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by

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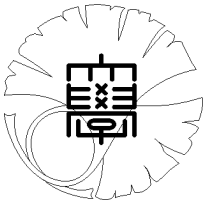

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TWISTED SECOND HOMOLOGY GROUPS OF THE AUTOMORPHISM GROUP OF A FREE GROUP

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Abstract. In this paper, we compute the second homology groups of the automorphism group of a free group with coefficients in the abelianization of the free group and its dual group except for the 2-torsion part, using combinatorial group theory.

1. Introduction

Let F_n be a free group of rank n, and let $\mathrm{Aut}\, F_n$ denote the automorphism group of F_n. There are several remarkable computation of the (co)homology groups of $\mathrm{Aut}\, F_n$ with trivial coefficients. For example, Gersten [2] showed $H_2(\mathrm{Aut}\ F_n, \mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$ for $n \geq 5$, and Hatcher and Vogtmann [3] showed $H_q(\mathrm{Aut}\, F_n, \mathbf{Q}) = 0$ for $n \geq 1$ and $1 \leq q \leq 6$, except for $H_4(\mathrm{Aut}\, F_4, \mathbf{Q}) = \mathbf{Q}$. In this paper we consider twisted (co)homology groups of $\mathrm{Aut}\, F_n$. Let H be the abelianization of F_n and $H^* := \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$ the dual group of H. The group $\mathrm{Aut}\, F_n$ naturally acts on H and H^*. The main interest of this paper is to compute the homology groups of $\mathrm{Aut}\, F_n$ with coefficients in H and H^* using combinatorial groups theory, in particular using a finite presentation for $\mathrm{Aut}\, F_n$.

In our previous paper [8], we computed the first homology groups of $\mathrm{Aut}\, F_n$ with coefficients in H and H^* for $n \geq 2$. In this paper, we show that the second homology groups $H_2(\mathrm{Aut}\, F_n, H)$ and $H_2(\mathrm{Aut}\, F_n, H^*)$ are trivial except for the $\mathbf{Z}/2\mathbf{Z}$-part for $n \geq 6$. Let L be the localization of the ring $\mathbf{Z}$ of integers by the prime ideal $2\mathbf{Z}$. The ring L is a principal ideal domain in which the element 2 is invertible, and naturally isomorphic to the subring $\mathbf{Z}[\frac{1}{2}]$ of $\mathbf{Q}$ which is obtained from the ring $\mathbf{Z}$ by attaching $1/2$. For any $\mathbf{Z}$-module M, we denote by M_L the L-module $M \otimes_{\mathbf{Z}} L$. Then our main theorem is

Theorem 1.1. *For $n \geq 6$,*

$$H_2(\mathrm{Aut}\, F_n, H_L) = 0, \qquad H_2(\mathrm{Aut}\, F_n, H_L^*) = 0.$$

Recently, Hatcher and Wahl [4] showed $H_i(\mathrm{Aut}\, F_n, H) = 0$ for $n \geq 3i + 9$ using the stability of the homology groups of the mapping class groups of certain 3-manifolds. If $n \geq 15$, one of our result $H_2(\mathrm{Aut}\, F_n, H_L) = 0$ is immediately follows from the results of them. Our computation, however, is based on combinatorial group theory, and quite different from that of them. Furthermore we remark that the computation of $H_2(\mathrm{Aut}\, F_n, H_L^*) = 0$, to which we cannot apply their method directly, is more complicated than that of $H_2(\mathrm{Aut}\, F_n, H_L) = 0$.

Here we summarize the proof of Theorem 1.1. To begin with, we review the computation of $H_1(\mathrm{Aut}\, F_n, H) = 0$ and $H_1(\mathrm{Aut}\, F_n, H^*) = \mathbf{Z}$ for $n \geq 4$ due to [8]. Let $\mathrm{Aut}^+ F_n$ be the index-2 subgroup of $\mathrm{Aut}\, F_n$ defined by the kernel of the composition

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map of a natural map $\rho : \mathrm{Aut}\, F_n \twoheadrightarrow \mathrm{Aut}\, H = GL(n, \mathbf{Z})$ and the determinant map $\det : GL(n, \mathbf{Z}) \longrightarrow \{\pm 1\}$. To compute the first homology groups of $\mathrm{Aut}\, F_n$, we computed those of $\mathrm{Aut}^+ F_n$ using a finite presentation for it due to Gersten [2]. Observing the Lyndon-Hochshild-Serre spectral sequence of

$$1 \to \mathrm{Aut}^+ F_n \to \mathrm{Aut}\, F_n \to \{\pm 1\} \to 1, \tag{1}$$

we obtain the results for $\mathrm{Aut}\, F_n$. We also computed the homology groups for $n = 2$ and 3, and showed they have non-trivial 2-torsions. For details, see [8].

Now, we show the outline of the computation of the second homology groups. First, we compute the second homology groups of $\mathrm{Aut}^+ F_n$, using a reduced finite presentation $\langle X \mid R \rangle$ for it introduced in Section 2, which is obtained from the Gersten's presentation using Tietze transformations. Let F and $\bar{R}$ be the free group on X and the normal closure of R in F respectively. Then, for $M = H_L$ and H_L^*, we have a five-term exact sequence

$$\begin{aligned} H_2(F, M) \to H_2(\mathrm{Aut}^+ F_n, M) \to H_1(\bar{R}, M)_{\mathrm{Aut}^+ F_n} \\ \xrightarrow{\varphi} H_1(F, M) \to H_1(\mathrm{Aut}^+ F_n, M) \to 0 \end{aligned}$$

of L-modules. Since F is a free group, $H_2(F, M) = 0$. Furthermore, we see $H_1(F, M) = L^{\oplus\{2n(n^2-n)-n\}}$, and we have obtained the rank r_M of the free L-module $H_1(\mathrm{Aut}^+ F_n, M)$ by the results of [8]. In Section 3, we show that

$$H_1(\bar{R}, M)_{\mathrm{Aut}^+ F_n} = L^{\oplus\{2n(n^2-n)-n-r_M\}}$$

by reducing generators of $H_1(\bar{R}, M)_{\mathrm{Aut}^+ F_n}$. This implies that the map φ is injective, and hence $H_2(\mathrm{Aut}^+ F_n, M) = 0$. Then, considering the homological Lyndon-Hochsild-Serre spectral sequence of (1), we obtain $H_2(\mathrm{Aut}\, F_n, M) = 0$.

In Section 2, we introduce some tools which we use in our computation in Section 3.

Contents

2. Tools for the computation

In this section, we prepare some tools to compute the second homology groups. First, we introduce some notation which we use throughout this paper. Then, we review a finite presentation for Aut^+F_n due to Gersten [2] and reduced finite presentation $\langle X \mid R\rangle$ for Aut^+F_n of the Gersten's presentation. Finally, we show some useful lemmas and equations which are used in Section 3 to reduce the generators of $H_1(\bar{R}, M)_{\mathrm{Aut}^+F_n}$ where $M = H_L$ and H_L^*, and $\bar{R}$ is the normal closure of R in the free group on the generating set X.

Let F_n be a free group of rank n with generators $\{x_1, \dots, x_n\}$. In this paper, the group $\mathrm{Aut}\, F_n$ acts on F_n on the right. For any $\sigma \in \mathrm{Aut}\, F_n$ and $x \in F_n$, the action of σ on x is denoted by x^σ. The elements $x_i^{\pm 1} \in F_n$, $(1 \le i \le n)$, are called letters of F_n. Let H be the abelianization of F_n and $H^* := \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$ the dual group of H. We remark that although the group H^* is isomorphic to H as a free abelian group, both group are not isomorphic as an $\mathrm{Aut}\, F_n$-module. For each generator $x_i \in F_n$, $(1 \le i \le n)$, set $e_i := [x_i] \in H$ where $[x]$ means the coset calss of x modulo the commutator subgroup of F_n. Then $\{e_1, \dots, e_n\}$ is a $\mathbf{Z}$-basis of H. Let denote $\{e_1^*, \dots, e_n^*\}$ the dual basis of it. In general, in group (co)homology theory, actions of groups on modules are understood to be left actions. So we consider the modules H and H^* as left $\mathrm{Aut}\, F_n$-modules in a way $\sigma \cdot x := x^{\sigma^{-1}}$ for $\sigma \in \mathrm{Aut}\, F_n$ and $x \in H$ or H^*.

Now, for any letters a and b such that $a \neq b^{\pm 1}$, let E_{ab} be an automorphism of F_n defined by the rule

$$E_{ab} : \begin{cases} a & \mapsto ab, \\ c & \mapsto c, \quad c \neq a^{\pm 1}. \end{cases}$$

Clearly, we see ${E_{ab}}^{-1} = E_{ab^{-1}}$. Automorphisms of F_n of E_{ab} type are called Nielsen automorphisms. In this paper, for simplicity, we write $E_{i^{\pm 1}j^{\pm 1}}$ for $E_{x_i^{\pm 1}x_j^{\pm 1}}$.

The actions of $E_{i^{\pm 1}j}$ on e_k and e_k^* are given by

$$E_{i^{\pm 1}j} \cdot e_k := [x_k^{E_{i^{\pm 1}j^{-1}}}] = \begin{cases} e_i \mp e_j, & k = i, \\ e_k, & k \neq i \end{cases}$$

and

$$E_{i^{\pm 1}j} \cdot e_k^* = \begin{cases} e_j^* \pm e_i^*, & k = j, \\ e_k^*, & k \neq j \end{cases}$$

respectively. An automorphism $w_{ab} := E_{ba}E_{a^{-1}b}E_{b^{-1}a^{-1}}$ is called a monomial automorphism $a \mapsto b^{-1}$, $b \mapsto a$. We see ${w_{ab}}^{-1} = w_{ab^{-1}}$, and write $w_{i^{\pm 1}j^{\pm 1}}$ for $w_{w_i^{\pm 1}x_j^{\pm 1}}$.

Let $\rho : \mathrm{Aut}\, F_n \twoheadrightarrow GL(n, \mathbf{Z})$ be a natural homomorphism induced from the action of $\mathrm{Aut}\, F_n$ on H, and $\det : GL(n, \mathbf{Z}) \longrightarrow \{\pm 1\}$ the determinant homomorphism. The kernel Aut^+F_n of the composition map $\det \circ \rho$ is called the special automorphism group of a free group. Here we review a finite presentation for Aut^+F_n due to Gersten. He [2] showed

Theorem 2.1 (Gersten [2])**.** *For $n \ge 3$, the group Aut^+F_n has a finite presentation whose generators are E_{ab} subject to relators:*

(R1): $E_{ab}E_{ab^{-1}}$,

(R2): $[E_{ab}, E_{cd}]$, for $a \neq c, d^{\pm 1}$ and $b \neq c^{\pm 1}$,
(R3): $[E_{ab}, E_{bc}]E_{ac^{-1}}$, for $a \neq c^{\pm 1}$,
(R4): $w_{ab}w_{a^{-1}b}$,
(R5): ${w_{ab}}^4$.

Here [,] denotes the commutator bracket defined by $[x, y] := xyx^{-1}y^{-1}$. In this paper we often use fundamental formulae of commutators

$$[x, yz] = [x, y][x, z][[z, x], y], \quad [xy, z] = [x, [y, z]][y, z][x, z]. \tag{2}$$

We call the relators above the Gersten's relators. In our paper [8], using Tietze transformations, we reduced the Gersten's presentation to

Lemma 2.1. *For $n \geq 3$, the group Aut^+F_n has a finite presentation whose generators are $E_{i^{\pm 1}j}$ subject to the relators:*

(R2-1): $[E_{ij}, E_{i^{-1}j}]$,
(R2-2): $[E_{ij}, E_{kj}]$,
(R2-3): $[E_{i^{-1}j}, E_{kj}]$,
(R2-4): $[E_{i^{-1}j}, E_{k^{-1}j}]$,
(R2-5): $[E_{ij}, E_{i^{-1}k}]$,
(R2-6): $[E_{ij}, E_{kl}]$,
(R2-7): $[E_{i^{-1}j}, E_{kl}]$,
(R2-8): $[E_{i^{-1}j}, E_{k^{-1}l}]$,
(R3-1): $[E_{ik}, E_{kj}]E_{ij^{-1}}$,
(R3-2): $[E_{ik^{-1}}, E_{k^{-1}j}]E_{ij^{-1}}$,
(R3-3): $[E_{i^{-1}k}, E_{kj}]E_{i^{-1}j^{-1}}$,
(R3-4): $[E_{i^{-1}k^{-1}}, E_{k^{-1}j}]E_{i^{-1}j^{-1}}$,
(R4-1): $w_{ij}w_{i^{-1}j}$,
(R5-1): ${w_{ij}}^4$

where i, j, k and l are distinct elements of $\{1, \ldots, n\}$.

In Section 3, we use this reduced presentation to compute the twisted second homology groups. In the computation of the second homology groups,

Let X and R be the set of generators and relators of the reduced presentation for Aut^+F_n introduced in Lemma 2.1 respectively. In the following, we study relations among the relators of the presentation $\langle X \mid R \rangle$, which is often required in the computation of the second homology groups. Let F be the free group on X, and $\bar{R}$ the normal closure of R in F. Here we define elements $r_{ac}(b)$ and h_{ab} of F to be

$$r_{ac}(b) := [E_{ab}, E_{bc}]\, E_{ac^{-1}} \quad \text{for } a \neq b^{\pm 1}, c^{\pm 1} \text{ and } b \neq c^{\pm 1}$$

and

$$h_{ab} := w_{ab}w_{a^{-1}b} \quad \text{for } a \neq b^{\pm 1}$$

respectively. Since $r_{ac}(b)$ and h_{ab} are the one of relators of the Gersten's presentation, we see that these elemets are in $\bar{R}$. In this paper, we write $r_{i^{\pm 1}j^{\pm 1}}(k^{\pm 1})$ and $h_{i^{\pm}j^{\pm 1}}$ for $r_{x_i^{\pm 1}x_j^{\pm 1}}(x_k^{\pm 1})$ and $h_{x_i^{\pm}x_j^{\pm 1}}$ respectively.

For letters a, b, c and d, we consider an element $({w_{ab}}^{-1}E_{cd}w_{ab})^{-1}E_{c^\sigma d^\sigma}$ of $\bar{R}$ where σ is the monomial map defined by w_{ab}. More precisely, we study how the elements $({w_{ab}}^{-1}E_{cd}w_{ab})^{-1}E_{c^\sigma d^\sigma}$ are rewritten with the relators of the Gersten's presentation. First, we consider the case $\sharp\{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}\} = 6$.

Lemma 2.2. *For letters a, b, c, d, we have*

(i) *if $c = a^{-1}$,*

$$\begin{aligned}(w_{ab}{}^{-1} E_{a^{-1}d} w_{ab})^{-1} E_{bd} &= E_{b^{-1}a} E_{a^{-1}b^{-1}}\, r_{bd^{-1}}(a^{-1})\, E_{a^{-1}b} E_{b^{-1}a^{-1}} \\ &\quad \cdot E_{b^{-1}a} E_{bd^{-1}} E_{a^{-1}b^{-1}}\, r_{a^{-1}d}(b)^{-1}\, E_{a^{-1}b} E_{bd} E_{b^{-1}a^{-1}} \\ &\quad \cdot [E_{b^{-1}a}, E_{bd^{-1}}].\end{aligned}$$

(ii) *if $c = b^{-1}$,*

$$\begin{aligned}(w_{ab}{}^{-1} E_{b^{-1}d} w_{ab})^{-1} E_{a^{-1}d} &= E_{b^{-1}a} E_{a^{-1}b^{-1}}\, [E_{ba^{-1}}, E_{b^{-1}d^{-1}}]\, E_{a^{-1}b} E_{b^{-1}a^{-1}} \\ &\quad \cdot E_{b^{-1}a}\, r_{a^{-1}d^{-1}}(b^{-1})\, E_{b^{-1}a^{-1}} \\ &\quad \cdot E_{a^{-1}d^{-1}} E_{b^{-1}a}\, r_{b^{-1}d}(a^{-1})\, E_{b^{-1}a^{-1}} E_{a^{-1}d}.\end{aligned}$$

(iii) *if $d = a$,*

$$\begin{aligned}(w_{ab}{}^{-1} E_{ca} w_{ab})^{-1} E_{cb^{-1}} &= E_{b^{-1}a} E_{a^{-1}b^{-1}}\, [E_{ba^{-1}}, E_{ca^{-1}}]\, E_{a^{-1}b} E_{b^{-1}a^{-1}} \\ &\quad \cdot E_{b^{-1}a} E_{cb}\, r_{cb^{-1}}(a^{-1})^{-1}\, E_{cb^{-1}} E_{b^{-1}a^{-1}} \\ &\quad \cdot E_{b^{-1}a} E_{cb}\, r_{ca^{-1}}(b^{-1})^{-1}\, E_{cb^{-1}} E_{b^{-1}a^{-1}}.\end{aligned}$$

(iv) *if $d = a^{-1}$,*

$$\begin{aligned}(w_{ab}{}^{-1} E_{ca^{-1}} w_{ab})^{-1} E_{cb} &= E_{b^{-1}a} E_{a^{-1}b^{-1}}\, [E_{ba^{-1}}, E_{ca}]\, E_{a^{-1}b} E_{b^{-1}a^{-1}} \\ &\quad \cdot E_{b^{-1}a} E_{ca}\, r_{cb^{-1}}(a^{-1})^{-1}\, E_{ca^{-1}} E_{b^{-1}a^{-1}} \\ &\quad \cdot E_{b^{-1}a} E_{ca}\, r_{ca^{-1}}(b^{-1})^{-1}\, E_{ca^{-1}} E_{b^{-1}a^{-1}}.\end{aligned}$$

(v) *if $d = b$,*

$$\begin{aligned}(w_{ab}{}^{-1} E_{cb} w_{ab})^{-1} E_{ca} &= E_{b^{-1}a} E_{a^{-1}b^{-1}} E_{cb^{-1}}\, r_{ca^{-1}}(b)\, E_{cb} E_{a^{-1}b} E_{b^{-1}a^{-1}} \\ &\quad \cdot E_{b^{-1}a} E_{a^{-1}b^{-1}} E_{cb^{-1}}\, r_{cb}(a^{-1})\, E_{cb} E_{a^{-1}b} E_{b^{-1}a^{-1}} \\ &\quad \cdot [E_{b^{-1}a}, E_{ca^{-1}}].\end{aligned}$$

(vi) *if $d = b^{-1}$,*

$$\begin{aligned}(w_{ab}{}^{-1} E_{cb^{-1}} w_{ab})^{-1} E_{ca^{-1}} &= E_{b^{-1}a} E_{a^{-1}b^{-1}} E_{ca}\, r_{ca^{-1}}(b)^{-1}\, E_{ca^{-1}} E_{a^{-1}b} E_{b^{-1}a^{-1}} \\ &\quad \cdot E_{b^{-1}a} E_{ca}\, r_{cb^{-1}}(a^{-1})\, E_{ca^{-1}} E_{b^{-1}a^{-1}} \\ &\quad \cdot E_{b^{-1}a} E_{ca}\, [E_{cb^{-1}}, E_{a^{-1}b^{-1}}]\, E_{ca^{-1}} E_{b^{-1}a^{-1}} \\ &\quad \cdot [E_{b^{-1}a}, E_{ca}].\end{aligned}$$

Since these equations follows from easy calculations, we omit the details. In the case where $c = a$ or $c = b$, observing

$$(3)\qquad \begin{aligned}(w_{ab}{}^{-1}E_{cd}w_{ab})^{-1}E_{c^\sigma d^\sigma} &= (w_{ab}{}^{-1}E_{cd^{-1}}h_{ab}E_{cd}w_{ab})\cdot(w_{ab}{}^{-1}h_{ab}{}^{-1}w_{ab})\\ &\quad\cdot(w_{a^{-1}b^{-1}}{}^{-1}E_{cd}w_{a^{-1}b^{-1}})^{-1}E_{c^\sigma d^\sigma},\end{aligned}$$

and Lemma 2.2 above, we see that $(w_{ab}{}^{-1}E_{cd}w_{ab})^{-1}E_{c^\sigma d^\sigma}$ is also rewritten with the relators of the Gersten's presentation. For the case $\sharp\{a^{\pm1}, b^{\pm1}, c^{\pm1}, d^{\pm1}\} = 8$, we have

Lemma 2.3.

$$\begin{aligned}(w_{ab}{}^{-1}E_{cd}w_{ab})^{-1}E_{cd} &= E_{b^{-1}a}E_{a^{-1}b^{-1}}\,[E_{ba^{-1}}, E_{cd^{-1}}]\,E_{a^{-1}b}E_{b^{-1}a^{-1}}\\ &\quad\cdot E_{b^{-1}a}\,[E_{a^{-1}b^{-1}}, E_{cd^{-1}}]\,E_{b^{-1}a^{-1}}\cdot[E_{b^{-1}a}, E_{cd^{-1}}].\end{aligned}$$

Next, we consider how rewrite the relators $[E_{ab}, E_{cd}]$, $r_{ac}(b)$ and h_{ab} of the Gersten's presentation with the relators (R2-1), ... , (R4-1) of the reduced presentation. First, by an easy calculation, we see that the (R2) type relator $[E_{ab}, E_{cd}]$ is rewritten as a conjugate of one of the relators (R2-1), ... , (R2-8). For example,

$$[E_{ij^{-1}}, E_{kj^{-1}}] = E_{ij^{-1}}E_{kj^{-1}}\,[E_{ij}, E_{kj}]\,E_{ij}E_{kj}.$$

For the relators $r_{ac}(b)$ and h_{ab}, we use

Lemma 2.4.

(i) $r_{ac^{-1}}(b) = (E_{bc^{-1}}E_{ac^{-1}}r_{ac}(b)^{-1}E_{ac}E_{bc})\cdot[E_{bc^{-1}}, E_{ac^{-1}}]$,
(ii) $h_{a^{-1}b} = w_{ab}{}^{-1}h_{ab}w_{ab}, \qquad h_{ab^{-1}} = w_{ab}{}^{-1}h_{ab}{}^{-1}w_{ab}$.

Finally, we consider two type of equations induced from elements of $\bar{R}$:

$$(w_{ab}{}^{-1}r_{cd}(e)w_{ab})^{-1}r_{c^\sigma d^\sigma}(e^\sigma) \quad\text{and}\quad (w_{ab}{}^{-1}h_{cd}w_{ab})^{-1}h_{c^\sigma d^\sigma}$$

where σ is the monomial map defined by w_{ab}. Observe

$$(4)\qquad \begin{aligned}r_{c^\sigma d^\sigma}(e^\sigma) = s_1^{-1}&\cdot(w_{ab}{}^{-1}E_{ce}w_{ab}\ s_2^{-1}\ w_{ab}{}^{-1}E_{ce}{}^{-1}w_{ab})\\ &\cdot(w_{ab}{}^{-1}r_{cd}(e)w_{ab})\\ &\cdot(w_{ab}{}^{-1}E_{cd}E_{ed}w_{ab}\ s_1\ w_{ab}{}^{-1}E_{ed}{}^{-1}E_{cd}{}^{-1}w_{ab})\\ &\cdot(w_{ab}{}^{-1}E_{cd}w_{ab}\ s_2\ w_{ab}{}^{-1}E_{cd}{}^{-1}w_{ab})\cdot s_3\end{aligned}$$

where

$$\begin{aligned}&s_1 := (w_{ab}{}^{-1}E_{ce^{-1}}w_{ab})^{-1}E_{c^\sigma e^\sigma}{}^{-1}, \quad s_2 := (w_{ab}{}^{-1}E_{ed^{-1}}w_{ab})^{-1}E_{e^\sigma d^\sigma}{}^{-1},\\ &s_3 := (w_{ab}{}^{-1}E_{cd^{-1}}w_{ab})^{-1}E_{c^\sigma d^\sigma}{}^{-1}.\end{aligned}$$

Considering the equation (4) in $\bar{R}^{\mathrm{ab}}$, and tensoring it with e_p in $\bar{R}^{\mathrm{ab}}\otimes_{\mathbf{Z}} H_L$, we obtain

$$\begin{aligned}r_{c^\sigma d^\sigma}(e^\sigma)\otimes e_p = s_1^{-1}\otimes e_p &+ (w_{ab}{}^{-1}E_{ce}w_{ab}\ s_2^{-1}\ w_{ab}{}^{-1}E_{ce}{}^{-1}w_{ab})\otimes e_p\\ &+ (w_{ab}{}^{-1}r_{cd}(e)w_{ab})\otimes e_p\\ &+ (w_{ab}{}^{-1}E_{cd}E_{ed}w_{ab}\ s_1\ w_{ab}{}^{-1}E_{ed}{}^{-1}E_{cd}{}^{-1}w_{ab})\otimes e_p\\ &+ (w_{ab}{}^{-1}E_{cd}w_{ab}\ s_2\ w_{ab}{}^{-1}E_{cd}{}^{-1}w_{ab})\otimes e_p + s_3\otimes e_p\end{aligned}$$

For convenience, we denote this equation by $(a, b, c, d, e)\otimes e_p$. Similarly, we define $(a, b, c, d, e)\otimes e_p^*$.

We also consider

$$
\begin{aligned}
h_{c^\sigma d^\sigma} = t_3 \cdot (&{w_{ab}}^{-1} E_{dc} w_{ab}\ t_2\ {w_{ab}}^{-1} {E_{dc}}^{-1} w_{ab}) \\
&\cdot ({w_{ab}}^{-1} E_{dc} E_{c^{-1}d} w_{ab}\ t_1\ {w_{ab}}^{-1} {E_{c^{-1}d}}^{-1} {E_{dc}}^{-1} w_{ab}) \\
&\cdot ({w_{ab}}^{-1} E_{d^{-1}c} {}^{-1} E_{cd}{}^{-1} w_{ab}\ t_4\ {w_{ab}}^{-1} E_{cd} E_{d^{-1}c} w_{ab}) \\
&\cdot ({w_{ab}}^{-1} E_{d^{-1}c} {}^{-1} w_{ab}\ t_5\ {w_{ab}}^{-1} E_{d^{-1}c} w_{ab}) \cdot t_6
\end{aligned} \tag{5}
$$

where

$$
\begin{aligned}
t_1 &:= E_{(d^{-1})^\sigma (c^{-1})^\sigma} (w_{ab}^{-1} E_{d^{-1}c^{-1}} w_{ab})^{-1}, & t_2 &:= E_{(c^{-1})^\sigma d^\sigma} (w_{ab}^{-1} E_{c^{-1}d} w_{ab})^{-1}, \\
t_3 &:= E_{d^\sigma c^\sigma} (w_{ab}^{-1} E_{dc} w_{ab})^{-1}, & t_4 &:= (w_{ab}^{-1} E_{dc^{-1}} w_{ab})^{-1} E_{d^\sigma (c^{-1})^\sigma}, \\
t_5 &:= (w_{ab}^{-1} E_{cd} w_{ab})^{-1} E_{c^\sigma d^\sigma}, & t_6 &:= (w_{ab}^{-1} E_{d^{-1}c} w_{ab})^{-1} E_{(d^{-1})^\sigma c^\sigma}.
\end{aligned}
$$

For convenience, we denote by $\{a, b, c, d\} \otimes e_p^*$ the equations obtained by considering (5) in $\bar{R}^{\mathrm{ab}}$, and tensoring it with e_p^* in $\bar{R}^{\mathrm{ab}} \otimes_{\mathbf{Z}} H_L^*$. We often use these equations in Section 3 to reduce the generators of $H_1(\bar{R}, M)_{\mathrm{Aut}^+ F_n}$ for $M = H_L$ and H_L^*.

3. The Proof of the main theorem

First we consider $H_2(\mathrm{Aut}^+ F_n, H_L)$ for $n \geq 6$. Let F, R and $\bar{R}$ be as above. Then we have an exact sequence

$$1 \to \bar{R} \to F \to \mathrm{Aut}^+ F_n \to 1.$$

This sequence induces a homological five-term exact sequence

$$
\begin{aligned}
H_2(F, H_L) \to H_2(\mathrm{Aut}^+ F_n, H_L) &\to H_1(\bar{R}, H_L)_{\mathrm{Aut}^+ F_n} \\
&\to H_1(F, H_L) \to H_1(\mathrm{Aut}^+ F_n, H_L) \to 0.
\end{aligned}
$$

of $\mathbf{Z}$-modules. Since a $\mathbf{Z}$-equivariant homomorphism between L-modules is naturally considered as a L-equivariant homomorphism, we see that this sequence is an L-equivariant exact sequence. Since F is a free group, $H_2(F, H_L) = 0$. Furthermore $H_1(\mathrm{Aut}^+ F_n, H_L) = 0$ from our results of [8]. Hence we have an L-equivariant short exact sequence

$$0 \to H_2(\mathrm{Aut}^+ F_n, H_L) \to H_1(\bar{R}, H_L)_{\mathrm{Aut}^+ F_n} \to H_1(F, H_L) \to 0.$$

Since F is a free group of rank $2(n^2 - n)$, and since H_L is a free L-module of rank n, we see $H^1(F, H_L)$ is a free L-module of rank $2n(n^2 - n) - n$. Hence, by universal coefficients theorem, we have $H_1(F, H_L) \simeq L^{\oplus\{2n(n^2-n)-n\}}$. Since L is a principal ideal domain, we can apply the structure theorem to any finitely generated L-modules. Therefore our required result $H_2(\mathrm{Aut}^+ F_n, H_L) = 0$ follows from

Proposition 3.1. *For $n \geq 6$,*

$$H_1(\bar{R}, H_L)_{\mathrm{Aut}^+ F_n} \simeq L^{\oplus\{2n(n^2-n)-n\}}.$$

We prove this proposition in Subsection 3.1. Then, observing the homological Lyndon-Hochschild-Serre spectral sequence of

$$1 \to \mathrm{Aut}^+ F_n \to \mathrm{Aut}\, F_n \to \{\pm 1\} \to 1, \tag{6}$$

we obtain $H_2(\mathrm{Aut}\, F_n, H_L) = 0$ for $n \geq 6$.

Next we consider $H_2(\mathrm{Aut}^+F_n, H_L^*)$ for $n \geq 6$. Similarly, we obtain a homological five-term exact sequence

$$H_2(F, H_L^*) \to H_2(\mathrm{Aut}^+F_n, H_L^*) \to H_1(\bar{R}, H_L^*)_{\mathrm{Aut}^+F_n} \\ \to H_1(F, H_L^*) \to H_1(\mathrm{Aut}^+F_n, H_L^*) \to 0,$$

of L-modules, and from the results of [8],

$$0 \to H_2(\mathrm{Aut}^+F_n, H_L^*) \to H_1(\bar{R}, H_L^*)_{\mathrm{Aut}^+F_n} \to H_1(F, H_L^*) \to L \to 0.$$

Since we have $H_1(F, H_L^*) \simeq L^{\oplus\{2n(n^2-n)-n\}}$, our required result $H_2(\mathrm{Aut}^+F_n, H_L^*) = 0$ follows from

Proposition 3.2. *For $n \geq 6$,*

$$H_1(\bar{R}, H_L^*)_{\mathrm{Aut}^+F_n} \simeq L^{\oplus\{2n(n^2-n)-n-1\}}.$$

We prove this proposition in Subsection 3.2. Then observing the homological Lyndon-Hochschild-Serre spectral sequence of (6), we obtain $H_2(\mathrm{Aut}\, F_n, H_L^*) = 0$ for $n \geq 6$.

3.1. **The proof of Proposotion 3.1.**

In this subsection, we prove Proposotion 3.1. Since the map $H_1(\bar{R}, H_L)_{\mathrm{Aut}^+F_n} \to H_1(F, H_L) = L^{\oplus\{2n(n^2-n)-n\}}$ is surjective, $H_1(\bar{R}, H_L)_{\mathrm{Aut}^+F_n}$ contains a free L-submodule which rank is greater than or equal to $2n(n^2-n)-n$. To show it is just $2n(n^2-n)-n$, it suffices to show that $H_1(\bar{R}, H_L)_{\mathrm{Aut}^+F_n}$ is generated by just $2n(n^2-n)-n$ elements. Let $\bar{R}^{\mathrm{ab}}$ be the abelianization of $\bar{R}$. We also denote by r the coset class of $r \in \bar{R}$. By definifition, we have $H_1(\bar{R}, H_L)_{\mathrm{Aut}^+F_n} = \bar{R}^{\mathrm{ab}} \otimes_{\mathrm{Aut}^+F_n} H_L$, and see that

$$\mathfrak{E} := \{r \otimes e_p \,|\, r \in R,\ 1 \leq p \leq n\}$$

is a generating set of $\bar{R}^{\mathrm{ab}} \otimes_{\mathrm{Aut}^+F_n} H_L$ as a L-module. In the following, we reduce the elements of $\mathfrak{E}$. We use $\equiv$ for the equality in $\bar{R}^{\mathrm{ab}} \otimes_{\mathrm{Aut}^+F_n} H_L$.

Step 0. In the reduction of the generators of $\bar{R}^{\mathrm{ab}} \otimes_{\mathrm{Aut}^+F_n} H_L$, we often use the following lemmas.

Lemma 3.1. *For $n \geq 3$,*

$$(E_{i^{\pm1}j}\, r\, E_{i^{\pm1}j^{-1}}) \otimes e_p \equiv \begin{cases} r \otimes e_p, & p \neq i, \\ r \otimes e_i \pm r \otimes e_j, & p = i, \end{cases}$$

$$(E_{i^{\pm1}j^{-1}}\, r\, E_{i^{\pm1}j}) \otimes e_p \equiv \begin{cases} r \otimes e_p, & p \neq i, \\ r \otimes e_i \mp r \otimes e_j, & p = i. \end{cases}$$

Proof of Lemma 3.1 For any $\sigma \in \mathrm{Aut}^+F_n$, $r \in \bar{R}^{\mathrm{ab}}$ and $h \in H$, we have $r \cdot \sigma \otimes h \equiv r \otimes \sigma \cdot h$. Then observing the equation $r \cdot \sigma \otimes h = \sigma^{-1} r \sigma \otimes h$ induced from the definition of the action of Aut^+F_n on $\bar{R}^{\mathrm{ab}}$, we obtain the required results by substituting $\sigma = E_{i^{\pm1}j^{\pm1}}$ and $h = e_p$. □

Corollary 3.1. *For $n \geq 3$,*

$$[E_{i^{\pm 1}j}, r] \otimes e_p \equiv \begin{cases} 0, & p \neq i, \\ \pm r \otimes e_j, & p = i, \end{cases}$$

$$[E_{i^{\pm 1}j^{-1}}, r] \otimes e_p \equiv \begin{cases} 0, & p \neq i, \\ \mp r \otimes e_j, & p = i \end{cases}$$

Proof of Corollary 3.1 Observing

$$[\sigma, r] \otimes e_p = \sigma r \sigma^{-1} r^{-1} \otimes e_p = (\sigma r \sigma^{-1}) \otimes e_p - r \otimes e_p$$

for $\sigma \in F$ and $r \in \bar{R}^{\mathrm{ab}}$, and Lemma 3.1, we immediately obtain the required results. □

Similarly, we have

Lemma 3.2. *For $n \geq 3$,*

$$(w_{i^{\pm 1}j}\, r\, {w_{i^{\pm 1}j}}^{-1}) \otimes e_p \equiv \begin{cases} r \otimes e_p, & p \neq i, j, \\ \mp r \otimes e_j, & p = i, \\ \pm r \otimes e_i, & p = j, \end{cases}$$

$$(w_{i^{\pm 1}j^{-1}}\, r\, w_{i^{\pm 1}j}) \otimes e_p \equiv \begin{cases} r \otimes e_p, & p \neq i, j, \\ \pm r \otimes e_j, & p = i, \\ \mp r \otimes e_i, & p = j. \end{cases}$$

Corollary 3.2. *For $n \geq 3$,*

$$[w_{i^{\pm 1}j}, r] \otimes e_p \equiv \begin{cases} 0, & p \neq i, j, \\ -r \otimes e_i \mp r \otimes e_j, & p = i, \\ \pm r \otimes e_i - r \otimes e_j, & p = j. \end{cases}$$

$$[{w_{i^{\pm 1}j}}^{-1}, r] \otimes e_p \equiv \begin{cases} 0, & p \neq i, j, \\ -r \otimes e_i \pm r \otimes e_j, & p = i, \\ \mp r \otimes e_i - r \otimes e_j, & p = j. \end{cases}$$

Considering any relator of (R2) of the Gersten's presentation is conjugate to one of the relator of (R2-1), ... , (R2-8), or considering Lemma 2.4, for any relator $r =$ (R2), (R3) and (R4), we can rewrite a element $r \otimes e_p$ with the relators (R2-1), ... , (R4-1) using Lemmas 3.1 and 3.2. The computation is easiest explained with examples, so we give three examples.

$$\begin{aligned} [E_{ij^{-1}}, E_{kj^{-1}}] \otimes e_p &= (E_{ij^{-1}} E_{kj^{-1}} [E_{ij}, E_{kj}] E_{ij} E_{kj}) \otimes e_p, \\ &\equiv \begin{cases} [E_{ij}, E_{kj}] \otimes e_p, & p \neq i, k, \\ [E_{ij}, E_{kj}] \otimes e_p - [E_{ij}, E_{kj}] \otimes e_j, & p = i, k. \end{cases} \end{aligned}$$

$$\begin{aligned} r_{ij^{-1}}(k) \otimes e_p &= \{(E_{kj^{-1}} E_{ij^{-1}} r_{ij}(k)^{-1} E_{ij} E_{kj}) \cdot [E_{kj^{-1}}, E_{ij^{-1}}]\} \otimes e_p, \\ &\equiv \begin{cases} -r_{ij}(k) \otimes e_p + [E_{ij}, E_{kj}] \otimes e_p, & p \neq i, k, \\ -r_{ij}(k) \otimes (e_p - e_j) + [E_{ij}, E_{kj}] \otimes (e_p - e_j), & p = i, k. \end{cases} \end{aligned}$$

$$h_{i^{-1}j} \otimes e_p = ({w_{ij}}^{-1} h_{ij} w_{ij}) \otimes e_p \equiv \begin{cases} h_{ij} \otimes e_p, & p \neq i, j, \\ h_{ij} \otimes e_j, & p = i, \\ -h_{ij} \otimes e_i, & p = j. \end{cases}$$

Step 1. First we consider the generators ${w_{ij}}^4 \otimes e_p$. Observing the Gersten's computation in [2], we see that for any a, b, c, and d, the element $({w_{ab}}^{-1} E_{cd}\ w_{ab})^{-1} E_{c^\sigma d^\sigma} \in \bar{R}$ is in the normal closure of (R2-1), ... , (R4-1) in F, Hence

$${w_{ij}}^8 = \{({w_{ij}}^4 {w_{jk}}^2 {w_{ij}}^{-4})^{-1} {w_{jk}}^2\} \cdot \{({w_{jk}}^{-2} {w_{ij}}^{-4} {w_{jk}}^2)^{-1} {w_{ij}}^4)\}$$

is also in it, and we see that

$${w_{ij}}^4 \otimes e_p = \frac{1}{2} (2\, {w_{ij}}^4 \otimes e_p) \equiv \frac{1}{2} ({w_{ij}}^8 \otimes e_p)$$

is rewritten as a sum of the generator $r \otimes e_p$ for $r =$ (R2-1), ... , (R4-1). Therefore we can remove the generators ${w_{ij}}^4 \otimes e_p$ from the generationg set $\mathfrak{E}$.

Step 2. Here we show that the generators $r \otimes e_p$ for $r =$ (R2-1), ... , (R2-8) is zero or equal to one of the generators $r_{i^{\pm 1}j}(k^{\pm 1}) \otimes e_p$. We have

Lemma 3.3. *For $n \geq 6$ and distinct i, j, k, l and m,*

(i) **(R2-2):**

$$[E_{ij}, E_{kj}] \otimes e_p \equiv \begin{cases} 0, & p \neq i, k, \\ -r_{kj}(l) \otimes e_j, & p = i, \\ r_{ij}(l) \otimes e_j, & p = k. \end{cases}$$

(ii) **(R2-3), (R2-4):**

$$[E_{i^{-1}j}, E_{k^{\pm 1}j}] \otimes e_p \equiv \begin{cases} 0, & p \neq i, k, \\ r_{k^{\pm 1}j}(l) \otimes e_j, & p = i, \\ \mp r_{i^{-1}j}(l) \otimes e_j, & p = k. \end{cases}$$

(iii) **(R2-1):**

$$[E_{ij}, E_{i^{-1}j}] \otimes e_p \equiv \begin{cases} 0, & p \neq i, \\ -r_{ij}(k) \otimes e_j - r_{i^{-1}j}(l) \otimes e_j, & p = i. \end{cases}$$

(iv) **(R2-6):**

$$[E_{ij}, E_{kl}] \otimes e_p \equiv \begin{cases} 0, & p \neq i, k, \\ -r_{kl}(m) \otimes e_j, & p = i, \\ r_{ij}(m) \otimes e_l, & p = k. \end{cases}$$

(v) **(R2-7), (R2-8):**

$$[E_{i^{-1}j}, E_{k^{\pm 1}l}] \otimes e_p \equiv \begin{cases} 0, & p \neq i, k, \\ r_{k^{\pm 1}l}(m) \otimes e_j, & p = i, \\ \pm r_{i^{-1}j}(m) \otimes e_l, & p = k. \end{cases}$$

(vi) **(R2-5):**

$$[E_{ij}, E_{i^{-1}k}] \otimes e_p \equiv \begin{cases} 0, & p \neq i, \\ -r_{ij}(l) \otimes e_k - r_{i^{-1}k}(m) \otimes e_j, & p = i. \end{cases}$$

Proof of Lemma 3.3 Here we prove (i). First we consider the case $p \neq i, k$. Since $n \geq 5$, we can choose a number $l \in \{1, \dots, n\}$ such that $l \neq i, j, k, p$. Set $r := E_{ij^{-1}}[E_{il}, E_{lj}] \in \bar{R}$. Since $[r, E_{kj}]$ is in $\bar{R}$, and since $p \neq i, k$, we have

$$[E_{ij}, [r, E_{kj}]] \otimes e_p \equiv 0, \quad [r, E_{kj}] \otimes e_p \equiv 0$$

by Corollary 3.1. Then, using the formula (2) repeatedly, we see

$$\begin{aligned}
[E_{ij}, E_{kj}] \otimes e_p &\equiv ([E_{ij}, [r, E_{kj}]][r, E_{kj}][E_{ij}, E_{kj}]) \otimes e_p, \\
&= [[E_{il}, E_{lj}], E_{kj}] \otimes e_p, \\
&= [E_{il}E_{lj}{E_{il}}^{-1}{E_{lj}}^{-1}, E_{kj}] \otimes e_p, \\
&= ([E_{il}, [E_{lj}{E_{il}}^{-1}{E_{lj}}^{-1}, E_{kj}]][E_{lj}{E_{il}}^{-1}{E_{lj}}^{-1}, E_{kj}][E_{il}, E_{kj}]) \otimes e_p, \\
&\equiv ([E_{lj}{E_{il}}^{-1}{E_{lj}}^{-1}, E_{kj}][E_{il}, E_{kj}]) \otimes e_p, \\
&\equiv \cdots\cdots, \\
&\equiv ([{E_{lj}}^{-1}, E_{kj}][{E_{il}}^{-1}, E_{kj}][E_{lj}, E_{kj}][E_{il}, E_{kj}]) \otimes e_p, \\
&\equiv [{E_{lj}}^{-1}, E_{kj}] \otimes e_p + [{E_{il}}^{-1}, E_{kj}] \otimes e_p + [E_{lj}, E_{kj}] \otimes e_p + [E_{il}, E_{kj}] \otimes e_p.
\end{aligned}$$

On the other hand, by (2) we have

$$1 = [E_{lj}{E_{lj}}^{-1}, E_{kj}] = [E_{lj}, [{E_{lj}}^{-1}, E_{kj}]][{E_{lj}}^{-1}, E_{kj}][E_{lj}, E_{kj}].$$

Since $p \neq l$, we see $[E_{lj}, [{E_{lj}}^{-1}, E_{kj}]] \otimes e_p \equiv 0$ by Corollary 3.1, and hence

$$[E_{lj}, E_{kj}] \otimes e_p + [{E_{lj}}^{-1}, E_{kj}] \otimes e_p \equiv 0.$$

Similarly, since $p \neq i$,

$$[E_{il}, E_{kj}] \otimes e_p + [{E_{il}}^{-1}, E_{kj}] \otimes e_p \equiv 0.$$

Therefore we obtain $[E_{ij}, E_{kj}] \otimes e_p \equiv 0$.

Next we consider the case $p = i$. Since $[[E_{kj}, E_{ij}], r_{kj}(l)] = 0$ in $\bar{R}^{\mathrm{ab}}$, and since $[E_{ij}, r_{kj}(l)] \otimes e_i \equiv r_{kj}(l) \otimes e_j$ by Corollary 3.1, we have

$$\begin{aligned}
&[E_{ij}, E_{kj}] \otimes e_i + r_{kj}(l) \otimes e_j \\
&\qquad \equiv ([E_{ij}, r_{kj}(l)][E_{ij}, E_{kj}][[E_{kj}, E_{ij}], r_{kj}(l)]) \otimes e_i, \\
&\qquad = [E_{ij}, [E_{kl}, E_{lj}]] \otimes e_i, \\
&\qquad = ([E_{ij}, E_{kl}][E_{ij}, E_{lj}E_{kl^{-1}}E_{lj^{-1}}][[E_{lj}E_{kl^{-1}}E_{lj^{-1}}, E_{ij}], E_{kl}]) \otimes e_i, \\
&\qquad \equiv ([E_{ij}, E_{kl}][E_{ij}, E_{lj}E_{kl^{-1}}E_{lj^{-1}}]) \otimes e_i, \\
&\qquad \equiv \cdots\cdots, \\
&\qquad \equiv ([E_{ij}, E_{kl}][E_{ij}, E_{kl^{-1}}][E_{ij}, E_{lj}][E_{ij}, E_{lj^{-1}}]) \otimes e_i \\
&\qquad \equiv ([E_{ij}, E_{kl}] + [E_{ij}, E_{kl^{-1}}]) \otimes e_i + ([E_{ij}, E_{lj}] + [E_{ij}, E_{lj^{-1}}]) \otimes e_i.
\end{aligned}$$

Since we have

$$1 = [E_{ij}, E_{kl}E_{kl^{-1}}] = [E_{ij}, E_{kl}][E_{ij}, E_{kl^{-1}}][[E_{kl^{-1}}, E_{ij}], E_{kl}],$$

and since $[[E_{kl^{-1}}, E_{ij}], E_{kl}] \otimes e_i \equiv 0$ by Corollary 3.1, we see

$$([E_{ij}, E_{kl}] + [E_{ij}, E_{kl^{-1}}]) \otimes e_i \equiv 0.$$

Similarly,

$$([E_{ij}, E_{lj}] + [E_{ij}, E_{lj^{-1}}]) \otimes e_i \equiv 0.$$

Hence we obtain $[E_{ij}, E_{kj}] \otimes e_i \equiv -r_{kj}(l) \otimes e_j$. Furthermore changing the role of i and k in the equation $[E_{kj}, E_{ij}] \otimes e_i \equiv r_{kj}(l) \otimes e_j$, we also obtain $[E_{ij}, E_{kj}] \otimes e_k \equiv r_{ij}(l) \otimes e_j$.

Similarly, we can show (ii), (iv) and (v). We remark that to show (iv) and (v), we need $n \geq 6$ since we use six distinct generators of the free group F_n. Then using these results, we obtain (iii) and (vi). Since the calculations are similar to that above, we leave it to the reader for exercise. (For details, see Subsection 5.1.) $\square$

By the lemma above, we can remove the generators $r \otimes e_p$ for $r =$ (R2-1), ..., (R2-8) from the generationg set $\mathfrak{E}$.

Step 3. Here we consider the generators $r_{i^{\pm 1}j}(k^{\pm 1}) \otimes e_p$ for $p \neq i$.

(3-a) The case $p \neq i, k$.

First we consider the case $p = j$. Observing (i) of Lemma 3.3, we see that $r_{ij}(l) \otimes e_j$ doesn't depend on the choice of a number l such that $l \neq i, j, k$. On the other hand, since $n \geq 5$, there exists another number m such that $m \neq i, j, k, l$. Similarly, we have $[E_{ij}, E_{mj}] \otimes e_m \equiv r_{ij}(k) \otimes e_j \equiv r_{ij}(l) \otimes e_j$ from (i) of Lemma 3.3. This shows that $r_{ij}(l) \otimes e_j$ doesn't depend on the choice of a number l such that $l \neq i, j$. Futhermore, using the relator $r_{kj}(l^{-1})$ instead of $r_{kj}(l)$ in the proof of (i) of Lemma 3.3, we also obtain

$$[E_{ij}, E_{kj}] \otimes e_p \equiv \begin{cases} 0, & p \neq i, k, \\ -r_{kj}(l^{-1}) \otimes e_j, & p = i, \\ r_{ij}(l^{-1}) \otimes e_j, & p = k. \end{cases}$$

Hence we can set

$$r_{ij}(\cdot) \otimes e_j :\equiv r_{ij}(k) \otimes e_j \equiv r_{ij}(k^{-1}) \otimes e_j$$

for distinct i and j. Similarly, observing (ii) of Lemma 3.3, we can set

$$r_{i^{-1}j}(\cdot) \otimes e_j :\equiv r_{i^{-1}j}(k) \otimes e_j \equiv r_{i^{-1}j}(k^{-1}) \otimes e_j$$

for distinct i and j.

For the case $p \neq j$, observing (iv) and (v) of Lemma 3.3, we can set

$$r_{ij}(\cdot) \otimes e_p :\equiv r_{ij}(k) \otimes e_p \equiv r_{ij}(k^{-1}) \otimes e_p,$$
$$r_{i^{-1}j}(\cdot) \otimes e_p :\equiv r_{i^{-1}j}(k) \otimes e_p \equiv r_{i^{-1}j}(k^{-1}) \otimes e_p.$$

(3-b) The case $p = k$**.**

Set

$$S_{ijk} := r_{ij}(k) \otimes e_k - r_{ij}(\cdot) \otimes e_k - r_{ik}(\cdot) \otimes e_j.$$

We show that $S_{ijk} \equiv 0$ in $\bar{R}^{\mathrm{ab}} \otimes_{\mathrm{Aut}^+ F_n} H_L$. For distinct i, j, k and l, the equation $(x_l, x_j, x_i, x_j, x_k) \otimes e_k$ is given by

$$\begin{aligned} r_{il}(k) \otimes e_k = {} & s_1^{-1} \otimes e_k + (w_{lj}{}^{-1} E_{ik} w_{lj}\ s_2^{-1}\ w_{lj}{}^{-1} E_{ik}{}^{-1} w_{lj}) \otimes e_k \\ & + (w_{lj}{}^{-1} r_{ij}(k) w_{lj}) \otimes e_k \\ & + (w_{lj}{}^{-1} E_{ij} E_{kj} w_{lj}\ s_1\ w_{lj}{}^{-1} E_{kj}{}^{-1} E_{ij}{}^{-1} w_{lj}) \otimes e_k \\ & + (w_{lj}{}^{-1} E_{ij} w_{lj}\ s_2\ w_{lj}{}^{-1} E_{ij}{}^{-1} w_{lj}) \otimes e_k + s_3 \otimes e_k \end{aligned} \tag{7}$$

where

$$\begin{aligned} & s_1 := (w_{lj}{}^{-1} E_{ik^{-1}} w_{lj})^{-1} E_{ik}{}^{-1}, \quad s_2 := (w_{lj}{}^{-1} E_{kj^{-1}} w_{lj})^{-1} E_{kl}{}^{-1}, \\ & s_3 := (w_{lj}{}^{-1} E_{ij^{-1}} w_{lj})^{-1} E_{il}{}^{-1}. \end{aligned}$$

Then using Lemmas 3.1 and 3.2 repeatedly, we obtain

$$r_{il}(k) \otimes e_k \equiv r_{ij}(k) \otimes e_k + s_1 \otimes e_l + s_3 \otimes e_k. \tag{8}$$

On the other hand, using Lemma 2.3, we see

$$\begin{aligned} s_1 &= (w_{lj}{}^{-1} E_{ik^{-1}} w_{lj})^{-1} E_{ik^{-1}} \\ &= E_{j^{-1}l} E_{l^{-1}j^{-1}} [E_{jl^{-1}}, E_{ik}] E_{l^{-1}j} E_{j^{-1}l^{-1}} \cdot E_{j^{-1}l} [E_{l^{-1}j^{-1}}, E_{ik}] E_{j^{-1}l^{-1}} \cdot [E_{j^{-1}l}, E_{ik}], \end{aligned}$$

and hence

$$s_1 \otimes e_l \equiv r_{ik}(\cdot) \otimes e_l - r_{ik}(\cdot) \otimes e_j.$$

Furthermore, applying (vi) of Lemma 2.2 to s_3, we have

$$\begin{aligned} s_3 &= (w_{lj}{}^{-1} E_{ij^{-1}} w_{lj})^{-1} E_{il^{-1}} \\ &= E_{j^{-1}l} E_{l^{-1}j^{-1}} E_{il}\, r_{il^{-1}}(j)^{-1}\, E_{il^{-1}} E_{l^{-1}j} E_{j^{-1}l^{-1}} \cdot E_{j^{-1}l} E_{il}\, r_{ij^{-1}}(l^{-1})\, E_{il^{-1}} E_{j^{-1}l^{-1}} \\ &\quad \cdot E_{j^{-1}l} E_{il}\, [E_{ij^{-1}}, E_{l^{-1}i^{-1}}]\, E_{il^{-1}} E_{j^{-1}l^{-1}} \cdot [E_{j^{-1}l}, E_{il}], \end{aligned}$$

and hence

$$s_3 \otimes e_k \equiv r_{il}(\cdot) \otimes e_k - r_{ij}(\cdot) \otimes e_k.$$

Substituting these results into (8), we obtain $S_{ijk} \equiv S_{ilk}$.

By the same argument, considering the equation $(x_l^{-1}, x_j, x_i, x_j, x_k) \otimes e_k$, we obtain $S_{ijk} \equiv -S_{ilk}$, and $2S_{ijk} \equiv 0$. Then 2 is invertible in L, we obtain $S_{ijk} \equiv 0$, i.e.,

$$r_{ij}(k) \otimes e_k \equiv r_{ij}(\cdot) \otimes e_k + r_{ik}(\cdot) \otimes e_j.$$

Similarly, considering the equations $(x_k^{-1}, x_l, x_i, x_j, x_l) \otimes e_k$ and $(x_i^{-1}, x_l, x_l, x_j, x_k^{\pm 1}) \otimes e_k$, we obtain

$$\begin{aligned} & r_{ij}(k^{-1}) \otimes e_k \equiv r_{ij}(\cdot) \otimes e_k + r_{ik}(\cdot) \otimes e_j, \\ & r_{i^{-1}j}(k^{\pm 1}) \otimes e_k \equiv r_{i^{-1}j}(\cdot) \otimes e_k + r_{i^{-1}k}(\cdot) \otimes e_j \end{aligned}$$

respectively. (For details, see Subsection 5.2.)

By the argument above, we can remove the generators $r_{i^{\pm 1}j}(k^{\pm 1}) \otimes e_k$ from the generationg set $\mathfrak{E}$.

Step 4. Here we consider the generators $h_{ij} \otimes e_p$.

First we consider the case $p \neq i, j$. From Lemma 2.3, we have

$$\begin{aligned}[w_{ij}{}^{-1}, E_{lk}] &= (w_{ij}{}^{-1}E_{lk^{-1}}w_{ij})^{-1}E_{lk^{-1}} \\ &= E_{j^{-1}i}E_{i^{-1}j^{-1}}\,[E_{ji^{-1}}, E_{lk}]\,E_{i^{-1}j}E_{j^{-1}i^{-1}} \\ &\quad \cdot E_{j^{-1}i}\,[E_{i^{-1}j^{-1}}, E_{lk}]\,E_{j^{-1}i^{-1}} \cdot [E_{j^{-1}i}, E_{lk}],\end{aligned}$$

and hence

$$[w_{ij}{}^{-1}, E_{lk}] \otimes e_l \equiv -r_{ji}(\cdot) \otimes e_k - r_{i^{-1}j}(\cdot) \otimes e_k + r_{j^{-1}i}(\cdot) \otimes e_k. \tag{9}$$

On the other hand, observing (3), we have

$$[w_{ij}{}^{-1}, E_{lk}] = (w_{ij}{}^{-1}E_{lk}h_{ij}{}^{-1}E_{lk^{-1}}w_{ij}) \cdot (w_{ij}{}^{-1}h_{ij}w_{ij}) \cdot [w_{i^{-1}j}, E_{lk}].$$

Using Lemmas 3.1 and 3.2, we have

$$\begin{aligned}&(w_{ij}{}^{-1}h_{ij}w_{ij}) \otimes e_l \equiv h_{ij} \otimes e_l, \\ &(w_{ij}{}^{-1}E_{lk}h_{ij}{}^{-1}E_{lk^{-1}}w_{ij}) \otimes e_l \equiv -h_{ij} \otimes e_l - h_{ij} \otimes e_k\end{aligned}$$

Furthermore, computing $[w_{i^{-1}j}, E_{lk}] \otimes e_l$ in a way similar to (9), we have

$$[w_{i^{-1}j}, E_{lk}] \otimes e_l \equiv r_{j^{-1}i}(\cdot) \otimes e_k + r_{ij}(\cdot) \otimes e_k - r_{ji}(\cdot) \otimes e_k.$$

Hence

$$[w_{ij}{}^{-1}, E_{lk}] \otimes e_l \equiv -h_{ij} \otimes e_k + r_{j^{-1}i}(\cdot) \otimes e_k + r_{ij}(\cdot) \otimes e_k - r_{ji}(\cdot) \otimes e_k. \tag{10}$$

Comparing (9) with (10), we obtain

$$h_{ij} \otimes e_k \equiv -r_{ij}(\cdot) \otimes e_k - r_{i^{-1}j}(\cdot) \otimes e_k.$$

Next we consider the case $p = i$. Applying (iv) of Lemma 2.2 to $\{(w_{ij}{}^{-1}E_{ki^{-1}}w_{ij})^{-1}E_{kj}\} \otimes e_k$, we see

$$\begin{aligned}(w_{ij}{}^{-1}E_{ki^{-1}}w_{ij})^{-1}E_{kj} &= E_{j^{-1}i}E_{i^{-1}j^{-1}}\,[E_{ji^{-1}}, E_{ki}]\,E_{i^{-1}j}E_{j^{-1}i^{-1}} \\ &\quad \cdot E_{j^{-1}i}E_{ki}\,r_{kj^{-1}}(i^{-1})^{-1}\,E_{ki^{-1}}E_{j^{-1}i^{-1}} \\ &\quad \cdot E_{j^{-1}i}E_{ki}\,r_{ki^{-1}}(j^{-1})^{-1}\,E_{ki^{-1}}E_{j^{-1}i^{-1}},\end{aligned}$$

and hence

$$\begin{aligned}\{(w_{ij}{}^{-1}E_{ki^{-1}}w_{ij})^{-1}E_{kj}\} \otimes e_k \equiv &-r_{ji}(\cdot) \otimes e_i + r_{kj}(i^{-1}) \otimes e_k + r_{kj}(i^{-1}) \otimes e_i \\ &+ r_{ki}(j^{-1}) \otimes e_k + r_{ki}(\cdot) \otimes e_i.\end{aligned} \tag{11}$$

On the other hand, using (3), we have

$$\begin{aligned}(w_{ij}{}^{-1}E_{ki^{-1}}w_{ij})^{-1}E_{kj} &= (w_{ij}{}^{-1}E_{ki}h_{ij}E_{ki^{-1}}w_{ij}) \cdot (w_{ij}{}^{-1}h_{ij}{}^{-1}w_{ij}) \\ &\quad \cdot (w_{i^{-1}j^{-1}}{}^{-1}E_{ki^{-1}}w_{i^{-1}j^{-1}})^{-1}E_{kj}.\end{aligned}$$

Tensoring both hands side of the equation above with e_k, we have

$$h_{ij} \otimes e_i \equiv \{(w_{ij}{}^{-1}E_{ki^{-1}}w_{ij})^{-1}E_{kj}\} \otimes e_k - \{(w_{i^{-1}j^{-1}}{}^{-1}E_{ki^{-1}}w_{i^{-1}j^{-1}})^{-1}E_{kj}\} \otimes e_k. \tag{12}$$

Applying (iii) of Lemma 2.2 to $(w_{i^{-1}j^{-1}}{}^{-1}E_{ki^{-1}}w_{i^{-1}j^{-1}})^{-1}E_{kj}$, we see

$$\begin{aligned}(w_{i^{-1}j^{-1}}{}^{-1}E_{ki^{-1}}w_{i^{-1}j^{-1}})^{-1}E_{kj} &= E_{ji^{-1}}E_{ij}\,[E_{j^{-1}i}, E_{ki}]\,E_{ij^{-1}}E_{ji} \\ &\quad \cdot E_{ji^{-1}}E_{kj^{-1}}\,r_{kj}(i)^{-1}\,E_{kj}E_{ji} \\ &\quad \cdot E_{ji^{-1}}E_{kj^{-1}}\,r_{ki}(j)^{-1}\,E_{kj}E_{ji}\end{aligned}$$

and

$$\{(w_{i^{-1}j^{-1}}{}^{-1}E_{ki^{-1}}w_{i^{-1}j^{-1}})^{-1}E_{kj}\}\otimes e_k \equiv r_{j^{-1}i}(\cdot)\otimes e_i - r_{kj}(i)\otimes e_k + r_{kj}(\cdot)\otimes e_j - r_{ki}(j)\otimes e_k + r_{ki}(j)\otimes e_j. \tag{13}$$

Substituiting (11) and (13) into (12), we obtain

$$\begin{aligned} h_{ij}\otimes e_i \equiv & -r_{j^{-1}i}(\cdot)\otimes e_i - r_{ji}(\cdot)\otimes e_i + r_{kj}(i)\otimes e_k + r_{kj}(i^{-1})\otimes e_k \\ & + r_{ki}(j)\otimes e_k + r_{ki}(j^{-1})\otimes e_k - r_{kj}(\cdot)\otimes e_j + r_{ki}(\cdot)\otimes e_i \\ & - r_{ki}(j)\otimes e_j + r_{kj}(i^{-1})\otimes e_i \end{aligned}$$

Similarly, considering $\{(w_{ij}E_{ki^{-1}}w_{ij^{-1}})^{-1}\ E_{kj^{-1}}\}\otimes e_k$, we have

$$\begin{aligned} h_{ij}\otimes e_j \equiv & \ r_{j^{-1}i}(\cdot)\otimes e_i + r_{ji}(\cdot)\otimes e_i + r_{kj}(i^{-1})\otimes e_k + r_{kj}(i)\otimes e_k \\ & - r_{ki}(j)\otimes e_k - r_{ki}(j^{-1})\otimes e_k - r_{ki}(\cdot)\otimes e_i + r_{kj}(\cdot)\otimes e_j \\ & + r_{kj}(i^{-1})\otimes e_j - r_{ki}(j^{-1})\otimes e_j \end{aligned}$$

By the argument above, we can remove the generators $h_{ij}\otimes e_p$ from the generationg set $\mathfrak{E}$.

Step 5. Here we consider the generators $r_{i^{\pm1}j}(k^{\pm1})\otimes e_i$.

For convenience, we use the following notation. Let V be the quotient L-module of $\bar{R}^{\mathrm{ab}}\otimes_{\mathrm{Aut}^+F_n}H_L$ by the L-submodule generated by the elements $r_{i^{\pm1}j}(\cdot)\otimes e_k$ for $k\neq i$. We use $\overset{\circ}{=}$ for the equality in V.

First we consider the equation $(x_l, x_k, x_i, x_j, x_k)\otimes e_i$ for distinct i, j, k and l. It is given by

$$\begin{aligned} r_{ij}(l)\otimes e_i = & \ s_1^{-1}\otimes e_i + (w_{lk}{}^{-1}E_{ik}w_{lk}\ s_2^{-1}\ w_{lk}{}^{-1}E_{ik}{}^{-1}w_{lk})\otimes e_i \\ & + (w_{lk}{}^{-1}r_{ij}(k)w_{lk})\otimes e_i \\ & + (w_{lk}{}^{-1}E_{ij}E_{kj}w_{lk}\ s_1\ w_{lk}{}^{-1}E_{kj}^{-1}E_{ij}{}^{-1}w_{lk})\otimes e_i \\ & + (w_{lk}{}^{-1}E_{ij}w_{lk}\ s_2\ w_{lk}{}^{-1}E_{ij}{}^{-1}w_{lk})\otimes e_i + s_3\otimes e_i \end{aligned}$$

where

$$\begin{aligned} & s_1 := (w_{lk}{}^{-1}E_{ik^{-1}}w_{lk})^{-1}E_{il}{}^{-1}, \quad s_2 := (w_{lk}{}^{-1}E_{kj^{-1}}w_{lk})^{-1}E_{lj}{}^{-1}, \\ & s_3 := (w_{lk}{}^{-1}E_{ij^{-1}}w_{lk})^{-1}E_{ij}{}^{-1}. \end{aligned}$$

Then using Lemmas 3.1 and 3.2, we obtain

$$r_{ij}(l)\otimes e_i \equiv r_{ij}(k)\otimes e_i + s_1\otimes e_j + s_2\otimes e_j - s_2\otimes e_l + s_3\otimes e_i. \tag{14}$$

By an argumet similar to that in **(3-b)**, we can compute

$$s_1\otimes e_j \equiv r_{il}(\cdot)\otimes e_j - r_{ik}(\cdot)\otimes e_j \overset{\circ}{=} 0$$

and

$$s_3\otimes e_i \equiv -r_{kl}(\cdot)\otimes e_j - r_{l^{-1}k}(\cdot)\otimes e_j + r_{k^{-1}l}(\cdot)\otimes e_j \overset{\circ}{=} 0.$$

On the other hand, using (3), we have

$$s_2 = (w_{lk}{}^{-1}E_{kj}h_{lk}E_{kj^{-1}}w_{lk})\cdot(w_{lk}{}^{-1}h_{lk}{}^{-1}w_{lk})\cdot(w_{l^{-1}k^{-1}}{}^{-1}E_{kj^{-1}}w_{l^{-1}k^{-1}})^{-1}E_{lj^{-1}},$$

and hence

$$\begin{aligned} s_2 \otimes e_j &\equiv \{(w_{l^{-1}k^{-1}}{}^{-1} E_{kj^{-1}} w_{l^{-1}k^{-1}})^{-1} E_{lj^{-1}}\} \otimes e_j, \\ s_2 \otimes e_l &\equiv \{(w_{l^{-1}k^{-1}}{}^{-1} E_{kj^{-1}} w_{l^{-1}k^{-1}})^{-1} E_{lj^{-1}}\} \otimes e_l + h_{lk} \otimes e_j, \\ &\equiv \{(w_{l^{-1}k^{-1}}{}^{-1} E_{kj^{-1}} w_{l^{-1}k^{-1}})^{-1} E_{lj^{-1}}\} \otimes e_l - r_{lk}(\cdot) \otimes e_j - r_{l^{-1}k}(\cdot) \otimes e_j. \end{aligned}$$

Then, applying (ii) of Lemma 2.2 to $(w_{l^{-1}k^{-1}}{}^{-1} E_{kj^{-1}} w_{l^{-1}k^{-1}})^{-1} E_{lj^{-1}}$, we see

$$\begin{aligned} (w_{l^{-1}k^{-1}}{}^{-1} E_{kj^{-1}} w_{l^{-1}k^{-1}})^{-1} E_{lj^{-1}} = E_{kl^{-1}} E_{lk} \, [E_{k^{-1}l}, E_{kj}] \, E_{lk^{-1}} E_{kl} \\ \cdot E_{kl^{-1}} \, r_{lj}(k) \, E_{kl} \cdot E_{lj} E_{kl^{-1}} \, r_{kj^{-1}}(l) \, E_{kl} E_{lj^{-1}}, \end{aligned}$$

and hence

$$\begin{aligned} s_2 \otimes e_j &\equiv r_{lj}(\cdot) \otimes e_j - r_{kj}(\cdot) \otimes e_j \stackrel{\circ}{=} 0, \\ s_2 \otimes e_l &\equiv r_{kj}(\cdot) \otimes e_l + r_{k^{-1}l}(\cdot) \otimes e_j + r_{lj}(k) \otimes e_l \\ &\qquad - r_{kj}(l) \otimes e_l - r_{kj}(\cdot) \otimes e_j - r_{lk}(\cdot) \otimes e_j - r_{l^{-1}k}(\cdot) \otimes e_j, \\ &\equiv r_{kj}(\cdot) \otimes e_l + r_{k^{-1}l}(\cdot) \otimes e_j + r_{lj}(k) \otimes e_l \\ &\qquad - r_{kj}(\cdot) \otimes e_l - r_{kl}(\cdot) \otimes e_j - r_{kj}(\cdot) \otimes e_j - r_{lk}(\cdot) \otimes e_j - r_{l^{-1}k}(\cdot) \otimes e_j, \\ &\stackrel{\circ}{=} r_{lj}(k) \otimes e_l. \end{aligned}$$

Substituting these results into (14), we obtain

$$r_{ij}(l) \otimes e_i \stackrel{\circ}{=} r_{ij}(k) \otimes e_i - r_{lj}(k) \otimes e_l. \tag{15}$$

Similarly, considering the equations $(x_l^{-1}, x_i, x_i, x_j, x_k) \otimes e_l$, $(x_k, x_l, x_i, x_j, x_k^{-1}) \otimes e_i$ and $(x_l^{-1}, x_k, x_i^{-1}, x_j, x_k^{-1}) \otimes e_i$, we obtain

$$\begin{aligned} r_{ij}(k) \otimes e_i &\stackrel{\circ}{=} r_{l^{-1}j}(i) \otimes e_l - r_{l^{-1}j}(k) \otimes e_l, \\ r_{ij}(k^{-1}) \otimes e_i &\stackrel{\circ}{=} r_{ij}(l) \otimes e_i - r_{k^{-1}j}(l) \otimes e_k, \\ r_{i^{-1}j}(k^{-1}) \otimes e_i &\stackrel{\circ}{=} r_{i^{-1}j}(l) \otimes e_i - r_{lj}(k^{-1}) \otimes e_l \end{aligned} \tag{16}$$

From the equations above, we see that V is generated by $r_{i^{\pm 1}j}(k) \otimes e_i$. We reduce these generators of V more. On the equation (15), exchanging the roles of k and l, we obtain

$$r_{ij}(k) \otimes e_i \stackrel{\circ}{=} r_{ij}(l) \otimes e_i - r_{kj}(l) \otimes e_k,$$

and hence

$$r_{kj}(l) \otimes e_k \stackrel{\circ}{=} -r_{lj}(k) \otimes e_l.$$

For any $j \in \{1, \ldots, n\}$, choose a number $\mu_j \in \{1, \ldots, n\}$ such that $\mu_j \neq j$ and fix it. Then we have

$$r_{\mu_j j}(k) \otimes e_{\mu_j} \stackrel{\circ}{=} -r_{kj}(\mu_j) \otimes e_k, \quad r_{ij}(k) \otimes e_i \stackrel{\circ}{=} r_{ij}(\mu_j) \otimes e_i - r_{kj}(\mu_j) \otimes e_k.$$

Furthermore, from (16), we have

$$r_{i^{-1}j}(k) \otimes e_i \stackrel{\circ}{=} r_{kj}(\mu_j) \otimes e_k + r_{i^{-1}j}(\mu_j) \otimes e_i.$$

This shows that the L-module V is generated by

$$r_{\alpha j}(\mu_j) \otimes e_\alpha, \quad (1 \leq \alpha \leq n, \ \ \alpha \neq j, \mu_j)$$

and

$$r_{\beta^{-1}j}(\mu_j) \otimes e_\beta, \quad (1 \leq \beta \leq n, \ \ \beta \neq j).$$

Therefore we conclude that the generating set $\mathfrak{E}$ of $\bar{R}^{\mathrm{ab}}\otimes_{\mathrm{Aut}^+F_n}H_L$ is reduced to

$$\{r_{i^{\pm 1}j}(\cdot)\otimes e_p \,|\, p\neq i\}\cup\{r_{\alpha j}(\mu_j)\otimes e_\alpha \,|\, 1\leq j\leq n\}\cup\{r_{\beta^{-1}j}(\mu_j)\otimes_\beta \,|\, 1\leq \beta\leq n\}.$$

The number of the generators above is just $2n(n^2-n)-n$. This completes the proof of Proposition 3.1. □

3.2. **The proof of Proposotion 3.2.**

In this subsection, we prove Proposotion 3.2. The outline of the proof is similar to that of Proposotion 3.1. Since the image of the map $H_1(\bar{R},H_L^*)_{\mathrm{Aut}^+F_n}\to H_1(F,H_L^*)$ is isomorphic to the free L-module of rank $2n(n^2-n)-n-1$, $H_1(\bar{R},H_L^*)_{\mathrm{Aut}^+F_n}$ contains a free L-submodule which rank is greater than or equal to $2n(n^2-n)-n-1$. To show it is just $2n(n^2-n)-n-1$, it suffices to show that $H_1(\bar{R},H_L)_{\mathrm{Aut}^+F_n}$ is generated by just $2n(n^2-n)-n-1$ elements. We have $H_1(\bar{R},H_L^*)_{\mathrm{Aut}^+F_n}=\bar{R}^{\mathrm{ab}}\otimes_{\mathrm{Aut}^+F_n}H_L^*$, and see that

$$\mathfrak{E}^*:=\{r\otimes e_p^* \,|\, r\in R,\ 1\leq p\leq n\}$$

is a generating set of $\bar{R}^{\mathrm{ab}}\otimes_{\mathrm{Aut}^+F_n}H_L^*$. In the following, we reduce the elemets of $\mathfrak{E}^*$. We also use $\equiv$ for the equality in $\bar{R}^{\mathrm{ab}}\otimes_{\mathrm{Aut}^+F_n}H_L^*$.

Step 0. By an argument similar to that of Step 0 in Subsection 3.1, we have

Lemma 3.4. *For $n\geq 3$,*

$$(E_{i^{\pm 1}j}\,r\,E_{i^{\pm 1}j^{-1}})\otimes e_p^*\equiv\begin{cases} r\otimes e_p^*, & p\neq j,\\ r\otimes e_j^*\mp r\otimes e_i^*, & p=j,\end{cases}$$

$$(E_{i^{\pm 1}j^{-1}}\,r\,E_{i^{\pm 1}j})\otimes e_p^*\equiv\begin{cases} r\otimes e_p^*, & p\neq j,\\ r\otimes e_j^*\pm r\otimes e_i^*, & p=j.\end{cases}$$

Corollary 3.3. *For $n\geq 3$,*

$$[E_{i^{\pm 1}j},r]\otimes e_p^*\equiv\begin{cases} 0, & p\neq j,\\ \mp r\otimes e_i^*, & p=j,\end{cases}$$

$$[E_{i^{\pm 1}j^{-1}},r]\otimes e_p^*\equiv\begin{cases} 0, & p\neq j,\\ \pm r\otimes e_i^*, & p=j.\end{cases}$$

Lemma 3.5. *For $n\geq 3$,*

$$(w_{i^{\pm 1}j}\,r\,{w_{i^{\pm 1}j}}^{-1})\otimes e_p^*\equiv\begin{cases} r\otimes e_p^*, & p\neq i,j,\\ \mp r\otimes e_j^*, & p=i,\\ \pm r\otimes e_i^*, & p=j,\end{cases}$$

$$({w_{i^{\pm 1}j}}^{-1}\,r\,w_{i^{\pm 1}j})\otimes e_p^*\equiv\begin{cases} r\otimes e_p^*, & p\neq i,j,\\ \pm r\otimes e_j^*, & p=i,\\ \mp r\otimes e_i^*, & p=j.\end{cases}$$

Corollary 3.4. *For $n \geq 3$,*

$$[w_{i^{\pm 1}j}, r] \otimes e_p^* \equiv \begin{cases} 0, & p \neq i, j, \\ -r \otimes e_i^* \mp r \otimes e_j^*, & p = i, \\ \pm r \otimes e_i^* - r \otimes e_j^*, & p = j, \end{cases}$$

$$[w_{i^{\pm 1}j^{-1}}, r] \otimes e_p^* \equiv \begin{cases} 0, & p \neq i, j, \\ -r \otimes e_i^* \pm r \otimes e_j^*, & p = i, \\ \mp r \otimes e_i^* - r \otimes e_j^*, & p = j. \end{cases}$$

Considering any relator of (R2) of the Gersten's presentation is conjugate to one of the relator of (R2-1), ... , (R2-8), or considering Lemma 2.4, for any relator $r =$ (R2), (R3) and (R4), we can rewrite a element $r \otimes e_p^*$ with the relators (R2-1), ... , (R4-1) using Lemmas 3.1 and 3.2.

Step 1. First we consider the generators ${w_{ij}}^4 \otimes e_p^*$. By the same argument as that of Step 1 in Subsection 3.1, we see

$${w_{ij}}^4 \otimes e_p^* = \frac{1}{2}\, 2\, {w_{ij}}^4 \otimes e_p^* \equiv \frac{1}{2}\, {w_{ij}}^8 \otimes e_p^*$$

is rewritten as a sum of the generators $r \otimes e_p^*$ for $r =$ (R2-1), ... , (R4-1). Therefore we can remove the generators ${w_{ij}}^4 \otimes e_p^*$ from the generating set $\mathfrak{E}^*$.

Step 2. Here we show that the generators $r \otimes e_p^*$ for $r =$ (R2-1), ... , (R2-8) is zero or equal to one of the generators $r_{i^{\pm 1}j}(k^{\pm 1}) \otimes e_p^*$. We have

Lemma 3.6. *For $n \geq 6$ and distinct i, j, k, l and m, we have*

(i) **(R2-6):**

$$[E_{ij}, E_{kl}] \otimes e_p^* \equiv \begin{cases} 0, & p \neq j, l, \\ r_{kl}(m) \otimes e_i^*, & p = j, \\ -r_{ij}(m) \otimes e_k^*, & p = l. \end{cases}$$

(ii) **(R2-7), (R2-8):**

$$[E_{i^{-1}j}, E_{k^{\pm 1}l}] \otimes e_p^* \equiv \begin{cases} 0, & p \neq j, l \\ -r_{k^{\pm 1}l}(m) \otimes e_i^*, & p = j, \\ \mp r_{i^{-1}j}(m) \otimes e_k^*, & p = l. \end{cases}$$

(iii) **(R2-5):**

$$[E_{ij}, E_{i^{-1}k}] \otimes e_p^* \equiv \begin{cases} 0, & p \neq j, k, \\ r_{i^{-1}k}(l) \otimes e_i^*, & p = j, \\ r_{ij}(l) \otimes e_i, & p = k. \end{cases}$$

(iv) **(R2-1):**

$$[E_{ij}, E_{i^{-1}j}] \otimes e_p^* \equiv \begin{cases} 0, & p \neq j, \\ r_{ij}(k) \otimes e_i^* - r_{i^{-1}j}(l) \otimes e_i, & p = j. \end{cases}$$

(v) **(R2-2):**

$$[E_{ij}, E_{kj}] \otimes e_p^* \equiv \begin{cases} 0, & p \neq j, \\ r_{kj}(l) \otimes e_i^* - r_{ij}(m) \otimes e_k^*, & p = j. \end{cases}$$

(vi) **(R2-3), (R2-4):**

$$[E_{i^{-1}j}, E_{k^{\pm 1}j}] \otimes e_p^* \equiv \begin{cases} 0, & p \neq j, \\ -r_{k^{\pm 1}j}(l) \otimes e_i^* \mp r_{i^{-1}j}(m) \otimes e_k^*, & p = j. \end{cases}$$

Since this Lemma is proved by an argument similar to that in Lemma 3.3, we omit the details. (For details, see Subsection 6.1.)

By the lemma above, we can remove the generators $r \otimes e_p^*$ for $r =$ (R2-1), ... , (R2-8) from the generationg set $\mathfrak{E}^*$.

Step 3. Here we consider the generators $r_{i^{\pm 1}j}(k^{\pm 1}) \otimes e_p^*$ for $p \neq j$.

(3-a) The case $p \neq j, k$.

First we consider the case $p \neq k$. By an argument similar to that of **(3-a)** in Subsection 3.1, observing the results of Lemma 3.6, we can set

$$r_{ij}(\cdot) \otimes e_p^* :\equiv r_{ij}(k^{\pm 1}) \otimes e_p^*,$$
$$r_{i^{-1}j}(\cdot) \otimes e_p^* :\equiv r_{i^{-1}j}(k^{\pm 1}) \otimes e_p^*$$

for $p \neq j, k$.

(3-b) The case $p = k$**.**

For the case $p = k$, set

$$S_{ijk}^* := r_{ij}(k) \otimes e_k^* - r_{ij}(\cdot) \otimes e_k^* - r_{kj}(\cdot) \otimes e_i^*.$$

By the same argument as that of **(3-b)** in Subsection 3.1, observing the equations $(x_l^{\pm 1}, x_j, x_i, x_j, x_k) \otimes e_k^*$, we obtain $S_{ijk}^* \equiv 0$. Furthermore, observing $(x_k^{-1}, x_l, x_i, x_j, x_l) \otimes e_k^*$ and $(x_i^{-1}, x_l, x_l, x_j, x_k^{\pm 1}) \otimes e_k^*$, we obtain

$$r_{ij}(k^{-1}) \otimes e_k^* \equiv r_{ij}(\cdot) \otimes e_k^* - r_{k^{-1}j}(\cdot) \otimes e_i^*,$$
$$r_{i^{-1}j}(k^{\pm 1}) \otimes e_k^* \equiv r_{i^{-1}j}(\cdot) \otimes e_k^* \mp r_{k^{\pm 1}j}(\cdot) \otimes e_i^*.$$

By the argument above, we can remove the generators $r_{i^{\pm 1}j}(k^{\pm 1}) \otimes e_k^*$ from the generationg set $\mathfrak{E}^*$.

Step 4. Here we consider the generators $h_{ij} \otimes e_p^*$ for $p \neq i, j$. By an argument similar to that of Step 4 in Subsection 3.1, considering the elements $[{w_{ij}}^{-1}, E_{kl}] \otimes e_l^*$, we obtain

$$h_{ij} \otimes e_k^* \equiv r_{ij}(\cdot) \otimes e_k^* + r_{i^{-1}j}(\cdot) \otimes e_k^*.$$

The cases where $p = i$ or j are mensioned in Step 6 later.

Step 5. Let V be the quotient L-module of $\bar{R}^{\mathrm{ab}} \otimes_{\mathrm{Aut}^+ F_n} H_L^*$ by the L-submodule generated by the elements $r_{i^{\pm 1}j}(\cdot) \otimes e_k^*$ for $k \neq j$. Then from the argument above,

the elements $r_{i^{\pm 1}j}(k^{\pm 1}) \otimes e_j^*$, $h_{ij} \otimes e_i^*$ and $h_{ij} \otimes e_j^*$ generate V. Here we reduce these generators of V. We use $\stackrel{\circ}{=}$ for the equality in V.

First, considering the equation $(x_l, x_k, x_i, x_j, x_k) \otimes e_j^*$ in a way similar to that of Step 5 in Subsection 3.1, we have

$$r_{ik}(l^{-1}) \otimes e_k^* \stackrel{\circ}{=} -r_{ij}(k) \otimes e_j^* + r_{ij}(l) \otimes e_j^*. \tag{17}$$

Similarly, considering $(x_l^{-1}, x_j, x_i, x_k, x_j) \otimes e_k^*$, $(x_l, x_k, x_i^{-1}, x_j, x_k) \otimes e_j^*$ and $(x_l^{-1}, x_j, x_i^{-1}, x_k, x_j) \otimes e_k^*$, we obtain

$$r_{ik}(l^{-1}) \otimes e_k^* \stackrel{\circ}{=} r_{ik}(j) \otimes e_k^* + r_{ij}(l) \otimes e_j^*, \tag{18}$$

$$r_{i^{-1}k}(l^{-1}) \otimes e_k^* \stackrel{\circ}{=} -r_{i^{-1}j}(k) \otimes e_j^* + r_{i^{-1}j}(l) \otimes e_j^*, \tag{19}$$

$$r_{i^{-1}k}(l^{-1}) \otimes e_k^* \stackrel{\circ}{=} r_{i^{-1}k}(j) \otimes e_k^* + r_{i^{-1}j}(l) \otimes e_j^* \tag{20}$$

respectively. Hence we see that the L-module V is generated by $r_{i^{\pm 1}j}(k) \otimes e_j^*$, $h_{ij} \otimes e_i^*$ and $h_{ij} \otimes e_j^*$.

Substituting (18) into (17), and substituting (20) into (19), we obtain

$$r_{i^{\pm 1}k}(j) \otimes e_k^* \stackrel{\circ}{=} -r_{i^{\pm 1}j}(k) \otimes e_j^*. \tag{21}$$

On the other hand, considering the equation $(x_l, x_k, x_i, x_j, x_k^{-1}) \otimes e_j^*$, we obtain

$$r_{ij}(l^{-1}) \otimes e_j^* \stackrel{\circ}{=} r_{ij}(k^{-1}) \otimes e_j^* + r_{ik}(l^{-1}) \otimes e_k^*.$$

Hence, rewriting each term of the equation above as a sum of $r_{\alpha\beta}(\gamma) \otimes e_\beta^*$, $(1 \geq \alpha, \beta, \gamma \geq n)$, using (17), and using (21), we have $2(r_{ij}(k) \otimes e_j^* + r_{ik}(l) \otimes e_k^* - r_{ij}(l) \otimes e_j^*) \stackrel{\circ}{=} 0$. Since 2 is invertible in V, we obtain

$$r_{ij}(k) \otimes e_j^* + r_{ik}(l) \otimes e_k^* - r_{ij}(l) \otimes e_j^* \stackrel{\circ}{=} 0. \tag{22}$$

Similarly, considering $(x_i, x_l^{-1}, x_l, x_j, x_k) \otimes e_j^*$, we have

$$\begin{aligned} r_{i^{-1}j}(k) \otimes e_j^* \stackrel{\circ}{=} {} & r_{lj}(k) \otimes e_j^* - r_{i^{-1}k}(l) \otimes e_k^* \\ & + r_{lk}(i^{-1}) \otimes e_k^* + r_{i^{-1}j}(l) \otimes e_j^* - r_{lj}(i^{-1}) \otimes e_j^*. \end{aligned}$$

Using (17) and (22), we can reduce the equation above to

$$r_{i^{-1}j}(k) \otimes e_j^* + r_{i^{-1}k}(l) \otimes e_k^* + r_{i^{-1}l}(j) \otimes e_l^* \stackrel{\circ}{=} 0.$$

Now, using the equations above, we show that each generator $r_{i^{\pm 1}j}(k) \otimes e_j^*$ is rewritten as a sum of the generators type of $r_{i^{\pm 1}j}(1) \otimes e_j^*$ and $r_{1^{\pm 1}j}(2) \otimes e_j^*$. For distinct $i, j, k \neq 1$, we have $r_{ij}(k) \otimes e_j^* \stackrel{\circ}{=} -r_{ik}(1) \otimes e_k^* + r_{ij}(1) \otimes e_j^*$. If $j = 1$, we have $r_{i1}(k) \otimes e_1^* \stackrel{\circ}{=} -r_{ik}(1) \otimes e_k^*$. If $i = 1$ and $j, k \neq 2$, then $r_{1j}(k) \otimes e_j^* \stackrel{\circ}{=} -r_{1k}(2) \otimes e_k^* + r_{ij}(2) \otimes e_j^*$. Finally, if $i = 1$ and $j = 2$, we have $r_{12}(k) \otimes e_2^* \stackrel{\circ}{=} -r_{1k}(2) \otimes e_k^*$. Hence any generator $r_{ij}(k) \otimes e_j^*$ is rewritten as a sum of the generators $r_{ij}(1) \otimes e_j^*$ and $r_{1j}(2) \otimes e_j^*$. Similarly we see that $r_{i^{-1}j}(k) \otimes e_j^*$ is rewritten as a sum of the generators $r_{i^{-1}j}(1) \otimes e_j^*$ and $r_{1^{-1}j}(2) \otimes e_j^*$.

From the argument above, we see that V is generated by $r_{i^{\pm 1}j}(1) \otimes e_j^*$, $r_{1^{\pm 1}j}(2) \otimes e_j^*$, $h_{ij} \otimes e_i^*$ and $h_{ij} \otimes e_j^*$, and hence $\bar{R}^{\mathrm{ab}} \otimes_{\mathrm{Aut}^+ F_n} H_L^*$ is generated by these elements and $r_{i^{\pm 1}j}(\cdot) \otimes e_k^*$.

Step 6. Finally we consider the generators $h_{ij} \otimes e_p^*$ for $p = i, j$. Let V' be the quotient L-module of $\bar{R}^{\mathrm{ab}} \otimes_{\mathrm{Aut}^+ F_n} H_L^*$ by the L-submodule generated by the elements $r_{i^{\pm 1}j}(\cdot) \otimes e_k^*$, $r_{i^{\pm 1}j}(1) \otimes e_j^*$ and $r_{1^{\pm 1}j}(2) \otimes e_j^*$. We use $\doteq$ for the equality in V'.

For distinct i, j and k, the equation $\{x_k, x_i^{-1}, x_i, x_j\} \otimes e_j^*$ is given by

$$\begin{aligned} h_{k^{-1}j} \otimes e_j^* = t_3 \otimes e_j^* &+ (w_{ki^{-1}}{}^{-1} E_{ji} w_{ki^{-1}}\ t_2\ w_{ki^{-1}}{}^{-1} E_{ji}{}^{-1} w_{ki^{-1}}) \otimes e_j^* \\ &+ (w_{ki^{-1}}{}^{-1} E_{ji} E_{i^{-1}j} w_{ki^{-1}}\ t_1\ w_{ki^{-1}}{}^{-1} E_{i^{-1}j}{}^{-1} E_{ji}{}^{-1} w_{ki^{-1}}) \otimes e_j^* \\ &+ (w_{ki^{-1}}{}^{-1} h_{ij} w_{ki^{-1}}) \otimes e_j^* \\ &+ (w_{ki^{-1}}{}^{-1} E_{j^{-1}i}{}^{-1} E_{ij}{}^{-1} w_{ki^{-1}}\ t_4\ w_{ki^{-1}}{}^{-1} E_{ij} E_{j^{-1}i} w_{ki^{-1}}) \otimes e_j^* \\ &+ (w_{ki^{-1}}{}^{-1} E_{j^{-1}i}{}^{-1} w_{ki^{-1}}\ t_5\ w_{ki^{-1}}{}^{-1} E_{j^{-1}i} w_{ki^{-1}}) \otimes e_j^* + t_6 \otimes e_j^* \end{aligned}$$

where

$$\begin{aligned} t_1 &:= E_{j^{-1}k}(w_{ki^{-1}}^{-1} E_{j^{-1}i^{-1}} w_{ki^{-1}})^{-1}, & t_2 &:= E_{kj}(w_{ki^{-1}}^{-1} E_{i^{-1}j} w_{ki^{-1}})^{-1}, \\ t_3 &:= E_{jk^{-1}}(w_{ki^{-1}}^{-1} E_{ji} w_{ki^{-1}})^{-1}, & t_4 &:= (w_{ki^{-1}}^{-1} E_{ji^{-1}} w_{ki^{-1}})^{-1} E_{jk}, \\ t_5 &:= (w_{ki^{-1}}^{-1} E_{ij} w_{ki^{-1}})^{-1} E_{k^{-1}j}, & t_6 &:= (w_{ki^{-1}}^{-1} E_{j^{-1}i} w_{ki^{-1}})^{-1} E_{j^{-1}k^{-1}}. \end{aligned}$$

Observing Lemma 2.2, we see that all t_m, $(1 \le m \le 6)$, except for t_2 belong to the normal closure of the relators of (R2-1), ... , (R3-4). Hence, using Lemmas 3.4 and Lemmas 3.5, we obtain

$$h_{k^{-1}j} \otimes e_j^* \doteq t_2 \otimes e_j^* + h_{ij} \otimes e_j^*. \tag{23}$$

From (3), we have

$$\begin{aligned} t_2^{-1} &= (w_{ki^{-1}}{}^{-1} E_{i^{-1}j^{-1}} w_{ki^{-1}})^{-1} E_{kj^{-1}} \\ &= (w_{ki^{-1}}{}^{-1} E_{i^{-1}j} h_{ki^{-1}} E_{i^{-1}j^{-1}} w_{ki^{-1}}) \cdot (w_{ki^{-1}}{}^{-1} h_{ki^{-1}}{}^{-1} w_{ki^{-1}}) \\ &\quad \cdot (w_{k^{-1}i}{}^{-1} E_{i^{-1}j^{-1}} w_{k^{-1}i})^{-1} E_{kj^{-1}}, \end{aligned}$$

and hence

$$t_2^{-1} \otimes e_j^* \equiv \{(w_{k^{-1}i}{}^{-1} E_{i^{-1}j^{-1}} w_{k^{-1}i})^{-1} E_{kj^{-1}}\} \otimes e_j^* - h_{ki^{-1}} \otimes e_i^*.$$

On the other hand, from Lemma 2.4, we have

$$\begin{aligned} h_{k^{-1}j} \otimes e_j^* &\equiv (w_{kj}{}^{-1} h_{kj}{}^{-1} w_{kj}) \otimes e_j^* \equiv -h_{kj} \otimes e_k^*, \\ h_{ki^{-1}} \otimes e_i^* &\equiv (w_{ki}{}^{-1} h_{ki}{}^{-1} w_{ki}) \otimes e_i^* \equiv -h_{ki} \otimes e_k^*, \end{aligned}$$

and see that the element $\{(w_{k^{-1}i}{}^{-1} E_{i^{-1}j^{-1}} w_{k^{-1}i})^{-1} E_{kj^{-1}}\} \in \bar{R}$ belongs to the normal closure of the relators of (R2-1), ... , (R3-4) by Lemma 2.2. Therefore we obtain

$$t_2 \otimes e_j^* \doteq -h_{ki} \otimes e_k^*.$$

Substituting these results into (23), we obtain

$$h_{ij} \otimes e_j^* \doteq h_{ki} \otimes e_k^* - h_{kj} \otimes e_k^*. \tag{24}$$

Similarly, considering $\{x_k, x_i, x_i, x_j\} \otimes e_j^*$ and $\{x_k, x_i^{\mp 1}, x_i, x_j\} \otimes e_k^*$, we obtain

$$h_{ij} \otimes e_j^* \doteq h_{kj} \otimes e_j^* - h_{ki} \otimes e_i^*, \tag{25}$$

$$h_{ij} \otimes e_i^* \doteq -h_{ki} \otimes e_k^* - h_{kj} \otimes e_j^*, \tag{26}$$

$$h_{ij} \otimes e_i^* \doteq h_{kj} \otimes e_k^* + h_{ki} \otimes e_i^*. \tag{27}$$

Now we show that all $h_{ij} \otimes e_j^*$ and $h_{ij} \otimes e_i^*$ are rewritten as a sum of $h_{1j} \otimes e_j^*$. First, from (24), we see $h_{ij} \otimes e_j^* \doteq -h_{ji} \otimes e_i^*$. Exchanging the roles of i and j on (26), we have $h_{ji} \otimes e_j^* \doteq -h_{kj} \otimes e_k^* - h_{ki} \otimes e_i^*$. Then substituting it into (27), we obtain $h_{ij} \otimes e_i^* \doteq -h_{ji} \otimes e_j^*$. Set $h(i,j) := h_{ij} \otimes e_j^* + h_{ij} \otimes e_i^*$. From (24) and (27), $h(i,j) \doteq h(k,i)$.

Similarly $h(i,j) \doteq -h(k,i)$ from (25) and (26). Since 2 is invertible in V', $h(i,j) \doteq 0$ and hence $h_{ij} \otimes e_j^* \doteq -h_{ij} \otimes e_i^*$. For dintinct $i, j \neq 1$, we have $h_{ij} \otimes e_j^* \doteq h_{1j} \otimes e_j^* - h_{1i} \otimes e_i^*$ from (25). Furthermore, if $j = 1$, $h_{i1} \otimes e_1^* \doteq h_{1i} \otimes e_i^*$. So we see that $h_{ij} \otimes e_i^*$ and $h_{ij} \otimes e_j^*$ are rewritten as a sum of the elements $h_{1i} \otimes e_i^*$ in V'.

From the argument above, we conclude that the generating set $\mathfrak{E}^*$ of $\bar{R}^{\mathrm{ab}} \otimes_{\mathrm{Aut}^+ F_n} H_L^*$ is reduced to

$$\{r_{i^{\pm 1}j}(\cdot) \otimes e_p \,|\, p \neq j\} \cup \{r_{i^{\pm 1}j}(1) \otimes e_j^* \,|\, i, j \neq 1\} \cup \{r_{1^{\pm 1}j}(2) \otimes e_j^* \,|\, j \neq 1, 2\} \cup \{h_{1j} \otimes e_j^* \,|\, j \neq 1\}.$$

The number of the generators above is just $2n(n^2 - n) - n - 1$. Hence it is a basis of $\bar{R}^{\mathrm{ab}} \otimes_{\mathrm{Aut}^+ F_n} H_L^*$ as a free L-module. This completes the proof of Proposition 3.2. □

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5. Appendix I

In this section, we show the details of the computation which are required in the proof of Proposition 3.1.

5.1. **The proof of Lemma 3.3.**

Here we prove (ii), (iv), (v) and (vi) of Lemma 3.3.

- **The proof of (ii): (R2-3), (R2-4)**

First, let $p \neq i, k$ and $l \in \{1, \dots, n\}$ such that $l \neq i, j, k, p$. Set $r = {E_{i^{-1}j}}^{-1}[E_{i^{-1}l}, E_{lj}]$. Then we have

$$[E_{i^{-1}j}, [r, E_{k^{\pm 1}j}]] \otimes e_p \equiv 0, \quad [r, E_{k^{\pm 1}j}] \otimes e_p \equiv 0$$

and

$$\begin{aligned}
&[E_{i^{-1}j}, E_{k^{\pm 1}j}] \otimes e_p \\
&\quad \equiv ([E_{i^{-1}j}, [r, E_{k^{\pm 1}j}]][r, E_{k^{\pm 1}j}][E_{i^{-1}j}, E_{k^{\pm 1}j}]) \otimes e_p \\
&\quad = [[E_{i^{-1}l}, E_{lj}], E_{k^{\pm 1}j}] \otimes e_p \\
&\quad = [E_{i^{-1}l}E_{lj}{E_{i^{-1}l}}^{-1}{E_{lj}}^{-1}, E_{k^{\pm 1}j}] \otimes e_p \\
&\quad = ([E_{i^{-1}l}, [E_{lj}{E_{i^{-1}l}}^{-1}{E_{lj}}^{-1}, E_{k^{\pm 1}j}]] \\
&\qquad\qquad [E_{lj}{E_{i^{-1}l}}^{-1}{E_{lj}}^{-1}, E_{k^{\pm 1}j}][E_{i^{-1}l}, E_{k^{\pm 1}j}]) \otimes e_p \\
&\quad \equiv ([E_{lj}{E_{i^{-1}l}}^{-1}{E_{lj}}^{-1}, E_{k^{\pm 1}j}][E_{i^{-1}l}, E_{k^{\pm 1}j}]) \otimes e_p \\
&\quad \equiv \cdots\cdots \\
&\quad \equiv ([{E_{lj}}^{-1}, E_{k^{\pm 1}j}][{E_{i^{-1}l}}^{-1}, E_{k^{\pm 1}j}][E_{lj}, E_{k^{\pm 1}j}][E_{i^{-1}l}, E_{k^{\pm 1}j}]) \otimes e_p.
\end{aligned}$$

Since $p \neq i, l$, we have

$$\begin{aligned}
&[E_{lj}, E_{k^{\pm 1}j}] \otimes e_p + [{E_{lj}}^{-1}, E_{k^{\pm 1}j}] \otimes e_p \equiv 0, \\
&[E_{i^{-1}l}, E_{k^{\pm 1}j}] \otimes e_p + [{E_{i^{-1}l}}^{-1}, E_{k^{\pm 1}j}] \otimes e_p \equiv 0,
\end{aligned}$$

and hence $[E_{i^{-1}j}, E_{k^{\pm 1}j}] \otimes e_p \equiv 0$.

Next, let $p = i$ and $l \neq i, j, k$. Since $[[E_{k^{-1}j}, E_{i^{-1}j}], r_{k^{\pm 1}j}(l)] = 0$ in $\bar{R}^{\mathrm{ab}}$ and $[E_{i^{-1}j}, r_{k^{\pm 1}j}(l)] \otimes e_i \equiv -r_{k^{\pm 1}j}(l) \otimes e_j$ by Corollary 3.1, we have

$$\begin{aligned}
&[E_{i^{-1}j}, E_{k^{\pm 1}j}] \otimes e_i - r_{k^{\pm 1}j}(l) \otimes e_j \\
&\quad \equiv ([E_{i^{-1}j}, r_{k^{\pm 1}j}(l)][E_{i^{-1}j}, E_{k^{\pm 1}j}][[E_{k^{\pm 1}j}, E_{i^{-1}j}], r_{k^{\pm 1}j}(l)]) \otimes e_i \\
&\quad = [E_{i^{-1}j}, [E_{k^{\pm 1}l}, E_{lj}]] \otimes e_i \\
&\quad = ([E_{i^{-1}j}, E_{k^{\pm 1}l}][E_{i^{-1}j}, E_{lj}{E_{k^{\pm 1}l}}^{-1}{E_{lj}}^{-1}] \\
&\qquad\qquad [[E_{lj}{E_{k^{\pm 1}l}}^{-1}{E_{lj}}^{-1}, E_{i^{-1}j}], E_{k^{\pm 1}l}]) \otimes e_i \\
&\quad \equiv \cdots\cdots \\
&\quad \equiv ([E_{i^{-1}j}, E_{k^{\pm 1}l}][E_{i^{-1}j}, {E_{k^{\pm 1}l}}^{-1}][E_{i^{-1}j}, E_{lj}][E_{i^{-1}j}, {E_{lj}}^{-1}]) \otimes e_i, \\
&\quad \equiv 0
\end{aligned}$$

and hence $[E_{i^{-1}j}, E_{k^{\pm 1}j}] \otimes e_i \equiv r_{k^{\pm 1}j}(l) \otimes e_j$.

From the result above, for $l \neq i,j,k$, we also see $[E_{i^{-1}j}, E_{k^{-1}j}] \otimes e_k \equiv -r_{i^{-1}j}(l) \otimes e_j$. To show $[E_{i^{-1}j}, E_{kj}] \otimes e_k \equiv r_{i^{-1}j}(l) \otimes e_j$, set $r_{i^{-1}j}(l) = [E_{i^{-1}l}, E_{lj}]{E_{i^{-1}j}}^{-1}$. Since $[E_{kj}, r_{i^{-1}j}(l)] \otimes e_k \equiv r_{i^{-1}j}(l) \otimes e_j$ by Corollary 3.1, we have

$$\begin{aligned}
&[E_{kj},E_{i^{-1}j}] \otimes e_k + r_{i^{-1}j}(l) \otimes e_j \\
&\quad \equiv ([E_{kj}, r_{i^{-1}j}(l)][E_{kj}, E_{i^{-1}j}][[E_{i^{-1}j}, E_{kj}], r_{i^{-1}j}(l)]) \otimes e_k \\
&\quad = [E_{kj}, [E_{i^{-1}l}, E_{lj}]] \otimes e_k \\
&\quad = ([E_{kj}, E_{i^{-1}l}][E_{kj}, E_{lj}{E_{i^{-1}l}}^{-1}{E_{lj}}^{-1}][[E_{lj}{E_{i^{-1}l}}^{-1}{E_{lj}}^{-1}, E_{kj}], E_{i^{-1}l}]) \otimes e_k \\
&\quad \equiv \cdots\cdots \\
&\quad \equiv ([E_{kj}, E_{i^{-1}l}][E_{kj}, {E_{i^{-1}l}}^{-1}][E_{kj}, E_{lj}][E_{kj}, {E_{lj}}^{-1}]) \otimes e_k \\
&\quad \equiv 0,
\end{aligned}$$

and hence $[E_{i^{-1}j}, E_{kj}] \otimes e_k \equiv r_{i^{-1}j}(l) \otimes e_j$.

• **The proof of (iv): (R2-6)**

First, let $p \neq i,k$. Since $n \geq 6$, we can choose a number $m \in \{1,\ldots,n\}$ such that $m \neq i,j,k,l,p$. Set $r = {E_{ij}}^{-1}[E_{im}, E_{mj}]$. Then we have

$$[E_{ij}, [r, E_{kl}]] \otimes e_p \equiv 0, \quad r, E_{kl} \otimes e_p \equiv 0$$

and

$$\begin{aligned}
&[E_{ij},E_{kl}] \otimes e_p \\
&\quad \equiv ([E_{ij}, [r, E_{kl}]][r, E_{kl}][E_{ij}, E_{kl}]) \otimes e_p \\
&\quad = [[E_{im}, E_{mj}], E_{kl}] \otimes e_p \\
&\quad = [E_{im}E_{mj}{E_{im}}^{-1}{E_{mj}}^{-1}, E_{kl}] \otimes e_p \\
&\quad = ([E_{im}, [E_{mj}{E_{im}}^{-1}{E_{mj}}^{-1}, E_{kl}]][E_{mj}{E_{im}}^{-1}{E_{mj}}^{-1}, E_{kl}][E_{im}, E_{kl}]) \otimes e_p \\
&\quad \equiv ([E_{mj}{E_{im}}^{-1}{E_{mj}}^{-1}, E_{kl}][E_{im}, E_{kl}]) \otimes e_p \\
&\quad \equiv \cdots\cdots \\
&\quad \equiv ([{E_{mj}}^{-1}, E_{kl}][{E_{im}}^{-1}, E_{kl}][E_{mj}, E_{kl}][E_{im}, E_{kl}]) \otimes e_p.
\end{aligned}$$

Since $p \neq i,m$, we have

$$\begin{aligned}
[E_{mj}, E_{kl}] \otimes e_p + [{E_{mj}}^{-1}, E_{kl}] \otimes e_p \equiv 0, \\
[E_{im}, E_{kl}] \otimes e_p + [{E_{im}}^{-1}, E_{kl}] \otimes e_p \equiv 0,
\end{aligned}$$

and hence $[E_{ij}, E_{kl}] \otimes e_p \equiv 0$.

Next, let $p = i$ and $m \neq i,j,k,l$.

Since $[[E_{kl}, E_{ij}], r_{kl}(m)] \otimes e_i \equiv 0$ in $\bar{R}^{\mathrm{ab}}$ and $[E_{ij}, r_{kl}(m)] \otimes e_i \equiv r_{kl}(m) \otimes e_j$ by Corollary 3.1, we have

$$\begin{aligned}
&[E_{ij}, E_{kl}] \otimes e_i + r_{kl}(m) \otimes e_j \\
&\qquad \equiv ([E_{ij}, r_{kl}(m)][E_{ij}, E_{kl}][[E_{kl}, E_{ij}], r_{kl}(m)]) \otimes e_i \\
&\qquad = [E_{ij}, [E_{km}, E_{ml}]] \otimes e_i \\
&\qquad \equiv \cdots\cdots \\
&\qquad \equiv 0
\end{aligned}$$

and hence $[E_{ij}, E_{kl}] \otimes e_i \equiv -r_{kl}(m) \otimes e_j$.

• **The proof of (v): (R2-7), (R2-8)**

First, let $p \neq i, k$ and $m \in \{1, \dots, n\}$ such that $m \neq i, j, k, l, p$. Set $r = {E_{i^{-1}j}}^{-1}[E_{i^{-1}m}, E_{mj}]$. Then we have

$$[E_{i^{-1}j}, [r, E_{k^{\pm 1}l}]] \otimes e_p \equiv 0, \quad [r, E_{k^{\pm 1}l}] \otimes e_p \equiv 0$$

and

$$\begin{aligned}
&[E_{i^{-1}j}, E_{k^{\pm 1}l}] \otimes e_p \\
&\quad \equiv ([E_{i^{-1}j}, [r, E_{k^{\pm 1}l}]][r, E_{k^{\pm 1}l}][E_{i^{-1}j}, E_{k^{\pm 1}l}]) \otimes e_p \\
&\quad = [[E_{i^{-1}m}, E_{mj}], E_{k^{\pm 1}l}] \otimes e_p \\
&\quad = [E_{i^{-1}m}E_{mj}{E_{i^{-1}m}}^{-1}{E_{mj}}^{-1}, E_{k^{\pm 1}l}] \otimes e_p \\
&\quad = ([E_{i^{-1}m}, [E_{mj}{E_{i^{-1}m}}^{-1}{E_{mj}}^{-1}, E_{k^{\pm 1}l}]] \\
&\qquad\qquad [E_{mj}{E_{i^{-1}m}}^{-1}{E_{mj}}^{-1}, E_{k^{\pm 1}l}][E_{i^{-1}m}, E_{k^{\pm 1}l}]) \otimes e_p \\
&\quad \equiv ([E_{mj}{E_{i^{-1}m}}^{-1}{E_{mj}}^{-1}, E_{k^{\pm 1}l}][E_{i^{-1}m}, E_{k^{\pm 1}l}]) \otimes e_p \\
&\quad \equiv \cdots\cdots \\
&\quad \equiv ([{E_{mj}}^{-1}, E_{k^{\pm 1}l}][{E_{i^{-1}m}}^{-1}, E_{k^{\pm 1}l}][E_{mj}, E_{k^{\pm 1}l}][E_{i^{-1}m}, E_{k^{\pm 1}l}]) \otimes e_p.
\end{aligned}$$

Since $p \neq i, m$, we have

$$\begin{aligned}
&[E_{mj}, E_{k^{\pm 1}l}] \otimes e_p + [{E_{mj}}^{-1}, E_{k^{\pm 1}l}] \otimes e_p \equiv 0, \\
&[E_{i^{-1}m}, E_{k^{\pm 1}l}] \otimes e_p + [{E_{i^{-1}m}}^{-1}, E_{k^{\pm 1}l}] \otimes e_p \equiv 0
\end{aligned}$$

and hence $[E_{i^{-1}j}, E_{k^{\pm 1}l}] \otimes e_p \equiv 0$.

Next, let $p = i$ and $m \neq i, j, k, l$. Since $[[E_{k^{\pm 1}l}, E_{i^{-1}j}], r_{k^{\pm 1}l}(m)] = 0$ in $\bar{R}^{\mathrm{ab}}$ and $[E_{i^{-1}j}, r_{k^{\pm 1}l}(m)] \otimes e_i \equiv -r_{k^{\pm 1}l}(m) \otimes e_j$ by Corollary 3.1, we have

$$\begin{aligned}
&[E_{i^{-1}j}, E_{k^{\pm 1}l}] \otimes e_i - r_{k^{\pm 1}l}(m) \otimes e_j \\
&\quad \equiv ([E_{i^{-1}j}, r_{k^{\pm 1}l}(m)][E_{i^{-1}j}, E_{k^{\pm 1}l}][[E_{k^{\pm 1}l}, E_{i^{-1}j}], r_{k^{\pm 1}l}(m)]) \otimes e_i \\
&\quad = [E_{i^{-1}j}, [E_{k^{\pm 1}m}, E_{ml}]] \otimes e_i \\
&\quad \equiv \cdots\cdots \\
&\quad \equiv 0
\end{aligned}$$

and hence $[E_{i^{-1}j}, E_{k^{\pm 1}l}] \otimes e_i \equiv r_{k^{\pm 1}l}(m) \otimes e_j$.

From the result above, for $m \neq i, j, k, l$, we also see $[E_{i^{-1}j}, E_{k^{-1}l}] \otimes e_k \equiv -r_{i^{-1}j}(m) \otimes e_l$. To show $[E_{i^{-1}j}, E_{kl}] \otimes e_k \equiv r_{i^{-1}j}(m) \otimes e_l$, set $r_{i^{-1}j}(m) = [E_{i^{-1}m}, E_{mj}]{E_{i^{-1}j}}^{-1}$. Since $[[E_{i^{-1}j}, E_{kl}], r_{i^{-1}j}(m)]$ in $\bar{R}^{\mathrm{ab}}$ and $[E_{kl}, r_{i^{-1}j}(m)] \otimes e_k \equiv r_{i^{-1}j}(m) \otimes e_l$ by Corollary 3.1, we have

$$\begin{aligned}
&[E_{kl}, E_{i^{-1}j}] \otimes e_k + r_{i^{-1}j}(m) \otimes e_l \\
&\quad \equiv ([E_{kl}, r_{i^{-1}j}(m)][E_{kl}, E_{i^{-1}j}][[E_{i^{-1}j}, E_{kl}], r_{i^{-1}j}(m)]) \otimes e_k \\
&\quad = [E_{kl}, [E_{i^{-1}m}, E_{mj}]] \otimes e_k \\
&\quad \equiv \cdots\cdots \\
&\quad \equiv 0
\end{aligned}$$

and hence $[E_{i^{-1}j}, E_{kl}] \otimes e_k \equiv r_{i^{-1}j}(m) \otimes e_l$.

• **The proof of (iii): (R2-1)**
First, let $p \neq i$ and $k \in \{1, \ldots, n\}$ such that $k \neq i, j, p$. Set $r = {E_{ij}}^{-1}[E_{ik}, E_{kj}]$. Then we have

$$[E_{ij}, [r, E_{i^{-1}j}]] \otimes e_p \equiv 0, \quad [r, E_{i^{-1}j}] \otimes e_p \equiv 0$$

and

$$\begin{aligned}
&[E_{ij}, E_{i^{-1}j}] \otimes e_p \\
&\quad \equiv ([E_{ij}, [r, E_{i^{-1}j}]][r, E_{i^{-1}j}][E_{ij}, E_{i^{-1}j}]) \otimes e_p \\
&\quad = [[E_{ik}, E_{kj}], E_{i^{-1}j}] \otimes e_p \\
&\quad = [E_{ik}E_{kj}{E_{ik}}^{-1}{E_{kj}}^{-1}, E_{i^{-1}j}] \otimes e_p \\
&\quad = ([E_{ik}, [E_{kj}{E_{ik}}^{-1}{E_{kj}}^{-1}, E_{i^{-1}j}]][E_{kj}{E_{ik}}^{-1}{E_{kj}}^{-1}, E_{i^{-1}j}][E_{ik}, E_{i^{-1}j}]) \otimes e_p \\
&\quad \equiv ([E_{kj}{E_{ik}}^{-1}{E_{kj}}^{-1}, E_{i^{-1}j}][E_{ik}, E_{i^{-1}j}]) \otimes e_p \\
&\quad \equiv \cdots\cdots \\
&\quad \equiv ([{E_{kj}}^{-1}, E_{i^{-1}j}][{E_{ik}}^{-1}, E_{i^{-1}j}][E_{kj}, E_{i^{-1}j}][E_{ik}, E_{i^{-1}j}]) \otimes e_p.
\end{aligned}$$

Since $p \neq i, k$, we have

$$\begin{aligned}
[E_{kj}, E_{i^{-1}j}] \otimes e_p + [{E_{kj}}^{-1}, E_{i^{-1}j}] \otimes e_p \equiv 0, \\
[E_{ik}, E_{i^{-1}j}] \otimes e_p + [{E_{ik}}^{-1}, E_{i^{-1}j}] \otimes e_p \equiv 0,
\end{aligned}$$

and hence $[E_{ij}, E_{i^{-1}j}] \otimes e_p \equiv 0$.

Next, let $p = i$ and $k \neq i, j$. Since $[r_{ij}(k), [E_{ij}, E_{i^{-1}j}]] = 0$ in $\bar{R}^{\mathrm{ab}}$ and $[r_{ij}(k), E_{i^{-1}j}] \otimes e_i \equiv r_{ij}(k) \otimes e_j$ by Corollary 3.1, we have

$$\begin{aligned}
&[E_{ij}, E_{i^{-1}j}] \otimes e_i + r_{ij}(k) \otimes e_j \\
&\quad \equiv ([r_{ij}(k), [E_{ij}, E_{i^{-1}j}]][E_{ij}, E_{i^{-1}j}][r_{ij}(k), E_{i^{-1}j}]) \otimes e_i \\
&\quad = [[E_{ik}, E_{kj}], E_{i^{-1}j}] \otimes e_i \\
&\quad = ([E_{ik}, [E_{kj}{E_{ik}}^{-1}{E_{kj}}^{-1}, E_{i^{-1}j}]][E_{kj}{E_{ik}}^{-1}{E_{kj}}^{-1}, E_{i^{-1}j}][E_{ik}, E_{i^{-1}j}]) \otimes e_i \\
&\quad \equiv \cdots\cdots \\
&\quad \equiv -[{E_{kj}}^{-1}, E_{i^{-1}j}] \otimes e_k \equiv [E_{kj}, E_{i^{-1}j}] \otimes e_k \equiv -r_{ij}(l) \otimes e_j
\end{aligned}$$

and hence $[E_{ij}, E_{i^{-1}j}] \otimes e_i \equiv -r_{ij}(k) \otimes e_j - r_{i^{-1}j}(l) \otimes e_j$.

• **The proof of (vi): (R2-5)**
First, let $p \neq i$ and $l \in \{1, \ldots, n\}$ such that $l \neq i, j, k, p$. Set $r = {E_{ij}}^{-1}[E_{il}, E_{lj}]$. Then we have

$$[E_{ij}, [r, E_{i^{-1}k}]] \otimes e_p \equiv 0, \quad r, E_{i^{-1}k} \otimes e_p \equiv 0$$

and

$$\begin{aligned}
&[E_{ij},E_{i^{-1}k}] \otimes e_p \\
&\quad \equiv ([E_{ij},[r,E_{i^{-1}k}]][r,E_{i^{-1}k}][E_{ij},E_{i^{-1}k}]) \otimes e_p \\
&\quad = [[E_{il},E_{lj}],E_{i^{-1}k}] \otimes e_p \\
&\quad = [E_{il}E_{lj}{E_{il}}^{-1}{E_{lj}}^{-1},E_{i^{-1}k}] \otimes e_p \\
&\quad = ([E_{il},[E_{lj}{E_{il}}^{-1}{E_{lj}}^{-1},E_{i^{-1}k}]][E_{lj}{E_{il}}^{-1}{E_{lj}}^{-1},E_{i^{-1}k}][E_{il},E_{i^{-1}k}]) \otimes e_p \\
&\quad \equiv ([E_{lj}{E_{il}}^{-1}{E_{lj}}^{-1},E_{i^{-1}k}][E_{il},E_{i^{-1}k}]) \otimes e_p \\
&\quad \equiv \cdots\cdots \\
&\quad \equiv ([{E_{lj}}^{-1},E_{i^{-1}k}][{E_{il}}^{-1},E_{i^{-1}k}][E_{lj},E_{i^{-1}k}][E_{il},E_{i^{-1}k}]) \otimes e_p
\end{aligned}$$

Since $p \neq i,l$, we have

$$\begin{aligned}
[E_{lj},E_{i^{-1}k}] \otimes e_p + [{E_{lj}}^{-1},E_{i^{-1}k}] \otimes e_p \equiv 0, \\
[E_{il},E_{i^{-1}k}] \otimes e_p + [{E_{il}}^{-1},E_{i^{-1}k}] \otimes e_p \equiv 0,
\end{aligned}$$

and hence $[E_{ij},E_{i^{-1}k}] \otimes e_p \equiv 0$.

Next, let $p = i$ and $l \neq i,j,k$. Since $[r_{ij}(l),[E_{ij},E_{i^{-1}k}]] = 0$ in $\bar{R}^{\mathrm{ab}}$ and $[r_{ij}(l),E_{i^{-1}k}] \otimes e_i \equiv r_{ij}(k) \otimes e_k$ by Corollary 3.1, we have

$$\begin{aligned}
&[E_{ij},E_{i^{-1}k}] \otimes e_i + r_{ij}(l) \otimes e_k \\
&\quad \equiv ([r_{ij}(l),[E_{ij},E_{i^{-1}k}]][E_{ij},E_{i^{-1}k}][r_{ij}(l),E_{i^{-1}k}]) \otimes e_i \\
&\quad \equiv [[E_{il},E_{lj}],E_{i^{-1}k}] \otimes e_i \\
&\quad = [E_{il}E_{lj}{E_{il}}^{-1}{E_{lj}}^{-1},E_{i^{-1}k}] \otimes e_i \\
&\quad = [E_{lj}{E_{il}}^{-1}{E_{lj}}^{-1},E_{i^{-1}k}] \otimes e_l + [E_{lj}{E_{il}}^{-1}{E_{lj}}^{-1},E_{i^{-1}k}] \otimes e_i + [E_{il},E_{i^{-1}k}] \otimes e_i \\
&\quad \equiv \cdots\cdots \\
&\quad \equiv [E_{lj},E_{i^{-1}k}] \otimes e_l \equiv -r_{i^{-1}k}(m) \otimes e_j
\end{aligned}$$

and hence $[E_{i^{-1}k},E_{ij}] \otimes e_i \equiv r_{ij}(l) \otimes e_k + r_{i^{-1}k}(m) \otimes e_j$ where m is a number such that $m \neq i,j,k,l$.

5.2. **The computation of $(a,b,c,d,e) \otimes e_p$.**

Here we give the details of the computation of the equation $(a,b,c,d,e) \otimes e_p$.

- $(x_l^{-1},x_j,x_i,x_j,x_k) \otimes e_k$

For distinct i,j,k and l, the equation $(x_l^{-1},x_j,x_i,x_j,x_k) \otimes e_k$ is given by

$$\begin{aligned}
r_{il^{-1}}(k) \otimes e_k = s_1^{-1} \otimes e_k &+ ({w_{l^{-1}j}}^{-1}E_{ik}w_{l^{-1}j}\ s_2^{-1}\ {w_{l^{-1}j}}^{-1}{E_{ik}}^{-1}w_{l^{-1}j}) \otimes e_k \\
&+ ({w_{l^{-1}j}}^{-1}r_{ij}(k)w_{l^{-1}j}) \otimes e_k \\
&+ ({w_{l^{-1}j}}^{-1}E_{ij}E_{kj}w_{l^{-1}j}\ s_1\ {w_{l^{-1}j}}^{-1}{E_{kj}}^{-1}{E_{ij}}^{-1}w_{l^{-1}j}) \otimes e_k \\
&+ ({w_{l^{-1}j}}^{-1}E_{ij}w_{l^{-1}j}\ s_2\ {w_{l^{-1}j}}^{-1}{E_{ij}}^{-1}w_{l^{-1}j}) \otimes e_k + s_3 \otimes e_k
\end{aligned}$$

where

$$\begin{aligned}
&s_1 := ({w_{l^{-1}j}}^{-1}E_{ik^{-1}}w_{l^{-1}j})^{-1}{E_{ik}}^{-1}, \quad s_2 := ({w_{l^{-1}j}}^{-1}E_{kj^{-1}}w_{l^{-1}j})^{-1}{E_{kl^{-1}}}^{-1}, \\
&s_3 := ({w_{l^{-1}j}}^{-1}E_{ij^{-1}}w_{l^{-1}j})^{-1}E_{il}.
\end{aligned}$$

Then using Lemmas 3.1 and 3.2, we obtain

$$r_{il^{-1}}(k)\otimes e_k \equiv r_{ij}(k)\otimes e_k - s_1\otimes e_l + s_3\otimes e_k. \tag{28}$$

From Lemma 2.4, we have $r_{il^{-1}}(k)\otimes e_k \equiv -r_{il}(k)\otimes e_k$. Using Lemma 2.3, we see

$$s_1 = E_{j^{-1}l}E_{lj^{-1}}\,[E_{jl},E_{ik}]\,E_{lj}E_{j^{-1}l}\cdot E_{j^{-1}l^{-1}}\,[E_{lj^{-1}},E_{ik}]\,E_{j^{-1}l}\cdot[E_{j^{-1}l^{-1}},E_{ik}],$$

and hence

$$s_1\otimes e_l \equiv -r_{ik}(\cdot)\otimes e_l - r_{ik}(\cdot)\otimes e_j.$$

On the other hand, applying (vi) of Lemma 2.2 to s_3, we have

$$\begin{aligned} s_3 = {} & E_{j^{-1}l^{-1}}E_{lj^{-1}}E_{il^{-1}}\,r_{il}(j)^{-1}\,E_{il}E_{lj}E_{j^{-1}l}\cdot E_{j^{-1}l^{-1}}E_{il^{-1}}\,r_{ij^{-1}}(l)\,E_{il}E_{j^{-1}l}\\ & \cdot E_{j^{-1}l^{-1}}E_{il^{-1}}\,[E_{ij^{-1}},E_{li^{-1}}]\,E_{il}E_{j^{-1}l}\cdot[E_{j^{-1}l^{-1}},E_{il^{-1}}], \end{aligned}$$

and hence

$$s_3\otimes e_k \equiv -r_{il}(\cdot)\otimes e_k - r_{ij}(\cdot)\otimes e_k.$$

Substituting these results into (28), we obtain $S_{ijk}\equiv -S_{ilk}$.

- $(x_k^{-1},x_l,x_i,x_j,x_l)\otimes e_k$

  For distinct i,j,k and l, the equation $(x_k^{-1},x_l,x_i,x_j,x_l)\otimes e_k$ is given by

$$\begin{aligned} r_{ij}(k^{-1})\otimes e_k = {} & s_1^{-1}\otimes e_k\\ & + (w_{k^{-1}l}{}^{-1}E_{il}w_{k^{-1}l}\ s_2^{-1}\ w_{k^{-1}l}{}^{-1}E_{il}{}^{-1}w_{k^{-1}l})\otimes e_k\\ & + (w_{k^{-1}l}{}^{-1}r_{ij}(l)w_{k^{-1}l})\otimes e_k\\ & + (w_{k^{-1}l}{}^{-1}E_{ij}E_{lj}w_{k^{-1}l}\ s_1\ w_{k^{-1}l}{}^{-1}E_{lj}{}^{-1}E_{ij}{}^{-1}w_{k^{-1}l})\otimes e_k\\ & + (w_{k^{-1}l}{}^{-1}E_{ij}w_{k^{-1}l}\ s_2\ w_{k^{-1}l}{}^{-1}E_{ij}{}^{-1}w_{k^{-1}l})\otimes e_k + s_3\otimes e_k \end{aligned}$$

where

$$\begin{aligned} & s_1 := (w_{k^{-1}l}{}^{-1}E_{il^{-1}}w_{k^{-1}l})^{-1}E_{ik},\quad s_2 := (w_{k^{-1}l}{}^{-1}E_{lj^{-1}}w_{k^{-1}l})^{-1}E_{k^{-1}j}{}^{-1},\\ & s_3 := (w_{k^{-1}l}{}^{-1}E_{ij^{-1}}w_{k^{-1}l})^{-1}E_{ij}{}^{-1}. \end{aligned}$$

Then using Lemmas 3.1 and 3.2, we obtain

$$r_{ij}(k^{-1})\otimes e_k \equiv -r_{ij}(l)\otimes e_l - s_1\otimes e_j + s_3\otimes e_k. \tag{29}$$

Applying (vi) of Lemma 2.2 to s_1, we have

$$\begin{aligned} s_1 = {} & E_{l^{-1}k^{-1}}E_{kl^{-1}}E_{ik^{-1}}\,r_{ik}(l)^{-1}\,E_{ik}E_{kl}E_{l^{-1}k}\\ & \cdot E_{l^{-1}k^{-1}}E_{ik^{-1}}\,r_{il^{-1}}(k)\,E_{ik}E_{l^{-1}k}\\ & \cdot E_{l^{-1}k^{-1}}E_{ik^{-1}}\,[E_{il^{-1}},E_{kl^{-1}}]\,E_{ik}E_{l^{-1}k}\\ & \cdot [E_{l^{-1}k^{-1}},E_{ik^{-1}}] \end{aligned}$$

and hence

$$s_1\otimes e_j \equiv -r_{il}(\cdot)\otimes e_j - r_{ik}(\cdot)\otimes e_j.$$

On the other hand, using Lemma 2.3, we see

$$s_3 = E_{l^{-1}k^{-1}}E_{kl^{-1}}\,[E_{lk},E_{ij}]\,E_{kl}E_{l^{-1}k}\cdot E_{l^{-1}k^{-1}}\,[E_{kl^{-1}},E_{ij}]\,E_{l^{-1}k}\cdot[E_{l^{-1}k^{-1}},E_{ij}]$$

and hence

$$s_3\otimes e_k \equiv r_{ij}(\cdot)\otimes e_k + r_{ij}(\cdot)\otimes e_l.$$

Substituting these results into (29), we obtain

$$r_{ij}(k^{-1}) \otimes e_k \equiv r_{ij}(\cdot) \otimes e_k + r_{ik}(\cdot) \otimes e_j.$$

- $(x_i^{-1}, x_l, x_l, x_j, x_k) \otimes e_k$

For distinct i, j, k and l, the equation $(x_i^{-1}, x_l, x_l, x_j, x_k^{-1}) \otimes e_k$ is given by

$$\begin{aligned} r_{i^{-1}j}(k) \otimes e_k = s_1^{-1} \otimes e_k &+ (w_{i^{-1}l}{}^{-1} E_{lk} w_{i^{-1}l}\ s_2^{-1}\ w_{i^{-1}l}{}^{-1} E_{lk}{}^{-1} w_{i^{-1}l}) \otimes e_k \\ &+ (w_{i^{-1}l}{}^{-1} r_{lj}(k) w_{i^{-1}l}) \otimes e_k \\ &+ (w_{i^{-1}l}{}^{-1} E_{lj} E_{kj} w_{i^{-1}l}\ s_1\ w_{i^{-1}l}{}^{-1} E_{kj}{}^{-1} E_{lj}{}^{-1} w_{i^{-1}l}) \otimes e_k \\ &+ (w_{i^{-1}l}{}^{-1} E_{lj} w_{i^{-1}l}\ s_2\ w_{i^{-1}l}{}^{-1} E_{lj}{}^{-1} w_{i^{-1}l}) \otimes e_k + s_3 \otimes e_k \end{aligned}$$

where

$$\begin{aligned} &s_1 := (w_{i^{-1}l}{}^{-1} E_{lk^{-1}} w_{i^{-1}l})^{-1} E_{i^{-1}k}{}^{-1}, \quad s_2 := (w_{i^{-1}l}{}^{-1} E_{kj^{-1}} w_{i^{-1}l})^{-1} E_{kj}{}^{-1}, \\ &s_3 := (w_{i^{-1}l}{}^{-1} E_{lj^{-1}} w_{i^{-1}l})^{-1} E_{i^{-1}j}{}^{-1}. \end{aligned}$$

Then using Lemmas 3.1 and 3.2, we obtain

$$r_{i^{-1}j}(k) \otimes e_k \equiv r_{lj}(k) \otimes e_k + s_1 \otimes e_j + s_3 \otimes e_k. \tag{30}$$

Applying (3) to s_1, we have

$$s_1 = (w_{i^{-1}l}{}^{-1} E_{lk} h_{i^{-1}l} E_{lk^{-1}} w_{i^{-1}l}) \cdot (w_{i^{-1}l}{}^{-1} h_{i^{-1}l}{}^{-1} w_{i^{-1}l}) \cdot (w_{il^{-1}}{}^{-1} E_{lk^{-1}} w_{il^{-1}})^{-1} E_{i^{-1}k^{-1}},$$

and hence

$$s_1 \otimes e_j \equiv \{(w_{il^{-1}}{}^{-1} E_{lk^{-1}} w_{il^{-1}})^{-1} E_{i^{-1}k^{-1}}\} \otimes e_j.$$

Then considering (ii) of Lemma 2.2, we have

$$\begin{aligned} (w_{il^{-1}}{}^{-1} E_{lk^{-1}} w_{il^{-1}})^{-1} E_{i^{-1}k^{-1}} = {} & E_{li} E_{i^{-1}l} \left[E_{l^{-1}i^{-1}}, E_{lk}\right] E_{i^{-1}l^{-1}} E_{li^{-1}} \\ & \cdot E_{li}\, r_{i^{-1}k}(l)\, E_{li^{-1}} \\ & \cdot E_{i^{-1}k} E_{li}\, r_{lk^{-1}}(i^{-1})\, E_{li^{-1}} E_{i^{-1}k^{-1}}, \end{aligned}$$

and hence

$$s_1 \otimes e_j \equiv r_{i^{-1}k}(\cdot) \otimes e_j - r_{lk}(\cdot) \otimes e_j.$$

By the same argument, we have

$$s_3 \otimes e_k \equiv r_{i^{-1}j}(\cdot) \otimes e_k - r_{lj}(\cdot) \otimes e_k.$$

Substituting these results into (30), we obtain

$$r_{i^{-1}j}(k) \otimes e_k \equiv r_{i^{-1}j}(\cdot) \otimes e_k + r_{i^{-1}k}(\cdot) \otimes e_j.$$

- $(x_i^{-1}, x_l, x_l, x_j, x_k^{-1}) \otimes e_k$

For distinct i, j, k and l, the equation $(x_i^{-1}, x_l, x_l, x_j, x_k^{-1}) \otimes e_k$ is given by

$$\begin{aligned} r_{i^{-1}j}(k^{-1}) \otimes e_k = s_1^{-1} \otimes e_k &+ (w_{i^{-1}l}{}^{-1} E_{lk^{-1}} w_{i^{-1}l}\ s_2^{-1}\ w_{i^{-1}l}{}^{-1} E_{lk} w_{i^{-1}l}) \otimes e_k \\ &+ (w_{i^{-1}l}{}^{-1} r_{lj}(k^{-1}) w_{i^{-1}l}) \otimes e_k \\ &+ (w_{i^{-1}l}{}^{-1} E_{lj} E_{k^{-1}j} w_{i^{-1}l}\ s_1\ w_{i^{-1}l}{}^{-1} E_{k^{-1}j}{}^{-1} E_{lj}{}^{-1} w_{i^{-1}l}) \otimes e_k \\ &+ (w_{i^{-1}l}{}^{-1} E_{lj} w_{i^{-1}l}\ s_2\ w_{i^{-1}l}{}^{-1} E_{lj}{}^{-1} w_{i^{-1}l}) \otimes e_k + s_3 \otimes e_k \end{aligned}$$

where

$$s_1 := (w_{i^{-1}l}{}^{-1} E_{lk} w_{i^{-1}l})^{-1} E_{i^{-1}k}, \quad s_2 := (w_{i^{-1}l}{}^{-1} E_{k^{-1}j^{-1}} w_{i^{-1}l})^{-1} E_{k^{-1}j}{}^{-1},$$
$$s_3 := (w_{i^{-1}l}{}^{-1} E_{lj^{-1}} w_{i^{-1}l})^{-1} E_{i^{-1}j^{\sigma}}{}^{-1}.$$

Then using Lemmas 3.1 and 3.2, we obtain

$$r_{i^{-1}j}(k^{-1}) \otimes e_k \equiv r_{lj}(k^{-1}) \otimes e_k - s_1 \otimes e_j + s_3 \otimes e_k. \tag{31}$$

Applying (3) to s_1, we have

$$s_1 = (w_{i^{-1}l}{}^{-1} E_{lk^{-1}} h_{i^{-1}l} E_{lk} w_{i^{-1}l}) \cdot (w_{i^{-1}l}{}^{-1} h_{i^{-1}l}{}^{-1} w_{i^{-1}l}) \cdot (w_{il^{-1}}{}^{-1} E_{lk} w_{il^{-1}})^{-1} E_{i^{-1}k},$$

and hence

$$s_1 \otimes e_j \equiv \{(w_{il^{-1}}{}^{-1} E_{lk} w_{il^{-1}})^{-1} E_{i^{-1}k}\} \otimes e_j.$$

Then considering (ii) of Lemma 2.2, we have

$$\begin{aligned}(w_{il^{-1}}{}^{-1} E_{lk} w_{il^{-1}})^{-1} E_{i^{-1}k} = {} & E_{li} E_{i^{-1}l} \, [E_{l^{-1}i^{-1}}, E_{lk^{-1}}] \, E_{i^{-1}l^{-1}} E_{li^{-1}} \\ & \cdot E_{li} \, r_{i^{-1}k^{-1}}(l) \, E_{li^{-1}} \\ & \cdot E_{i^{-1}k^{-1}} E_{li} \, r_{lk}(i^{-1}) \, E_{li^{-1}} E_{i^{-1}k},\end{aligned}$$

and hence

$$s_1 \otimes e_j \equiv -r_{i^{-1}k}(\cdot) \otimes e_j + r_{lk}(\cdot) \otimes e_j.$$

By the same argument, we have

$$s_3 \otimes e_j \equiv r_{i^{-1}j}(\cdot) \otimes e_k - r_{lj}(\cdot) \otimes e_k.$$

Substituting these results into (31), we obtain

$$r_{i^{-1}j}(k^{-1}) \otimes e_k \equiv r_{i^{-1}j}(\cdot) \otimes e_k + r_{i^{-1}k}(\cdot) \otimes e_j.$$

- $(x_l, x_k, x_i, x_j, x_k) \otimes e_i$

  For distinct i, j, k and l, the equation $(x_l, x_k, x_i, x_j, x_k) \otimes e_i$ is given by

$$\begin{aligned}r_{ij}(l) \otimes e_i = {} & s_1^{-1} \otimes e_i + (w_{lk}{}^{-1} E_{ik} w_{lk} \; s_2^{-1} \; w_{lk}{}^{-1} E_{ik}{}^{-1} w_{lk}) \otimes e_i \\ & + (w_{lk}{}^{-1} r_{ij}(k) w_{lk}) \otimes e_i \\ & + (w_{lk}{}^{-1} E_{ij} E_{kj} w_{lk} \; s_1 \; w_{lk}{}^{-1} E_{kj}^{-1} E_{ij}{}^{-1} w_{lk}) \otimes e_i \\ & + (w_{lk}{}^{-1} E_{ij} w_{lk} \; s_2 \; w_{lk}{}^{-1} E_{ij}{}^{-1} w_{lk}) \otimes e_i + s_3 \otimes e_i\end{aligned}$$

where

$$s_1 := (w_{lk}{}^{-1} E_{ik^{-1}} w_{lk})^{-1} E_{il}{}^{-1}, \quad s_2 := (w_{lk}{}^{-1} E_{kj^{-1}} w_{lk})^{-1} E_{lj}{}^{-1},$$
$$s_3 := (w_{lk}{}^{-1} E_{ij^{-1}} w_{lk})^{-1} E_{ij}{}^{-1}.$$

Then using Lemmas 3.1 and 3.2, we obtain

$$r_{ij}(l) \otimes e_i \equiv r_{ij}(k) \otimes e_i + s_1 \otimes e_j + s_2 \otimes e_j - s_2 \otimes e_l + s_3 \otimes e_i. \tag{32}$$

Applying (vi) of Lemma 2.2 to s_1, we have

$$\begin{aligned}s_1 = {} & E_{k^{-1}l} E_{l^{-1}k^{-1}} E_{il} \, r_{il^{-1}}(k)^{-1} \, E_{il^{-1}} E_{l^{-1}k} E_{k^{-1}l^{-1}} \cdot E_{k^{-1}l} E_{il} \, r_{ik^{-1}}(l^{-1}) \, E_{il^{-1}} E_{k^{-1}l^{-1}} \\ & \cdot E_{k^{-1}l} E_{il} \, [E_{ik^{-1}}, E_{l^{-1}k^{-1}}] \, E_{il^{-1}} E_{k^{-1}l^{-1}} \cdot [E_{k^{-1}l}, E_{il}],\end{aligned}$$

and hence

$$s_1 \otimes e_j \equiv r_{il}(\cdot) \otimes e_j - r_{ik}(\cdot) \otimes e_j \stackrel{\circ}{=} 0.$$

Using (3), we have

$$s_2 = ({w_{lk}}^{-1} E_{kj} h_{lk} E_{kj^{-1}} w_{lk}) \cdot ({w_{lk}}^{-1} {h_{lk}}^{-1} w_{lk}) \cdot ({w_{l^{-1}k^{-1}}}^{-1} E_{kj^{-1}} w_{l^{-1}k^{-1}})^{-1} E_{lj^{-1}},$$

and hence

$$\begin{aligned} s_2 \otimes e_j &\equiv \{({w_{l^{-1}k^{-1}}}^{-1} E_{kj^{-1}} w_{l^{-1}k^{-1}})^{-1} E_{lj^{-1}}\} \otimes e_j, \\ s_2 \otimes e_l &\equiv \{({w_{l^{-1}k^{-1}}}^{-1} E_{kj^{-1}} w_{l^{-1}k^{-1}})^{-1} E_{lj^{-1}}\} \otimes e_l + h_{lk} \otimes e_j, \\ &\equiv \{({w_{l^{-1}k^{-1}}}^{-1} E_{kj^{-1}} w_{l^{-1}k^{-1}})^{-1} E_{lj^{-1}}\} \otimes e_l - r_{lk}(\cdot) \otimes e_j - r_{l^{-1}k}(\cdot) \otimes e_j. \end{aligned}$$

On the other hand, applying (ii) of Lemma 2.2 to $({w_{l^{-1}k^{-1}}}^{-1} E_{kj^{-1}} w_{l^{-1}k^{-1}})^{-1} E_{lj^{-1}}$, we see

$$\begin{aligned} ({w_{l^{-1}k^{-1}}}^{-1} E_{kj^{-1}} w_{l^{-1}k^{-1}})^{-1} E_{lj^{-1}} = {} & E_{kl^{-1}} E_{lk} \, [E_{k^{-1}l}, E_{kj}] \, E_{lk^{-1}} E_{kl} \\ & \cdot E_{kl^{-1}} \, r_{lj}(k) \, E_{kl} \cdot E_{lj} E_{kl^{-1}} \, r_{kj^{-1}}(l) \, E_{kl} E_{lj^{-1}}, \end{aligned}$$

and hence

$$\begin{aligned} s_2 \otimes e_j &\equiv r_{lj}(\cdot) \otimes e_j - r_{kj}(\cdot) \otimes e_j \stackrel{\circ}{=} 0, \\ s_2 \otimes e_l &\equiv r_{kj}(\cdot) \otimes e_l + r_{k^{-1}l}(\cdot) \otimes e_j + r_{lj}(k) \otimes e_l \\ &\qquad - r_{kj}(l) \otimes e_l - r_{kj}(\cdot) \otimes e_j - r_{lk}(\cdot) \otimes e_j - r_{l^{-1}k}(\cdot) \otimes e_j, \\ &\equiv r_{kj}(\cdot) \otimes e_l + r_{k^{-1}l}(\cdot) \otimes e_j + r_{lj}(k) \otimes e_l \\ &\qquad - r_{kj}(\cdot) \otimes e_l - r_{kl}(\cdot) \otimes e_j - r_{kj}(\cdot) \otimes e_j - r_{lk}(\cdot) \otimes e_j - r_{l^{-1}k}(\cdot) \otimes e_j, \\ &\stackrel{\circ}{=} r_{lj}(k) \otimes e_l. \end{aligned}$$

Finally from Lemma 2.3, we see

$$s_3 = E_{k^{-1}l} E_{l^{-1}k^{-1}} \, [E_{kl^{-1}}, E_{ij}] \, E_{l^{-1}k} E_{k^{-1}l^{-1}} \cdot E_{k^{-1}l} \, [E_{l^{-1}k^{-1}}, E_{ij}] \, E_{k^{-1}l^{-1}} \cdot [E_{k^{-1}l}, E_{ij}],$$

and hence

$$s_3 \otimes e_i \equiv -r_{kl}(\cdot) \otimes e_j - r_{l^{-1}k}(\cdot) \otimes e_j + r_{k^{-1}l}(\cdot) \otimes e_j \stackrel{\circ}{=} 0.$$

Substituting these results into (32), we obtain

$$r_{ij}(l) \otimes e_i \stackrel{\circ}{=} r_{ij}(k) \otimes e_i - r_{lj}(k) \otimes e_l.$$

.

- $(x_l^{-1}, x_i, x_i, x_j, x_k) \otimes e_l$

For distinct i, j, k and l, the equation $(x_l^{-1}, x_i, x_i, x_j, x_k) \otimes e_l$ is given by

$$\begin{aligned} r_{l^{-1}j}(k) \otimes e_l = {} & s_1^{-1} \otimes e_l \\ & + ({w_{l^{-1}i}}^{-1} E_{ik} w_{l^{-1}i} \; s_2^{-1} \; {w_{l^{-1}i}}^{-1} {E_{ik}}^{-1} w_{l^{-1}i}) \otimes e_l \\ & + ({w_{l^{-1}i}}^{-1} r_{ij}(k) w_{l^{-1}i}) \otimes e_l \\ & + ({w_{l^{-1}i}}^{-1} E_{ij} E_{kj} w_{l^{-1}i} \; s_1 \; {w_{l^{-1}i}}^{-1} E_{kj}^{-1} {E_{ij}}^{-1} w_{l^{-1}i}) \otimes e_l \\ & + ({w_{l^{-1}i}}^{-1} E_{ij} w_{l^{-1}i} \; s_2 \; {w_{l^{-1}i}}^{-1} {E_{ij}}^{-1} w_{l^{-1}i}) \otimes e_l + s_3 \otimes e_l \end{aligned}$$

where

$$\begin{aligned} s_1 &:= ({w_{l^{-1}i}}^{-1} E_{ik^{-1}} w_{l^{-1}i})^{-1} {E_{l^{-1}k}}^{-1}, \quad s_2 := ({w_{l^{-1}i}}^{-1} E_{kj^{-1}} w_{l^{-1}i})^{-1} {E_{kj}}^{-1}, \\ s_3 &:= ({w_{l^{-1}i}}^{-1} E_{ij^{-1}} w_{l^{-1}i})^{-1} {E_{l^{-1}j}}^{-1}. \end{aligned}$$

Then using Lemmas 3.1 and 3.2, we obtain

$$r_{l^{-1}j}(k)\otimes e_l \equiv -r_{ij}(k)\otimes e_i - s_1\otimes e_j - s_2\otimes e_j + s_2\otimes e_k + s_3\otimes e_l. \tag{33}$$

Using (3), we have

$$s_1 = ({w_{l^{-1}i}}^{-1}E_{ik}h_{l^{-1}i}E_{ik^{-1}}w_{l^{-1}i})\cdot({w_{l^{-1}i}}^{-1}{h_{l^{-1}i}}^{-1}w_{l^{-1}i})\cdot({w_{li^{-1}}}^{-1}E_{ik^{-1}}w_{li^{-1}})^{-1}E_{l^{-1}k^{-1}},$$

and hence

$$s_1\otimes e_j \equiv \{({w_{li^{-1}}}^{-1}E_{ik^{-1}}w_{li^{-1}})^{-1}E_{l^{-1}k^{-1}}\}\otimes e_j.$$

On the other hand, applying (ii) of Lemma 2.2 to $({w_{li^{-1}}}^{-1}E_{ik^{-1}}w_{li^{-1}})^{-1}E_{l^{-1}k^{-1}}$, we have

$$\begin{aligned}({w_{li^{-1}}}^{-1}E_{ik^{-1}}w_{li^{-1}})^{-1}E_{l^{-1}k^{-1}} &= E_{il}E_{l^{-1}i}\,[E_{i^{-1}l^{-1}},E_{ik}]\,E_{l^{-1}i^{-1}}E_{il^{-1}}\\ &\quad\cdot E_{il}\,r_{l^{-1}k}(i)\,E_{il^{-1}}\cdot E_{l^{-1}k}E_{il}\,r_{ik^{-1}}(l^{-1})\,E_{il^{-1}}E_{l^{-1}k^{-1}},\end{aligned}$$

and hence

$$s_1\otimes e_j \equiv r_{l^{-1}k}(\cdot)\otimes e_j - r_{ik}(\cdot)\otimes e_j \mathrel{\mathring{=}} 0.$$

By the same arguent, we have

$$s_3\otimes e_l \equiv \{({w_{li^{-1}}}^{-1}E_{ij^{-1}}w_{li^{-1}})^{-1}E_{l^{-1}j^{-1}}\}\otimes e_l,$$

and from (ii) of Lemma 2.2

$$\begin{aligned}({w_{li^{-1}}}^{-1}E_{ij^{-1}}w_{li^{-1}})^{-1}E_{l^{-1}j^{-1}} &= E_{il}E_{l^{-1}i}\,[E_{i^{-1}l^{-1}},E_{ij}]\,E_{l^{-1}i^{-1}}E_{il^{-1}}\\ &\quad\cdot E_{il}\,r_{l^{-1}j}(i)\,E_{il^{-1}}\cdot E_{l^{-1}j}E_{il}\,r_{ij^{-1}}(l^{-1})\,E_{il^{-1}}E_{l^{-1}j^{-1}}.\end{aligned}$$

Hence we obtain

$$\begin{aligned}s_3\otimes e_l &\equiv r_{ij}(\cdot)\otimes e_l + r_{i^{-1}l}(\cdot)\otimes e_j + r_{l^{-1}j}(i)\otimes e_l + r_{ij}(\cdot)\otimes e_j - r_{ij}(l^{-1})\otimes e_l,\\ &\equiv r_{ij}(\cdot)\otimes e_l + r_{i^{-1}l}(\cdot)\otimes e_j + r_{l^{-1}j}(i)\otimes e_l + r_{ij}(\cdot)\otimes e_j - r_{ij}(\cdot)\otimes e_l - r_{il}(\cdot)\otimes e_j,\\ &\mathrel{\mathring{=}} r_{l^{-1}j}(i)\otimes e_l.\end{aligned}$$

Finally, applying Lemma 2.3 to s_2, we have

$$s_2 = E_{i^{-1}l^{-1}}E_{li^{-1}}\,[E_{il},E_{kj}]\,E_{li}E_{i^{-1}l}\cdot E_{i^{-1}l^{-1}}\,[E_{li^{-1}},E_{kj}]\,E_{i^{-1}l}\cdot[E_{i^{-1}l^{-1}},E_{kj}],$$

and hence

$$s_2\otimes e_j \equiv 0,\quad s_2\otimes e_k \equiv r_{il}(\cdot)\otimes e_j - r_{li}(\cdot)\otimes e_j - r_{i^{-1}l}(\cdot)\otimes e_j \mathrel{\mathring{=}} 0.$$

Substituting these results into (33), we obtain

$$r_{l^{-1}j}(k)\otimes e_l \mathrel{\mathring{=}} -r_{ij}(k)\otimes e_i + r_{l^{-1}j}(i)\otimes e_l.$$

- $(x_k,x_l,x_i,x_j,x_k^{-1})\otimes e_i$

  For distinct i,j,k and l, the equation $(x_k,x_l,x_i,x_j,x_k^{-1})\otimes e_i$ is given by

$$\begin{aligned}r_{ij}(l)\otimes e_i = s_1^{-1}\otimes e_i &+ ({w_{kl}}^{-1}E_{ik^{-1}}w_{kl}\ s_2^{-1}\ {w_{kl}}^{-1}E_{ik}w_{kl})\otimes e_i\\ &+ ({w_{kl}}^{-1}r_{ij}(k^{-1})w_{kl})\otimes e_i\\ &+ ({w_{kl}}^{-1}E_{ij}E_{k^{-1}j}w_{kl}\ s_1\ {w_{kl}}^{-1}{E_{k^{-1}j}}^{-1}{E_{ij}}^{-1}w_{kl})\otimes e_i\\ &+ ({w_{kl}}^{-1}E_{ij}w_{kl}\ s_2\ {w_{kl}}^{-1}{E_{ij}}^{-1}w_{kl})\otimes e_i + s_3\otimes e_i\end{aligned}$$

where

$$\begin{aligned}&s_1 := ({w_{kl}}^{-1}E_{ik}w_{kl})^{-1}{E_{il}}^{-1},\quad s_2 := ({w_{kl}}^{-1}E_{k^{-1}j^{-1}}w_{kl})^{-1}{E_{lj}}^{-1},\\ &s_3 := ({w_{kl}}^{-1}E_{ij^{-1}}w_{kl})^{-1}{E_{ij}}^{-1}.\end{aligned}$$

Then using Lemmas 3.1 and 3.2, we obtain

$$r_{ij}(l)\otimes e_i \equiv r_{ij}(k^{-1})\otimes e_i + s_1\otimes e_j - s_2\otimes e_l + s_2\otimes e_j + s_3\otimes e_i. \tag{34}$$

Applying (v) of Lemma 2.2 to s_1, we see

$$\begin{aligned} s_1 = {} & E_{l^{-1}k}E_{k^{-1}l^{-1}}E_{il^{-1}}\, r_{ik^{-1}}(l)\, E_{il}E_{k^{-1}l}E_{l^{-1}k^{-1}} \\ & \cdot E_{l^{-1}k}E_{k^{-1}l^{-1}}E_{il^{-1}}\, r_{il}(k^{-1})\, E_{il}E_{k^{-1}l}E_{l^{-1}k^{-1}}\cdot[E_{l^{-1}k}, E_{ik^{-1}}], \end{aligned}$$

and hence

$$s_1\otimes e_j \equiv -r_{ik}(\cdot)\otimes e_j + r_{il}(\cdot)\otimes e_j \mathrel{\mathring{=}} 0.$$

Similarly, applying (i) of Lemma 2.2 to s_2, we see

$$\begin{aligned} s_2 = {} & E_{l^{-1}k}E_{k^{-1}l^{-1}}\, r_{lj}(k^{-1})\, E_{k^{-1}l}E_{l^{-1}k^{-1}} \\ & \cdot E_{l^{-1}k}E_{lj}E_{k^{-1}l^{-1}}\, r_{k^{-1}j^{-1}}(l)^{-1}\, E_{k^{-1}l}E_{lj^{-1}}E_{l^{-1}k^{-1}}\cdot[E_{l^{-1}k}, E_{lj}] \end{aligned}$$

and hence

$$\begin{aligned} s_2\otimes e_l &\equiv -r_{lj}(k^{-1})\otimes e_k + r_{k^{-1}j}(\cdot)\otimes e_j - r_{k^{-1}j}(l)\otimes e_k + r_{l^{-1}k}(\cdot)\otimes e_j + r_{lj}(\cdot)\otimes e_k, \\ &\mathrel{\mathring{=}} -r_{k^{-1}j}(l)\otimes e_k, \\ s_2\otimes e_j &\equiv r_{lj}(\cdot)\otimes e_j + r_{k^{-1}j}(\cdot)\otimes e_j \mathrel{\mathring{=}} 0. \end{aligned}$$

Finally, using Lemma 2.3, we have

$$s_3 = E_{l^{-1}k}E_{k^{-1}l^{-1}}\,[E_{lk^{-1}}, E_{ij}]\,E_{k^{-1}l}E_{l^{-1}k^{-1}}\cdot E_{l^{-1}k}\,[E_{k^{-1}l^{-1}}, E_{ij}]\,E_{l^{-1}k^{-1}}\cdot[E_{l^{-1}k}, E_{ij}],$$

and hence

$$s_3\otimes e_i \equiv -r_{lk}(\cdot)\otimes e_j - r_{k^{-1}l}(\cdot)\otimes e_j + r_{l^{-1}k}(\cdot)\otimes e_j \mathrel{\mathring{=}} 0.$$

Substituting these results into (34), we obtain

$$r_{ij}(l)\otimes e_i \mathrel{\mathring{=}} r_{ij}(k^{-1})\otimes e_i + r_{k^{-1}j}(l)\otimes e_k.$$

- $(x_l^{-1}, x_k, x_i^{-1}, x_j, x_k^{-1})\otimes e_i$

  For distinct i, j, k and l, the equation $(x_l^{-1}, x_k, x_i^{-1}, x_j, x_k^{-1})\otimes e_i$ is given by

$$\begin{aligned} r_{i^{-1}j}(l)\otimes e_i = {} & s_1^{-1}\otimes e_i + (w_{l^{-1}k}{}^{-1}E_{i^{-1}k^{-1}}w_{l^{-1}k}\ s_2^{-1}\ w_{l^{-1}k}{}^{-1}E_{i^{-1}k}w_{l^{-1}k^{-1}})\otimes e_i \\ & + (w_{l^{-1}k}{}^{-1}r_{i^{-1}j}(k^{-1})w_{l^{-1}k})\otimes e_i \\ & + (w_{l^{-1}k}{}^{-1}E_{i^{-1}j}E_{k^{-1}j}w_{l^{-1}k}\ s_1\ w_{l^{-1}k}{}^{-1}E_{k^{-1}j}{}^{-1}E_{i^{-1}j}{}^{-1}w_{l^{-1}k})\otimes e_i \\ & + (w_{l^{-1}k}{}^{-1}E_{i^{-1}j}w_{l^{-1}k}\ s_2\ w_{l^{-1}k}{}^{-1}E_{i^{-1}j}{}^{-1}w_{l^{-1}k})\otimes e_i + s_3\otimes e_i \end{aligned}$$

where

$$\begin{aligned} & s_1 := (w_{l^{-1}k}{}^{-1}E_{i^{-1}k}w_{l^{-1}k})^{-1}E_{i^{-1}l}{}^{-1}, \quad s_2 := (w_{l^{-1}k}{}^{-1}E_{k^{-1}j^{-1}}w_{l^{-1}k})^{-1}E_{lj}{}^{-1}, \\ & s_3 := (w_{l^{-1}k}{}^{-1}E_{i^{-1}j^{-1}}w_{l^{-1}k})^{-1}E_{i^{-1}j}{}^{-1}. \end{aligned}$$

Then using Lemmas 3.1 and 3.2, we obtain

$$r_{i^{-1}j}(l)\otimes e_i \equiv r_{i^{-1}j}(k^{-1})\otimes e_i - s_1\otimes e_j - s_2\otimes e_j + s_2\otimes e_l + s_3\otimes e_i. \tag{35}$$

Applying (v) of Lemma 2.3 to s_1, we see

$$\begin{aligned} s_1 = {} & E_{k^{-1}l^{-1}}E_{lk^{-1}}E_{i^{-1}k^{-1}}\, r_{i^{-1}l}(k)\, E_{i^{-1}k}E_{lk}E_{k^{-1}l} \\ & \cdot E_{k^{-1}l^{-1}}E_{lk^{-1}}E_{i^{-1}k^{-1}}\, r_{i^{-1}k}(l)\, E_{i^{-1}k}E_{lk}E_{k^{-1}l}\cdot[E_{k^{-1}l^{-1}}, E_{i^{-1}l}] \end{aligned}$$

and hence

$$s_1 \otimes e_j \equiv r_{i^{-1}l}(\cdot) \otimes e_j + r_{i^{-1}k}(\cdot) \otimes e_j \stackrel{\circ}{=} 0.$$

Similarly, applying (ii) of Lemma 2.3 to s_2, we see

$$\begin{aligned} s_2 = {} & E_{k^{-1}l^{-1}} E_{lk^{-1}} \left[E_{kl}, E_{k^{-1}j}\right] E_{lk} E_{k^{-1}l} \\ & \cdot E_{k^{-1}l^{-1}}\, r_{lj}(k^{-1})\, E_{k^{-1}l} \cdot E_{lj} E_{k^{-1}l^{-1}}\, r_{k^{-1}j^{-1}}(l)\, E_{k^{-1}l} E_{lj^{-1}}, \end{aligned}$$

and hence

$$\begin{aligned} s_2 \otimes e_j &\equiv r_{lj}(\cdot) \otimes e_j - r_{k^{-1}j}(\cdot) \otimes e_j \stackrel{\circ}{=} 0, \\ s_2 \otimes e_l &\equiv r_{k^{-1}j}(\cdot) \otimes e_l + r_{kl}(\cdot) \otimes e_j + r_{lj}(k^{-1}) \otimes e_l - r_{k^{-1}j}(l) \otimes e_l - r_{k^{-1}j}(\cdot) \otimes e_j, \\ &\stackrel{\circ}{=} r_{lj}(k^{-1}) \otimes e_l. \end{aligned}$$

Finally, using Lemma 2.3, we have

$$\begin{aligned} s_3 = {} & E_{k^{-1}l^{-1}} E_{lk^{-1}} \left[E_{kl}, E_{i^{-1}j}\right] E_{lk} E_{k^{-1}l} \\ & \cdot E_{k^{-1}l^{-1}} \left[E_{lk^{-1}}, E_{i^{-1}j}\right] E_{k^{-1}l} \cdot \left[E_{k^{-1}l^{-1}}, E_{i^{-1}j}\right], \end{aligned}$$

and hence

$$s_3 \otimes e_i \equiv -r_{kl}(\cdot) \otimes e_j + r_{lk}(\cdot) \otimes e_j + r_{k^{-1}l}(\cdot) \otimes e_j \stackrel{\circ}{=} 0.$$

Substituting these results into (35), we obtain

$$r_{i^{-1}j}(l) \otimes e_i \stackrel{\circ}{=} r_{i^{-1}j}(k^{-1}) \otimes e_i + r_{lj}(k^{-1}) \otimes e_l.$$

5.3. **The reduction of $h_{ij} \otimes e_j$.**

Here we prove

$$\begin{aligned} h_{ij} \otimes e_j \equiv {} & r_{j^{-1}i}(\cdot) \otimes e_i + r_{ji}(\cdot) \otimes e_i + r_{kj}(i^{-1}) \otimes e_k + r_{kj}(i) \otimes e_k \\ & - r_{ki}(j) \otimes e_k - r_{ki}(j^{-1}) \otimes e_k - r_{ki}(\cdot) \otimes e_i + r_{kj}(\cdot) \otimes e_j \\ & + r_{kj}(i^{-1}) \otimes e_j - r_{ki}(j^{-1}) \otimes e_j. \end{aligned}$$

Applying (iv) of Lemma 2.2 to $\{({w_{ij^{-1}}}^{-1} E_{ki^{-1}} w_{ij^{-1}})^{-1} E_{kj^{-1}}\} \otimes e_k$, we see

$$\begin{aligned} ({w_{ij^{-1}}}^{-1} E_{ki^{-1}} w_{ij^{-1}})^{-1} E_{kj^{-1}} = {} & E_{ji} E_{i^{-1}j} \left[E_{j^{-1}i^{-1}}, E_{ki}\right] E_{i^{-1}j^{-1}} E_{ji^{-1}} \\ & \cdot E_{ji} E_{ki}\, r_{kj}(i^{-1})^{-1}\, E_{ki^{-1}} E_{ji^{-1}} \\ & \cdot E_{ji} E_{ki}\, r_{ki^{-1}}(j)^{-1}\, E_{ki^{-1}} E_{ji^{-1}}, \end{aligned}$$

and hence

(36)

$$\begin{aligned} \{({w_{ij^{-1}}}^{-1} E_{ki^{-1}} w_{ij^{-1}})^{-1} E_{kj^{-1}}\} \otimes e_k \equiv {} & -r_{j^{-1}i}(\cdot) \otimes e_i - r_{kj}(i^{-1}) \otimes e_k - r_{kj}(i^{-1}) \otimes e_i \\ & + r_{ki}(j) \otimes e_k + r_{ki}(\cdot) \otimes e_i. \end{aligned}$$

On the other hand, using (3), we have

$$\begin{aligned} ({w_{ij^{-1}}}^{-1} E_{ki^{-1}} w_{ij^{-1}})^{-1} E_{kj^{-1}} = {} & ({w_{ij^{-1}}}^{-1} E_{ki} h_{ij^{-1}} E_{ki^{-1}} w_{ij^{-1}}) \cdot ({w_{ij^{-1}}}^{-1} {h_{ij^{-1}}}^{-1} w_{ij^{-1}}) \\ & \cdot ({w_{i^{-1}j}}^{-1} E_{ki^{-1}} w_{i^{-1}j})^{-1} E_{kj^{-1}}, \end{aligned}$$

and hence, tensoring each term of the equation above with e_k,

$$h_{ij^{-1}} \otimes e_i \equiv \{({w_{ij^{-1}}}^{-1} E_{ki^{-1}} w_{ij^{-1}})^{-1} E_{kj^{-1}}\} \otimes e_k - \{({w_{i^{-1}j}}^{-1} E_{ki^{-1}} w_{i^{-1}j})^{-1} E_{kj^{-1}}\} \otimes e_k.$$

Since $h_{ij^{-1}} = w_{ij^{-1}} {h_{ij}}^{-1} w_{ij}$ from Lemma 2.4, we see

$$h_{ij^{-1}} \otimes e_i \equiv -h_{ij} \otimes e_j$$

and

$$h_{ij} \otimes e_j \equiv -\{({w_{ij^{-1}}}^{-1} E_{ki^{-1}} w_{ij^{-1}})^{-1} E_{kj^{-1}}\} \otimes e_k + \{({w_{i^{-1}j}}^{-1} E_{ki^{-1}} w_{i^{-1}j})^{-1} E_{kj^{-1}}\} \otimes e_k. \tag{37}$$

Applying (iii) of Lemma 2.2 to $({w_{i^{-1}j}}^{-1} E_{ki^{-1}} w_{i^{-1}j})^{-1} E_{kj^{-1}}$, we see

$$\begin{aligned}({w_{i^{-1}j}}^{-1} E_{ki^{-1}} w_{i^{-1}j})^{-1} E_{kj^{-1}} = {} & E_{j^{-1}i^{-1}} E_{ij^{-1}} \, [E_{ji}, E_{ki}] \, E_{ij} E_{j^{-1}i} \\ & \cdot E_{j^{-1}i^{-1}} E_{kj} \, r_{kj^{-1}}(i)^{-1} \, E_{kj^{-1}} E_{j^{-1}i} \\ & \cdot E_{j^{-1}i^{-1}} E_{kj} \, r_{ki}(j^{-1})^{-1} \, E_{kj^{-1}} E_{j^{-1}i}\end{aligned}$$

and

$$\begin{aligned}\{({w_{i^{-1}j}}^{-1} E_{ki^{-1}} w_{i^{-1}j})^{-1} E_{kj^{-1}}\} \otimes e_k \equiv {} & r_{ji}(\cdot) \otimes e_i + r_{kj}(i) \otimes e_k + r_{kj}(\cdot) \otimes e_j \\ & - r_{ki}(j^{-1}) \otimes e_k - r_{ki}(j^{-1}) \otimes e_j.\end{aligned} \tag{38}$$

Substituiting (36) and (38) into (37), we obtain the required result.

6. Appendix II

In this section, we show the details of the computation which are required in the proof of Proposition 3.2.

6.1. **The proof of Lemma 3.6.**

Here we prove Lemma 3.6.

• The proof of (i): (R2-6)

First, let $p \neq j, l$ and $m \in \{1, \dots, n\}$ such that $m \neq i, j, k, l, p$. Set $r = {E_{ij}}^{-1}[E_{im}, E_{mj}]$. Then we have

$$[E_{ij}, [r, E_{kl}]] \otimes e_p^* \equiv 0, \quad [r, E_{kl}] \otimes e_p^* \equiv 0$$

and

$$\begin{aligned}
&[E_{ij},E_{kl}] \otimes e_p^* \\
&\quad \equiv ([E_{ij}, [r, E_{kl}]][r, E_{kl}][E_{ij}, E_{kl}]) \otimes e_p^* \\
&\quad = [[E_{im}, E_{mj}], E_{kl}] \otimes e_p^* \\
&\quad = [E_{im}E_{mj}{E_{im}}^{-1}{E_{mj}}^{-1}, E_{kl}] \otimes e_p^* \\
&\quad = ([E_{im}, [E_{mj}{E_{im}}^{-1}{E_{mj}}^{-1}, E_{kl}]][E_{mj}{E_{im}}^{-1}{E_{mj}}^{-1}, E_{kl}][E_{im}, E_{kl}]) \otimes e_p^* \\
&\quad \equiv ([E_{mj}{E_{im}}^{-1}{E_{mj}}^{-1}, E_{kl}][E_{im}, E_{kl}]) \otimes e_p^* \\
&\quad \equiv \cdots\cdots \\
&\quad \equiv ([{E_{mj}}^{-1}, E_{kl}][{E_{im}}^{-1}, E_{kl}][E_{mj}, E_{kl}][E_{im}, E_{kl}]) \otimes e_p^*
\end{aligned}$$

Since $p \neq j, l$, we have

$$\begin{aligned}
[E_{mj}, E_{kl}] \otimes e_p^* + [{E_{mj}}^{-1}, E_{kl}] \otimes e_p^* \equiv 0, \\
[E_{im}, E_{kl}] \otimes e_p^* + [{E_{im}}^{-1}, E_{kl}] \otimes e_p^* \equiv 0,
\end{aligned}$$

and hence $[E_{ij}, E_{kl}] \otimes e_p^* \equiv 0$.

Next, let $p = j$ and $m \neq i, j, k, l$. Since $[[E_{kl}, E_{ij}], r_{kl}(m)] = 0$ in $\bar{R}^{\mathrm{ab}}$ and $[E_{ij}, r_{kl}(m)] \otimes e_j^* \equiv -r_{kl}(m) \otimes e_i^*$ by Corollary 3.3, we have

$$\begin{aligned}
&[E_{ij},E_{kl}] \otimes e_j^* - r_{kl}(m) \otimes e_i^* \\
&\quad \equiv ([E_{ij}, r_{kl}(m)][E_{ij}, E_{kl}][[E_{kl}, E_{ij}], r_{kl}(m)]) \otimes e_j^* \\
&\quad = [E_{ij}, [E_{km}, E_{ml}]] \otimes e_j^* \\
&\quad \equiv \cdots\cdots \\
&\quad \equiv 0
\end{aligned}$$

and hence $[E_{ij}, E_{kl}] \otimes e_j^* \equiv r_{kl}(m) \otimes e_i^*$.

From the results above, for $m \neq i, j, k, l$, we also see $[E_{ij}, E_{kl}] \otimes e_l^* \equiv -r_{ij}(m) \otimes e_k^*$.

• The proof of (ii): (R2-7), (R2-8)

First, let $p \neq j, l$ and $m \in \{1, \dots, n\}$ such that $m \neq i, j, k, l, p$. Set $r = {E_{i^{-1}j}}^{-1}[E_{i^{-1}m}, E_{mj}]$. Then we have

$$[E_{i^{-1}j}, [r, E_{k^{\pm 1}l}]] \otimes e_p^* \equiv 0, \quad [r, E_{k^{\pm 1}l}] \otimes e_p^* \equiv 0$$

and

$$\begin{aligned}
&[E_{i^{-1}j},E_{k^{\pm 1}l}] \otimes e_p^* \\
&\quad \equiv ([E_{i^{-1}j},[r,E_{k^{\pm 1}l}]][r,E_{k^{\pm 1}l}][E_{i^{-1}j},E_{k^{\pm 1}l}]) \otimes e_p^* \\
&\quad = [[E_{i^{-1}m},E_{mj}],E_{k^{\pm 1}l}] \otimes e_p^* \\
&\quad = [E_{i^{-1}m}E_{mj}{E_{i^{-1}m}}^{-1}{E_{mj}}^{-1},E_{k^{\pm 1}l}] \otimes e_p^* \\
&\quad = ([E_{i^{-1}m},[E_{mj}{E_{i^{-1}m}}^{-1}{E_{mj}}^{-1},E_{k^{\pm 1}l}]][E_{mj}{E_{i^{-1}m}}^{-1}{E_{mj}}^{-1},E_{k^{\pm 1}l}][E_{i^{-1}m},E_{k^{\pm 1}l}]) \otimes e_p^* \\
&\quad \equiv ([E_{mj}{E_{i^{-1}m}}^{-1}{E_{mj}}^{-1},E_{k^{\pm 1}l}][E_{i^{-1}m},E_{k^{\pm 1}l}]) \otimes e_p^* \\
&\quad \equiv \cdots\cdots \\
&\quad \equiv ([{E_{mj}}^{-1},E_{k^{\pm 1}l}][{E_{i^{-1}m}}^{-1},E_{k^{\pm 1}l}][E_{mj},E_{k^{\pm 1}l}][E_{i^{-1}m},E_{k^{\pm 1}l}]) \otimes e_p^*
\end{aligned}$$

Since $p \neq j,l$, we have

$$\begin{aligned}
&[E_{mj},E_{k^{\pm 1}l}] \otimes e_p^* + [{E_{mj}}^{-1},E_{k^{\pm 1}l}] \otimes e_p^* \equiv 0, \\
&[E_{i^{-1}m},E_{k^{\pm 1}l}] \otimes e_p^* + [{E_{i^{-1}m}}^{-1},E_{k^{\pm 1}l}] \otimes e_p^* \equiv 0
\end{aligned}$$

and hence $[E_{i^{-1}j},E_{k^{\pm 1}l}] \otimes e_p^* \equiv 0$.

Next, let $p = j$ and $m \neq i,j,k,l$. Since $[[E_{k^{\pm 1-1}l},E_{i^{-1}j}],r_{k^{\pm 1}l}(m)] = 0$ in $\bar{R}^{\mathrm{ab}}$ and $[E_{i^{-1}j},r_{k^{\pm 1}l}(m)] \otimes e_j^* \equiv r_{k^{\pm 1}l}(m) \otimes e_i^*$ by Corollary 3.3, we have

$$\begin{aligned}
&[E_{i^{-1}j},E_{k^{\pm 1}l}] \otimes e_j^* + r_{k^{\pm 1}l}(m) \otimes e_i^* \\
&\quad \equiv ([E_{i^{-1}j},r_{k^{\pm 1}l}(m)][E_{i^{-1}j},E_{k^{\pm 1}l}][[E_{k^{\pm 1}l},E_{i^{-1}j}],r_{k^{\pm 1}l}(m)]) \otimes e_j^* \\
&\quad = [E_{i^{-1}j},[E_{k^{\pm 1}m},E_{ml}]] \otimes e_j^* \\
&\quad \equiv \cdots\cdots \\
&\quad \equiv 0
\end{aligned}$$

and hence $[E_{i^{-1}j},E_{k^{\pm 1}l}] \otimes e_j^* \equiv -r_{k^{\pm 1}l}(m) \otimes e_i^*$.

From the result above, for $m \neq i,j,k,l$, we also see $[E_{i^{-1}j},E_{k^{-1}l}] \otimes e_l^* \equiv r_{i^{-1}j}(m) \otimes e_k^*$. To show $[E_{i^{-1}j},E_{kl}] \otimes e_l^* \equiv -r_{i^{-1}j}(m) \otimes e_k^*$, set $r_{i^{-1}j}(m) = [E_{i^{-1}m},E_{mj}]{E_{i^{-1}j}}^{-1}$. Since $[[E_{i^{-1}j},E_{kl}],r_{i^{-1}j}(m)]$ in $\bar{R}^{\mathrm{ab}}$ and $[E_{kl},r_{i^{-1}j}(m)] \otimes e_l^* \equiv -r_{i^{-1}j}(m) \otimes e_k^*$ by Corollary 3.3, we have

$$\begin{aligned}
&[E_{kl},E_{i^{-1}j}] \otimes e_l^* - r_{i^{-1}j}(m) \otimes e_k^* \\
&\quad \equiv ([E_{kl},r_{i^{-1}j}(m)][E_{kl},E_{i^{-1}j}][[E_{i^{-1}j},E_{kl}],r_{i^{-1}j}(m)]) \otimes e_l^* \\
&\quad = [E_{kl},[E_{i^{-1}m},E_{mj}]] \otimes e_l^* \\
&\quad \equiv \cdots\cdots \\
&\quad \equiv 0
\end{aligned}$$

and hence $[E_{i^{-1}j},E_{kl}] \otimes e_l^* \equiv -r_{i^{-1}j}(m) \otimes e_k^*$.

• The proof of (iii): (R2-5)

First, let $p \neq j,k$ and $l \in \{1,\ldots,n\}$ such that $l \neq i,j,k,p$. Set $r = {E_{ij}}^{-1}[E_{il},E_{lj}]$. Then we have

$$[E_{ij},[r,E_{i^{-1}k}]] \otimes e_p^* \equiv 0, \quad [r,E_{i^{-1}k}] \otimes e_p^* \equiv 0$$

and

$$\begin{aligned}
&[E_{ij},E_{i^{-1}k}] \otimes e_p^* \\
&\quad \equiv ([E_{ij},[r,E_{i^{-1}k}]][r,E_{i^{-1}k}][E_{ij},E_{i^{-1}k}]) \otimes e_p^* \\
&\quad = [[E_{il},E_{lj}],E_{i^{-1}k}] \otimes e_p^* \\
&\quad = [E_{il}E_{lj}{E_{il}}^{-1}{E_{lj}}^{-1},E_{i^{-1}k}] \otimes e_p^* \\
&\quad = ([E_{il},[E_{lj}{E_{il}}^{-1}{E_{lj}}^{-1},E_{i^{-1}k}]][E_{lj}{E_{il}}^{-1}{E_{lj}}^{-1},E_{i^{-1}k}][E_{il},E_{i^{-1}k}]) \otimes e_p^* \\
&\quad \equiv ([E_{lj}{E_{il}}^{-1}{E_{lj}}^{-1},E_{i^{-1}k}][E_{il},E_{i^{-1}k}]) \otimes e_p^* \\
&\quad \equiv \cdots\cdots \\
&\quad \equiv ([{E_{lj}}^{-1},E_{i^{-1}k}][{E_{il}}^{-1},E_{i^{-1}k}][E_{lj},E_{i^{-1}k}][E_{il},E_{i^{-1}k}]) \otimes e_p^*.
\end{aligned}$$

Since $p \neq j,k$, we have

$$\begin{aligned}
[E_{lj},E_{i^{-1}k}] \otimes e_p^* + [{E_{lj}}^{-1},E_{i^{-1}k}] \otimes e_p^* \equiv 0, \\
[E_{il},E_{i^{-1}k}] \otimes e_p^* + [{E_{il}}^{-1},E_{i^{-1}k}] \otimes e_p^* \equiv 0
\end{aligned}$$

and hence $[E_{i^{-1}j},E_{k^{\pm 1}l}] \otimes e_p^* \equiv 0$.

Next, let $p = k$ and $l \neq i,j,k$. Since $[r_{ij}(l),[E_{ij},E_{i^{-1}k}]] = 0$ in $\bar{R}^{\mathrm{ab}}$ and $[r_{ij}(l),E_{i^{-1}k}] \otimes e_k^* \equiv -r_{ij}(l) \otimes e_i^*$ by Corollary 3.3, we have

$$\begin{aligned}
&[E_{ij},E_{i^{-1}k}] \otimes e_k^* - r_{ij}(l) \otimes e_i^* \\
&\quad \equiv ([r_{ij}(l),[E_{ij},E_{i^{-1}k}]][E_{ij},E_{i^{-1}k}][r_{ij}(l),E_{i^{-1}k}]) \otimes e_k^* \\
&\quad \equiv [[E_{il},E_{lj}],E_{i^{-1}k}] \otimes e_k^* \\
&\quad = [E_{il}E_{lj}{E_{il}}^{-1}{E_{lj}}^{-1},E_{i^{-1}k}] \otimes e_k^* \\
&\quad \equiv ([E_{il},E_{i^{-1}k}][{E_{il}}^{-1},E_{i^{-1}k}][E_{lj},E_{i^{-1}k}][{E_{lj}}^{-1},E_{i^{-1}k}]) \otimes e_k^* \\
&\quad \equiv 0
\end{aligned}$$

and hence $[E_{ij},E_{i^{-1}k}] \otimes e_k^* \equiv [r_{ij}(l)] \otimes e_i^*$.

Finally, let $p = j$ and $l \neq i,j,k$. Since $[[[E_{i^{-1}k},E_{ij}],r_{i^{-1}k}(l)] = 0$ in $\bar{R}^{\mathrm{ab}}$ and $[E_{ij},r_{i^{-1}k}(l)] \otimes e_j^* \equiv -r_{i^{-1}k}(l) \otimes e_i^*$ by Corollary 3.3, we have

$$\begin{aligned}
&[E_{ij},E_{i^{-1}k}] \otimes e_j^* - [r_{i^{-1}k}(l)] \otimes e_i^* \\
&\quad \equiv ([E_{ij},r_{i^{-1}k}(l)][E_{ij},E_{i^{-1}k}][[E_{i^{-1}k},E_{ij}],r_{i^{-1}k}(l)]) \otimes e_j^* \\
&\quad \equiv [E_{ij},[E_{i^{-1}l},E_{lk}]] \otimes e_j^* \\
&\quad = [E_{ij},E_{i^{-1}l}E_{lk}{E_{i^{-1}l}}^{-1}{E_{lk}}^{-1}] \otimes e_j^* \\
&\quad \equiv ([E_{ij},E_{i^{-1}l}][E_{ij},{E_{i^{-1}l}}^{-1}][E_{ij},E_{lk}][E_{ij},{E_{lk}}^{-1}]) \otimes e_j^* \\
&\quad \equiv 0
\end{aligned}$$

• The proof of (iv): (R2-1)

First, let $p \neq j$ and $k \in \{1,\ldots,n\}$ such that $k \neq i,j,p$. Set $r = {E_{ij}}^{-1}[E_{ik},E_{kj}]$. Then we have

$$[E_{ij},[r,E_{i^{-1}j}]] \otimes e_p^* \equiv 0, \quad [r,E_{i^{-1}j}] \otimes e_p^* \equiv 0$$

and

$$\begin{aligned}
&[E_{ij},E_{i^{-1}j}] \otimes e_p^* \\
&\quad \equiv ([E_{ij},[r,E_{i^{-1}j}]][r,E_{i^{-1}j}][E_{ij},E_{i^{-1}j}]) \otimes e_p^* \\
&\quad = [[E_{ik},E_{kj}],E_{i^{-1}j}] \otimes e_p^* \\
&\quad = [E_{ik}E_{kj}{E_{ik}}^{-1}{E_{kj}}^{-1},E_{i^{-1}j}] \otimes e_p^* \\
&\quad = ([E_{ik},[E_{kj}{E_{ik}}^{-1}{E_{kj}}^{-1},E_{i^{-1}j}]][E_{kj}{E_{ik}}^{-1}{E_{kj}}^{-1},E_{i^{-1}j}][E_{ik},E_{i^{-1}j}]) \otimes e_p^* \\
&\quad \equiv ([E_{kj}{E_{ik}}^{-1}{E_{kj}}^{-1},E_{i^{-1}j}][E_{ik},E_{i^{-1}j}]) \otimes e_p^* \\
&\quad \equiv \cdots\cdots \\
&\quad \equiv ([{E_{kj}}^{-1},E_{i^{-1}j}][{E_{ik}}^{-1},E_{i^{-1}j}][E_{kj},E_{i^{-1}j}][E_{ik},E_{i^{-1}j}]) \otimes e_p^*.
\end{aligned}$$

Since $p \neq j$, we have

$$\begin{aligned}
[E_{kj},E_{i^{-1}j}] \otimes e_p^* + [{E_{kj}}^{-1},E_{i^{-1}j}] \otimes e_p^* \equiv 0, \\
[E_{ik},E_{i^{-1}j}] \otimes e_p^* + [{E_{ik}}^{-1},E_{i^{-1}j}] \otimes e_p^* \equiv 0
\end{aligned}$$

and hence $[E_{ij},E_{i^{-1}j}] \otimes e_p^* \equiv 0, \quad p \neq j.$

Next, let $p = j$ and $k \neq i, j$. Since $[r_{ij}(k),[E_{ij},E_{i^{-1}j}]] = 0$ in $\bar{R}^{\mathrm{ab}}$ and $[r_{ij}(k),E_{i^{-1}j}] \otimes e_j^* \equiv -r_{ij}(k) \otimes e_i^*$ by Corollary 3.3, we have

$$\begin{aligned}
&[E_{ij},E_{i^{-1}j}] \otimes e_j^* - r_{ij}(k) \otimes e_i^* \\
&\quad \equiv ([r_{ij}(k),[E_{ij},E_{i^{-1}j}]][E_{ij},E_{i^{-1}j}][r_{ij}(k),E_{i^{-1}j}]) \otimes e_j^* \\
&\quad = [[E_{ik},E_{kj}],E_{i^{-1}j}] \otimes e_j^* \\
&\quad = ([E_{ik},[E_{kj}{E_{ik}}^{-1}{E_{kj}}^{-1},E_{i^{-1}j}]][E_{kj}{E_{ik}}^{-1}{E_{kj}}^{-1},E_{i^{-1}j}][E_{ik},E_{i^{-1}j}]) \otimes e_j^* \\
&\quad \equiv ([E_{kj}{E_{ik}}^{-1}{E_{kj}}^{-1},E_{i^{-1}j}][E_{ik},E_{i^{-1}j}]) \otimes e_j^* \\
&\quad \equiv \cdots\cdots \\
&\quad \equiv ([E_{ik},E_{i^{-1}j}][{E_{ik}}^{-1},E_{i^{-1}j}]) \otimes e_j^* + ([E_{kj},E_{i^{-1}j}][{E_{kj}}^{-1},E_{i^{-1}j}]) \otimes e_j^* \\
&\qquad\qquad - [{E_{ik}}^{-1}{E_{kj}}^{-1},E_{i^{-1}j}] \otimes e_k^* \\
&\quad \equiv -r_{i^{-1}j}(l) \otimes e_i^*
\end{aligned}$$

and hence

$$[E_{ij},E_{i^{-1}j}] \otimes e_j^* \equiv r_{ij}(k) \otimes e_i^* - r_{i^{-1}j}(l) \otimes e_i^*.$$

• **The proof of (v): (R2-2)**

First, let $p \neq j$ and $l \in \{1,\ldots,n\}$ such that $l \neq i,j,k,p$. Set $r = {E_{ij}}^{-1}[E_{il},E_{lj}]$. Then we have

$$[E_{ij},[r,E_{kj}]] \otimes e_p^* \equiv 0, \quad [r,E_{kj}] \otimes e_p^* \equiv 0$$

and

$$\begin{aligned}
&[E_{ij},E_{kj}] \otimes e_p^* \\
&\quad \equiv ([E_{ij}, [r, E_{kj}]][r, E_{kj}][E_{ij}, E_{kj}]) \otimes e_p^* \\
&\quad = [[E_{il}, E_{lj}], E_{kj}] \otimes e_p^* \\
&\quad = [E_{il}E_{lj}{E_{il}}^{-1}{E_{lj}}^{-1}, E_{kj}] \otimes e_p^* \\
&\quad = ([E_{il}, [E_{lj}{E_{il}}^{-1}{E_{lj}}^{-1}, E_{kj}]][E_{lj}{E_{il}}^{-1}{E_{lj}}^{-1}, E_{kj}][E_{il}, E_{kj}]) \otimes e_p^* \\
&\quad \equiv ([E_{lj}{E_{il}}^{-1}{E_{lj}}^{-1}, E_{kj}][E_{il}, E_{kj}]) \otimes e_p^* \\
&\quad \equiv \cdots\cdots \\
&\quad \equiv ([{E_{lj}}^{-1}, E_{kj}][{E_{il}}^{-1}, E_{kj}][E_{lj}, E_{kj}][E_{il}, E_{kj}]) \otimes e_p^*
\end{aligned}$$

Since $p \neq j$, we have

$$\begin{aligned}
[E_{lj}, E_{kj}] \otimes e_p^* + [{E_{lj}}^{-1}, E_{kj}] \otimes e_p^* \equiv 0, \\
[E_{il}, E_{kj}] \otimes e_p^* + [{E_{il}}^{-1}, E_{kj}] \otimes e_p^* \equiv 0
\end{aligned}$$

and hence

$$[E_{ij}, E_{kj}] \otimes e_p^* \equiv 0, \quad p \neq j.$$

Next, let $p = j$ and $l \neq i, j, k$. Since $[[E_{kj}, E_{ij}], r_{kj}(l)] = 0$ in $\bar{R}^{\mathrm{ab}}$ and $[E_{ij}, r_{kj}(l)] \otimes e_j^* \equiv -r_{kj}(l) \otimes e_i^*$ by Corollary 3.3, we have

$$\begin{aligned}
&[E_{ij},E_{kj}] \otimes e_j^* - [r_{kj}(l)] \otimes e_i^* \\
&\quad \equiv ([E_{ij}, r_{kj}(l)][E_{ij}, E_{kj}][[E_{kj}, E_{ij}], r_{kj}(l)]) \otimes e_j^* \\
&\quad = [E_{ij}, [E_{kl}, E_{lj}]] \otimes e_j^* \\
&\quad = ([E_{ij}, E_{kl}][E_{ij}, E_{lj}{E_{kl}}^{-1}{E_{lj}}^{-1}][[E_{lj}{E_{kl}}^{-1}{E_{lj}}^{-1}, E_{ij}], E_{kl}]) \otimes e_j^* \\
&\quad \equiv ([E_{ij}, E_{kl}][E_{ij}, E_{lj}{E_{kl}}^{-1}{E_{lj}}^{-1}]) \otimes e_j^* \\
&\quad \equiv \cdots\cdots \\
&\quad \equiv ([E_{ij}, E_{kl}][E_{ij}, {E_{kl}}^{-1}]) \otimes e_j^* + ([E_{ij}, E_{lj}][E_{ij}, {E_{lj}}^{-1}]) \otimes e_j^* + [{E_{kl}}^{-1}{E_{lj}}^{-1}, E_{ij}] \otimes e_l^* \\
&\quad \equiv [{E_{kl}}^{-1}{E_{lj}}^{-1}, E_{ij}] \otimes e_l^* \\
&\quad \equiv ([{E_{kl}}^{-1}, [{E_{lj}}^{-1}, E_{ij}]][{E_{lj}}^{-1}, E_{ij}][{E_{kl}}^{-1}, E_{ij}]) \otimes e_l^* \\
&\quad \equiv -r_{ij}(m) \otimes e_k^*
\end{aligned}$$

and hence

$$[E_{ij}, E_{kj}] \otimes e_j^* \equiv r_{kj}(l) \otimes e_i^* - r_{ij}(m) \otimes e_k^*.$$

• The proof of (vi): (R2-3), (R2-4)

First, let $p \neq j$ and $l \in \{1, \ldots, n\}$ such that $l \neq i, j, k, p$. Set $r = {E_{i^{-1}j}}^{-1}[E_{i^{-1}l}, E_{lj}]$. Then we have

$$[E_{i^{-1}j}, [r, E_{k^{\pm 1}j}]] \otimes e_p^* \equiv 0, \quad [r, E_{k^{\pm 1}j}] \otimes e_p^* \equiv 0$$

and

$$\begin{aligned}
&[E_{i^{-1}j},E_{k^{\pm1}j}] \otimes e_p^* \\
&\quad\equiv ([E_{i^{-1}j},[r,E_{k^{\pm1}j}]][r,E_{k^{\pm1}j}][E_{i^{-1}j},E_{k^{\pm1}j}]) \otimes e_p^* \\
&\quad= [[E_{i^{-1}l},E_{lj}],E_{k^{\pm1}j}] \otimes e_p^* \\
&\quad= [E_{i^{-1}l}E_{lj}{E_{i^{-1}l}}^{-1}{E_{lj}}^{-1},E_{k^{\pm1}j}] \otimes e_p^* \\
&\quad= ([E_{i^{-1}l},[E_{lj}{E_{i^{-1}l}}^{-1}{E_{lj}}^{-1},E_{k^{\pm1}j}]][E_{lj}{E_{i^{-1}l}}^{-1}{E_{lj}}^{-1},E_{k^{\pm1}j}][E_{i^{-1}l},E_{k^{\pm1}j}]) \otimes e_p^* \\
&\quad\equiv ([E_{lj}{E_{i^{-1}l}}^{-1}{E_{lj}}^{-1},E_{k^{\pm1}j}][E_{i^{-1}l},E_{k^{\pm1}j}]) \otimes e_p^* \\
&\quad\equiv \cdots\cdots \\
&\quad\equiv ([{E_{lj}}^{-1},E_{k^{\pm1}j}][{E_{i^{-1}l}}^{-1},E_{k^{\pm1}j}][E_{lj},E_{k^{\pm1}j}][E_{i^{-1}l},E_{k^{\pm1}j}]) \otimes e_p^*
\end{aligned}$$

Since $p \neq j$, we have

$$\begin{aligned}
&[E_{lj},E_{k^{\pm1}j}] \otimes e_p^* + [{E_{lj}}^{-1},E_{k^{\pm1}j}] \otimes e_p^* \equiv 0, \\
&[E_{i^{-1}l},E_{k^{\pm1}j}] \otimes e_p^* + [{E_{i^{-1}l}}^{-1},E_{k^{\pm1}j}] \otimes e_p^* \equiv 0
\end{aligned}$$

and hence

$$[E_{i^{-1}j},E_{k^{\pm1}j}] \otimes e_p^* \equiv 0.$$

Next, let $p = j$ and $l \neq i,j,k$. Since $[[E_{k^{\pm}j},E_{i^{-1}j}],r_{k^{\pm}j}(l)] = 0$ in $\bar{R}^{\mathrm{ab}}$ and $[E_{i^{-1}j},r_{k^{\pm1}j}(l)] \otimes e_j^* \equiv r_{k^{\pm1}j}(l) \otimes e_i^*$ by Corollary 3.3, we have

$$\begin{aligned}
&[E_{i^{-1}j},E_{k^{\pm1}j}] \otimes e_j^* + r_{k^{\pm1}j}(l) \otimes e_i^* \\
&\quad\equiv ([E_{i^{-1}j},r_{k^{\pm1}j}(l)][E_{i^{-1}j},E_{k^{\pm1}j}][[E_{k^{\pm1}j},E_{i^{-1}j}],r_{k^{\pm1}j}(l)]) \otimes e_j^* \\
&\quad= [E_{i^{-1}j},[E_{k^{\pm1}l},E_{lj}]] \otimes e_j^* \\
&\quad= ([E_{i^{-1}j},E_{k^{\pm1}l}][E_{i^{-1}j},E_{lj}{E_{k^{\pm1}l}}^{-1}{E_{lj}}^{-1}][[E_{lj}{E_{k^{\pm1}l}}^{-1}{E_{lj}}^{-1},E_{i^{-1}j}],E_{k^{\pm1}l}]) \otimes e_j^* \\
&\quad\equiv ([E_{i^{-1}j},E_{k^{\pm1}l}][E_{i^{-1}j},E_{lj}{E_{k^{\pm1}l}}^{-1}{E_{lj}}^{-1}]) \otimes e_j^* \\
&\quad\equiv \cdots\cdots \\
&\quad\equiv ([E_{i^{-1}j},E_{k^{\pm1}l}][E_{i^{-1}j},{E_{k^{\pm1}l}}^{-1}]) \otimes e_j^* + ([E_{i^{-1}j},E_{lj}][E_{i^{-1}j},{E_{lj}}^{-1}]) \otimes e_j^* \\
&\qquad\qquad + [{E_{k^{\pm1}l}}^{-1}{E_{lj}}^{-1},E_{i^{-1}j}] \otimes e_l^* \\
&\quad\equiv [{E_{k^{\pm1}l}}^{-1}{E_{lj}}^{-1},E_{i^{-1}j}] \otimes e_l^* \equiv [{E_{k^{\pm1}l}}^{-1},E_{i^{-1}j}] \otimes e_l^* \\
&\quad\equiv \mp r_{i^{-1}j}(m) \otimes e_k^*
\end{aligned}$$

for $m \neq i,j,k,l.$ and hence

$$[E_{i^{-1}j},E_{k^{\pm1}j}] \otimes e_j^* \equiv -r_{k^{\pm1}j}(l) \otimes e_i^* \mp r_{i^{-1}j}(m) \otimes e_k^*.$$

6.2. **The computation of** $(a,b,c,d,e) \otimes e_p^*$**.**

Here we give the details of the computation of the equation $(a,b,c,d,e) \otimes e_p^*$.

• $(x_l,x_j,x_i,x_j,x_k) \otimes e_k^*$

For distinct i,j,k and l, the equation $(x_l,x_j,x_i,x_j,x_k) \otimes e_k^*$ is given by

$$
\begin{aligned}
r_{il}(k)\otimes e_k^* = {} & s_1^{-1}\otimes e_k^* + (w_{lj}{}^{-1}E_{ik}w_{lj}\ s_2^{-1}\ w_{lj}{}^{-1}E_{ik}{}^{-1}w_{lj})\otimes e_k^* \\
& + (w_{lj}{}^{-1}r_{ij}(k)w_{lj})\otimes e_k^* \\
& + (w_{lj}{}^{-1}E_{ij}E_{kj}w_{lj}\ s_1\ w_{lj}{}^{-1}E_{kj}{}^{-1}E_{ij}{}^{-1}w_{lj})\otimes e_k^* \\
& + (w_{lj}{}^{-1}E_{ij}w_{lj}\ s_2\ w_{lj}{}^{-1}E_{ij}{}^{-1}w_{lj})\otimes e_k^* + s_3\otimes e_k^*
\end{aligned} \tag{39}
$$

where

$$
\begin{aligned}
& s_1 := (w_{lj}{}^{-1}E_{ik^{-1}}w_{lj})^{-1}E_{ik}{}^{-1}, \quad s_2 := (w_{lj}{}^{-1}E_{kj^{-1}}w_{lj})^{-1}E_{kl}{}^{-1}, \\
& s_3 := (w_{lj}{}^{-1}E_{ij^{-1}}w_{lj})^{-1}E_{il}{}^{-1}.
\end{aligned}
$$

Then using Lemmas 3.4 and 3.5, we obtain

$$
r_{il}(k)\otimes e_k^* \equiv r_{ij}(k)\otimes e_k^* + s_2\otimes e_i^* + s_3\otimes e_k^*. \tag{40}
$$

Using (vi) of Lemma 2.2, we see

$$
\begin{aligned}
s_2 = {} & E_{j^{-1}l}E_{l^{-1}j^{-1}}E_{kl}\,r_{kl^{-1}}(j)^{-1}\,E_{kl^{-1}}E_{l^{-1}j}E_{j^{-1}l^{-1}}\cdot E_{j^{-1}l}E_{kl}\,r_{kj^{-1}}(l^{-1})\,E_{kl^{-1}}E_{j^{-1}l^{-1}} \\
& \cdot E_{j^{-1}l}E_{kl}\,[E_{kj^{-1}},E_{l^{-1}j^{-1}}]\,E_{kl^{-1}}E_{j^{-1}l^{-1}}\cdot[E_{j^{-1}l},E_{kl}],
\end{aligned}
$$

and hence

$$
s_2\otimes e_i^* \equiv r_{kl}(\cdot)\otimes e_i^* - r_{kj}(\cdot)\otimes e_i^*.
$$

By the same argument as above, we have

$$
s_3\otimes e_k^* \equiv r_{il}(\cdot)\otimes e_k^* - r_{ij}(\cdot)\otimes e_k^*.
$$

Substituting these results into (40), we obtain $S_{ijk}^* \equiv S_{ilk}^*$.

- $(x_l^{-1}, x_j, x_i, x_j, x_k)\otimes e_k$

  For distinct i, j, k and l, the equation $(x_l^{-1}, x_j, x_i, x_j, x_k)\otimes e_k$ is given by

$$
\begin{aligned}
r_{il^{-1}}(k)\otimes e_k^* = {} & s_1^{-1}\otimes e_k^* + (w_{l^{-1}j}{}^{-1}E_{ik}w_{l^{-1}j}\ s_2^{-1}\ w_{l^{-1}j}{}^{-1}E_{ik}{}^{-1}w_{l^{-1}j})\otimes e_k^* \\
& + (w_{l^{-1}j}{}^{-1}r_{ij}(k)w_{l^{-1}j})\otimes e_k^* \\
& + (w_{l^{-1}j}{}^{-1}E_{ij}E_{kj}w_{l^{-1}j}\ s_1\ w_{l^{-1}j}{}^{-1}E_{kj}{}^{-1}E_{ij}{}^{-1}w_{l^{-1}j})\otimes e_k^* \\
& + (w_{l^{-1}j}{}^{-1}E_{ij}w_{l^{-1}j}\ s_2\ w_{l^{-1}j}{}^{-1}E_{ij}{}^{-1}w_{l^{-1}j})\otimes e_k^* + s_3\otimes e_k^*
\end{aligned}
$$

where

$$
\begin{aligned}
& s_1 := (w_{l^{-1}j}{}^{-1}E_{ik^{-1}}w_{l^{-1}j})^{-1}E_{ik}{}^{-1}, \quad s_2 := (w_{l^{-1}j}{}^{-1}E_{kj^{-1}}w_{l^{-1}j})^{-1}E_{kl^{-1}}{}^{-1}, \\
& s_3 := (w_{l^{-1}j}{}^{-1}E_{ij^{-1}}w_{l^{-1}j})^{-1}E_{il}.
\end{aligned}
$$

Then using Lemmas 3.4 and 3.5, we obtain

$$
r_{il^{-1}}(k)\otimes e_k^* \equiv r_{ij}(k)\otimes e_k^* + s_2\otimes e_i^* + s_3\otimes e_k^*. \tag{41}
$$

From Lemma 2.4, we have $r_{il^{-1}}(k)\otimes e_k^* \equiv -r_{il}(k)\otimes e_k^*$. Using (vi) of Lemma 2.2, we see

$$
\begin{aligned}
s_2 = {} & E_{j^{-1}l^{-1}}E_{lj^{-1}}E_{kl^{-1}}\,r_{kl}(j)^{-1}\,E_{kl}E_{lj}E_{j^{-1}l}\cdot E_{j^{-1}l^{-1}}E_{kl^{-1}}\,r_{kj^{-1}}(l)\,E_{kl}E_{j^{-1}l} \\
& \cdot E_{j^{-1}l^{-1}}E_{kl^{-1}}\,[E_{kj^{-1}},E_{lj^{-1}}]\,E_{kl}E_{j^{-1}l}\cdot[E_{j^{-1}l^{-1}},E_{kl^{-1}}],
\end{aligned}
$$

and hence

$$
s_2\otimes e_i^* \equiv -r_{kl}(\cdot)\otimes e_i^* - r_{kj}(\cdot)\otimes e_i^*.
$$

By the same argument as above, we have

$$s_3^* \otimes e_k^* \equiv -r_{il}(\cdot) \otimes e_k^* + r_{ij}(\cdot) \otimes e_k^*.$$

Substituting these results into (41), we obtain $S_{ijk}^* \equiv -S_{ilk}^*$.

- $(x_k^{-1}, x_l, x_i, x_j, x_l) \otimes e_k^*$

For distinct i, j, k and l, the equation $(x_k^{-1}, x_l, x_i, x_j, x_l) \otimes e_k^*$ is given by

$$\begin{aligned} r_{ij}(k^{-1}) \otimes e_k^* = s_1^{-1} \otimes e_k^* \\ + (w_{k^{-1}l}{}^{-1} E_{il} w_{k^{-1}l}\ s_2^{-1}\ w_{k^{-1}l}{}^{-1} E_{il}{}^{-1} w_{k^{-1}l}) \otimes e_k^* \\ + (w_{k^{-1}l}{}^{-1} r_{ij}(l) w_{k^{-1}l}) \otimes e_k^* \\ + (w_{k^{-1}l}{}^{-1} E_{ij} E_{lj} w_{k^{-1}l}\ s_1\ w_{k^{-1}l}{}^{-1} E_{lj}{}^{-1} E_{ij}{}^{-1} w_{k^{-1}l}) \otimes e_k^* \\ + (w_{k^{-1}l}{}^{-1} E_{ij} w_{k^{-1}l}\ s_2\ w_{k^{-1}l}{}^{-1} E_{ij}{}^{-1} w_{k^{-1}l}) \otimes e_k^* + s_3 \otimes e_k^* \end{aligned}$$

where

$$\begin{aligned} &s_1 := (w_{k^{-1}l}{}^{-1} E_{il^{-1}} w_{k^{-1}l})^{-1} E_{ik}, \quad s_2 := (w_{k^{-1}l}{}^{-1} E_{lj^{-1}} w_{k^{-1}l})^{-1} E_{k^{-1}j}{}^{-1}, \\ &s_3 := (w_{k^{-1}l}{}^{-1} E_{ij^{-1}} w_{k^{-1}l})^{-1} E_{ij}{}^{-1}. \end{aligned}$$

Then using Lemmas 3.4 and 3.5, we obtain

$$r_{ij}(k^{-1}) \otimes e_k^* \equiv -r_{ij}(l) \otimes e_l^* - s_2 \otimes e_i^* + s_3 \otimes e_k^*. \tag{42}$$

From (3), we have

$$s_2 = (w_{k^{-1}l}{}^{-1} E_{lj} h_{k^{-1}l} E_{lj^{-1}} w_{k^{-1}l}) \cdot (w_{k^{-1}l}{}^{-1} h_{k^{-1}l}{}^{-1} w_{k^{-1}l}) \cdot (w_{kl^{-1}}{}^{-1} E_{lj^{-1}} w_{kl^{-1}})^{-1} E_{k^{-1}j^{-1}},$$

and hence

$$s_2 \otimes e_i^* \equiv \{(w_{kl^{-1}}{}^{-1} E_{lj^{-1}} w_{kl^{-1}})^{-1} E_{k^{-1}j^{-1}}\} \otimes e_i^*.$$

Applying (ii) of Lemma 2.2 to $(w_{kl^{-1}}{}^{-1} E_{lj^{-1}} w_{kl^{-1}})^{-1} E_{k^{-1}j^{-1}}$, we have

$$\begin{aligned} (w_{kl^{-1}}{}^{-1} E_{lj^{-1}} w_{kl^{-1}})^{-1} E_{k^{-1}j^{-1}} = E_{lk} E_{k^{-1}l}\, [E_{l^{-1}k^{-1}}, E_{lj}]\, E_{k^{-1}l^{-1}} E_{lk^{-1}} \\ \cdot E_{lk}\, r_{k^{-1}j}(l)\, E_{lk^{-1}} \\ \cdot E_{k^{-1}j} E_{lk}\, r_{lj^{-1}}(a^{-1})\, E_{lk^{-1}} E_{k^{-1}j^{-1}} \end{aligned}$$

and hence

$$s_2 \otimes e_i^* \equiv r_{k^{-1}j}(\cdot) \otimes e_i^* - r_{lj}(\cdot) \otimes e_i^*.$$

On the other hand, using Lemma 2.3, we see

$$s_3 = E_{l^{-1}k^{-1}} E_{kl^{-1}}\, [E_{lk}, E_{ij}]\, E_{kl} E_{l^{-1}k} \cdot E_{l^{-1}k^{-1}}\, [E_{kl^{-1}}, E_{ij}]\, E_{l^{-1}k} \cdot [E_{l^{-1}k^{-1}}, E_{ij}]$$

and hence

$$s_3 \otimes e_k^* \equiv r_{ij}(\cdot) \otimes e_k^* + r_{ij}(\cdot) \otimes e_l^*.$$

Substituting these results into (42), we obtain

$$r_{ij}(k^{-1}) \otimes e_k^* \equiv r_{ij}(\cdot) \otimes e_k^* - r_{k^{-1}j}(\cdot) \otimes e_i^*.$$

- $(x_i^{-1}, x_l, x_l, x_j, x_k) \otimes e_k^*$

For distinct i, j, k and l, the equation $(x_i^{-1}, x_l, x_l, x_j, x_k) \otimes e_k^*$ is given by

$$\begin{aligned} r_{i^{-1}j}(k)\otimes e_k^* = s_1^{-1}\otimes e_k^* &+ (w_{i^{-1}l}{}^{-1}E_{lk}w_{i^{-1}l}\ s_2^{-1}\ w_{i^{-1}l}{}^{-1}E_{lk}{}^{-1}w_{i^{-1}l})\otimes e_k^* \\ &+ (w_{i^{-1}l}{}^{-1}r_{lj}(k)w_{i^{-1}l})\otimes e_k^* \\ &+ (w_{i^{-1}l}{}^{-1}E_{lj}E_{kj}w_{i^{-1}l}\ s_1\ w_{i^{-1}l}{}^{-1}E_{kj}{}^{-1}E_{lj}{}^{-1}w_{i^{-1}l})\otimes e_k^* \\ &+ (w_{i^{-1}l}{}^{-1}E_{lj}w_{i^{-1}l}\ s_2\ w_{i^{-1}l}{}^{-1}E_{lj}{}^{-1}w_{i^{-1}l})\otimes e_k^* + s_3\otimes e_k^* \end{aligned}$$

where

$$s_1 := (w_{i^{-1}l}{}^{-1}E_{lk^{-1}}w_{i^{-1}l})^{-1}E_{i^{-1}k}{}^{-1},\quad s_2 := (w_{i^{-1}l}{}^{-1}E_{kj^{-1}}w_{i^{-1}l})^{-1}E_{kj}{}^{-1},$$
$$s_3 := (w_{i^{-1}l}{}^{-1}E_{lj^{-1}}w_{i^{-1}l})^{-1}E_{i^{-1}j}{}^{-1}.$$

Then using Lemmas 3.4 and 3.5, we obtain

$$r_{i^{-1}j}(k)\otimes e_k^* \equiv r_{lj}(k)\otimes e_k^* - s_2\otimes e_i^* + s_3\otimes e_k^*. \tag{43}$$

Applying Lemma 2.3 to s_2, we have

$$s_2 = E_{l^{-1}i^{-1}}E_{il^{-1}}\,[E_{li},E_{kj}]\,E_{il}E_{l^{-1}i}\cdot E_{l^{-1}i^{-1}}\,[E_{il^{-1}},E_{kj}]\,E_{l^{-1}i}\cdot[E_{l^{-1}i^{-1}},E_{kj}],$$

and hence

$$s_2\otimes e_i^* \equiv r_{kj}(\cdot)\otimes e_i^* + r_{kj}(\cdot)\otimes e_l^*.$$

On the other hand, Applying (3) to s_3, we have

$$s_3 = (w_{i^{-1}l}{}^{-1}E_{lj}h_{i^{-1}l}E_{lj^{-1}}w_{i^{-1}l})\cdot(w_{i^{-1}l}{}^{-1}h_{i^{-1}l}{}^{-1}w_{i^{-1}l})\cdot(w_{il^{-1}}{}^{-1}E_{lj^{-1}}w_{il^{-1}})^{-1}E_{i^{-1}j^{-1}},$$

and hence

$$s_3\otimes e_k^* \equiv \{(w_{il^{-1}}{}^{-1}E_{lj^{-1}}w_{il^{-1}})^{-1}E_{i^{-1}j^{-1}}\}\otimes e_k^*.$$

Then considering (ii) of Lemma 2.2, we have

$$\begin{aligned} (w_{il^{-1}}{}^{-1}E_{lj^{-1}}w_{il^{-1}})^{-1}E_{i^{-1}j^{-1}} = E_{li}E_{i^{-1}l}\,[E_{l^{-1}i^{-1}},E_{lj}]\,E_{i^{-1}l^{-1}}E_{li^{-1}} \\ \cdot E_{li}\,r_{i^{-1}j}(l)\,E_{li^{-1}} \\ \cdot E_{i^{-1}j}E_{li}\,r_{lj^{-1}}(i^{-1})\,E_{li^{-1}}E_{i^{-1}j^{-1}}, \end{aligned}$$

and hence

$$s_3\otimes e_k^* \equiv r_{i^{-1}j}(\cdot)\otimes e_k^* - r_{lj}(\cdot)\otimes e_k^*.$$

Substituting these results into (43), we obtain

$$r_{i^{-1}j}(k)\otimes e_k^* \equiv r_{i^{-1}j}(\cdot)\otimes e_k^* - r_{kj}(\cdot)\otimes e_i^*.$$

- $(x_i^{-1},x_l,x_l,x_j,x_k^{-1})\otimes e_k^*$

For distinct i,j,k and l, the equation $(x_i^{-1},x_l,x_l,x_j,x_k^{-1})\otimes e_k^*$ is given by

$$\begin{aligned} r_{i^{-1}j}(k^{-1})\otimes e_k^* = s_1^{-1}\otimes e_k^* &+ (w_{i^{-1}l}{}^{-1}E_{lk^{-1}}w_{i^{-1}l}\ s_2^{-1}\ w_{i^{-1}l}{}^{-1}E_{lk}w_{i^{-1}l})\otimes e_k^* \\ &+ (w_{i^{-1}l}{}^{-1}r_{lj}(k^{-1})w_{i^{-1}l})\otimes e_k^* \\ &+ (w_{i^{-1}l}{}^{-1}E_{lj}E_{k^{-1}j}w_{i^{-1}l}\ s_1\ w_{i^{-1}l}{}^{-1}E_{k^{-1}j}{}^{-1}E_{lj}{}^{-1}w_{i^{-1}l})\otimes e_k^* \\ &+ (w_{i^{-1}l}{}^{-1}E_{lj}w_{i^{-1}l}\ s_2\ w_{i^{-1}l}{}^{-1}E_{lj}{}^{-1}w_{i^{-1}l})\otimes e_k + s_3\otimes e_k^* \end{aligned}$$

where

$$s_1 := (w_{i^{-1}l}{}^{-1}E_{lk}w_{i^{-1}l})^{-1}E_{i^{-1}k},\quad s_2 := (w_{i^{-1}l}{}^{-1}E_{k^{-1}j^{-1}}w_{i^{-1}l})^{-1}E_{k^{-1}j}{}^{-1},$$
$$s_3 := (w_{i^{-1}l}{}^{-1}E_{lj^{-1}}w_{i^{-1}l})^{-1}E_{i^{-1}j}{}^{-1}.$$

Then using Lemmas 3.4 and 3.5, we obtain

$$r_{i^{-1}j}(k^{-1}) \otimes e_k^* \equiv r_{lj}(k^{-1}) \otimes e_k^* + s_2 \otimes e_i^* + s_3 \otimes e_k^*. \tag{44}$$

Applying Lemma 2.3 to s_2, we have

$$s_2 = E_{l^{-1}i^{-1}} E_{il^{-1}} \left[E_{li}, E_{k^{-1}j}\right] E_{il} E_{l^{-1}i} \cdot E_{l^{-1}i^{-1}} \left[E_{il^{-1}}, E_{k^{-1}j}\right] E_{l^{-1}i} \cdot \left[E_{l^{-1}i^{-1}}, E_{k^{-1}j}\right],$$

and hence

$$s_2 \otimes e_i^* \equiv r_{k^{-1}j}(\cdot) \otimes e_i^* + r_{k^{-1}j}(\cdot) \otimes e_l^*.$$

On the other hand, Applying (3) to s_3, we have

$$s_3 = (w_{i^{-1}l}{}^{-1} E_{lj} h_{i^{-1}l} E_{lj^{-1}} w_{i^{-1}l}) \cdot (w_{i^{-1}l}{}^{-1} h_{i^{-1}l}{}^{-1} w_{i^{-1}l}) \cdot (w_{il^{-1}}{}^{-1} E_{lj^{-1}} w_{il^{-1}})^{-1} E_{i^{-1}j^{-1}},$$

and hence

$$s_3 \otimes e_k^* \equiv \{(w_{il^{-1}}{}^{-1} E_{lj^{-1}} w_{il^{-1}})^{-1} E_{i^{-1}j^{-1}} \otimes e_k^*.$$

Then considering (ii) of Lemma 2.2, we have

$$\begin{aligned}(w_{il^{-1}}{}^{-1} E_{lj^{-1}} w_{il^{-1}})^{-1} E_{i^{-1}j^{-1}} = {} & E_{li} E_{i^{-1}l} \left[E_{l^{-1}i^{-1}}, E_{lj}\right] E_{i^{-1}l^{-1}} E_{li^{-1}} \\ & \cdot E_{li}\, r_{i^{-1}j}(l)\, E_{li^{-1}} \\ & \cdot E_{i^{-1}j} E_{li}\, r_{lj^{-1}}(i^{-1})\, E_{li^{-1}} E_{i^{-1}j^{-1}},\end{aligned}$$

and hence

$$s_3 \otimes e_k^* \equiv r_{i^{-1}j}(\cdot) \otimes e_k^* - r_{lj}(\cdot) \otimes e_k^*.$$

Substituting these results into (44), we obtain

$$r_{i^{-1}j}(k^{-1}) \otimes e_k^* \equiv r_{i^{-1}j}(\cdot) \otimes e_k^* + r_{k^{-1}j}(\cdot) \otimes e_i^*.$$

- $(x_l, x_k, x_i, x_j, x_k) \otimes e_j^*$

  For distinct i, j, k and l, the equation $(x_l, x_k, x_i, x_j, x_k) \otimes e_j^*$ is given by

$$\begin{aligned}r_{ij}(l) \otimes e_j^* = {} & s_1^{-1} \otimes e_j^* + (w_{lk}{}^{-1} E_{ik} w_{lk}\ s_2^{-1}\ w_{lk}{}^{-1} E_{ik}{}^{-1} w_{lk}) \otimes e_j^* \\ & + (w_{lk}{}^{-1} r_{ij}(k) w_{lk}) \otimes e_j^* \\ & + (w_{lk}{}^{-1} E_{ij} E_{kj} w_{lk}\ s_1\ w_{lk}{}^{-1} E_{kj}^{-1} E_{ij}{}^{-1} w_{lk}) \otimes e_j^* \\ & + (w_{lk}{}^{-1} E_{ij} w_{lk}\ s_2\ w_{lk}{}^{-1} E_{ij}{}^{-1} w_{lk}) \otimes e_j^* + s_3 \otimes e_j^*\end{aligned}$$

where

$$\begin{aligned}& s_1 := (w_{lk}{}^{-1} E_{ik^{-1}} w_{lk})^{-1} E_{il}{}^{-1}, \quad s_2 := (w_{lk}{}^{-1} E_{kj^{-1}} w_{lk})^{-1} E_{lj}{}^{-1}, \\ & s_3 := (w_{lk}{}^{-1} E_{ij^{-1}} w_{lk})^{-1} E_{ij}{}^{-1}.\end{aligned}$$

Then using Lemmas 3.4 and 3.5, we obtain

$$r_{ij}(l) \otimes e_j^* \equiv r_{ij}(k) \otimes e_j^* - s_1 \otimes e_l^* - s_1 \otimes e_i^* - s_2 \otimes e_i^* + s_3 \otimes e_j^*. \tag{45}$$

Applying (vi) of Lemma 2.2 to s_1, we have

$$\begin{aligned}s_1 = {} & E_{k^{-1}l} E_{l^{-1}k^{-1}} E_{il}\, r_{il^{-1}}(k)^{-1}\, E_{il^{-1}} E_{l^{-1}k} E_{k^{-1}l^{-1}} \cdot E_{k^{-1}l} E_{il}\, r_{ik^{-1}}(l^{-1})\, E_{il^{-1}} E_{k^{-1}l^{-1}} \\ & \cdot E_{k^{-1}l} E_{il} \left[E_{ik^{-1}}, E_{l^{-1}k^{-1}}\right] E_{il^{-1}} E_{k^{-1}l^{-1}} \cdot \left[E_{k^{-1}l}, E_{il}\right],\end{aligned}$$

and hence

$$s_1 \otimes e_l^* \stackrel{\circ}{=} -r_{ik}(l^{-1}) \otimes e_k^*, \qquad s_1 \otimes e_i^* \stackrel{\circ}{=} 0.$$

Using (3), we have

$$s_2 = (w_{lk}{}^{-1}E_{kj}h_{lk}E_{kj^{-1}}w_{lk}) \cdot (w_{lk}{}^{-1}h_{lk}{}^{-1}w_{lk}) \cdot (w_{l^{-1}k^{-1}}{}^{-1}E_{kj^{-1}}w_{l^{-1}k^{-1}})^{-1}E_{lj^{-1}},$$

and hence

$$s_2 \otimes e_i^* \equiv \{(w_{l^{-1}k^{-1}}{}^{-1}E_{kj^{-1}}w_{l^{-1}k^{-1}})^{-1}E_{lj^{-1}}\} \otimes e_i^*.$$

On the other hand, applying (ii) of Lemma 2.2 to $(w_{l^{-1}k^{-1}}{}^{-1}E_{kj^{-1}}w_{l^{-1}k^{-1}})^{-1}E_{lj^{-1}}$, we see

$$\begin{aligned}(w_{l^{-1}k^{-1}}{}^{-1}E_{kj^{-1}}w_{l^{-1}k^{-1}})^{-1}E_{lj^{-1}} &= E_{kl^{-1}}E_{lk}\,[E_{k^{-1}l}, E_{kj}]\,E_{lk^{-1}}E_{kl} \\ &\quad \cdot E_{kl^{-1}}\,r_{lj}(k)\,E_{kl} \cdot E_{lj}E_{kl^{-1}}\,r_{kj^{-1}}(l)\,E_{kl}E_{lj^{-1}},\end{aligned}$$

and hence $s_2 \otimes e_i^* \overset{\circ}{=} 0$. From Lemma 2.3, we see

$$s_3 = E_{k^{-1}l}E_{l^{-1}k^{-1}}\,[E_{kl^{-1}}, E_{ij}]\,E_{l^{-1}k}E_{k^{-1}l^{-1}} \cdot E_{k^{-1}l}\,[E_{l^{-1}k^{-1}}, E_{ij}]\,E_{k^{-1}l^{-1}} \cdot [E_{k^{-1}l}, E_{ij}],$$

and hence $s_3 \otimes e_j^* \overset{\circ}{=} 0$. Substituting these results into (45), we obtain

$$r_{ik}(l^{-1}) \otimes e_k^* \overset{\circ}{=} -r_{ij}(k) \otimes e_j^* + r_{ij}(l) \otimes e_j^*.$$

- $(x_l^{-1}, x_j, x_i, x_k, x_j) \otimes e_k^*$

  For distinct i, j, k and l, the equation $(x_l^{-1}, x_j, x_i, x_k, x_j) \otimes e_k^*$ is given by

$$\begin{aligned}r_{ik}(l^{-1}) \otimes e_k^* = {} & s_1^{-1} \otimes e_k^* \\ & + (w_{l^{-1}j}{}^{-1}E_{ij}w_{l^{-1}j}\ s_2^{-1}\ w_{l^{-1}j}{}^{-1}E_{ij}{}^{-1}w_{l^{-1}j}) \otimes e_k^* \\ & + (w_{l^{-1}j}{}^{-1}r_{ik}(j)w_{l^{-1}j}) \otimes e_k^* \\ & + (w_{l^{-1}j}{}^{-1}E_{ik}E_{jk}w_{l^{-1}j}\ s_1\ w_{l^{-1}j}{}^{-1}E_{jk}{}^{-1}E_{ik}{}^{-1}w_{l^{-1}j}) \otimes e_k^* \\ & + (w_{l^{-1}j}{}^{-1}E_{ik}w_{l^{-1}j}\ s_2\ w_{l^{-1}j}{}^{-1}E_{ik}{}^{-1}w_{l^{-1}j}) \otimes e_k^* + s_3 \otimes e_k^*\end{aligned}$$

where

$$\begin{aligned}&s_1 := (w_{l^{-1}j}{}^{-1}E_{ij^{-1}}w_{l^{-1}j})^{-1}E_{il}, \quad s_2 := (w_{l^{-1}j}{}^{-1}E_{jk^{-1}}w_{l^{-1}j})^{-1}E_{l^{-1}k}{}^{-1}, \\ &s_3 := (w_{l^{-1}j}{}^{-1}E_{ik^{-1}}w_{l^{-1}j})^{-1}E_{ik}{}^{-1}.\end{aligned}$$

Then using Lemmas 3.4 and 3.5, we obtain

$$r_{ik}(l^{-1}) \otimes e_k^* \equiv r_{ik}(j) \otimes e_k^* - s_1 \otimes e_i^* + s_1 \otimes e_l^* - s_2 \otimes e_i^* + s_3 \otimes e_k^*. \tag{46}$$

Applying (vi) of Lemma 2.2 to s_1, we have

$$\begin{aligned}s_1 = {} & E_{j^{-1}l^{-1}}E_{lj^{-1}}E_{il^{-1}}\,r_{il}(j)^{-1}\,E_{il}E_{lj}E_{j^{-1}l} \\ & \cdot E_{j^{-1}l^{-1}}E_{il^{-1}}\,r_{ij^{-1}}(l)\,E_{il}E_{j^{-1}l} \\ & \cdot E_{j^{-1}l^{-1}}E_{il^{-1}}\,[E_{ij^{-1}}, E_{lj^{-1}}]\,E_{il}E_{j^{-1}l} \\ & \cdot [E_{j^{-1}l^{-1}}, E_{il^{-1}}]\end{aligned}$$

and hence

$$s_1 \otimes e_l^* \overset{\circ}{=} r_{ij}(l) \otimes e_j^*, \qquad s_1 \otimes e_i^* \overset{\circ}{=} 0.$$

Using (3), we have

$$s_2 = (w_{l^{-1}j}{}^{-1}E_{jk}h_{l^{-1}j}E_{jk^{-1}}w_{l^{-1}j}) \cdot (w_{l^{-1}j}{}^{-1}h_{l^{-1}j}{}^{-1}w_{l^{-1}j}) \cdot (w_{lj^{-1}}{}^{-1}E_{jk^{-1}}w_{lj^{-1}})^{-1}E_{l^{-1}k^{-1}},$$

and hence

$$s_2 \otimes e_i^* \equiv \{(w_{lj^{-1}}{}^{-1} E_{jk^{-1}} w_{lj^{-1}})^{-1} E_{l^{-1}k^{-1}}\} \otimes e_i^*.$$

On the other hand, applying (ii) of Lemma 2.2 to $(w_{lj^{-1}}{}^{-1} E_{jk^{-1}} w_{lj^{-1}})^{-1} E_{l^{-1}k^{-1}}$, we see

$$\begin{aligned}(w_{lj^{-1}}{}^{-1} E_{jk^{-1}} w_{lj^{-1}})^{-1} E_{l^{-1}k^{-1}} &= E_{jl} E_{l^{-1}j} \, [E_{j^{-1}l^{-1}}, E_{jk}] \, E_{l^{-1}j^{-1}} E_{jl^{-1}} \\ &\quad \cdot E_{jl} \, r_{l^{-1}k}(j) \, E_{jl^{-1}} \cdot E_{l^{-1}k} E_{jl} \, r_{jk^{-1}}(l^{-1}) \, E_{jl^{-1}} E_{l^{-1}k^{-1}},\end{aligned}$$

and hence $s_2 \otimes e_i^* \overset{\circ}{=} 0$. From Lemma 2.3, we see

$$s_3 = E_{j^{-1}l^{-1}} E_{lj^{-1}} \, [E_{jl}, E_{ik}] \, E_{lj} E_{j^{-1}l} \cdot E_{j^{-1}l^{-1}} \, [E_{lj^{-1}}, E_{ik}] \, E_{j^{-1}l} \cdot [E_{j^{-1}l^{-1}}, E_{ik}]$$

and hence $s_3 \otimes e_k^* \overset{\circ}{=} 0$. Substituting these results into (46), we obtain

$$r_{ik}(l^{-1}) \otimes e_k^* \equiv r_{ik}(j) \otimes e_k^* + r_{ij}(l) \otimes e_j^*.$$

- $(x_l, x_k, x_i^{-1}, x_j, x_k) \otimes e_j^*$

  For distinct i, j, k and l, the equation $(x_l, x_k, x_i^{-1}, x_j, x_k) \otimes e_j^*$ is given by

$$\begin{aligned}r_{i^{-1}j}(l) \otimes e_j^* = s_1^{-1} \otimes e_j^* &+ (w_{lk}{}^{-1} E_{i^{-1}k} w_{lk} \; s_2^{-1} \; w_{lk}{}^{-1} E_{i^{-1}k}{}^{-1} w_{lk}) \otimes e_j^* \\ &+ (w_{lk}{}^{-1} r_{i^{-1}j}(k) w_{lk}) \otimes e_j^* \\ &+ (w_{lk}{}^{-1} E_{i^{-1}j} E_{kj} w_{lk} \; s_1 \; w_{lk}{}^{-1} E_{kj}^{-1} E_{i^{-1}j}{}^{-1} w_{lk}) \otimes e_j^* \\ &+ (w_{lk}{}^{-1} E_{i^{-1}j} w_{lk} \; s_2 \; w_{lk}{}^{-1} E_{i^{-1}j}{}^{-1} w_{lk}) \otimes e_j^* + s_3 \otimes e_j^*\end{aligned}$$

where

$$\begin{aligned}&s_1 := (w_{lk}{}^{-1} E_{i^{-1}k^{-1}} w_{lk})^{-1} E_{i^{-1}l}{}^{-1}, \quad s_2 := (w_{lk}{}^{-1} E_{kj^{-1}} w_{lk})^{-1} E_{lj}{}^{-1}, \\ &s_3 := (w_{lk}{}^{-1} E_{i^{-1}j^{-1}} w_{lk})^{-1} E_{i^{-1}j}{}^{-1}.\end{aligned}$$

Then using Lemmas 3.4 and 3.5, we obtain

$$r_{i^{-1}j}(l) \otimes e_j^* \equiv r_{i^{-1}j}(k) \otimes e_j^* - s_1 \otimes e_l^* + s_1 \otimes e_i^* + s_2 \otimes e_i^* + s_3 \otimes e_j^*. \tag{47}$$

Applying (vi) of Lemma 2.2 to s_1, we have

$$\begin{aligned}s_1 = {} & E_{k^{-1}l} E_{l^{-1}k^{-1}} E_{i^{-1}l} \, r_{i^{-1}l^{-1}}(k)^{-1} \, E_{i^{-1}l^{-1}} E_{l^{-1}k} E_{k^{-1}l^{-1}} \cdot E_{k^{-1}l} E_{i^{-1}l} \, r_{i^{-1}k^{-1}}(l^{-1}) \, E_{i^{-1}l^{-1}} E_{k^{-1}l^{-1}} \\ & \cdot E_{k^{-1}l} E_{i^{-1}l} \, [E_{i^{-1}k^{-1}}, E_{l^{-1}k^{-1}}] \, E_{i^{-1}l^{-1}} E_{k^{-1}l^{-1}} \cdot [E_{k^{-1}l}, E_{i^{-1}l}],\end{aligned}$$

and hence

$$s_1 \otimes e_l^* \overset{\circ}{=} -r_{i^{-1}k}(l^{-1}) \otimes e_k^*, \qquad s_1 \otimes e_i^* \overset{\circ}{=} 0.$$

Using (3), we have

$$s_2 = (w_{lk}{}^{-1} E_{kj} h_{lk} E_{kj^{-1}} w_{lk}) \cdot (w_{lk}{}^{-1} h_{lk}{}^{-1} w_{lk}) \cdot (w_{l^{-1}k^{-1}}{}^{-1} E_{kj^{-1}} w_{l^{-1}k^{-1}})^{-1} E_{lj^{-1}},$$

and hence

$$s_2 \otimes e_i^* \equiv \{(w_{l^{-1}k^{-1}}{}^{-1} E_{kj^{-1}} w_{l^{-1}k^{-1}})^{-1} E_{lj^{-1}}\} \otimes e_i^*.$$

On the other hand, applying (ii) of Lemma 2.2 to $(w_{l^{-1}k^{-1}}{}^{-1} E_{kj^{-1}} w_{l^{-1}k^{-1}})^{-1} E_{lj^{-1}}$, we see

$$\begin{aligned}(w_{l^{-1}k^{-1}}{}^{-1} E_{kj^{-1}} w_{l^{-1}k^{-1}})^{-1} E_{lj^{-1}} &= E_{kl^{-1}} E_{lk} \, [E_{k^{-1}l}, E_{kj}] \, E_{lk^{-1}} E_{kl} \\ &\quad \cdot E_{kl^{-1}} \, r_{lj}(k) \, E_{kl} \cdot E_{lj} E_{kl^{-1}} \, r_{kj^{-1}}(l) \, E_{kl} E_{lj^{-1}},\end{aligned}$$

and hence $s_2 \otimes e_i^* \doteq 0$. From Lemma 2.3, we see

$$s_3 = E_{k^{-1}l}E_{l^{-1}k^{-1}}\,[E_{kl^{-1}}, E_{ij}]\,E_{l^{-1}k}E_{k^{-1}l^{-1}} \cdot E_{k^{-1}l}\,[E_{l^{-1}k^{-1}}, E_{i^{-1}j}]\,E_{k^{-1}l^{-1}} \cdot [E_{k^{-1}l}, E_{i^{-1}j}],$$

and hence $s_3 \otimes e_j^* \doteq 0$. Substituting these results into (47), we obtain

$$r_{i^{-1}k}(l^{-1}) \otimes e_k^* \doteq -r_{i^{-1}j}(k) \otimes e_j^* + r_{i^{-1}j}(l) \otimes e_j^*.$$

- $(x_l^{-1}, x_j, x_i^{-1}, x_k, x_j) \otimes e_k^*$

For distinct i, j, k and l, the equation $(x_l^{-1}, x_j, x_i^{-1}, x_k, x_j) \otimes e_k^*$ is given by

$$\begin{aligned} r_{i^{-1}k}(l^{-1}) \otimes e_k^* = {} & s_1^{-1} \otimes e_k^* \\ & + (w_{l^{-1}j}{}^{-1} E_{i^{-1}j} w_{l^{-1}j}\ s_2^{-1}\ w_{l^{-1}j}{}^{-1} E_{i^{-1}j}{}^{-1} w_{l^{-1}j}) \otimes e_k^* \\ & + (w_{l^{-1}j}{}^{-1} r_{i^{-1}k}(j) w_{l^{-1}j}) \otimes e_k^* \\ & + (w_{l^{-1}j}{}^{-1} E_{i^{-1}k} E_{jk} w_{l^{-1}j}\ s_1\ w_{l^{-1}j}{}^{-1} E_{jk}{}^{-1} E_{i^{-1}k}{}^{-1} w_{l^{-1}j}) \otimes e_k^* \\ & + (w_{l^{-1}j}{}^{-1} E_{i^{-1}k} w_{l^{-1}j}\ s_2\ w_{l^{-1}j}{}^{-1} E_{i^{-1}k}{}^{-1} w_{l^{-1}j}) \otimes e_k^* + s_3 \otimes e_k^* \end{aligned}$$

where

$$s_1 := (w_{l^{-1}j}{}^{-1} E_{i^{-1}j^{-1}} w_{l^{-1}j})^{-1} E_{i^{-1}l}, \quad s_2 := (w_{l^{-1}j}{}^{-1} E_{jk^{-1}} w_{l^{-1}j})^{-1} E_{l^{-1}k}{}^{-1},$$
$$s_3 := (w_{l^{-1}j}{}^{-1} E_{i^{-1}k^{-1}} w_{l^{-1}j})^{-1} E_{i^{-1}k}{}^{-1}.$$

Then using Lemmas 3.4 and 3.5, we obtain

$$r_{i^{-1}k}(l^{-1}) \otimes e_k^* \equiv r_{i^{-1}k}(j) \otimes e_k^* + s_1 \otimes e_i^* + s_1 \otimes e_l^* + s_2 \otimes e_i^* + s_3 \otimes e_k^*. \tag{48}$$

Applying (vi) of Lemma 2.2 to s_1, we have

$$\begin{aligned} s_1 = {} & E_{j^{-1}l^{-1}} E_{lj^{-1}} E_{i^{-1}l^{-1}}\, r_{i^{-1}l}(j)^{-1}\, E_{i^{-1}l} E_{lj} E_{j^{-1}l} \\ & \cdot E_{j^{-1}l^{-1}} E_{i^{-1}l^{-1}}\, r_{i^{-1}j^{-1}}(l)\, E_{i^{-1}l} E_{j^{-1}l} \\ & \cdot E_{j^{-1}l^{-1}} E_{i^{-1}l^{-1}}\, [E_{i^{-1}j^{-1}}, E_{lj^{-1}}]\, E_{i^{-1}l} E_{j^{-1}l} \\ & \cdot [E_{j^{-1}l^{-1}}, E_{i^{-1}l^{-1}}] \end{aligned}$$

and hence

$$s_1 \otimes e_l^* \doteq r_{i^{-1}j}(l) \otimes e_j^*, \qquad s_1 \otimes e_i^* \doteq 0.$$

Using (3), we have

$$s_2 = (w_{l^{-1}j}{}^{-1} E_{jk} h_{l^{-1}j} E_{jk^{-1}} w_{l^{-1}j}) \cdot (w_{l^{-1}j}{}^{-1} h_{l^{-1}j}{}^{-1} w_{l^{-1}j}) \cdot (w_{lj^{-1}}{}^{-1} E_{jk^{-1}} w_{lj^{-1}})^{-1} E_{l^{-1}k^{-1}},$$

and hence

$$s_2 \otimes e_i^* \equiv \{(w_{lj^{-1}}{}^{-1} E_{jk^{-1}} w_{lj^{-1}})^{-1} E_{l^{-1}k^{-1}}\} \otimes e_i^*.$$

On the other hand, applying (ii) of Lemma 2.2 to $(w_{lj^{-1}}{}^{-1} E_{jk^{-1}} w_{lj^{-1}})^{-1} E_{l^{-1}k^{-1}}$, we see

$$\begin{aligned} (w_{lj^{-1}}{}^{-1} E_{jk^{-1}} w_{lj^{-1}})^{-1} E_{l^{-1}k^{-1}} = {} & E_{jl} E_{l^{-1}j}\, [E_{j^{-1}l^{-1}}, E_{jk}]\, E_{l^{-1}j^{-1}} E_{jl^{-1}} \\ & \cdot E_{jl}\, r_{l^{-1}k}(j)\, E_{jl^{-1}} \cdot E_{l^{-1}k} E_{jl}\, r_{jk^{-1}}(l^{-1})\, E_{jl^{-1}} E_{l^{-1}k^{-1}}, \end{aligned}$$

and hence $s_2 \otimes e_i^* \doteq 0$. From Lemma 2.3, we see

$$s_3 = E_{j^{-1}l^{-1}} E_{lj^{-1}}\, [E_{jl}, E_{i^{-1}k}]\, E_{lj} E_{j^{-1}l} \cdot E_{j^{-1}l^{-1}}\, [E_{lj^{-1}}, E_{i^{-1}k}]\, E_{j^{-1}l} \cdot [E_{j^{-1}l^{-1}}, E_{i^{-1}k}]$$

and hence $s_3 \otimes e_k^* \doteq 0$. Substituting these results into (48), we obtain

$$r_{i^{-1}k}(l^{-1}) \otimes e_k^* \equiv r_{i^{-1}k}(j) \otimes e_k^* + r_{i^{-1}j}(l) \otimes e_j^*.$$

• $(x_i, x_l^{-1}, x_l, x_j, x_k) \otimes e_j^*$

For distinct i, j, k and l, the equation $(x_i, x_l^{-1}, x_l, x_j, x_k) \otimes e_j^*$ is given by

$$\begin{aligned} r_{i^{-1}j}(k) \otimes e_j^* = s_1^{-1} \otimes e_j^* &+ ({w_{il^{-1}}}^{-1} E_{lk} w_{il^{-1}}\ s_2^{-1}\ {w_{il^{-1}}}^{-1} {E_{lk}}^{-1} w_{il^{-1}}) \otimes e_j^* \\ &+ ({w_{il^{-1}}}^{-1} r_{lj}(k) w_{il^{-1}}) \otimes e_j^* \\ &+ ({w_{il^{-1}}}^{-1} E_{lj} E_{kj} w_{il^{-1}}\ s_1\ {w_{il^{-1}}}^{-1} {E_{kj}}^{-1} {E_{lj}}^{-1} w_{il^{-1}}) \otimes e_j^* \\ &+ ({w_{il^{-1}}}^{-1} E_{lj} w_{il^{-1}}\ s_2\ {w_{il^{-1}}}^{-1} {E_{lj}}^{-1} w_{il^{-1}}) \otimes e_j^* + s_3 \otimes e_j^* \end{aligned}$$

where

$$s_1 := ({w_{il^{-1}}}^{-1} E_{lk^{-1}} w_{il^{-1}})^{-1} {E_{i^{-1}k}}^{-1}, \quad s_2 := ({w_{il^{-1}}}^{-1} E_{kj^{-1}} w_{il^{-1}})^{-1} {E_{kj}}^{-1},$$
$$s_3 := ({w_{il^{-1}}}^{-1} E_{lj^{-1}} w_{il^{-1}})^{-1} {E_{i^{-1}j}}^{-1}.$$

Then using Lemmas 3.4 and 3.5, we obtain

$$r_{i^{-1}j}(k) \otimes e_j^* \equiv r_{lj}(k) \otimes e_j^* + s_1 \otimes e_i^* - s_1 \otimes e_k^* + s_2 \otimes e_i^* + s_3 \otimes e_j^*. \tag{49}$$

First applying (ii) of Lemma 2.2 to s_1, we have

$$\begin{aligned} s_1 &= ({w_{il^{-1}}}^{-1} E_{lk^{-1}} w_{il^{-1}})^{-1} E_{i^{-1}k^{-1}} \\ &= E_{li} E_{i^{-1}l} \left[E_{l^{-1}i^{-1}}, E_{lk}\right] E_{i^{-1}l^{-1}} E_{li^{-1}} \cdot E_{li}\, r_{i^{-1}k}(l)\, E_{li^{-1}} \\ &\quad \cdot E_{i^{-1}k} E_{li}\, r_{lk^{-1}}(i^{-1})\, E_{li^{-1}} E_{i^{-1}k^{-1}}, \end{aligned}$$

and hence

$$s_1 \otimes e_k^* \doteq r_{i^{-1}k}(l) \otimes e_k^* - r_{lk}(i^{-1}) \otimes e_k^*, \qquad s_1 \otimes e_i^* \doteq 0.$$

By the same argument, we have

$$s_3 \otimes e_j^* \doteq r_{i^{-1}j}(l) \otimes e_j^* - r_{lj}(i^{-1}) \otimes e_j^*.$$

On the other hand, applying Lemma 2.3 to s_2, we have

$$\begin{aligned} s_2 &= ({w_{il^{-1}}}^{-1} E_{kj^{-1}} w_{il^{-1}})^{-1} E_{kj^{-1}} \\ &= E_{li} E_{i^{-1}i} \left[E_{l^{-1}i^{-1}}, E_{kj}\right] E_{i^{-1}l^{-1}} E_{li^{-1}} \cdot E_{li} \left[E_{i^{-1}l}, E_{kj}\right] E_{li^{-1}} \cdot \left[E_{li}, E_{kj}\right], \end{aligned}$$

and hence

$$s_2 \otimes e_i^* \doteq 0.$$

Substituting these results into (49), we obtain

$$\begin{aligned} r_{i^{-1}j}(k) \otimes e_j^* \doteq r_{lj}(k) \otimes e_j^* &- r_{i^{-1}k}(l) \otimes e_k^* \\ &+ r_{lk}(i^{-1}) \otimes e_k^* + r_{i^{-1}j}(l) \otimes e_j^* - r_{lj}(i^{-1}) \otimes e_j^*. \end{aligned}$$

6.3. **The reduction of $h_{ij} \otimes e_p^*$.**

Here we give the details with respect to the reduction of $h_{ij} \otimes e_p^*$.

• $h_{ij} \otimes e_p^*$ for $p \neq i, j$

From Lemma 2.3, we have

$$\begin{aligned} \left[{w_{ij}}^{-1}, E_{kl}\right] &= ({w_{ij}}^{-1} E_{kl^{-1}} w_{ij})^{-1} E_{kl^{-1}} \\ &= E_{j^{-1}i} E_{i^{-1}j^{-1}} \left[E_{ji^{-1}}, E_{kl}\right] E_{i^{-1}j} E_{j^{-1}i^{-1}} \\ &\quad \cdot E_{j^{-1}i} \left[E_{i^{-1}j^{-1}}, E_{kl}\right] E_{j^{-1}i^{-1}} \cdot \left[E_{j^{-1}i}, E_{kl}\right], \end{aligned}$$

and hence, tensoring each terms of the equation above with e_l^*,

$$[{w_{ij}}^{-1}, E_{kl}] \otimes e_l^* \equiv r_{ji}(\cdot) \otimes e_k^* + r_{i^{-1}j}(\cdot) \otimes e_k^* - r_{j^{-1}i}(\cdot) \otimes e_k^*. \tag{50}$$

On the other hand, observing (3), we have

$$[{w_{ij}}^{-1}, E_{kl}] = ({w_{ij}}^{-1} E_{kl} {h_{ij}}^{-1} E_{kl^{-1}} w_{ij}) \cdot ({w_{ij}}^{-1} h_{ij} w_{ij}) \cdot [w_{i^{-1}j}, E_{kl}].$$

Then

$$\begin{aligned} ({w_{ij}}^{-1} h_{ij} w_{ij}) \otimes e_l^* &\equiv h_{ij} \otimes e_l^*, \\ ({w_{ij}}^{-1} E_{kl} {h_{ij}}^{-1} E_{kl^{-1}} w_{ij}) \otimes e_l^* &\equiv -h_{ij} \otimes e_l^* + h_{ij} \otimes e_k^*, \end{aligned}$$

From Lemma 2.3, we have

$$\begin{aligned} [w_{i^{-1}j}, E_{kl}] &= ({w_{i^{-1}j^{-1}}}^{-1} E_{kl^{-1}} w_{i^{-1}j^{-1}})^{-1} E_{kl^{-1}} \\ &= E_{ji^{-1}} E_{ij} \, [E_{j^{-1}i}, E_{kl}] \, E_{ij^{-1}} E_{ji} \\ &\qquad \cdot E_{ji^{-1}} \, [E_{ij}, E_{kl}] \, E_{ji} \cdot [E_{ji^{-1}}, E_{kl}], \end{aligned}$$

and hence, tensoring each terms of the equation above with e_l^*,

$$[w_{i^{-1}j}, E_{kl}] \otimes e_l^* \equiv -r_{j^{-1}i}(\cdot) \otimes e_k^* - r_{ij}(\cdot) \otimes e_k^* + r_{ji}(\cdot) \otimes e_k^*.$$

Therefore

$$[{w_{ij}}^{-1}, E_{kl}] \otimes e_l^* \equiv h_{ij} \otimes e_k^* - r_{j^{-1}i}(\cdot) \otimes e_k^* - r_{ij}(\cdot) \otimes e_k^* + r_{ji}(\cdot) \otimes e_k^*. \tag{51}$$

Comparing (50) with (51), we have

$$h_{ij} \otimes e_k^* \equiv r_{ij}(\cdot) \otimes e_k^* + r_{i^{-1}j}(\cdot) \otimes e_k^*.$$

- $\{x_k, x_i, x_i, x_j\} \otimes e_j^*$

  For distinct i, j and k, the equation $\{x_k, x_i, x_i, x_j\} \otimes e_j^*$ is given by

$$\begin{aligned} h_{kj} \otimes e_j^* = t_3 \otimes e_j^* &+ ({w_{ki}}^{-1} E_{ji} w_{ki} \; t_2 \; {w_{ki}}^{-1} {E_{ji}}^{-1} w_{ki}) \otimes e_j^* \\ &+ ({w_{ki}}^{-1} E_{ji} E_{i^{-1}j} w_{ki} \; t_1 \; {w_{ki}}^{-1} {E_{i^{-1}j}}^{-1} {E_{ji}}^{-1} w_{ki}) \otimes e_j^* \\ &+ ({w_{ki}}^{-1} h_{ij} w_{ki}) \otimes e_j^* \\ &+ ({w_{ki}}^{-1} {E_{j^{-1}i}}^{-1} {E_{ij}}^{-1} w_{ki} \; t_4 \; {w_{ki}}^{-1} E_{ij} E_{j^{-1}i} w_{ki}) \otimes e_j^* \\ &+ ({w_{ki}}^{-1} {E_{j^{-1}i}}^{-1} w_{ki} \; t_5 \; {w_{ki}}^{-1} E_{j^{-1}i} w_{ki}) \otimes e_j^* + t_6 \otimes e_j^* \end{aligned}$$

where

$$\begin{aligned} t_1 &:= E_{j^{-1}k^{-1}} (w_{ki}^{-1} E_{j^{-1}i^{-1}} w_{ki})^{-1}, & t_2 &:= E_{k^{-1}j} (w_{ki}^{-1} E_{i^{-1}j} w_{ki})^{-1}, \\ t_3 &:= E_{jk} (w_{ki}^{-1} E_{ji} w_{ki})^{-1}, & t_4 &:= (w_{ki}^{-1} E_{ji^{-1}} w_{ki})^{-1} E_{jk^{-1}}, \\ t_5 &:= (w_{ki}^{-1} E_{ij} w_{ki})^{-1} E_{kj}, & t_6 &:= (w_{ki}^{-1} E_{j^{-1}i} w_{ki})^{-1} E_{j^{-1}k}. \end{aligned}$$

Observing Lemma 2.2, we see that all t_m, $(1 \leq m \leq 6)$, except for t_5 belong to the normal closure of the relators of (R2-1), ... , (R3-4). Hence, using Lemmas 3.4 and Lemmas 3.5, we obtain

$$h_{kj} \otimes e_j^* \doteq t_5 \otimes e_j^* + h_{ij} \otimes e_j^*. \tag{52}$$

On the other hand, from (3), we have

$$\begin{aligned} t_5 &= ({w_{ki}}^{-1}E_{ij}w_{ki})^{-1}E_{kj} \\ &= ({w_{ki}}^{-1}E_{ij^{-1}}h_{ki}E_{ij}w_{ki})\cdot({w_{ki}}^{-1}{h_{ki}}^{-1}w_{ki})\cdot({w_{k^{-1}i^{-1}}}^{-1}E_{ij}w_{k^{-1}i^{-1}})^{-1}E_{kj}, \end{aligned}$$

and hence

$$t_5\otimes e_j^* \equiv \{({w_{k^{-1}i^{-1}}}^{-1}E_{ij}w_{k^{-1}i^{-1}})^{-1}E_{kj}\}\otimes e_j^* + h_{ki}\otimes e_i^*.$$

Since the element $\{({w_{k^{-1}i^{-1}}}^{-1}E_{ij}w_{k^{-1}i^{-1}})^{-1}E_{kj}\}\in\bar{R}$ belongs to the normal closure of the relators of (R2-1), ... , (R3-4) by Lemma 2.2, we see $t_5\otimes e_j^*\doteq h_{ki}\otimes e_i^*$. Substituting this results into (52), we have

$$h_{ij}\otimes e_j^* \doteq h_{kj}\otimes e_j^* - h_{ki}\otimes e_i^*.$$

- $\{x_k, x_i^{-1}, x_i, x_j\}\otimes e_k^*$

For distinct i, j and k, the equation $\{x_k, x_i^{-1}, x_i, x_j\}\otimes e_k^*$ is given by

$$\begin{aligned} h_{k^{-1}j}\otimes e_k^* = t_3\otimes e_k^* &+ ({w_{ki^{-1}}}^{-1}E_{ji}w_{ki^{-1}}\ t_2\ {w_{ki^{-1}}}^{-1}{E_{ji}}^{-1}{w_{ki^{-1}}})\otimes e_k^* \\ &+ ({w_{ki^{-1}}}^{-1}E_{ji}E_{i^{-1}j}w_{ki^{-1}}\ t_1\ {w_{ki^{-1}}}^{-1}{E_{i^{-1}j}}^{-1}{E_{ji}}^{-1}{w_{ki^{-1}}})\otimes e_k^* \\ &+ ({w_{ki^{-1}}}^{-1}h_{ij}w_{ki^{-1}})\otimes e_k^* \\ &+ ({w_{ki^{-1}}}^{-1}E_{j^{-1}i}{}^{-1}{E_{ij}}^{-1}{w_{ki^{-1}}}\ t_4\ {w_{ki^{-1}}}^{-1}E_{ij}E_{j^{-1}i}w_{ki^{-1}})\otimes e_k^* \\ &+ ({w_{ki^{-1}}}^{-1}E_{j^{-1}i}{}^{-1}w_{ki^{-1}}\ t_5\ {w_{ki^{-1}}}^{-1}E_{j^{-1}i}w_{ki^{-1}})\otimes e_k^* + t_6\otimes e_k^* \end{aligned}$$

where

$$\begin{aligned} t_1 &:= E_{j^{-1}k}(w_{ki^{-1}}^{-1}E_{j^{-1}i^{-1}}w_{ki^{-1}})^{-1}, & t_2 &:= E_{kj}(w_{ki^{-1}}^{-1}E_{i^{-1}j}w_{ki^{-1}})^{-1}, \\ t_3 &:= E_{jk^{-1}}(w_{ki^{-1}}^{-1}E_{ji}w_{ki^{-1}})^{-1}, & t_4 &:= (w_{ki^{-1}}^{-1}E_{ji^{-1}}w_{ki^{-1}})^{-1}E_{jk}, \\ t_5 &:= (w_{ki^{-1}}^{-1}E_{ij}w_{ki^{-1}})^{-1}E_{k^{-1}j}, & t_6 &:= (w_{ki^{-1}}^{-1}E_{j^{-1}i^{-1}}w_{ki^{-1}})^{-1}E_{j^{-1}k^{-1}}. \end{aligned}$$

Observing Lemma 2.2, we see that all t_m, $(1\le m\le 6)$, except for t_2 belong to the normal closure of the relators of (R2-1), ... , (R3-4). Hence, using Lemmas 3.4 and Lemmas 3.5, we obtain

$$h_{k^{-1}j}\otimes e_k^* \doteq t_2\otimes e_k^* + t_2\otimes e_j^* - h_{ij}\otimes e_i^*. \tag{53}$$

From (3), we have

$$\begin{aligned} t_2^{-1} &= ({w_{ki^{-1}}}^{-1}E_{i^{-1}j^{-1}}w_{ki^{-1}})^{-1}E_{kj^{-1}} \\ &= ({w_{ki^{-1}}}^{-1}E_{i^{-1}j}h_{ki^{-1}}E_{i^{-1}j^{-1}}w_{ki^{-1}})\cdot({w_{ki^{-1}}}^{-1}{h_{ki^{-1}}}^{-1}w_{ki^{-1}}) \\ &\quad\cdot({w_{k^{-1}i}}^{-1}E_{i^{-1}j^{-1}}w_{k^{-1}i})^{-1}E_{kj^{-1}}, \end{aligned}$$

and hence

$$\begin{aligned} t_2^{-1}\otimes e_k^* &\equiv \{({w_{k^{-1}i}}^{-1}E_{i^{-1}j^{-1}}w_{k^{-1}i})^{-1}E_{kj^{-1}}\}\otimes e_k^*, \\ t_2^{-1}\otimes e_j^* &\equiv \{({w_{k^{-1}i}}^{-1}E_{i^{-1}j^{-1}}w_{k^{-1}i})^{-1}E_{kj^{-1}}\}\otimes e_j^* - h_{ki^{-1}}\otimes e_i^*. \end{aligned}$$

Here, from Lemma 2.4, we have

$$\begin{aligned} h_{ki^{-1}}\otimes e_i^* &\equiv ({w_{ki^{-1}}}^{-1}{h_{ki}}^{-1}w_{ki})\otimes e_i^* \equiv -h_{ki}\otimes e_k^*, \\ h_{k^{-1}j}\otimes e_k^* &\equiv ({w_{kj}}^{-1}k_{kj}w_{kj})\otimes e_k^* = h_{kj}\otimes e_j^*, \end{aligned}$$

and see that the element $\{(w_{k^{-1}i}{}^{-1}E_{i^{-1}j^{-1}}w_{k^{-1}i})^{-1}E_{kj^{-1}}\} \in \bar{R}$ belongs to the normal closure of the relators of (R2-1), ... , (R3-4) by Lemma 2.2. Therefore we obtain

$$t_2 \otimes e_j^* \doteq -h_{ki} \otimes e_k^*, \qquad t_2 \otimes e_k^* \doteq 0.$$

Substituting this results into (53), we obtain

$$h_{ij} \otimes e_i^* \doteq -h_{ki} \otimes e_k^* - h_{kj} \otimes e_j^*.$$

- $\{x_k, x_i, x_i, x_j\} \otimes e_k^*$

For distinct i, j and k, the equation $\{x_k, x_i, x_i, x_j\} \otimes e_k^*$ is given by

$$\begin{aligned} h_{kj} \otimes e_k^* = t_3 \otimes e_k^* &+ (w_{ki}{}^{-1}E_{ji}w_{ki}\ t_2\ w_{ki}{}^{-1}E_{ji}{}^{-1}w_{ki}) \otimes e_k^* \\ &+ (w_{ki}{}^{-1}E_{ji}E_{i^{-1}j}w_{ki}\ t_1\ w_{ki}{}^{-1}E_{i^{-1}j}{}^{-1}E_{ji}{}^{-1}w_{ki}) \otimes e_k^* \\ &+ (w_{ki}{}^{-1}h_{ij}w_{ki}) \otimes e_k^* \\ &+ (w_{ki}{}^{-1}E_{j^{-1}i}{}^{-1}E_{ij}{}^{-1}w_{ki}\ t_4\ w_{ki}{}^{-1}E_{ij}E_{j^{-1}i}w_{ki}) \otimes e_k^* \\ &+ (w_{ki}{}^{-1}E_{j^{-1}i}{}^{-1}w_{ki}\ t_5\ w_{ki}{}^{-1}E_{j^{-1}i}w_{ki}) \otimes e_k^* + t_6 \otimes e_k^* \end{aligned}$$

where

$$\begin{aligned} t_1 &:= E_{j^{-1}k^{-1}}(w_{ki}^{-1}E_{j^{-1}i^{-1}}w_{ki})^{-1}, & t_2 &:= E_{k^{-1}j}(w_{ki}^{-1}E_{i^{-1}j}w_{ki})^{-1}, \\ t_3 &:= E_{jk}(w_{ki}^{-1}E_{ji}w_{ki})^{-1}, & t_4 &:= (w_{ki}^{-1}E_{ji^{-1}}w_{ki})^{-1}E_{jk^{-1}}, \\ t_5 &:= (w_{ki}^{-1}E_{ij}w_{ki})^{-1}E_{kj}, & t_6 &:= (w_{ki}^{-1}E_{j^{-1}i}w_{ki})^{-1}E_{j^{-1}k}. \end{aligned}$$

Observing Lemma 2.2, we see that all t_m, $(1 \leq m \leq 6)$, except for t_5 belong to the normal closure of the relators of (R2-1), ... , (R3-4). Hence, using Lemmas 3.4 and Lemmas 3.5, we obtain

$$h_{kj} \otimes e_k^* \doteq t_5 \otimes e_k^* - t_5 \otimes e_j^* + h_{ij} \otimes e_i^*. \tag{54}$$

On the other hand, from (3), we have

$$\begin{aligned} t_5 &= (w_{ki}{}^{-1}E_{ij}w_{ki})^{-1}E_{kj} \\ &= (w_{ki}{}^{-1}E_{ij^{-1}}h_{ki}E_{ij}w_{ki}) \cdot (w_{ki}{}^{-1}h_{ki}{}^{-1}w_{ki}) \cdot (w_{k^{-1}i^{-1}}{}^{-1}E_{ij}w_{k^{-1}i^{-1}})^{-1}E_{kj}, \end{aligned}$$

and hence

$$\begin{aligned} t_5 \otimes e_k^* &\equiv \{(w_{k^{-1}i^{-1}}{}^{-1}E_{ij}w_{k^{-1}i^{-1}})^{-1}E_{kj}\} \otimes e_k^* \\ t_5 \otimes e_j^* &\equiv \{(w_{k^{-1}i^{-1}}{}^{-1}E_{ij}w_{k^{-1}i^{-1}})^{-1}E_{kj}\} \otimes e_j^* + h_{ki} \otimes e_i^*. \end{aligned}$$

Since the element $\{(w_{k^{-1}i^{-1}}{}^{-1}E_{ij}w_{k^{-1}i^{-1}})^{-1}E_{kj}\} \in \bar{R}$ belongs to the normal closure of the relators of (R2-1), ... , (R3-4) by Lemma 2.2, we see

$$\begin{aligned} t_5 \otimes e_k^* &\doteq 0 \\ t_5 \otimes e_j^* &\doteq h_{ki} \otimes e_i^*. \end{aligned}$$

Substituting this results into (54), we have

$$h_{ij} \otimes e_i^* \doteq h_{kj} \otimes e_k^* + h_{ki} \otimes e_i^*.$$

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