d}{dt}y(t))\Big| \end{aligned}$$

if $|t| \le 2\tau$ and $0 \le m^{1/2}t + s \le T \wedge \sigma(\omega)$. Also, by (1), $y(-\tau) = \frac{d}{dt}y(-\tau) = 0$. Let $g(t) = |(y(t-\tau), \frac{d}{dt}y(t-\tau))|$, then we have $g(0) = 0$ and

$$\begin{aligned} & \left|\frac{d}{dt}g(t)\right| \le \Big|\frac{d}{dt}\Big|\Big(y(t-\tau),\frac{d}{dt}y(t-\tau)\Big)\Big|\Big| \\ \le\ & m^{1/2}\Big(\sum_{i=1}^N \|\nabla^2 U_i\|_\infty n\Big)(2\tau+|a|) + \left(1+\sum_{i=1}^N \|\nabla^2 U_i\|_\infty\right) g(t), \end{aligned}$$

if $-\tau \le t \le 3\tau$ and $0 \le m^{1/2}(t-\tau) + s \le T \wedge \sigma(\omega)$. (Notice that $t = 0$ satisfies these conditions since $0 \le s - m^{1/2}\tau \le T \wedge \sigma(\omega)$ under our assumption). Therefore, if $0 \le t \le 3\tau$ and $0 \le m^{1/2}(t-\tau) + s \le T \wedge \sigma(\omega)$, then

$$g(t) \le m^{1/2}\Big(\sum_{i=1}^N \|\nabla^2 U_i\|_\infty n\Big)(2\tau+|a|)3\tau + \Big(1+\sum_{i=1}^N \|\nabla^2 U_i\|_\infty\Big)\int_0^t g(s)ds,$$

so by Gronwall's inequality, we get

$$g(t) \le m^{1/2}\Big(\sum_{i=1}^N \|\nabla^2 U_i\|_\infty n\Big)(2\tau+|a|)3\tau e^{(1+\sum_{i=1}^N \|\nabla^2 U_i\|_\infty)t}.$$

The assertion for $t \in [-\tau, 0]$ satisfying $0 \le m^{1/2}(t-\tau) + s \le T \wedge \sigma(\omega)$ is proved in the same way. This completes the proof. ∎

Now, choose any $x, v \in \mathbf{R}^d$ with $|v| > 2C_0 + 1$, and $\vec{X}, \vec{V} \in (\mathbf{R}^d)^N$ with $|X_i| \le |X_{i,0}| + nT$, $i = 1, \cdots, N$. For any $a \in \mathbf{R}$, let $Z(t) = Z(t; x, v, \vec{X}, \vec{V}, a) \in \mathbf{R}^d$ be the solution of the equation

$$\begin{cases} \dfrac{d^2}{dt^2} Z(t) = -\displaystyle\sum_{i=1}^N \nabla^2 U_i\Big(\psi^0(t, x, v, \vec{X}) - X_i\Big)(Z(t) - (t+a)V_i), \\ Z(-\tau) = \dfrac{d}{dt} Z(-\tau) = 0. \end{cases}$$

It is easy to see that $Z(t; x, v, \vec{X}, \vec{V}, a)$ is linear with respect to $\vec{V}$.

PROPOSITION 5.4.4 *Let $a \in \mathbf{R}$. Suppose that $t \ge -a$, $0 \le s - m^{1/2}\tau \le T \wedge \sigma(\omega)$, $-\tau \le t \le 2\tau$ and $0 \le s - am^{1/2} \le s + m^{1/2}t \le T \wedge \sigma(\omega)$. Also, let $|v| > 2C_0 + 1$. Then*

$$\begin{aligned} &\Big| x(m^{1/2}t + s, \Psi(s, x, m^{-1/2}v)) \\ &\quad - \Big(\psi^0(t, x, v, \vec{X}(s - am^{1/2})) + m^{1/2} Z(t; x, v, \vec{X}(s - am^{1/2}), \vec{V}(s - am^{1/2}), a)\Big)\Big| \\ \le\ & Cm^{1/2}\Big\{(1+|a|)^2 m^{1/2} + m^{-1/2} \int_{s-am^{1/2}}^{s+m^{1/2}t} |\vec{V}(r) - \vec{V}(s - am^{1/2})| dr\Big\}. \end{aligned}$$

Here C is a constant depending only on τ, n, $\sum_{i=1}^N \|\nabla^3 U_i\|_\infty$ and $\sum_{i=1}^N \|\nabla^2 U_i\|_\infty$.

Proof. Let

$$y(t) = x(m^{1/2}t + s, \Psi(s, x, m^{-1/2}v)) - \psi^0(t, x, v, \vec{X}(s - am^{1/2}, \omega))$$

as before, and let

$$\xi(t) = y(t) - m^{1/2} Z(t; x, v, \vec{X}(s - am^{1/2}), \vec{V}(s - am^{1/2}), a).$$

We need to estimate $|\xi(t)|$. By a simply calculation,

$$\begin{aligned} &\frac{d^2}{dt^2} y(t) \\ =\ & -\sum_{i=1}^N \Big\{\nabla U_i(y(t) + \psi^0(t, x, v; \vec{X}(s - am^{1/2})) - X_i(m^{1/2}t + s)) \\ &\qquad - \nabla U_i(\psi^0(t, x, v; \vec{X}(s - am^{1/2}) - X_i(s - am^{1/2}))\Big\} \\ =\ & -\sum_{i=1}^N \int_0^1 \nabla^2 U_i\Big(\eta\Big[y(t) - \Big\{X_i(m^{1/2}t + s) - X_i(s - m^{1/2}a)\Big\}\Big] \\ &\qquad + \psi^0(t, x, v, \vec{X}(s - am^{1/2})) - X_i(s - am^{1/2})\Big) \\ &\qquad \Big[y(t) - \Big\{X_i(m^{1/2}t + s) - X_i(s - m^{1/2}a)\Big\}\Big] d\eta, \end{aligned}$$

and so

$$
\begin{aligned}
&\frac{d^2}{dt^2}\xi(t) \\
= \quad & -\sum_{i=1}^{N}\int_0^1 d\eta\Big\{\nabla^2 U_i\Big(\eta\Big[y(t)-\Big\{X_i(m^{1/2}t+s)-X_i(s-m^{1/2}a)\Big\}\Big] \\
&\qquad +\psi^0(t,x,v;\vec{X}(s-am^{1/2}))-X_i(s-am^{1/2})\Big) \\
&\quad -\nabla^2 U_i\Big(\psi^0(t,x,v;\vec{X}(s-am^{1/2}))-X_i(s-am^{1/2})\Big)\Big\} \\
&\qquad \cdot\Big(y(t)-\Big\{X_i(m^{1/2}t+s)-X_i(s-m^{1/2}a)\Big\}\Big) \\
& -\sum_{i=1}^{N}\nabla^2 U_i\Big(\psi^0(t,x,v,\vec{X}(s-am^{1/2}))-X_i(s-am^{1/2})\Big) \\
&\qquad \Big(\xi(t)-\Big\{X_i(m^{1/2}t+s)-X_i(s-m^{1/2}a)-m^{1/2}(t+a)V_i(s-m^{1/2}a)\Big\}\Big).
\end{aligned}
$$

Therefore, since $|X_i(m^{1/2}t+s)-X_i(s-m^{1/2}a)|\le n(t+|a|)m^{1/2}$ in our domain, and $X_i(m^{1/2}t+s)-X_i(s-m^{1/2}a)=\int_{s-am^{1/2}}^{s+m^{1/2}t}V_i(r)dr$, we get

$$
\Big|\frac{d^2}{dt^2}\xi(t)\Big| \tag{5.4.4}
$$

$$
\le \sum_{i=1}^{N}\|\nabla^3 U_i\|_\infty(|y(t)|+n(t+|a|)m^{1/2})^2+(\sum_{i=1}^{N}\|\nabla^2 U_i\|_\infty)|\xi(t)| + \sum_{i=1}^{N}\|\nabla^2 U_i\|_\infty\int_{s-am^{1/2}}^{s+m^{1/2}t}|V_i(r)-V_i(s-m^{1/2}a)|dr. \tag{5.4.5}
$$

Let $\widetilde{C}$ be the constant in Proposition 5.4.3, and let

$$
\begin{aligned}
C_1 &= \sum_{i=1}^{N}\|\nabla^3 U_i\|_\infty(\widetilde{C}+n)^2(2\tau+1)^2, \\
C_2 &= \sum_{i=1}^{N}\|\nabla^2 U_i\|_\infty.
\end{aligned}
$$

Then (5.4.5) combined with Proposition 5.4.3 gives us

$$
\Big|\frac{d^2}{dt^2}\xi(t)\Big|\le C_1 m(1+|a|)^2+C_2\int_{s-am^{1/2}}^{s+m^{1/2}t}|V_i(r)-V_i(s-m^{1/2}a)|dr+C_2|\xi(t)|,
$$

if $0\le m^{1/2}t+s\le T\wedge\sigma(\omega)$, $|t|\le 2\tau$ and $t\ge -a$. Let

$$
g(t)=\Big|(\xi(t-\tau),\frac{d}{dt}\xi(t-\tau))\Big|.
$$

Then the estimate above gives us

$$\begin{aligned}\Big|\frac{d}{dt}g(t)\Big| &\le \Big|\frac{d}{dt}\xi(t-\tau)\Big| + \Big|\frac{d^2}{dt^2}\xi(t-\tau)\Big| \\ &\le C_1 m(1+|a|)^2 + C_2\int_{s-am^{1/2}}^{s+m^{1/2}(t-\tau)} |V_i(r)-V_i(s-m^{1/2}a)|dr + (C_2+1)g(t),\end{aligned}$$

if $t-\tau \ge -a$, $|t-\tau| \le 2\tau$ and $0 \le m^{1/2}(t-\tau)+s \le T\wedge\sigma(\omega)$. Since $\xi(-\tau) = \frac{d}{dt}\xi(-\tau) = 0$, we have $g(0) = 0$. Also, $\int_{s-am^{1/2}}^{s+m^{1/2}(t-\tau)} |V_i(r)-V_i(s-m^{1/2}a)|dr$ is monotone non-decreasing with respect to t. So if $t-\tau \ge -a$ and $0 \le t \le 3\tau$, then

$$\begin{aligned}g(t) \le 3\tau\Big(C_1 m(1+|a|)^2 + C_2\int_{s-am^{1/2}}^{s+m^{1/2}(t-\tau)} |V_i(r)-V_i(s-m^{1/2}a)|dr\Big) \\ +(C_2+1)\int_0^t g(u)du.\end{aligned}$$

Therefore, by Gronwall's inequality and the monotonicity of $\int_{s-am^{1/2}}^{s+m^{1/2}t} |V_i(r)-V_i(s-m^{1/2}a)|dr$ again, the above implies

$$g(t) \le 3\tau e^{(C_2+1)3\tau}\Big(C_1 m(1+|a|)^2 + C_2\int_{s-am^{1/2}}^{s+m^{1/2}(t-\tau)} |V_i(r)-V_i(s-m^{1/2}a)|dr\Big),$$

if $t-\tau \ge -a$, $-\tau \le t-\tau \le 2\tau$ and $0 \le m^{1/2}(t-\tau)+s \le T\wedge\sigma(\omega)$. This completes our proof. ∎

PROPOSITION 5.4.5 *Let $|v| > 2C_0+1$. Suppose that $0 \le m^{1/2}t+s \le T\wedge\sigma(\omega)$ and that either $t < -\tau$ or $t > 2\tau$. Then*

$$\nabla U_i\Big(x(m^{1/2}t+s,\Psi(s,x,m^{-1/2}v)) - X_i(m^{1/2}t+s,\omega)\Big) = 0.$$

Proof. Let $\eta = |v|^{-1}v$. First notice that $|X_i(m^{1/2}t+s,\omega)| \le |X_{0,i}| + nT$ if $0 \le m^{1/2}t+s \le T\wedge\sigma(\omega)$. So we only need to show that $|x(m^{1/2}t+s,\Psi(s,x,m^{-1/2}v))| \ge R_0$ for t satisfying our condition.

We show it from now on. First notice that by (5.4.2), if $t < -\tau = -C_0^{-1}R_0$, then $|x(m^{1/2}t+s,\Psi(s,x,m^{-1/2}v))| = |x+tv| \ge |t||v| \ge C_0^{-1}R_0(2C_0+1) > R_0$. For $t > 2\tau$, notice that since

$$\frac{d}{dt}x(m^{1/2}t+s,\Psi(s,x,m^{-1/2}v)) = m^{1/2}v(m^{1/2}t+s,\Psi(s,x,m^{-1/2}v)),$$

and $0 \le m^{1/2}t+s \le \sigma(\omega)$ by assumption, we have by Proposition 5.4.1 that

$$\frac{d}{dt}\Big(\eta\cdot x(m^{1/2}t+s,\Psi(s,x,m^{-1/2}v))\Big) > C_0. \tag{5.4.6}$$

In particular, $\eta\cdot x(m^{1/2}t+s,\Psi(s,x,m^{-1/2}v))$ is monotone increasing with respect to t if $0 \le m^{1/2}t+s \le \sigma(\omega)$. So if $\eta\cdot x(s,\Psi(s,x,m^{-1/2}v)) > R_0$, then for any

$t \geq 2\tau > 0$, we have $\eta \cdot x(m^{1/2}t+s, \Psi(s,x,m^{-1/2}v)) \geq \eta \cdot x(s, \Psi(s,x,m^{-1/2}v)) > R_0$. Also, if $\eta \cdot x(s, \Psi(s,x,m^{-1/2}v)) \leq R_0$, then by (5.4.6), we get that for any $t > 2\tau$,

$$\begin{aligned}
& \left|\eta \cdot x(m^{1/2}t + s, \Psi(s,x,m^{-1/2}v))\right| \\
= \ & \Big|\int_0^t \frac{d}{du}\Big(\eta \cdot x(m^{1/2}u + s, \Psi(s,x,m^{-1/2}v))\Big)du + \eta \cdot x(s, \Psi(s,x,m^{-1/2}v))\Big| \\
\geq \ & C_0 t - R_0 \geq C_0 \cdot 2\tau - R_0 \geq 2R_0 - R_0 = R_0,
\end{aligned}$$

hence $|x(m^{1/2}t+s, \Psi(s,x,m^{-1/2}v))| \geq R_0$. This completes the proof of our assertion, hence the lemma. ∎

Before closing this section, let us discuss a little bit more about the new potential $\widetilde{U}$. As in Section 5.3, let

$$\widetilde{\rho}(t) = -\int_t^\infty \rho(s)ds, \qquad t \in \mathbf{R}.$$

Then $\frac{d}{dt}\widetilde{\rho}(t) = \rho(t)$. Also, let

$$\widetilde{U}(\vec{X}) = \widetilde{U}(X_1, \cdots, X_N) = \int_{\mathbf{R}^{2d}} \Big(\widetilde{\rho}\left(\frac{1}{2}|v|^2 + \sum_{i=1}^N U_i(x - X_i)\right) - \widetilde{\rho}(\frac{1}{2}|v|^2)\Big)dxdv.$$

Then it is easy that

$$\nabla_i \widetilde{U}(\vec{X}) = \int_{\mathbf{R}^{2d}} \nabla U_i(X_i - x)\rho\left(\frac{1}{2}|v|^2 + \sum_{i=1}^N U_i(x - X_i)\right)dxdv. \tag{5.4.7}$$

Let

$$f(s) = \int_{\mathbf{R}^d} \widetilde{\rho}(\frac{1}{2}|v|^2 + s)dv.$$

Then by a simple calculation, there exists a global constant C_d such that

$$f(s) = C_d \int_0^\infty \widetilde{\rho}(r + s)r^{\frac{d}{2}-1}dr.$$

Also,

$$\widetilde{U}(\vec{X}) = \int_{\mathbf{R}^d} \Big(f\left(\sum_{i=1}^N U_i(x - X_i)\right) - f(0)\Big)dx. \tag{5.4.8}$$

It is easy that if $|X_i - X_j| > R_i + R_j$ for any $i \neq j$, then

$$\widetilde{U}(\vec{X}) = \sum_{i=1}^N \int_{\mathbf{R}^d} \Big(f(U_i(x)) - f(0)\Big)dx,$$

therefore,

$$\nabla_i \widetilde{U}(\vec{X}) = 0, \qquad \text{if } |X_i - X_j| > R_i + R_j, i \neq j. \tag{5.4.9}$$

Moreover, we have

$$\begin{aligned} f'(s) &= C_d \int_0^\infty \rho(r+s) r^{\frac{d}{2}-1} dr \\ &= C_d \int_s^\infty \rho(t)(t-s)^{\frac{d}{2}-1} dt. \end{aligned}$$

So if $s < e_0$, then

$$f'(s) = C_d \int_0^\infty \rho(t)(t-s)^{\frac{d}{2}-1} dt, \tag{5.4.10}$$

$$f''(s) = C_d(1-\frac{d}{2}) \int_0^\infty \rho(t)(t-s)^{\frac{d}{2}-2} dt, \tag{5.4.11}$$

$$f'''(s) = C_d(1-\frac{d}{2})(2-\frac{d}{2}) \int_0^\infty \rho(t)(t-s)^{\frac{d}{2}-3} dt.$$

Also notice that under the condition $s < e_0$, if $0 \leq t < s$, then $t < e_0$, hence $\rho(t) = 0$. Therefore, we get that

$$f''(s) \begin{cases} < 0, & \text{if } d \geq 3, \\ = 0, & \text{if } d = 2, \\ > 0, & \text{if } d = 1. \end{cases} \tag{5.4.12}$$

We remark that in reality, we have $\rho(t) = e^{-t}$, so $\tilde{\rho}(t) = -e^{-t}$ and $f(s) = -Ce^{-s}$ with some constant $C > 0$, so $f''(s) < 0$.

Chapter 6

Proof of Lemmas 5.3.1 $\sim$ 5.3.3

We give the proofs of Lemmas 5.3.1 $\sim$ 5.3.3 in this section. Sections 6.1 $\sim$ 6.4 give the proof of Lemma 5.3.1, sections 6.6 and 6.5 prove Lemmas 5.3.2 and 5.3.3, respectively. The proof of Lemma 5.3.4 will be given in the next chapter.

6.1 First decomposition

First, we have by (5.2.1)

$$\begin{aligned} & M_i(V_i(t) - V_i(0)) \\ = & -\int_0^t ds \int_{\mathbf{R}\times E} \nabla U_i(X_i(s,\omega) - x(s, \Psi(r, x, m^{-1/2}v)))\mu_\omega(dr, dx, dv). \end{aligned}$$

Let $\sigma(\omega) = \sigma_n(\omega) = \inf\{t \geq 0; \max_{i=1,\cdots,N} |V_i(t,\omega)| \geq n\}$, and $\tau = C_0^{-1}R_0$ as before.

Notice that under our condition, we have $|v| \geq 2C_0 + 1$, P_m-*a.s.*. So for any $s \in [0, T\wedge\sigma(\omega))$, we have by Proposition 5.4.5 that $\nabla U_i(X_i(s,\omega) - x(s, \Psi(r, x, m^{-1/2}v))) = 0$ if $|s - r| > 2m^{1/2}\tau$.

For any $t \leq T$, we can decompose

$$-M_i(V_i(t\wedge\sigma_n) - V_i(0)) = V_i^0(t) + V_i^1(t),$$

with

$$\begin{aligned} V_i^0(t) &= \int_0^{t\wedge\sigma_n} 1_{[4m^{1/2}\tau,\infty)}(s)ds \\ & \quad \int_{\mathbf{R}\times E} \nabla U_i(X_i(s,\omega) - x(s, \Psi(r, x, m^{-1/2}v)))\mu_\omega(dr, dx, dv), \\ V_i^1(t) &= \int_0^{t\wedge\sigma_n} 1_{[0,4m^{1/2}\tau)}(s)ds \\ & \quad \int_{\mathbf{R}\times E} \nabla U_i(X_i(s,\omega) - x(s, \Psi(r, x, m^{-1/2}v)))\mu_\omega(dr, dx, dv). \end{aligned}$$

6.2 The term $V_i^1(t)$

Let us deal with $V_i^1(t)$ in this section. We will show that it is negligible. Decompose it into

$$V_i^1(t) = V_i^{10}(t) + V_i^{11}(t),$$

with

$$\begin{aligned} V_i^{10}(t) &= \int_0^{t\wedge\sigma_n} 1_{[0,4m^{1/2}T)}(s)ds \\ &\quad \int_{\mathbf{R}\times E} \Big\{\nabla U_i(X_i(s,\omega) - x(s,\Psi(r,x,m^{-1/2}v))) \\ &\quad -\nabla U_i(X_i(0) - \tilde{\varphi}^0(m^{-1/2}s, \Psi(m^{-1/2}r,x,v); \vec{X}(0)))\Big\}\mu_\omega(dr,dx,dv), \\ V_i^{11}(t) &= \int_0^{t\wedge\sigma_n} 1_{[0,4m^{1/2}T)}(s)ds \\ &\quad \int_{\mathbf{R}\times E} \nabla U_i(X_i(0) - \tilde{\varphi}^0(m^{-1/2}s, \Psi(m^{-1/2}r,x,v); \vec{X}(0)))\mu_\omega(dr,dx,dv). \end{aligned}$$

Before discussing the behavior of $V_i^{10}(t)$, let us prepare the following result. Fix any $t_0 > 0$. Then we have the following:

Lemma 6.2.1 *For any $s \in [0,t_0]$ satisfying $0 \le m^{1/2}s \le T \wedge \sigma_n(\omega)$, we have that*

$$\begin{aligned} &\Big|x(m^{1/2}s, \Psi(r,x,m^{-1/2}v)) - \tilde{\varphi}^0(s, \Psi(m^{-1/2}r,x,v); \vec{X}(0)))\Big| \\ &\le nm^{1/2}s\sum_{i=1}^N \|\nabla^2 U_i\|_\infty t_0 e^{(\sum_{i=1}^N \|\nabla^2 U_i\|_\infty + 1)t_0}. \end{aligned}$$

Proof. First notice that under our condition, $|X_i(m^{1/2}s) - X_i(0)| \le nm^{1/2}s$. Let

$$\xi(s) = x(m^{1/2}s, \Psi(r,x,m^{-1/2}v)) - \tilde{\varphi}^0(s, \Psi(m^{-1/2}r,x,v); \vec{X}(0))).$$

Then we have

$$\begin{aligned} \frac{d^2}{ds^2}\xi(s) &= \sum_{i=1}^N \Big\{ -\nabla U_i(x(m^{1/2}s, \Psi(r,x,m^{-1/2}v)) - X_i(m^{1/2}s)) \\ &\quad +\nabla U_i(\tilde{\varphi}^0(s, \Psi(m^{-1/2}r,x,v); \vec{X}(0))) - X_i(0))\Big\}. \end{aligned}$$

Therefore, since $\nabla^2 U_i$, $i = 1, \cdots, N$, are bounded, we have that

$$\begin{aligned} \left|\frac{d^2}{ds^2}\xi(s)\right| &\le \sum_{i=1}^N \|\nabla^2 U_i\|_\infty(|\xi(s)| + |X_i(m^{1/2}s) - X_i(0)|) \\ &\le \sum_{i=1}^N \|\nabla^2 U_i\|_\infty(|\xi(s)| + nm^{1/2}s). \end{aligned}$$

Let $g(s) = |(\xi(s), \frac{d}{ds}\xi(s))|$. Then the above implies that

$$\Big|\frac{d}{ds}g(s)\Big| \le \Big|\frac{d}{ds}\xi(s)\Big| + \Big|\frac{d^2}{ds^2}\xi(s)\Big| \le nm^{1/2}s\sum_{i=1}^N \|\nabla^2 U_i\|_\infty + \Big(\sum_{i=1}^N \|\nabla^2 U_i\|_\infty + 1\Big)g(s).$$

Also, $g(0) = 0$. So for any $0 \le s \le t_0$, we get that

$$g(s) \le nm^{1/2}s\sum_{i=1}^N \|\nabla^2 U_i\|_\infty t_0 + \Big(\sum_{i=1}^N \|\nabla^2 U_i\|_\infty + 1\Big)\int_0^s g(u)du.$$

Therefore, by Gronwall's Lemma, we have

$$g(s) \le nm^{1/2}s\sum_{i=1}^N \|\nabla^2 U_i\|_\infty t_0 e^{(\sum_{i=1}^N \|\nabla^2 U_i\|_\infty + 1)s}.$$

This gives us our assertion. ∎

In particular, applying Lemma 6.2.1 to $t_0 = 4\tau$, we get that

$$\begin{aligned}&\Big|x(s, \Psi(r, x, m^{-1/2}v)) - \tilde{\varphi}^0(m^{-1/2}s, \Psi(m^{-1/2}r, x, v); \vec{X}(0)))\Big| \\ &\quad\le ns\sum_{i=1}^N \|\nabla^2 U_i\|_\infty 4\tau e^{(\sum_{i=1}^N \|\nabla^2 U_i\|_\infty + 1)4\tau}, \\ &|X_i(s) - X_i(0)| \le ns, \qquad \forall s \in [0, 4m^{1/2}\tau \wedge T \wedge \sigma(\omega)). \end{aligned} \tag{6.2.1}$$

We use this to proof the next lemma.

Lemma 6.2.2 $E^{P_m}[\sup_{0\le t\le T} |V_i^{10}(t)|] \to 0$ *as* $m \to 0$.

Proof. First notice that in the definition of V_i^{10}, we are taking integral for $s \in [0, 4m^{1/2}\tau \wedge T \wedge \sigma(\omega))$, so if $r > 6m^{1/2}\tau$ or $r < -2m^{1/2}\tau$, then we have $|u| > 2m^{1/2}\tau$ for any $u \in [r - s, r]$, so since $x \cdot v = 0$, we get by definition

$$\begin{aligned} &\Big|\tilde{\varphi}^0(m^{-1/2}s, \Psi(m^{-1/2}r, x, v); \vec{X}(0)))\Big| \\ = &\ |x - m^{-1/2}(r - s)v| \ge m^{-1/2}|r - s||v| \ge 2\tau|v| \\ \ge &\ R_0. \end{aligned}$$

Therefore, for any $s \in [0, 4m^{1/2}\tau \wedge T \wedge \sigma(\omega))$, we have

$$\nabla U_i(X_i(0) - \tilde{\varphi}^0(m^{-1/2}s, \Psi(m^{-1/2}r, x, v); \vec{X}(0))) = 0 \tag{6.2.2}$$

if $r > 6m^{1/2}\tau$ or $r < -2m^{1/2}\tau$. Also, (6.2.2) holds if $|x| \ge R_0 + 1$. Similarly, the same holds with $X(0)$ substituted by $X(s)$ (since $0 \le s \le \sigma$). Let

$$C_1 = \|\nabla^2 U_i\|_\infty\Big(\sum_{j=1}^N \|\nabla^2 U_j\|_\infty 4\tau e^{(\sum_{j=1}^N \|\nabla^2 U_j\|_\infty + 1)4\tau} + 1\Big).$$

Then by combining the above with (6.2.1), we get that for any $s \in [0, 4m^{1/2}\tau \wedge T \wedge \sigma(\omega))$,

$$\begin{aligned}
&\Big|\nabla U_i(X_i(s,\omega) - x(s, \Psi(r,x,m^{-1/2}v))) \\
&\quad -\nabla U_i(X_i(0) - \widetilde{\varphi}^0(m^{-1/2}s, \Psi(m^{-1/2}r,x,v); \vec{X}(0)))\Big| \\
\leq\ & 1_{[0,R_0+1)}(|x|)1_{[-2m^{1/2}\tau, 6m^{1/2}\tau]}(r)nsC_1.
\end{aligned}$$

Therefore, by the definition of $V_i^{10}(t)$, we get that

$$\begin{aligned}
&|V_i^{10}(t)| \\
\leq\ & \int_0^{t\wedge\sigma_n} 1_{[0,4m^{1/2}\tau)}(s)ds \int_{\mathbf{R}\times E} C_1 ns 1_{[0,R_0+1)}(|x|)1_{[-2m^{1/2}\tau, 6m^{1/2}\tau]}(r)\mu_\omega(dr,dx,dv) \\
\leq\ & \frac{C_1}{2} n(4m^{1/2}\tau)^2 \int_{\mathbf{R}\times E} 1_{[0,R_0+1)}(|x|)1_{[-2m^{1/2}\tau, 6m^{1/2}\tau]}(r)\mu_\omega(dr,dx,dv).
\end{aligned} \tag{6.2.3}$$

We need to discuss the expection of the integral on the right hand side above. Let $c = \sum_{j=1}^N \|U_j\|_\infty$, and let

$$C_2 = 8\tau(2(R_0+1))^{d-1}\int_{\mathbf{R}^d} \widetilde{\rho_c}(\frac{1}{2}|v|^2)|v|dv,$$

which is finite by our assumption. Then we have by definition that

$$\begin{aligned}
&\int_{\mathbf{R}\times E} 1_{[0,R_0+1)}(|x|)1_{[-2m^{1/2}\tau, 6m^{1/2}\tau]}(r)\lambda(dr,dx,dv) \\
=\ & \int_{\mathbf{R}\times E} 1_{[0,R_0+1)}(|x|)1_{[-2m^{1/2}\tau, 6m^{1/2}\tau]}(r)m^{-1}\rho_0(x - m^{-1/2}rv, v)dr\nu(dx,dv) \\
\leq\ & \int_{\mathbf{R}\times E} 1_{[0,R_0+1)}(|x|)1_{[-2m^{1/2}\tau, 6m^{1/2}\tau]}(r)m^{-1}\widetilde{\rho_c}(\frac{1}{2}|v|^2)dr\nu(dx,dv) \\
\leq\ & 8m^{1/2}\tau m^{-1}(2(R_0+1))^{d-1}\int_{\mathbf{R}^d}\widetilde{\rho_c}(\frac{1}{2}|v|^2)|v|dv \\
=\ & C_2 m^{-1/2}.
\end{aligned}$$

Therefore, by the definition of Poisson point process, we have

$$\begin{aligned}
&E^{P_m}\Big[\Big(\int_{\mathbf{R}\times E} 1_{[0,R_0+1)}(|x|)1_{[-2m^{1/2}\tau, 6m^{1/2}\tau]}(r)\mu_\omega(dr,dx,dv)\Big)^2\Big] \\
\leq\ & \int_{\mathbf{R}\times E} 1_{[0,R_0+1)}(|x|)1_{[-2m^{1/2}\tau, 6m^{1/2}\tau]}(r)\lambda(dr,dx,dv) \\
&+\Big(\int_{\mathbf{R}\times E} 1_{[0,R_0+1)}(|x|)1_{[-2m^{1/2}\tau, 6m^{1/2}\tau]}(r)\lambda(dr,dx,dv)\Big)^2 \\
\leq\ & C_2 m^{-1/2} + C_2^2 m^{-1}.
\end{aligned} \tag{6.2.4}$$

This combined with (6.2.3) gives us that

$$E^{P_m}[\sup_{0\leq t\leq T}|V_i^{10}(t)|] \leq \frac{1}{2}C_1 n(4m^{1/2}\tau)^2(C_2 m^{-1/2} + C_2^2 m^{-1})^{1/2}.$$

The right hand side above converges to 0 as $m \to 0$. This completes the proof of our assertion. ∎

For the term $V_i^{11}(t)$, we have the following:

Lemma 6.2.3 $E^{P_m}[\sup_{0\le t\le T} |V_i^{11}(t)|] \to 0$ *as* $m \to 0$.

Proof. We first notice that

$$\int_{\mathbf{R}\times E} \nabla U_i(X_i(0) - \widetilde{\varphi}^0(m^{-1/2}s, \Psi(m^{-1/2}r, x, v); \vec{X}(0)))\lambda(dr, dx, dv) = 0 \qquad (6.2.5)$$

for any $s \in [0, 4m^{1/2}\tau \wedge T \wedge \sigma)$ and $|v| \ge C_0$. Actually, since $|X_i(0) - X_j(0)| > R_i + R_j$ for any $i \ne j$, we have by (5.4.7) and (5.4.9) that

$$\int_{\mathbf{R}^{2d}} \nabla U_i(X_i(0) - x)\rho\Big(\frac{1}{2}|v|^2 + \sum_{j=1}^{N} U_j(x - X_j(0))\Big)dxdv = 0.$$

So by Theorem 3.2.2 (with $t = m^{-1/2}s$, $N = 1$, and $f(x, v) = \nabla U_i(X_i(0) - x)$),

$$\int_{\mathbf{R}^{2d}} \nabla U_i\Big(X_i(0) - \widetilde{\varphi}^0(m^{-1/2}s, x, v; \vec{X}(0))\Big)\rho\Big(\frac{1}{2}|v|^2 + \sum_{j=1}^{N} U_j(x - X_j(0)\Big)dxdv = 0,$$

which, by Theorem 3.3.1, means that

$$\int_{\mathbf{R}\times E} \nabla U_i\Big(X_i(0) - \widetilde{\varphi}^0(m^{-1/2}s, \Psi(r, x, v); \vec{X}(0))\Big) \times\rho\Big(\frac{1}{2}|v|^2 + \sum_{j=1}^{N} U_j(\Psi^0(r, x, v) - X_j(0))\Big)dr\nu(dx, dv) = 0.$$

Changing variable $r' = m^{-1/2}r$, we get (6.2.5).

By (6.2.5), we get that

$$\begin{aligned} V_i^{11}(t) &= \int_0^{t\wedge\sigma_n} 1_{[0,4m^{1/2}\tau)}(s)ds \\ &\quad \int_{\mathbf{R}\times E} \nabla U_i(X_i(0) - \widetilde{\varphi}^0(m^{-1/2}s, \Psi(m^{-1/2}r, x, v); \vec{X}(0))) \\ &\quad\quad (\mu_\omega(dr, dx, dv) - \lambda(dr, dx, dv)). \end{aligned} \qquad (6.2.6)$$

As in the proof of Lemma 6.2.2, (6.2.2) holds if $r > 6m^{1/2}\tau$ or $r < -2m^{1/2}\tau$, or if $|x| \ge R_0 + 1$. Let

$$C_3 = 8\tau(2(R_0 + 1))^{d-1}\|\nabla U_i\|_\infty^2 \int_{\mathbf{R}^d} \widetilde{\rho_c}(\frac{1}{2}|v|^2)|v|dv,$$

which is finite by our assumption. Then we have that

$$\begin{aligned}
&E^{P_m}\Big[\Big|\int_{\mathbf{R}\times E}\nabla U_i(X_i(0)-\widetilde{\varphi}^0(m^{-1/2}s,\Psi(m^{-1/2}r,x,v);\vec{X}(0)))\\
&\qquad(\mu_\omega(dr,dx,dv)-\lambda(dr,dx,dv))\Big|^2\Big]\\
=&\int_{\mathbf{R}\times E}\Big|\nabla U_i(X_i(0)-\widetilde{\varphi}^0(m^{-1/2}s,\Psi(m^{-1/2}r,x,v);\vec{X}(0)))\Big|^2\lambda(dr,dx,dv)\\
\le&\int_{\mathbf{R}\times E}1_{[0,R_0+1)}(|x|)1_{[-2m^{1/2}\tau,6m^{1/2}\tau]}(r)\|\nabla U_i\|_\infty^2\lambda(dr,dx,dv)\\
=&\|\nabla U_i\|_\infty^2\int_{\mathbf{R}\times E}1_{[0,R_0+1)}(|x|)1_{[-2m^{1/2}\tau,6m^{1/2}\tau]}(r)\\
&\times m^{-1}\rho\Big(\frac{1}{2}|v|^2+\sum_{j=1}^N U_j(X_{j,0}-(x-m^{-1/2}rv))\Big)dr\nu(dx,dv)\\
\le& m^{-1}8m^{1/2}\tau(2(R_0+1))^{d-1}\|\nabla U_i\|_\infty^2\int_{\mathbf{R}^d}\widetilde{\rho_c}(\frac{1}{2}|v|^2)|v|dv\\
=&C_3m^{-1/2}. \qquad (6.2.7)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&E^{P_m}[\sup_{0\le t\le T}|V_i^{11}(t)|]\\
\le&\int_0^T 1_{[0,4m^{1/2}\tau)}(s)ds\\
&\times E^{P_m}\Big[\Big|\int_{\mathbf{R}\times E}\nabla U_i(X_i(0)-\widetilde{\varphi}^0(m^{-1/2}s,\Psi(m^{-1/2}r,x,v);\vec{X}(0)))\\
&\qquad(\mu_\omega(dr,dx,dv)-\lambda(dr,dx,dv))\Big|^2\Big]^{1/2}\\
\le&\int_0^{4m^{1/2}\tau}\Big(C_3m^{-1/2}\Big)^{1/2}ds=4C_3^{1/2}\tau m^{1/4},
\end{aligned}$$

which converges to 0 as $m\to 0$. This completes the proof of our assertion. ∎

By Lemmas 6.2.2 and 6.2.3, we get the following main result of this section.

Lemma 6.2.4 $E^{P_m}\Big[\sup_{0\le t\le T}|V_i^1(t)|\Big]\to 0$ *as* $m\to 0$.

6.3 The term $V_i^0(t)$

Let us discuss the term $V_i^0(t)$ in this section. For any $r\in\mathbf{R}$, let $\widetilde{r}=\widetilde{r}(\omega)=((r-2m^{1/2}\tau)\vee 0)\wedge T\wedge\sigma(\omega)$.

We first decompose

$$V_i^0(t)=V_i^{02}(t)+\widetilde{V_i^{01}}(t)+\widetilde{V_i^{05}}(t)+\widetilde{V_i^{03}}(t)-V_i^{04}(t),$$

with

$$
\begin{aligned}
V_i^{02}(t) &= \int_0^{t\wedge\sigma_n} 1_{[4m^{1/2}\tau,\infty)}(s)ds \\
&\quad \int_{\mathbf{R}\times E} \Big\{\nabla U_i\Big(X_i(s) - x(s,\Psi(r,x,m^{-1/2}v))\Big) \\
&\qquad -\nabla U_i\Big(X_i(\widetilde{r}) - \psi^0(m^{-1/2}(s-r),x,v;\vec{X}(\widetilde{r}))\Big)\Big\} \\
&\qquad\quad \mu_\omega(dr,dx,dv), \\
\widetilde{V}_i^{01}(t) &= \int_0^{t\wedge\sigma_n} 1_{[4m^{1/2}\tau,\infty)}(s)ds \\
&\quad \int_{\mathbf{R}\times E} \nabla U_i(X_i(s) - \psi^0(m^{-1/2}(s-r),x,v;\vec{X}(s)))\lambda(dr,dx,dv), \\
\widetilde{V}_i^{05}(t) &= \int_0^{t\wedge\sigma_n} 1_{[4m^{1/2}\tau,\infty)}(s)ds \int_{\mathbf{R}\times E} \widetilde{F_i^{05}}(s,r,x,v)\lambda(dr,dx,dv), \\
\widetilde{V}_i^{03}(t) &= \int_0^{t\wedge\sigma_n} ds \int_{(2m^{1/2}\tau,\infty)\times E} \nabla U_i(X_i(\widetilde{r}) - \psi^0(m^{-1/2}(s-r),x,v;\vec{X}(\widetilde{r}))) \\
&\quad (\mu_\omega(dr,dx,dv) - \lambda(dr,dx,dv)), \\
V_i^{04}(t) &= \int_0^{t\wedge\sigma_n} 1_{[0,4m^{1/2}\tau)}(s)ds \int_{[2m^{1/2}\tau,\infty)\times E} \\
&\quad \nabla U_i(X_i(\widetilde{r}) - \psi^0(m^{-1/2}(s-r),x,v;\vec{X}(\widetilde{r}))) \\
&\quad (\mu_\omega(dr,dx,dv) - \lambda(dr,dx,dv)),
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{F_i^{05}}(s,r,x,v) &= -\Big\{\nabla U_i(X_i(s) - \psi^0(m^{-1/2}(s-r),x,v;\vec{X}(s))) \\
&\qquad -\nabla U_i(X_i(\widetilde{r}) - \psi^0(m^{-1/2}(s-r),x,v;\vec{X}(\widetilde{r})))\Big\}.
\end{aligned}
$$

Actually, to show that this decomposition is correct, we only need to notice that

$$
\begin{aligned}
&\nabla U_i(X_i(\widetilde{r}) - \psi^0(m^{-1/2}(s-r),x,v;\vec{X}(\widetilde{r}))) \neq 0 \\
&\Rightarrow |m^{-1/2}(s-r)| \leq 2\tau.
\end{aligned}
$$

Therefore, for $s \in [4m^{1/2}\tau,\infty)$,

$$
r < 2m^{1/2}\tau \Rightarrow \nabla U_i(X_i(\widetilde{r}) - \psi^0(m^{-1/2}(s-r),x,v;\vec{X}(\widetilde{r}))) = 0.
$$

So

$$
\begin{aligned}
&\widetilde{V}_i^{03}(t) - V_i^{04}(t) \\
&= \int_0^{t\wedge\sigma_n} 1_{[4m^{1/2}\tau,\infty)}(s)ds \int_{(2m^{1/2}\tau,\infty)\times E} \\
&\qquad \nabla U_i(X_i(\widetilde{r}) - \psi^0(m^{-1/2}(s-r),x,v;\vec{X}(\widetilde{r})))(\mu_\omega(dr,dx,dv) - \lambda(dr,dx,dv)) \\
&= \int_0^{t\wedge\sigma_n} 1_{[4m^{1/2}\tau,\infty)}(s)ds \int_{\mathbf{R}\times E} \\
&\qquad \nabla U_i(X_i(\widetilde{r}) - \psi^0(m^{-1/2}(s-r),x,v;\vec{X}(\widetilde{r})))(\mu_\omega(dr,dx,dv) - \lambda(dr,dx,dv)).
\end{aligned}
$$

We discuss each term in the above decomposition in the following. First, for the term $V_i^{02}(t)$, we have by definition

$$\frac{d}{dt}V_i^{02}(t) = 1_{(4m^{1/2}\tau,\sigma)}(t)\int_{\mathbf{R}\times E}\widetilde{f}_i(t,r,x,v;\omega)\mu_\omega(dr,dx,dv),$$

where

$$\begin{aligned}\widetilde{f}_i(t,r,x,v) &= \nabla U_i\Big(X_i(t)-x(t,\Psi(r,x,m^{-1/2}v))\Big)\\ &\qquad -\nabla U_i\Big(X_i(\widetilde{r})-\psi^0(m^{-1/2}(t-r),x,v;\vec{X}(\widetilde{r}))\Big).\end{aligned}$$

By definition and assumption, we have that $\lambda_m(dr,dx,dv)=0$ if $|v|\le 2C_0+1$. Also, by Proposition 5.4.5 and Corollary 3.4.2, $\widetilde{f}_i(t,r,x,v)=0$ if $|r-t|\ge 2m^{1/2}\tau$. So we only need to consider the case when $t\in[4m^{1/2}\tau,T\wedge\sigma)$, $r\in[2m^{1/2}\tau,T\wedge\sigma(\omega)+2m^{1/2}\tau]$ and $|v|\ge 2C_0+1$. We first show the following:

Lemma 6.3.1 *Assume that $t\in[4m^{1/}\tau,T\wedge\sigma]$, $|r-2m^{1/2}\tau|\le T\wedge\sigma(\omega)$ and $|v|\ge 2C_0+1$. Then*

$$|\widetilde{f}_i(t,r,x,v)|\le 1_{[0,R_0+1)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau)}(t-r)\cdot Cm^{1/2}.$$

Proof. First, since $t\in[0,T\wedge\sigma_n)$, we have by Proposition 5.4.5 that $\nabla U_i\Big(X_i(t)-x(t,\Psi(r,x,m^{-1/2}v))\Big)=0$ if $t-r>2m^{1/2}\tau$ or $t-r<-m^{1/2}\tau$. Also, since $\widetilde{r}\in[0,T\wedge\sigma_n)$ by definition, we have $|X_i(\widetilde{r})|\le|X_{i,0}|+nT$, so by Corollary 3.4.2, $\nabla U_i\Big(X_i(\widetilde{r})-\psi^0(m^{-1/2}(t-r),x,v;\vec{X}(\widetilde{r}))\Big)=0$ if $t-r\ge 2m^{1/2}\tau$ or $t-r\le -m^{1/2}\tau$. Combining the above, we get that $\widetilde{f}_i(t,r,x,v)=0$ if $r\notin[t-2m^{1/2}\tau,t+m^{1/2}\tau]$.

Next, for $r\in[t-2m^{1/2}\tau,t+m^{1/2}\tau]$, if $|x|\ge R_0+1$, since $x\cdot v=0$, we get easily that $|x(t,\Psi(r,x,m^{-1/2}v))|=|x-(r-t)m^{1/2}v|\ge|x|\ge R_0+1$, hence both of the terms of $\widetilde{f}_i(t,r,x,v)$ equal to 0.

Finally, we show, for $|x|<R_0+1$ and $r\in[t-2m^{1/2}\tau,t+m^{1/2}\tau]$, that $|\widetilde{f}_i(t,r,x,v)|\le Cm^{1/2}$. For this kind of x and r, since $t\in[4m^{1/}\tau,T\wedge\sigma(\omega)]$, we have by definition $2m^{1/2}\tau\le r\le T\wedge\sigma+m^{1/2}\tau$, so $\widetilde{r}=r-2m^{1/2}\tau$. We have

$$\begin{aligned}&|\widetilde{f}_i(t,r,x,v)|\\ \le\ & \|\nabla^2U_i\|_\infty\Big(|X_i(t)-X_i(\widetilde{r})|+|x(t,\Psi(r,x,m^{-1/2}v))-\psi^0(m^{-1/2}(t-r),x,v;\vec{X}(\widetilde{r}))|\Big).\end{aligned}$$

The term of X is easy. Actually, since $t,\widetilde{r}\in[0,T\wedge\sigma(\omega)]$, we have by definition $|X_i(t)-X_i(\widetilde{r})|\le n|t-\widetilde{r}|=n|t-(r-2m^{1/2}\tau)|\le n(|t-r|+2m^{1/2}\tau)\le n4m^{1/2}\tau$.

We next deal with the second term. Notice that by assumption, $0\le r-2m^{1/2}\tau\le T\wedge\sigma(\omega)$, $0\le r-m^{1/2}\tau\le T\wedge\sigma(\omega)$ and $0\le t\le T\wedge\sigma(\omega)$. Therefore, by Proposition 5.4.3 (3) (with (t,s,a) given by $(m^{-1/2}(t-r),r,2\tau)$), there exists a constant $\widetilde{C}$ such that

$$\begin{aligned}&|x(t,\Psi(r,x,m^{-1/2}v))-\psi^0(m^{-1/2}(t-r),x,v;\vec{X}(r-2m^{1/2}\tau))|\\ \le\ & m^{1/2}\widetilde{C}(2\tau+2\tau).\end{aligned}$$

This completes the proof of our assertion. ∎

Now we are ready to show that the term $V_i^{02}(t)$ is tight.

Lemma 6.3.2 $\left\{\{V_i^{02}(t)\}_{t\in[0,T]}\right\}_{m>0}$ *is tight in* D. *Here* D *is the space defined in Chapter 5.1.*

Proof. By Lemma 6.3.1, we have

$$\Big|\frac{d}{dt}V_i^{02}(t)\Big| \le Cm^{1/2}\int_{\mathbf{R}\times E} 1_{[0,R_0+1)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau)}(t-r)\mu_\omega(dr,dx,dv).$$

Notice that in general, it is easy by the definition of Poisson point process and a simply calculation that $E^{P_m}[(\int gd\mu_\omega)^2] \le \int g^2 d\lambda_m + (\int gd\lambda_m)^2$. Therefore,

$$\begin{aligned} & E^{P_m}\left[\Big|\frac{d}{dt}V_i^{02}(t)\Big|^2\right] \\ \le\ & C^2 m\int_{\mathbf{R}\times E} 1_{[0,R_0+1)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau)}(t-r)\lambda_m(dr,dx,dv) \\ & +\Big(Cm^{1/2}\int_{\mathbf{R}\times E} 1_{[0,R_0+1)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau)}(t-r)\lambda_m(dr,dx,dv)\Big)^2. \end{aligned}$$

Here, let $c := \sum_{i=1}^N \|U_i\|_\infty$, then

$$\begin{aligned} & \int_{\mathbf{R}\times E} 1_{[0,R_0+1)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau)}(t-r)\lambda_m(dr,dx,dv) \\ =\ & \int_{\mathbf{R}\times E} 1_{[0,R_0+1)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau)}(t-r) \\ & \qquad \times m^{-1}\rho\Big(\frac{1}{2}|v|^2+\sum_{i=1}^N U_i(x-m^{1/2}rv-X_{i,0})\Big)dr|v|\widetilde{\nu}(dx;v)dv \\ \le\ & m^{-1}3m^{1/2}\tau\int_E 1_{[0,R_0+1)}(|x|)\widetilde{\rho_c}(1+\frac{1}{2}|v|^2)|v|\widetilde{\nu}(dx;v)dv \\ \le\ & 3m^{-1/2}\tau[2(R_0+1)]^{d-1}\int_{\mathbf{R}^d}\widetilde{\rho_c}(1+\frac{1}{2}|v|^2)|v|dv, \end{aligned}$$

which, by assumption, is dominated by $Cm^{-1/2}$ with some constant C.

Therefore,

$$\widetilde{C} := \sup_{m\in(0,1]}\sup_{0\le t\le T} E^{P_m}\left[\Big|\frac{d}{dt}V_i^{02}(t)\Big|^2\right] < \infty.$$

So

$$E[|V_i^{02}(t)-V_i^{02}(t')|^2] \le \widetilde{C}|t-t'|^2,$$

hence by Theorem 5.1.7 (with $\beta=\varepsilon=\gamma=1$), $\left\{\{V_i^{02}(t)\}_{t\in[0,T]}\right\}_{m>0}$ is tight in D.

The next result is about the term $\widetilde{V}_i^{01}(t)$.

Lemma 6.3.3 *There exists an* $m_0>0$ *(depending on* $\vec{X}_0$, n, T *and* $U_i, i=1,\cdots,N$*) such that for any* $m\le m_0$,

$$\widetilde{V}_i^{01}(t) = m^{-1/2}\int_0^{t\wedge\sigma}\nabla_i\widetilde{U}(\vec{X}(s))ds,$$

where $\widetilde{U}$ *is as defined in Sections 5.3 and 5.4.*

Proof. Suppose that $\nabla U_i(X_i(s) - \psi^0(m^{-1/2}(s-r), x, v; \vec{X}(s)) \neq 0$. Then $s - r < 2m^{1/2}\tau$ by Proposition 5.4.5, this combined with $s \geq 4m^{1/2}\tau$ implies that $r > 2m^{1/2}\tau = 2m^{1/2}C_0^{-1}R$. Since $|v| \geq 2C_0 + 1$ and $x \cdot v = 0$, $\lambda(dr, dx, dv)$-*a.e.*, this implies $|x - m^{-1/2}rv| \geq m^{-1/2}r|v| \geq R_0$, hence $U_i(X_{i,0} - (x - m^{-1/2}rv)) = 0$.

Therefore, by definition, Proposition 3.4.4 and (5.4.7),

$$\begin{aligned}
\widetilde{V}_i^{01}(t) &= \int_0^{t\wedge\sigma_n} 1_{[4m^{1/2}\tau,\infty)}(s)ds \int_{\mathbf{R}\times E} \nabla U_i(X_i(s) - \psi^0(m^{-1/2}(s-r), x, v; \vec{X}(s))) \\
&\qquad m^{-1}\rho\Big(\frac{1}{2}|v|^2 + \sum_{i=1}^N U_i(x - m^{-1/2}rv - X_{i,0})\Big) dr\nu(dx, dv) \\
&= \int_0^{t\wedge\sigma_n} 1_{[4m^{1/2}\tau,\infty)}(s)ds \int_{\mathbf{R}\times E} \nabla U_i(X_i(s) - \psi^0(m^{-1/2}(s-r), x, v; \vec{X}(s))) \\
&\qquad m^{-1}\rho\Big(\frac{1}{2}|v|^2\Big) dr\nu(dx, dv) \\
&= \int_0^{t\wedge\sigma_n} 1_{[4m^{1/2}\tau,\infty)}(s)ds \\
&\qquad m^{-1/2}\int_{\mathbf{R}^{2d}} \nabla U_i(X_i(s) - x)\rho\Big(\frac{1}{2}|v|^2 + \sum_{k=1}^N U_k(x - X_{k,0})\Big) dxdv \\
&= \int_0^{t\wedge\sigma_n} 1_{[4m^{1/2}\tau,\infty)}(s) m^{-1/2}\nabla_i\widetilde{U}(\vec{X}(s))ds,
\end{aligned}$$

where we used Proposition 3.4.4 in passing to the third equality, and used (5.4.7) in passing to the last equality.

So in order to complete the proof of our assertion, it is sufficient to show that $\nabla_i\widetilde{U}(\vec{X}(s)) = 0$ for any $s \in [0, 4m^{1/2}\tau \wedge \sigma]$, if m is small enough. We show it from now on. Notice that since $|X_{i,0} - X_{j,0}| > R_i + R_j$ $(i \neq j)$ for any $i, j = 1, \cdots, N$ by assumption, there exists a $m_0 > 0$ (small enough) such that for any $m \leq m_0$, $|X_{i,0} - X_{j,0}| > R_i + R_j + 8m^{1/2}\tau n$ for any $i \neq j$. Also, by definition, we have $|X_i(s) - X_{i,0}| \leq sn \leq 4m^{1/2}\tau n$ for any $s \in [0, 4m^{1/2}\tau \wedge \sigma]$ and $i = 1, \cdots, N$. Therefore,

$$\begin{aligned}
|X_i(s) - X_j(s)| &\geq |X_{i,0} - X_{j,0}| - |X_i(s) - X_{i,0}| - |X_j(s) - X_{j,0}| \\
&> R_i + R_j + 8m^{1/2}\tau n - 4m^{1/2}\tau n - 4m^{1/2}\tau n \\
&= R_i + R_j,
\end{aligned}$$

so by (5.4.9), $\nabla_i\widetilde{U}(\vec{X}(s)) = 0$ for any $s \in [0, 4m^{1/2}\tau \wedge \sigma]$. This completes our proof. ∎

Before discussing the term $\widetilde{V}_i^{05}(t)$, let us first prepare the following continuity of $\psi^0(t, x, v; \vec{X})$ with respect to $\vec{X}$:

Lemma 6.3.4 *For any $Y > 0$, there exists a constant $\widetilde{C}$ (depending on $\max_{i=1}^N R_i + Y$, τ, C_0 and $\sum_{i=1}^N \|\nabla^2 U_i\|_\infty$) such that*

$$\Big|\psi^0(t, x, v; \vec{X}^1) - \psi^0(t, x, v; \vec{X}^2)\Big| \leq \widetilde{C}\|\vec{X}^1 - \vec{X}^2\|_{\mathbf{R}^d},$$

for any $x \in \mathbf{R}^d, |v| \geq 2C_0+1, |t| \leq 2\tau$ *and* $|\vec{X}^1|, |\vec{X}^2| \leq Y$.

Proof. Choose and fix any $v \in \mathbf{R}^d$ with $|v| \geq 2C_0+1$, and let $s_0 = \frac{\max_{i=1}^N R_i + Y}{|v|} \vee 2\tau$. Let $g(t) = \psi^0(t,x,v;\vec{X}^1) - \psi^0(t,x,v;\vec{X}^2)$. Then by definition,

$$g(t) = \varphi^0(t+s_0, x-s_0v, v; \vec{X}^1) - \varphi^0(t+s_0, x-s_0v, v; \vec{X}^2),$$

so

$$\begin{aligned}\frac{d^2}{dt^2}g(t) &= -\sum_{i=1}^N \nabla U_i\left(\varphi_i^0(t+s_0, x-s_0v, v; \vec{X}^1) - X_i^1\right) \\ &\quad + \sum_{i=1}^N \nabla U_i\left(\varphi_i^0(t+s_0, x-s_0v, v; \vec{X}^2) - X_i^2\right).\end{aligned}$$

Let $C = \sum_{i=1}^N \|\nabla^2 U_i\|_\infty$, then

$$\left|\frac{d^2}{dt^2}g(t)\right| \leq \sum_{i=1}^N \|\nabla^2 U_i\|_\infty(|g(t)| + |X_i^1 - X_i^2|) \leq C(|g(t)| + \|\vec{X}^1 - \vec{X}^2\|_{\mathbf{R}^d}),$$

therefore,

$$\left|\frac{d}{dt}\left|(g(t), \frac{d}{dt}g(t))\right|\right| \leq \left|\frac{d}{dt}g(t)\right| + \left|\frac{d^2}{dt^2}g(t)\right| \leq C\|\vec{X}^1 - \vec{X}^2\|_{\mathbf{R}^d} + (1+C)\left|(g(t), \frac{d}{dt}g(t))\right|.$$

Also, $g(-s_0) = \frac{d}{dt}g(-s_0) = 0$. Let $h(t) = \left|(g(t-s_0), \frac{d}{dt}g(t-s_0))\right|$. Then $h(0) = 0$, and for any $t \in [0, s_0+2\tau]$,

$$h(t) \leq C\|\vec{X}^1 - \vec{X}^2\|_{\mathbf{R}^d}(s_0+2\tau) + (1+C)\int_0^t h(s)ds,$$

so by Gronwall's Lemma,

$$h(t) \leq C\|\vec{X}^1 - \vec{X}^2\|_{\mathbf{R}^d}(s_0+2\tau)e^{(1+C)(s_0+2\tau)}, \qquad t \in [0, s_0+2\tau].$$

Notice that since $|v| \geq 2C_0+1$, we have $2\tau \leq s_0 \leq \frac{\max_{i=1}^N R_i + Y}{2C_0+1} \vee 2\tau$. Therefore,

$$\begin{aligned} g(t) &\leq h(t+s_0) \\ &\leq C\left(\frac{\max_{i=1}^N R_i + Y}{2C_0+1} \vee 2\tau + 2\tau\right) e^{(1+C)\left(\frac{\max_{i=1}^N R_i + Y}{2C_0+1} \vee 2\tau + 2\tau\right)} \|\vec{X}^1 - \vec{X}^2\|_{\mathbf{R}^d}, \end{aligned}$$

for any $t \in [-2\tau, 2\tau]$. This completes the proof of our assertion. ∎

We use Lemma 6.3.4 to prove the following:

Lemma 6.3.5 *There exists a constant* C *(which may different from before) such that*

$$\left|\widetilde{F_i^{05}}(s,r,x,v)\right| \leq Cm^{1/2}1_{[0,2m^{1/2}\tau]}(|s-r|)1_{[0,R_0+1)}(|x|)$$

in the corresponding integral domain.

Proof. First, since $s, \widetilde{r} \in [0, T \wedge \sigma(\omega)]$ in our integral domain, it is easy to see that $\left|\widetilde{F_i^{05}}(s,r,x,v)\right| = 0$ if $|x| \geq R_0 + 1$. Also, by Corollary 3.4.2, $\left|\widetilde{F_i^{05}}(s,r,x,v)\right| = 0$ if $|m^{-1/2}(s-r)| \geq 2\tau$. Finally, for $|x| \leq R_0 + 1$ and $|s-r| \leq 2m^{1/2}\tau$, by definition and Lemma 6.3.4, we only need to show the following

$$|X_i(s) - X_i(\widetilde{r})| \leq Cm^{1/2}, \quad s \geq 4m^{1/2}\tau. \tag{6.3.1}$$

To show (6.3.1), again, notice that in the present setting, $0 \leq r - 2m^{1/2}\tau \leq T \wedge \sigma$, so $\widetilde{r} = r - 2m^{1/2}\tau$. So (LHS) of (6.3.1) $= |X_i(s) - X_i(r - 2m^{1/2}\tau)| \leq n|s - (r - 2m^{1/2}\tau)| \leq n(|s-r| + 2m^{1/2}\tau) \leq n4m^{1/2}\tau$.

This completes the proof of our assertion. ∎

By Lemma 6.3.5, we get the following in exactly the same way as when deriving Lemma 6.3.2 from Lemma 6.3.1.

Lemma 6.3.6 *1.* $\sup_{m\in(0,1]} \sup_{0\leq t\leq T} E^{P_m}\Big[\Big|\frac{d}{dt}\widetilde{V}_i^{05}(t)\Big|^2\Big] < \infty,$

2. $\Big\{\{\widetilde{V}_i^{05}(t)\}_{t\in[0,T]}\Big\}_{m>0}$ *is tight in D.*

∎

Let us discuss the term V_i^{04} before $\widetilde{V}_i^{03}$. We have the following:

Lemma 6.3.7 $E^{P_m}[\sup_{0\leq t\leq T} |V_i^{04}(t)|] \to 0$ *as* $m \to 0$.

Proof. The proof is similar to the ones up to now. We have for any $s \in [0, 4m^{1/2}\tau]$ that

$$\begin{aligned} & \left|\nabla U_i(X_i(\widetilde{r}) - \psi^0(m^{-1/2}(s-r), x, v; \vec{X}(\widetilde{r})))\right| \\ \leq \ & \|\nabla U_i\|_\infty 1_{[0,R_0+1)}(|x|) 1_{[0,2m^{1/2}\tau)}(|s-r|) \\ \leq \ & \|\nabla U_i\|_\infty 1_{[0,R_0+1)}(|x|) 1_{[-2m^{1/2}\tau, 6m^{1/2}\tau]}(r). \end{aligned}$$

Let $C_4 = 8\|\nabla U_i\|_\infty^2 \tau(2(R_0+1))^{d-1} \int_{\mathbf{R}^d} \widetilde{\rho_c}(\frac{1}{2}|v|^2)|v|dv$, which is finite. Then we have by the definition of λ and assumption

$$\begin{aligned} & E^{P_m}\Big[\Big|\int_{[2m^{1/2}\tau,\infty)\times E} \nabla U_i(X_i(\widetilde{r}) - \psi^0(m^{-1/2}(s-r), x, v; \vec{X}(\widetilde{r}))) \\ & \qquad (\mu_\omega(dr,dx,dv) - \lambda(dr,dx,dv))\Big|^2\Big] \\ = \ & \int_{[2m^{1/2}\tau,\infty)\times E} \nabla U_i\Big(X_i(\widetilde{r}) - \psi^0(m^{-1/2}(s-r), x, v; \vec{X}(\widetilde{r}))\Big)^2 \lambda(dr,dx,dv) \\ \leq \ & \int_{[2m^{1/2}\tau,\infty)\times E} \|\nabla U_i\|_\infty^2 1_{[0,R_0+1)}(|x|) 1_{[-2m^{1/2}\tau, 6m^{1/2}\tau]}(r) \\ & \qquad m^{-1}\rho\Big(\frac{1}{2}|v|^2 + \sum_{j=1}^N U_j(x - m^{-1/2}rv - X_{j,0})\Big) dr\nu(dx,dv) \\ \leq \ & \|\nabla U_i\|_\infty^2 8m^{1/2}\tau(2(R_0+1))^{d-1} m^{-1} \int_{\mathbf{R}^d} \widetilde{\rho_c}(\frac{1}{2}|v|^2)|v|dv \\ = \ & C_4 m^{-1/2}. \end{aligned} \tag{6.3.2}$$

Therefore,

$$\begin{aligned}
& E^{P_m}[\sup_{0\le t\le T}|V_i^{04}(t)|] \\
\le\ & \int_0^{4m^{1/2}\tau} E^{P_m}\Big[\Big|\int_{[2m^{1/2}\tau,\infty)\times E}\nabla U_i(X_i(\tilde{r})-\psi^0(m^{-1/2}(s-r),x,v;\vec{X}(\tilde{r})))\\
& \quad (\mu_\omega(dr,dx,dv)-\lambda(dr,dx,dv))\Big|^2\Big]^{1/2}ds\\
\le\ & C_4^{1/2}m^{-1/4}4m^{1/2}\tau,
\end{aligned}$$

which converges to 0 as $m\to 0$. This completes our proof. ∎

Now, the only term left to be discussed is $\widetilde{V}_i^{03}$. We deal with it in the next section.

6.4 The term $\widetilde{V}_i^{03}$

We deal with the term $\widetilde{V}_i^{03}$ in this section. More precisely, we show that it is equal to a martingale plus a negligible term. We first prepare some notations. Let $\mathcal{F}_t=\mathcal{F}_t^{(m,n)}=\mathcal{F}_{(-\infty,2m^{1/2}\tau+t)\times E}\vee\aleph$. Then $\mathcal{F}_t$ is increasing and right continuous. Let

$$N((0,t]\times A):=\mu_\omega((2m^{1/2}\tau,2m^{1/2}\tau+t]\times A)$$

for any $A\in\mathcal{B}(E)$. Notice that if $\rho(\frac{1}{2}|v|^2+\sum_{j=1}^N U_j(X_{j,0}-(x-m^{-1/2}rv)))>0$ and $r\ge m^{1/2}\tau$, then $|v|\ge 2C_0+1$, hence $|x-m^{-1/2}rv|\ge\tau|v|>R_0$, so $\rho(\frac{1}{2}|v|^2+\sum_{j=1}^N U_j(X_{j,0}-(x-m^{-1/2}rv)))=\rho(\frac{1}{2}|v|^2)$. Therefore, if we let

$$\overline{\nu}(dx,dv)=\rho\Big(\frac{1}{2}|v|^2\Big)\nu(dx,dv),$$

then N is the $\mathcal{F}_t$-adapted Poisson point process with intensity measure $\overline{\lambda}(dt,dx,dv)=m^{-1}dt\overline{\nu}(dx,dv)=m^{-1}dt\rho(\frac{1}{2}|v|^2)\nu(dx,dv)$. Notice that $N((s,t]\times A)$ is independent to $\mathcal{F}_s$ for any $s<t$ and $A\in\mathcal{B}(E)$. Let

$$\overline{N}(dt,dx,dv)=N(dt,dx,dv)-m^{-1}dt\overline{\nu}(dx,dv).$$

Notice that $X_i(t\wedge\sigma)$ and $V_i(t\wedge\sigma)$ are $\mathcal{F}_t$-measurable. Also, since $\nabla U_i\Big(X_i(\tilde{r})-\psi^0(m^{-1/2}(s-r),x,v;\vec{X}(\tilde{r}))\neq 0$ only if $|m^{-1/2}(s-r)|\le 2\tau$, which combined with $r\ge 2m^{1/2}\tau$ and $s\le T\wedge\sigma$ implies $\tilde{r}=r-2m^{1/2}\tau$, we get by definition

$$\begin{aligned}
\widetilde{V}_i^{03}(t) &= \int_0^{t\wedge\sigma}ds\int_{[2m^{1/2}\tau,2m^{1/2}\tau+(T\wedge\sigma))\times E}\\
&\qquad \nabla U_i\Big(X_i(r-2m^{1/2}\tau)-\psi^0(m^{-1/2}(s-r),x,v;\vec{X}(r-2m^{1/2}\tau))\Big)\\
&\qquad\quad (\mu_\omega(dr,dx,dv)-\lambda(dr,dx,dv))\\
&= \int_0^{t\wedge\sigma}ds\int_{[0,T\wedge\sigma)\times E}\\
&\qquad \nabla U_i(X_i(r)-\psi^0(m^{-1/2}(s-r)-2\tau,x,v;\vec{X}(r)))\overline{N}(dr,dx,dv).
\end{aligned}$$

In the last expression, if $r > t\wedge\sigma$, then since $s \le t\wedge\sigma$, we get $m^{-1/2}(s-r)-2\tau < -\tau$, hence $\nabla U_i(X_i(r) - \psi^0(m^{-1/2}(s-r) - 2\tau, x, v; \vec{X}(r))) = 0$. Therefore,

$$\begin{aligned}\widetilde{V}_i^{03}(t) &= \int_0^{t\wedge\sigma} ds \int_{[0,t\wedge\sigma)\times E} \\ &\qquad \nabla U_i(X_i(r) - \psi^0(m^{-1/2}(s-r) - 2\tau, x, v; \vec{X}(r)))\overline{N}(dr,dx,dv).\end{aligned}$$

Let

$$\begin{aligned}\widetilde{\widetilde{V_i^{03}}}(t) &= \int_{(0,t]\times E} \overline{N}(dr,dx,dv)\int_0^t ds \\ &\qquad \nabla U_i(X_i(r\wedge\sigma) - \psi^0(m^{-1/2}(s-r) - 2\tau, x, v; \vec{X}(r\wedge\sigma))).\end{aligned}$$

Then

$$\widetilde{V}_i^{03}(t) = \widetilde{\widetilde{V_i^{03}}}(t\wedge\sigma).$$

By Corollary 3.4.2, $\nabla U_i(X_i(r\wedge\sigma) - \psi^0(u, x, v; \vec{X}(r\wedge\sigma))) = 0$ if $|u| \ge 2\tau$. So the integral domain $s \in [0,t]$ in the definition of $\widetilde{\widetilde{V_i^{03}}}(t)$, *i.e.*, $s - r \in [-r, t-r]$, can be substituted by $s-r \in [0, (t-r)\wedge 4m^{1/2}\tau] = [0, 4m^{1/2}\tau]\setminus[(t-r)\wedge(4m^{1/2}\tau), 4m^{1/2}\tau]$. Therefore, $\widetilde{\widetilde{V_i^{03}}}(t)$ can be decomposed into

$$\widetilde{\widetilde{V_i^{03}}}(t) = \widetilde{M^i}(t) + \widetilde{\eta}_i(t),$$

where

$$\begin{aligned}\widetilde{M^i}(t) &= \int_{(0,t]\times E} \overline{N}(dr,dx,dv)\int_0^{4m^{1/2}\tau} ds \\ &\qquad \nabla U_i(X_i(r\wedge\sigma) - \psi^0(m^{-1/2}s - 2\tau, x, v; \vec{X}(r\wedge\sigma))), \\ \widetilde{\eta}_i(t) &= -\int_{(0,t]\times E} \overline{N}(dr,dx,dv)\int_{(t-r)\wedge(4m^{1/2}\tau)}^{4m^{1/2}\tau} ds \\ &\qquad \nabla U_i(X_i(r\wedge\sigma) - \psi^0(m^{-1/2}s - 2\tau, x, v; \vec{X}(r\wedge\sigma))).\end{aligned}$$

By definition, (notice that the integral domain $(0,t]\times E$ in the definition of $\widetilde{\widetilde{V_i^{03}}}(t)$ can always be converted into $(0,T]\times E$ whenever necessary, and the vice verse)

$$\begin{aligned}\frac{d}{dt}\widetilde{\widetilde{V_i^{03}}}(t) &= \int_{(0,t]\times E} \overline{N}(dr,dx,dv) \\ &\qquad \nabla U_i(X_i(r\wedge\sigma) - \psi^0(m^{-1/2}(t-r) - 2\tau, x, v; \vec{X}(r\wedge\sigma))),\end{aligned}$$

so there exists a constant $C > 0$ such that

$$E^{P_m}\Big[\Big|\frac{d}{dt}\widetilde{\widetilde{V_i^{03}}}(t)\Big|^2\Big]$$

$$
\begin{aligned}
&= \int_{(0,t]\times E} \Big|\nabla U_i(X_i(r\wedge\sigma) - \psi^0(m^{-1/2}(t-r) - 2\tau, x, v; \vec{X}(r\wedge\sigma)))\Big|^2 \\
&\qquad m^{-1} dr \rho\Big(\frac{1}{2}|v|^2\Big)\nu(dx,dv) \\
&\le \int_{(0,t]\times E} \|\nabla U_i\|_\infty^2 1_{[0,R_0+1)}(|x|) 1_{[0,2\tau]}\Big(|m^{-1/2}(t-r) - 2\tau|\Big) m^{-1} dr \rho\Big(\frac{1}{2}|v|^2\Big)\nu(dx,dv) \\
&\le 4m^{1/2}\tau \|\nabla U_i\|_\infty^2 (2(R_0+1))^{d-1} m^{-1} \int_{\mathbf{R}^d} \rho\Big(\frac{1}{2}|v|^2\Big)|v| dv \\
&= Cm^{-1/2}. \qquad (6.4.1)
\end{aligned}
$$

This fact will be used later.

Let us investigate the term $\widetilde{M^i}(t)$ now. First, it is easy to see by definition that $\widetilde{M^i}(t)$ is a $\mathcal{F}_t$-martingale, with its jumps satisfying $|\Delta\widetilde{M^i}| \le 4m^{1/2}\tau\|\nabla U_i\|_\infty$, and there exists a constant $C > 0$ independent of n such that for any $0 \le s \le t \le T$,

$$
\begin{aligned}
& E^{P_m}\Big[|\widetilde{M^i}(t) - \widetilde{M^i}(s)|^2 \Big| \mathcal{F}_s\Big] \\
&= E^{P_m}\Big[\int_{(s,t)\times E} \Big| \int_0^{4m^{1/2}\tau} \nabla U_i(X_i(r) - \psi^0(m^{-1/2}u - \tau, x, v; \vec{X}(r))) du\Big|^2 \\
&\qquad 1_{[0,R_0+1)}(|x|) m^{-1} dr \overline{\nu}(dx,dv) \Big| \mathcal{F}_s\Big] \\
&\le C|t-s|, \qquad (6.4.2)
\end{aligned}
$$

hence for any $0 \le r \le s \le t \le T$,

$$
E^{P_m}\Big[|\widetilde{M^i}(t) - \widetilde{M^i}(s)|^2 |\widetilde{M^i}(s) - \widetilde{M^i}(r)|^2\Big] \le C^2|t-s||s-r|. \qquad (6.4.3)
$$

Also, by Doob's inequality and (6.4.2), we get

$$
\begin{aligned}
E^{P_m}\Big[\sup_{t\in[0,T]} |\widetilde{M^i}(t)|\Big] &\le E^{P_m}\Big[\Big(\sup_{t\in[0,T]} |\widetilde{M^i}(t)|\Big)^2\Big]^{1/2} \\
&\le 2 \sup_{t\in[0,T]} E^{P_m}\Big[|\widetilde{M^i}(t)|^2\Big]^{1/2} \\
&\le 2 \sup_{t\in[0,T]} \sqrt{Ct} = 2\sqrt{CT} < \infty. \qquad (6.4.4)
\end{aligned}
$$

By Theorem 5.1.7 (with $\varepsilon = 1$, $\beta = 2$ and $\gamma = 1/2$), (6.4.2), (6.4.3) and (6.4.4) imply the following.

Lemma 6.4.1 *$\{$the distribution of $\{\widetilde{M^i}(t)\}_{t\in[0,T]}$ under $P_m\}_{m\in(0,1)}$ is tight.*

We next show that any of its cluster points must be continuous processes. We first make the following preparation.

Lemma 6.4.2 *For any $\varepsilon \in (0,1]$, let*

$$\begin{aligned} A &= \cap_{\delta \geq 0}\{\omega \in D^d([0,T]) : \sup_{|t-s|\leq \delta} |\omega(t)-\omega(s)| > \varepsilon\}, \\ B &= \cap_{\delta \geq 0}\{\omega \in D^d([0,T]) : \sup_{|t-s|\leq e\delta} |\omega(t)-\omega(s)| > \frac{\varepsilon}{2}\}. \end{aligned}$$

Then

$$A \subset \overline{A} \subset B^o \subset B.$$

Here $\overline{A}$ and B^o means the closure of A and the interior of B in (D^d, d^0), respectively.

Proof. For any $\omega_0 \in A$ and $\omega \in D^d([0,T])$ with $d^0(\omega,\omega_0) < \frac{\varepsilon}{5}$, we have that $\omega \in B$. Actually, by definition, we have that there exists a continuous non-decreasing function $\lambda : [0,T] \to [0,T]$ such that $\lambda(0) = 0$, $\lambda(T) = T$, and

$$\begin{aligned} &\sup_{0\leq s<t\leq T} |\lambda(t)-\lambda(s)| \leq e^{\varepsilon/4}|t-s| \leq e|t-s|, \\ &\sup_{0\leq t\leq T} |\omega_0(t)-\omega(\lambda(t))| \leq \varepsilon/4. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sup_{|t-s|\leq e\delta} |\omega(t)-\omega(s)| \\ = &\sup_{|\lambda(t)-\lambda(s)|\leq e\delta} |\omega(\lambda(t))-\omega(\lambda(s))| \geq \sup_{|t-s|\leq\delta} |\omega(\lambda(t))-\omega(\lambda(s))| \\ \geq &\sup_{|t-s|\leq\delta} |\omega_0(t)-\omega_0(s)| - \sup_{0\leq t\leq T} |\omega_0(t)-\omega(\lambda(t))| - \sup_{0\leq s\leq T} |\omega_0(s)-\omega(\lambda(s))| \\ > &\ \varepsilon - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} = \varepsilon/2, \end{aligned}$$

which means that $\omega \in B$. This completes our proof. ∎

Now, we are ready to prove the continuity of process under cluster points of $\{\widetilde{M^i}(t)\}_{t\in[0,T]}$ under $P_m\}_{m\in(0,1)}$.

Lemma 6.4.3 *Any cluster point of $\{\{\widetilde{M^i}(t)\}_{t\in[0,T]}$ under $P_m\}_{m\in(0,1)}$ in D must have continuous canonical processes.*

Proof. Suppose there exists a sequence $m_n \to 0$ (as $n \to 0$) such that $P_{m_n} \circ (\widetilde{M^i})^{-1}$ (which we write as Q_n for the sake of simplicity) converge to Q_∞ as $n \to \infty$. we show that the canonical processes are continuous under Q_∞. Suppose not. Then there exists a constant $\varepsilon > 0$ such that

$$Q_\infty\Big(\cap_{\delta\geq 0} \{\omega \in D^d([0,T]) : \sup_{|t-s|\leq\delta} |\omega(t)-\omega(s)| > \varepsilon\}\Big) = a > 0.$$

Without loss of generality, we assume that $\varepsilon \leq 1$. Let A and B be the sets defined in Lemma 6.4.2. Then $Q_\infty(A) = a > 0$, so by Lemma 6.4.2, $Q_\infty(B^o) \geq a > 0$. Also,

B^0 is an open set, and $Q_n \to Q_\infty$ weakly in $\mathcal{P}(D^d)$, so we have $\liminf_{n\to\infty} Q_n(B^o) \geq Q_\infty(B^o)$. Therefore, there exists an $N \in \mathbf{N}$ such that for any $n \geq N$, $Q_n(B^o) \geq \frac{a}{2}$, hence $Q_n(B) \geq \frac{a}{2}$, which means that $P_{m_n}(\widetilde{M^i}$ has a jump greater than $\varepsilon/2) \geq \frac{a}{2}$. Since $m_n \to 0$ as $n \to \infty$, this makes a contradiction with the fact that all of the jumps of $\widetilde{M^i}$ under P_{m_n} are small than $4m_n^{1/2}\tau\|\nabla U_i\|_\infty$.

This completes the proof of our assertion. ∎

We next use Lemma 6.4.3 to show the following, which will be used later.

Lemma 6.4.4 *For any $\varepsilon > 0$, we have that*

$$\limsup_{\delta\to 0}\limsup_{m\to 0} P_m\Big(\sup_{0\leq s\leq t\leq T, |s-t|\leq\delta} |\widetilde{M^i}(t) - \widetilde{M^i}(s)| > \varepsilon\Big) = 0. \tag{6.4.5}$$

Proof. Let $a(m,\delta) = P_m\left(\sup_{0\leq s\leq t\leq T,|s-t|\leq\delta} |\widetilde{M^i}(t) - \widetilde{M^i}(s)| > \varepsilon\right)$. If

$$\limsup_{\delta\to 0}\limsup_{m\to 0} a(m,\delta) > 0,$$

then there exists a constant $a > 0$ and sequences $\delta_n \to 0$, $m_n \to 0$ (as $n \to \infty$) such that

$$P_{m_n}\Big(\sup_{0\leq s\leq t\leq T,|s-t|\leq\delta_n} |\widetilde{M^i}(t) - \widetilde{M^i}(s)| > \varepsilon\Big) \geq a \tag{6.4.6}$$

for any $n \in \mathbf{N}$. As before, let $Q_n = P_{m_n} \circ (\widetilde{M^i})^{-1}$, $n \in \mathbf{N}$. Also, let

$$\begin{aligned} A_n &= \Big\{\omega \in D^d([0,T]) : \sup_{0\leq s\leq t\leq T,|t-s|\leq\delta_n} |\omega(t)-\omega(s)| > \varepsilon\Big\}, \\ B_n &= \Big\{\omega \in D^d([0,T]) : \sup_{0\leq s\leq t\leq T,|t-s|\leq e\delta_n} |\omega(t)-\omega(s)| > \frac{\varepsilon}{2}\Big\}. \end{aligned}$$

Then $Q_n(A_n) > a$ by assumption, and by the same argument as in the proof of Lemma 6.4.2, we get that $A_n \subset \overline{A_n} \subset B_n^o \subset B_n$ for any $n \in \mathbf{N}$. Also, A_n is monotone decreasing with respect to n, hence for any $k \geq n$, we have that $Q_k(A_n) \geq Q_k(A_k) > a$. Therefore, since $\overline{A_n}$ is a closed set, we get that

$$Q_\infty(B_n) \geq Q_\infty(\overline{A_n}) \geq \limsup_{k\to\infty} Q_k(\overline{A_n}) \geq a.$$

This is true for any $n \in \mathbf{N}$, so since B_n is monotone decreasing with respect to n, we get that

$$Q_\infty(\cap_{n=1}^\infty B_n) \geq a,$$

which means that $Q_\infty(\{\text{canonical process has jump } \geq \varepsilon/2\}) \geq a$, and contradicts Lemma 6.4.3. This completes the proof of our assertion. ∎

We next deal with $\widetilde{\eta}_i(t)$. We first show that exists a constant independent of n such that

$$E^{P_m}[|\widetilde{\eta}_i(t)|^6] \leq Cm^{3/2}, \qquad t \in [0,T], m \in (0,1]. \tag{6.4.7}$$

In fact, notice that $\widetilde{\eta}_i(t)$ can be expressed as

$$\begin{aligned}\widetilde{\eta}_i(t) &= -\int_{[(t-4m^{1/2}\tau)\vee 0,t]\times E} \overline{N}(dr,dx,dv)\int_{(t-r)\wedge(4m^{1/2}\tau)}^{4m^{1/2}\tau} ds \\ & \qquad \nabla U_i(X_i(r\wedge\sigma)-\psi^0(m^{-1/2}s-2\tau,x,v;\vec{X}(r\wedge\sigma))).\end{aligned}$$

Also, in general, if Z is a Poisson random variable with mean a, then we have $E[Z-a]=0$, $E[(Z-a)^2]=E[(Z-a)^3]=a$, $E[(Z-a)^4]=3a^2+a$, and $E[(Z-a)^6]=15a^3+25a^2+a$. Therefore, by definition of Poisson point process and a simple calculation, there exists a global constant C such that

$$E\Big[\Big|\int fd\overline{N}\Big|^6\Big]\le C\Big[\Big(\int f^2d\overline{\lambda}\Big)^3+\Big(\int f^3d\overline{\lambda}\Big)^2+\Big(\int f^2d\overline{\lambda}\Big)\Big(\int f^4d\overline{\lambda}\Big)+\int f^6d\overline{\lambda}\Big]$$

for any measurable function f. Actually, for any simple function $f=\sum a_i 1_{A_i}$ with A_i mutually disjoint and the summation finite, we have

$$\begin{aligned}& E\Big[\Big|\int fd\overline{N}\Big|^6\Big]=E\Big[\Big|\int\sum a_i 1_{A_i}d\overline{N}\Big|^6\Big] \\ =& E\Big[\Big\{\sum a_i(N(A_i)-\overline{\lambda}(A_i))\Big\}^6\Big] \\ =& \sum_{i_1,\cdots,i_6} a_{i_1}\cdots a_{i_6}E\Big[\{N(A_{i_1})-\overline{\lambda}(A_{i_1})\}\cdots\{N(A_{i_6})-\overline{\lambda}(A_{i_6})\}\Big].\end{aligned}$$

Notice that $E\Big[N(A_{i_1})-\overline{\lambda}(A_{i_1})\Big]=0$, and $N(A_i)$ and $N(A_j)$ are independent if $i\neq j$. So the terms above are not 0 only if $\{i_1,\cdots,i_6\}$ has the forms $\{j_1,j_1,j_2,j_2,j_3,j_3\}$, $\{j_1,j_1,j_2,j_2,j_2,j_2\}$, $\{j_1,j_1,j_1,j_2,j_2,j_2\}$ or $\{j_1,j_1,j_1,j_1,j_1,j_1\}$ with $j_1\neq j_2\neq j_3$. Therefore,

$$\begin{aligned}& E\Big[\Big|\int fd\overline{N}\Big|^6\Big] \\ =& \sum_{i_1\neq i_2\neq i_3} a_{i_1}^2a_{i_2}^2a_{i_3}^2\overline{\lambda}(A_{i_1})\overline{\lambda}(A_{i_2})\overline{\lambda}(A_{i_3})+\sum_{i_1\neq i_2}a_{i_1}^2a_{i_2}^4\overline{\lambda}(A_{i_1})(3\overline{\lambda}(A_{i_2})^2+\overline{\lambda}(A_{i_2})) \\ & +\sum_{i_1\neq i_2}a_{i_1}^3a_{i_2}^3\overline{\lambda}(A_{i_1})\overline{\lambda}(A_{i_2})+\sum_{i_1}a_{i_1}^6\Big(15\overline{\lambda}(A_{i_1})^3+25\overline{\lambda}(A_{i_1})^2+\overline{\lambda}(A_{i_1})\Big).\end{aligned}$$

By choosing the constant $C>0$ properly, this gives us our assertion for simple functions. The assertion for general function f is now a easy result by approximation.

Let $A=\Big|\int_{(t-r)\wedge(4m^{1/2}\tau)}^{4m^{1/2}\tau}\nabla U_i(X_i(r\wedge\sigma)-\psi^0(m^{-1/2}s-2\tau,x,v;\vec{X}(r\wedge\sigma)))ds\Big|$. Then since $t-r\ge 0$, we get that $A\le 4m^{1/2}\tau\|\nabla U_i\|_\infty$. Therefore,

$$\begin{aligned}& E^{P_m}[|\widetilde{\eta}_i(t)|^6] \\ \le& C\Big[\Big(\int_{[(t-4m^{1/2}\tau)\vee 0,t]\times E}A^2m^{-1}\rho(\frac{1}{2}|v|^2)dr\nu(dx,dv)\Big)^3 \\ & +\Big(\int_{[(t-4m^{1/2}\tau)\vee 0,t]\times E}A^3m^{-1}\rho(\frac{1}{2}|v|^2)dr\nu(dx,dv)\Big)^2\end{aligned}$$

$$
\begin{aligned}
&+\Big(\int_{[(t-4m^{1/2}\tau)\vee 0,t]\times E} A^2 m^{-1}\rho(\frac{1}{2}|v|^2)dr\nu(dx,dv)\Big)\\
&\times\Big(\int_{[(t-4m^{1/2}\tau)\vee 0,t]\times E} A^4 m^{-1}\rho(\frac{1}{2}|v|^2)dr\nu(dx,dv)\Big)\\
&+\Big(\int_{[(t-4m^{1/2}\tau)\vee 0,t]\times E} A^6 m^{-1}\rho(\frac{1}{2}|v|^2)dr\nu(dx,dv)\Big)\Big]\\
\le\ & C\Big[\Big(4m^{1/2}\tau(4m^{1/2}\tau\|\nabla U_i\|_\infty)^2 m^{-1}(2(R_0+1))^d\int_{\mathbf{R}^d}\rho(\frac{1}{2}|v|^2)|v|dv\Big)^3\\
&+\Big(4m^{1/2}\tau(4m^{1/2}\tau\|\nabla U_i\|_\infty)^3 m^{-1}(2(R_0+1))^d\int_{\mathbf{R}^d}\rho(\frac{1}{2}|v|^2)|v|dv\Big)^2\\
&+\Big(4m^{1/2}\tau(4m^{1/2}\tau\|\nabla U_i\|_\infty)^2 m^{-1}(2(R_0+1))^d\int_{\mathbf{R}^d}\rho(\frac{1}{2}|v|^2)|v|dv\Big)\\
&\times\Big(4m^{1/2}\tau(4m^{1/2}\tau\|\nabla U_i\|_\infty)^4 m^{-1}(2(R_0+1))^d\int_{\mathbf{R}^d}\rho(\frac{1}{2}|v|^2)|v|dv\Big)\\
&+\Big(4m^{1/2}\tau(4m^{1/2}\tau\|\nabla U_i\|_\infty)^6 m^{-1}(2(R_0+1))^d\int_{\mathbf{R}^d}\rho(\frac{1}{2}|v|^2)|v|dv\Big)\Big],
\end{aligned}
$$

which gives us our assertion.

By (6.4.7),

$$
E^{P_m}\left[\sum_{k=0}^{[m^{-\frac{4}{3}}T]}|\widetilde{\eta}_i(km^{4/3})|^6\right]\le Cm^{3/2}m^{-4/3}T\to 0,\qquad \text{as } m\to 0.
$$

In particular,

$$
E^{P_m}\left[\max_{0\le k\le[m^{-\frac{4}{3}}T]}|\widetilde{\eta}_i(km^{4/3})|^6\right]\to 0,\qquad \text{as } m\to 0. \tag{6.4.8}
$$

Since $\widetilde{\eta}_i(t)$ is a cadlag process (with jumps $|\Delta\widetilde{\eta}_i(t)|\le 4m^{1/2}\tau\|\nabla U_i\|_\infty$), there exists a measurable $\xi=\xi_m:\Omega\to[0,T]$ such that

$$
|\widetilde{\eta}_i(\xi)|\vee|\widetilde{\eta}_i(\xi-)|=\sup_{0\le t\le T}|\widetilde{\eta}_i(t)|. \tag{6.4.9}
$$

Let $\widetilde{\xi}=m^{4/3}[m^{-4/3}\xi]$. Then $0\le\xi-\widetilde{\xi}\le m^{4/3}$. Therefore, by (6.4.1),

$$
\begin{aligned}
&E^{P_m}\Big[\Big|\widetilde{\widetilde{V_i^{03}}}(\xi)-\widetilde{\widetilde{V_i^{03}}}(\widetilde{\xi})\Big|\Big]\\
\le\ & E^{P_m}\Big[\int_0^T 1_{[\xi,\widetilde{\xi}]}(t)\left|\frac{d}{dt}\widetilde{\widetilde{V_i^{03}}}(t)\right|dt\Big]\\
\le\ & E^{P_m}\Big[\int_0^T 1_{[\xi,\widetilde{\xi}]}(t)dt\Big]^{1/2}\cdot\left\{\int_0^T E^{P_m}\left[\left|\frac{d}{dt}\widetilde{\widetilde{V_i^{03}}}(t)\right|^2\right]dt\right\}^{1/2}\\
\le\ & m^{2/3}(TCm^{-1/2})^{1/2}\to 0,\qquad \text{as } m\to 0.
\end{aligned} \tag{6.4.10}
$$

So we have for any $\varepsilon > 0$

$$\lim_{m\to 0} P_m(\left|\widetilde{\widetilde{V_i^{03}}}(\xi_m) - \widetilde{\widetilde{V_i^{03}}}(\widetilde{\xi_m})\right| > \varepsilon) = 0.$$

This combined with

$$\lim_{m\to 0} P_m(\left|\widetilde{M^i}(\xi_m) - \widetilde{M^i}(\widetilde{\xi_m})\right| > \varepsilon) = 0,$$

which came from Lemma 6.4.4 and the fact that $0 \le \xi - \widetilde{\xi} \le m^{4/3}$, implies that

$$\lim_{m\to 0} P_m(\left|\widetilde{\eta}_i(\xi_m) - \widetilde{\eta}_i(\widetilde{\xi_m})\right| > \varepsilon) = 0.$$

On the other hand, $\lim_{m\to 0} P_m(|\widetilde{\eta}_i(\widetilde{\xi_m})|) = 0$ by (6.4.8) and the definition of $\widetilde{\xi_m}$. Also, since the jumps of $\widetilde{\eta}_i$ satisfy $|\Delta\widetilde{\eta}_i| \le 4m^{1/2}\tau\|\nabla U_i\|_\infty$, we have $|\widetilde{\eta}_i(\xi_m-)| \le |\widetilde{\eta}_i(\xi_m)| + 4m^{1/2}\tau\|\nabla U_i\|_\infty$. These combined with (6.4.9) give us the following.

Lemma 6.4.5 $\lim_{m\to 0} P_m(\sup_{0\le t\le T}\left|\widetilde{\eta}_i(t)\right| > \varepsilon) = 0.$

We can show the following further more.

Lemma 6.4.6 $\lim_{m\to 0} E^{P_m}\Big[\sup_{0\le t\le T}\left|\widetilde{\eta}_i(t)\right| > \varepsilon\Big] = 0.$

Proof. The calculations used essentially are the same as in Lemma 6.4.5.

Since the jumps of $\widetilde{\eta}_i$ satisfy $|\Delta\widetilde{\eta}_i| \le 4m^{1/2}\tau\|\nabla U_i\|_\infty$, we have $|\widetilde{\eta}_i(\xi_m-)| \le |\widetilde{\eta}_i(\xi_m)| + 4m^{1/2}\tau\|\nabla U_i\|_\infty$. So by (6.4.9) and the definition of $\widetilde{\xi_m}$, we get that

$$\begin{aligned}
& E^{P_m}[\sup_{0\le t\le T}|\widetilde{\eta}_i(t)|] = E^{P_m}[|\widetilde{\eta}_i(\xi_m)| \vee |\widetilde{\eta}_i(\xi_m-)|] \\
\le\ & 4m^{1/2}\tau\|\nabla U_i\|_\infty + E^{P_m}[|\widetilde{\eta}_i(\xi_m)|] \\
\le\ & 4m^{1/2}\tau\|\nabla U_i\|_\infty + E^{P_m}\left[\max_{0\le k\le[m^{-\frac{4}{3}}T]}|\widetilde{\eta}_i(km^{4/3})|\right] + E^{P_m}[|\widetilde{\eta}_i(\xi_m) - \widetilde{\eta}_i(\widetilde{\xi_m})|].
\end{aligned}$$

The first term above converges to 0 as $m \to 0$ evidently. By (6.4.8), the second term above is also converging to 0 as $m \to 0$. So in order to show that $E^{P_m}[\sup_{0\le t\le T}|\widetilde{\eta}_i(t)|] \to 0$, it is sufficient to show that the third term $E^{P_m}[|\widetilde{\eta}_i(\xi_m) - \widetilde{\eta}_i(\widetilde{\xi_m})|]$ converges to 0.

We have

$$E^{P_m}[|\widetilde{\eta}_i(\xi_m) - \widetilde{\eta}_i(\widetilde{\xi_m})|] \le E^{P_m}\Big[\Big|\widetilde{\widetilde{V_i^{03}}}(\xi) - \widetilde{\widetilde{V_i^{03}}}(\widetilde{\xi})\Big|\Big] + E^{P_m}\Big[\Big|\widetilde{M^i}(\xi_m) - \widetilde{M^i}(\widetilde{\xi_m})\Big|\Big].$$

We already showed that $E^{P_m}\Big[\Big|\widetilde{\widetilde{V_i^{03}}}(\xi) - \widetilde{\widetilde{V_i^{03}}}(\widetilde{\xi})\Big|\Big] \to 0$ in (6.4.10). For the term $E^{P_m}\Big[\Big|\widetilde{M^i}(\xi_m) - \widetilde{M^i}(\widetilde{\xi_m})\Big|\Big]$, we first notice that since $0 \le \xi - \widetilde{\xi} \le m^{4/3}$ by definition, (6.4.5) gives us that

$$\lim_{m\to 0} P_m(\left|\widetilde{M^i}(\xi_m) - \widetilde{M^i}(\widetilde{\xi_m})\right| > \varepsilon) = 0. \tag{6.4.11}$$

This is true for any $\varepsilon > 0$. We have by (6.4.4) that for any $\varepsilon > 0$,

$$\begin{aligned}
& E^{P_m}\Big[\Big|\widetilde{M^i}(\xi_m) - \widetilde{M^i}(\widetilde{\xi_m})\Big|\Big] \\
\leq\ & E^{P_m}\Big[\Big|\widetilde{M^i}(\xi_m) - \widetilde{M^i}(\widetilde{\xi_m})\Big|, \Big|\widetilde{M^i}(\xi_m) - \widetilde{M^i}(\widetilde{\xi_m})\Big| > \varepsilon\Big] + \varepsilon \\
\leq\ & E^{P_m}\Big[\Big|\widetilde{M^i}(\xi_m) - \widetilde{M^i}(\widetilde{\xi_m})\Big|^2\Big]^{1/2} P(\Big|\widetilde{M^i}(\xi_m) - \widetilde{M^i}(\widetilde{\xi_m})\Big| > \varepsilon)^{1/2} + \varepsilon \\
\leq\ & 2E^{P_m}\Big[\Big(\sup_{t\in[0,T]} |\widetilde{M^i}(t)|\Big)^2\Big]^{1/2} P(\Big|\widetilde{M^i}(\xi_m) - \widetilde{M^i}(\widetilde{\xi_m})\Big| > \varepsilon)^{1/2} + \varepsilon \\
\leq\ & 4\sqrt{CT} P(\Big|\widetilde{M^i}(\xi_m) - \widetilde{M^i}(\widetilde{\xi_m})\Big| > \varepsilon)^{1/2} + \varepsilon.
\end{aligned}$$

This combined with (6.4.11) gives us that

$$\lim_{m\to 0} E^{P_m}\Big[\Big|\widetilde{M^i}(\xi_m) - \widetilde{M^i}(\widetilde{\xi_m})\Big|\Big] = 0,$$

and completes the proof of the fact that

$$\lim_{m\to 0} E^{P_m}[|\tilde{\eta}_i(\xi_m) - \tilde{\eta}_i(\widetilde{\xi_m})|] = 0,$$

hence completes the proof of our assertion. ∎

Combining all of the results in Sections 6.1 ∼ 6.6, we get Lemma 5.3.1, with

$$\begin{aligned}
M^i(t) &= -\widetilde{M^i}(t\wedge\sigma), \\
P_i^{*1}(t) &= -V_i^{02}(t) - \widetilde{V_i^{05}}(t), \\
\eta_i(t) &= -V_i^1(t) + V_i^{04}(t) - \tilde{\eta}_i(t\wedge\sigma).
\end{aligned}$$

6.5 Proof of Lemma 5.3.3

Let b_i be as in Lemma 5.3.3. First notice that

$$\begin{aligned}
& \sum_{i=1}^N M_i b_i(\vec{X}(t\wedge\sigma)) V_i(t\wedge\sigma) - \sum_{i=1}^N M_i b_i(\vec{X}(0)) V_i(0) \\
=\ & \sum_{i=1}^N M_i \int_0^{t\wedge\sigma} (\nabla b_i(\vec{X}(s))\vec{V}(s)) \cdot V_i(s) ds + \sum_{i=1}^N M_i \int_0^{t\wedge\sigma} b_i(\vec{X}(s)) \frac{d}{ds} V_i(s) ds.
\end{aligned}$$

It is trivial that

$$\Big|(\nabla b_i(\vec{X}(t\wedge\sigma))\vec{V}(t\wedge\sigma)) \cdot V_i(t\wedge\sigma)\Big| \leq \|\nabla b_i\|_\infty n^2,$$

so the first term above is tight in D^d by Theorem 5.1.7. For the second integral, we have by Lemma 5.3.1 and our assumption that

$$\begin{aligned}
& \sum_{i=1}^N M_i \int_0^{t\wedge\sigma} b_i(\vec{X}(s)) \frac{d}{ds} V_i(s) ds \\
=\ & \sum_{i=1}^N \int_0^{t\wedge\sigma} b_i(\vec{X}(s)) \frac{d}{ds} P^{*1}(s) ds + \int_0^{t\wedge\sigma} b_i(\vec{X}(s)) dM_i(s) + \int_0^{t\wedge\sigma} b_i(\vec{X}(s)) d\eta_i(s).
\end{aligned}$$

We discuss these three terms in the following.

Since b_i is bounded, we have by Lemma 5.3.1 (2) that

$$\sup_{m\in(0,1]}\sup_{0\le t\le T} E^{P_m}\Big[|b_i(\vec{X}(t))\frac{d}{dt}P_i^{*1}(t)|\Big] < \infty.$$

Therefore, the first term $\int_0^{t\wedge\sigma} b_i(\vec{X}(s))\frac{d}{ds}P^{*1}(s)ds$ is tight.

Let $N_i(t)$ be the second term,

$$N_i(t) = \int_0^{t\wedge\sigma} b_i(\vec{X}(s))dM_i(s).$$

Then since $\{M_i(t)\}_{t\ge0}$ is a martingale and b_i is bounded, we have that $\{N_i(t)\}_{t\ge0}$ is also a $\mathcal{F}_t$-martingale. Also, since $|\Delta M_i(t)| \le Cm^{1/2}$ and $|d\langle M_i^k, M_j^\ell\rangle_t| \le Cdt$ P_m-*a.s.* by Lemma 5.3.1 (3), we have that $|\Delta N_i(t)| \le Cm^{1/2}\|b_i\|_\infty$ and $|d\langle N_i^k, N_j^l\rangle_t| \le \|b_i\|_\infty\|b_j\|_\infty Cdt$. Therefore, same as in the proof of the proof of Lemmas 6.4.1 and 6.4.3, we get that the second term $\{N_i(t)\}$ is also tight, with the limit processes continuous.

For the third term, we have that

$$\begin{aligned}
&\int_0^{t\wedge\sigma} b_i(\vec{X}(s))d\eta_i(s)\\
=\ & b_i(\vec{X}(t\wedge\sigma))\cdot\eta_i(t\wedge\sigma) - \int_0^{t\wedge\sigma}(\nabla b_i(\vec{X}(s))\vec{V}(s))\cdot\eta_i(s)ds.
\end{aligned}$$

Therefore,

$$\sup_{0\le t\le T}\Big|\int_0^{t\wedge\sigma} b_i(\vec{X}(s))d\eta_i(s)\Big| \le (\|b_i\|_\infty + n\|\nabla b_i\|_\infty T)\sup_{0\le t\le T}|\eta_i(t)|.$$

So by Lemma 5.3.1 (4),

$$E^{P_m}\Big[\sup_{0\le t\le T}\Big|\int_0^{t\wedge\sigma} b_i(\vec{X}(s))d\eta_i(s)\Big|\Big] \to 0, \quad \text{as } m\to 0.$$

This completes the proof of Lemma 5.3.3.

6.6 Proof of Lemma 5.3.2

To prove the first assertion, notice that for any $t\ge0$, we have by assumption and integration by parts formula that

$$\begin{aligned}
&\int_0^{t\wedge\widetilde{\sigma_D}} m^{-1/2}|\nabla_i\widetilde{U}(\vec{X}(s))|ds\\
=\ & \int_0^{t\wedge\widetilde{\sigma_D}} g(\vec{X}(s))\cdot\Big(m^{-1/2}\nabla_i\widetilde{U}(\vec{X}(s))\Big)ds\\
=\ & g(\vec{X}(t\wedge\widetilde{\sigma_D}))\int_0^{t\wedge\widetilde{\sigma_D}} m^{-1/2}\nabla_i\widetilde{U}(\vec{X}(s))ds\\
&\quad -\int_0^{t\wedge\widetilde{\sigma_D}} ds(\nabla g(\vec{X}(s))\vec{V}(s))\int_0^s m^{1/2}\nabla_i\widetilde{U}(\vec{X}(r))dr.
\end{aligned}$$

Therefore, by Lemma 5.3.1 (1), we get

$$\begin{aligned}
&\int_0^{T\wedge\sigma\wedge\widetilde{\sigma_D}} m^{-1/2}|\nabla_i\widetilde{U}(\vec{X}(s))|ds \\
= \;& g(\vec{X}(T\wedge\widetilde{\sigma_D}))\Big(- M_i(V_i(T\wedge\sigma\wedge\widetilde{\sigma_D}) - V_i(0)) + P_i^{*0}(T\wedge\widetilde{\sigma_D}) + P_i^{*1}(T\wedge\widetilde{\sigma_D})\Big) \\
& - \int_0^{T\wedge\sigma\wedge\widetilde{\sigma_D}} (\nabla g(\vec{X}(t))\vec{V}(t)) \\
& \quad \times\Big\{ - M_i(V_i(t\wedge\sigma\wedge\widetilde{\sigma_D}) - V_i(0)) + P_i^{*0}(t\wedge\widetilde{\sigma_D}) + P_i^{*1}(t\wedge\widetilde{\sigma_D})\Big\}dt \\
\le \;& (\|g\|_\infty + \|\nabla g\|_\infty\cdot dnT)\Big\{2M_i n + \sup_{0\le t\le T}|P_i^{*0}(t)| + \sup_{0\le t\le T}|P_i^{*1}(t)|\Big\}.
\end{aligned}$$

Therefore, we get our first assertion by Lemma 5.3.1 (2), (4) and (6.4.4).

Before giving the proof of the second assertion, let us first prepare the following.

Lemma 6.6.1 *1.* $\lim_{m\to 0} E^{P_m}\Big[\sup_{t\in[0,T]}|\eta_i(t)|^2\Big] = 0$,

2. $\lim_{m\to 0} E^{P_m}\Big[\sup_{t\in[0,T]}|\int_0^t \eta_i(s)dM_i(s)|^2\Big] = 0$

Proof. We use the same notations as in Sections 6.1 ~ 6.4. Then $\eta_i(t) = -V_i^1(t) + V_i^{04}(t) - \widetilde{\eta}_i(t)$. So to show the first assertion, we only need to show that the sup of each of these three terms converges to 0 in $L^2(P_m)$.

As before, $V_i^1(t) = V_i^{10}(t) + V_i^{11}(t)$. For $V_i^{10}(t)$, we have by (6.2.3) and (6.2.4) that

$$E^{P_m}\Big[\sup_{t\in[0,T]}|V_i^{10}(t)|^2\Big] \le \Big(\frac{1}{2}C_1 n(4m^{1/2}\tau)^2\Big)^2(C_2 m^{-1/2} + C_2^2 m^{-1}),$$

which converges to 0 as $m\to 0$. For $V_i^{11}(t)$, by (6.2.6) and (6.2.7), we have

$$\begin{aligned}
& E^{P_m}\Big[\sup_{t\in[0,T]}|V_i^{11}(t)|^2\Big] \\
\le \;& E^{P_m}\Big[\Big\{\int_0^{T\wedge\sigma} 1_{[0,4m^{1/2}\tau)}(s)\Big|\int_{\mathbf{R}\times E}\nabla U_i(X_i(0) - \widetilde{\varphi}^0(m^{-1/2}s, \Psi(m^{-1/2}r,x,v);\vec{X}(0))) \\
& \quad (\mu_\omega(dr,dx,dv) - \lambda(dr,dx,dv))\Big|ds\Big\}^2\Big] \\
\le \;& E^{P_m}\Big[4m^{1/2}\tau\int_0^{T\wedge\sigma} 1_{[0,4m^{1/2}\tau)}(s)\Big|\int_{\mathbf{R}\times E}\nabla U_i(X_i(0) - \widetilde{\varphi}^0(m^{-1/2}s, \Psi(m^{-1/2}r,x,v);\vec{X}(0))) \\
& \quad (\mu_\omega(dr,dx,dv) - \lambda(dr,dx,dv))\Big|^2 ds\Big] \\
\le \;& (4m^{1/2}\tau)^2\int_0^{4m^{1/2}\tau} dsE^{P_m}\Big[\Big|\int_{\mathbf{R}\times E}\nabla U_i(X_i(0) - \widetilde{\varphi}^0(m^{-1/2}s, \Psi(m^{-1/2}r,x,v);\vec{X}(0))) \\
& \quad (\mu_\omega(dr,dx,dv) - \lambda(dr,dx,dv))\Big|^2\Big] \\
\le \;& (4m^{1/2}\tau)^2 C_3 m^{-1/2},
\end{aligned}$$

which converges to 0 as $m\to 0$. This completes the discussion about the term $V_i^1(t)$.

The discussion about the term $V_i^{04}(t)$ is similar. By definition and (6.3.2), we get

$$
\begin{aligned}
& E^{P_m}[\sup_{0\le t\le T}|V_i^{04}(t)|^2] \\
\le\ & E^{P_m}\Big[4m^{1/2}\tau\int_0^{4m^{1/2}\tau}\Big|\int_{[2m^{1/2}\tau,\infty)\times E}\nabla U_i(X_i(\widetilde{r})-\psi^0(m^{-1/2}(s-r),x,v;\vec{X}(\widetilde{r}))) \\
& (\mu_\omega(dr,dx,dv)-\lambda(dr,dx,dv))\Big|^2 ds\Big] \\
\le\ & (4m^{1/2}\tau)^2 C_4 m^{-1/2},
\end{aligned}
$$

which converges to 0 as $m\to 0$.

Finally, we deal with the term $\widetilde{\eta}_i(t)$. With the same notations as before, we have

$$
\begin{aligned}
& E^{P_m}\Big[\sup_{t\in[0,T]}|\widetilde{\eta}_i(t)|^2\Big] \\
=\ & E^{P_m}\Big[|\widetilde{\eta}_i(\xi_m)|^2\vee|\widetilde{\eta}_i(\xi_m-)|^2\Big] \\
\le\ & 2\Big(4m^{1/2}\tau\|\nabla u_i\|_\infty\Big)^2+2E^{P_m}\Big[|\widetilde{\eta}_i(\xi_m)|^2\Big] \\
\le\ & 2\Big(4m^{1/2}\tau\|\nabla u_i\|_\infty\Big)^2+4E^{P_m}\left[\max_{0\le k\le[m^{-\frac{4}{3}}T]}|\widetilde{\eta}_i(km^{4/3})|^2\right]+4E^{P_m}[|\widetilde{\eta}_i(\xi_m)-\widetilde{\eta}_i(\widetilde{\xi_m})|^2].
\end{aligned}
$$

By (6.4.8), the second term above converges to 0 as $m\to 0$. So it suffices to show that the third term $E^{P_m}[|\widetilde{\eta}_i(\xi_m)-\widetilde{\eta}_i(\widetilde{\xi_m})|^2]$ also converge to 0. We have

$$
E^{P_m}[|\widetilde{\eta}_i(\xi_m)-\widetilde{\eta}_i(\widetilde{\xi_m})|^2]\le 2E^{P_m}\Big[\Big|\widetilde{\widetilde{V_i^{03}}}(\xi)-\widetilde{\widetilde{V_i^{03}}}(\widetilde{\xi})\Big|^2\Big]+2E^{P_m}\Big[\Big|M^i(\xi_m)-M^i(\widetilde{\xi_m})\Big|^2\Big].
$$

By (6.4.1),

$$
\begin{aligned}
& E^{P_m}\Big[\Big|\widetilde{\widetilde{V_i^{03}}}(\xi)-\widetilde{\widetilde{V_i^{03}}}(\widetilde{\xi})\Big|^2\Big] \\
\le\ & E^{P_m}\Big[\Big(\int_0^T 1_{[\xi,\widetilde{\xi}]}(t)\left|\frac{d}{dt}\widetilde{\widetilde{V_i^{03}}}(t)\right|dt\Big)^2\Big] \\
\le\ & E^{P_m}\Big[\Big(\int_0^T 1_{[\xi,\widetilde{\xi}]}(t)dt\Big)\cdot\Big(\int_0^T\Big|\frac{d}{dt}\widetilde{\widetilde{V_i^{03}}}(t)\Big|^2dt\Big)\Big] \\
\le\ & m^{4/3}\int_0^T E^{P_m}\Big[\Big|\frac{d}{dt}\widetilde{\widetilde{V_i^{03}}}(t)\Big|^2\Big]dt \\
\le\ & m^{4/3}TCm^{-1/2}\to 0, \qquad \text{as } m\to 0.
\end{aligned}
$$

Finally, for the term $E^{P_m}\Big[\Big|M^i(\xi_m)-M^i(\widetilde{\xi_m})\Big|^2\Big]$, we first prepare the following result: There exists a constant $C>0$ (not depending on m) such that

$$
E^{P_m}\Big[\sup_{t\in[0,T]}|M^i(t)|^4\Big]\le C.
$$

Actually, by the general fact that $E\Big[\Big|\int f d\overline{N}\Big|^4\Big] \le 3\Big(\int f^2 d\overline{\lambda}\Big)^2 + \int f^4 d\overline{\lambda}$, we get with the help of Doob's inequality that

$$
\begin{aligned}
& E^{P_m}\Big[\sup_{t\in[0,T]} |M^i(t)|^4\Big] \le (4/3)^4 E^{P_m}\Big[|M^i(T)|^4\Big] \\
= \;& (4/3)^4\Big[3\Big\{\int_{(0,T]\times E} \overline{\lambda}(dr,dx,dv)\Big(\int_0^{4m^{1/2}\tau} ds \\
& \nabla U_i\Big(X_i(r\wedge\sigma) - \psi(m^{-1/2}s - 2\tau, x, v; \vec{X}(r\wedge\sigma))\Big)\Big)^2\Big\}^2 \\
& + \int_{(0,T]\times E} \overline{\lambda}(dr,dx,dv)\Big(\int_0^{4m^{1/2}\tau} ds \\
& \nabla U_i\Big(X_i(r\wedge\sigma) - \psi(m^{-1/2}s - 2\tau, x, v; \vec{X}(r\wedge\sigma))\Big)\Big)^4\Big] \\
\le \;& (4/3)^4\Big[3\Big\{\int_{(0,T]\times E} m^{-1}\rho\Big(\frac{1}{2}|v|^2\Big)dr\nu(dx,dv)\Big(4m^{1/2}\tau\|\nabla U_i\|_\infty 1_{[0,R_0+1)}(|x|)\Big)^2\Big\}^2 \\
& + \int_{(0,T]\times E} m^{-1}\rho\Big(\frac{1}{2}|v|^2\Big)dr\nu(dx,dv)\Big(4m^{1/2}\tau\|\nabla U_i\|_\infty 1_{[0,R_0+1)}(|x|)\Big)^4\Big] \\
\le \;& (4/3)^4\Big[3(4\tau\|\nabla U_i\|_\infty)^4\Big(T(2(R_0+1))^{d-1}\int_{\mathbf{R}^d}\rho\Big(|\frac{1}{2}|v|^2\Big)|v|dv\Big)^2 \\
& + (4\tau\|\nabla U_i\|_\infty)^4 mT(2(R_0+1))^{d-1}\int_{\mathbf{R}^d}\rho\Big(|\frac{1}{2}|v|^2\Big)|v|dv\Big],
\end{aligned}
$$

with the right hand side above bounded by a finite global constant $C > 0$.

Therefore,

$$
\begin{aligned}
& E^{P_m}\Big[\Big|M^i(\xi_m) - M^i(\widetilde{\xi_m})\Big|^2\Big] \\
\le \;& E^{P_m}\Big[\Big|M^i(\xi_m) - M^i(\widetilde{\xi_m})\Big|^2, \Big|M^i(\xi_m) - M^i(\widetilde{\xi_m})\Big| > \varepsilon\Big] + \varepsilon^2 \\
\le \;& E^{P_m}\Big[\Big|M^i(\xi_m) - M^i(\widetilde{\xi_m})\Big|^4\Big]^{1/2} P(\Big|M^i(\xi_m) - M^i(\widetilde{\xi_m})\Big| > \varepsilon)^{1/2} + \varepsilon^2 \\
\le \;& 4E^{P_m}\Big[\sup_{t\in[0,T]} |M^i(t)|^4\Big]^{1/2} P(\Big|M^i(\xi_m) - M^i(\widetilde{\xi_m})\Big| > \varepsilon)^{1/2} + \varepsilon^2 \\
\le \;& 4C^{1/2} P(\Big|M^i(\xi_m) - M^i(\widetilde{\xi_m})\Big| > \varepsilon)^{1/2} + \varepsilon.
\end{aligned}
$$

With the help of Lemma 6.4.4, by taking first $\varepsilon > 0$ small enough then $m > 0$ small enough, this implies $E^{P_m}\Big[\Big|M^i(\xi_m) - M^i(\widetilde{\xi_m})\Big|^2\Big] \to 0$ as $m \to 0$, so completes the proof of the fact that $E^{P_m}\Big[\sup_{t\in[0,T]} |\widetilde{\eta}_i(t)|^2\Big]$ converge to 0 as $m \to 0$, hence completes the proof of the first assertion of this lemma.

We next show the second assertion. Since M_i is a martingale, the first assertion implies that $\int_0^t \eta_i(s)dM_i(s)$ is also a martingale. By the definition of $M_i(t)$, (see Lemma 7.3.1 for the details and the proof), if we let

$$
A_i(r) = A_i(r,x,v) = \int_{-2\tau}^{2\tau} \nabla U_i(X_i(r) - \psi^0(u,x,v;\vec{X}(r)))du,
$$

then we have that

$$\Big[M_i, M_i\Big]_s = m \int_{[0,s\wedge\sigma]\times E} A_i(r,x,v)^2 N(dr,dx,dv).$$

Therefore, with the help of Doob's inequality, we get that

$$\begin{aligned}
& E^{P_m}\Big[\sup_{t\in[0,T]}\Big|\int_0^t \eta_i(s)dM_i(s)\Big|^2\Big] \le 4E^{P_m}\Big[\Big|\int_0^T \eta_i(s)dM_i(s)\Big|^2\Big] \\
= \;& 4E^{P_m}\Big[\int_0^T \eta_i(s)^2 d[M_i,M_i]_s\Big] \\
= \;& 4mE^{P_m}\Big[\int_0^{T\wedge\sigma} \eta_i(s)^2 \int_E A_i(r,x,v)^2 N(ds,dx,dv)\Big] \\
\le \;& 4(4\tau\|\nabla U_i\|_\infty)^2 \int_{[0,T]\times E} E^{P_m}\Big[\eta_i(s)^2\Big] 1_{[0,R_0+1)}(|x|)\rho\Big(\frac{1}{2}|v|^2\Big)\nu(dx,dv)dr \\
= \;& 4(4\tau\|\nabla U_i\|_\infty)^2 T(2(R_0+1))^{d-1}\int_{\mathbf{R}^d}\rho\Big(|\frac{1}{2}|v|^2\Big)|v|dv E^{P_m}\Big[\eta_i(s)^2\Big].
\end{aligned}$$

This combined with the first assertion of this lemma completes the proof of our second assertion. ∎

We next show the second assertion of Lemma 5.3.2. First, by Lemma 5.3.1, we have

$$\begin{aligned}
& m^{-1/2}\Big(\tilde U(\vec X(t\wedge\sigma)) - \tilde U(\vec X(0))\Big) + \sum_{i=1}^N \Big\{\frac{M_i}{2}|V_i(t\wedge\sigma)|^2 + \frac{1}{M_i}\int_0^{t\wedge\sigma}\eta_i(s)dM_i(s)\Big\} \\
= \;& \sum_{i=1}^N \frac{M_i}{2}|V_i(0)|^2 + \sum_{i=1}^N \Big\{\int_0^{t\wedge\sigma} m^{-1/2}\nabla_i \tilde U(\vec X(s))\cdot V_i(s)ds \\
& \qquad + \int_0^{t\wedge\sigma} M_i\Big(\frac{d}{ds}V_i(s)\Big)\cdot V_i(s)ds + \frac{1}{M_i}\int_0^{t\wedge\sigma}\eta_i(s)dM_i(s)\Big\} \\
= \;& \sum_{i=1}^N \frac{M_i}{2}|V_i(0)|^2 + \sum_{i=1}^N \Big\{\int_0^{t\wedge\sigma} V_i(s)\frac{d}{ds}P_i^{*1}(s)ds \\
& \qquad + \int_0^{t\wedge\sigma} V_i(s)\frac{d}{ds}P_i^{*0}(s)ds + \frac{1}{M_i}\int_0^{t\wedge\sigma}\eta_i(s)dM_i(s)\Big\} \\
= \;& \sum_{i=1}^N \frac{M_i}{2}|V_i(0)|^2 + \sum_{i=1}^N \Big\{\int_0^{t\wedge\sigma} V_i(s)\frac{d}{ds}P_i^{*1}(s)ds + \int_0^{t\wedge\sigma} V_i(s)dM_i(s) \\
& \qquad + \int_0^{t\wedge\sigma} V_i(s)d\eta_i(s) + \frac{1}{M_i}\int_0^{t\wedge\sigma}\eta_i(s)dM_i(s)\Big\}.
\end{aligned}$$

Since $|V_i(t\wedge\sigma)| \le n$ by the definition of σ, we have by Lemma 5.3.1 (2) that

$$\sup_{m\in(0,1]}\sup_{0\le t\le T} E^{P_m}\Big[|\vec V_i(t\wedge\sigma)\frac{d}{dt}P_i^{*1}(t)|^2\Big] < \infty.$$

Therefore, by Theorem 5.1.7, we get that $\int_0^{t\wedge\sigma}\vec V_i(s)\frac{d}{ds}P_i^{*1}(s)ds$ is tight for $m\in(0,1]$.

For the term $\int_0^t 1_{[0,\sigma]}(s)V_i(s)dM_i(s)$, recall that $\sigma = \inf\{t > 0; |\vec{V}_i(t)| = n\}$, so σ is a $\mathcal{F}_t$-stopping time. Therefore, since $\{M_i(s)\}_s$ is a martingale, we get that

$$N_i(t) \equiv \int_0^t 1_{[0,\sigma]}(s)V_i(s)dM_i(s)$$

is also a $\mathcal{F}_t$-martingale. Notice that

$$\langle N_i\rangle_t = \sum_{k,l=1}^{d} \int_0^{t\wedge\sigma} V_i^k(s\wedge\sigma)V_i^l(s\wedge\sigma)d\langle M_i^k, M_i^l\rangle(s).$$

So by Lemma 5.3.1 (3), we get that

$$d\langle N_i\rangle_t \le n^2d^2Cdt, \qquad |\Delta N_t| \le n^2d^2Cm^{1/2}.$$

Therefore, same as in the proof of Lemmas 6.4.1 and 6.4.3, we get that $\{N_i(t)\}_t$ is tight for $m \in (0, 1]$, with canonical limit processes continuous.

We next show that $\int_0^{t\wedge\sigma} V_i(s)d\eta_i(s) + \int_0^{t\wedge\sigma} \eta_i(s)dM_i(s)$ is negligible. Notice that by Lemma 5.3.1 (3),

$$\begin{aligned}
&\int_0^{t\wedge\sigma} V_i(s)d\eta_i(s) + \frac{1}{M_i}\int_0^{t\wedge\sigma} \eta_i(s)dM_i(s)\\
= {}& V_i(t\wedge\sigma)\eta_i(t) - \int_0^{t\wedge\sigma} \eta_i(s)dV_i(s) + \frac{1}{M_i}\int_0^{t\wedge\sigma} \eta_i(s)dM_i(s)\\
= {}& V_i(t\wedge\sigma)\eta_i(t) - \frac{1}{M_i}\int_0^{t\wedge\sigma} \eta_i(s)\Big(\frac{d}{ds}P_i^{*1}(s)\Big)ds - \frac{1}{M_i}\int_0^{t\wedge\sigma} \eta_i(s)d\eta_i(s)\\
&\quad + \frac{1}{M_i}\int_0^{t\wedge\sigma} \eta_i(s)m^{-1/2}\nabla_i\widetilde{U}(\vec{X}(s))ds\\
= {}& V_i(t\wedge\sigma)\eta_i(t) - \frac{1}{M_i}\eta_i(t)^2 + \frac{1}{M_i}[\eta_i,\eta_i]_t + \frac{1}{M_i}\int_0^{t\wedge\sigma} \eta_i(s)\Big(m^{-1/2}\nabla_i\widetilde{U}(\vec{X}(s)) - \frac{d}{ds}P_i^{*1}(s)\Big)ds.
\end{aligned}$$

Since $|V_i(t\wedge\sigma)| \le n$, Lemma 6.6.1 (1) gives us that

$$\lim_{m\to 0} E^{P_m}\Big[\sup_{t\in[0,T\wedge\sigma]}\Big|V_i(t\wedge\sigma)\eta_i(t) - \frac{1}{M_i}\eta_i(t)^2 + \frac{1}{M_i}[\eta_i,\eta_i]_t\Big|\Big] = 0.$$

Also, for any $\varepsilon > 0$, we have for any $A > 0$,

$$\begin{aligned}
&P_m\Big(\sup_{t\in[0,T\wedge\sigma]}\Big|\int_0^{t\wedge\sigma} \eta_i(s)\Big(m^{-1/2}\nabla_i\widetilde{U}(\vec{X}(s)) - \frac{d}{ds}P_i^{*1}(s)\Big)ds\Big| > \varepsilon\Big)\\
\le {}& P_m\Big(\sup_{s\in[0,T\wedge\sigma]}|\eta_i(s)| > A\Big)\\
&\quad + P_m\Big(\sup_{s\in[0,T\wedge\sigma]}\int_0^{t\wedge\sigma}\Big(\Big|m^{-1/2}\nabla_i\widetilde{U}(\vec{X}(s))\Big| + \Big|\frac{d}{ds}P_i^{*1}(s)\Big|\Big)ds > \frac{\varepsilon}{A}\Big)\\
\le {}& \frac{1}{A}E^{P_m}\Big[\sup_{s\in[0,T\wedge\sigma]}|\eta_i(s)|\Big]\\
&\quad + \frac{A}{\varepsilon}E^{P_m}\Big[\int_0^{T\wedge\sigma}\Big(\Big|m^{-1/2}\nabla_i\widetilde{U}(\vec{X}(s))\Big| + \Big|\frac{d}{ds}P_i^{*1}(s)\Big|\Big)ds\Big].
\end{aligned}$$

Combining this with Lemma 5.3.1 (2) (4) and Lemma 5.3.2 (1), by taking first $A > 0$ small enough then $m > 0$ small enough, we get that

$$\lim_{m\to 0} P_m\Big(\sup_{t\in[0,T\wedge\sigma]}\Big|\int_0^{t\wedge\sigma}\eta_i(s)\Big(m^{-1/2}\nabla_i\widetilde{U}(\vec{X}(s)) - \frac{d}{ds}P_i^{*1}(s)\Big)ds\Big| > \varepsilon\Big) = 0$$

for any $\varepsilon > 0$. This completes the proof of the fact that $m^{-1/2}\Big(\widetilde{U}(\vec{X}(t\wedge\sigma)) - \widetilde{U}(\vec{X}(0))\Big) + \sum_{i=1}^N \frac{M_i}{2}|V_i(t\wedge\sigma)|^2 + \frac{1}{M_i}\int_0^{t\wedge\sigma}\eta_i(s)dM_i(s)$ under P_m is tight, with canonical limit processes continuous. This combined with Lemma 6.6.1 (2) gives us our second assertion of Lemma 5.3.2. ∎

Chapter 7

Convergence Until "Near"

In Chapter 6, we showed that $M_iV_i(t\wedge\sigma_n) + m^{-1/2}\int_0^{t\wedge\sigma_n}\nabla_i\widetilde{U}(\vec{X}_s)ds$ under P_m is tight in $\mathcal{P}(C([0,T];\mathbf{R}^d))$.

Let $\sigma_0(\omega) = \inf\{t > 0; \min_{i\neq j}\{|X_i(t) - X_j(t)| - (R_i + R_j)\} \leq 0\}$. Then by (5.4.9), $\nabla_i\widetilde{U}(\vec{X}_s) = 0$ for any $s \leq \sigma_0$. Therefore, $V_i(t\wedge\sigma_n\wedge\sigma_0)$ under P_m have (at least one) subsequence that converges in $\mathcal{P}(C([0,T];\mathbf{R}^d))$.

In this section, we give the proof of the fact that, any cluster point of it is the stopped diffusion process as given in Chapter 1. For sake of simplicity, in this chapter, we let $\sigma = \sigma_n \wedge \sigma_0$. We use the notation $D_0 = (supp\widetilde{U})^C \subset \mathbf{R}^{dN}$.

7.1 Decomposition

First, since we do not have enough information about the term $\eta_i(t)$, we use the following to convert the problem to the one without $\eta_i(t)$. Let

$$\begin{aligned} Y_i(t) &= V_i(0) + M_i^{-1}\Big(M_i(t) + P_i^{*1}(t) - m^{-1/2}\int_0^{t\wedge\sigma_n}\nabla_i\widetilde{U}(\vec{X}_s)ds\Big) \\ &= V_i(t) - M_i^{-1}\eta_i(t), \qquad i = 1,\cdots,N, \end{aligned}$$

and let $Y(t) = (Y_1(t),\cdots,Y_N(t))$. Then we have the following.

Lemma 7.1.1 *For any $f \in C_0^\infty(D_0\times\mathbf{R}^{dN})$, we have that $\{f(X_{t\wedge\sigma_n}, V_{t\wedge\sigma_n})\}_t$ and $\{f(X_{t\wedge\sigma_n}, Y_{t\wedge\sigma_n})\}_t$ converge or not for $m\to 0$ at the same time, and when converge, they have the same limit.*

proof. Just notice that if we let f_V denote the partial differential of f with respect to the second variables, then $\|f_V\|_\infty < \infty$ and

$$|f(X_{t\wedge\sigma_n}, V_{t\wedge\sigma_n}) - f(X_{t\wedge\sigma_n}, Y_{t\wedge\sigma_n})| \leq \|f_V\|_\infty \max_{i=1,\cdots,N}\frac{1}{M_i}\sup_{s\in[0,T]}|\eta_i(s)|,$$

hence

$$
\begin{aligned}
& E^{P_m}\Big[\sup_{0\le t\le T}|f(X_{t\wedge\sigma_n},V_{t\wedge\sigma_n})-f(X_{t\wedge\sigma_n},Y_{t\wedge\sigma_n})|\Big]\\
\le\ & \|f_V\|_\infty \max_{i=1,\cdots,N}\frac{1}{M_i}E^{P_m}\Big[\sup_{s\in[0,T]}|\eta_i(s)|\Big],
\end{aligned}
$$

which, by Lemma 5.3.1, converges to 0 as $m\to 0$. ∎

In the same way, we can also convert between $f_V(X_s,Y_s)$ and $f_V(X_s,V_s)$ when considering limit.

By Lemma 7.1.1, we only need to consider the problem with V_t substituted by Y_t. For any $f\in C_0^\infty(D_0\times\mathbf{R}^{dN})$, (notice that all the terms involved except $M_i(t)$ are continuous with respect to t), since

$$
\int_0^{t\wedge\sigma} f_V(X_s,Y_s)\cdot\nabla\widetilde{U}(\vec{X}_s)ds=0,
$$

we have by Ito's formula that

$$
\begin{aligned}
& f(X_{t\wedge\sigma},Y_{t\wedge\sigma})-f(X_0,Y_0)\\
=\ & \int_0^{t\wedge\sigma} f_X(X_s,Y_s)\cdot V_s ds+\sum_{i=1}^N\frac{1}{M_i}\int_0^{t\wedge\sigma} f_{V_i}(X_s,Y_s)\cdot dM_i(s)+(II)+(III)+(IV),
\end{aligned}
$$

with

$$
\begin{aligned}
(II) &= \sum_{i=1}^N\frac{1}{M_i}\int_0^{t\wedge\sigma} f_V(X_s,Y_s)\cdot dP_i^{*1}(s),\\
(III) &= \sum_{l_1,l_2=1}^N\sum_{k_1,k_2=1}^d\frac{1}{2M_{l_1}M_{l_2}}\int_{0+}^{t\wedge\sigma} f_{V_{l_1}^{k_1}V_{l_2}^{k_2}}(X_s,Y_s)d[M_{l_1}^{k_1},M_{l_2}^{k_2}]_s,\\
(IV) &= \sum_{0<s\le t\wedge\sigma}\Big\{f(X_s,Y_s)-f(X_s,Y_{s-})-\sum_{l=1}^N f_V(X_s,Y_{s-})\cdot\Delta M_l(s)\frac{1}{M_l}\\
&\quad -\sum_{l_1,l_2=1}^N\frac{1}{2}f_{V_{l_1}V_{l_2}}(X_s,Y_{s-})(\Delta M_{l_1}(s))(\Delta M_{l_2}(s))\frac{1}{M_{l_1}M_{l_2}}\Big\}.
\end{aligned}
$$

It is easy to see that after taking limit $m\to 0$, the term $\int_0^{t\wedge\sigma} f_X(X_s,Y_s)\cdot V_s ds$ above gives us the term $\sum_{i=1}^N\sum_{k=1}^d V_i^k\frac{\partial}{\partial X_i^k}$ of the generator L. Also, the term $\sum_{i=1}^N\frac{1}{M_i}\int_0^{t\wedge\sigma} f_{V_i}(X_s,Y_s)\cdot dM_i(s)$ is a martingale since $\{M_i(t)\}_t$, $i=1,\cdots,N$, are martingales and f_V is bounded. In the following sections, we study what do the terms (II), (III), (IV) correspond to, respectively.

7.2 The term (IV)

Lemma 7.2.1 $\lim_{m\to 0}E^{P_m}\Big[\sup_{0\le t\le T}|(IV)|\Big]=0.$

Proof. Since $f \in C_0^\infty(D_0 \times \mathbf{R}^{dN})$, we have that the third partial derivatives $f_{V_{l_1},V_{l_2},V_{l_3}}$, $l_1, l_2, l_3 = 1, \cdots, N$, are bounded. Also, the jumps satisfy $|\Delta M_i(s)| \leq Cm^{1/2}$. Therefore, by Taylor's expansion, there exists a constant $C > 0$ (depending on f) such that

$$\begin{aligned} |(IV)| &\leq \sum_{l_1,l_2,l_3=1}^{N} \sum_{k_1,k_2,k_3=1}^{d} \|f_{V_{l_1}^{k_1},V_{l_2}^{k_2},V_{l_3}^{k_3}}\|_\infty |\Delta M_{l_1}^{k_1}(s)||\Delta M_{l_2}^{k_2}(s)||\Delta M_{l_3}^{k_3}(s)| \\ &\leq Cm^{1/2} \sum_{l=1}^{N} |\Delta M_l(s)|^2. \end{aligned}$$

Therefore, to complete the proof of this lemma, it is sufficient to show that $E^{P_m}[\sum_{0<s\leq T\wedge\sigma} |\Delta M_i(s)|^2]$ is bounded for $m > 0$. We do it from now on.

We have by the definition of $\{M_i(t)\}$ that

$$\begin{aligned} M_i(t) = &- \int_{(0,t\wedge\sigma]\times E} \overline{N}(dr, dx, dv) \int_0^{4m^{1/2}\tau} du \\ &\nabla U_i\Big(X_i(r\wedge\sigma) - \psi^0(m^{-1/2}u - 2\tau, x, v; \vec{X}(r\wedge\sigma))\Big), \end{aligned}$$

so

$$\begin{aligned} \sum_{0<s\leq t\wedge\sigma} |\Delta M_i(s)|^2 &= \int_{(0,t\wedge\sigma]\times E} N(dr, dx, dv)\Big(\int_0^{4m^{1/2}\tau} du \\ &\qquad \nabla U_i\Big(X_i(r\wedge\sigma) - \psi^0(m^{-1/2}u - 2\tau, x, v; \vec{X}(r\wedge\sigma))\Big)\Big)^2. \end{aligned}$$

Notice that by definition, N is the Poisson point process with intensity $m^{-1}\rho(\frac{1}{2}|v|^2)dr\nu(dx, dv)$. Therefore, since

$$|\nabla U_i\Big(X_i(r\wedge\sigma) - \psi^0(m^{-1/2}u - 2\tau, x, v; \vec{X}(r\wedge\sigma))\Big)| \leq \|\nabla U_i\|_\infty 1_{[0,R_0+1)}(|x|),$$

we get that

$$\begin{aligned} &E^{P_m}[\sum_{0<s\leq T\wedge\sigma} |\Delta M_i(s)|^2] \\ = \; &E^{P_m}\Big[\int_{(0,T\wedge\sigma]\times E} N(dr, dx, dv)\Big(\int_0^{4m^{1/2}\tau} du \\ &\qquad \nabla U_i\Big(X_i(r\wedge\sigma) - \psi^0(m^{-1/2}u - 2\tau, x, v; \vec{X}(r\wedge\sigma))\Big)\Big)^2\Big] \\ \leq \; &\int_{[0,T]\times E} m^{-1}\rho\Big(\frac{1}{2}|v|^2\Big)1_{[0,R_0+1)}(|x|)(4m^{1/2}\tau)^2\|\nabla U_i\|_\infty^2 dr\nu(dx, dv) \\ \leq \; &16\tau^2\|\nabla U_i\|_\infty^2 T(2(R_0+1))^{d-1}\int_{\mathbf{R}^d} \rho\Big(\frac{1}{2}|v|^2\Big)|v|dv, \end{aligned}$$

which is finite by our assumption.

This completes the proof of our assertion. ∎

7.3 The term (III)

For the term (III), we will show that after taking $m \to 0$ it is corresponding to the term $\sum_{i,j=1}^{N}\sum_{k,l=1}^{d} a_{ik,jl}(\vec{X})\frac{\partial^2}{\partial V_i^k \partial V_j^l}$.

We first calculate the quadratic variance of M_i. For $l = 1, \cdots, N$ and $k = 1, \cdots, d$, let

$$A_{lk}(r) = A_{lk}(r,x,v) = \int_{-2\tau}^{2\tau} \nabla_k U_l(X_l(r) - \psi^0(u,x,v;\vec{X}(r)))du.$$

Then we have the following.

Lemma 7.3.1 *For any $l_1, l_2 = 1, \cdots, N$ and $k_1, k_2 = 1, \cdots, d$, we have that*

$$\left[M_{l_1}^{k_1}, M_{l_2}^{k_2}\right]_s = m\int_{[0,s\wedge\sigma]\times E} A_{l_1k_1}(r,x,v)A_{l_2k_2}(r,x,v)N(dr,dx,dv).$$

Proof. Since the methods are totally the same, for the sake of simplicity, we give the proof only for the case $l_1 = l_2 = i$ and $k_1 = k_2 = k$. Write $A_{ik}(r,x,v) = A(r,x,v)$.

By the definition of quadratic variance, we have that $\{[M_i^k, M_i^k]_t\}_t$ is the only process such that $(M_i^k)^2 - [M_i^k, M_i^k]_t$ is a martingale with jumps $\Delta[M_i^k, M_i^k]_t = (\Delta M_i^k(t))^2$. Using variable change $u' = m^{-1/2}u - 2\tau$, it is easy to see that $A(r \wedge \sigma, x, v) = m^{-1/2}\int_0^{4m^{1/2}\tau} du \nabla_k U_i\Big(X_i(r\wedge\sigma) - \psi^0(m^{-1/2}u - 2\tau, x, v; \vec{X}(r\wedge\sigma))\Big)$. So

$$M_i^k(t) = -m^{1/2}\int_{(0,t\wedge\sigma]\times E} \overline{N}(dr,dx,dv)A(r,x,v),$$

with $\overline{N}(dr,dx,dv) = N(dr,dx,dv) - \overline{\lambda}(dr,dx,dv)$, $N(dr,dx,dv)$ the Poisson point process with intensity $\overline{\lambda}(dr,dx,dv) = m^{-1}\rho(\frac{1}{2}|v|^2)dr\nu(dx,dv)$. Hence

$$E^{P_m}[M_i^k(t)^2] = mE^{P_m}\Big[\int_{(0,t\wedge\sigma]\times E} \overline{\lambda}(dr,dx,dv)A(r,x,v)^2\Big],$$

which is a continuous process. Also,

$$\sum_{0<s\le t}(\Delta M_i^k(s))^2 = m\int_{(0,t\wedge\sigma]\times E} N(dr,dx,dv)A(r,x,v)^2,$$

hence

$$E^{P_m}\Big[\sum_{0<s\le t}(\Delta M_i^k(s))^2\Big] = mE^{P_m}\Big[\int_{(0,t\wedge\sigma]\times E}\overline{\lambda}(dr,dx,dv)A(r,x,v)^2\Big].$$

Combining the above, we get that

$$\begin{aligned}[M_i^k, M_i^k]_t &= E^{P_m}[M_i^k(t)^2] + \sum_{0<s\le t}(\Delta M_i^k(s))^2 - E^{P_m}\Big[\sum_{0<s\le t}(\Delta M_i^k(s))^2\Big] \\ &= m\int_{(0,t\wedge\sigma]\times E} N(dr,dx,dv)A(r,x,v)^2.\end{aligned}$$

This completes the proof of our assertion. ∎

By Lemma 7.3.1, we get that

$$\begin{aligned}&\int_0^{t\wedge\sigma} f_{V_{l_1}^{k_1}V_{l_2}^{k_2}}(X_s,Y_s)d\Big[M_{l_1}^{k_1},M_{l_2}^{k_2}\Big]_s\\&= m\int_0^{t\wedge\sigma} f_{V_{l_1}^{k_1}V_{l_2}^{k_2}}(X_s,Y_s)A_{l_1k_1}(s,x,v)A_{l_2k_2}(s,x,v)N(ds,dx,dv).\end{aligned}$$

Let

$$\begin{aligned}(III') &= \sum_{l_1,l_2=1}^{N}\sum_{k_1,k_2=1}^{d}\frac{1}{2M_{l_1}M_{l_2}}\int_{0+}^{t\wedge\sigma} f_{V_{l_1}^{k_1}V_{l_2}^{k_2}}(X_s,Y_s)\\&\qquad\Big(\int_E A_{l_1k_1}(s,x,v)A_{l_2k_2}(s,x,v)\rho(\frac{1}{2}|v|^2)\nu(dx,dv)\Big)ds.\end{aligned}$$

Then we have the following.

Lemma 7.3.2 $\lim_{m\to 0} E^{P_m}\Big[\sup_{0\le t\le T}\Big|(III)-(III')\Big|\Big]=0.$

Proof. We have by definition that $N(ds,dx,dv)$ is the Poisson point process with intensity $\overline{\lambda}(ds,dx,dv)=m^{-1}\rho(\frac{1}{2}|v|^2)ds\nu(dx,dv)$. Also, notice that there exists a constant $C>0$ such that $|A_{lk}(s\wedge\sigma,x,v)|\le C1_{[0,R_0+1]}(|x|)$. So by Doob's inequality, for any $l_1,l_2=1,\cdots,N$ and $k_1,k_2=1,\cdots,d$, there exist constants $C_1,C_2>0$ such that

$$\begin{aligned}&E^{P_m}\Big[\sup_{0\le t\le T}\Big|\int_0^{t\wedge\sigma}\int_E mf_{V_{l_1}^{k_1}V_{l_2}^{k_2}}(X_s,Y_s)A_{l_1k_1}(s)A_{l_2k_2}(s)(N-\overline{\lambda})(ds,dx,dv)\Big|\Big]\\ \le\ & E^{P_m}\Big[\sup_{0\le t\le T}\Big|\int_0^{t\wedge\sigma}\int_E mf_{V_{l_1}^{k_1}V_{l_2}^{k_2}}(X_s,Y_s)A_{l_1k_1}(s)A_{l_2k_2}(s)(N-\overline{\lambda})(ds,dx,dv)\Big|^2\Big]^{1/2}\\ \le\ & 2E^{P_m}\Big[\Big(\int_0^{T\wedge\sigma}\int_E mf_{V_{l_1}^{k_1}V_{l_2}^{k_2}}(X_s,Y_s)A_{l_1k_1}(s)A_{l_2k_2}(s)(N-\overline{\lambda})(ds,dx,dv)\Big)^2\Big]^{1/2}\\ =\ & 2E^{P_m}\Big[\int_0^{T\wedge\sigma}\int_E(mf_{V_{l_1}^{k_1}V_{l_2}^{k_2}}(X_s,Y_s)A_{l_1k_1}(r)A_{l_2k_2}(s))^2\overline{\lambda}(ds,dx,dv)\Big|\Big]^{1/2}\\ \le\ & 2C_1m\Big(\int_0^T\int_E 1_{[0,R_0+1]}(|x|)m^{-1}\rho\Big(\frac{1}{2}|v|^2\Big)ds\nu(dx,dv)\Big)^{1/2}\\ \le\ & C_2m^{1/2}.\end{aligned}$$

This completes the proof of our assertion. ∎

Lemma 7.3.2 implies that after taking limit $m\to 0$, (III) is corresponding to the term $\sum_{i,j=1}^N\sum_{k,l=1}^d a_{ik,jl}(\vec{X})\frac{\partial^2}{\partial V_i^k\partial V_j^l}$.

7.4 The term (II)

In this section, we deal with the term (II). We use the same notations as in Chapter 6. Then P_i^{*1} is given by $P_i^{*1}(t)=-V_i^{02}(t)-\widetilde{V}_i^{05}(t)$. Recall that $\widetilde{f}_i(s,r,x,v)=$

$\nabla U_i\Big(X_i(s) - x(s,\Psi(r,x,m^{-1/2}v))\Big) - \nabla U_i\Big(X_i(\tilde{r}) - \psi^0(m^{-1/2}(s-r),x,v;\vec{X}(\tilde{r})\Big)$. So we have the decomposition

$$
\begin{aligned}
& -\int_0^{t\wedge\sigma} f_V(X_s,Y_s)\cdot dP_i^{*1}(s) \\
= \; & \int_0^{t\wedge\sigma} f_V(X_s,Y_s)1_{[4m^{1/2}\tau,\sigma)}(s)\Big(\int_{\mathbf{R}\times E}\tilde{f}_i(s,r,x,v)\mu_\omega(dr,dx,dv)\Big)ds \\
& +\int_0^{t\wedge\sigma} f_V(X_s,Y_s)1_{[4m^{1/2}\tau,\sigma)}(s)\Big(\int_{\mathbf{R}\times E}\widetilde{F_i^{05}}(s,r,x,v)\lambda(dr,dx,dv)\Big)ds \\
= \; & \int_0^{t\wedge\sigma} f_V(X_s,Y_s)1_{[4m^{1/2}\tau,\sigma)}(s) \\
& \qquad \Big(\int_{\mathbf{R}\times E}\Big(\tilde{f}_i(s,r,x,v)+\widetilde{F_i^{05}}(s,r,x,v)\Big)\mu_\omega(dr,dx,dv)\Big)ds \\
& +\int_0^{t\wedge\sigma} f_V(X_s,Y_s)1_{[4m^{1/2}\tau,\sigma)}(s) \\
& \qquad \Big(\int_{\mathbf{R}\times E}\widetilde{F_i^{05}}(s,r,x,v)(\lambda(dr,dx,dv)-\mu_\omega(dr,dx,dv))\Big)ds. \qquad (7.4.1)
\end{aligned}
$$

We first show in the following lemma that the second term on the right hand side above is negligible.

Lemma 7.4.1

$$
\begin{aligned}
& \lim_{m\to 0} E^{P_m}\Big[\sup_{0\le t\le T}\Big|\int_0^{t\wedge\sigma} f_V(X_s,Y_s)1_{[4m^{1/2}\tau,\sigma)}(s) \\
& \qquad \Big(\int_{\mathbf{R}\times E}\widetilde{F_i^{05}}(s,r,x,v)(\lambda(dr,dx,dv)-\mu_\omega(dr,dx,dv))\Big)ds\Big|\Big] = 0.
\end{aligned}
$$

Proof. Let

$$
\begin{aligned}
& R(s,r,x,v) \\
& = -\widetilde{F_i^{05}}(s,r,x,v) - \nabla^2 U_i(X_i(\tilde{r}) - \psi^0(m^{-1/2}(s-r),x,v;\vec{X}(\tilde{r}))) \\
& \quad \times\Big\{X_i(s) - X_i(\tilde{r}) - \psi^0(m^{-1/2}(s-r),x,v;\vec{X}(s)) + \psi^0(m^{-1/2}(s-r),x,v;\vec{X}(\tilde{r}))\Big\}.
\end{aligned}
$$

Then we have the decomposition

$$
\begin{aligned}
& \int_0^{t\wedge\sigma} f_V(X_s,Y_s)1_{[4m^{1/2}\tau,\sigma)}(s) \\
& \qquad \Big(\int_{\mathbf{R}\times E}\widetilde{F_i^{05}}(s,r,x,v)(\lambda(dr,dx,dv)-\mu_\omega(dr,dx,dv))\Big)ds \\
= \; & (5I)+(5II)+(5III),
\end{aligned}
$$

where

$$
(5I) \; = \; \int_0^{t\wedge\sigma} f_V(X_s,Y_s)1_{[4m^{1/2}\tau,\sigma)}(s)
$$

$$
\begin{aligned}
& \times\Big(\int_{\mathbf{R}\times E} R(s,r,x,v)(\mu_\omega(dr,dx,dv)-\lambda(dr,dx,dv))\Big)ds,\\
(5II) \;=\; & \int_0^{t\wedge\sigma} f_V(X_s,Y_s)1_{[4m^{1/2}\tau,\sigma)}(s)\\
& \Big(\int_{\mathbf{R}\times E}\nabla^2 U_i(X_i(\tilde r)-\psi^0(m^{-1/2}(s-r),x,v;\vec X(\tilde r)))\\
& \Big\{(X_i(s)-X_i(\tilde r)-(s-r)V_i(\tilde r))-\Big(\psi^0(m^{-1/2}(s-r),x,v;\vec X(s))\\
& -\psi^0(m^{-1/2}(s-r),x,v;\vec X(\tilde r)+(s-\tilde r)\vec V(\tilde r))\Big)\Big\}\\
& \qquad (\mu_\omega(dr,dx,dv)-\lambda(dr,dx,dv))\Big)ds,\\
(5III) \;=\; & \int_0^{t\wedge\sigma} f_V(X_s,Y_s)1_{[4m^{1/2}\tau,\sigma)}(s)\\
& \Big(\int_{\mathbf{R}\times E} g(r,s,x,v)(\mu_\omega(dr,dx,dv)-\lambda(dr,dx,dv))\Big)ds,
\end{aligned}
$$

with

$$
\begin{aligned}
g(r,s,x,v) \;=\; & \nabla^2 U_i(X_i(\tilde r)-\psi^0(m^{-1/2}(s-r),x,v;\vec X(\tilde r)))\\
& \Big\{(s-r)V_i(\tilde r)-\psi^0(m^{-1/2}(s-r),x,v;\vec X(\tilde r)+(s-\tilde r)\vec V(\tilde r))\\
& \quad +\psi^0(m^{-1/2}(s-r),x,v;\vec X(\tilde r))\Big\}.
\end{aligned}
$$

So Lemma 7.4.1 follows from the following three lemmas.

Lemma 7.4.2 $\lim_{m\to 0} E^{P_m}[\sup_{0\le t\le T}|(5III)|]=0$.

Proof. First notice that $0\le\tilde r\le\sigma$, hence $|\vec V(\tilde r)|\le n$. Therefore, we have by Lemma 6.3.4 that there exists a constants $C, C_1>0$ such that

$$
\begin{aligned}
|g(r,s,x,v)| &\le \|\nabla^2 U_i\|_\infty 1_{[0,2m^{1/2}\tau)}(|s-r|)C|s-r||\vec V(\tilde r)|1_{[0,R_0+1)}(|x|)\\
&\le C_1 1_{[0,2m^{1/2}\tau)}(|s-r|)2m^{1/2}\tau 1_{[0,R_0+1)}(|x|).
\end{aligned}
$$

Also, it is easy to see that $g(r,s,x,v)$ is $\mathcal{F}_r$-measurable. Therefore, there exists a constant $C>0$ such that

$$
\begin{aligned}
& E^{P_m}[\sup_{0\le t\le T}|(5III)|]\\
\le\; & \|f_V\|_\infty E^{P_m}\Big[\int_0^{T\wedge\sigma} ds\Big|\int_{\mathbf{R}\times E} g(r,s,x,v)(\mu_\omega(dr,dx,dv)-\lambda(dr,dx,dv))\Big|\Big]\\
\le\; & \|f_V\|_\infty\int_0^T dsE^{P_m}\Big[\Big|\int_{\mathbf{R}\times E} g(r,s,x,v)(\mu_\omega(dr,dx,dv)-\lambda(dr,dx,dv))\Big|^2\Big]^{1/2}\\
=\; & \|f_V\|_\infty\int_0^T ds\Big(\int_{\mathbf{R}\times E} E^{P_m}[|g(r,s,x,v)|^2]\lambda(dr,dx,dv)\Big)^{1/2}\\
\le\; & \|f_V\|_\infty\int_0^T ds\Big\{\int_{\mathbf{R}\times E}\Big(C_1 1_{[0,2m^{1/2}\tau)}(|s-r|)2m^{1/2}\tau 1_{[0,R_0+1)}(|x|)\Big)^2
\end{aligned}
$$

$$m^{-1}\rho\Big(\frac{1}{2}|v|^2+\sum_{i=1}^N U_i(x-m^{-1/2}rv-X_{i,0})\Big)dr\nu(dx,dv)\Big\}^{1/2}$$
$$\leq\ Cm^{1/4},$$

which converges to 0 as $m\to 0$. ∎

Lemma 7.4.3 $\lim_{m\to 0}E^{P_m}[\sup_{0\leq t\leq T}|(5I)|]=0$.

Proof. By the definition of $R(s,r,x,v)$, Taylor expansion and Lemma 6.3.4, we get that there exists a constant $C>0$ such that

$$\begin{aligned}
&|R(s,r,x,v)|\\
\leq\ &\|\nabla^3U_i\|_\infty\Big|\Big(X_i(s)-X_i(\widetilde{r})\Big)\\
&\quad-\Big(\psi^0(m^{-1/2}(s-r),x,v;\vec{X}(s))-\psi^0(m^{-1/2}(s-r),x,v;\vec{X}(\widetilde{r}))\Big)\Big|^2\\
\leq\ &C|X_i(s)-X_i(\widetilde{r})|^2 1_{[0,2m^{1/2}\tau)}(|s-r|)1_{[0,R_0+1)}(|x|).
\end{aligned}$$

Notice that when $|s-r|\leq 2m^{1/2}\tau$, since $s,\widetilde{r}\in[0,\sigma]$, we get that $|X_i(s)-X_i(\widetilde{r})|\leq n|s-\widetilde{r}|\leq n4m^{1/2}\tau$. So the above gives us that

$$|R(s,r,x,v)|\leq(4n\tau)^2Cm1_{[0,2m^{1/2}\tau)}(|s-r|)1_{[0,R_0+1)}(|x|).$$

Therefore, there exists a constant $C>0$ such that

$$\begin{aligned}
&E^{P_m}[\sup_{0\leq t\leq T}|(5I)|]\\
\leq\ &2\|f_V\|_\infty\int_0^T ds\int_{\mathbf{R}\times E}E^{P_m}\Big[1_{[0,\sigma]}(s)|R(s,r,x,v)|\Big]\lambda(dr,dx,dv)\\
\leq\ &2\|f_V\|_\infty\int_0^T ds\int_{\mathbf{R}\times E}(4n\tau)^2Cm1_{[0,2m^{1/2}\tau)}(|s-r|)1_{[0,R_0+1)}(|x|)\\
&\qquad m^{-1}\rho\Big(\frac{1}{2}|v|^2+\sum_{i=1}^N U_i(x-m^{-1/2}rv-X_{i,0})\Big)dr\nu(dx,dv)\\
\leq\ &Cm^{1/2},
\end{aligned}$$

which converges to 0 as $m\to 0$. ∎

Lemma 7.4.4 $\lim_{m\to 0}E^{P_m}[\sup_{0\leq t\leq T}|(5II)|]=0$.

Proof. First, by Lemma 6.3.4, there exists a constant $C>0$ such that

$$\begin{aligned}
&\Big|\psi^0(m^{-1/2}(s-r),x,v;\vec{X}(s))-\psi^0(m^{-1/2}(s-r),x,v;\vec{X}(\widetilde{r})+(s-\widetilde{r})\vec{V}(\widetilde{r}))\Big|\\
&\leq C|\vec{X}(s)-\vec{X}(\widetilde{r})-(s-\widetilde{r})\vec{V}(\widetilde{r}))|.
\end{aligned}$$

Notice that if $s \geq 4m^{1/2}\tau$ and $|s-r| \leq 2m^{1/2}\tau$, then by the definition of $\widetilde{r}$, we always have that $\widetilde{r} \leq s$. Therefore,

$$\vec{X}(s) - \vec{X}(\widetilde{r}) - (s-\widetilde{r})\vec{V}(\widetilde{r}) = \int_{\widetilde{r}}^{s} (V_i(u) - V_i(\widetilde{r}))du.$$

For any $l \leq s \leq \sigma$, we have that $|X_i(l) - X_j(l)| \geq R_i + R_j$, $i \neq j$, which implies that $\nabla_i \widetilde{U}(\vec{X}(l)) = 0$, $i = 1, \cdots, N$. Therefore, We have by Lemma 5.3.1 that

$$\begin{aligned}
& V_i(u) - V_i(\widetilde{r}) \\
= \ & \frac{1}{M_i}\Big(\int_{\widetilde{r}}^{u} \frac{d}{dl}P_i^{*1}(l)dl + \eta_i(u) - \eta_i(\widetilde{r}) + M_i(u) - M_i(\widetilde{r}) - m^{-1/2}\int_{\widetilde{r}}^{u} \nabla_i \widetilde{U}(\vec{X}(l))dl\Big) \\
= \ & \frac{1}{M_i}\Big(\int_{\widetilde{r}}^{u} \frac{d}{dl}P_i^{*1}(l)dl + \eta_i(u) - \eta_i(\widetilde{r}) + M_i(u) - M_i(\widetilde{r})\Big).
\end{aligned}$$

Let

$$a_m = 4m^{1/2}\tau + E^{P_m}[\sup_{0\leq u\leq T} |\eta_i(u)|] + (4m^{1/2}\tau)^{1/2}.$$

Then by Lemma 5.3.1, we have $a_m \to 0$ as $m \to 0$. Notice that for $s \in [0,\sigma]$, $|s-r| \leq 2m^{1/2}\tau$ implies $|s-\widetilde{r}| \leq 4m^{1/2}\tau$. Therefore, we get by Lemma 5.3.1 and (6.4.2) that there exists a contant $C > 0$ such that

$$\begin{aligned}
& E^{P_m}\Big[|\vec{X}(s) - \vec{X}(\widetilde{r}) - (s-\widetilde{r})\vec{V}(\widetilde{r}))|\Big] \\
\leq \ & \sum_{i=1}^{N} E^{P_m}\Big[\Big|\int_{\widetilde{r}}^{s} du \int_{\widetilde{r}}^{u} \frac{d}{dl}P_i^{*1}(l)dl\Big| \\
& \qquad + \Big|\int_{\widetilde{r}}^{s} du(\eta_i(u) - \eta_i(\widetilde{r}))\Big| + \Big|\int_{\widetilde{r}}^{s} du(M_i(u) - M_i(\widetilde{r}))\Big|\Big] \\
\leq \ & N(4m^{1/2}\tau)^2 \sup_{m\leq 1}\sup_{t\in[0,T]} E^{P_m}\Big[\Big|\frac{d}{dl}P_i^{*1}(l)\Big|^2\Big]^{1/2} \\
& \qquad + N4m^{1/2}\tau 2\sup_{0\leq u\leq T} E^{P_m}[|\eta_i(u)|] + N\int_{\widetilde{r}}^{s} du E^{P_m}\Big[\Big|M_i(u) - M_i(\widetilde{r})\Big|^2\Big]^{1/2} \\
\leq \ & C(4m^{1/2}\tau)a_m.
\end{aligned}$$

Before going further, we notice that $E^{P_m}[\int f d\mu_\omega] = E^{P_m}[\int f d\lambda_m] = \int E[f]d\lambda_m$ by Corollary 4.2.5. Actually, for any $A \in \mathcal{E}_0$ and $B \in \mathcal{B}(M)$, taking $S(\omega) = 1_A(\omega)$ and $f(x) = 1_B(x)$, Corollary 4.2.5 implies that

$$\int\int 1_B(x)1_A(\omega)\mu_\omega(dx)P(d\omega) = \int\int 1_B(x)1_A(\omega)\nu(dx)P(d\omega).$$

Therefore, by the linearity and limit convergence theorem, we get our assertion.

So there exist constants $C_1, C_2, C_3 > 0$ such that

$$E^{P_m}[\sup_{0\leq t\leq T} |(5II)|]$$

$$
\begin{aligned}
&\leq \|f_V\|_\infty \int_0^T ds C_1 E^{P_m}\Big[1_{[0,\sigma)}(s)\int_{\mathbf{R}\times E}\|\nabla^2 U_i\|_\infty |\vec{X}(s)-\vec{X}(\tilde{r})-(s-\tilde{r})\vec{V}(\tilde{r}))| \\
&\qquad 1_{[0,2m^{1/2}\tau)}(|s-r|)1_{[0,R_0+1)}(|x|)(\mu_\omega(dr,dx,dv)+\lambda_m(dr,dx,dv))\Big] \\
&= 2C_1\|f_V\|_\infty\|\nabla^2 U_i\|_\infty \int_0^T ds\int_{\mathbf{R}\times E} E^{P_m}\Big[1_{[0,\sigma)}(s)|\vec{X}(s)-\vec{X}(\tilde{r})-(s-\tilde{r})\vec{V}(\tilde{r}))|\Big] \\
&\qquad \times 1_{[0,2m^{1/2}\tau)}(|s-r|)1_{[0,R_0+1)}(|x|)\lambda_m(dr,dx,dv)) \\
&\leq C_2\int_0^T ds\int_{\mathbf{R}\times E}(6m^{1/2}\tau)a_m 1_{[0,2m^{1/2}\tau)}(|s-r|)1_{[0,R_0+1)}(|x|) \\
&\qquad \times m^{-1}\rho\Big(\frac{1}{2}|v|^2+\sum_{i=1}^N U_i(x-m^{-1/2}rv-X_{i,0})\Big)dr\nu(dx,dv) \\
&\leq C_3 a_m,
\end{aligned}
$$

which converges to 0 as $m\to 0$. This completes the proof of Lemma 7.4.4. ∎

Lemmas 7.4.2, 7.4.3 and 7.4.4 complete the proof of Lemma 7.4.1. ∎

We next deal first term on the right hand side of (7.4.1). We first make the decomposition

$$
\begin{aligned}
&\widetilde{f_i}(s,r,x,v)+\widetilde{F_i^{05}}(s,r,x,v) \\
&= \nabla U_i(X_i(s)-x(s,\Psi(r,x,m^{-1/2}v)))-\nabla U_i(X_i(s)-\psi^0(m^{-1/2}(s-r),x,v;\vec{X}(s))) \\
&= \widetilde{f_i^1}(s,r,x,v)+\widetilde{f_i^2}(s,r,x,v)+\widetilde{f_i^3}(s,r,x,v),
\end{aligned}
$$

with

$$
\begin{aligned}
&\widetilde{f_i^1}(s,r,x,v) \\
&= \nabla U_i(X_i(s)-x(s,\Psi(r,x,m^{-1/2}v)))-\nabla U_i(X_i(s)-\psi^0(m^{-1/2}(s-r),x,v;\vec{X}(s))) \\
&\quad -\frac{1}{2}\nabla^2 U_i(X_i(s)-\psi^0(m^{-1/2}(s-r),x,v;\vec{X}(s))) \\
&\qquad \cdot\Big(x(s,\Psi(r,x,m^{-1/2}v))-\psi^0(m^{-1/2}(s-r),x,v;\vec{X}(s))\Big), \\
&\widetilde{f_i^2}(s,r,x,v) \\
&= \frac{1}{2}\nabla^2 U_i(X_i(s)-\psi^0(m^{-1/2}(s-r),x,v;\vec{X}(s))) \\
&\qquad \cdot\Big(x(s,\Psi(r,x,m^{-1/2}v))-\psi^0(m^{-1/2}(s-r),x,v;\vec{X}(s)) \\
&\qquad\quad -m^{1/2}Z(m^{-1/2}(s-r),x,v,\vec{X}(s),\vec{V}(s),-m^{-1/2}(s-r))\Big), \\
&\widetilde{f_i^3}(s,r,x,v) \\
&= \frac{1}{2}\nabla^2 U_i(X_i(s)-\psi^0(m^{-1/2}(s-r),x,v;\vec{X}(s))) \\
&\qquad \cdot m^{1/2}Z(m^{-1/2}(s-r),x,v,\vec{X}(s),\vec{V}(s),-m^{-1/2}(s-r)).
\end{aligned}
$$

We show in the following that $\widetilde{f_i^1}(s,r,x,v)$ and $\widetilde{f_i^2}(s,r,x,v)$ are negligible.

Lemma 7.4.5 *We have that*

$$\lim_{m\to 0} E^{P_m}\Big[\sup_{0\le t\le T}\Big|\int_0^{t\wedge\sigma} f_V(X_s,Y_s)1_{[4m^{1/}\tau,\sigma]}(s)$$
$$\Big(\int_{\mathbf{R}\times E}\widetilde{f_i^k}(s,r,x,v)\mu_\omega(dr,dx,dv)\Big)ds\Big|\Big] = 0, \qquad k=1,2.$$

Proof. We first show the assertion for $k=1$. First notice that for $s\in[0,T\wedge\sigma]$, $\widetilde{f_i^1}(s,r,x,v)\neq 0$ only if $|x|\le R_0+1$ and $s-r\in[-m^{1/2}\tau,2m^{1/2}\tau]$. Since $s\in[4m^{1/2}\tau,T\wedge\sigma]$, this implies that $r-m^{1/2}\tau\in[0,T\wedge\sigma]$. So in this region, we have by Proposition 5.4.3 that there exists a constant $\widetilde{C}>0$ such that

$$\begin{aligned}&\Big|x(s,\Psi(r,x,m^{-1/2}v))-\psi^0(m^{-1/2}(s-r),x,v;\vec{X}(s))\Big|\\ \le\ & m^{1/2}\widetilde{C}(2\tau+|m^{-1/2}(r-s)|)\le 4\widetilde{C}\tau m^{1/2}.\end{aligned}$$

So there exists a constant $C>0$ such that

$$\begin{aligned}&|\widetilde{f_i^1}(s,r,x,v)|\\ \le\ & \|\nabla^3U_i\|_\infty\Big|x(s,\Psi(r,x,m^{-1/2}v))-\psi^0(m^{-1/2}(s-r),x,v;\vec{X}(s))\Big|^2\\ \le\ & Cm1_{[0,2m^{1/2}\tau]}(|s-r|)1_{[0,R_0+1]}(|x|).\end{aligned}$$

Therefore, by the definition of λ, there exists a constant $C_1>0$ such that

$$\begin{aligned}&E^{P_m}\Big[\sup_{0\le t\le T}\Big|\int_0^{t\wedge\sigma} f_V(X_s,Y_s)1_{[4m^{1/}\tau,\sigma]}(s)\Big(\int_{\mathbf{R}\times E}\widetilde{f_i^1}(s,r,x,v)\mu_\omega(dr,dx,dv)\Big)ds\Big|\Big]\\ \le\ & \int_0^T ds\|f_V\|_\infty Cm\int_{\mathbf{R}\times E}1_{[0,2m^{1/2}\tau]}(|s-r|)1_{[0,R_0+1]}(|x|)\\ &\quad\times m^{-1}\rho\Big(\frac{1}{2}|v|^2+\sum_{i=1}^N U_i(x-m^{-1/2}rv-X_{i,0})\Big)dr\nu(dx,dv)\\ \le\ & C_1m^{1/2},\end{aligned}$$

which converges to 0 as $m\to 0$.

The assertion for $k=2$ is similar. Again, for $s\in[0,T\wedge\sigma]$, $\widetilde{f_i^2}(s,r,x,v)\neq 0$ only if $|x|\le R_0+1$ and $s-r\in[-m^{1/2}\tau,2m^{1/2}\tau]$. For any s,r satisfying $|s-r|\le 2m^{1/2}\tau$, we have by Proposition 5.4.4 that

$$\begin{aligned}&\Big|x(s,\Psi(r,x,m^{-1/2}v))-\psi^0(m^{-1/2}(s-r),x,v;\vec{X}(s))\\ &\quad -m^{1/2}Z(m^{-1/2}(s-r),x,v,\vec{X}(s),\vec{V}(s),-m^{-1/2}(s-r))\Big|\\ \le\ & Cm^{1/2}(1+2\tau)^2m^{1/2}.\end{aligned}$$

Therefore, there exist constants $C_1,C_2>0$ such that

$$E^{P_m}\Big[\sup_{0\le t\le T}\Big|\int_0^{t\wedge\sigma} f_V(X_s,Y_s)1_{[4m^{1/}\tau,\sigma]}(s)\Big(\int_{\mathbf{R}\times E}\widetilde{f_i^2}(s,r,x,v)\mu_\omega(dr,dx,dv)\Big)ds\Big|\Big]$$

$$
\begin{aligned}
&\leq \int_0^T ds \|f_V\|_\infty \|\nabla^2 U_i\|_\infty C_1 m \int_{\mathbf{R}\times E} 1_{[0,2m^{1/2}\tau]}(|s-r|)1_{[0,R_0+1]}(|x|) \\
&\qquad \times m^{-1}\rho\Big(\frac{1}{2}|v|^2 + \sum_{i=1}^N U_i(x - m^{-1/2}rv - X_{i,0})\Big) dr\nu(dx,dv) \\
&\leq C_2 m^{1/2},
\end{aligned}
$$

which converges to 0 as $m \to 0$.

This completes the proof of our lemma. ∎

Before dealing with the main term $f_i^3(s,r,x,v)$, let us prepare the following continuity of $Z(t,x,v,\vec{X},\vec{V},a)$ with respect to $\vec{X}$ and $\vec{V}$.

Lemma 7.4.6 *For any $T_1 > 0$ and $n, A, B > 0$, there exists a constant $C = C(T_1, n, A, B)$ such that*

$$|Z(t,x,v,\vec{X}^1,\vec{V}^1,a) - Z(t,x,v,\vec{X}^2,\vec{V}^2,a)| \leq C(\|\vec{X}^1 - \vec{X}^2\| + \|\vec{V}^1 - \vec{V}^2\|).$$

for any $t \in [-\tau, T_1]$, $|a| \leq A$, $\|\vec{X}^1\|, \|\vec{X}^2\| \leq B$, $\|\vec{V}^1\|, \|\vec{V}^2\| \leq n$.

Proof. First notice that for any $a, x, v, \vec{X}, \vec{V}$, by using the same method as in the proofs of Lemmas 5.4.3, 6.3.4, *etc.*, with the help of Gronwall's Lemma, we get easily that for any $T_0 > 0$,

$$|Z(t)| \vee |Z'(t)| \leq (T_0 + |a|)\|\vec{V}\| T_0 e^{(1+\sum_{i=1}^N \|\nabla^2 U_i\|_\infty)T_0}, \qquad |t| \leq T_0. \tag{7.4.2}$$

For the sake of simplicity, we write $Z^k(t) = Z(t,x,v,\vec{X}^k,\vec{V}^k,a)$, $k = 1,2$, and $\xi(t) = Z^1(t) - Z^2(t)$. Then we have that in our domain, there exists a constant $C = C(T,n,A,B) > 0$ such that $|Z^1(t)| \leq C$. So by definition and Lemma 6.3.4, there exist constants $\widetilde{C}, C > 0$ such that

$$
\begin{aligned}
&\Big|\frac{d^2}{dt^2}\xi(t)\Big| \\
&= \Big| -\sum_{i=1}^N \nabla^2 U_i(\psi^0(t,x,v;\vec{X}^1) - X_i^1)(Z^1(t) - (t+a)\vec{V}^1) \\
&\qquad + \sum_{i=1}^N \nabla^2 U_i(\psi^0(t,x,v;\vec{X}^2) - X_i^2)(Z^2(t) - (t+a)\vec{V}^2)\Big| \\
&= \Big| -\sum_{i=1}^N \Big\{\nabla^2 U_i(\psi^0(t,x,v;\vec{X}^1) - X_i^1) - \nabla^2 U_i(\psi^0(t,x,v;\vec{X}^2) - X_i^2)\Big\}(Z^1(t) - (t+a)\vec{V}^1) \\
&\qquad - \sum_{i=1}^N \nabla^2 U_i(\psi^0(t,x,v;\vec{X}^2) - X_i^2)(Z^1(t) - Z^2(t) - (t+a)(\vec{V}^1 - \vec{V}^2))\Big| \\
&\leq \sum_{i=1}^N \|\nabla^3 U_i\|_\infty (\widetilde{C}+1)\|\vec{X}^1 - \vec{X}^2\|(\|Z^1\| + (T+|a|)\|\vec{V}^1\|)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{N}\|\nabla^2 U_i\|_\infty(\|Z^1-Z^2\|+(T+|a|)\|\vec{V}^1-\vec{V}^2\|)\\
\le\ & C(\|\vec{X}^1-\vec{X}^2\|+\|\vec{V}^1-\vec{V}^2\|)+C\|Z^1-Z^2\|\\
=\ & C(\|\vec{X}^1-\vec{X}^2\|+\|\vec{V}^1-\vec{V}^2\|)+C|\xi(t)|.
\end{aligned}
$$

Let $g(t)=\Big|(\xi(t),\frac{d}{dt}\xi(t))\Big|$. Then

$$
\begin{aligned}
\Big|\frac{d}{dt}g(t)\Big| &\le \Big|\frac{d}{dt}\xi(t)\Big|+\Big|\frac{d^2}{dt^2}\xi(t)\Big|\\
&\le C(\|\vec{X}^1-\vec{X}^2\|+\|\vec{V}^1-\vec{V}^2\|)+(C+1)g(t).
\end{aligned}
$$

Hence if we let $\widetilde{g}(t)=g(t-\tau)$, then $\widetilde{g}(0)=\frac{d}{dt}\widetilde{g}(0)=0$ by definition, and the above gives us that

$$
\frac{d}{dt}\widetilde{g}(t)\le C(\|\vec{X}^1-\vec{X}^2\|+\|\vec{V}^1-\vec{V}^2\|)+(C+1)\widetilde{g}(t).
$$

So we have for any $t\in[0,T_1+\tau]$

$$
\widetilde{g}(t)\le Ct(\|\vec{X}^1-\vec{X}^2\|+\|\vec{V}^1-\vec{V}^2\|)+(C+1)\int_0^t\widetilde{g}(s)ds.
$$

This combined with Gronwall's Lemma implies that

$$
\widetilde{g}(t)\le C(T_1+\tau)(\|\vec{X}^1-\vec{X}^2\|+\|\vec{V}^1-\vec{V}^2\|)e^{(C+1)(T_1+\tau)},\qquad t\in[0,T_1+\tau],
$$

which completes the proof of our assertion. ∎

Now, we come back to deal with the term corresponding to $f_i^3(s,r,x,v)$. Again, we make decomposition

$$
\int_0^{t\wedge\sigma}f_V(X_s,Y_s)1_{[4m^{1/\tau},\sigma]}(s)\Big(\int_{\mathbf{R}\times E}\widetilde{f_i^3}(s,r,x,v)\mu_\omega(dr,dx,dv)\Big)ds=(V1)+(V2),
$$

with

$$
\begin{aligned}
(V1) &= \int_0^{t\wedge\sigma}f_V(X_s,Y_s)1_{[4m^{1/\tau},\sigma]}(s)\Big(\int_{\mathbf{R}\times E}\widetilde{f_i^3}(s,r,x,v)\lambda(dr,dx,dv)\Big)ds,\\
(V2) &= \int_0^{t\wedge\sigma}f_V(X_s,Y_s)1_{[4m^{1/\tau},\sigma]}(s)\Big(\int_{\mathbf{R}\times E}\widetilde{f_i^3}(s,r,x,v)(\mu_\omega-\lambda)(dr,dx,dv)\Big)ds.
\end{aligned}
$$

Notice that up to σ_n, $\vec{V}(t)$ and $\vec{X}(t)$ are bounded. Also, $m^{1/2}|s-r|\le 2\tau$ and $|x|\le R_0+1$ if $\nabla^2U_i(X_i(s)-\psi^0(m^{-1/2}(s-r),x,v;\vec{X}(s)))\neq 0$. So by (7.4.2), in this case, $Z(m^{-1/2}(s-r),x,v,\vec{X}(s),\vec{V}(s),-m^{-1/2}(s-r))$ is bounded. So by the definition of $\widetilde{f_i^3}$ and the boundedness of ∇^2U_i, we get that there exists a constant $C>0$ such that

$$
|\widetilde{f_i^3}(s,r,x,v)|\le Cm^{1/2}1_{[0,2m^{1/2}\tau]}(|s-r|)1_{[0,R_0+1]}(|x|).
$$

As the following shows, the term (V2) is also negligible.

Lemma 7.4.7 $\lim_{m\to 0} E^{P_m}\Big[\sup_{0\le t\le T}|(V2)|\Big] = 0.$

Proof. Let

$$\begin{aligned} R^3(s,r,x,v) &= \widetilde{f_i^3}(s,r,x,v) - \frac{1}{2}\nabla^2 U_i(X_i(\widetilde{r}) - \psi^0(m^{-1/2}(s-r),x,v;\vec{X}(\widetilde{r}))) \\ &\qquad \cdot m^{1/2} Z(m^{-1/2}(s-r),x,v,\vec{X}(\widetilde{r}),\vec{V}(\widetilde{r}),-m^{-1/2}(s-r)). \end{aligned}$$

Then

$$(V2) = (V21) + (V22),$$

with

$$\begin{aligned} (V21) &= \int_0^{t\wedge\sigma} f_V(X_s,Y_s) 1_{[4m^{1/}\tau,\sigma]}(s)\Big(\int_{\mathbf{R}\times E} R^3(s,r,x,v)(\mu_\omega - \lambda)(dr,dx,dv)\Big)ds, \\ (V22) &= \int_0^{t\wedge\sigma} f_V(X_s,Y_s) 1_{[4m^{1/}\tau,\sigma]}(s) \\ &\qquad \Big(\int_{\mathbf{R}\times E} \frac{1}{2}\nabla^2 U_i(X_i(\widetilde{r}) - \psi^0(m^{-1/2}(s-r),x,v;\vec{X}(\widetilde{r}))) \\ &\qquad \cdot m^{1/2} Z(m^{-1/2}(s-r),x,v,\vec{X}(\widetilde{r}),\vec{V}(\widetilde{r}),-m^{-1/2}(s-r)) \\ &\qquad\qquad (\mu_\omega - \lambda)(dr,dx,dv)\Big)ds. \end{aligned}$$

We first deal with $(V21)$. For $s \in [0, T\wedge\sigma]$ and $|s-r| \le 2m^{1/2}\tau$, we have by definition $|s-\widetilde{r}| \le 4m^{1/2}\tau$, so by (7.4.2), Lemmas 7.4.6 and 6.3.4, there exists a constant $C > 0$ such that

$$\begin{aligned} & |R^3(s,r,x,v)| \\ = \ & \Big|\frac{1}{2}\nabla^2 U_i(X_i(s) - \psi^0(m^{-1/2}(s-r),x,v;\vec{X}(s))) \\ & \quad \cdot m^{1/2}\Big\{Z(m^{-1/2}(s-r),x,v,\vec{X}(s),\vec{V}(s),-m^{-1/2}(s-r)) \\ & \qquad -Z(m^{-1/2}(s-r),x,v,\vec{X}(\widetilde{r}),\vec{V}(\widetilde{r}),-m^{-1/2}(s-r))\Big\} \\ & +\frac{1}{2}\Big\{\nabla^2 U_i(X_i(s) - \psi^0(m^{-1/2}(s-r),x,v;\vec{X}(s))) \\ & \qquad -\nabla^2 U_i(X_i(\widetilde{r}) - \psi^0(m^{-1/2}(s-r),x,v;\vec{X}(\widetilde{r})))\Big\} \\ & \quad \times m^{1/2} Z(m^{-1/2}(s-r),x,v,\vec{X}(\widetilde{r}),\vec{V}(\widetilde{r}),-m^{-1/2}(s-r))\Big| \\ \le \ & \frac{1}{2}\|\nabla^2 U_i\|_\infty m^{1/2}\Big|Z(m^{-1/2}(s-r),x,v,\vec{X}(s),\vec{V}(s),-m^{-1/2}(s-r)) \\ & \qquad -Z(m^{-1/2}(s-r),x,v,\vec{X}(\widetilde{r}),\vec{V}(\widetilde{r}),-m^{-1/2}(s-r))\Big| \\ & +\frac{1}{2}\|\nabla^3 U_i\|_\infty\Big(|X_i(s) - X_i(\widetilde{r})| \\ & \qquad +|\psi^0(m^{-1/2}(s-r),x,v;\vec{X}(s)) - \psi^0(m^{-1/2}(s-r),x,v;\vec{X}(\widetilde{r}))|\Big) \\ & \quad \times m^{1/2}|Z(m^{-1/2}(s-r),x,v,\vec{X}(\widetilde{r}),\vec{V}(\widetilde{r}),-m^{-1/2}(s-r))| \\ \le \ & Cm^{1/2}(\|\vec{X}(s) - \vec{X}(\widetilde{r})\| + \|\vec{V}(s) - \vec{V}(\widetilde{r})\|). \end{aligned}$$

Since $|\vec{V}| \le n$ until σ_n, we have $|\vec{X}(s) - \vec{X}(\tilde{r})| \le n|s-\tilde{r}| \le 4m^{1/2}\tau n$. To estimate the term for $\vec{V}$, let

$$a_m = 4m^{1/2}\tau + E^{P_m}\Big[\sup_{0\le u\le T}|\eta_i(u)|\Big] + (4m^{1/2}\tau)^{1/2}$$

as before. Then by Lemma 5.3.1 (4), $b_m \to 0$ as $m \to 0$. Also, there exists a constant $C > 0$ such that

$$E^{P_m}\Big[|\vec{V}(s) - \vec{V}(\tilde{r})|\Big] \le Cb_m, \qquad |s-r| \le 2m^{1/2}\tau.$$

Actually, since $s, \tilde{r} \in [0, \sigma_0 \wedge \sigma_n]$, we have by Lemma 5.3.1 (1)

$$V_i(s) - V_i(\tilde{r}) = \frac{1}{M_i}\Big(\int_{\tilde{r}}^{s}\frac{d}{dl}P_i^{*1}(l)dl + \eta_i(s) - \eta_i(\tilde{r}) + M_i(s) - M_i(\tilde{r})\Big),$$

hence by Lemma 5.3.1 (2) and (6.4.2)

$$\begin{aligned}
&E^{P_m}\Big[|\vec{V}(s) - \vec{V}(\tilde{r})|\Big]\\
\le\ & C_1\Big(|s-\tilde{r}| + E^{P_m}\Big[\sup_{0\le u\le T}|\eta_i(u)|\Big] + E^{P_m}[|M(s) - M(\tilde{r})|]\Big)\\
\le\ & C_2\Big(|s-\tilde{r}| + E^{P_m}\Big[\sup_{0\le u\le T}|\eta_i(u)|\Big] + |s-\tilde{r}|^{1/2}\Big)\\
\le\ & C_2 b_m,
\end{aligned}$$

which gives us our assertion.

Combining the above and the definition of λ, we get that

$$\begin{aligned}
&E^{P_m}\Big[\sup_{0\le t\le T}|(V21)|\Big]\\
\le\ & \int_0^T ds\|f_V\|_\infty E^{P_m}\Big[1_{[0,T\wedge\sigma]}(s)\int_{\mathbf{R}\times E}(Cm + Cm^{1/2}|\vec{V}(s) - \vec{V}(\tilde{r})|)\\
&\quad \times 1_{[0,2m^{1/2}\tau]}(|s-r|)1_{[0,R_0+1]}(|x|)(\mu_\omega + \lambda)(dr,dx,dv)\Big]\\
=\ & 2\int_0^T ds\|f_V\|_\infty\int_{\mathbf{R}\times E}E^{P_m}\Big[1_{[0,T\wedge\sigma]}(s)(Cm + Cm^{1/2}|\vec{V}(s) - \vec{V}(\tilde{r})|)\Big]\\
&\quad \times 1_{[0,2m^{1/2}\tau]}(|s-r|)1_{[0,R_0+1]}(|x|)\lambda(dr,dx,dv)\\
\le\ & \tilde{C}(m^{1/2} + b_m) \to 0, \qquad \text{as } m \to 0.
\end{aligned}$$

The term $(V22)$ is easier. We have $\Big|\nabla^2 U_i(X_i(\tilde{r}) - \psi^0(m^{-1/2}(s-r), x, v; \vec{X}(\tilde{r})))\Big| \le \|\nabla^2 U_i\|_\infty 1_{[0,2m^{1/2}\tau]}(|s-r|)1_{[0,R_0+1]}(|x|)$. Also, for $s \in [0,T]$ and $|s-r| \le 2m^{1/2}\tau$, we have that $Z(m^{-1/2}(s-r), x, v, \vec{X}(\tilde{r}), \vec{V}(\tilde{r}), -m^{-1/2}(s-r))$ is bounded. Therefore, since $\vec{X}(\tilde{r})$ is $\mathcal{F}_r$-measurable, by the definition of Poisson point processes and the definition of λ, we have

$$E^{P_m}\Big[\sup_{0\le t\le T}|(V21)|\Big]$$

$$
\begin{aligned}
\leq\; & \int_0^T ds \|f_V\|_\infty E^{P_m}\Big[\Big|\int_{\mathbf{R}\times E}\frac{1}{2}\nabla^2 U_i(X_i(\tilde{r})-\psi^0(m^{-1/2}(s-r),x,v;\vec{X}(\tilde{r}))\\
&\quad m^{1/2}Z(m^{-1/2}(s-r),x,v,\vec{X}(\tilde{r}),\vec{V}(\tilde{r}),-m^{-1/2}(s-r))\\
&\quad (\mu_\omega(dr,dx,dv)-\lambda(dr,dx,dv))\Big|^2\Big]^{1/2}\\
=\; & \int_0^T ds\|f_V\|_\infty\Big\{\int_{\mathbf{R}\times E}E^{P_m}\Big[\Big(\frac{1}{2}\nabla^2 U_i(X_i(\tilde{r})-\psi^0(m^{-1/2}(s-r),x,v;\vec{X}(\tilde{r}))\\
&\quad m^{1/2}Z(m^{-1/2}(s-r),x,v,\vec{X}(\tilde{r}),\vec{V}(\tilde{r}),-m^{-1/2}(s-r))\Big)^2\Big]\lambda(dr,dx,dv)\Big\}^{1/2}\\
\leq\; & \int_0^T ds\|f_V\|_\infty\int_{\mathbf{R}\times E}\Big(\|\nabla^2 U_i\|_\infty m^{1/2}C\Big)^2 1_{[0,R_0+1)}(|x|)1_{[0,2m^{1/2}\tau)}(|s-r|)\lambda(dr,dx,dv)\\
\leq\; & \widetilde{C}m^{1/2}\to 0, \qquad \text{as } m\to 0.
\end{aligned}
$$

This completes the proof of Lemma 7.4.7. ∎

Up to now, we have shown that all of the terms of $-(II)$ except $\sum_{i=1}^N \frac{1}{M_i}(V1)$ are negligible. Notice that in the integral domain of $(V1)$, we have $s\geq 4m^{1/2}\tau$. So if $\nabla^2 U_i(X_i(s)-\psi^0(m^{-1/2}(s-r),x,v;\vec{X}(s))\neq 0$, then $r\geq 2m^{1/2}\tau$. If $\rho(\frac{1}{2}|v|^2+\sum_{i=1}^N U_i(x-m^{-1/2}rv-X_{i,0}))\neq 0$ in addition, then $|v|\geq 2C_0+1$. Therefore, in this case, $|m^{-1/2}rv|\geq 2\tau(2C_0+1)\geq R_0$, hence since $x\cdot v=0$, we get $|x-m^{-1/2}rv|\geq R_0$, which in turn gives us that $\rho(\frac{1}{2}|v|^2+\sum_{i=1}^N U_i(x-m^{-1/2}rv-X_{i,0}))=\rho(\frac{1}{2}|v|^2)$. Therefore, by definition,

$$
\begin{aligned}
& (V1)\\
=\; & \int_0^{t\wedge\sigma} f_V(X_s,Y_s)1_{[4m^{1/}\tau,\sigma]}(s)\Big(\int_{\mathbf{R}\times E}\\
&\quad \frac{1}{2}\nabla^2 U_i(X_i(s)-\psi^0(m^{-1/2}(s-r),x,v;\vec{X}(s)))\\
&\quad \cdot m^{1/2}Z(m^{-1/2}(s-r),x,v,\vec{X}(s),\vec{V}(s),-m^{-1/2}(s-r))m^{-1}\rho\Big(\frac{1}{2}|v|^2\Big)dr\nu(dx,dv)\\
=\; & \int_0^{t\wedge\sigma} f_V(X_s,Y_s)1_{[4m^{1/}\tau,\sigma]}(s)\\
&\quad \times\Big(\int_E\Big(\int_{-\infty}^{+\infty}du\frac{1}{2}\nabla^2 U_i(X_i(s)-\psi^0(u,x,v;\vec{X}(s)))\\
&\qquad Z(u,x,v,\vec{X}(s),\vec{V}(s),-u)\Big)\rho\Big(\frac{1}{2}|v|^2\Big)\nu(dx,dv)\Big),
\end{aligned}
$$

where in the last equality, we used the change of variable $u=m^{-1/2}(s-r)$ for every s fixed.

We divide this last expression into two parts again

$$(V1)=(V11)+(V12),$$

with

$$(V11) \;=\; \frac{1}{2}\int_0^{t\wedge\sigma} f_V(X_s,Y_s)$$

$$\Big(\int_E \Big(\int_{-\infty}^{+\infty} du \nabla^2 U_i(X_i(s) - \psi^0(u,x,v;\vec{X}(s)))$$
$$Z(u,x,v,\vec{X}(s),\vec{V}(s),-u)\Big)\rho\Big(\frac{1}{2}|v|^2\Big)\nu(dx,dv)\Big),$$

$$(V12) \quad = \quad -\frac{1}{2}\int_0^{t\wedge\sigma} f_V(X_s,Y_s)1_{[0,4m^{1/2}\tau]}(s)$$
$$\Big(\int_E \Big(\int_{-\infty}^{+\infty} du \nabla^2 U_i(X_i(s) - \psi^0(u,x,v;\vec{X}(s)))$$
$$Z(u,x,v,\vec{X}(s),\vec{V}(s),-u)\Big)\rho\Big(\frac{1}{2}|v|^2\Big)\nu(dx,dv)\Big).$$

Notice that for $s \in [0, T\wedge\sigma]$, $\nabla^2 U_i(X_i(s) - \psi^0(u,x,v;\vec{X}(s))) \neq 0$ only if $|u| \leq 2\tau$, and $Z(u,x,v,\vec{X}(s),\vec{V}(s),-u)$ is bounded in this domain. So

$$\int_E \Big(\int_{-\infty}^{+\infty} du \nabla^2 U_i(X_i(s)-\psi^0(u,x,v;\vec{X}(s)))Z(u,x,v,\vec{X}(s),\vec{V}(s),-u)\Big)\rho\Big(\frac{1}{2}|v|^2\Big)\nu(dx,dv)$$

is bounded. Therefore, there exists a constant $C > 0$ such that

$$|(V12)| \leq Cm^{1/2}.$$

This completes the proof that the term (II) is converging to $-\sum_{i=1}^N \frac{1}{M_i}(V11)$ as $m \to 0$.

7.5 Result

Combining Sections 7.1, 7.2, 7.3 and 7.4, and take the limit $n \to \infty$ at last (notice that $\sigma_n \to \infty$ *a.s.*), we get the desired Results 1 and 2 in Chapter 1.

Notice that this also gives us Lemma 5.3.4, by considering each time interval $[\eta_{n-1}, \xi_n]$, with η_n, ξ_n given by the following. $\eta_0 = 0$,

$$\xi_n = \inf\{t \geq \eta_{n-1}; \vec{X}(t) \in B(supp\widetilde{U}, \frac{\varepsilon_f}{2})\},$$
$$\eta_n = \inf\{t \geq \xi_n; \vec{X}(t) \notin B(supp\widetilde{U}, \varepsilon_f)\}, \qquad n \geq 1.$$

Here $\varepsilon_f > 0$ is chosen such that $suppf \subset \Big(B(supp\widetilde{U}, 2\varepsilon_f) \times \mathbf{R}^{dN}\Big)^C$.

Remark 2 *In this chapter, we stopped the process at* $\sigma_0(\omega) = \inf\{t > 0; \min_{i\neq j}\{|X_i(t) - X_j(t)| - (R_i + R_j)\} \leq 0\}$, *only because we wanted to keep the drift term* $m^{-1/2}\int_0^{t\wedge\sigma} \nabla_i \widetilde{U}(\vec{X}(s))ds$ *equal to* 0. *However, as shown in the following, if* $d = 2$, *then* $\nabla_i\widetilde{U}(\vec{X})$ *is always* 0, *no matter whether* $\min_{i\neq j}\{|X_i - X_j| - (R_i + R_j)\}$ *is positive or not. So for* $d = 2$, *we do not need to stop the process at* $\sigma_0(\omega)$, *and the same argument as in this chapter gives us the convergence to the diffusion process until any* $T > 0$.

Actually, for $d = 2$*, by using the same notation as at the end of Section 5.4, since* $\sum_{i=1}^{N} \|U_i\|_\infty < e_0$*, we have by (5.4.10) that for any* $x, X_i \in \mathbf{R}^2$*,*

$$f'\Big(\sum_{i=1}^{N} U_i(X_i - x)\Big) = C_2 \int_0^\infty \rho(t)dt,$$

which is a constant. We write it as $\widetilde{C_2}$*. So by (5.4.8),*

$$\begin{aligned}
\nabla_i \widetilde{U}(\vec{X}) &= \int_{\mathbf{R}^2} f'\Big(\sum_{i=1}^{N} U_i(X_i - x)\Big) \nabla U_i(X_i - x)dx \\
&= \widetilde{C_2} \int_{\mathbf{R}^2} \nabla U_i(X_i - x)dx \\
&= \widetilde{C_2} \nabla \Big(\int_{\mathbf{R}^2} U_i(X_i - x)dx \Big) = 0.
\end{aligned}$$

∎

Chapter 8

Case of Two Atoms

In this chapter, we consider a special case with two atoms and special potential functions U_1, U_2, for $d \geq 3$. Precisely, in addition to all of the assumptions in Chapters 5 $\sim$ 7, we assume in further from now on that $d \geq 3$, and there exist functions $h_1, h_2 : [0,\infty) \to \mathbf{R}$ such that

$$U_i(x) = h_i(|x|), \qquad i = 1, 2,$$

and there exists a constant $\varepsilon_0 > 0$ such that

$$(-1)^{i-1} h_i(s) > 0, \quad (-1)^{i-1} h_i''(s) > 0, \qquad s \in (R_i - \varepsilon_0, R_i), i = 1, 2.$$

We show that in this special case, as announced in Chapter 1, $\{(\vec{X}(t), \vec{V}(t))\}_t$ converges to a Markov process as $m \to 0$.

We first show that in our present setting, the condition of Lemma 5.3.2 is satisfied, and that when $m \to 0$, the distribution of $\{(\vec{X}(t \wedge \sigma_n), \vec{V}(t \wedge \sigma_n))\}_t$ under P_m is tight in $\wp(\widetilde{W^d})$ with the metric function $\widetilde{dis}$ of $\widetilde{W^d} = C([0,\infty); \mathbf{R}^d) \times D([0,\infty); \mathbf{R}^d)$ given by

$$\widetilde{dis}(\omega_1, \omega_2) = \sum_{n=1}^{\infty} 2^{-n} \Big(1 \wedge \Big(\max_{t \in [0,n]} |x_1(t) - x_2(t)| + (\int_0^n |v_1(t) - v_2(t)|^n)^{1/n} \Big)\Big)$$

for $\omega_i = (x_i(\cdot), v_i(\cdot)) \in \widetilde{W^d}$, $i = 1, 2$. We then discuss a little bit more about the new potentials $\widetilde{U}$. Finally, we use these to show the desired convergence.

8.1 Preparation

Same as before, we only need to make the discussion under condition $|V_i| \leq n$, *i.e.*, for $t \wedge \sigma_n$, and finally take $n \to \infty$.

We first show that the condition of Lemma 5.3.2 is satisfied. Actually, by assumption, we have $U_i(x) = h_i(|x|)$, so $\nabla U_i(x) = \frac{x}{|x|} h_i'(|x|)$, hence

$$\nabla_i \widetilde{U}(\vec{X})$$

$$
\begin{aligned}
&= \int_{\mathbf{R}^{2d}} \nabla U_i(X_i - x)\rho\Big(\frac{1}{2}|v|^2 + U_1(X_1 - x) + U_2(X_2 - x)\Big)dxdv \\
&= \int_{\mathbf{R}^{2d}} \frac{X_i - x}{|X_i - x|} h_i'(|X_i - x|)\rho\Big(\frac{1}{2}|v|^2 + U_1(X_1 - x) + U_2(X_2 - x)\Big)dxdv.
\end{aligned}
$$

From this, we see easily that for $\vec{X}$ with $|X_1 - X_2| < R_1 + R_2$ big enough, $\nabla_i \widetilde{U}(\vec{X})$ is parallel to $X_1 - X_2$ in $\mathbf{R}^d$. Moreover, $\nabla_1 \widetilde{U}(\vec{X})$ has the opposite direction as $X_1 - X_2$, and $\nabla_2 \widetilde{U}(\vec{X})$ has the same direction as $X_1 - X_2$ (see Lemma 8.2.2 below for details).

Therefore, if we let $g_1(\vec{X}) = \frac{X_2 - X_1}{|X_2 - X_1|}$, $g_2(\vec{X}) = \frac{X_1 - X_2}{|X_1 - X_2|}$, and let $\overline{D} = \{(X_1, X_2)\big||X_i| \le |X_{i,0}| + nT, |X_1 - X_2| \ge R_1 + R_2 - \varepsilon_0\}$. Then since $R_1 + R_2 - \varepsilon_0 > 0$, we have that $g_1, g_2 \in C_b^1(\overline{D})$ and $g_i(\vec{X}) \cdot \nabla_i \widetilde{U}(\vec{X}) = |\nabla_i \widetilde{U}(\vec{X})|$ for any $x \in \overline{D}$. *i.e.*, the condition of Lemma 5.3.2 is satisfied.

We next give a brief proof of the tightness of $\{(\vec{X}(t \wedge \sigma_n), \vec{V}(t \wedge \sigma_n))\}_t$ under P_m as $m \to 0$. The only difficulty is for $\vec{V}$. We deal with it now. Let $A_k = \{Y_t : \int_0^T |dY_t| \le k\}$, $k \in \mathbf{N}$. Then we have by Kusuoka [9, Corollary 8] that A_k is compact in $L^p([0,T]; \mathbf{R}^d)$ with cluster points in $D([0,T]; \mathbf{R}^d)$ for any $k \in \mathbf{N}$. Also, by Lemma 5.3.2, there exists a constant $C > 0$ such that

$$
\begin{aligned}
& \Big(P_m \circ \Big(m^{-1/2} \int_0^{\cdot \wedge \sigma_n} \nabla_i \widetilde{U}(\vec{X}(s))ds\Big)^{-1}\Big)(A_k) \\
= \;& 1 - P_m\Big(\int_0^{T \wedge \sigma} m^{-1/2} |\nabla_i \widetilde{U}(\vec{X}(s))|ds > k\Big) \\
\ge \;& 1 - \frac{1}{k} E^{P_m}\Big[\int_0^{T \wedge \sigma} m^{-1/2} |\nabla_i \widetilde{U}(\vec{X}(s))|ds\Big] \\
\ge \;& 1 - \frac{C}{k},
\end{aligned}
$$

which converges to 1 as $k \to \infty$. Therefore, for $m \to 0$, $m^{-1/2} \int_0^{\cdot \wedge \sigma_n} \nabla_i \widetilde{U}(\vec{X}(s))ds$ under P_m is tight in $\wp(D([0,\infty); \mathbf{R}^d))$ with metric of $D([0,\infty); \mathbf{R}^d)$ derived by $\widetilde{dis}$. Therefore, since by Lemma 5.3.1,

$$
M_i(V_i(t \wedge \sigma) - V_i(0)) = P_i^{*0}(t) + P_i^{*1} - m^{-1/2} \int_0^{t \wedge \sigma} \nabla_i \widetilde{U}(\vec{X}(s))ds,
$$

and the distributions of $P_i^{*0}(t)$ and P_i^{*1} under P_m are tight in $\wp(D([0,\infty); \mathbf{R}^d))$, we get the conclusion that for $m \to 0$, $\{V_i(t \wedge \sigma_n)\}_t$ under P_m is tight in $\wp(D([0,\infty); \mathbf{R}^d))$.

8.2 The new potential $\widetilde{U}$

As in Section 5.3, let

$$
\begin{aligned}
\widetilde{\rho}(t) &= -\int_t^\infty \rho(s)ds, \\
f(s) &= \int_{\mathbf{R}^d} \widetilde{\rho}\Big(\frac{1}{2}|v|^2 + s\Big)dv.
\end{aligned}
$$

Then for any $d \in \mathbf{N}$, there exists a constant $C_d > 0$ such that $f(s) = C_d \int_0^\infty \widetilde{\rho}(r+s) r^{\frac{d}{2}-1} dr$. Section 5.4 also showed that

$$\widetilde{U}(X_1, X_2) = \int_{\mathbf{R}^d} \Big(f\Big(U_1(X_1 - x) + U_2(X_2 - x) \Big) - f(0) \Big) dx.$$

Also, let $\widetilde{U}_0$ be the constant

$$\widetilde{U}_0 = \sum_{i=1}^{2} \int_{\mathbf{R}^d} \Big(f(U_i(X_i - x)) - f(0) \Big) dx,$$

which, as claimed in Section 5.4, is the value of the potential $\widetilde{U}$ when X_1 and X_2 are far enough, precisely, when $|X_1 - X_2| \geq R_1 + R_2$. Then

$$\begin{aligned}
& \widetilde{U}(X_1, X_2) - \widetilde{U}_0 \\
= & \int_{\mathbf{R}^d} \Big\{ \Big[f\Big(U_1(X_1 - x) + U_2(X_2 - x) \Big) - f(0) \Big] \\
& \qquad - \Big[\Big(f(U_1(X_1 - x)) - f(0) \Big) + \Big(f(U_2(X_2 - x)) - f(0) \Big) \Big] \Big\} dx \\
= & \int_{\mathbf{R}^d} dx \Big\{ \int_0^{U_1(X_1 - x) + U_2(X_2 - x)} f'(s) ds - \int_0^{U_1(X_1 - x)} f'(s) ds - \int_0^{U_2(X_2 - x)} f'(s) ds \Big\} \\
= & \int_{\mathbf{R}^d} dx \Big\{ \int_{U_2(X_2 - x)}^{U_1(X_1 - x) + U_2(X_2 - x)} f'(s) ds - \int_0^{U_1(X_1 - x)} f'(s) ds \Big\} \\
= & \int_{\mathbf{R}^d} dx \int_0^{U_1(X_1 - x)} \Big(f'(s + U_2(X_2 - x)) - f'(s) \Big) ds \\
= & \int_{\mathbf{R}^d} dx \int_0^{U_1(X_1 - x)} ds \int_0^{U_2(X_2 - x)} f''(s + u) du.
\end{aligned}$$

Therefore,

$$\nabla_1 \widetilde{U}(X_1, X_2) = \int_{\mathbf{R}^d} dx \int_0^{U_2(X_2 - x)} f''(U_1(X_1 - x) + u) du \nabla U_1(X_1 - x). \qquad (8.2.1)$$

Also, notice that the integrand in (8.2.1) is 0 out of

$$B_2 = B_{X_1, X_2} = \{ x \in \mathbf{R}^d ; |x - X_1| \leq R_1, |x - X_2| \leq R_2 \},$$

therefore,

$$\nabla_1 \widetilde{U}(X_1, X_2) = \int_{B_2} dx \int_0^{U_2(X_2 - x)} f''(U_1(X_1 - x) + u) du \nabla U_1(X_1 - x). \qquad (8.2.2)$$

We will use this expression in the following calculations. First, we have the following.

Lemma 8.2.1 *Let $\varepsilon \in (0, \varepsilon_0]$. Then there exists a $C_\varepsilon > 0$ such that for any $X_1, X_2 \in \mathbf{R}^d$ satisfying $|X_1 - X_2| \in [R_1 + R_2 - \varepsilon, R_1 + R_2 - \frac{\varepsilon}{2})$, we have that*

$$(X_1 - X_2) \cdot \nabla_1 \widetilde{U}(X_1, X_2) \leq -C_\varepsilon, \quad (X_1 - X_2) \cdot \nabla_2 \widetilde{U}(X_1, X_2) \geq C_\varepsilon.$$

Proof. Since the proofs are the same, we only show the first assertion.

Notice that for any $x \in B_2$, since $|X_1 - X_2| \geq R_1 + R_2 - \varepsilon$, we have that $|X_1 - x| \geq R_1 - \varepsilon$, $|X_2 - x| \geq R_2 - \varepsilon$, hence by our assumption, $U_1(X_1 - x) = h_1(|X_1 - x|)$, $U_2(X_2 - x) = h_2(|X_2 - x|)$.

Therefore, by (8.2.2),

$$\nabla_1 \widetilde{U}(X_1, X_2) = \int_{B_2} dx \int_0^{h_2(|X_2 - x|)} f''(h_1(|X_1 - x|) + u) du h_1'(|X_1 - x|) \frac{X_1 - x}{|X_1 - x|}.$$

Notice that in this integral domain, $(X_1 - X_2) \cdot \frac{X_1 - x}{|X_1 - x|} > 0$, and by assumption,

$$\begin{array}{ll} h_1(|X_1 - x|) > 0, & h_2(|X_2 - x|) < 0, \\ h_1'(|X_1 - x|) < 0, & h_2'(|X_2 - x|) > 0. \end{array}$$

Also, since $d \geq 3$, we have by (5.4.12) that $f''(s) < 0$ for any $|s| < e_0$. Therefore, if we let

$$\widetilde{B_2} = \{x; |X_1 - x| \leq R_1 - \frac{\varepsilon}{6}, |X_2 - x| \leq R_2 - \frac{\varepsilon}{6}\} \subset B_2,$$

then

$$\begin{aligned} & -(X_1 - X_2) \cdot \nabla_1 \widetilde{U}(X_1, X_2) \\ \geq \ & -\int_{\widetilde{B_2}} dx \int_0^{h_2(|X_2 - x|)} f''(h_1(|X_1 - x|) + u) du h_1'(|X_1 - x|)(X_1 - X_2) \cdot \frac{X_1 - x}{|X_1 - x|}. \end{aligned}$$

For any $|s| \leq \|U_1\|_\infty + \|U_2\|_\infty$, we have by (5.4.11) that

$$-f''(s) = C_d(\frac{d}{2} - 1) \int_{e_0}^\infty \rho(t)(t - s)^{\frac{d}{2} - 2} dt > 0,$$

also, $-f''(\cdot)$ is continuous in this closed interval. Therefore, there exists a constant $C_0 > 0$ such that

$$\inf \Big\{ -f''(s); |s| \leq \|U_1\|_\infty + \|U_2\|_\infty \Big\} \geq C_0.$$

Also, for any $x \in \widetilde{B_\varepsilon}$, we have that

$$|X_1 - x| \geq |X_1 - X_2| - |X_2 - x| \geq (R_1 + R_2 - \varepsilon) - (R_2 - \frac{\varepsilon}{6}) = R_1 - \frac{5}{6}\varepsilon,$$

i.e., $|X_1 - x| \in [R_1 - \frac{5}{6}\varepsilon, R_1 - \frac{\varepsilon}{6}]$. In the same way, $|X_2 - x| \in [R_2 - \frac{5}{6}\varepsilon, R_2 - \frac{\varepsilon}{6}]$. So by assumption, there exists a constant $C_\varepsilon^1 > 0$ (which does not depend on x) such that

$$\begin{array}{ll} h_1(|X_1 - x|) \geq C_\varepsilon^1, & h_2(|X_2 - x|) \leq -C_\varepsilon^1, \\ h_1'(|X_1 - x|) \leq -C_\varepsilon^1, & h_2'(|X_2 - x|) \geq C_\varepsilon^1. \end{array}$$

Also, we have that

$$(X_1 - X_2) \cdot \frac{X_1 - x}{|X_1 - x|} \geq \frac{(R_1 + R_2 - \varepsilon)(R_1 - \varepsilon)}{R_1}.$$

Actually, if we decompose $X_1 - x$ into

$$X_1 - x = X_1 - \widetilde{x} + (x - \widetilde{x})$$

with $X_1 - \widetilde{x} \parallel X_1 - X_2$ and $x - \widetilde{x} \perp X_1 - X_2$, then $X_2 - x = X_2 - \widetilde{x} + (x - \widetilde{x})$ is also a perpendicular decomposition. So $R_2^2 \geq |X_2 - x|^2 = |X_2 - \widetilde{x}|^2 + |x - \widetilde{x}|^2$ implies that $|X_2 - \widetilde{x}| \leq R_2$. Also, $|X_1 - X_2| \geq R_1 + R_2 - \varepsilon$, So $|X_1 - \widetilde{x}| \geq |X_1 - X_2| - |X_2 - \widetilde{x}| \geq (R_1 + R_2 - \varepsilon) - R_2 = R_1 - \varepsilon$. Therefore,

$$(X_1 - X_2) \cdot \frac{X_1 - x}{|X_1 - x|} \geq \frac{|X_1 - X_2|\ |X_1 - \widetilde{x}|}{R_1} \geq \frac{(R_1 + R_2 - \varepsilon)(R_1 - \varepsilon)}{R_1}.$$

Combining these, we get that

$$\begin{aligned}
& -(X_1 - X_2) \cdot \nabla_1 \widetilde{U}(X_1, X_2) \\
\geq \ & -\int_{\widetilde{B_2}} dx \int_0^{h_2(|X_2 - x|)} f''(h_1(|X_1 - x|) + u) du h_1'(|X_1 - x|)(X_1 - X_2) \cdot \frac{X_1 - x}{|X_1 - x|} \\
\geq \ & C_\varepsilon^1 C_0 C_\varepsilon^1 \frac{(R_1 + R_2 - \varepsilon)(R_1 - \varepsilon)}{R_1} \int_{\widetilde{B_\varepsilon}} dx,
\end{aligned}$$

which gives us our first assertion. ∎

As a corollary of Lemma 8.2.1, we have the following.

Lemma 8.2.2 *Let $\varepsilon \in (0, \varepsilon_0]$, and let $X_1, X_2 \in \mathbf{R}^d$ satisfying $|X_1 - X_2| \in [R_1 + R_2 - \varepsilon, R_1 + R_2)$. Then we have that*

$$(X_1 - X_2) \cdot \nabla_1 \widetilde{U}(X_1, X_2) < 0, \quad (X_1 - X_2) \cdot \nabla_2 \widetilde{U}(X_1, X_2) > 0.$$

Also, by Lemma 8.2.1, we get the following as an easy corollary.

COROLLARY 8.2.3 *Assume that $t_1, t_2 \in [0, \sigma_n]$ satisfy $|t_1 - t_2| \leq \frac{\varepsilon}{4n}$, and $|X_1(t_1) - X_2(t_1)| \in [R_1 + R_2 - \varepsilon, R_1 + R_2 - \frac{\varepsilon}{2})$. Then*

$$-(X_1(t_2) - X_2(t_2)) \cdot \nabla_1 \widetilde{U}(X_1(t_1), X_2(t_1)) \geq C_\varepsilon \Big(1 - \frac{\varepsilon/2}{R_1 + R_2 - \varepsilon}\Big).$$

Proof. By using the general fact that $\frac{(a,b)}{|b|^2} \geq 1 - \frac{|a-b|}{|b|}$ for any $a, b \in \mathbf{R}^d$ and the fact that $\nabla_1 \widetilde{U}(X_1, X_2)$ is parallel to $X_1 - X_2$, we get by Lemma 8.2.1 that for $(\widetilde{X_1}, \widetilde{X_2})$ near to (X_1, X_2),

$$\begin{aligned}
& -(\widetilde{X_1} - \widetilde{X_2}) \cdot \nabla_1 \widetilde{U}(X_1, X_2) \\
= \ & -(X_1 - X_2) \cdot \nabla_1 \widetilde{U}(X_1, X_2) \ \frac{(\widetilde{X_1} - \widetilde{X_2}, X_1 - X_2)}{|X_1 - X_2|^2} \\
\geq \ & C_\varepsilon \Big(1 - \frac{|(\widetilde{X_1} - \widetilde{X_2}) - (X_1 - X_2)|}{|X_1 - X_2|}\Big).
\end{aligned}$$

In particular, under our assumption, we have $|X_1(t_1) - X_1(t_2)| \le n|t_1 - t_2| \le \frac{\varepsilon}{4}$, similarly, $|X_2(t_1) - X_2(t_2)| \le \frac{\varepsilon}{4}$. Therefore, by the argument above,

$$\begin{aligned} & -(X_1(t_2) - X_2(t_2)) \cdot \nabla_1 \widetilde{U}(X_1(t_1), X_2(t_1)) \\ \ge \;& C_\varepsilon \Big(1 - \frac{|(X_1(t_2) - X_2(t_2)) - (X_1(t_1) - X_2(t_1))|}{|X_1(t_1) - X_2(t_1)|}\Big) \\ \ge \;& C_\varepsilon \Big(1 - \frac{\varepsilon/2}{R_1 + R_2 - \varepsilon}\Big). \end{aligned}$$

∎

8.3 Convergence to Markov process

Let us first recall the following existence and uniqueness theorem of Kusuoka [9, Theorem 1]. Let $\Phi : \mathbf{R}^d \times \partial D \to \mathbf{R}^d$ be a smooth map satisfying the following.

(1) $\Phi(\cdot, x) : \mathbf{R}^d \to \mathbf{R}^d$ is linear for all $x \in \partial D$,

(2) $\Phi(v, x) = v$ for any $x \in \partial D$ and $v \in T_x(\partial D)$, *i.e.*, $\Phi(v, x) = v$ if $x \in \partial D$, $v \in \mathbf{R}^d$ and $v \cdot n(x) = 0$,

(3) $\Phi(\Phi(v, x), x) = v$ for all $v \in \mathbf{R}^d$ and $x \in \partial D$,

(4) $\Phi(n(x), x) \ne n(x)$ for any $x \in \partial D$.

Then Kusuoka [9, Theorem 1] showed the following.

THEOREM 8.3.1 *Let $(x_0, v_0) \in (\overline{D})^C \times \mathbf{R}^d$. Then there exists a unique probability measure μ over $\widetilde{W}^d$ satisfying the following.*

(1) $\mu(\omega(0) = (x_0, v_0)) = 1$,

(2) $\mu(\omega(t) \in D^C \times \mathbf{R}^d, t \in [0, \infty)) = 1$,

(3) *For any $f \in C_0^\infty((\overline{D})^C \times \mathbf{R}^d)$, $\{f(\omega(t)) - \int_0^t L_0 f(w(s))ds; t \ge 0\}$ is a martingale under $\mu(\omega)$,*

(4) $\mu\Big(1_{\partial D}(x(t))(v(t) - \Phi(v(t-), x(t))) = 0$ *for all* $t \in [0, \infty)\Big) = 1$.

Here $\omega(\cdot) = (x(\cdot), v(\cdot)) \in \widetilde{W}^d$.

By using this, we get the following, which is just a slight variation. Recall that $D_0 = \{(X_1, X_2) \in \mathbf{R}^{2d}; |X_1 - X_2| > R_1 + R_2\}$ in our present setting.

THEOREM 8.3.2 *There exists a unique probability measure $P_{\infty,0}$ over $D([0, \infty); \mathbf{R}^{4d})$ satisfying the following.*

(1) $P_{\infty,0}(\omega(0) = (x_0, v_0)) = 1$,

(2) $P_{\infty,0}(\vec{X}(t) \in \overline{D_0}, t \in [0,\infty)) = 1$,

(3) *For any* $f \in C_0^\infty(D_0 \times \mathbf{R}^{2d})$, $\{f(\vec{X}(t), \vec{V}(t)) - \int_0^t (Lf)(\vec{X}(s), \vec{V}(s))ds; t \geq 0\}$ *is a martingale under* $P_{\infty,0}$,

(4) *If* $f \in C_0^\infty(\mathbf{R}^{4d})$ *satisfies*

$$M_1^{-1}(\nabla_{V_1} f)(\vec{X}, \vec{V}) \cdot (X_1 - X_2) + M_2^{-1}(\nabla_{V_2} f)(\vec{X}, \vec{V}) \cdot (X_2 - X_1) = 0$$

for any $(\vec{X}, \vec{V}) \in \partial D_0 \times \mathbf{R}^{2d}$, *then* $f(\vec{X}(t), \vec{V}(t))$ *is continuous in* t, $P_{\infty,0}$*-a.s.*,

(5) $M_1|V_1(t)|^2 + M_2|V_2(t)|^2$ *is continuous in* t, $P_{\infty,0}$*-a.s.*.

We have already shown in Section 8.1 that $\{(\vec{X}(t \wedge \sigma_n), \vec{V}(t \wedge \sigma_n))\}_t$ under P_m is tight as $m \to 0$. We show from now on that, any cluster point of it satisfies all of the conditions of Theorem 8.3.2.

That it satisfies (1) is trivial. The fact that it satisfies (3) is nothing but Lemma 5.3.4. So we only need to show that it satisfies (2), (4) and (5).

We show (2) first. Fix any $\varepsilon > 0$ for a while, and let

$$\xi = \xi_\varepsilon = \inf\{t > 0; |X_1(t) - X_2(t)| \leq R_1 + R_2 - \frac{3}{4}\varepsilon\} \wedge \sigma_n \wedge T.$$

Then (2) is implied if we can show the following.

Lemma 8.3.3 *Let* $\varepsilon \in (0, \varepsilon_0]$ *and let* ξ *be as defined above. Then*

$$\lim_{m \to 0} P_m(\xi < T \wedge \sigma_n) = 0.$$

Proof. Notice that if $\xi < T \wedge \sigma_n$, then $|X_1(\xi) - X_2(\xi)| = R_1 + R_2 - \frac{3}{4}\varepsilon$, hence

$$|X_1(t) - X_2(t)| \in [R_1 + R_2 - \varepsilon, R_1 + R_2 - \frac{\varepsilon}{2}], \qquad \text{for any } t \in [\xi - \frac{\varepsilon}{8n}, \xi].$$

We have by Ito's formula and Lemma 5.3.1 that

$$\begin{aligned}
&|X_1(t) - X_2(t)|^2 \\
= \;& |X_1(0) - X_2(0)|^2 + 2\int_0^t (X_1(s) - X_2(s)) \\
&\quad \cdot \Big[M_1(s) - M_2(s) + \eta_1(s) - \eta_2(s) + P_1^{*1}(s) - P_2^{*1}(s) \\
&\qquad -m^{-1/2} \int_0^s \Big(\nabla_1 \widetilde{U}(X_1(u), X_2(u)) - \nabla_2 \widetilde{U}(X_1(u), X_2(u))\Big) du\Big] ds,
\end{aligned}$$

so

$$\begin{aligned}
& (R_1+R_2-\frac{3}{4}\varepsilon)^2-(R_1+R_2-\varepsilon)^2 \\
\geq\ & |X_1(\xi)-X_2(\xi)|^2-|X_1(\xi-\frac{\varepsilon}{8n})-X_2(\xi-\frac{\varepsilon}{8n})|^2 \\
=\ & 2\int_{\xi-\frac{\varepsilon}{8n}}^{\xi}(X_1(s)-X_2(s)) \\
& \quad\cdot\Big[M_1(s)-M_2(s)+\eta_1(s)-\eta_2(s)+P_1^{*1}(s)-P_2^{*1}(s) \\
& \qquad -m^{-1/2}\int_0^{\xi-\frac{\varepsilon}{8n}}(\nabla_1\widetilde{U}(X_1(u),X_2(u))-\nabla_2\widetilde{U}(X_1(u),X_2(u)))du \\
& \qquad -m^{-1/2}\int_{\xi-\frac{\varepsilon}{8n}}^{s}(\nabla_1\widetilde{U}(X_1(u),X_2(u))-\nabla_2\widetilde{U}(X_1(u),X_2(u)))du\Big]ds \\
\geq\ & -2\int_{\xi-\frac{\varepsilon}{8n}}^{\xi}\Big\{(R_1+R_2-\frac{\varepsilon}{2})\Big(|M_1(s)|+|M_2(s)|+|\eta_1(s)|+|\eta_2(s)|+|P_1^{*1}(s)|+|P_2^{*1}(s)| \\
& \qquad +m^{-1/2}\int_0^{T\wedge\sigma_n}(|\nabla_1\widetilde{U}(X_1(u),X_2(u))|+|\nabla_2\widetilde{U}(X_1(u),X_2(u)))|)du\Big)\Big\}ds \\
& +2m^{-1/2}\int_{\xi-\frac{\varepsilon}{8n}}^{\xi}ds\int_{\xi-\frac{\varepsilon}{8n}}^{s} \\
& \quad\Big[-(X_1(s)-X_2(s))\cdot(\nabla_1\widetilde{U}(X_1(u),X_2(u))-\nabla_2\widetilde{U}(X_1(u),X_2(u)))\Big]du.
\end{aligned}$$

Let $C_1=(R_1+R_2-\varepsilon)^2-(R_1+R_2-\frac{3}{4}\varepsilon)^2$ and $C_2=(\frac{\varepsilon}{8n})^2C_\varepsilon\Big(1-\frac{\varepsilon/2}{R_1+R_2-\varepsilon}\Big)>0$, where C_ε is the constant given in Lemma 8.2.1 and Corollary 8.2.3. Notice that C_1 and C_2 depend only on R_1+R_2, ε and n, and do not depend on m. Also, write $Y_s=|M_1(s)|+|M_2(s)|+|\eta_1(s)|+|\eta_2(s)|+|P_1^{*1}(s)|+|P_2^{*1}(s)|$. Then with the help of Corollary 8.2.3, we get

$$\begin{aligned}
& \xi<T\wedge\sigma_n \\
\Longrightarrow\ & \\
& 2(R_1+R_2-\frac{\varepsilon}{2})\int_{\xi-\frac{\varepsilon}{8n}}^{\xi}Y_s ds+\frac{\varepsilon}{4n}(R_1+R_2-\frac{\varepsilon}{2}) \\
& \quad\times\int_0^{T\wedge\sigma_n}m^{-1/2}(|\nabla_1\widetilde{U}(X_1(u),X_2(u))|+|\nabla_2\widetilde{U}(X_1(u),X_2(u)))|)du \\
\geq\ & (R_1+R_2-\varepsilon)^2-(R_1+R_2-\frac{3}{4}\varepsilon)^2 \\
& +2m^{-1/2}\int_{\xi-\frac{\varepsilon}{8n}}^{\xi}ds\int_{\xi-\frac{\varepsilon}{8n}}^{s}\Big[-(X_1(s)-X_2(s)) \\
& \quad\cdot(\nabla_1\widetilde{U}(X_1(u),X_2(u))-\nabla_2\widetilde{U}(X_1(u),X_2(u)))\Big]du \\
\geq\ & C_1+2m^{-1/2}C_\varepsilon\Big(1-\frac{\varepsilon/2}{R_1+R_2-\varepsilon}\Big)\int_{\xi-\frac{\varepsilon}{8n}}^{\xi}ds\int_{\xi-\frac{\varepsilon}{8n}}^{s}du \\
=\ & C_1+2m^{-1/2}C_\varepsilon\Big(1-\frac{\varepsilon/2}{R_1+R_2-\varepsilon}\Big)\frac{1}{2}(\frac{\varepsilon}{8n})^2
\end{aligned}$$

$$= \quad C_1 + m^{-1/2} C_2.$$

Let $C_3 = 2\int_0^T E^{P_m}[Y_s]ds + \frac{\varepsilon}{4n}\sum_{i=1}^2 E^{P_m}\Big[\int_0^{T\wedge\sigma_n} m^{-1/2}|\nabla_i \widetilde{U}(X_1(u),X_2(u))|du\Big]$, which is finite by Lemma 5.3.1 and Lemma 5.3.2. Then the above implies that

$$\begin{aligned}
& P_m(\xi < T\wedge\sigma_n) \\
\le\ & P_m\Big(2(R_1+R_2-\frac{\varepsilon}{2})\int_{\xi-\frac{\varepsilon}{8n}}^{\xi} Y_s ds + \frac{\varepsilon}{4n}(R_1+R_2-\frac{\varepsilon}{2}) \\
& \quad \times \int_0^{T\wedge\sigma_n} m^{-1/2}(|\nabla_1\widetilde{U}(X_1(u),X_2(u))| + |\nabla_2\widetilde{U}(X_1(u),X_2(u)))|)du \\
& \ge C_1 + m^{-1/2}C_2\Big) \\
\le\ & \frac{1}{C_1+m^{-1/2}C_2} E^{P_m}\Big[2(R_1+R_2-\frac{\varepsilon}{2})\int_{\xi-\frac{\varepsilon}{8n}}^{\xi\wedge\sigma_n} Y_s ds + \frac{\varepsilon}{4n}(R_1+R_2-\frac{\varepsilon}{2}) \\
& \quad \times \int_0^{T\wedge\sigma_n} m^{-1/2}(|\nabla_1\widetilde{U}(X_1(u),X_2(u))| + |\nabla_2\widetilde{U}(X_1(u),X_2(u)))|)du\Big] \\
\le\ & \frac{1}{C_1+m^{-1/2}C_2}(R_1+R_2-\frac{\varepsilon}{2})C_3,
\end{aligned}$$

which converges to 0 as $m \to 0$.

This completes the proof of our assertion. ∎

We next show that the condition (5) of Theorem 8.3.2 is satisfied, *i.e.*, $M_1|V_1(t)|^2 + M_2|V_2(t)|^2$ is continuous in t, *a.s.*'ly, under the limit probability.

We first prepare the following.

Lemma 8.3.4 $-\nabla_1\widetilde{U}(Y_1,Y_2)\cdot\frac{Y_1-Y_2}{|Y_1-Y_2|}$ *is monotone non-increasing with respect to* $|Y_1-Y_2|$ *for* $|Y_1-Y_2| \in [R_1+R_2-\varepsilon_0, R_1+R_2]$.

Proof. As in the proof of Lemma 8.2.1, by (8.2.2), we have that in our present setting,

$$\begin{aligned}
& -\nabla_1\widetilde{U}(Y_1,Y_2)\cdot\frac{Y_1-Y_2}{|Y_1-Y_2|} \\
=\ & -\int_{B_{Y_1,Y_2}} dx \int_0^{h_2(|Y_2-x|)} f''(h_1(|Y_1-x|)+u)du\, h_1'(|Y_1-x|)\frac{Y_1-x}{|Y_1-x|}\cdot\frac{Y_1-Y_2}{|Y_1-Y_2|}.
\end{aligned}$$

Let $\widehat{B_{Y_1,Y_2}} = \{(s,t)\big|\exists x \in B_{Y_1,Y_2}, s=|Y_1-x|, t=|Y_2-x|\}$, and for any $(s,t)\in\widehat{B_{Y_1,Y_2}}$, let α, β, θ be the angles between Y_1Y_2 and Y_1x, Y_2Y_1 and Y_2x, xY_1 and xY_2, respectively. Write $A = |Y_1-Y_2|$. Finally, let $l(s,t)$ denote the length of the hyper-circle $\{x\in\mathbf{R}^d; |Y_1-x|=s, |Y_2-x|=t\}$ in $\mathbf{R}^{d-2}$. Then by using variable change,

$$\begin{aligned}
& -\nabla_1\widetilde{U}(Y_1,Y_2)\cdot\frac{Y_1-Y_2}{|Y_1-Y_2|} \\
=\ & \int_{\widehat{B_{Y_1,Y_2}}} dsdt \int_{h_2(t)}^0 \Big(-f''(h_1(s)+u)\Big)du\Big(-h_1'(s)\Big)l(s,t)\cos\alpha\sin\theta.
\end{aligned}$$

Notice that all of the terms above are positive. The integration domain $\widehat{B_{Y_1,Y_2}}$ is decreasing with respect to $|Y_1 - Y_2|$. Also, for any fixed s and t, the term $l(s,t)$ is also decreasing with respect to $|Y_1 - Y_2|$. Therefore, it is sufficient to show that for any s, t fixed, $\cos\alpha \sin\theta$ is decreasing with respect to $A = |Y_1 - Y_2|$. We shall show it from now on.

By sine formula, $\cos\alpha \sin\theta = \frac{A}{t} \sin\alpha \cos\alpha$. So it suffices to show that $A \sin\alpha \cos\alpha$ is monotone decreasing with respect to A, or equivalent, is monotone increasing with respec to α, for $\alpha > 0$ small enough. It is easy to see that $A = s\cos\alpha + \sqrt{t^2 - s^2 \sin^2\alpha}$. So

$$\begin{aligned} & A \sin\alpha \cos\alpha \\ = \;& s \sin\alpha \cos^2\alpha + \sqrt{t^2 - s^2 \sin^2\alpha}\, \sin\alpha \cos\alpha \\ = \;& s \sin\alpha (1 - \sin^2\alpha) + \sqrt{(t^2 - s^2 \sin^2\alpha)(1 - \sin^2\alpha)\sin^2\alpha}. \end{aligned}$$

Since $\alpha > 0$ is small enough, we have $\sin^2\alpha > 0$ small enough and monotone increasing with respect to α. Also, since s/t is near to $\frac{R_1}{R_2} > 0$, there exists an $\varepsilon_1 > 0$ such that functions $f_1(x) = sx(1 - x^2)$ and $f_2(x) = (t^2 - s^2 x)(1 - x)x = t^2 x(x-1)(x - \frac{s^2}{t^2})$ are monotone increasing in $x \in [0, \varepsilon_1]$. Combining these, we get the desired increasity of $A \sin\alpha \cos\alpha$ with respect to α for $\alpha > 0$ small enough, equivalent, the decreasity with respect to A.

This completes the proof of our assertion. ∎

Let $\xi_0 = \xi_{\varepsilon_0} = \inf\{t > 0; |X_1(t) - X_2(t)| \le R_1 + R_2 - \frac{3}{4}\varepsilon_0\} \wedge \sigma_n \wedge T$. Then as a easy result of Lemma 8.3.3, $P_m(\xi_0 < T \wedge \sigma_n) \to 0$ as $m \to \infty$. We next use Lemma 8.3.4 to show the following.

Lemma 8.3.5 $\lim_{m\to 0} P_m\Big(\int_0^{T\wedge\sigma_n\wedge\xi_0} m^{-1/2}\big(\widetilde{U}(\vec{X}(s)) - \widetilde{U}_0\big)ds > \delta\Big) = 0$ *for any* $\delta > 0$.

Proof. By Lemma 8.2.2, we have that $-\nabla_1 \widetilde{U}(Y_1, Y_2) \cdot \frac{Y_1 - Y_2}{|Y_1 - Y_2|}$ is positive for $|Y_1 - Y_2| \in [R_1 + R_2 - \varepsilon_0, R_1 + R_2)$. Also, by Lemma 8.3.4, it is monotone non-increasing with respec to $|Y_1 - Y_2|$. Notice that $\widetilde{U}(X_1, X_2) = \widetilde{U}(X_1 - X_2, 0)$. So with a little bit abuse of notation, we can write $\widetilde{U}(X_1, X_2) = \widetilde{U}(X_1 - X_2)$. Then we have that $\widetilde{U}(X_1, X_2) - \widetilde{U}_0 = 0$ if $|X_1 - X_2| \ge R_1 + R_2$. Also, for any $|X_1 - X_2| < R_1 + R_2$, we have $\widetilde{U}\Big(\frac{R_1+R_2}{|X_1-X_2|}(X_1 - X_2)\Big) = \widetilde{U}_0$, and $\frac{R_1+R_2}{|X_1-X_2|} + t\Big(1 - \frac{R_1+R_2}{|X_1-X_2|}\Big) \ge 1$ for $t \in [0,1]$, hence

$$\begin{aligned} & \widetilde{U}(X_1, X_2) - \widetilde{U}_0 = \widetilde{U}(X_1 - X_2) - \widetilde{U}\Big(\frac{R_1+R_2}{|X_1 - X_2|}(X_1 - X_2)\Big) \\ = \;& \int_0^1 -\nabla_1 \widetilde{U}\Big(\frac{R_1+R_2}{|X_1-X_2|}(X_1 - X_2) + t\Big(1 - \frac{R_1+R_2}{|X_1-X_2|}\Big)(X_1 - X_2)\Big) \\ & \qquad \cdot\Big(-1 + \frac{R_1+R_2}{|X_1-X_2|}\Big)(X_1 - X_2)dt \\ \le \;& \int_0^1 -\nabla_1 \widetilde{U}(X_1 - X_2) \cdot (X_1 - X_2)\Big(-1 + \frac{R_1+R_2}{|X_1-X_2|}\Big)dt \end{aligned}$$

$$
\begin{aligned}
&= -\nabla_1\widetilde{U}(X_1-X_2)\cdot(X_1-X_2)\Big(-1+\frac{R_1+R_2}{|X_1-X_2|}\Big)\\
&\le \Big|\nabla_1\widetilde{U}(X_1-X_2)\Big||X_1-X_2|\frac{R_1+R_2-|X_1-X_2|}{|X_1-X_2|}\\
&= \Big|\nabla_1\widetilde{U}(X_1-X_2)\Big|(R_1+R_2-|X_1-X_2|).
\end{aligned}
$$

The first equation in the calculation above also gives us that $\widetilde{U}(X_1,X_2)-\widetilde{U}_0$ is non-negative. Therefore, by Lemma 5.3.2, there exists a constant $C>0$ such that for any $\varepsilon\in(0,\frac{3}{4}\varepsilon_0)$,

$$
\begin{aligned}
&P_m\Big(\int_0^{T\wedge\sigma_n\wedge\xi_0} m^{-1/2}\Big(\widetilde{U}(\vec{X}(s))-\widetilde{U}_0\Big)ds>\delta\Big)\\
\le\ & P_m\Big(\int_0^{T\wedge\sigma_n\wedge\xi_0} m^{-1/2}|\nabla_1\widetilde{U}(X_1(s)-X_2(s))|\\
&\qquad\times(R_1+R_2-|X_1(s)-X_2(s)|)1_{\{|X_1(s)-X_2(s)|<R_1+R_2\}}ds>\delta\Big)\\
\le\ & P_m\Big(\inf_{s\in[0,T\wedge\sigma_n]}|X_1(s)-X_2(s)|\le R_1+R_2-\varepsilon\Big)\\
&\qquad+P_m\Big(\int_0^{T\wedge\sigma_n\wedge\xi_0} m^{-1/2}|\nabla_1\widetilde{U}(X_1(s)-X_2(s))|ds>\frac{\delta}{\varepsilon}\Big)\\
\le\ & P_m(\xi_{\frac{4}{3}\varepsilon}<T\wedge\sigma_n)+\frac{\varepsilon}{\delta}E^{P_m}\Big[\int_0^{T\wedge\sigma_n\wedge\xi_0} m^{-1/2}|\nabla_1\widetilde{U}(X_1(s)-X_2(s))|ds\Big]\\
\le\ & P_m(\xi_{\frac{4}{3}\varepsilon}<T\wedge\sigma_n)+\frac{\varepsilon}{\delta}C.
\end{aligned}
$$

But by Lemma 8.3.3, $P_m(\xi_{\frac{4}{3}\varepsilon}<T\wedge\sigma_n)\to 0$ as $m\to 0$ for any $\varepsilon>0$. Therefore, taking first $\varepsilon>0$ small enough and then $m>0$ small enough, we get our assertion.
∎

We are now ready to show that the condition (5) of Theorem 8.3.2 is satisfied.

Lemma 8.3.6 *$M_1|V_1(t)|^2+M_2|V_2(t)|^2$ is continuous in t almost suely, under the limit probability.*

Proof. Write the limit probability measure as P_∞. Let

$$
H_s^m=m^{-1/2}\Big(\widetilde{U}(\vec{X}(s))-\widetilde{U}_0\Big)+\frac{1}{2}\sum_{i=1}^{2}M_i|V_i(s)|^2.
$$

Then we have by Lemma 5.3.2 that under our present setting, $(H^m_{t\wedge\sigma_n\wedge\xi_0})_t$ under P_m is tight in $\wp(C([0,T];\mathbf{R}^d))$. *i.e.*, there exists a $H_s\in C([0,T];\mathbf{R}^d)$ such that

$$
(H_s^m)_s \text{ under } P_m\to(H_s)_s \text{ under } P_\infty
$$

in $\wp(C([0,T];\mathbf{R}^d))$ as $m\to 0$. Also, as we have shown at the beginning of this Chapter,

$$
(V_i^2(s))_s \text{ under } P_m\to(V_i^2(s))_s \text{ under } P_\infty.
$$

in $\wp(D([0,T];\mathbf{R}^d),\widetilde{dist})$ as $m \to 0$. So

$$(H_s^m - \frac{1}{2}\sum_{i=1}^{2} M_i V_i(s)^2)_s \text{ under } P_m \to (H_s - \frac{1}{2}\sum_{i=1}^{2} M_i V_i(s)^2)_s \text{ under } P_\infty$$

in $\wp(D([0,T];\mathbf{R}^d))$ as $m \to 0$. However, for any $\delta > 0$, we have by Lemma 8.3.5 that

$$P_m\Big(\int_0^{T\wedge\sigma_n\wedge\xi_0} |H_s^m - \frac{1}{2}\sum_{i=1}^{2} M_i V_i(s)^2| ds > \delta\Big) \to 0, \qquad \text{as } m \to 0.$$

So

$$P_\infty\Big(\int_0^{T\wedge\sigma_n\wedge\xi_0} |H_s - \frac{1}{2}\sum_{i=1}^{2} M_i V_i(s)^2| ds > \delta\Big) = 0$$

for any $\delta > 0$. Also, $\xi_0 \to \infty$ as $m \to 0$, hence

$$\int_0^{T\wedge\sigma_n} |H_s - \frac{1}{2}\sum_{i=1}^{2} M_i V_i(s)^2| ds = 0, \quad P_\infty - a.e.$$

This combined with the continuity of H_s and the fact that $\sigma_n \to \infty$ *a.e.* gives us that $M_1|V_1(t)|^2 + M_2|V_2(t)|^2$ is continuous in t P_∞-almost surely. ∎

We finally show that the condition (4) of Theorem 8.3.2 is satisfied. The method is similar to that of the proof of (5).

Let $Y_i(t) = V_i(t) - M_i^{-1}\eta_i(t)$, $i = 1, 2$, where $\eta_i(t)$ is as given in Lemma 5.3.1. Let $\vec{Y}(t) = (Y_1(t), Y_2(t))$, and let

$$\begin{aligned} G_t &= m^{-1/2}\int_0^t \Big\{M_1^{-1} f_{V_1}(\vec{X}(s),\vec{Y}(s)) \cdot \nabla_1 \widetilde{U}(\vec{X}(s)) \\ &\quad + M_2^{-1} f_{V_2}(\vec{X}(s),\vec{Y}(s)) \cdot \nabla_2 \widetilde{U}(\vec{X}(s))\Big\} ds + f(\vec{X}(t),\vec{V}(t)). \end{aligned}$$

We first show the following.

Lemma 8.3.7 *$(G_{t\wedge\sigma_n})_t$ under P_m is tight in $\wp(C([0,T];\mathbf{R}^d))$.*

Proof. Let

$$\widetilde{G}_t = G_t - f(\vec{X}(t),\vec{V}(t)) + f(\vec{X}(t),\vec{Y}(t)).$$

Then

$$|G_t - \widetilde{G}_t| \le \|f_{V_1}\|_\infty M_1^{-1}|\eta_1(t)| + \|f_{V_2}\|_\infty M_2^{-1}|\eta_2(t)|.$$

Therefore, by Lemma 5.3.1 (4), the tightnessness of $(G_{t\wedge\sigma_n})_t$ under P_m in $\wp(C([0,T];\mathbf{R}^d))$ is equivalent to the tightnessness of $(\widetilde{G}_{t\wedge\sigma_n})_t$ under P_m in $\wp(C([0,T];\mathbf{R}^d))$.

On the other hand, we have by Lemma 5.3.1 and Ito's formula that

$$\begin{aligned} d\widetilde{G}_t &= f_{X_1}(\vec{X}(t),\vec{Y}(t)) \cdot V_1(t)dt + f_{X_2}(\vec{X}(t),\vec{Y}(t)) \cdot V_2(t)dt \\ &\quad + M_1^{-1} f_{V_1}(\vec{X}(t),\vec{Y}(t)) \cdot (dM_1(t) + dP_1^{*1}(t)) \\ &\quad + M_2^{-1} f_{V_2}(\vec{X}(t),\vec{Y}(t)) \cdot (dM_2(t) + dP_2^{*1}(t)). \end{aligned}$$

So by Lemma 5.3.1 (2), (6.4.2) and Theorem 5.1.7, $(\widetilde{G}_{t\wedge\sigma_n})_t$ under P_m is tight in $\wp(C([0,T];\mathbf{R}^d))$. This completes the proof of our assertion. ∎

Lemma 8.3.8 *Suppose that $f \in C_0^\infty(\mathbf{R}^{4d})$ satisfies the condition in (4) of Theorem 8.3.2. Then for any $\delta > 0$, we have that*

$$\lim_{m\to 0} P_m\Big(\int_0^{T\wedge\sigma_n\wedge\xi_0} \Big|m^{-1/2}\int_0^t \Big\{M_1^{-1} f_{V_1}(\vec X(s),\vec Y(s))\cdot\nabla_1\widetilde U(\vec X(s)) \\ +M_2^{-1} f_{V_2}(\vec X(s),\vec Y(s))\cdot\nabla_2\widetilde U(\vec X(s))\Big\}ds\Big|dt > \delta\Big) = 0.$$

Proof. First notice that $\nabla_i\widetilde U(X_1,X_2) = 0$ if $|X_1 - X_2| > R_1 + R_2$. For any $X_1, X_2 \in \mathbf{R}^d$ with $|X_1 - X_2| \le R_1 + R_2$, let $\widetilde X_i = \frac{R_1+R_2}{|X_1-X_2|}X_i$, $i = 1, 2$. Then $|\widetilde X_1 - \widetilde X_2| = R_1 + R_2$, *i.e.*, $\widetilde X = (\widetilde X_1, \widetilde X_2) \in \partial D_0$. Also, as shown before, $-\nabla_1\widetilde U(X_1,X_2) = \nabla_2\widetilde U(X_1,X_2)$ is parallel with same direction to $X_1 - X_2$, so

$$\begin{aligned}
\nabla_1\widetilde U(X_1,X_2) &= -\frac{|\nabla_1\widetilde U(X_1,X_2)|}{|X_1-X_2|}(X_1-X_2) = -\frac{|\nabla_1\widetilde U(X_1,X_2)|}{R_1+R_2}(\widetilde X_1-\widetilde X_2),\\
\nabla_2\widetilde U(X_1,X_2) &= +\frac{|\nabla_2\widetilde U(X_1,X_2)|}{|X_1-X_2|}(X_1-X_2) = +\frac{|\nabla_1\widetilde U(X_1,X_2)|}{R_1+R_2}(\widetilde X_1-\widetilde X_2).
\end{aligned}$$

So by assumption, for any $Y \in \mathbf{R}^{2d}$,

$$\begin{aligned}
& M_1^{-1} f_{V_1}(\widetilde X, Y)\cdot\nabla_1\widetilde U(X_1,X_2) + M_2^{-1} f_{V_2}(\widetilde X, Y)\cdot\nabla_2\widetilde U(X_1,X_2)\\
=\ & \frac{|\nabla_1\widetilde U(X_1,X_2)|}{R_1+R_2}\Big(- M_1^{-1} f_{V_1}(\widetilde X,Y)\cdot(\widetilde X_1-\widetilde X_2) + M_2^{-1} f_{V_2}(\widetilde X,Y)\cdot(\widetilde X_1-\widetilde X_2)\Big)\\
=\ & 0,
\end{aligned}$$

hence if we let $C_1 = M_1^{-1}\|f_{XV_1}\|_\infty \vee M_2^{-1}\|f_{XV_2}\|_\infty$, then

$$\begin{aligned}
& \Big|M_1^{-1} f_{V_1}(X,Y)\cdot\nabla_1\widetilde U(X_1,X_2) + M_2^{-1} f_{V_2}(X,Y)\cdot\nabla_2\widetilde U(X_1,X_2)\Big|\\
=\ & \Big|M_1^{-1}\Big(f_{V_1}(X,Y) - f_{V_1}(\widetilde X,Y)\Big)\cdot\nabla_1\widetilde U(X_1,X_2)\\
& \quad +M_2^{-1}\Big(f_{V_2}(X,Y) - f_{V_2}(\widetilde X,Y)\Big)\cdot\nabla_2\widetilde U(X_1,X_2)\Big|\\
\le\ & M_1^{-1}\|f_{XV_1}\|_\infty|X-\widetilde X||\nabla_1\widetilde U(X_1,X_2)| + M_2^{-1}\|f_{XV_2}\|_\infty|X-\widetilde X||\nabla_2\widetilde U(X_1,X_2)|\\
\le\ & C_1(|\nabla_1\widetilde U(X_1,X_2)| + |\nabla_2\widetilde U(X_1,X_2)|)\Big(\frac{R_1+R_2}{|X_1-X_2|} - 1\Big)|X|.
\end{aligned}$$

Let $C_2 = 2(|\vec X_0| + 2nT)(R_1+R_2)^{-1}$, and let

$$C_3 = C_1C_2E^{P_m}\Big[\int_0^{T\wedge\sigma_n} m^{-1/2}(|\nabla_1\widetilde U(\vec X(s))| + |\nabla_2\widetilde U(\vec X(s))|)ds\Big],$$

which is finite by Lemma 5.3.2. Then by the calculation above, we have for any $\varepsilon \in [0, \frac34\varepsilon_0 \wedge \frac12(R_1+R_2))$, (hence $R_1 + R_2 - \varepsilon > \frac12(R_1+R_2)$),

$$P_m\Big(\int_0^{T\wedge\sigma_n\wedge\xi_0} \Big|m^{-1/2}\int_0^t \Big\{M_1^{-1} f_{V_1}(\vec X(s),\vec Y(s))\cdot\nabla_1\widetilde U(\vec X(s))$$

$$
\begin{aligned}
&+M_2^{-1}f_{V_2}(\vec{X}(s),\vec{Y}(s))\cdot\nabla_2\widetilde{U}(\vec{X}(s))\Big\}ds\Big|dt>\delta\Big)\\
\le\;& P_m\Big(\int_0^{T\wedge\sigma_n\wedge\xi_0} m^{-1/2}C_1(|\nabla_1\widetilde{U}(\vec{X}(s))|+|\nabla_2\widetilde{U}(\vec{X}(s))|)\\
&(|\vec{X}_0|+2nT)\Big(\frac{R_1+R_2}{|X_1(s)-X_2(s)|}-1\Big)1_{\{|X_1(s)-X_2(s)|<R_1+R_2\}}ds>\delta\Big)\\
\le\;& P_m\Big(\inf_{s\in[0,T\wedge\sigma_n]}|X_1(s)-X_2(s)|\le R_1+R_2-\varepsilon\Big)\\
&+P_m\Big(\int_0^{T\wedge\sigma_n\wedge\xi_0} m^{-1/2}C_1(|\nabla_1\widetilde{U}(\vec{X}(s))|+|\nabla_2\widetilde{U}(\vec{X}(s))|)ds\\
&>\delta\Big(|\vec{X}_0|+2nT)\frac{\varepsilon}{(R_1+R_2)/2}\Big)^{-1}\Big)\\
\le\;& P_m(\xi_{\frac{4}{3}\varepsilon}<T\wedge\sigma_n)+C_1C_2\cdot\frac{\varepsilon}{\delta}E^{P_m}\Big[\int_0^{T\wedge\sigma_n\wedge\xi_0} m^{-1/2}C_1\Big(\textstyle\sum_{i=1}^2|\nabla_i\widetilde{U}(\vec{X}(s))|\Big)ds\Big]\\
\le\;& P_m(\xi_{\frac{4}{3}\varepsilon}<T\wedge\sigma_n)+\frac{\varepsilon}{\delta}C_3.
\end{aligned}
$$

Since $P_m(\xi_{\frac{4}{3}\varepsilon}<T\wedge\sigma_n)\to 0$ as $m\to 0$ for any $\varepsilon>0$ by Lemma 8.3.3, we get our assertion by taking first $\varepsilon>0$ small enough and then $m>0$ small enough. ∎

By using the same argument as in the proof of Lemma 8.3.6, from Lemma 8.3.7 and Lemma 8.3.8, we get the following, which means that the condition (4) of Theorem 8.3.2 is also satisfied.

Lemma 8.3.9 *Assume that $f\in C_0^\infty(\mathbf{R}^{4d})$ satisfies*

$$M_1^{-1}(\nabla_{V_1}f)(\vec{X},\vec{V})\cdot(X_1-X_2)+M_2^{-1}(\nabla_{V_2}f)(\vec{X},\vec{V})\cdot(X_2-X_1)=0$$

for any $(\vec{X},\vec{V})\in\partial D_0\times\mathbf{R}^{2d}$, then $f(\vec{X}(t),\vec{V}(t))$ is continuous in t almost suely, under the limit probability.

This completes the proof of the fact that under our present setting, any cluster point of the distribution of $(X_t,V_t)_t$ under P_m as $m\to 0$ satisfies all of the conditions of Theorem 8.3.2. Therefore, by the uniqueness, the distribution of $(X_t,V_t)_t$ under P_m converges to $P_{\infty,0}$ as $m\to 0$.

Remark 3 *The result of this chapter holds only for d ≥ 3. Actually, as remarked at the end of Chapter 7, when d $=2$, the drift term $m^{-1/2}\int_0^{t\wedge\sigma}\nabla_i\widetilde{U}(\vec{X}(s))ds$ is always 0, so the force of replusion between two atoms, given by the limit of $m^{-1/2}\nabla_i\widetilde{U}(\vec{X}(s))$, is always 0.*

The case for d $=1$ is even more different. Notice that to keep the two atoms replusive when $|X_1-X_2|\in(R_1+R_2-\varepsilon_0,R_1+R_2)$, the only way is to keep the force $-\nabla_1\widetilde{U}(X_1,X_2)\cdot(X_1-X_2)$ positive in this domain. This was also the main idea of this chapter. However, when d $=1$, if we make the same assumption as for d ≥ 3, then for $|X_1-X_2|\in(R_1+R_2-\varepsilon_0,R_1+R_2)$, as shown below, the "limit force between two atoms", given by $m^{-1/2}\nabla_i\widetilde{U}(X_1,X_2)$, becomes "attractive force" *instead of "repulsive force". This will certainly cause problem. Actually, by (8.2.2) (with the set B_{X_1,X_2} now given by $B_{X_1,X_2} = (X_1 - R_1, X_2 + R_2)$ for any $X_1 > X_2$ and $|X_1 - X_2| \in (R_1 + R_2 - \varepsilon_0, R_1 + R_2)$), we have that*

$$
\begin{aligned}
&-\nabla_1 \widetilde{U}(X_1, X_2) \cdot (X_1 - X_2) \\
&= -\int_{B_{X_1,X_2}} dx \int_0^{h_2(|X_2-x|)} f''\Big(h_1(|X_1 - x|) + u\Big) du h_1'(|X_1 - x|) \frac{X_1 - x}{|X_1 - x|} \cdot (X_1 - X_2).
\end{aligned}
$$

Since in the present setting, $h_2(|X_2 - x|) < 0$, $h_1'(|X_1 - x|) < 0$, $\frac{X_1 - x}{|X_1 - x|} \cdot (X_1 - X_2) > 0$ and $f'' > 0$ by (5.4.12), we get that $-\nabla_1 \widetilde{U}(X_1, X_2) \cdot (X_1 - X_2)$ is negative, i.e., X_1 get the force $-\nabla_i \widetilde{U}(\vec{X})$ towards X_2.

However, for $d = 1$, e.g., if we assume that $U_i(x) = h_i(|x|)$, and there exists a constant $\varepsilon_0 > 0$ such that $h_i(s) > 0$, $h_i''(s) > 0$, $s \in (R_i - \varepsilon_0, R_i)$, $i = 1, 2$, then by the calculation above, the force will keep "repulsive".

Remark 4 *Let us consider a little bit more intuitively. Notice that*

$$
\begin{aligned}
-\nabla U_i(X_i - x) \cdot (X_i - x) &= -h_i'(|X_i - x|) \frac{X_i - x}{|X_i - x|} \cdot (X_i - x) \\
&= -|X_i - x| h_i'(|X_i - x|).
\end{aligned}
$$

So intuitively, for X_i and x with distance in a certain domain, the assumption of this chapter means the following: one of the two atoms has "repulsive force" with the small particles, and the other one has "attractive force" with the small particles. The "attractive" one actually is more troublesome. When the space dimension d is big enough, since the velocity of a certain small particle could be any direction, even though it get attracted strongly by a atom when they are near, it can "escape" within a short time. In space dimension 2, although not as nice as in the case $d \geq 3$, it still can "escape" safely. However, when $d = 1$, all of the particles stay in and move on only a line, so if there exists a attractive force, it will get no hope to escape. In order to avoid this situation, we have to have that both of the two forces are repulsive. This is actually the case we introduced at the end of the last remark.

Certainly, this problem might also be caused by our cutoffs U_i and ρ.

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