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LIPSCHITZ STABILITY IN DETERMINING DENSITY AND TWO LAMÉ COEFFICIENTS

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Abstract. We consider an inverse problem of determining spatially varying density and two Lamé coefficients in a non-stationary isotropic elastic equation by a single measuerment of data on the whole lateral boundary. We prove the Lipschitz stability provided that initial data are suitablty chosen. The proof is based on a Carleman estimate which can be obtained by the decomposition of the Lamé system into the rotation and the divergence components.

§1. Introduction and the main result.

We consider the three dimensional isotropic non-stationary Lamé system:

$$\rho(x)\partial_t^2\mathbf{u}(x,t) - (L_{\lambda,\mu}\mathbf{u})(x,t) = \mathbf{f}(x,t),$$

$$(x,t) \in Q \equiv \Omega \times (-T,T), \tag{1.1}$$

where

$$(L_{\lambda,\mu}\mathbf{v})(x) \equiv \mu(x)\Delta\mathbf{v}(x) + (\mu(x)+\lambda(x))\nabla \mathrm{div}\,\mathbf{v}(x)$$

$$+(\mathrm{div}\,\mathbf{v}(x))\nabla\lambda(x) + (\nabla\mathbf{v} + (\nabla\mathbf{v})^T)\nabla\mu(x), \qquad x\in\Omega \tag{1.2}$$

(e.g., Gurtin [12]). Throughout this paper, $\Omega\subset\mathbb{R}^3$ is a bounded domain whose boundary $\partial\Omega$ is of class C^3, t and $x=(x_1,x_2,x_3)$ denote the time variable and the

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spatial variable respectively, and $\mathbf{u} = (u_1, u_2, u_3)^T$ where $\cdot^T$ denotes the transpose of matrices,

$$\partial_j \phi = \frac{\partial \phi}{\partial x_j}, \quad j = 1, 2, 3, \quad \partial_t \phi = \frac{\partial \phi}{\partial t}.$$

For $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \{\mathbb{N} \cup \{0\}\}^3$, we set $\partial_x^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3}$ and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, and $\partial_{x,t}^\alpha$ is similarly defined. We set $\nabla \mathbf{v} = (\partial_k v_j)_{1 \le j,k \le 3}$, $\nabla_{x,t} \mathbf{v} = (\nabla \mathbf{v}, \partial_t \mathbf{v})$ for a vector function $\mathbf{v} = (v_1, v_2, v_3)^T$. Moreover the coefficients ρ, λ, μ under consideration, satisfy

$$\rho, \lambda, \mu \in C^2(\overline{\Omega}), \quad \rho(x) > 0, \ \mu(x) > 0, \ \lambda(x) + \mu(x) > 0 \quad \text{for } x \in \overline{\Omega}. \tag{1.3}$$

Let $\mathbf{u} = \mathbf{u}(\lambda, \mu, \rho; \mathbf{p}, \mathbf{q})(x, t)$ be sufficiently smooth and satisfy

$$\rho(x)(\partial_t^2 \mathbf{u})(x, t) = (L_{\lambda,\mu} \mathbf{u})(x, t), \quad (x, t) \in Q, \tag{1.4}$$

$$\mathbf{u}(x, 0) = \mathbf{p}(x), \quad (\partial_t \mathbf{u})(x, 0) = \mathbf{q}(x), \quad x \in \Omega. \tag{1.5}$$

We consider

Inverse problem with finite measurements. Let $\omega \subset \Omega$ be a suitable subdomain and let $\mathbf{p}_j, \mathbf{q}_j$, $1 \le j \le \mathcal{N}$, be appropriately given. Then determine $\lambda(x)$, $\mu(x)$, $\rho(x)$, $x \in \Omega$, by

$$\mathbf{u}(\lambda, \mu, \rho; \mathbf{p}_j, \mathbf{q}_j)|_{\omega \times (-T,T)}. \tag{1.6}$$

As for the inverse problem of determining some (or all) of λ, μ and ρ with finite measurements, we can first refer to:

Isakov [26] where the author proved the uniqueness in determining a single coefficient $\rho(x)$, using four measurements (i.e., $\mathcal{N} = 4$).

Ikehata, Nakamura and Yamamoto [14] which reduced the number $\mathcal{N}$ of measurements to three for determining ρ.

Imanuvilov, Isakov and Yamamoto [16] which proved conditional stability and the uniqueness in the determination of the three functions $\lambda(x)$, $\mu(x)$, $\rho(x)$, $x \in \Omega$, with two measurements (i.e., $\mathcal{N} = 2$). See also Isakov [30].

Imanuvilov and Yamamoto [23] - [25] which reduced $\mathcal{N} = 2$ to $\mathcal{N} = 1$ (i.e., a single measurement) in determining all of λ, μ, ρ by a single measurement $\mathbf{u}|_{\omega\times(-T,T)}$, and established conditional stability of Hölder type by means of an H^{-1}-Carleman estimate. See also [21].

As for similar inverse problems for the Lamé system with residual stress, see Isakov, Wang and Yamamoto [31], Lin and Wang [44].

Our method is based on the tool of Carleman estimates, which was originally introduced in the field of coefficient inverse problems by Bukhgeim and Klibanov [8] simultaneously and independently on each other for the proofs of global uniqueness and stability theorems for these problems. Also see Klibanov [36]. In particular, for the Lamé system, we use a modification of the method in [8] by Imanuvilov and Yamamoto [23]. In [23], only a Hölder stability estimate is proved, but by the ideas in Klibanov and Timonov [40], Klibanov and Yamamoto [41], we can prove the Lipschitz stability for our inverse problem with $\mathcal{N} = 1$. For a related technique, see Chapter 3.5 in Klibanov and Timonov [39]. In [16] and [23], an H^{-1}-Carleman estimate is a key but requires more technical details. Here we will use a Carleman estimate for the Lamé system which is derived from a usual L^2-Carleman estimate for a scalar hyperbolic equation.

Thus the advantages of this paper are:

(1) the Lipschitz stability in our inverse problem with $\mathcal{N} = 1$.

(2) use of a conventional Carleman estimate.

On the other hand, for (2) we have to choose a neighbourhood ω of $\partial\Omega$, although it is sufficient that ω is a neighbourhood of a sufficiently large subboundary ([23] - [25]). Then, $\mathbf{u}(\lambda,\mu,\rho;\mathbf{p},\mathbf{q})(\cdot,t)$, $t\in(-T,T)$, is given in a neighbourhood of $\partial\Omega$, so that we do not directly assign boundary values but the observation data in $\omega\times(-T,T)$ include information of boundary values.

For the statement of the main result, we introduce notations and an admissible set of unknown coefficients λ, μ, ρ. Set

$$d=\Big(\sup_{x\in\Omega}|x-x_0|^2-\inf_{x\in\Omega}|x-x_0|^2\Big)^{\frac{1}{2}}, \tag{1.7}$$

where $x_0\notin\overline{\Omega}$ is arbitrarily fixed. Let $M_0\ge 0$, $0<\theta_0\le 1$ and $\theta_1>0$ be arbitrarily fixed and let us introduce the conditions on a scalar function β:

$$\begin{cases} \beta(x)\ge\theta_1>0, \quad x\in\overline{\Omega}, \\ \|\beta\|_{C^3(\overline{\Omega})}\le M_0, \quad \dfrac{(\nabla\beta(x)\cdot(x-x_0))}{2\beta(x)}\le 1-\theta_0, \quad x\in\overline{\Omega\setminus\omega}. \end{cases} \tag{1.8}$$

For fixed functions $a^{(\ell)}$, $a_j^{(\ell)}$, $a_{jk}^{(\ell)}$, b, b_j, $1\le\ell\le 2$, $1\le j,k\le 3$ on $\partial\Omega$, we set

$$\mathcal{W}=\mathcal{W}_{M_0,\theta_0,\theta_1}=\Big\{(\lambda,\mu,\rho)\in\{C^3(\overline{\Omega})\}^3;\lambda=a^{(1)},\ \partial_j\lambda=a_j^{(1)},\ \partial_j\partial_k\lambda=a_{jk}^{(1)},$$
$$\mu=a^{(2)},\ \partial_j\mu=a_j^{(2)},\ \partial_j\partial_k\mu=a_{jk}^{(2)}\quad \text{on } \partial\Omega,\quad \frac{\lambda+2\mu}{\rho},\ \frac{\mu}{\rho}\ \text{satisfy (1.8)}\Big\}. \tag{1.9}$$

We choose $\theta>0$ such that

$$\theta+\frac{M_0 d}{\sqrt{\theta_1}}\sqrt{\theta}<\theta_0\theta_1, \quad \theta_1\inf_{x\in\Omega}|x-x_0|^2-\theta\sup_{x\in\Omega}|x-x_0|^2>0. \tag{1.10}$$

Here we note that since $x_0\notin\overline{\Omega}$, such $\theta>0$ exists.

Let E_3 the 3×3 identity matrix. We note that $(L_{\lambda,\mu}\mathbf{p})(x)$ is a 3-column vector for 3-column vector $\mathbf{p}$. Moreover by $\{\mathbf{a}\}_j$ we denote the matrix (or vector) obtained from $\mathbf{a}$ after deleting the j-th row and $\det_j A$ means $\det\{A\}_j$ for a square matrix A. Let (λ,μ,ρ) be an arbitrary element of $\mathcal{W}$.

Now we are ready to state

Theorem. *Let $\omega \subset \Omega$ be a subdomain such that $\partial\omega \supset \partial\Omega$. For $\mathbf{p} = (p_1, p_2, p_3)^T$ and $\mathbf{q} = (q_1, q_2, q_3)^T$, we assume that there exist $j_1, j_2 \in \{1,2,3,4,5,6\}$ such that*

$$det_{j_1} \begin{pmatrix} (L_{\lambda,\mu}\mathbf{p})(x) & (div\,\mathbf{p}(x))E_3 & (\nabla\mathbf{p}(x) + (\nabla\mathbf{p}(x))^T)(x-x_0) \\ (L_{\lambda,\mu}\mathbf{q})(x) & (div\,\mathbf{q}(x))E_3 & (\nabla\mathbf{q}(x) + (\nabla\mathbf{q}(x))^T)(x-x_0) \end{pmatrix} \neq 0, \quad \forall x \in \overline{\Omega}, \tag{1.11}$$

$$det_{j_2} \begin{pmatrix} (L_{\lambda,\mu}\mathbf{p})(x) & \nabla\mathbf{p}(x) + (\nabla\mathbf{p}(x))^T & (div\,\mathbf{p})(x-x_0) \\ (L_{\lambda,\mu}\mathbf{q})(x) & \nabla\mathbf{q}(x) + (\nabla\mathbf{q}(x))^T & (div\,\mathbf{q})(x-x_0) \end{pmatrix} \neq 0,\ \forall x \in \overline{\Omega}, \tag{1.12}$$

and that

$$T > \frac{1}{\sqrt{\theta}} d. \tag{1.13}$$

Then, for any $M_1 > 0$, there exists a constant $C_1 = C_1(\mathcal{W}, M_1, \omega, \Omega, T, \lambda, \mu, \rho) > 0$ such that

$$\begin{aligned} &\|\widetilde{\lambda} - \lambda\|_{H^2(\Omega)} + \|\widetilde{\mu} - \mu\|_{H^2(\Omega)} + \|\widetilde{\rho} - \rho\|_{H^1(\Omega)} \\ \le & C_1 \Bigg(\|\mathbf{u}(\lambda,\mu,\rho;\mathbf{p},\mathbf{q}) - \mathbf{u}(\widetilde{\lambda},\widetilde{\mu},\widetilde{\rho};\mathbf{p},\mathbf{q})\|_{H^5(-T,T;H^2(\omega))} \\ & + \|\mathbf{u}(\lambda,\mu,\rho;\mathbf{p},\mathbf{q}) - \mathbf{u}(\widetilde{\lambda},\widetilde{\mu},\widetilde{\rho};\mathbf{p},\mathbf{q})\|_{H^4(-T,T;H^{\frac{5}{2}}(\omega))} \Bigg), \end{aligned} \tag{1.14}$$

provided that $(\widetilde{\lambda},\widetilde{\mu},\widetilde{\rho}) \in \mathcal{W}$ and

$$\|\mathbf{u}(\lambda,\mu,\rho;\mathbf{p},\mathbf{q})\|_{W^{7,\infty}(Q)}, \quad \|\mathbf{u}(\widetilde{\lambda},\widetilde{\mu},\widetilde{\rho};\mathbf{p},\mathbf{q})\|_{W^{7,\infty}(Q)} \le M_1. \tag{1.15}$$

Inequality (1.14) gives the Lipschitz stability by a single measurement in a neighbourhood of the whole boundary, and after artificial choice (1.11) and (1.12) of initial values, a single measurement yields such stability. Moreover conditions (1.11) and (1.12) depend on a fixed (λ,μ,ρ), so that in our conclusion (1.14), we can not change both (λ,μ,ρ), $(\widetilde{\lambda},\widetilde{\mu},\widetilde{\rho}) \in \mathcal{W}$.

As the following example shows, we can take such $\mathbf{p}$ and $\mathbf{q}$.

Example of p, q satisfying (1.11) and (1.12). For simplicity, we assume that λ, μ are positive constants. Noting that the fifth columns of the matrices in (1.11) and (1.12) have $x-x_0$ as factors, we will take quadratic functions in x. For example, we take

$$\mathbf{p}(x)=\begin{pmatrix}0\\ x_1x_2\\ 0\end{pmatrix},\quad \mathbf{q}(x)=\begin{pmatrix}x_2^2\\ 0\\ x_2^2\end{pmatrix}.$$

Then, by choosing $j_1=6$ and $j_2=5$, we can satisfy (1.11) and (1.12).

We conclude this section with the references to other publications concerning inverse problems by Carleman estimates after the originating paper Bukhgeim and Klibanov [8].

(1) Baudouin and Puel [2], Bukhgeim [6] for an inverse problem of determining potentials in Schrödinger equations,

(2) Imanuvilov and Yamamoto [17], [20], Isakov [27], [28], Klibanov [37] for the corresponding inverse problems for parabolic equations,

(3) Amirov and Yamamoto [1], Bellassoued [3], [4], Bellassoued and Yamamoto [5], Bukhgeim, Cheng, Isakov and Yamamoto [7], Imanuvilov and Yamamoto [18], [19], [22] (especially for conditional stability), Isakov [27] - [29], Isakov and Yamamoto [32], Khaĭdarov [34], [35], Klibanov [36], [37], Klibanov and Timonov [39], [40], Klibanov and Yamamoto [41], Puel and Yamamoto [45], [46], Yamamoto [48] for inverse problems of determining potentials, damping coefficients or the principal terms in scalar hyperbolic equations.

(4) Li [42], Li and Yamamoto [43] for Maxwell's equations.

(5) Yuan and Yamamoto [49] for plate equations.

§2. Proof of Theorem.

We set

$$\psi(x,t) = |x - x_0|^2 - \theta t^2, \quad \varphi(x,t) = e^{\tau\psi(x,t)}, \quad (x,t) \in Q$$

and

$$Q_\omega = \omega \times (-T,T).$$

First, in terms of (1.9) and (1.10), we can deduce the following lemma in the same way as in [23]. Henceforth C, C_j denote constants which are independent of s but dependent on Ω, ω, T and the choice of fixed λ, μ, ρ.

Lemma 2.1. *Let $(\lambda,\mu,\rho) \in \mathcal{W}$ and let (1.10) and (1.13) hold. There exists $\widehat{\tau} > 0$ such that for any $\tau > \widehat{\tau}$, we can choose $s_0 = s_0(\tau) > 0$ and $C_1 = C_1(s_0, \tau_0, \Omega, \omega, T) > 0$ such that*

$$\begin{aligned}&\int_Q (s^4|\mathbf{y}|^2 + s^2|\nabla_{x,t}\mathbf{y}|^2 + \sum_{|\alpha|=2} |\partial_x^\alpha \mathbf{y}|^2\\ &+s^2|\nabla_{x,t}(rot\,\mathbf{y})|^2 + s^4|rot\,\mathbf{y}|^2 + s^2|\nabla_{x,t}(div\,\mathbf{y})|^2 + s^4|div\,\mathbf{y}|^2)e^{2s\varphi}dxdt\\ &\le C_1\int_Q (s|div\,\mathbf{f}|^2 + s|rot\,\mathbf{f}|^2 + s|\mathbf{f}|^2)e^{2s\varphi}dxdt + Ce^{Cs}\|\mathbf{y}\|^2_{H^1(-T,T;H^2(\omega))}, \quad s \ge s_0\end{aligned} \tag{2.1}$$

for any $\mathbf{y} \in H^3(Q)$ such that

$$\rho\partial_t^2\mathbf{y} - L_{\lambda,\mu}\mathbf{y} = \mathbf{f}, \qquad \partial_t^j\mathbf{y}(\cdot, \pm T) = 0, \qquad j = 0,1.$$

The constants in (2.1) can be taken uniformly as long as $(\lambda,\mu,\rho) \in \mathcal{W}$.

Proof. Let us set $v = \mathrm{div}\,\mathbf{y}$ and $\mathbf{w} = \mathrm{rot}\,\mathbf{y}$. Then we have (e.g., Eller, Isakov, Nakamura and Tataru [11], Imanuvilov and Yamamoto [23]):

$$\rho\partial_t^2\mathbf{y} - \mu\Delta\mathbf{y} + Q_1(\mathbf{y}, v) = \mathbf{f} \quad \text{in } Q,$$

$$\rho\partial_t^2 v - (\lambda + 2\mu)\Delta v + Q_2(\mathbf{y}, v, \mathbf{w}) = \mathrm{div}\,\mathbf{f} \quad \text{in } Q$$

and

$$\rho\partial_t^2\mathbf{w} - \mu\Delta\mathbf{w} + Q_3(\mathbf{y}, v, \mathbf{w}) = \mathrm{rot}\,\mathbf{f} \quad \text{in } Q,$$

where $Q_1(\mathbf{y}, v) = \sum_{|\alpha|=1} a_\alpha^{(1)}(x)\partial_x^\alpha \mathbf{y} + \sum_{|\alpha|\le 1} b_\alpha^{(1)}(x)\partial_x^\alpha v$, $Q_j(\mathbf{y}, v, \mathbf{w}) = \sum_{|\alpha|=1} a_\alpha^{(j)}(x)\partial_x^\alpha \mathbf{y} + \sum_{|\alpha|=1} b_\alpha^{(j)}(x)\partial_x^\alpha v + \sum_{|\alpha|=1} c_\alpha^{(j)}(x)\partial_x^\alpha \mathbf{w}$, $j = 2, 3$ and $a_\alpha^{(j)}$, $b_\alpha^{(j)}$, $c_\alpha^{(j)} \in L^\infty(Q)$. Therefore we apply a Carleman estimate by Imanuvilov [15] to the system, so that

$$\begin{aligned}&\int_Q \{s^3(|\mathrm{rot}\,\mathbf{y}|^2 + |\mathrm{div}\,\mathbf{y}|^2 + |\mathbf{y}|^2) + s(|\nabla_{x,t}(\mathrm{rot}\,\mathbf{y})|^2 + |\nabla_{x,t}(\mathrm{div}\,\mathbf{y})|^2 + |\nabla_{x,t}\mathbf{y}|^2)\}e^{2s\varphi}dxdt\\ \le& C\int_Q (|\mathrm{div}\,\mathbf{f}|^2 + |\mathrm{rot}\,\mathbf{f}|^2 + |\mathbf{f}|^2)e^{2s\varphi}dxdt + Ce^{Cs}\|\mathbf{y}\|_{H^1(-T,T;H^1(\omega))} \qquad (2.2)\end{aligned}$$

and

$$\begin{aligned}&\int_Q (s^4|\mathbf{y}|^2 + s^2|\nabla_{x,t}\mathbf{y}|^2)e^{2s\varphi}dxdt\\ \le& C\int_Q (s|\mathrm{div}\,\mathbf{f}|^2 + s|\mathrm{rot}\,\mathbf{f}|^2 + s|\mathbf{f}|^2)e^{2s\varphi}dxdt + Ce^{Cs}\|\mathbf{y}\|^2_{H^1(-T,T;H^1(\omega))}. \qquad (2.3)\end{aligned}$$

Next for all large $s > 0$, we have

$$\begin{aligned}&\Delta(\mathbf{y}e^{s\varphi}) = \nabla(\mathrm{div}\,(\mathbf{y}e^{s\varphi})) - \mathrm{rot}\,(\mathrm{rot}\,(\mathbf{y}e^{s\varphi}))\\ =&\sum_{j=1}^3 \{\{\nabla(\partial_j e^{s\varphi})\}y_j + (\partial_j e^{s\varphi})\nabla y_j\} + s(\nabla\varphi)e^{s\varphi}\mathrm{div}\,\mathbf{y} + e^{s\varphi}\nabla(\mathrm{div}\,\mathbf{y})\\ &+(\mathbf{y}\cdot\nabla)(\nabla e^{s\varphi}) - ((\nabla e^{s\varphi})\cdot\nabla)\mathbf{y}\\ &+(\nabla e^{s\varphi})\mathrm{div}\,\mathbf{y} - \mathbf{y}\mathrm{div}\,(\nabla e^{s\varphi}) - (\nabla e^{s\varphi})\times\mathrm{rot}\,\mathbf{y} - e^{s\varphi}\mathrm{rot}\,(\mathrm{rot}\,\mathbf{y})\\ =&e^{s\varphi}\nabla(\mathrm{div}\,\mathbf{y}) + O(s^2)K_1(\mathbf{y})e^{s\varphi} + O(s)K_2(\nabla\mathbf{y})e^{s\varphi} - (\mathrm{rot}\,(\mathrm{rot}\,\mathbf{y}))e^{s\varphi},\end{aligned}$$

where K_1, K_2 are linear operators. Therefore

$$|\Delta(\mathbf{y}e^{s\varphi})| \le Ce^{s\varphi}\{s^2|\mathbf{y}| + s|\nabla\mathbf{y}| + |\nabla(\mathrm{div}\,\mathbf{y})| + |\nabla(\mathrm{rot}\,\mathbf{y})|\},$$

so that

$$\begin{aligned}&\int_\Omega |\Delta(\mathbf{y}(x,t)e^{s\varphi(x,t)})|^2dx\\ \le& C\int_\Omega (s^4|\mathbf{y}|^2 + s^2|\nabla\mathbf{y}|^2 + |\nabla(\mathrm{div}\,\mathbf{y})|^2 + |\nabla(\mathrm{rot}\,\mathbf{y})|^2)e^{2s\varphi}dx \qquad (2.4)\end{aligned}$$

for any $t \in [-T, T]$. The elliptic regularity and (2.4) yield

$$
\begin{aligned}
&\sum_{|\alpha|=2} \int_\Omega |\partial_x^\alpha (\mathbf{y}(x,t) e^{s\varphi(x,t)})|^2 dx \\
\le & C \int_\Omega (|\Delta(\mathbf{y} e^{s\varphi})|^2 + |\mathbf{y} e^{s\varphi}|^2) dx + C \|\mathbf{y} e^{s\varphi}\|^2_{H^{\frac{3}{2}}(\partial\Omega)} \\
\le & C \int_\Omega (s^4 |\mathbf{y}|^2 + s^2 |\nabla \mathbf{y}|^2 + |\nabla(\operatorname{div} \mathbf{y})|^2 + |\nabla(\operatorname{rot} \mathbf{y})|^2) e^{2s\varphi} dx + C \|\mathbf{y} e^{s\varphi}\|^2_{H^2(\omega)}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\sum_{|\alpha|=2} \int_Q |(\partial_x^\alpha \mathbf{y})(x,t) e^{s\varphi(x,t)}|^2 dxdt - Cs^2 \sum_{j=1}^3 \int_Q |\partial_j \mathbf{y}|^2 e^{2s\varphi} dxdt - Cs^4 \int_Q |\mathbf{y}|^2 e^{2s\varphi} dxdt \\
\le & C \int_Q (s^4 |\mathbf{y}|^2 + s^2 |\nabla \mathbf{y}|^2 + |\nabla(\operatorname{div} \mathbf{y})|^2 + |\nabla(\operatorname{rot} \mathbf{y})|^2) e^{2s\varphi} dxdt \\
& + Ce^{Cs} \|\mathbf{y}\|^2_{L^2(-T,T;H^2(\omega))}.
\end{aligned} \tag{2.5}
$$

Thus, in terms of (2.2), (2.3) and (2.5), we have

$$
\begin{aligned}
&\int_Q (s^4 |\mathbf{y}|^2 + s^2 |\nabla_{x,t} \mathbf{y}|^2 + \sum_{|\alpha|=2} |\partial_x^\alpha \mathbf{y}|^2 \\
&+ s^2 |\nabla_{x,t}(\operatorname{rot} \mathbf{y})|^2 + s^4 |\operatorname{rot} \mathbf{y}|^2 + s^2 |\nabla_{x,t}(\operatorname{div} \mathbf{y})|^2 + s^4 |\operatorname{div} \mathbf{y}|^2) e^{2s\varphi} dxdt \\
\le & C \int_Q (s |\operatorname{div} \mathbf{f}|^2 + s |\operatorname{rot} \mathbf{f}|^2 + s |\mathbf{f}|^2) e^{2s\varphi} dxdt + Ce^{Cs} \|\mathbf{u}\|^2_{H^1(-T,T;H^2(\omega))}.
\end{aligned}
$$

Thus the proof of Lemma 2.1 is complete.

As for Carleman estimates, see also Hörmander [13], Triggiani and Yao [47].

Next we consider a first order partial differential operator

$$
(P_0 g)(x) = B(x) \cdot \nabla g(x) + B_0(x) g(x), \qquad x \in \Omega, \tag{2.6}
$$

where $B = (b_1, b_2, b_3) \in \{W^{2,\infty}(\Omega)\}^3$ and $B_0 \in W^{2,\infty}(\Omega)$. Then

Lemma 2.2. *We assume*

$$
|(B(x) \cdot (x - x_0))| > 0, \qquad x \in \overline{\Omega}. \tag{2.7}
$$

Then there exists a constant $\tau_0 > 0$ *such that for all* $\tau > \tau_0$*, there exist* $s_0 = s_0(\tau) > 0$ *and* $C_2 = C_2(s_0, \tau_0, \Omega, \omega) > 0$ *such that*

$$s^2 \int_\Omega \left(\sum_{|\alpha| \le 2} |\partial_x^\alpha g(x)|^2 \right) e^{2s\varphi(x,0)} dx \le C_2 \int_\Omega \left(\sum_{|\alpha| \le 2} |\partial_x^\alpha (P_0 g)(x)|^2 \right) e^{2s\varphi(x,0)} dx \tag{2.8}$$

for all $s > s_0$ *and* $g \in C_0^2(\Omega)$.

Proof of Lemma 2.2. We set $F = P_0 g$ and $\varphi_0(x) = \varphi(x,t)$. By integration by parts, we can prove

$$s^2 \int_\Omega |g|^2 e^{2s\varphi_0} dx \le C_2 \int_\Omega |F|^2 e^{2s\varphi_0} dx \tag{2.9}$$

(e.g., [23]). Since $P_0(\partial_j g) = \partial_j F - (\partial_j P_0) g$ and $\partial_j g|_{\partial\Omega} = 0$, we apply (2.9) to $\partial_j g$, so that

$$\begin{aligned} &s^2 \int_\Omega |\partial_j g|^2 e^{2s\varphi_0} dx \\ \le& C_2 \int_\Omega (|g|^2 + |\nabla g|^2) e^{2s\varphi_0} dx + C_2 \int_\Omega |\partial_j F|^2 e^{2s\varphi_0} dx \\ \le& C_2 \int_\Omega (|F|^2 + |\partial_j F|^2) e^{2s\varphi_0} dx + C_2 \int_\Omega |\nabla g|^2 e^{2s\varphi_0} dx. \end{aligned}$$

Therefore

$$\begin{aligned} &s^2 \int_\Omega |\nabla g|^2 e^{2s\varphi_0} dx \\ \le& C_2 \int_\Omega (|F|^2 + |\nabla F|^2) e^{2s\varphi_0} dx + C_2 \int_\Omega |\nabla g|^2 e^{2s\varphi_0} dx. \end{aligned}$$

Taking $s_0 > 0$ sufficiently large, we have

$$s^2 \int_\Omega |\nabla g|^2 e^{2s\varphi_0} dx \le C_2 \int_\Omega (|F|^2 + |\nabla F|^2) e^{2s\varphi_0} dx. \tag{2.10}$$

Next we have

$$\begin{aligned} &P_0(\partial_k \partial_\ell g) = \partial_k \partial_\ell F \\ -&\sum_{j=1}^3 \{(\partial_k b_j)(\partial_\ell \partial_j g) + (\partial_\ell b_j)(\partial_k \partial_j g) + K(g, \nabla g), \end{aligned}$$

where K is a linear operator of g and ∇g. Noting that $\partial_k\partial_\ell g = 0$ on $\partial\Omega$, we apply (2.9) to $\partial_k\partial_\ell g$, and, similarly to (2.10), we can complete the proof of Lemma 2.2.

Finally we show an observability inequality, which may be an independent interest.

Lemma 2.3. *Let $(\lambda, \mu, \rho) \in \mathcal{W}$ and let us assume (1.13). Let $\mathbf{u} \in H^3(Q)$ satisfy $(\rho\partial_t^2 - L_{\lambda,\mu})\mathbf{u} = \mathbf{f}$. Then there exists a constant $C_3 > 0$ such that*

$$\begin{aligned}&\int_\Omega (\sum_{|\alpha|\le 2} |\partial_x^\alpha \mathbf{u}(x,t)|^2 + |\partial_t \mathbf{u}(x,t)|^2)dx + \int_Q |\nabla\partial_t \mathbf{u}(x,t)|^2 dxdt\\ \le& C_3\int_Q (|div\mathbf{f}|^2 + |rot\mathbf{f}|^2 + |\mathbf{f}|^2)dxdt + C_3(\|\mathbf{u}\|^2_{H^1(-T,T;H^2(\omega))} + \|\mathbf{u}\|^2_{L^2(-T,T;H^{\frac{5}{2}}(\omega))})\end{aligned}$$

for all $t \in [-T, T]$.

Starting from works of Klibanov and Malinsky [38] and Kazemi and Klibanov [33], this kind of of inequality is usually proved by Carleman estimate. See e.g., Cheng, Isakov, Yamamoto and Zou [9], and we will prove it in Appendix for completeness.

Now we proceed to

Proof of Theorem. The proof is similar to Imanuvilov and Yamamoto [23]. Henceforth, for simplicity, we set

$$\mathbf{u} = \mathbf{u}(\lambda, \mu, \rho; \mathbf{p}, \mathbf{q}), \quad \mathbf{v} = \mathbf{u}(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\rho}; \mathbf{p}, \mathbf{q}) \tag{2.11}$$

and

$$\mathbf{y} = \mathbf{u} - \mathbf{v}, \quad f = \rho - \widetilde{\rho}, \quad g = \lambda - \widetilde{\lambda}, \quad h = \mu - \widetilde{\mu}. \tag{2.12}$$

Then

$$\widetilde{\rho}\partial_t^2 \mathbf{y} = L_{\widetilde{\lambda},\widetilde{\mu}}\mathbf{y} + G\mathbf{u} \quad \text{in } Q \tag{2.13}$$

and

$$\mathbf{y}(x, 0) = \partial_t \mathbf{y}(x, 0) = 0, \qquad x \in \Omega. \tag{2.14}$$

Here we set

$$G\mathbf{u}(x,t)=-f(x)\partial_t^2\mathbf{u}(x,t)+(g+h)(x)\nabla(\operatorname{div}\mathbf{u})(x,t)+h(x)\Delta\mathbf{u}(x,t)$$
$$+(\operatorname{div}\mathbf{u})(x,t)\nabla g(x)+(\nabla\mathbf{u}(x,t)+(\nabla\mathbf{u}(x,t))^T)\nabla h(x). \tag{2.15}$$

By (1.13), we have the inequality $\theta T^2>d^2$. Therefore, by the definition of d and the definition of the function φ, we have

$$\varphi(x,0)\ge d_1,\quad \varphi(x,T)=\varphi(x,-T)<d_1,\qquad x\in\overline{\Omega}$$

with $d_1=\exp(\tau\inf_{x\in\Omega}|x-x_0|^2)$. Thus, for given $\varepsilon>0$, we can choose a sufficiently small $\delta=\delta(\varepsilon)>0$ such that

$$\varphi(x,t)\ge d_1-\varepsilon,\qquad (x,t)\in\overline{\Omega}\times[-\delta,\delta] \tag{2.16}$$

and

$$\varphi(x,t)\le d_1-2\varepsilon,\quad x\in\overline{\Omega},\ t\in[-T,-T+2\delta]\cup[T-2\delta,T]. \tag{2.17}$$

In order to apply Lemma 2.1, it is necessary to introduce a cut-off function χ satisfying $0\le\chi\le1$, $\chi\in C^\infty(\mathbb{R})$ and

$$\chi=\begin{cases}0 & \text{on } [-T,-T+\delta]\cup[T-\delta,T],\\ 1 & \text{on } [-T+2\delta,T-2\delta].\end{cases} \tag{2.18}$$

In the sequel, $C_j>0$ denote generic constants depending on s_0, τ, M_0, M_1, θ_0, θ_1, Ω, T, x_0, ω, χ and $\mathbf{p}$, $\mathbf{q}$, ε, δ, but independent of $s>s_0$.

Setting $\mathbf{z}_1=\chi\partial_t^2\mathbf{y}$, $\mathbf{z}_2=\chi\partial_t^3\mathbf{y}$ and $\mathbf{z}_3=\chi\partial_t^4\mathbf{y}$, we have

$$\begin{cases}\widetilde{\rho}\partial_t^2\mathbf{z}_1=L_{\widetilde{\lambda},\widetilde{\mu}}\mathbf{z}_1+\chi G(\partial_t^2\mathbf{u})+2\widetilde{\rho}(\partial_t\chi)\partial_t^3\mathbf{y}+\widetilde{\rho}(\partial_t^2\chi)\partial_t^2\mathbf{y},\\ \widetilde{\rho}\partial_t^2\mathbf{z}_2=L_{\widetilde{\lambda},\widetilde{\mu}}\mathbf{z}_2+\chi G(\partial_t^3\mathbf{u})+2\widetilde{\rho}(\partial_t\chi)\partial_t^4\mathbf{y}+\widetilde{\rho}(\partial_t^2\chi)\partial_t^3\mathbf{y},\\ \widetilde{\rho}\partial_t^2\mathbf{z}_3=L_{\widetilde{\lambda},\widetilde{\mu}}\mathbf{z}_3+\chi G(\partial_t^4\mathbf{u})+2\widetilde{\rho}(\partial_t\chi)\partial_t^5\mathbf{y}+\widetilde{\rho}(\partial_t^2\chi)\partial_t^4\mathbf{y}\quad \text{in } Q.\end{cases} \tag{2.19}$$

We set

$$\mathcal{D} = \|\mathbf{y}\|^2_{H^5(-T,T;H^2(\omega))} + \|\mathbf{y}\|^2_{H^4(-T,T;H^{\frac{5}{2}}(\omega))}.$$

Noting that $\mathbf{u} \in W^{7,\infty}(Q)$, in view of (2.18) and Lemma 2.1, we can Carleman estimate (2.1) to (2.19), so that

$$\begin{aligned}
&\sum_{j=2}^{4} \int_Q \left(s^4 |\partial_t^j \mathbf{y}|^2 \chi^2 + s^2 |\nabla \partial_t^j \mathbf{y}|^2 \chi^2 + \sum_{|\alpha|=2} |\partial_x^\alpha \partial_t^j \mathbf{y}|^2 \chi^2 \right) e^{2s\varphi} dxdt \\
\le & Cs \int_Q \sum_{j=2}^{4} (\chi^2 |\nabla (G(\partial_t^j \mathbf{u}))|^2 + \chi^2 |G(\partial_t \mathbf{u})|^2) e^{2s\varphi} dxdt \\
+ & Cs \int_Q (|\partial_t \chi|^2 + |\partial_t^2 \chi|^2) \left\{ \sum_{j=2}^{5} (|\mathrm{div}\,(\partial_t^j \mathbf{y})|^2 + |\mathrm{rot}\,(\partial_t^j \mathbf{y})|^2 + |\partial_t^j \mathbf{y}|^2) \right\} e^{2s\varphi} dxdt \\
+ & Ce^{Cs} \mathcal{D}. \qquad (2.20)
\end{aligned}$$

Here we used $\mathrm{div}\,(\widetilde{\rho}(\partial_t \chi)\partial_t^j \mathbf{y}) = \nabla(\widetilde{\rho}\partial_t \chi)\cdot \partial_t^j \mathbf{y} + \widetilde{\rho}(\partial_t \chi)\mathrm{div}\,(\partial_t^j \mathbf{y})$ and $\mathrm{rot}\,(\widetilde{\rho}(\partial_t \chi)\partial_t^j \mathbf{y}) = \nabla(\widetilde{\rho}\partial_t \chi) \times \partial_t^j \mathbf{y} + \widetilde{\rho}(\partial_t \chi)\mathrm{rot}\,(\partial_t^j \mathbf{y})$ for $j = 2, 3, 4, 5$.

Moreover by (1.15) we see that

$$\begin{aligned}
&|\nabla (G(\partial_t^j \mathbf{u}))| \le C \left\{ |\nabla f| + |f| + \sum_{|\alpha| \le 2} |\partial_x^\alpha g(x)| + \sum_{|\alpha| \le 2} |\partial_x^\alpha h(x)| \right\} \quad \text{in } Q, \\
&|G(\partial_t^j \mathbf{u})| \le C(|f| + |\nabla g| + |\nabla h| + |g| + |h|) \quad \text{in } Q \qquad (2.21)
\end{aligned}$$

and

$$|\partial_t \chi|, |\partial_t^2 \chi| \ne 0 \text{ only for } t \in (T - 2\delta, T - \delta) \cup (-T + \delta, -T + 2\delta). \qquad (2.22)$$

On the other hand, (2.13) implies

$$\widetilde{\rho} \partial_t^2 (\partial_t^j \mathbf{y}) = L_{\widetilde{\lambda},\widetilde{\mu}} \partial_t^j \mathbf{y} + G(\partial_t^j \mathbf{u}) \quad \text{in } Q, \quad j = 0, 1, 2, 3, 4.$$

Therefore Lemma 2.3 and (2.21) yield

$$\begin{aligned}
&\int_Q \left\{ \sum_{k=2}^{5} (|\nabla \partial_t^k \mathbf{y}|^2 + |\partial_t^k \mathbf{y}|^2) + \sum_{j=2}^{4} \sum_{|\alpha|=2} |\partial_t^j \partial_x^\alpha \mathbf{y}|^2 \right\} dxdt \\
\le & C \sum_{j=0}^{4} \left(\|G(\partial_t^j \mathbf{u})\|^2_{L^2(Q)} + \|\nabla G(\partial_t^j \mathbf{u})\|^2_{L^2(Q)} \right) + C\mathcal{D} \\
\le & C \int_Q \left(\sum_{|\alpha|\le 2} |\partial_x^\alpha g(x)|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha h(x)|^2 + |f(x)|^2 + |\nabla f(x)|^2 \right) dxdt + C\mathcal{D} \\
\le & C(\|f\|^2_{H^1(\Omega)} + \|g\|^2_{H^2(\Omega)} + \|h\|^2_{H^2(\Omega)}) + C\mathcal{D}.
\end{aligned} \tag{2.23}$$

Hence inequalities (2.20), (2.21), (2.22) and (2.23) yield

$$\begin{aligned}
&\sum_{j=2}^{4} \int_Q \left(s^4 |\partial_t^j \mathbf{y}|^2 \chi^2 + s^2 |\nabla \partial_t^j \mathbf{y}|^2 \chi^2 + \sum_{|\alpha|=2} |\partial_x^\alpha \partial_t^j \mathbf{y}|^2 \chi^2 \right) e^{2s\varphi} dxdt \\
\le & Cs \int_Q \left(|f|^2 + |\nabla f(x)|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha g(x)|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha h(x)|^2 \right) e^{2s\varphi} dxdt \\
& + Cse^{2s(d_1-2\varepsilon)} (\|f\|^2_{H^1(\Omega)} + \|g\|^2_{H^2(\Omega)} + \|h\|^2_{H^2(\Omega)}) + Ce^{Cs}\mathcal{D} \\
\equiv & Cs\mathcal{E} + Cs(\|f\|^2_{H^1(\Omega)} + \|g\|^2_{H^2(\Omega)} + \|h\|^2_{H^2(\Omega)}) e^{2s(d_1-2\varepsilon)} + Ce^{Cs}\mathcal{D}.
\end{aligned} \tag{2.24}$$

On the other hand, for $|\alpha| = 2$, we use (2.23) and

$$\begin{aligned}
&\int_\Omega |(\partial_t^2 \partial_x^\alpha \mathbf{y})(x,0)|^2 e^{2s\varphi(x,0)} dx \\
= & \int_{-T}^0 \frac{\partial}{\partial t} \left(\int_\Omega |(\partial_t^2 \partial_x^\alpha \mathbf{y})(x,t)|^2 \chi(t)^2 e^{2s\varphi} dx \right) dt \\
= & \int_{-T}^0 \int_\Omega 2((\partial_t^3 \partial_x^\alpha \mathbf{y}) \cdot (\partial_t^2 \partial_x^\alpha \mathbf{y})) \chi^2 e^{2s\varphi} dxdt \\
& + 2s \int_{-T}^0 \int_\Omega |\partial_t^2 \partial_x^\alpha \mathbf{y}|^2 \chi^2 (\partial_t \varphi) e^{2s\varphi} dxdt + \int_{-T}^0 \int_\Omega |\partial_t^2 \partial_x^\alpha \mathbf{y}|^2 (\partial_t(\chi^2)) e^{2s\varphi} dxdt \\
\le & C \int_Q s\chi^2 (|\partial_t^3 \partial_x^\alpha \mathbf{y}|^2 + |\partial_t^2 \partial_x^\alpha \mathbf{y}|^2) e^{2s\varphi} dx \\
& + Ce^{2s(d_1-2\varepsilon)} (\|f\|^2_{H^1(\Omega)} + \|g\|^2_{H^2(\Omega)} + \|h\|^2_{H^2(\Omega)}) + C\mathcal{D}e^{Cs}.
\end{aligned}$$

Therefore (2.24) yields

$$\begin{aligned}
&\sum_{|\alpha|=2} \int_\Omega |(\partial_t^2 \partial_x^\alpha \mathbf{y})(x,0)|^2 e^{2s\varphi(x,0)} dx \\
\le & Cs^2 (\|f\|^2_{H^1(\Omega)} + \|g\|^2_{H^2(\Omega)} + \|h\|^2_{H^2(\Omega)}) e^{2s(d_1-2\varepsilon)} + Cs^2\mathcal{E} + Ce^{Cs}\mathcal{D}
\end{aligned}$$

for all large $s > 0$. Similarly we can estimate $\sum_{|\alpha|=2}\int_\Omega |(\partial_t^3\partial_x^\alpha \mathbf{y})(x,0)|^2 e^{2s\varphi(x,0)}dx$ to obtain

$$\begin{aligned}&\sum_{j=2}^{3}\sum_{|\alpha|=2}\int_\Omega |\partial_x^\alpha\partial_t^j\mathbf{y}(x,0)|^2e^{2s\varphi(x,0)}dx\\ \le&Cs^2e^{2s(d_1-2\varepsilon)}(\|f\|^2_{H^1(\Omega)}+\|g\|^2_{H^2(\Omega)}+\|h\|^2_{H^2(\Omega)})+Cs^2\mathcal{E}+Ce^{Cs}\mathcal{D}\end{aligned} \tag{2.25}$$

for all large $s > 0$.

Now we will consider first order partial differential equations satisfied by h, g and f. That is, by (2.13), (2.14) and (1.15), we have

$$\widetilde{\rho}\partial_t^2\mathbf{y}(x,0) = G\mathbf{u}(x,0), \quad \widetilde{\rho}\partial_t^3\mathbf{y}(x,0) = G\partial_t\mathbf{u}(x,0). \tag{2.26}$$

For simplicity, we set

$$\begin{cases}\mathbf{a} = \begin{pmatrix}-\frac{1}{\rho}L_{\lambda,\mu}\mathbf{p}\\ -\frac{1}{\rho}L_{\lambda,\mu}\mathbf{q}\end{pmatrix},\\ \mathbf{b}_1 = \begin{pmatrix}\operatorname{div}\mathbf{p}\\0\\0\\\operatorname{div}\mathbf{q}\\0\\0\end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix}0\\\operatorname{div}\mathbf{p}\\0\\0\\\operatorname{div}\mathbf{q}\\0\end{pmatrix}, \quad \mathbf{b}_3 = \begin{pmatrix}0\\0\\\operatorname{div}\mathbf{p}\\0\\0\\\operatorname{div}\mathbf{q}\end{pmatrix},\\ (\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3) = \begin{pmatrix}\nabla\mathbf{p}+(\nabla\mathbf{p})^T\\ \nabla\mathbf{q}+(\nabla\mathbf{q})^T\end{pmatrix},\\ \mathbf{G} = \begin{pmatrix}\widetilde{\rho}\partial_t^2\mathbf{y}(x,0)-(g+h)\nabla(\operatorname{div}\mathbf{p})-h\Delta\mathbf{p}\\ \widetilde{\rho}\partial_t^3\mathbf{y}(x,0)-(g+h)\nabla(\operatorname{div}\mathbf{q})-h\Delta\mathbf{q}\end{pmatrix} \quad \text{on } \overline{\Omega}.\end{cases} \tag{2.27}$$

Then we can rewrite (2.26) as

$$\mathbf{a}f+\mathbf{b}_1\partial_1 g+\mathbf{b}_2\partial_2 g+\mathbf{b}_3\partial_3 g = \mathbf{G}-\mathbf{d}_1\partial_1 h-\mathbf{d}_2\partial_2 h-\mathbf{d}_3\partial_3 h.$$

Therefore for $j_1 \in \{1,2,3,4,5,6\}$, we have

$$\begin{aligned}&\{\mathbf{a}\}_{j_1}f+\{\mathbf{b}_1\}_{j_1}\partial_1 g+\{\mathbf{b}_2\}_{j_1}\partial_2 g+\{\mathbf{b}_3\}_{j_1}\partial_3 g\\ =&\{\mathbf{G}\}_{j_1}-\{\mathbf{d}_1\}_{j_1}\partial_1 h-\{\mathbf{d}_2\}_{j_1}\partial_2 h-\{\mathbf{d}_3\}_{j_1}\partial_3 h \quad \text{on } \overline{\Omega}.\end{aligned} \tag{2.28}$$

Equality (2.28) is a system of five linear equations with respect to four unknowns f, $\partial_1 g$, $\partial_2 g$, $\partial_3 g$, and so for the existence of solutions, we need the consistency of the coefficients, that is,

$$\det_{j_1}(\mathbf{a}, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{G} - \mathbf{d}_1\partial_1 h - \mathbf{d}_2\partial_2 h - \mathbf{d}_3\partial_3 h) = 0 \quad \text{on } \overline{\Omega},$$

that is,

$$\sum_{k=1}^{3} \det_{j_1}(\mathbf{a}, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{d}_k)\partial_k h = \det_{j_1}(\mathbf{a}, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{G}) \quad \text{on } \overline{\Omega} \tag{2.29}$$

by the linearity of the determinant. Here by (1.15) we note that $\mathbf{p}, \mathbf{q} \in W^{5,\infty}(\Omega)$ and

$$\begin{aligned}\sum_{|\alpha|\le 2} |\partial_x^\alpha \mathbf{G}(x)| \le{}& C\sum_{j=2}^{3}\sum_{|\alpha|\le 2} |\partial_x^\alpha \partial_t^j \mathbf{y}(x,0)| \\ &+C\sum_{|\alpha|\le 2}|\partial_x^\alpha g(x)| + C\sum_{|\alpha|\le 2}|\partial_x^\alpha h(x)|.\end{aligned}$$

In terms of condition (1.11) and $h \equiv \mu - \widetilde{\mu} \in C_0^2(\Omega)$, considering (2.29) as a first order partial differential operator in h, we can apply Lemma 2.2 to obtain

$$\begin{aligned}&s^2\int_\Omega \sum_{|\alpha|\le 2}|\partial_x^\alpha h|^2 e^{2s\varphi(x,0)}dx \le C\int_\Omega \sum_{j=2}^{3}\sum_{|\alpha|\le 2}|\partial_x^\alpha\partial_t^j \mathbf{y}(x,0)|^2 e^{2s\varphi(x,0)}dx \\ &+C\int_\Omega\left(\sum_{|\alpha|\le 2}|\partial_x^\alpha g(x)|^2 + \sum_{|\alpha|\le 2}|\partial_x^\alpha h(x)|^2\right)e^{2s\varphi(x,0)}dx \\ &\le Cs^2 e^{2s(d_1-2\varepsilon)}(\|f\|^2_{H^1(\Omega)} + \|g\|^2_{H^2(\Omega)} + \|h\|^2_{H^2(\Omega)}) + Cs^2\mathcal{E} + Ce^{Cs}\mathcal{D} \\ &+C\int_\Omega\left(\sum_{|\alpha|\le 2}|\partial_x^\alpha g(x)|^2 + \sum_{|\alpha|\le 2}|\partial_x^\alpha h(x)|^2\right)e^{2s\varphi(x,0)}dx\end{aligned} \tag{2.30}$$

for all large $s > 0$. Here we used (2.25). Similarly to (2.30), in terms of (1.12), we can argue for g. Hence with (2.30), we have

$$
\begin{aligned}
&\int_\Omega \left(\sum_{|\alpha|\le 2} |\partial_x^\alpha g(x)|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha h(x)|^2 \right) e^{2s\varphi(x,0)} dx \\
\le & Ce^{2s(d_1-2\varepsilon)}(\|f\|^2_{H^1(\Omega)} + \|g\|^2_{H^2(\Omega)} + \|h\|^2_{H^2(\Omega)}) \\
+ & C\int_Q \left(|f(x)|^2 + |\nabla f(x)|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha g(x)|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha h(x)|^2 \right) e^{2s\varphi} dxdt + Ce^{Cs}\mathcal{D} \\
+ & \frac{C}{s^2}\int_\Omega \left(\sum_{|\alpha|\le 2} |\partial_x^\alpha g(x)|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha h(x)|^2 \right) e^{2s\varphi(x,0)} dx
\end{aligned}
$$

for all large $s > 0$. Here we recall the definition of $\mathcal{E}$ in (2.24). Taking $s > 0$ sufficiently large, we can absorb the last term into the left hand side:

$$
\begin{aligned}
&\int_\Omega \left(\sum_{|\alpha|\le 2} |\partial_x^\alpha g(x)|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha h(x)|^2 \right) e^{2s\varphi(x,0)} dx \\
\le & Ce^{2s(d_1-2\varepsilon)}(\|f\|^2_{H^1(\Omega)} + \|g\|^2_{H^2(\Omega)} + \|h\|^2_{H^2(\Omega)}) \\
+ & C\int_Q \left(|f(x)|^2 + |\nabla f(x)|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha g(x)|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha h(x)|^2 \right) e^{2s\varphi} dxdt + Ce^{Cs}\mathcal{D}.
\end{aligned} \tag{2.31}
$$

Finally, by (2.28), we have

$$
\mathbf{a}f = -\mathbf{b}_1\partial_1 g - \mathbf{b}_2\partial_2 g - \mathbf{b}_3\partial_3 g + \mathbf{G} - \mathbf{d}_1\partial_1 h - \mathbf{d}_2\partial_2 h - \mathbf{d}_3\partial_3 h \qquad \text{in } \Omega.
$$

Moreover, by (1.11) or (1.12), we see that $|\mathbf{a}(x)| > 0$ for $x \in \overline{\Omega}$, so that

$$
f(x) = \widetilde{K_1}\mathbf{G} + \widetilde{K_2}(\nabla g, \nabla h) \quad \text{in } \Omega,
$$

where $\widetilde{K_1}, \widetilde{K_2}$ are linear operators with $W^{1,\infty}$-coefficients. Thus

$$
\begin{aligned}
&|\nabla f(x)| \le C\left(|\nabla \mathbf{G}(x)| + \sum_{|\alpha|\le 2} |\partial_x^\alpha g(x)| + \sum_{|\alpha|\le 2} |\partial_x^\alpha h(x)| \right) \\
\le & C\left\{ \sum_{j=2}^3 (|\nabla(\partial_t^j \mathbf{y})(x,0)| + |\partial_t^j \mathbf{y}(x,0)|) + \sum_{|\alpha|\le 2} |\partial_x^\alpha g(x)| + \sum_{|\alpha|\le 2} |\partial_x^\alpha h(x)| \right\},
\end{aligned}
$$

and

$$
|f(x)| \le C\left\{ \sum_{j=2}^3 |\partial_t^j \mathbf{y}(x,0)| + \sum_{|\alpha|\le 2} |\partial_x^\alpha g(x)| + \sum_{|\alpha|\le 2} |\partial_x^\alpha h(x)| \right\}
$$

for $x \in \Omega$. Hence

$$\int_\Omega (|\nabla f(x)|^2 + |f(x)|^2) e^{2s\varphi(x,0)} dx$$
$$\le C \int_\Omega \left\{ \sum_{j=2}^{3} \sum_{|\alpha|\le 1} |\partial_x^\alpha \partial_t^j \mathbf{y}(x,0)|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha g(x)|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha h(x)|^2 \right\} e^{2s\varphi(x,0)} dx. \tag{2.32}$$

On the other hand, for $j = 2, 3$, we have by (2.23) and (2.24),

$$\begin{aligned}
&\int_\Omega |\nabla(\partial_t^j \mathbf{y})(x,0)|^2 e^{2s\varphi(x,0)} dx \\
=& \int_{-T}^0 \frac{\partial}{\partial t} \int_\Omega \chi^2 |\nabla(\partial_t^j \mathbf{y})|^2 e^{2s\varphi} dx dt \\
\le& C \int_Q (s|\nabla(\partial_t^j \mathbf{y})|^2 \chi^2 + |\nabla(\partial_t^{j+1} \mathbf{y})|^2 \chi^2) e^{2s\varphi} dx dt \\
+& C e^{2s(d_1 - 2\varepsilon)} (\|f\|_{H^1(\Omega)}^2 + \|g\|_{H^2(\Omega)}^2 + \|h\|_{H^2(\Omega)}^2) + C e^{Cs} \mathcal{D} \\
\le& C e^{2s(d_1 - 2\varepsilon)} (\|f\|_{H^1(\Omega)}^2 + \|g\|_{H^2(\Omega)}^2 + \|h\|_{H^2(\Omega)}^2) + C e^{Cs} \mathcal{D} + C\mathcal{E}
\end{aligned} \tag{2.33}$$

and

$$\begin{aligned}
&\int_\Omega |(\partial_t^j \mathbf{y})(x,0)|^2 e^{2s\varphi(x,0)} dx = \int_{-T}^0 \frac{\partial}{\partial t} \int_\Omega \chi^2 |(\partial_t^j \mathbf{y})|^2 e^{2s\varphi} dx dt \\
\le& C \int_Q (s|\partial_t^j \mathbf{y}|^2 \chi^2 + |\partial_t^{j+1} \mathbf{y}|^2 \chi^2) e^{2s\varphi} dx dt \\
+& C e^{2s(d_1 - 2\varepsilon)} (\|f\|_{H^1(\Omega)}^2 + \|g\|_{H^2(\Omega)}^2 + \|h\|_{H^2(\Omega)}^2) + C e^{Cs} \mathcal{D} \\
\le& C e^{2s(d_1 - 2\varepsilon)} (\|f\|_{H^1(\Omega)}^2 + \|g\|_{H^2(\Omega)}^2 + \|h\|_{H^2(\Omega)}^2) + C e^{Cs} \mathcal{D} + C\mathcal{E}
\end{aligned} \tag{2.34}$$

for all large $s > 0$.

Substituting (2.31), (2.33) and (2.34) into (2.32), we obtain

$$\begin{aligned}
&\int_\Omega (|\nabla f(x)|^2 + |f(x)|^2) e^{2s\varphi(x,0)} dx \\
\le& C e^{2s(d_1 - 2\varepsilon)} (\|f\|_{H^1(\Omega)}^2 + \|g\|_{H^2(\Omega)}^2 + \|h\|_{H^2(\Omega)}^2) + C\mathcal{E} + C e^{Cs} \mathcal{D}.
\end{aligned}$$

Here with (2.31), we have

$$\begin{aligned}&\int_\Omega \left(\sum_{|\alpha|\le 2} |\partial_x^\alpha g|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha h|^2 + |\nabla f|^2 + |f|^2\right) e^{2s\varphi(x,0)}dx\\ \le& Ce^{2s(d_1-2\varepsilon)}(\|f\|^2_{H^1(\Omega)} + \|g\|^2_{H^2(\Omega)} + \|h\|^2_{H^2(\Omega)}) + Ce^{Cs}\mathcal{D}\\ &+C\int_Q \left(\sum_{|\alpha|\le 2} |\partial_x^\alpha g|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha h|^2 + |\nabla f|^2 + |f|^2\right) e^{2s\varphi}dxdt.\end{aligned}$$

Since

$$\begin{aligned}&\int_Q \left(\sum_{|\alpha|\le 2} |\partial_x^\alpha g|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha h|^2 + |\nabla f|^2 + |f|^2\right) e^{2s\varphi}dxdt\\ =&\int_\Omega \left(\sum_{|\alpha|\le 2} |\partial_x^\alpha g|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha h|^2 + |\nabla f|^2 + |f|^2\right) e^{2s\varphi(x,0)} \left(\int_{-T}^T e^{2s(\varphi(x,t)-\varphi(x,0))}dt\right) dx\\ =&o(1)\int_\Omega \left(\sum_{|\alpha|\le 2} |\partial_x^\alpha g|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha h|^2 + |\nabla f|^2 + |f|^2\right) e^{2s\varphi(x,0)}dx,\end{aligned}$$

as $s \longrightarrow \infty$ by the Lebesgue theorem and $\varphi(x,t) < \varphi(x,0)$ for $t \ne 0$, we can absorb the last term at the right hand side into the left hand side, and

$$\begin{aligned}&\int_\Omega \left(\sum_{|\alpha|\le 2} |\partial_x^\alpha g|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha h|^2 + |\nabla f|^2 + |f|^2\right) e^{2s\varphi(x,0)}dx\\ \le& Ce^{2s(d_1-2\varepsilon)}(\|f\|^2_{H^1(\Omega)} + \|g\|^2_{H^2(\Omega)} + \|h\|^2_{H^2(\Omega)}) + Ce^{Cs}\mathcal{D}\end{aligned}$$

for all large $s > 0$. By $\varphi(x,0) \ge d_1$, we divide the both sides by e^{2sd_1}, we have

$$\begin{aligned}&\int_\Omega \left(\sum_{|\alpha|\le 2} |\partial_x^\alpha g|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha h|^2 + |\nabla f|^2 + |f|^2\right) dx\\ \le& Ce^{-4s\varepsilon}(\|f\|^2_{H^1(\Omega)} + \|g\|^2_{H^2(\Omega)} + \|h\|^2_{H^2(\Omega)}) + Ce^{Cs}\mathcal{D}\end{aligned}$$

for all large $s > 0$. Choosing $s > 0$ sufficiently large, we can absorb the first term at the right hand side into the left hand side, so that we have conclusion (1.14).

Appendix. Proof of Lemma 2.3.

Let us set $v = \operatorname{div}\mathbf{u}$ and $\mathbf{w} = \operatorname{rot}\mathbf{u}$. Then, as in the proof of Lemma 2.1, we have

$$\rho\partial_t^2\mathbf{u} - \mu\Delta\mathbf{u} + Q_1(\mathbf{u}, v) = \mathbf{f} \quad \text{in } Q, \tag{1}$$

$$\rho\partial_t^2 v - (\lambda+2\mu)\Delta v + Q_2(\mathbf{u}, v, \mathbf{w}) = \operatorname{div}\mathbf{f} \quad \text{in } Q \tag{2}$$

and

$$\rho\partial_t^2 \mathbf{w} - \mu\Delta\mathbf{w} + Q_3(\mathbf{u}, v, \mathbf{w}) = \operatorname{rot}\mathbf{f} \quad \text{in } Q, \tag{3}$$

where $Q_1(\mathbf{u}, v) = \sum_{|\alpha|=1} a_\alpha^{(1)}(x)\partial_x^\alpha \mathbf{u} + \sum_{|\alpha|\le 1} b_\alpha^{(1)}(x)\partial_x^\alpha v$, $Q_j(\mathbf{u}, v, \mathbf{w}) = \sum_{|\alpha|=1} a_\alpha^{(j)}(x)\partial_x^\alpha \mathbf{u} + \sum_{|\alpha|=1} b_\alpha^{(j)}(x)\partial_x^\alpha v + \sum_{|\alpha|=1} c_\alpha^{(j)}(x)\partial_x^\alpha \mathbf{w}$, $j = 2, 3$ and $a_\alpha^{(j)}$, $b_\alpha^{(j)}$, $c_\alpha^{(j)} \in L^\infty(Q)$.

Let $t \ge 0$. We set

$$E_1(t) \equiv \int_\Omega (|\nabla_{x,t}\mathbf{u}(x,t)|^2 + |\nabla_{x,t}v(x,t)|^2 + |\nabla_{x,t}\mathbf{w}(x,t)|^2)dx.$$

Taking the scalar products of (1) and (3) with $\partial_t\mathbf{u}$ and $\partial_t\mathbf{w}$ respectively and multiplying (2) with $\partial_t v$, we integrate by parts to have

$$\begin{aligned} E_1(t) &\le CE_1(0) + C\left(\int_0^t E_1(\xi)d\xi + \|\mathbf{f}\|^2_{L^2(Q)} + \|\operatorname{div}\mathbf{f}\|^2_{L^2(Q)} + \|\operatorname{rot}\mathbf{f}\|^2_{L^2(Q)}\right) \\ &+ C(\|\partial_t\mathbf{u}\|^2_{L^2(\partial\Omega\times(-T,T))} + \|\partial_\nu\mathbf{u}\|^2_{L^2(\partial\Omega\times(-T,T))} + \|\partial_t v\|^2_{L^2(\partial\Omega\times(-T,T))} + \|\partial_\nu v\|^2_{L^2(\partial\Omega\times(-T,T))} \\ &+ \|\partial_t\mathbf{w}\|^2_{L^2(\partial\Omega\times(-T,T))} + \|\partial_\nu\mathbf{w}\|^2_{L^2(\partial\Omega\times(-T,T))}). \end{aligned}$$

Applying the trace theorem, we have

$$E_1(t) \le CE_1(0) + CF + C\int_0^t E_1(\xi)d\xi, \quad 0 \le t \le T.$$

Here we set

$$F = \|\mathbf{u}\|^2_{H^1(-T,T;H^2(\omega))} + \|\mathbf{u}\|^2_{L^2(-T,T;H^{\frac{5}{2}}(\omega))} + \|\operatorname{div}\mathbf{f}\|^2_{L^2(Q)} + \|\operatorname{rot}\mathbf{f}\|^2_{L^2(Q)} + \|\mathbf{f}\|^2_{L^2(Q)}.$$

The Gronwall inequality implies $E_1(t) \le C(F + E_1(0))$, $0 \le t \le T$. Similarly we can prove $C^{-1}E_1(0) \le E_1(t) + CF$, $0 \le t \le T$. For $-T \le t \le 0$, we can simiarly argue to obtain

$$E_1(t_1) \le CE_1(t_2) + CF, \qquad -T \le t_1, t_2 \le T. \tag{4}$$

Next we will include $\|\mathbf{u}(\cdot,t)\|^2_{L^2(\Omega)}$ into $E_1(t)$. By the Sobolev extension theorem and the trace theorem, we can find $\mathbf{u}^*(\cdot,t) \in H^1(\Omega)$ such that $\mathbf{u}^*(\cdot,t) = \mathbf{u}(\cdot,t)$ on $\partial\Omega$ and $\|\mathbf{u}^*(\cdot,t)\|_{H^1(\Omega)} \le C\|\mathbf{u}(\cdot,t)\|_{H^{\frac{1}{2}}(\partial\Omega)} \le C\|\mathbf{u}(\cdot,t)\|_{H^1(\omega)}$ for $-T \le t \le T$. Then $(\mathbf{u}-\mathbf{u}^*)(\cdot,t) \in H^1_0(\Omega)$ and the Poincaré inequality yield

$$\|(\mathbf{u}-\mathbf{u}^*)(\cdot,t)\|_{L^2(\Omega)} \le C\|(\nabla\mathbf{u}-\nabla\mathbf{u}^*)(\cdot,t)\|_{L^2(\Omega)} \le C\|\nabla\mathbf{u}(\cdot,t)\|_{L^2(\Omega)} + C\|\mathbf{u}(\cdot,t)\|_{H^1(\omega)}.$$

Moreover the Sobolev embedding theorem implies

$$\|\mathbf{u}(\cdot,t)\|^2_{H^1(\omega)} \le C\|\mathbf{u}\|^2_{H^1(-T,T;H^1(\omega))} \le CF.$$

That is,

$$\|\mathbf{u}(\cdot,t)\|^2_{L^2(\Omega)} \le C\|\nabla\mathbf{u}(\cdot,t)\|^2_{L^2(\Omega)} + CF, \quad -T \le t \le T.$$

Therefore

$$\begin{aligned} E_1(t) \le E(t) \equiv \int_\Omega (|\mathbf{u}(x,t)|^2 + |\nabla_{x,t}\mathbf{u}(x,t)|^2 + |\nabla_{x,t}\mathrm{div}\,\mathbf{u}(x,t)|^2 + |\nabla_{x,t}\mathrm{rot}\,\mathbf{u}(x,t)|^2)dx \\ \le CE_1(t) + CF, \qquad -T \le t \le T, \end{aligned}$$

so that (4) implies

$$E(t_1) \le CE(t_2) + CF, \qquad -T \le t_1, t_2 \le T. \tag{5}$$

Let $\chi \in C^\infty_0(\mathbb{R})$ satisfy $0 \le \chi \le 1$ and (2.18). We set $\mathbf{v} = \chi\mathbf{u}$. Then $\partial_t^j\mathbf{v}(\cdot,\pm T) = 0$, $j = 0,1$ and

$$\rho\partial_t^2\mathbf{v} - L_{\lambda,\mu}\mathbf{v} = \chi\mathbf{f} - \rho(2(\partial_t\chi)\partial_t\mathbf{u} + (\partial_t^2\chi)\mathbf{u}).$$

We can apply (2.1) to $\mathbf{v}$:

$$\begin{aligned} &\int_Q (|\mathbf{v}|^2 + |\nabla_{x,t}\mathbf{v}|^2 + |\nabla_{x,t}\mathrm{div}\,\mathbf{v}|^2 + |\nabla_{x,t}\mathrm{rot}\,\mathbf{v}|^2)e^{2s\varphi}dxdt \\ \le &Ce^{Cs}F + C\int_Q (|\partial_t\chi|^2 + |\partial_t^2\chi|^2)\sum_{j=0}^{1}(|\partial_t^j\mathbf{u}|^2 + |\mathrm{div}\,\partial_t^j\mathbf{u}|^2 + |\mathrm{rot}\,\partial_t^j\mathbf{u}|^2)e^{2s\varphi}dxdt. \end{aligned}$$

Taking $\delta > 0$ small and shrinking the domain Q into $\Omega \times (-\delta, \delta)$ at the left hand side and using (2.16) and (2.18), we have

$$
\begin{aligned}
&e^{2s(d_1-\varepsilon)} \int_{-\delta}^{\delta} \int_{\Omega} (|\mathbf{u}|^2 + |\nabla_{x,t}\mathbf{u}|^2 + |\nabla_{x,t}\mathrm{div}\,\mathbf{u}|^2 + |\nabla_{x,t}\mathrm{rot}\,\mathbf{u}|^2) dxdt \\
\le& Ce^{Cs}F + Ce^{2s(d_1-2\varepsilon)} \int_Q \sum_{j=0}^{1} (|\partial_t^j \mathbf{u}|^2 + |\mathrm{div}\,\partial_t^j \mathbf{u}|^2 + |\mathrm{rot}\,\partial_t^j \mathbf{u}|^2) dxdt.
\end{aligned}
$$

Therefore by (5), we have

$$
2\delta e^{2s(d_1-\varepsilon)}(E(0) - CF) \le Ce^{Cs}F + 2TCe^{2s(d_1-2\varepsilon)}(E(0)+F),
$$

that is,

$$
E(0)(2\delta - 2CTe^{-2s\varepsilon}) \le Ce^{Cs}F + CF.
$$

Taking $s > 0$ sufficiently large, we obtain $E(0) \le CF$. By (5), we have

$$
E(t) \le CF, \qquad -T \le t \le T. \tag{6}
$$

By the Sobolev extension theorem, we can find $\mathbf{u}^*(\cdot,t) \in H^2(\Omega)$ such that $\|\mathbf{u}^*(\cdot,t)\|_{H^2(\Omega)} \le C\|u(\cdot,t)\|_{H^2(\omega)}$ and $\mathbf{u}^*(\cdot,t) = \mathbf{u}(\cdot,t)$ on $\partial\Omega$. Set $\mathbf{v} = \mathbf{u} - \mathbf{u}^*$. Then $\Delta\mathbf{v} = \Delta\mathbf{u} - \Delta\mathbf{u}^* = \nabla(\mathrm{div}\,\mathbf{u}) - \mathrm{rot}\,(\mathrm{rot}\,\mathbf{u}) - \Delta\mathbf{u}^*$ and $\mathbf{v}|_{\partial\Omega} = 0$. Hence the a priori estimate for the boundary value problem for Δ implies

$$
\|\mathbf{v}(\cdot,t)\|_{H^2(\Omega)} \le C(\|\nabla\mathrm{div}\,\mathbf{u}(\cdot,t)\|_{L^2(\Omega)} + \|\mathrm{rot}\,(\mathrm{rot}\,\mathbf{u})(\cdot,t)\|_{L^2(\Omega)} + \|\Delta\mathbf{u}^*(\cdot,t)\|_{L^2(\Omega)}).
$$

Since $\mathbf{u} = \mathbf{v} + \mathbf{u}^*$, we have

$$
\begin{aligned}
&\|\mathbf{u}(\cdot,t)\|_{H^2(\Omega)} \le C(\|\nabla\mathrm{div}\,\mathbf{u}(\cdot,t)\|_{L^2(\Omega)} + \|\mathrm{rot}\,(\mathrm{rot}\,\mathbf{u})(\cdot,t)\|_{L^2(\Omega)} + \|\mathbf{u}^*(\cdot,t)\|_{H^2(\Omega)}) \\
\le& C(\|\nabla\mathrm{div}\,\mathbf{u}(\cdot,t)\|_{L^2(\Omega)} + \|\mathrm{rot}\,(\mathrm{rot}\,\mathbf{u})(\cdot,t)\|_{L^2(\Omega)} + \|\mathbf{u}(\cdot,t)\|_{H^2(\omega)}).
\end{aligned}
$$

Since $\|\mathbf{u}(\cdot,t)\|_{H^2(\omega)} \le C\|\mathbf{u}\|_{H^1(-T,T;H^2(\omega))}$, we have

$$
\|\mathbf{u}(\cdot,t)\|^2_{H^2(\Omega)} \le C(\|\nabla\mathrm{div}\,\mathbf{u}(\cdot,t)\|^2_{L^2(\Omega)} + \|\mathrm{rot}\,(\mathrm{rot}\,\mathbf{u})(\cdot,t)\|^2_{L^2(\Omega)} + F),
$$

with which (6) yields

$$\int_\Omega \left(\sum_{|\alpha|\le 2} |\partial_x^\alpha \mathbf{u}(x,t)|^2 + |\partial_t \mathbf{u}(x,t)|^2 + |\partial_t \mathrm{div}\, \mathbf{u}(x,t)|^2 + |\partial_t \mathrm{rot}\, \mathbf{u}(x,t)|^2 \right) dx \le CF. \tag{7}$$

Finally we will estimate $\nabla(\partial_t \mathbf{u})$ at the right hand side of the conclusion. By the Sobolev extension theorem and the trace theorem, for $-T \le t \le T$ we can find $\mathbf{u}_1^*$ such that $\mathbf{u}_1^*(\cdot,t) = \partial_t \mathbf{u}(\cdot,t)$ on $\partial\Omega$ and $\|\mathbf{u}_1^*(\cdot,t)\|_{H^1(\Omega)} \le C\|\partial_t \mathbf{u}(\cdot,t)\|_{H^{\frac{1}{2}}(\partial\Omega)} \le C\|\partial_t \mathbf{u}(\cdot,t)\|_{H^1(\omega)}$. Applying Theorem 6.1 (pp.358-359) in Duvaut and Lions [10], we have

$$\|\partial_t \mathbf{u}(\cdot,t) - \mathbf{u}_1^*(\cdot,t)\|_{H^1(\Omega)} \le C\|\partial_t \mathbf{u}(\cdot,t) - \mathbf{u}_1^*(\cdot,t)\|_{L^2(\Omega)}$$

$$+C\|\mathrm{div}\,(\partial_t \mathbf{u}(\cdot,t) - \mathbf{u}_1^*(\cdot,t))\|_{L^2(\Omega)} + C\|\mathrm{rot}\,(\partial_t \mathbf{u}(\cdot,t) - \mathbf{u}_1^*(\cdot,t))\|_{L^2(\Omega)},$$

that is,

$$\int_\Omega |\partial_t \nabla \mathbf{u}(x,t)|^2 dx$$

$$\le C\int_\Omega (|\partial_t \mathbf{u}(x,t)|^2 + |\partial_t \mathrm{div}\, \mathbf{u}(x,t)|^2 + |\partial_t \mathrm{rot}\, \mathbf{u}(x,t)|^2) dx + C\|\partial_t \mathbf{u}(\cdot,t)\|_{H^1(\omega)}^2.$$

Hence by (7), we obtain

$$\int_Q |\partial_t \nabla \mathbf{u}|^2 dxdt$$

$$\le C\int_Q (|\partial_t \mathbf{u}(x,t)|^2 + |\partial_t \mathrm{div}\, \mathbf{u}(x,t)|^2 + |\partial_t \mathrm{rot}\, \mathbf{u}(x,t)|^2) dxdt + CF \le CF. \tag{8}$$

Inequality (7) and (8) completes the proof of Lemma 2.3.

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