UTMS 2005–43 November 29, 2005

Detection of irregular points by regularization in numerical differentiation and an application to edge detection

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DETECTION OF IRREGULAR POINTS BY REGULARIZATION IN NUMERICAL DIFFERENTIATION AND AN APPLICATION TO EDGE DETECTION

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Abstract. Numerical differentiation is a typical ill-posed problem which can be treated by the Tikhonov regularization. In this paper, we prove that the L^2-norms of the second order derivatives of the regularized solutions blow up in any small interval I where the exact solution is not in $H^2(I)$. One application in the image edge detection is presented.

1. Introduction

A numerical differentiation arises in many applications and engineering computations such as the determination of the underground water, the Dupire formulae in financial mathematics([25]), image edge detection([13]), etc. One of the main difficulties for this problem is the ill-posedness, which means that the small errors of measurement may cause huge errors in computed derivatives ([7], [12], [22]). Therefore one needs regularizing techniques for reasonable computations. Several numerical algorithms have been proposed for overcoming the instabilities ([5], [7], [8], [9], [10], [11], [12], [16],[19], [23]). It has been shown that the Tikhonov regularization for treating the numerical differentiation problem is one of the effective methods.

In [24], the authors discussed the numerical differentiation by using the Tikhonov regularization method. It is shown that, if the exact solution is smooth, then the

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Date: November 29, 2005.

Key words and phrases. Ill-posed problem, Numerical differentiation, Tikhonov regularization,Image edge detection.

This work is partially support by NSF of China (No. 10271032, No. 10431030) and the Laboratory of Nonlinear Sciences in Fudan University, Shanghai, China.

The third author was supported partly by Grant 15340027 from the Japan Society for the Promotion of Science and Grant 17654019 from the Ministry of Education, Cultures, Sports and Technology.

regularized solutions converge to the exact solution, and if the exact solution is not smooth, then the L^2-norms of the corresponding high order derivatives of the regularized solutions will blow up in the whole interval. The numerical results indicate that this kind of blow-up happens only near the irregular points. However to the authors' knowledge, this property has not been proved up to now.

In this paper we discuss this local property of the regularized solution based on the work of [24]. We prove that, if the exact solution is not in $H^2(I)$ for an interval I, then the norms of the second order derivatives of the regularized solutions in I blow up. Although we can develop a similar method also in multidimensional cases, we will discuss only one dimensional case, which is sufficient in many practical cases. In particular, for the edge detection, we use line-by-line scans to apply the numerical differentiation for functions in a single variable (section 4).

This paper is organized as follows: We give the formulation of the problem and show the theoretic results in section 2. In section 3 we will give numerical examples to verify this property. An application in image edge detection is presented in section 4. Some conclusions are given in section 5.

2. Formulation of the problem and theoretical result

Hereforth we set

$$g^{'} = \frac{dg}{dx}, \quad g^{''} = \frac{d^2 g}{dx^2}$$

Suppose that $\Delta = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ is a uniform grid of $[0, 1]$. Here $x_j = \frac{j}{n}$, $j = 0, 1, \cdots, n$, and we denote $h = \frac{1}{n}$. Let $y = y(x)$ be a continuous function defined on [0,1]. A noisy value of $y(x)$ at point x_j is given as y_j^δ which satisfies

$$|y(x_j) - y_j^\delta| < \delta, j = 0, 1, \cdots, n, \tag{2.1}$$

where δ is a given constant called the level of noise in the data. The numerical differentiation is then to approximate $y'(x)$ from the value of $y_j^\delta, j = 0, 1, \cdots, n$. Without loss of generality, we assume that there are no errors at the boundary for the sample data, i. e., $y_0^\delta = y(0)$, $y_n^\delta = y(1)$. Otherwise we can use a new function

$$Y(x) = y(x) + y_0^\delta - y(0) + (y_n^\delta - y(1) + y(0) - y_0^\delta)x.$$

It can be easily proved that $Y(0) = y_0^\delta$, $Y(1) = y_n^\delta$. Since

$$\left|y^{'}(x) - Y^{'}(x)\right| = \left|y_n^\delta - y(1) + y(0) - y_0^\delta\right| \leq 2\delta$$

Therefore the approximation of $Y^{'}(x)$ is also an approximation of $y^{'}(x)$ (see further [12], [24]).

The **numerical differentiation** is to find a function f from the data $\{y_j^\delta\}_{j=0}^n$ such that $f^{'}$ approximates $y^{'}$. We will solve this problem by the Tikhonov regularization method as in [24]. The following spaces and norms will be used in this paper:

$$L^2(0,1) = \left\{ g \,|\, (\int_0^1 g^2(x)dx)^{1/2} < \infty \right\},$$

$$H^2(0,1) = \left\{ g \,|\, g \in L^2(0,1),\, g^{''} \in L^2(0,1) \right\},$$

$$C[0,1] = \{g \,|\, g \text{ is a continuous function on [0,1]}\},$$

$$\|g\|_{L^2(0,1)} = \left(\int_0^1 |g(x)|^2 dx \right)^{1/2},$$

$$\|g\|_{H^2(0,1)} = \left(\|g\|^2_{L^2(0,1)} + \|g^{''}\|^2_{L^2(0,1)} \right)^{\frac{1}{2}},$$

$$\|g\|_{C[0,1]} = \max_{x\in[0,1]} |g(x)|.$$

Define a cost functional by

$$\Phi(f) = \frac{1}{n}\sum_{i=1}^{n-1}(f(x_i) - y_i^\delta)^2 + \alpha\|f^{''}\|^2_{L^2(0,1)} \tag{2.2}$$

for all $f \in H^2(0,1)$ with $f(0) = y(0)$, $f(1) = y(1)$, where α is a regularization parameter.

Then we can prove (e.g. [24]):

Theorem 2.1. *Let $y \in C[0,1]$. There exists a unique minimizer $f_* = f_*(\delta, \alpha, h)$ of functional* (2.2)*.*

The minimizer f_* is called the regularized solution, and we have the following error estimate ([24]).

Theorem 2.2. *Suppose that $y \in H^2(0,1)$. Then we have the following error estimation for the regularized solution $f_*(\delta,\alpha,h)$:*

$$\|f_*^{'}(\delta,\alpha,h) - y^{'}\|_{L^2(0,1)} \leq \left(2h + 4\alpha^{\frac{1}{4}} + \frac{h}{\pi}\right)\|y^{''}\|_{L^2(0,1)} + h\sqrt{\frac{\delta^2}{\alpha} + \frac{2\delta}{\alpha^{\frac{1}{4}}}}.$$

If we choose $\alpha = \delta^2$, then

$$\|f_*^{'}(\delta,\delta^2,h) - y^{'}\|_{L^2(0,1)} \leq (2h + 4\sqrt{\delta} + \frac{h}{\pi})\|y^{''}\|_{L^2(0,1)} + h + 2\sqrt{\delta}.$$

It is practically important to detect subintervals where a state function f is not smooth. Thus we are more interested in an interval $(a,b) \subset (0,1)$ where $y^{''} \notin L^2(a,b)$. Therefore, it is necessary to know the behaviour of the regularized solution f_* in a subinterval as short as possible. In particular, if $(a,b) = (0,1)$, then the following theorem is proved in [24].

Theorem 2.3. *If $y \in C[0,1] \setminus H^2(0,1)$, and we choose the regularization parameter $\alpha = \delta^2$, then*

$$\|f_*^{''}(\delta,\delta^2,h)\|_{L^2(0,1)} \longrightarrow \infty, \qquad as \quad \delta, h \to 0. \tag{2.3}$$

Remark 2.4. The general study corresponding to Theorem 2.3 is found in [3].

This result indicates only that, if the exact solution is not in $H^2(0,1)$, then the L^2-norms of the second order derivatives of the regularized solutions on the whole interval $[0,1]$ blow up. In this paper we will give a localized version in $(a,b) \subset (0,1)$ of this theorem. We want to know if the exact solution is not smooth on a small interval, namely, $y \in C[0,1] \setminus H^2(a,b)$, what will happen to $\|f_*^{''}\|_{L^2(a,b)}$?

We state our theoretical result on which our numerical method is based.

Theorem 2.5. *Suppose that $y \in C[0,1]$, $(a,b) \subset (0,1)$, and we choose the regularization parameter $\alpha = \delta^2$. If $y \notin H^2(a,b)$, then the regularized solution $f_*(\delta,\delta^2,h)$ satisfies*

$$\lim_{\delta,h\to 0} \|f_*^{''}(\delta,\delta^2,h)\|_{L^2(a,b)} = \infty.$$

Remark 2.6. For a piecewise continuous function, the conclusion in Theorem 2.5 is still true. We can prove it by the same method with some small modifications.

In the proof of this theorem, we will use an interpolation inequality. (e.g., Theorem 4.14 (p.75) in [1]):

Lemma 2.7. *Let $-\infty \le a < b \le \infty$, $1 \le p < \infty$, and $0 < \epsilon_0 < \infty$, f have the m-th order derivative $f^{(m)}$ in (a,b). There exists a constant $K > 0$ which depends on ϵ_0, p, m, and $b-a$ such that for every ϵ, $0 < \epsilon \le \epsilon_0$, $0 \le j < m$, we have*

$$(\int_a^b |f^{(j)}|^p dt)^{\frac{1}{p}} \le K\epsilon(\int_a^b |f^{(m)}|^p dt)^{\frac{1}{p}} + K\epsilon^{\frac{-j}{m-j}}(\int_a^b |f|^p dt)^{\frac{1}{p}}. \tag{2.4}$$

Proof of Theorem 2.5: Henceforth for simplicity, we set

$$f_*(\delta, n)(x) = f_*(\delta, \delta^2, h)(x)$$

where we recall $h = \frac{1}{n}$.

Assume contrarily that the conclusion of the theorem is not correct. This means that there exist two sequences $\{\delta_k\}, \{h_k\}$, $k = 1, 2, \cdots$, such that

$$\lim_{k\to\infty} \delta_k = \lim_{k\to\infty} h_k = 0$$

and

$$\|f_*^{''}(\delta_k, n_k)\|_{L^2(a,b)} \le M, \qquad k = 1, 2, \cdots \tag{2.5}$$

where M is a positive constant and $n_k = \frac{1}{h_k}$.

By $y \in C[0,1]$, we can take a sequence of functions $y_m \in H^2(0,1)$ satisfying

$$y_m(0) = y(0), \qquad y_m(1) = y(1),$$

and

$$\|y_m - y\|_{C[0,1]} \le \frac{1}{m}, \quad \sup_m \|y_m\|_{L^2(a,b)} < \infty. \tag{2.6}$$

In fact, y_m can be constructed for example by suitable interpolated polynomials.

Next we will prove

$$\sup_m \|y_m^{''}\|_{L^2(a,b)} = \infty. \tag{2.7}$$

Assume contrarily that $\sup_m \|y_m^{''}\|_{L^2(a,b)} < \infty$. Then from the definition of the norm, we know that $\sup_m \|y_m\|_{H^2(a,b)} < \infty$. By the reflexiveness of $H^2(a,b)$, there exists a subsequence $y_{m_k} \in H^2(a,b)$ and $\widetilde{y} \in H^2(a,b)$ so that $y_{m_k} \to \widetilde{y}$ weakly in $H^2(a,b)$. On the other hand, we see from (2.6) that $y_m \to y$ strongly in $L^2(a,b)$, so $\widetilde{y} = y$ and thus we have $y \in H^2(a,b)$. This is a contradiction. Thus the proof of (2.7) is complete.

Moreover, for each $k \in N$, we set

$$\ell(k) = \min\{j \in N; \delta_j \|y_k^{''}\|_{L^2(0,1)} < 1\}.$$

Such an $\ell(k)$ exists uniquely because $\delta_j \to 0$ as $j \to \infty$. By (2.7), we note that $\lim_{k\to\infty} \ell(k) = \infty$. For simplicity we denote $\delta_{\ell(k)}$ and $n_{\ell(k)}$ again by δ_k and n_k respectively. Hence

$$\delta_k \|y_k^{''}\|^2_{L^2(0,1)} < 1. \tag{2.8}$$

We set $f_{*k} = f_*(\delta_k, n_k)$. Since $\Phi(f_{*k}) \le \Phi(y_k)$, we have

$$\begin{aligned}
&\frac{1}{n_k}\sum_{j=1}^{n_k-1}\left(f_{*k}(x_j) - y_j^{\delta_k}\right)^2 \le \Phi(f_{*k}) \\
&\le \Phi(y_k) = \frac{1}{n_k}\sum_{j=1}^{n_k-1}(y_k(x_j) - y_j^{\delta_k})^2 + \delta_k^2\|y_k^{''}\|^2_{L^2(0,1)} \\
&\le \frac{2}{n_k}\sum_{j=1}^{n_k-1}\left((y_k(x_j) - y(x_j))^2 + (y(x_j) - y_j^{\delta_k})^2\right) + \delta_k(\delta_k\|y_k^{''}\|^2_{L^2(0,1)}) \\
&\le \frac{2}{k^2} + 2\delta_k^2 + \delta_k.
\end{aligned} \tag{2.9}$$

At the last inequality, we used (2.8) and

$$|y_k(x_j) - y(x_j)| \le \frac{1}{k}, \quad 1 \le j \le n_k - 1,$$

by (2.6).

Suppose that $x_{i_0-1} \le a < x_{i_0}$ and $x_{j_0} < b \le x_{j_0+1}$. Then by (2.9), we have

$$\begin{aligned}
\frac{1}{n_k}\sum_{j=i_0}^{j_0}\left(f_{*k}(x_j) - y_j^{\delta_k}\right)^2 &\le \frac{1}{n_k}\sum_{j=1}^{n_k-1}\left(f_{*k}(x_j) - y_j^{\delta_k}\right)^2 \\
&\le \frac{2}{k^2} + 2\delta_k^2 + \delta_k.
\end{aligned}$$

Moreover

$$\begin{aligned}
|y_j^{\delta_k}|^2 = (y_j^{\delta_k} - y(x_j) + y(x_j))^2 &\le 2((y_j^{\delta_k} - y(x_j))^2 + y(x_j)^2) \\
&\le 2(\delta_k^2 + \|y\|^2_{L^\infty(0,1)}).
\end{aligned}$$

Therefore by (2.1) and (2.9)

$$\frac{1}{n_k}\sum_{j=i_0}^{j_0} f_{*k}^2(x_j) \le \frac{2}{n_k}\sum_{j=i_0}^{j_0}\Big((f_{*k}(x_j)-y_j^{\delta_k})^2+|y_j^{\delta_k}|^2\Big) \tag{2.10}$$
$$\le \frac{4}{k^2}+4\delta_k^2+2\delta_k+4(\delta_k^2+\|y\|^2_{L^\infty(0,1)}) \le A.$$

Here $A>0$ is a constant which is independent of $k\in N$. Moreover, denote

$$\eta_{i_0}=a, \qquad \eta_i=\frac{x_{i-1}+x_i}{2}, i=i_0+1,\cdots,j_0, \qquad \eta_{j_0+1}=b.$$

By the mean value theorem, we can choose $\xi_j\in(\eta_j,\eta_{j+1})$, and

$$\begin{aligned}\|f_{*k}\|^2_{L^2(a,b)} &= \int_{\eta_{i_0}}^{\eta_{j_0+1}} f_{*k}^2(x)dx=\sum_{j=i_0}^{j_0}\int_{\eta_j}^{\eta_{j+1}} f_{*k}^2(x)dx\le \frac{2}{n_k}\sum_{j=i_0}^{j_0} f_*^2(\xi_j)\\ &= \frac{2}{n_k}\sum_{j=i_0}^{j_0}(f_{*k}^2(\xi_j)-f_{*k}^2(x_j))+\frac{2}{n_k}\sum_{j=i_0}^{j_0} f_{*k}^2(x_j)\\ &\le \frac{2}{n_k}\int_a^b\Big|2f_{*k}^{'}(x)f_{*k}(x)\Big|\,dx+\frac{2}{n_k}\sum_{j=i_0}^{j_0} f_{*k}^2(x_j).\end{aligned}$$

Here we used

$$\left|\sum_{j=i_0}^{j_0}(f_{*k}^2(\xi_j)-f_{*k}^2(x_j))\right| = \left|\sum_{j=i_0}^{j_0}\int_{x_j}^{\xi_j}\frac{d}{dx}(f_{*k}(x))^2dx\right|$$
$$\le \sum_{j=i_0}^{j_0}\int_{x_j}^{\xi_j} 2|f_{*k}^{'}(x)f_{*k}(x)|dx \le \int_{\eta_{i_0}}^{\eta_{j_0+1}} 2|f_{*k}^{'}(x)f_{*k}(x)|dx.$$

Hence (2.10) and the Schwarz inequality yield

$$\|f_{*k}\|^2_{L^2(a,b)} \le \frac{4}{n_k}\|f_{*k}\|_{L^2(a,b)}\|f_{*k}^{'}\|_{L^2(a,b)}+2A. \tag{2.11}$$

Choosing parameters as $p=2, m=2,\ j=1, \epsilon=1$ in Lemma 2.7, we have

$$\|f_{*k}^{'}\|_{L^2(a,b)} \le K(\|f_{*k}\|_{L^2(a,b)}+\|f_{*k}^{''}\|_{L^2(a,b)}). \tag{2.12}$$

Therefore, (2.11) yields

$$(1-\frac{4K}{n_k})\|f_{*k}\|^2_{L^2(a,b)} \le \frac{4K}{n_k}\|f_{*k}\|_{L^2(a,b)}\|f_{*k}^{''}\|_{L^2(a,b)}+2A.$$

Substituting

$$\|f_{*k}\|_{L^2(a,b)}\|f_{*k}^{''}\|_{L^2(a,b)} \le \frac{1}{2}\|f_{*k}\|^2_{L^2(a,b)}+\frac{1}{2}\|f_{*k}^{''}\|^2_{L^2(a,b)}$$

and choosing $k_0\in N$ large, we have

$$\|f_{*k}\|_{L^2(a,b)} \le K_1\|f_{*k}^{''}\|_{L^2(a,b)} + K_1\sqrt{A}$$

for all $k \ge k_0$. Here the constant $K_1 > 0$ is independent of k.

By using this inequality and (2.5), we have

$$\sup_{k\ge k_0} \|f_{*k}\|_{L^2(a,b)} < \infty.$$

Therefore, by (2.12) we have

$$\sup_k \|f_{*k}\|_{H^2(a,b)} < \infty, \quad k \ge k_0.$$

Since $H^2(a,b)$ is reflexive, there exist a subsequence f_{*k}, which is denoted by the same letter, and $\widetilde{f} \in H^2(a,b)$ such that

$$f_{*k} \to \widetilde{f} \quad \text{weakly in} \quad H^2(a,b).$$

Since the embedding from $H^2(a,b)$ to $C[a,b]$ is compact, we see that

$$f_{*k} \to \widetilde{f} \quad \text{strongly in} \quad C[a,b].$$

That is,

$$\lim_{k\to\infty} \|f_{*k} - \widetilde{f}\|_{C[a,b]} = 0. \tag{2.13}$$

By the definition of the integral, for any $\epsilon > 0$, there exists $k = k(\epsilon) \in N$ such that by (2.1) and (2.9) we obtain

$$\begin{aligned}
&\|y - \widetilde{f}\|^2_{L^2(a,b)} \\
\le\ & \frac{1}{n_k}\sum_{j=i_0}^{j_0}(y(x_j) - \widetilde{f}(x_j))^2 + \epsilon \\
\le\ & \frac{3}{n_k}\sum_{j=i_0}^{j_0}\{(y(x_j) - y_j^{\delta_k})^2 + (y_j^{\delta_k} - f_{*k}(x_j))^2 + (f_{*k}(x_j) - \widetilde{f}(x_j))^2\} + \epsilon \\
\le\ & 3\delta_k^2 + \frac{6}{k^2} + 6\delta_k^2 + 3\delta_k + 3\|f_{*k} - \widetilde{f}\|_{C[a,b]} + \epsilon
\end{aligned}$$

for any $k \ge k_1(\epsilon)$.

Hence letting $k \to \infty$, we have

$$\|y - \widetilde{f}\|_{L^2(a,b)} \le \epsilon.$$

Since ϵ is arbitrary, we have

$$\|y - \widetilde{f}\|^2_{L^2(a,b)} = 0,$$

that is,

$$y(x) = \widetilde{f}(x) \qquad \text{for almost all } x \in [a, b].$$

Since $\widetilde{f} \in H^2(a, b)$, this implies that $y \in H^2(a, b)$, which contradicts the assumption $y \notin H^2(a, b)$. Thus the proof is complete.

3. Numerical Example

In this section we make two numerical tests on the basis of Theorem 2.5.

First we will consider a continuous function but with irregular points. Let

$$y(x) = \begin{cases} 2x, & 0 \le x < 0.4 \\ -3x + 2, & 0.4 \le x < 0.6 \\ -5(x-1)^2 + 1. & 0.6 \le x < 1 \end{cases}$$

which is shown as Figure 1. This function has two irregular points: 0.4 and 0.6.

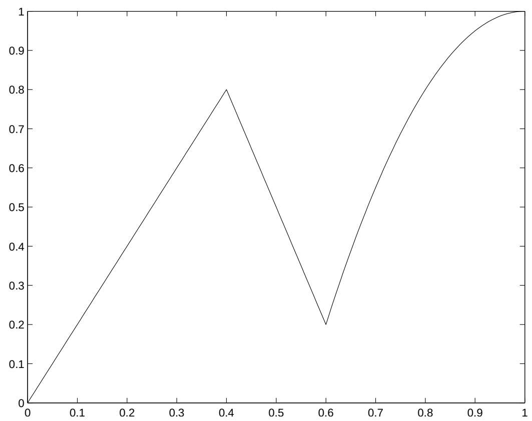

Figure 1: $y(x)$

We set $\delta = 0.001, n = 200$ that is, $x_i = \frac{i}{200}$, $0 \le i \le 200$, and the regularized solution f_* can be obtained by the algorithm in [24].

We set

$$F(x_i) = \|f_*^{''}(\delta, \delta^2, \frac{1}{n})\|^2_{L^2(x_{i-1}, x_i)}$$

and $F(x_i)$ is shown as Figure 2.

Choose the different parameters $\delta = 0.0001, n = 400$, then $F(x_i)$ is shown as Figure 3.

By Theorem 2.5, we know that this value will grow up near the irregular points. From Figure 2 and Figure 3 we can see that $F(x_i)$ is large near 0.4 and 0.6. The numerical results are consistent with the theoretic results. Thus by our method, we can reconstruct the irregular points.

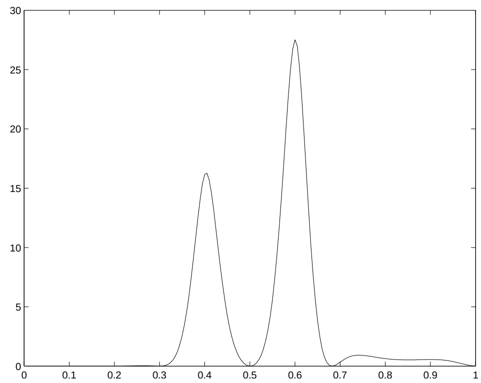

Figure 2: $F(x_i)$,$\delta = 0.001, n = 200$

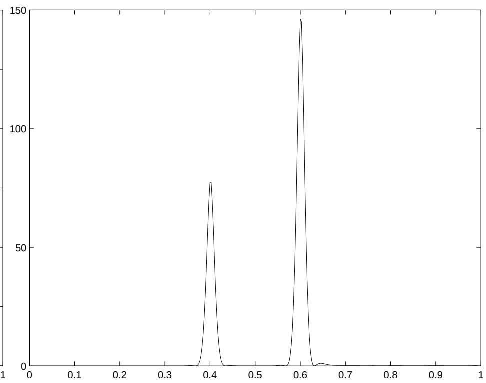

Figure 3: $F(x_i)$,$\delta = 0.0001, n = 400$

Next we will show that our result still works for a discontinuous function, although Theorem 2.5 does not cover discontinuous functions. Let

$$y(x) = \begin{cases} 1, & 0 \le x < 0.2 \\ x - 1, & 0.2 \le x < 0.5 \\ -x + 1, & 0.5 \le x < 0.7 \\ -0.5. & 0.7 \le x < 1 \end{cases}$$

which is shown as Figure 4. This function has three discontinuous points: 0.2, 0.5 and 0.7.

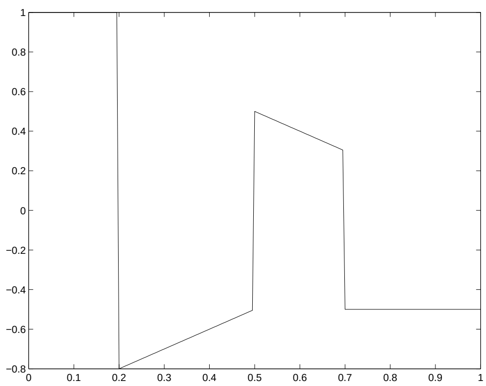

Figure 4: $y(x)$

The numerical results are shown in Figure 5 and Figure 6.

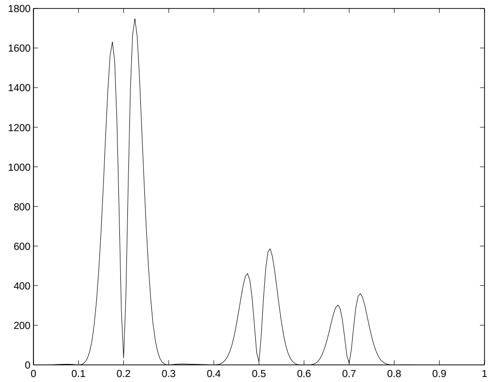

Figure 5: $F(x_i)$,$\delta = 0.0005, n = 200$

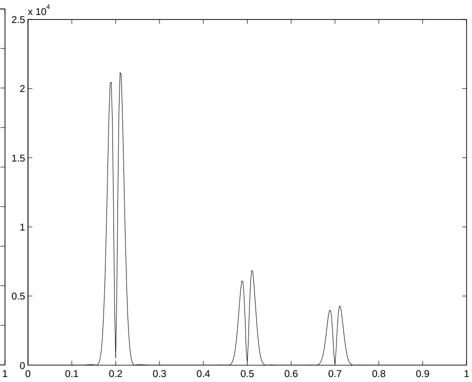

Figure 6: $F(x_i)$,$\delta = 0.0001, n = 400$

There is a difference in the reconstruction from the first example of a continuous function, and the numerical results show extra two peaks around the true discontinuous points. However our numerical method can still localize discontinuous points.

4. Application

In this section we will give one application of the numerical differentiation in the image edge detection. In the image processing, the reconstruction of a good image relies heavily on how to locate accurately the discontinuous points of an unknown source function. This depends also on how to reconstruct the derivatives of the source function from scattered measured data. This is called the edge detection problem which can become very complicated if the source function is non-smooth. At present, several numerical methods for solving the edge detection problem have been proposed. These include the logarithmic image processing method ([4], [17]), which originated from the theory of functional analysis; a gradient operator method in which the ratio of the local gradient magnitude is used to compute the local mean intensity value of the source function ([13], [21]); and the genetic approach ([2], [18]). Most of the existing edge detection techniques are based on finding the maxima of the first-order derivative or the zero-crossing of the second order derivatives of the source function and see also [6],[14],[15],[20].

Here we will use a new method for edge detection which is based on Theorem 2.5. Numerical results show that our method is effective. Here we give the outline of our method:

(1) Consider a line-by-line scan of an image with size $N_1 \times N_2$ where N_1 and N_2 are positive integers. For simplicity we only consider gray level pictures. The two examples presented here are all 8-bit gray level pictures, which means the colour value will vary from 0 to $2^8 - 1 = 255$.

(2) The colour values of an digital image are noisy observation data at discrete points. Here we denote the colour value of a pixel at (i, j) as $c(i, j)$.

(3) Let $M = 2^8$ and define $y_i^\delta(x_j) = \frac{c(i,j)}{M}$, where $x_j = \frac{j-1}{N_2-1}$, $j = 1, 2, \cdots, N_2$. Then $0 \le y_i^\delta(x_j) < 1$ and the value of the grid length h in Theorem 2.5 is $\frac{1}{N_2-1}$.

(4) The numerical differentiation algorithm developed in [24] is then applied to find the approximation of the real colour function $y_i(x), \quad 0 \le x \le 1$ from the noisy values $y_i^\delta(x_j)$ on each line. The approximation is denoted by f_{*i}.

(5) The result of Theorem 2.5 is then applied to locate the discontinuous points of the function $y_i(x)$. In image processing, the edge of an image is usually defined as the area where a rapid change of magnitude occurs. This rapid change is then detected by the magnitude of the H^2 norm of the minimizer f_{*i}. A large value can be set as the threshold for detecting the edge of the image.

(6) It usually takes two different directions of line-by-line scans to obtain an acceptable approximation result. Note that the directions are not required to be perpendicular.

(7) The pictures are taken by a camera and we will add some random noise to test our method.

The original pictures and pictures with 1%, 5%, 10%, 20% and 30% random noises are displayed left and the computed edges are given right respectively. Table 4.1 gives the parameters and the computational times for these two images. It can be observed from both the figures and the table that the numerical differentiation algorithm successfully reconstructs the edges of the images in a short period of CPU time.

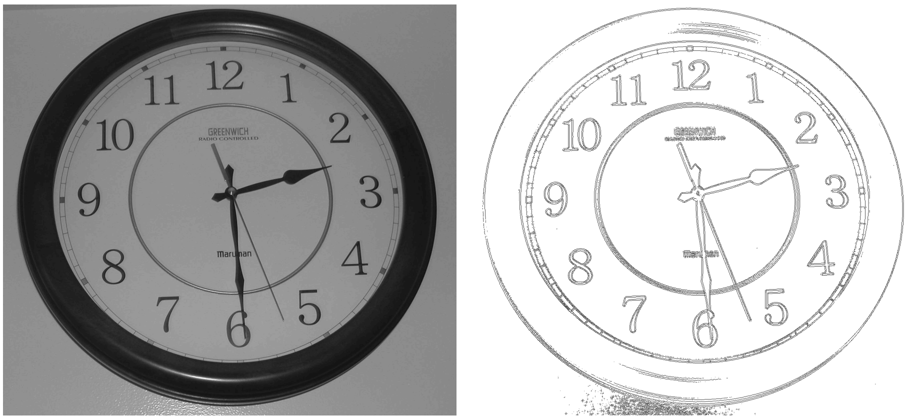

Figure 7a: without noise

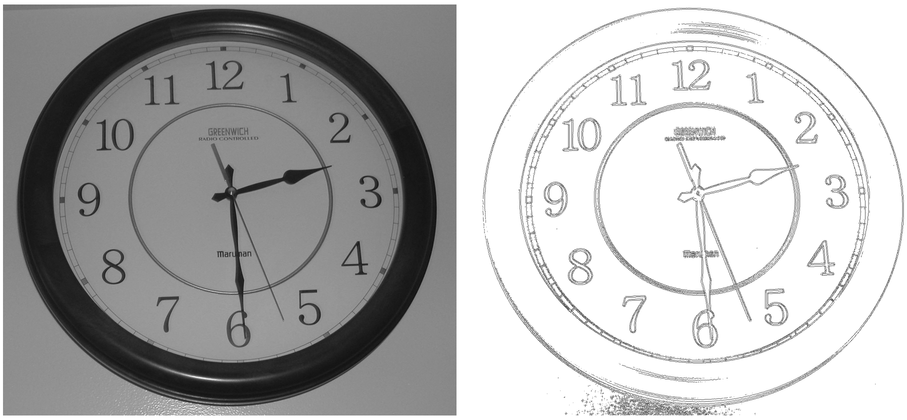

Figure 7b: with 1% random noise

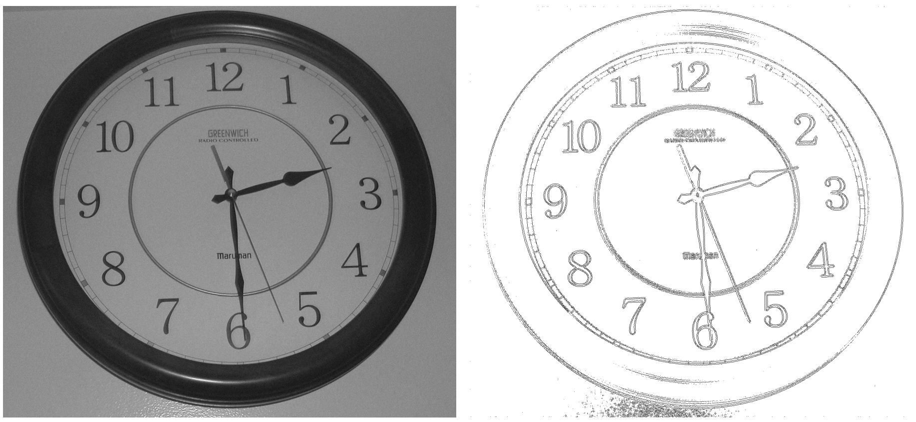

Figure 7c: with 5% random noise

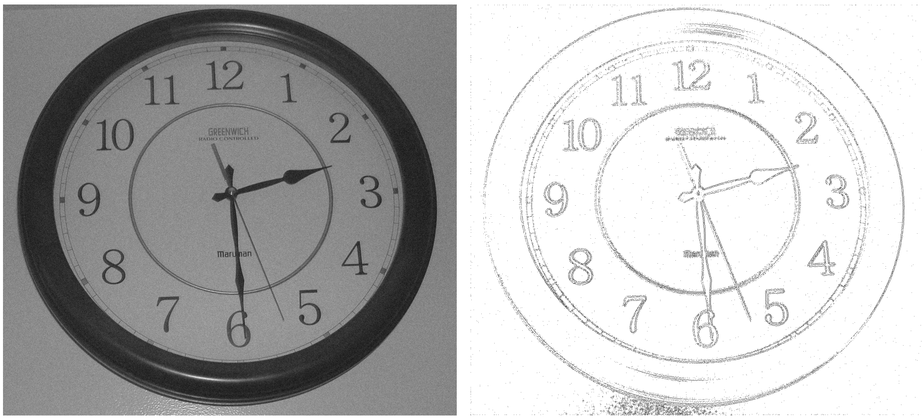

Figure 7d: with 10% random noise

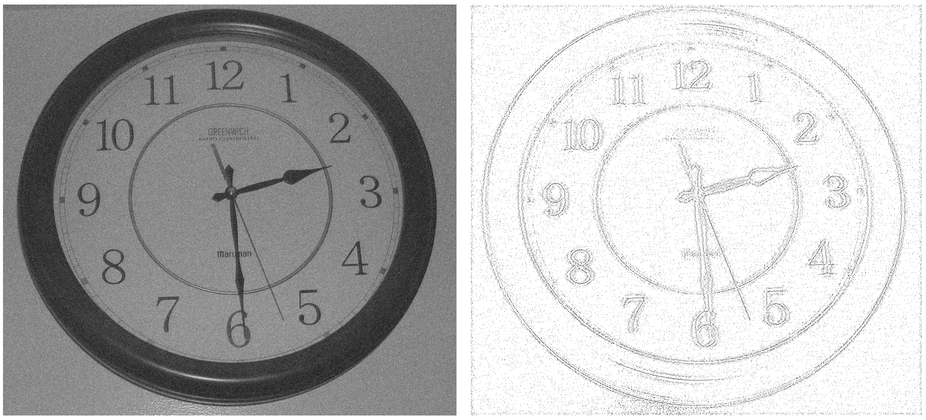

Figure 7e: with 20% random noise

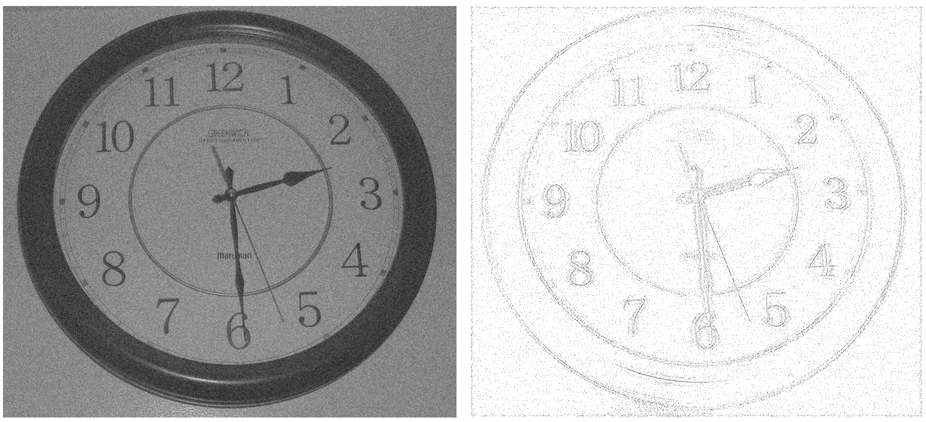

Figure 7f: with 30% random noise

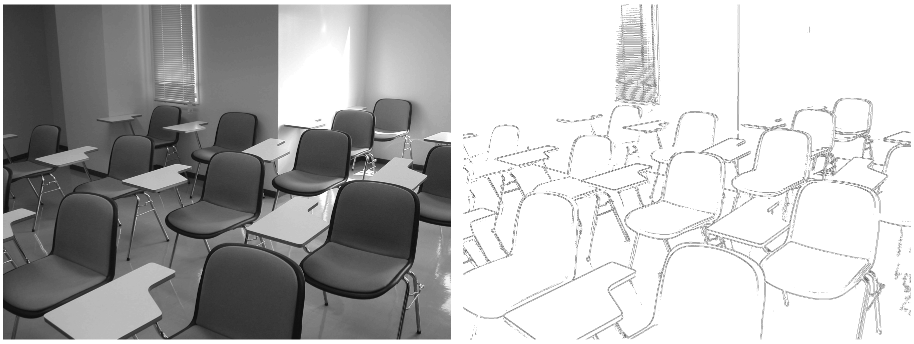

Figure 8a: without noise

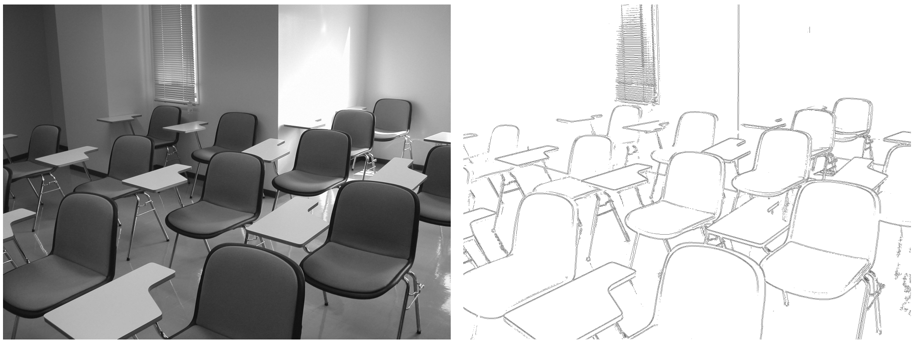

Figure 8b: with 1% random noise

Figure 8c: with 5% random noise

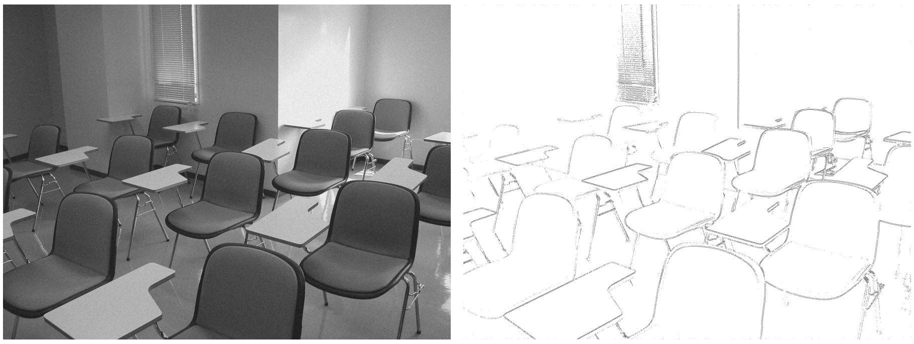

Figure 8d: with 10% random noise

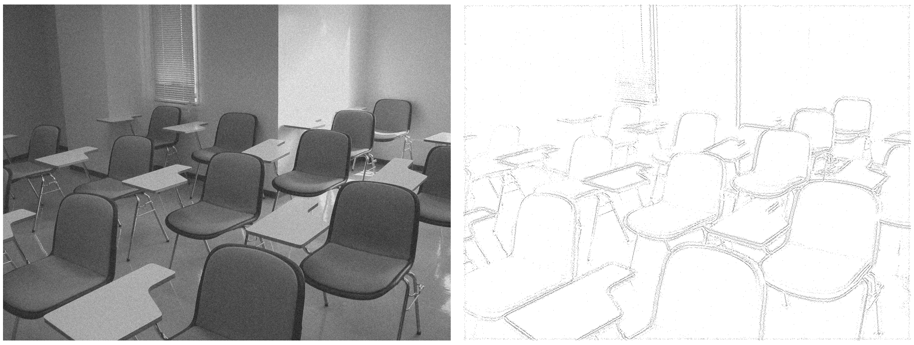

Figure 8e: with 20% random noise

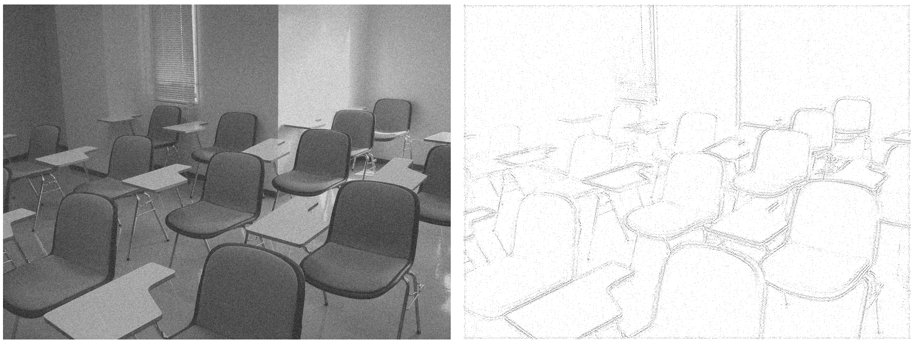

Figure 8f: with 30% random noise

Table 4.1: Parameters and CPU time for the computations

	size of the image	α	time used for computation
Figure 7	1360×1236	10^{-12}	218 seconds
Figure 8	2048×1536	10^{-12}	548 seconds

*The computations were performed on a HP DX200 P4 2800MHz with 512M RAM.

5. Conclusion

In this paper, near irregular points we established the behaviour of the Tikhononv regularized solutions in the numerical differentiation. We proved that, if the exact solution has irregular points, then the norms of the second order derivatives of the regularized solutions in any small interval which contains the irregular points, will blow up. Unlike the result of [24], we proved that the blow-up will take place near the irregular points. Our numerical example and the application to edge detection show that irregular points can be located very easily by using our method.

References

1. R. A. Adams, Sobolev Spaces. Pure and Applied Mathematics, Vol. 65, Academic Press, New York-London, 1975.
2. J. Canny, A computational approach to edge detection, IEEE Trans. Patt. Anal. Mach. Intell., Vol. PAMI-8 (1986), pp. 679-698.
3. J. Cheng and M. Yamamoto, Regularity of the exact solution and the boundedness of the Tikhonov regularized solution. Preprint 2001.
4. G. Deng and J. C. Pinoli, Differentiation-based edge detection using the logarithmic image processing model, Journal of Mathematical Imaging and Vision, Vol. 8 (1998), pp. 161-180.
5. H. W. Engl, M. Hanke and A. Neubauer, Regularization of Inverse Problems. Kluwer Academic Publishers, Dordrecht Hardbound, 1996.
6. S. Godbole and A. Amin, Mathematical morphology for edge and overlap detection for medical images, Real-Time Imaging 1 (1995), pp. 191-201.
7. C. W. Groetsch, The Theory of Tikhonov Regularization for Fredholm Equations of the First Kind. Research Notes in Mathematics, 105. Pitman (Advanced Publishing Program), Boston, 1984.
8. C. W. Groetsch, Differentiation of approximately specified functions. Amer. Math. Monthly 98 (1991), pp. 847-850.
9. C. W. Groetsch, Lanczos' generalized derivative. Amer. Math. Montyly 105 (1998), No. 4, pp. 320-326.

10. C. W. Groetsch, Optimal order of accuracy in Vasin's method for differentiation of noisy functions. Journal of Optimization Theory and Applications 74 (1992), No. 2, pp. 373-378.
11. C. W. Groetsch and O. Scherzer, The optimal order of convergence for stable evaluation of differential operators. Electronic Journal of Differential Equations 1993 (1993), No. 4, pp. 1-10.
12. M. Hanke and O. Scherzer, Inverse problems light: numerical differentiation. Amer. Math. Monthly 108 (2001), No. 6, pp. 512-521.
13. R. P. Johnson, Contrast-based edge detection, Pattern Recogn., Vol. 23 (1990), pp. 311-318.
14. D. Marr and E. Hildreth, Theory of edge detection, Proc. R. Soc. London B207 (1980), pp. 187-217.
15. R. Mehrotra and S. M. Zhan, A computational approach to zero-crossing-based two-dimensional edge detection, Graphical Models and Image Processing, Vol 58, No, 1, January (1996), pp. 1-17.
16. D. A. Murio, Automatic numerical differentiation by discrete mollification, Comppputers and Mathematics with Applications 13 (1987), pp. 381-386.
17. J. C. Pinoli, A contrast definition for logarithmic images in the continuous setting, Acta Stereol., Vol. 10 (1991), pp. 85-96.
18. W. K. Pratt, Digital Image Processing, 2nd edition, John Wiley, New York, 1991.
19. A. G. Ramm and A. B. Smirnova, On stable numerical differentiation. Math. Comp. 70 (2001), pp. 1131-1153.
20. L. G. Roberts, Machine perception of three-dimentional solids, Optical and Electro-Optical Information Processing, MIT Press, Cambridge, MA, 1965.
21. G. S. Robinson, Edge detection by compass gradient masks, Comput. Graph. Image Proc. Vol. 6 (1977), pp. 492-501.
22. A. N. Tikhonov and V. Y. Arsenin, Solutions of Ill-posed Problems. Winston and Sons, Washington, 1977.
23. V. V. Vasin, The stable evaluation of a derivative in the space $C(-\infty, \infty)$. USSR Computational Mathematics and Mathematical Physics 13 (1973), pp. 16-24.
24. Y. B. Wang, X. Z. Jia and J. Cheng, A numerical differentiation method and its application to reconstruction of discontinuity. Inverse Problems 18 (2002) No. 6, pp. 1461-1476.
25. B. B. Yin and Y. Z. Ye, Recovering the Local volatility in Black-Scholes model by numerical differentiation. Preprint 2005.

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