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Abstract

X. Tolsa defined the space of type BMO for a positive Radon measure satisfying some growth condition on $\mathbf{R}^d$. This space is very suitable for the Calderón-Zygmund theory with non-doubling measures. Especially, the John-Nirenberg type inequality remains true. In this paper we introduce the localized and weighted version of this inequality and, as an application, we obtain some vector-valued inequalities.

1 Introduction

In this paper we discuss the (weighted) John-Nirenberg type inequality for the sharp maximal operator defined by X. Tolsa.

By "cube" $Q \subset \mathbf{R}^d$ we mean a compact cube whose edges are parallel to the coordinate axes. Its center will be denoted by z_Q and its side length by $\ell(Q)$. For $\rho > 0$, ρQ will denote a cube concentric to Q of sidelength $\rho\,\ell(Q)$. $Q(x,l)$ will denote the cube centered at x and of sidelength l. Throughout this paper we assume that μ is a positive Radon measure satisfying the growth condition :

$$\mu(Q(x,l)) \le C_0\, l^n \text{ for all } x \in \operatorname{supp}(\mu) \text{ and } l > 0, \tag{1}$$

where C_0 and $n \in (0,d]$ are some fixed numbers. We emphasize that we do not assume μ is doubling. By $\mathcal{Q}(\mu)$ we will denote the set of all cubes $Q \subset \mathbf{R}^d$ with positive μ-measures.

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It is well known that the doubling property of the underlying measure is a basic condition in the classical Calderón-Zygmund theory of harmonic analysis. Recently, more attention has been paid to non-doubling measures. It has been shown that many results of this theory still hold without the doubling property.

To investigate the analytic capacity on the complex plane Nazarov, Treil and Volberg developed the theory of the singular integrals for the measures with growth condition [4], [5]. Tolsa proved subadditivity and bi-Lipschitz invariance of the analytic capacity, which had been left open for a long time, [13], [14]. The research, which was started from their pioneer works using the modified maximal operator, has been developed in many ways : Tolsa defined for the growth measures RBMO (regular bounded mean oscillation), the Hardy space $H^1(\mu)$ and the Littlewood-Paley decomposition [10], [11]. He also gave his $H^1(\mu)$ space in terms of the grand maximal operator [12]. Deng, Han and Yang defined for the growth measures the Besov spaces and the Triebel-Lizorkin spaces [1], [2]. The authors defined for such measures the Morrey spaces and established some inequalities [8], [9]. The aim of this paper is to introduce the (weighted) John-Nirenberg type inequality for the growth measures, which can be applied to the vector-valued sharp maximal inequality for the Morrey spaces.

Given two cubes $Q \subset R$, we denote

$$\delta(Q,R) := \int_{\ell(Q)}^{\ell(Q_R)} \frac{\mu(Q(z_Q,l))}{l^n}\,\frac{dl}{l},$$

where Q_R denotes the smallest cube concentric to Q containing R. We say that Q is a doubling cube if $\mu(2Q) \le 2^{d+1}\,\mu(Q)$. By $\mathcal{Q}(\mu,2)$ we will denote the set of all doubling cubes. Given $Q \in \mathcal{Q}(\mu)$, we set Q^* as the smallest doubling cube R of the form $R = 2^j Q$ with $j = 0, 1, \ldots$.[2]

Our BMO here is RBMO (regular bounded mean oscillation) introduced by Tolsa [10], which are the suitable substitutes for the classical spaces. Denoting the average of f over the cube Q by $m_Q(f) := \dfrac{1}{\mu(Q)}\displaystyle\int_Q f\,d\mu$, we say that $f \in L^1_{loc}(\mu)$ is an element of RBMO if it satisfies

$$\|f\|_* := \sup_{Q\in\mathcal{Q}(\mu)} \frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q |f(x) - m_{Q^*}(f)|\,d\mu(x) + \sup_{\substack{Q\subset R\\ Q,R\in\mathcal{Q}(\mu,2)}} \frac{|m_Q(f) - m_R(f)|}{1+\delta(Q,R)} < \infty.$$

(Many other equivalent norms may be found in [10, Section 2].) The advantage of RBMO would be the following John-Nirenberg lemma due to Tolsa.

Theorem 1 *Let $f \in$RBMO and $Q \in \mathcal{Q}(\mu)$.*

[2] By the growth condition (1) there are a lot of big doubling cubes. Precisely speaking, given any cube $Q \in \mathcal{Q}(\mu)$, we can find $j \in \mathbf{N}$ with $2^j Q \in \mathcal{Q}(\mu,2)$. Meanwhile, for μ-a.e. $x \in \mathbf{R}^d$, there exists a sequence of doubling cubes $\{Q_k\}_k$ centered at x with $\ell(Q_k) \to 0$ as $k \to \infty$. So we can say that there are a lot of small doubling cubes too (see [10]).

(1) *There exist positive constants C and C' independent of f so that, for every $\lambda > 0$,*

$$\mu\{x \in Q \,|\, |f(x) - m_{Q^*}(f)| > \lambda\} \le C\,\mu\left(\frac{3}{2}Q\right)\,\exp\left(-\frac{C'\lambda}{\|f\|_*}\right).$$

(2) *Let $q \in [1,\infty)$. Then there exists a constant C independent of f so that, for every cube $Q \in \mathcal{Q}(\mu)$,*

$$\left(\frac{1}{\mu\left(\frac{3}{2}Q\right)}\int_Q |f(x) - m_{Q^*}(f)|^q\,d\mu(x)\right)^{\frac{1}{q}} \le C\,\|f\|_*.$$

The purpose of this paper is to establish the localized and weighted version of Theorem 1 (2) (Theorem 2). And, as a corollary, we obtain a vector-valued extension of Theorem 1 (2) (Corollary 13).

For $f \in L^1_{loc}(\mu)$, we define two maximal operators also due to Tolsa : The sharp maximal operator $M^\sharp f(x)$ is defined as

$$M^\sharp f(x) := \sup_{x\in Q\in\mathcal{Q}(\mu)} \frac{1}{\mu\left(\frac{3}{2}Q\right)}\int_Q |f(x) - m_{Q^*}(f)|\,d\mu(x) + \sup_{\substack{x\in Q\subset R\\ Q,R\in\mathcal{Q}(\mu,2)}} \frac{|m_Q(f) - m_R(f)|}{1+\delta(Q,R)}$$

and $Nf(x)$ is defined as $Nf(x) := \sup_{x\in Q\in\mathcal{Q}(\mu,2)} m_Q(|f|)$. It is well known that N is a bounded operator on $L^p(\mu)$ with $p > 1$ and by $\|N\|_p$ we will denote the operator norm. Since there are a lot of small doubling cubes, we have also a pointwise estimate : $|f(x)| \le Nf(x)$ for μ-a.e. $x \in \mathbf{R}^d$. In this paper the weight w will be a non-negative function on $\mathbf{R}^d$ satisfying a (mild) condition :

$$w \in L^{p_0}(\mu) \text{ for some } p_0 > 1 \tag{2}$$

and $w(A)$, $A \subset \mathbf{R}^d$, will denote $\int_A w(x)\,d\mu(x)$. We shall prove the following theorem.

Theorem **2** *Suppose that w satisfies* (2)*. For every $f \in L^1_{loc}(\mu)$, $Q_0 \in \mathcal{Q}(\mu)$, $q \in [1,\infty)$ and $\alpha \in (0,1)$, there exists a constant C independent of f such that*

$$\left(\int_{Q_0} |f(x) - m_{(Q_0)^*}(f)|^q\,w(x)^\alpha\,d\mu(x)\right)^{\frac{1}{q}} \le C\left(\int_{\frac{3}{2}Q_0} M^\sharp f(x)^q\,W(x)^\alpha\,d\mu(x)\right)^{\frac{1}{q}}.$$

Here, denoting N^j as the j-th composition of the operator N, we put

$$W(x) := \sum_{j=1}^\infty (2\,\beta)^{1-j} N^j w(x), \quad \beta \ge \|N\|_{p_0}. \tag{3}$$

For the weighted inequalities on nonhomogeneous spaces, we refer to [3] and [7].

2 Proof of Theorem 2

The letter C will be used for constants that may change from one occurrence to another. Constants with subscripts, such as C_0 and C_1, do not change in different occurrences.

The cubes with generation In the sequel we follow [12] with minor modifications.

LEMMA **3** *The following properties hold* :

(1) *For* $\rho > 1$ *and* $Q \in \mathcal{Q}(\mu)$, *we have* $\delta(Q, \rho Q) \le C_0 \log \rho$.

(2) *Let* $Q \in \mathcal{Q}(\mu)$. *Then* $\delta(Q, Q^*) \le C_0\, 2^{n+1} \log 2$.

(3) *Let* $Q \in \mathcal{Q}(\mu)$, $k_0 \in \mathbf{N}$ *and* $\alpha > 0$. *Suppose that, for some* $\theta > 0$,

$$\alpha \le \mu(Q) \le \mu(2^{k_0} Q) \le \theta\, \alpha.$$

Then $\delta(Q, 2^{k_0}Q) \le 2^n \log 2 \cdot \theta\, C_0\, c_n$, *where* $c_n := \sum_{k=0}^{\infty} 2^{-nk}$.

Proof. (1) follows easily from the growth condition. We prove (2) first. Let $Q^* = 2^{k_1}Q$. The dyadic argument yields that $\delta(Q, 2^{k_1}Q) = \int_{\ell(Q)}^{\ell(2^{k_1}Q)} \frac{\mu(Q(z_Q, l))}{l^n} \frac{dl}{l} \le 2^n \log 2 \sum_{k=1}^{k_1} \frac{\mu(2^k Q)}{\ell(2^k Q)^n}$. By the growth condition we have $d - n \ge 0$. The definitions of Q^* and the doubling cubes imply $2^{d+1}\, \mu(2^{k-1}Q) \le \mu(2^k Q)$, $\quad k = 1, 2, \dots, k_1$. These observations yield

$$\delta(Q, 2^{k_1}Q) \le 2^n \log 2\, \frac{\mu(2^{k_1}Q)}{\ell(2^{k_1}Q)^n} \sum_{k=1}^{k_1} (2^{n-d-1})^{k_1 - k} \le C_0\, 2^{n+1} \log 2.$$

What remains to be done is (3). It follows from the dyadic argument and the assumption that

$$\delta(Q, 2^{k_0}Q) \le 2^n \log 2 \sum_{k=1}^{k_0} \frac{\mu(2^k Q)}{\ell(2^k Q)^n} \le 2^n \log 2 \cdot \theta\, \frac{\alpha}{\ell(Q)^n} \sum_{k=1}^{k_0} 2^{-nk} \le 2^n \log 2 \cdot \theta\, C_0\, c_n.$$

The proof of the lemma is concluded. ∎

Given two cubes $Q \subset R$, we denote

$$\tilde{\delta}(Q, R) := \int_{\ell(Q)}^{\ell(Q^R)} \frac{\mu(Q(z_Q, l))}{l^n} \frac{dl}{l},$$

where Q^R denotes the largest cube concentric to Q contained in R. We will treat points $x \in \operatorname{supp}(\mu)$ as if they were cubes (with $\ell(x) = 0$). So, for $x \in \operatorname{supp}(\mu)$ and some cube $R \ni x$, the notations $\tilde{\delta}(x, R)$ and x^R make sense.

Let $C_1 = C_0\, 2^{n+1} \log 2$. Fix $Q_0 \in \mathcal{Q}(\mu)$ and let $Q_1 = \dfrac{3}{2} Q_0$.

Lemma **4** *If $\alpha > 3C_1$, then, for each $x \in Q_0 \cap \operatorname{supp}(\mu)$ with $\tilde{\delta}(x, Q_1) > \alpha$, there exists some doubling cube $Q \subset Q_1$ centered at x satisfying*

$$|\tilde{\delta}(Q, Q_1) - \alpha| \le 2C_1.$$

Proof. Let R be a unique cube of the form $2^{-k} x^{Q_1}$, $k = 1, 2, \ldots$, such that

$$\tilde{\delta}(2R, Q_1) \le \alpha < \tilde{\delta}(R, Q_1).$$

Then $\alpha < \tilde{\delta}(R, Q_1) = \tilde{\delta}(R, 2R) + \tilde{\delta}(2R, Q_1) \le C_0 \log 2 + \tilde{\delta}(2R, Q_1)$. This implies $2C_1 \le \tilde{\delta}(2R, Q_1)$ and hence, by Lemma 3 (2), $Q := (2R)^* \subset Q_1$. It follows by Lemma 3 again that $\alpha < \tilde{\delta}(R, Q_1) = \tilde{\delta}(R, Q) + \tilde{\delta}(Q, Q_1) \le 2C_1 + \tilde{\delta}(Q, Q_1)$, and that $\tilde{\delta}(Q, Q_1) \le \tilde{\delta}(2R, Q_1) \le \alpha$. Thus, we have $|\tilde{\delta}(Q, Q_1) - \alpha| \le 2C_1$. ∎

Fix $A > 3C_1$. Let $m \ge 1$ be some fixed integer and $x \in Q_0 \cap \operatorname{supp}(\mu)$. If $\tilde{\delta}(x, Q_1) > mA$, we denote by $Q_{x,m}$ a doubling cube centered at x and contained in Q_1 such that

$$|\tilde{\delta}(Q_{x,m}, Q_1) - mA| \le 2C_1.$$

The cubes $Q_{x,m}$, $x \in Q_0 \cap \operatorname{supp}(\mu)$, are called cubes of the m-th generation. The set of all cubes with m-th generation will be denoted by D_m and the set $\bigcup_m D_m$ will be denoted by D.

Lemma **5** *Assume that A is big enough.*

(1) *For every $Q_{x,m}$, $Q_{x,m+1} \in D$, we have $100\, Q_{x,m+1} \subset Q_{x,m}$.*

(2) *If $x, y \in \operatorname{supp}(\mu)$ are such that $Q_{x,m} \cap Q_{y,m+1} \ne \emptyset$, then $\ell(Q_{y,m+1}) \le \dfrac{1}{8}\, \ell(Q_{x,m})$.*

Proof. To prove (1) we resort to the reduction-to-the-absurdity argument. Suppose that $Q_{x,m} \subset 100\, Q_{x,m+1}$. Then

$$\begin{aligned} (m+1)A - 2C_1 &\le \tilde{\delta}(Q_{x,m+1}, Q_1) = \tilde{\delta}(Q_{x,m+1}, Q_{x,m}) + \tilde{\delta}(Q_{x,m}, Q_1) \\ &\le \tilde{\delta}(Q_{x,m+1}, 100 Q_{x,m+1}) + mA + 2C_1 \le C_0 \log 100 + mA + 2C_1. \end{aligned}$$

This implies $A \le C_0 \log 100 + 4C_1$. Thus, we have $Q_{x,m} \supset 100\, Q_{x,m+1}$, if $A > C_0 \log 100 + 4C_1$.

We now turn to the proof of (2). Put $P = Q_{y,m+1}$ and $P' = Q_{x,m}$. If $\ell(P) > \dfrac{1}{8} \ell(P')$, then $P' \subset 24P$. As a result, defining $R := Q(x, 48\ell(P))$, we have $P, P' \subset 24P \subset R \subset 72P \subset Q_1$ and hence

$$\tilde{\delta}(P, R) \le \delta(P, 72P) \le C. \tag{4}$$

We now claim that

$$S := |\tilde{\delta}(P^R, Q_1) - \tilde{\delta}(R, Q_1)| \le C. \tag{5}$$

We decompose S as

$$\begin{aligned} S &= \left| \int_{\ell(P^R)}^{\ell(P^{Q_1})} \frac{\mu(Q(y,l))}{l^n} \frac{dl}{l} - \int_{\ell(R)}^{\ell(R^{Q_1})} \frac{\mu(Q(x,l))}{l^n} \frac{dl}{l} \right| \\ &\le \int_{\ell(P^R)}^{\ell(R)} \frac{\mu(Q(y,l))}{l^n} \frac{dl}{l} + \left| \int_{\ell(R)}^{\min\{\ell(P^{Q_1}),\ell(R^{Q_1})\}} (\mu(Q(y,l)) - \mu(Q(x,l))) \frac{dl}{l^{n+1}} \right| \\ &\quad + \int_{\min\{\ell(P^{Q_1}),\ell(R^{Q_1})\}}^{\max\{\ell(P^{Q_1}),\ell(R^{Q_1})\}} \left(\frac{\mu(Q(y,l))}{l^n} + \frac{\mu(Q(x,l))}{l^n} \right) \frac{dl}{l} =: S_1 + S_2 + S_3. \end{aligned}$$

The integrals S_1 and S_3 are easily estimated from above by some constant C. What remains to be majorized is S_2. We bound S_2 from above by

$$S_2 \le \int_{\ell(R)}^{\infty} \mu(Q(y,l) \Delta Q(x,l)) \frac{dl}{l^{n+1}} = \int_{\ell(R)}^{\infty} \int_{\mathbf{R}^d} \chi_{Q(y,l)\Delta Q(x,l)}(z)\, d\mu(z) \frac{dl}{l^{n+1}},$$

where χ_A is the indicator function of a set $A \subset \mathbf{R}^d$. Let $|z|_\infty := \max\{|z_1|, |z_2|, \ldots, |z_d|\}$. Then a simple geometric observation tells us that

$$\chi_{Q(y,l)\Delta Q(x,l)}(z) = 0 \text{ if } l \notin [\min\{|z-x|_\infty, |z-y|_\infty\}, \max\{|z-x|_\infty, |z-y|_\infty\}].$$

This observation and Fubini's theorem yield

$$\begin{aligned} S_2 &\le C \int_{\mathbf{R}^d \setminus P} \left| \frac{1}{|z-x|_\infty{}^n} - \frac{1}{|z-y|_\infty{}^n} \right| d\mu(z) \\ &\le C \int_{|z-y|_\infty \ge \frac{\ell(P)}{2}} \frac{|y-x|_\infty\, d\mu(z)}{|z-y|_\infty{}^{n+1}} \le C \frac{|y-x|_\infty}{\ell(P)} \le C. \end{aligned}$$

This proves (5).

From (4) and (5) we have

$$\begin{aligned} \tilde{\delta}(P, Q_1) &= \tilde{\delta}(P, R) + \tilde{\delta}(P^R, Q_1) \\ &\le \tilde{\delta}(P, R) + |\tilde{\delta}(P^R, Q_1) - \tilde{\delta}(R, Q_1)| + \tilde{\delta}(R, Q_1) \le \tilde{\delta}(P', Q_1) + C \end{aligned}$$

and hence $(m+1)A \le mA + 4C_1 + C$. Thus, if $A > 4C_1 + C$, we see that $\ell(Q_{y,m+1}) \le \frac{1}{8}\ell(Q_{x,m})$. ∎

The weight W Now we shall show the simple properties of the weight W defined by (3). To begin with we notice that N is subadditive and W satisfies (so called) A_1 condition :

$$NW(x) \le 2\beta\, W(x) \text{ for } \mu\text{-a.e. } x \in \mathbf{R}^d. \tag{6}$$

Indeed,

$$NW(x) \le \sum_{j=1}^{\infty} (2\beta)^{1-j} N^{j+1} w(x) = 2\beta \left\{ \sum_{j=1}^{\infty} (2\beta)^{1-j} N^j w(x) - Nw(x) \right\} \le 2\beta\, W(x).$$

This implies the following lemma.

LEMMA **6** *Let $\alpha \in (0,1)$ and $Q \in \mathcal{Q}(\mu, 2)$. Then, for any μ-measurable subset $A \subset Q$, we have*

$$\frac{W^\alpha(A)}{W^\alpha(Q)} \le (2\beta)^\alpha \left(\frac{\mu(A)}{\mu(Q)}\right)^{1-\alpha}.$$

Proof. It follows by Hölder's inequality and (6) that

$$\begin{aligned} W^\alpha(A) &= \int_A W^\alpha(x)\, d\mu(x) \le \left(\int_A W\, d\mu\right)^\alpha \mu(A)^{1-\alpha} \\ &\le W(Q)^\alpha \mu(A)^{1-\alpha} = \mu(Q) \left(\frac{W(Q)}{\mu(Q)}\right)^\alpha \left(\frac{\mu(A)}{\mu(Q)}\right)^{1-\alpha} \\ &\le \left(\int_Q NW(x)^\alpha\, d\mu(x)\right) \cdot \left(\frac{\mu(A)}{\mu(Q)}\right)^{1-\alpha} \le (2\beta)^\alpha\, W^\alpha(Q) \cdot \left(\frac{\mu(A)}{\mu(Q)}\right)^{1-\alpha}. \end{aligned}$$

This proves the lemma. ∎

Proof of Theorem 2 Choose A large enough so that Lemma 5 holds and fix D_m and D. Letting $F(x) := |f(x) - m_{(2Q_0)^*}(f)|$, we consider a maximal function

$$N_D F(x) := \sup_{x \in Q \in D} m_Q(F), \quad x \in Q_1.$$

If $x \in Q_1 \setminus \bigcup_{Q \in D} Q$, it will be understood that $N_D F(x)$ is equal to zero.

CLAIM **7** *For μ-a.e. $x \in Q_0 \cap \operatorname{supp}(\mu)$ we have*

$$|f(x) - m_{(Q_0)^*}(f)| \le C\, M^\sharp f(x) + F(x), \quad F(x) \le C\left(M^\sharp f(x) + N_D F(x)\right).$$

Proof. Since $\delta((Q_0)^*, (2Q_0)^*) \le C$, the first inequality is obvious. So we prove the second one. To begin with, we notice that, for μ-a.e. $x \in Q_0 \cap \operatorname{supp}(\mu)$, there exists a sequence of doubling cubes $\{Q_k\}_k$ centered at x with $\ell(Q_k) \to 0$ as $k \to \infty$ and (see [10])

$$\lim_{k \to \infty} m_{Q_k}(F) = F(x). \tag{7}$$

Fix $x \in Q_0 \cap \operatorname{supp}(\mu)$. If $\tilde{\delta}(x, Q_1) = \infty$, $\{Q_{x,m}\}$ satisfies $\ell(Q_{x,m}) \to 0$ as $m \to \infty$ and hence $F(x) \le N_D F(x)$. If $\tilde{\delta}(x, Q_1) \in (mA, (m+1)A]$, for sufficiently small doubling cube Q centered at x and contained in $Q_{x,m}$, we have $\delta(Q, Q_{x,m}) \le C$. Thus, we see that

$$\begin{aligned} m_Q(F) &= m_Q(|f - m_{(2Q_0)^*}(f)|) \\ &\le m_Q(|f - m_Q(f)|) + |m_Q(f) - m_{Q_{x,m}}(f)| + |m_{Q_{x,m}}(f) - m_{(2Q_0)^*}(f)| \\ &\le C\left(m_Q(|f - m_Q(f)|) + \frac{|m_Q(f) - m_{Q_{x,m}}(f)|}{1 + \delta(Q, Q_{x,m})}\right) + m_{Q_{x,m}}(|f - m_{(2Q_0)^*}(f)|) \\ &\le C\left(M^\sharp f(x) + N_D F(x)\right). \end{aligned}$$

If $\tilde{\delta}(x, Q_1) \le A$, for sufficiently small doubling cube Q centered at x and contained in Q_1, we have $\tilde{\delta}(Q, (2Q_0)^*) \le C$ and hence

$$m_Q(F) \le m_Q(|f - m_Q(f)|) + |m_Q(f) - m_{(2Q_0)^*}(f)| \le C\, M^\sharp f(x).$$

These observations and (7) yield the claim. ∎

From Claim 7 and the fact that $w(x) \le W(x)$ for proving the theorem it suffices to show the following claim.

Claim **8** *We have*

$$\left(\int_{Q_1} N_D F(x)^q\, W(x)^\alpha\, d\mu(x)\right)^{\frac{1}{q}} \le C \left(\int_{Q_1} M^\sharp f(x)^q\, W(x)^\alpha\, d\mu(x)\right)^{\frac{1}{q}}.$$

We shall prove Claim 8 by means of the good-λ inequality.

Lemma **9** *If $\eta > 0$ is sufficiently small, there exists a constant C so that, for every $\lambda > 0$,*

$$W^\alpha\{x \in Q_1 \,|\, N_D F(x) > 2\lambda,\ M^\sharp f(x) \le \eta\lambda\} \le C\, \eta^{1-\alpha}\, W^\alpha\{x \in Q_1 \,|\, N_D F(x) > \lambda\}.$$

Proof. Choose $\eta > 0$ sufficiently small. We set

$$E_\lambda := \{x \in Q_1 \,|\, N_D F(x) > 2\lambda,\ M^\sharp f(x) \le \eta\lambda\} \text{ and } \Omega_\lambda := \{x \in Q_1 \,|\, N_D F(x) > \lambda\}.$$

We may assume that E_λ is not empty. For all $x \in E_\lambda$, we can select a doubling cube $Q_x = Q_{z(x),m(x)} \in D$, $Q_x \ni x$, that satisfies $m_{Q_x}(F) > \frac{3}{2}\lambda$. If $m(x) = 1$, we have $\delta(Q_x, (2Q_0)^*) < C$ and hence

$$m_{Q_x}(F) \le m_{Q_x}(|f - m_{Q_x}(f)|) + |m_{Q_x}(f) - m_{(2Q_0)^*}(f)| \le C\, M^\sharp f(x) \le C\, \eta\lambda.$$

As a result we obtain $C\,\eta\lambda > \frac{3}{2}\lambda$, which is not possible for sufficiently small η. By replacing younger one, if necessary, we may assume that $m_{Q_{z,m}}(F) < \frac{3}{2}\lambda$ for any cube $Q_{z,m} \ni x$ with $m < m(x)$.

Let $S_x = Q_{z(x),m(x)-1}$. We claim that if η is small enough, we have $m_{S_x}(F) > \lambda$. Indeed, noticing $\delta(Q_x, S_x) \le 2A$, we see that

$$\begin{aligned} &m_{Q_x}(F) \\ &\quad\le\ m_{Q_x}(|f - m_{Q_x}(f)|) + |m_{Q_x}(f) - m_{S_x}(f)| + |m_{S_x}(f) - m_{(2Q_0)^*}(f)| \\ &\quad\le\ C\, M^\sharp f(x) + m_{S_x}(F) \le C\, \eta\, \lambda + m_{S_x}(F). \end{aligned}$$

This yields $m_{S_x}(F) \ge \frac{3}{2}\lambda - C\,\eta\lambda > \lambda$. Thus, we have

$$\frac{3}{2}\lambda > m_{S_x}(F) > \lambda \tag{8}$$

for sufficient small η.

Notice that by Lemma 5 (1) $Q_x \subset \dfrac{1}{100} S_x$. By Besicovitch's covering lemma there exists a countable subset $\{x_j\}_{j\in J} \subset E_\lambda$ such that

$$E_\lambda \subset \bigcup_{j\in J} S_{x_j} \text{ and } \sum_{j\in J} \chi_{S_{x_j}} \leq C\,\chi_{\Omega_\lambda}. \tag{9}$$

To simplify the notation, we write $S_j = S_{x_j}$ and $Q_j = Q_{x_j}$. Now we claim the following:

Claim **10** *If η is small enough, then*

$$W^\alpha(S_j \cap E_\lambda) \leq C\,\eta^{1-\alpha}\, W^\alpha(S_j) \text{ for all } j \in J.$$

Let us temporarily assume Claim 10. Then (9) and the claim lead us to

$$W^\alpha(E_\lambda) \leq \sum_{j\in J} W^\alpha(S_j \cap E_\lambda) \leq C\,\eta^{1-\alpha} \sum_{j\in J} W^\alpha(S_j) \leq C\,\eta^{1-\alpha}\, W^\alpha(\Omega_\lambda).$$

Consequently, we are left with the task of proving the claim.

Proof of Claim 10. By Lemma 6 it suffices to show that

$$\mu(S_j \cap E_\lambda) \leq C\,\eta\,\mu(S_j).$$

Let $y \in S_j \cap E_\lambda$. There exists a doubling cube $R_y = Q_{z(y),m(y)} \in D$, $R_y \ni y$, that satisfies $m_{R_y}(F) > 2\lambda$. We show that for sufficiently small η, $\ell(R_y) \leq \dfrac{1}{8}\ell(S_j)$. From Lemma 5 (2) we may assume that $m(y) < m(x_j)$. By Lemma 5 (1) if $\ell(R_y) > \dfrac{1}{8}\ell(S_j)$, then $Q_{z(y),m(y)-1} \supset S_j \supset Q_j$. This and $m(y)-1 < m(x_j)$ imply $\dfrac{3}{2}\lambda > m_{Q_{z(y),m(y)-1}}(F)$. Notice that $m_{R_y}(F)$ can be bounded from above by

$$m_{R_y}(|f - m_{R_y}(f)|) + |m_{R_y}(f) - m_{Q_{z(y),m(y)-1}}(f)| + m_{Q_{z(y),m(y)-1}}(|f - m_{(2Q_0)^*}(f)|)$$

and, as a consequence, it can be bounded by $C\,M^\sharp f(y) + m_{Q_{z(y),m(y)-1}}(F)$. Thus, it follows that $\dfrac{3}{2}\lambda > m_{Q_{z(y),m(y)-1}}(F) \geq m_{R_y}(F) - C\,M^\sharp f(y) \geq 2\lambda - C\,\eta\lambda$. Hence, if $\eta < \dfrac{1}{3C}$, we must have $\ell(R_y) \leq \dfrac{1}{8}\ell(S_j)$. Thus,

$$N_D\left(\chi_{\frac{5}{4}S_j} F\right)(y) > 2\lambda \text{ for all } y \in S_j \cap E_\lambda.$$

From (8) we obtain that $|m_{S_j}(f) - m_{(2Q_0)^*}(f)| \leq \dfrac{3}{2}\lambda$, and that

$$N_D\left(\chi_{\frac{5}{4}S_j}(f - m_{S_j}(f))\right)(y) > \frac{\lambda}{2} \text{ for all } y \in S_j \cap E_\lambda.$$

It follows by using the weak-$(1,1)$ boundedness of N_D that

$$\mu(S_j \cap E_\lambda) \le \mu\left\{y \,|\, N_D\left(\chi_{\frac{5}{4}S_j}(f - m_{S_j}(f))\right)(y) > \frac{\lambda}{2}\right\} \le \frac{C}{\lambda}\int_{\frac{5}{4}S_j} |f - m_{S_j}(f)|\, d\mu.$$

Noticing that

$$\begin{aligned}
&\frac{1}{\mu\left(\frac{15}{8}S_j\right)}\int_{\frac{5}{4}S_j} |f - m_{S_j}(f)|\, d\mu \\
\le\ & \frac{1}{\mu\left(\frac{15}{8}S_j\right)}\int_{\frac{5}{4}S_j} |f - m_{(\frac{5}{4}S_j)^*}(f)|\, d\mu + \left|m_{(\frac{5}{4}S_j)^*}(f) - m_{S_j}(f)\right| \le C\,\eta\lambda,
\end{aligned}$$

we see that $\mu(S_j \cap E_\lambda) \le C\,\eta\,\mu(2S_j) \le C\,\eta\,\mu(S_j)$. ∎

We return to the proof of Claim 8.

Proof of Claim 8. Using Lemma 9 with $\eta > 0$ sufficiently small, for $L \ge 1$ we see that

$$\begin{aligned}
&\frac{1}{2}\left(\int_0^L q\lambda^{q-1} W^\alpha\{x \in Q_1 \,|\, N_D F(x) > \lambda\}\, d\lambda\right)^{\frac{1}{q}} \\
=\ & \left(\int_0^{L/2} q\lambda^{q-1} W^\alpha\{x \in Q_1 \,|\, N_D F(x) > 2\lambda\}\, d\lambda\right)^{\frac{1}{q}} \\
\le\ & \left(\int_0^{L/2} q\lambda^{q-1} W^\alpha\{x \in Q_1 \,|\, N_D F(x) > 2\lambda,\ M^\sharp f(x) \le \eta\lambda\}\, d\lambda\right)^{\frac{1}{q}} \\
+\ & \left(\int_0^{L/2} q\lambda^{q-1} W^\alpha\{x \in Q_1 \,|\, N_D F(x) > 2\lambda,\ M^\sharp f(x) > \eta\lambda\}\, d\lambda\right)^{\frac{1}{q}} \\
\le\ & \left(C\,\eta^{1-\alpha}\int_0^L q\lambda^{q-1} W^\alpha\{x \in Q_1 \,|\, N_D f(x) > \lambda\}\, d\lambda\right)^{\frac{1}{q}} \\
+\ & \left(\int_0^\infty q\lambda^{q-1} W^\alpha\{x \in Q_1 \,|\, M^\sharp f(x) > \eta\lambda\}\, d\lambda\right)^{\frac{1}{q}} \\
=\ & \left(C\,\eta^{1-\alpha}\int_0^L q\lambda^{q-1} W^\alpha\{x \in Q_1 \,|\, N_D F(x) > \lambda\}\, d\lambda\right)^{\frac{1}{q}} \\
+\ & \eta^{-1}\left(\int_{Q_1} (M^\sharp f)^q\, W^\alpha\, d\mu\right)^{\frac{1}{q}}.
\end{aligned}$$

When η is sufficiently small, we can bring the first term of the right side to the left. The result is

$$\left(\int_0^L q\lambda^{q-1} W^\alpha\{x \in Q_1 \,|\, N_D F(x) > \lambda\}\, d\lambda\right)^{\frac{1}{q}} \le C\left(\int_{Q_1} (M^\sharp f)^q\, W^\alpha\, d\mu\right)^{\frac{1}{q}}.$$

Letting $L \to \infty$, we obtain the claim. ∎

3 Application to vector-valued inequalities

Applying Theorem 2, we can obtain some vector-valued inequalities. To denote the vector-valued inequality we adopt the following notation. For a sequence of μ-measurable functions $\{f_j\}_{j=1}^{\infty}$ and $q, r \geq 1$, we denote

$$\|f_j(x) \,|\, l^r\| := \left(\sum_{j=1}^{\infty} |f_j(x)|^r \right)^{\frac{1}{r}}, \quad \|f_j \,|\, L^q(l^r, \mu)\| := \left(\int_{\mathbf{R}^d} \|f_j(x) \,|\, l^r\|^q \, d\mu(x) \right)^{\frac{1}{q}}.$$

First, we need the following lemma (see [6]).

Lemma **11** *If $q, r \in (1, \infty)$, then*

$$\|Nf_j \,|\, L^q(l^r, \mu)\| \leq C_{q,r} \; \|f_j \,|\, L^q(l^r, \mu)\|.$$

Proposition **12** *Let $q, r \in (1, \infty)$ and let $\{f_j\}_{j=1}^{\infty}$ be a sequence of $L^1_{loc}(\mu)$ functions. Then there exists a constant C independent of $\{f_j\}$ so that, for every $Q_0 \in \mathcal{Q}(\mu)$,*

$$\left(\int_{Q_0} \|f_j(x) - m_{(Q_0)^*}(f_j) \,|\, l^r\|^q \, d\mu(x) \right)^{\frac{1}{q}} \leq C \left(\int_{\frac{3}{2}Q_0} \|M^{\sharp} f_j(x) \,|\, l^r\|^q \, d\mu(x) \right)^{\frac{1}{q}}.$$

Proof. Passage to the limit allows us to assume $f_j \equiv 0$ for large j, say $j \geq N$, as long as we obtain the constants independent of N. Take $s \in (1, \min(q, r))$ and let $t = q/s$, $u = r/s$. Take α slightly smaller than 1 so that $1 < 1/\alpha < \min(t', u')$. By using a duality argument we shall estimate

$$\left(\int_{Q_0} \|f_j(x) - m_{(Q_0)^*}(f_j) \,|\, l^r\|^q \, d\mu(x) \right)^{\frac{s}{q}} = \left(\int_{Q_0} \||f_j(x) - m_{(Q_0)^*}(f_j)|^s \,|\, l^u\|^t \, d\mu(x) \right)^{\frac{1}{t}}.$$

Take a vector-valued weight $(w_1, w_2, \ldots)$ such that $\operatorname{supp} w_j \subset Q_0$ and

$$\left\| {w_j}^{\alpha} \,|\, L^{t'}(l^{u'}, \mu) \right\| = 1. \tag{10}$$

Then it follows by Theorem 2 and Hölder's inequality that

$$\begin{aligned}
&\int_{Q_0} \||f_j(x) - m_{(Q_0)^*}(f_j)|^s \, {w_j}^{\alpha}(x) \,|\, l^1\| \, d\mu(x) \\
&\leq \int_{\frac{3}{2}Q_0} \|M^{\sharp} f_j(x)^s \, {W_j}^{\alpha}(x) \,|\, l^1\| \, d\mu(x) \\
&\leq \left(\int_{\frac{3}{2}Q_0} \|M^{\sharp} f_j(x)^s \,|\, l^u\|^t \, d\mu(x) \right)^{\frac{1}{t}} \times \left(\int_{\frac{3}{2}Q_0} \|{W_j}^{\alpha}(x) \,|\, l^{u'}\|^{t'} \, d\mu(x) \right)^{\frac{1}{t'}},
\end{aligned}$$

where $W_j(x) := \sum_{k=1}^{\infty}(2\,\beta)^{1-k}N^k w_j(x)$. Choose β as the constant $C_{\alpha t',\alpha u'}$ in Lemma 11. Then Lemma 11, the definition of W_j and (10) yield

$$\left(\int_{\frac{3}{2}Q_0}\|W_j^\alpha(x)\,|\,l^{u'}\|^{t'}\,d\mu(x)\right)^{\frac{1}{\alpha t'}} = \left(\int_{\frac{3}{2}Q_0}\|W_j(x)\,|\,l^{\alpha u'}\|^{\alpha t'}\,d\mu(x)\right)^{\frac{1}{\alpha t'}} \leq C.$$

These prove the proposition. ∎

The following corollary is a vector-valued extension of Theorem 1 (2).

Corollary **13** *Let $f_j \in$RBMO. For any cube $Q_0 \in \mathcal{Q}(\mu)$ and $q, r \in (1,\infty)$, there exists a constant C independent of f_j such that*

$$\left(\frac{1}{\mu\left(\frac{3}{2}Q_0\right)}\int_{Q_0}\|f_j(x) - m_{(Q_0)^*}(f_j)\,|\,l^r\|^q\,d\mu(x)\right)^{\frac{1}{q}} \leq C\sup_{x\in\mathbf{R}^d}\|M^\sharp f_j(x)\,|\,l^r\|.$$

We apply Proposition 12 to obtain a sharp maximal inequality on the Morrey spaces.

Let $k > 1$ and $1 \leq q \leq p < \infty$. We define the Morrey space $\mathcal{M}_q^p(k,\mu)$ as

$$\mathcal{M}_q^p(k,\mu) := \left\{f \in L_{loc}^q(\mu)\,|\,\|f\,|\,\mathcal{M}_q^p(k,\mu)\| < \infty\right\},$$

where

$$\|f\,|\,\mathcal{M}_q^p(k,\mu)\| := \sup_{Q\in\mathcal{Q}(\mu)}\mu(k\,Q)^{\frac{1}{p}-\frac{1}{q}}\left(\int_Q |f|^q\,d\mu\right)^{\frac{1}{q}}. \tag{11}$$

By applying Hölder's inequality to (11) it is easy to see that

$$L^p(\mu) = \mathcal{M}_p^p(k,\mu) \subset \mathcal{M}_{q_1}^p(k,\mu) \subset \mathcal{M}_{q_2}^p(k,\mu) \tag{12}$$

for $1 \leq q_2 \leq q_1 \leq p < \infty$. Let $k_1 > k_2 > 1$. Then $\mathcal{M}_q^p(k_1,\mu)$ and $\mathcal{M}_q^p(k_2,\mu)$ coincide as a set and their norms are mutually equivalent. More precisely, we have (see [8])

$$\|f\,|\,\mathcal{M}_q^p(k_1,\mu)\| \leq \|f\,|\,\mathcal{M}_q^p(k_2,\mu)\| \leq C_d\left(\frac{k_1-1}{k_2-1}\right)^d\|f\,|\,\mathcal{M}_q^p(k_1,\mu)\|. \tag{13}$$

Nevertheless, for definiteness, we will assume $k = 2$ in the definition and denote $\mathcal{M}_q^p(2,\mu)$ by $\mathcal{M}_q^p(\mu)$. For a sequence of μ-measurable functions $\{f_j\}_{j=1}^\infty$, we also denote

$$\left\|f_j\,|\,\mathcal{M}_q^p(l^r,\mu)\right\| := \left\|\,\|f_j\,|\,l^r\|\,|\,\mathcal{M}_q^p(\mu)\right\|.$$

The following proposition is a vector-valued extension of [9, Corollary 1.5].

Proposition **14** *Let $\{f_j\}_{j=1}^\infty$ be a sequence of $L_{loc}^1(\mu)$ functions. Suppose that $1 < q \leq p < \infty$, $r \in (1,\infty)$ and there exists an increasing sequence of concentric doubling cubes $I_1 \subset I_2 \subset \ldots$ such that*

$$\lim_{k\to\infty} m_{I_k}(f_j) = 0 \textit{ for all } j \in \mathbf{N} \textit{ and } \bigcup_{k=1}^\infty I_k = \mathbf{R}^d. \tag{14}$$

Then there exists a constant C independent of $\{f_j\}$ such that

$$\left\| f_j \,|\, \mathcal{M}^p_q(l^r,\mu) \right\| \le C \left\| M^\sharp f_j \,|\, \mathcal{M}^p_q(l^r,\mu) \right\|.$$

Proof. We may assume that $f_j \equiv 0$ for sufficiently large j. Letting $R \in \mathcal{Q}(\mu)$, we shall estimate $\mu(2R)^{\frac{1}{p}-\frac{1}{q}} \left(\int_R \|f_j(x) \,|\, l^r\|^q \, d\mu(x) \right)^{\frac{1}{q}}$. It follows by Proposition 12 that

$$\begin{aligned}
&\mu(2R)^{\frac{1}{p}-\frac{1}{q}} \left(\int_R \|f_j \,|\, l^r\|^q \, d\mu \right)^{\frac{1}{q}} \\
&\le \mu(2R)^{\frac{1}{p}-\frac{1}{q}} \left(\int_R \|f_j - m_{R^*}(f_j) \,|\, l^r\|^q \, d\mu \right)^{\frac{1}{q}} + \mu(R)^{\frac{1}{p}} \|m_{R^*}(f_j) \,|\, l^r\| \\
&\le C\, \mu(2R)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{\frac{3}{2}R} \|M^\sharp f_j \,|\, l^r\|^q \, d\mu \right)^{\frac{1}{q}} + \mu(R)^{\frac{1}{p}} \|m_{R^*}(f_j) \,|\, l^r\| \\
&\le C \left\| M^\sharp f_j \,|\, \mathcal{M}^p_q(l^r,\mu) \right\| + \mu(R)^{\frac{1}{p}} \|m_{R^*}(f_j) \,|\, l^r\|.
\end{aligned}$$

So we concentrate on estimating :

$$\mu(R)^{\frac{1}{p}} \|m_{R^*}(f_j) \,|\, l^r\|. \tag{15}$$

We choose doubling cubes inductively. Let $R_1 = R^*$ and $R_{m+1} = (2R_m)^*$, $m \in \mathbf{N}$. Let d be the distance between the center of R_1 and that of I_1. We select $m_1 \in \mathbf{N}$ so big that $\ell(R_{m_1}) \ge 2d$ and there exists some I_κ such that $R_{m_1} \subset I_\kappa$, $R_{m_1+1} \not\subset I_\kappa$ and

$$\mu(R)^{\frac{1}{p}} \|m_{I_\kappa}(f_j) \,|\, l^r\| \le \left\| \|M^\sharp f_j \,|\, l^r\| \,|\, \mathcal{M}^p_q(\mu) \right\|.$$

Then simple geometric observation shows that $R_{m_1} \subset I_\kappa \subset R_{m_1+3}$ and hence

$$\delta(R_{m_1}, I_\kappa) \le \delta(R_{m_1}, R_{m_1+3}) \le C. \tag{16}$$

We put for $i = 1, 2, \ldots$

$$M_i := \left\{ m \in \mathbf{N} \cap [1, m_1] \,|\, 2^{i-1}\mu(R) \le \mu(R_m) < 2^i \mu(R) \right\}.$$

Deleting all emptysets from $\{M_i\}_{i=1,2,\ldots}$, we obtain $\{M_i\}_{i=i_1,i_2,\ldots,i_{\kappa'}}$. Set $a(i_k) := \min M_{i_k}$ and $b(i_k) := \max M_{i_k}$. Then we notice that $R_{b(i_{\kappa'})} = R_{m_1}$.

For $k = 1, 2, \ldots, \kappa' - 1$, from Lemma 3 we see that

$$\delta(R_{a(i_k)}, R_{b(i_k)}),\ \delta(R_{b(i_k)}, R_{a(i_{k+1})}) \le C$$

and hence $\delta(R_{a(i_k)}, R_{a(i_{k+1})}) \le C$. This implies that

$$\begin{aligned}
&\mu(R)^{\frac{1}{p}} \left\| m_{R_{a(i_k)}}(f_j) - m_{R_{a(i_{k+1})}}(f_j) \,|\, l^r \right\| \\
&\quad \le C\, 2^{-\frac{i_k}{p}} \mu(R_{a(i_k)})^{\frac{1}{p}-\frac{1}{q}} \left(\int_{R_{a(i_k)}} \|M^\sharp f_j \,|\, l^r\|^q \, d\mu(x) \right)^{\frac{1}{q}} \le C\, 2^{-\frac{i_k}{p}} \left\| M^\sharp f_j \,|\, \mathcal{M}^p_q(l^r,\mu) \right\|.
\end{aligned}$$

Similarly, from (16) we also have

$$\mu(R)^{\frac{1}{p}} \left\| m_{R_{a(i_{\kappa'})}}(f_j) - m_{I_\kappa}(f_j) \,|\, l^r \right\| \le C\, 2^{-\frac{i_{\kappa'}}{p}} \left\| M^\sharp f_j \,|\, \mathcal{M}^p_q(l^r,\mu) \right\|.$$

Using triangle inequality to (15), we finally have

$$\begin{aligned}
&\mu(R)^{\frac{1}{p}} \|m_{R^*}(f_j) \,|\, l^r\| \\
&\le \mu(R)^{\frac{1}{p}} \sum_{k=1}^{\kappa'-1} \left\| m_{R_{a(i_k)}}(f_j) - m_{R_{a(i_{k+1})}}(f_j) \,|\, l^r \right\| \\
&+ \mu(R)^{\frac{1}{p}} \left\{ \left\| m_{R_{a(i_{\kappa'})}}(f_j) - m_{I_\kappa}(f_j) \,|\, l^r \right\| + \|m_{I_\kappa}(f_j) \,|\, l^r\| \right\} \\
&\le C \left(\sum_{k=1}^{\kappa'} 2^{-\frac{i_k}{p}} \right) \left\| M^\sharp f_j \,|\, \mathcal{M}^p_q(l^r,\mu) \right\| + \mu(R)^{\frac{1}{p}} \|m_{I_\kappa}(f_j) \,|\, l^r\| \\
&\le C \left\| M^\sharp f_j \,|\, \mathcal{M}^p_q(l^r,\mu) \right\|.
\end{aligned}$$

The proof is completed. ∎

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