The pointed harmonic volumes of hyperelliptic curves with Weierstrass base points

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ABSTRACT. We give a explicit computation of the pointed harmonic volumes of hyperelliptic curves with Weierstrass base points, which are paraphrased into a combinatorial formula.

1. INTRODUCTION

Let X be a compact Riemann surface of genus $g \geq 2$ and p a point on X. By Pulte [5], the pointed harmonic volume of (X, p) was defined to be the homomorphism $K \otimes H \to \mathbb{R}/\mathbb{Z}$, using Harris' method for the harmonic volume of X [4]. Here, we denote by $H = H^1(X; \mathbb{Z})$ the first cohomology group of X and K the kernel of the intersection pairing $H \otimes H \to \mathbb{Z}$. In this paper, we compute the pointed harmonic volume of any hyperelliptic curve C with any Weierstrass point p. In theorem 5.6, we compute that of some special hyperelliptic curve C_0 with Weierstrass points in an analytic way, by the explicit computation of Chen's iterated integrals [2]. Using Proposition 4.1, we can compute the the pointed harmonic volumes of all the hyperelliptic curves with Weierstrass base points from those of C_0. These results are paraphrased from a combinatorial viewpoint as follows. Let $\{P_j\}_{j=0,1,\ldots,2g+1}$ denote the set of Weierstrass points on C, and fix a Weierstrass point P_ν, $0 \leq \nu \leq 2g + 1$. A certain homomorphism $\kappa_\nu \colon K \otimes H \to \frac{1}{2}\mathbb{Z}/\mathbb{Z} = \{0, 1/2\}$ is defined in §6, which depends on the choice of P_ν.

Theorem 6.2. *For any hyperelliptic curve C and $A \in K \otimes H$, we have*

$$I_{P_\nu}(A) \equiv \kappa_\nu(A) \bmod \mathbb{Z}.$$

The author [6] computed the harmonic volumes of hyperelliptic curves. But the computation of the pointed ones of (X, p) is more complicated than that of X. For any hyperelliptic curve C, it is tedious to compute I_p in the case $p \in C \setminus \{P_j\}_{j=0,1,\ldots,2g+1}$. But we have $I_p \equiv 0$ or $1/2$ mod $\mathbb{Z}$ in the case $p \in \{P_j\}_{j=0,1,\ldots,2g+1}$. It has been still unknown which elements of $K \otimes H$ and Weierstrass points p have nontrivial I_p or not. In this paper, we compute them completely.

As an application of the pointed harmonic volume of (X, p), Pulte proved the pointed Torelli theorem [5]. We denote by $\pi_1(X, p)$ the fundamental group of X at the base point $p \in X$ and J_p the augmentation ideal of the group ring $\mathbb{Z}\pi_1(X, p)$.

Theorem 1.1. (*the pointed Torelli theorem* [5])
Suppose that X and Y are compact Riemann surfaces and that $p \in X$ and $q \in Y$. With the exception of two points p in X, if there is a ring isomorphism

$$\mathbb{Z}\pi_1(X, p)/J_p^3 \to \mathbb{Z}\pi_1(Y, q)/J_q^3$$

which preserves the mixed Hodge structure, then there is a biholomorphism $\varphi\colon X \to Y$ such that $\varphi(p) = q$.

If X is generic (e.g. X is hyperelliptic), then there are no exceptional points. The pointed harmonic volumes determine the choice of the base points. In the proof of this theorem, the classical Torelli theorem follows from the preservation of the mixed Hodge structure and we obtain the biholomorphism $X \cong Y$. When we choose the base points, the pointed harmonic volume plays an important role. Theorem 6.2 also tells the choice of Weierstrass base points on C.

Now we describe the contents of this paper briefly. In §2, we define the pointed harmonic volume of (X, p), using Chen's iterated integrals [2]. In §3, we give a basis of the first homology group $H_1(C; \mathbb{Z}/2\mathbb{Z})$ of the hyperelliptic curve C. In §4, we prove $I_{P_\nu} \in H^0(\Delta_g^1; \mathrm{Hom}(K \otimes H, \mathbb{Z}/2\mathbb{Z}))$. In §5, the pointed harmonic volume of some special hyperelliptic curve C_0 with Weierstrass base points is computed in an analytic way. This result can be extended to all the hyperelliptic curves with Weierstrass base points and interpreted from a combinatorial viewpoint. In §6, we obtain a simple combinatorial formula of the pointed harmonic volume of (C, P_ν).

2. The pointed harmonic volume

We recall the definition of the pointed harmonic volume of a pointed Riemann surface (X, p). Here X is a compact Riemann surface of genus $g \geq 2$ and p a point on X. We identify the first integral homology group $H_1(X; \mathbb{Z})$ of X with the first integral cohomology group by Poincaré duality, and denote it by H. For closed 1-forms $\omega_{1,i}$ and $\omega_{2,i}$, $i = 1, 2, \ldots, m$, on X such that $\int_X \sum_{i=1}^{m} \omega_{1,i} \wedge \omega_{2,i} = 0$, we obtain the 1-form η such that $d\eta = \sum_{i=1}^{m} \omega_{1,i} \wedge \omega_{2,i}$ and $\int_X \eta \wedge *\alpha = 0$ for any closed 1-form α on X. Here, $*$ is the Hodge star operator which depends only on the complex structure and not the choice of Hermitian metric. We identify H with the space of all the real harmonic 1-forms on X with integral periods by the Hodge theorem. We denote by K the kernel of the intersection pairing $(\ ,\ )\colon H \otimes H \to \mathbb{Z}$.

Definition 2.1. (The pointed harmonic volume [5])
For $\sum_{i=1}^{m} a_i \otimes b_i \in K$ and $c \in H$, the pointed harmonic volume is defined to be

$$I_p\left(\left(\sum_{i=1}^{m} a_i \otimes b_i\right) \otimes c\right) := \sum_{i=1}^{m} \int_\gamma a_i b_i - \int_\gamma \eta \quad \mathrm{mod}\ \mathbb{Z}.$$

Here η is the 1-form on X which is associated to $\sum_{i=1}^{m} a_i \otimes b_i$ in the way stated above and $\gamma\colon [0, 1] \to X$ is a loop in X at the base point p whose homology class is equal to c. The integral $\int_\gamma a_i b_i$ is Chen's iterated integral [2], that is, $\int_\gamma a_i b_i =$

$\int_{0 \le t_1 \le t_2 \le 1} f_i(t_1) g_i(t_2) dt_1 dt_2$ for $\gamma^* a_i = f_i(t) dt$ and $\gamma^* b_i = g_i(t) dt$. Here t is the coordinate in the unit interval $[0, 1]$. See Chen [2] for iterated integrals and Harris [4], Pulte [5] for the (pointed) harmonic volume.

Remark 2.2. By the definition of I_p, we have $I_p((\sum_{i=1}^m a_i \otimes b_i) \otimes c) \equiv -I_p((\sum_{i=1}^m b_i \otimes a_i) \otimes c) \bmod \mathbb{Z}$.

3. Hyperelliptic curves

Let C be a hyperelliptic curve and $\mathbb{Z}_2$ the field $\mathbb{Z}/2\mathbb{Z}$. In this section, we explain the first homology group of C with $\mathbb{Z}_2$-coefficient.

We define the hyperelliptic curve C as follows. It is the compactification of the plane curve in the (z, w) plane $\mathbb{C}^2$

$$w^2 = \prod_{i=0}^{2g+1} (z - p_i),$$

where $p_0, p_1, \ldots, p_{2g+1}$ are some distinct points on $\mathbb{C}$. It admits the hyperelliptic involution given by $\iota : (z, w) \mapsto (z, -w)$. Let π be the 2-sheeted covering $C \to \mathbb{C}P^1, (z, w) \mapsto z$, branched over $2g + 2$ branch points $\{p_i\}_{i=0,1,\cdots,2g+1}$ and $P_i \in C$ a ramification point such that $\pi(P_i) = p_i$. It is known that $\{P_i\}_{i=0,1,\ldots,2g+1}$ is just the set of all the Weierstrass points on any hyperelliptic curve C.

For points p_i and p_j, we denote by $p_i p_j$ a simple path joining p_i and p_j. We draw simple paths $p_0 p_1, p_1 p_2, \ldots, p_{2g} p_{2g+1}$ and $p_{2g+1} p_0$ such that all the $2g + 2$ arcs do not intersect except for endpoints of them. We take a disk $D \subset \mathbb{C}P^1$ whose boundary is $\left(\bigcup_{j=0}^{2g} p_j p_{j+1} \right) \bigcup p_{2g+1} p_0$ (Figure 1, $g = 2$). We picture two copies of $\mathbb{C}P^1$ as above and call them Ω_0 and Ω_1. We make crosscuts along $p_{2k} p_{2k+1}, k = 0, 1, \ldots, g$ and construct the hyperelliptic curve C by joining every $p_{2k} p_{2k+1}$ on Ω_0 to the corresponding one on Ω_1 for $k = 0, 1, \ldots, g$. See 102-103 in [3] for example. We may consider $\Omega_i \subset C$ for $i = 0, 1$.
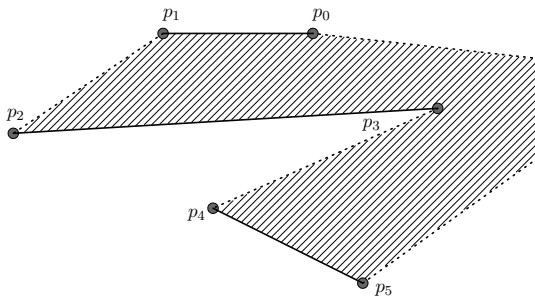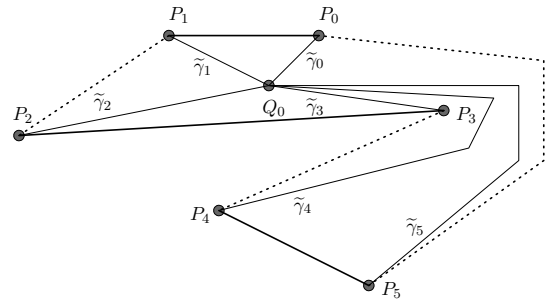

FIGURE 1. $D \subset \mathbb{C}P^1$ FIGURE 2. $\Omega_0 \subset C$

The hyperelliptic involution ι interchanges a point on Ω_0 and the corresponding one on Ω_1, and fixes $P_i, i = 0, 1, \ldots, 2g + 1$. We choose a base point $Q_0 \in \Omega_0$ and denote $Q_1 = \iota(Q_0) \in \Omega_1$. Let $\gamma_j, j = 0, 1, \ldots, 2g + 1$, be a simple path in D joining $\pi(Q_0)$ and p_j. We denote by $\widetilde{\gamma}_j$ the lift of γ_j in Ω_0 from Q_0 to P_j (Figure 2, $g = 2$). Set $e_j = \widetilde{\gamma}_j \cdot \iota(\widetilde{\gamma}_j)^{-1}$, where the product $\widetilde{\gamma}_j \cdot \iota(\widetilde{\gamma}_j)^{-1}$ indicates that we traverse $\widetilde{\gamma}_j$ first, then

$\iota(\widetilde{\gamma}_j)^{-1}$. It is a path in C which is to be followed from Q_0 to P_j and go to Q_1 in Figure 3. It is clear that $e_{j_1} \cdot \iota(e_{j_2})$ is a loop in C at the base point Q_0. Moreover we have the
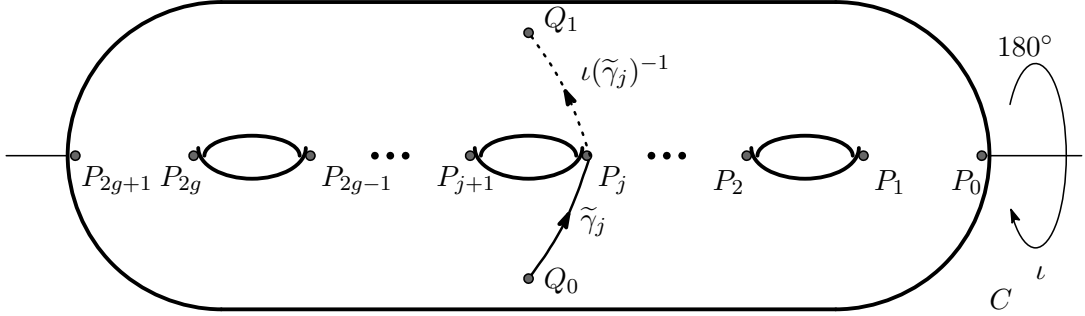

FIGURE 3. $e_j = \widetilde{\gamma}_j \cdot \iota(\widetilde{\gamma}_j)^{-1}$

homotopy equivalences relative to the base point Q_0

$$e_j \cdot \iota(e_j) \sim 1 \quad \text{and} \quad e_0 \cdot \iota(e_1) \cdot \cdots \cdot e_{2g} \cdot \iota(e_{2g+1}) \sim 1.$$

We set $a_i = e_{2i-1} \cdot \iota(e_{2i})$ and $b_i = e_{2i-1} \cdot \iota(e_{2i-2}) \cdot \cdots \cdot e_1 \cdot \iota(e_0)$, and denote by x_i and y_i the homology classes of a_i and b_i respectively. Then we have $\{x_i, y_i\}_{i=1,2,\dots,g}$ is a symplectic basis of $H_1(C; \mathbb{Z})$ in Figure 1 in [6].

Let $H_{\mathbb{Z}_2}$ denote $H_1(C; \mathbb{Z}_2)$ and B branch locus $\{p_i\}_{i=0,1,\dots,2g+1}$. We deform the path e_i in C and denote it by e_i' in $C \setminus \pi^{-1}(B)$ as follows. The path e_i' avoids P_i in a sufficiently small neighborhood at P_i so that $\pi(e_i')$ goes around p_i which does not any p_k with $k \neq i$ (Figure 4) and the set of homology classes $\{\pi(e_i')\}_{i=0,1,\dots,2g}$ is a basis of $H_1(\mathbb{C}P^1 \setminus B; \mathbb{Z}_2)$. Moreover we have $\pi(e_0') + \pi(e_1') + \cdots + \pi(e_{2g+1}') = 0 \in H_1(\mathbb{C}P^1 \setminus B; \mathbb{Z}_2)$. Since the coefficients are in $\mathbb{Z}_2$, the homology class of e_i' is independent of the choice of it. From the 2-sheeted covering π, we have the well-defined homomorphism $v_0 \colon H \to$
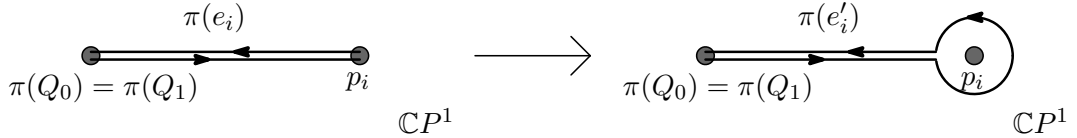

FIGURE 4. A deformation of e_i

$H_1(\mathbb{C}P^1 \setminus B; \mathbb{Z}_2)$ which factors through $H_1(C \setminus \pi^{-1}(B); \mathbb{Z})$ (Arnol'd [1]). We obtain the linear map $v \colon H_{\mathbb{Z}_2} \to H_1(\mathbb{C}P^1 \setminus B; \mathbb{Z}_2)$ induced naturally by v_0. It immediately follows that $v(x_i \bmod 2) = \pi(e_{2i-1}') + \pi(e_{2i}')$, $v(y_i \bmod 2) = \pi(e_0') + \pi(e_1') + \cdots + \pi(e_{2i-1}')$, and v is injective. The map v gives the short exact sequence

$$0 \longrightarrow H_{\mathbb{Z}_2} \xrightarrow{\ v\ } H_1(\mathbb{C}P^1 \setminus B; \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

Here the map $H_1(\mathbb{C}P^1 \setminus B; \mathbb{Z}_2) \to \mathbb{Z}_2$ is the augmentation map $\pi(e_i') \mapsto 1$. Fix a Weierstrass point P_ν. Let f_i denote $\pi(e_\nu') + \pi(e_i')$ for $i = 0, 1, \dots, 2g+1$. We remark that f_i may be considered as an element of $H_{\mathbb{Z}_2}$ and $f_\nu = 0$. For $i = 1, 2, \dots, g$, we have the identification

(3.1)
$$\begin{cases} x_i = f_{2i-1} + f_{2i}, \\ y_i = f_0 + f_1 + \cdots + f_{2i-1}, \end{cases}$$

in $H_{\mathbb{Z}_2}$. It is clear that $f_0 + f_1 + \cdots + f_{2g+1} = 0$.

For any hyperelliptic curve C and Weierstrass point $P_j \in C$, the hyperelliptic involution ι fixes I_{P_j} and acts on $H_{\mathbb{Z}_2}$ as (-1)-times. Then we have the value of I_{P_j} is 0 or $1/2$ mod $\mathbb{Z}$ from the equation $I_{P_j} \equiv (-1)^3 I_{P_j}$ mod $\mathbb{Z}$. We may consider $I_{P_j} \in \mathrm{Hom}((K \otimes H)_{\mathbb{Z}_2}, \mathbb{Z}_2)$, where $(K \otimes H)_{\mathbb{Z}_2}$ denotes $(K \otimes H) \otimes \mathbb{Z}_2$.

4. Pointed harmonic volumes of hyperelliptic curves and the moduli space of compact Riemann surfaces

We recall some results about the moduli space of compact Riemann surfaces. Let Σ_g be a closed oriented surface of genus g. Its mapping class group, denoted here by Γ_g^s, is the group of isotopy classes of orientation preserving diffeomorphisms of Σ_g which fix s points on Σ_g for $s = 0, 1$. We denote $\Gamma_g = \Gamma_g^0$. The group Γ_g^1 acts on the Teichmüller space $\mathcal{T}_g^1$ of Σ_g with a marked point and the quotient space $\mathcal{M}_g^1$ is the moduli space of Riemann surfaces of genus g with a marked point. The group Γ_g^1 acts naturally on the first homology group $H_1(\Sigma_g; \mathbb{Z})$ of Σ_g.

Let $\mathcal{H}_g^1 \subset \mathcal{M}_g^1$ be the moduli space of hyperelliptic curves of genus g with a marked Weierstrass point P_ν. For the rest of this paper, we suppose that a marked point is a Weierstrass point. The hyperelliptic mapping class group Δ_g^1 is the subgroup of Γ_g defined by

$$\{\varphi \in \Gamma_g; \varphi\iota = \iota\varphi, \ \varphi(P_\nu) = P_\nu\},$$

where ι is the hyperelliptic involution of Σ_g. We have $\Delta_g^1 \subset \Gamma_g^1$. The moduli space $\mathcal{H}_g^1$ is known to be connected and has a natural structure of a quasi-projective orbifold. The group Δ_g^1 can be considered as its orbifold fundamental group. For any $\mathbb{Z}\Delta_g^1$-module M, we may consider the dual $M^* = \mathrm{Hom}(M, \mathbb{Z}_2)$ as a $\mathbb{Z}_2\Delta_g^1$-module in a natural way. We denote $I_\nu = I_{P_\nu}$.

Proposition 4.1. *We have*

$$I_\nu \in H^0(\Delta_g^1; (K \otimes H)^*),$$

i.e. I_ν is a Δ_g^1-invariant in the dual $(K \otimes H)^*$.

Proof. Let $\mathcal{L}$ be a locally constant sheaf with a stalk $\mathrm{Hom}_{\mathbb{Z}}(K \otimes H, \mathbb{Z}_2)$. In a similar way to Harris' method [4], I_ν varies in $\mathcal{H}_g^1$ continuously. For any hyperelliptic curves, $I_\nu \equiv 0$ or $1/2$ modulo $\mathbb{Z}$. We remark that the pointed harmonic volume is uniquely determined for any point on $\mathcal{H}_g^1$. The locally constant sheaf $\mathcal{L}$ has a global section $\widetilde{I_\nu}$ associated to I_ν. Moreover $\mathcal{H}_g^1$ is arcwise connected. Therefore $\widetilde{I_\nu}$ is a constant section of $\mathcal{L}$ and I_ν is invariant under the action of the orbifold fundamental group Δ_g^1 of $\mathcal{H}_g^1$. □

5. Pointed harmonic volumes of a hyperelliptic curve C_0

We compute the pointed harmonic volume of a pointed hyperelliptic curve (C_0, P_ν). See §3 and 4 in [6] for details. We define the hyperelliptic curve C_0 by the equation $w^2 = z^{2g+2} - 1$. We take $Q_i = (0, (-1)^i \sqrt{-1}), i = 0, 1,$ and $P_j = (\zeta^j, 0), j = 0, 1, \ldots, 2g+1,$ where $\zeta = \exp\left(2\pi\sqrt{-1}/(2g+2)\right)$. We define a path $e_j \colon [0, 1] \to C_0, j = 0, 1, \ldots, 2g+1,$

by

$$\begin{cases} \left(2t\zeta^j, \sqrt{-1}\sqrt{1-(2t)^{2g+2}}\right) & \text{for} \quad 0 \le t \le 1/2, \\ \left((2-2t)\zeta^j, -\sqrt{-1}\sqrt{1-(2-2t)^{2g+2}}\right) & \text{for} \quad 1/2 \le t \le 1. \end{cases}$$

For $i = 1, 2, \ldots, g$, we denote by ω_i a holomorphic 1-form $z^{i-1}dz/w$ on C_0. It is known that $\{\omega_i\}_{i=1,2,\ldots,g}$ is a basis of the space of holomorphic 1-forms on C_0. Let $B(u,v)$ denote the beta function $\int_0^1 x^{u-1}(1-x)^{v-1}dx$ for $u, v > 0$. For the normalization, we set $\omega_i' = \dfrac{(2g+2)\sqrt{-1}}{2B(i/(2g+2), 1/2)}\omega_i$. Then we have

$$\int_{a_j} \omega_i' = \zeta^{i(2j-1)}(1 - \zeta^i) \quad \text{and} \quad \int_{b_j} \omega_i' = \frac{\zeta^{2ij} - 1}{\zeta^i + 1},$$

where $i, j \in \{1, 2, \ldots, g\}$. The integral $\int_\gamma \omega_i'$ depends only on the homology class of γ, since ω_i' is a closed 1-form.

We compute the iterated integrals of real harmonic 1-forms of C_0 with integral periods. Let Ω_a and Ω_b be the non-singular matrices whose (i,j)-entries are

$$\int_{a_j} \omega_i' \quad \text{and} \quad \int_{b_j} \omega_i'$$

respectively. We define real harmonic 1-forms α_i and $\beta_i, i = 1, 2, \ldots, g$, by

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_g \end{pmatrix} = \Re \left((\Omega_b)^{-1} \begin{pmatrix} \omega_1' \\ \vdots \\ \omega_g' \end{pmatrix} \right) \quad \text{and} \quad \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_g \end{pmatrix} = -\Re \left((\Omega_a)^{-1} \begin{pmatrix} \omega_1' \\ \vdots \\ \omega_g' \end{pmatrix} \right)$$

respectively. It is clear that $\int_{a_j} \alpha_i = \int_{b_j} \beta_i = 0$ and $\int_{b_j} \alpha_i = \delta_{ij} = -\int_{a_j} \beta_i$. Let $\Theta \colon H_1(C_0; \mathbb{Z}) \to H^1(C_0; \mathbb{Z})$ denote the Poincaré dual. We have $\Theta(x_i) = \alpha_i$ and $\Theta(y_i) = \beta_i$ for $i = 1, 2, \ldots, g$. Hence, $\{\alpha_i, \beta_i\}_{i=1,2,\ldots,g}$ is a symplectic basis of $H^1(C_0; \mathbb{Z})$.

Let t_u be a complex number $\sum_{p=1}^g \zeta^{up}$ for any integer u. It is obvious that

$$t_u = \begin{cases} g & \text{for} \quad u \in (2g+2)\mathbb{Z}, \\ -1 & \text{for} \quad u \in 2\mathbb{Z} \setminus (2g+2)\mathbb{Z}, \\ \dfrac{1+\zeta^u}{1-\zeta^u} & \text{for} \quad u \in 2\mathbb{Z}+1. \end{cases}$$

Furthermore, t_u is pure imaginary and $t_{-u} = -t_u$ when u is odd. In addition to the formulas (1),(2),(3) and (4) of Lemma 3.8 in [6], it is to show

Lemma 5.1. *On the curve C_0, we have*

$$(5) \int_{a_k} \alpha_i \beta_j = \frac{-1}{2(g+1)^2} t_{2k-2i}(t_{2k-2j} - t_{2k}),$$

$$(6) \int_{b_k} \alpha_i \beta_j = \frac{-1}{2(g+1)^2} \sum_{u=1}^k \left\{ (t_{2u-2i-2} - t_{2u-2i}) \sum_{v=1}^j t_{2v+2u-2j-2} \right\}.$$

Here $i, j, k \in \{1, 2, \ldots, g\}$.

Proof. We compute the case (5) in the following way. Let $A_{j,m}$ and $B_{i,l}$ be (j,m) and (i,l)-entries of $(\Omega_a)^{-1}$ and $(\Omega_b)^{-1}$ respectively.

$$\int_{a_k} \alpha_i \beta_j = \int_{a_k} -\Re\left(\sum_{l=1}^{g} B_{i,l}\omega'_l\right)\Re\left(\sum_{m=1}^{g} A_{j,m}\omega'_m\right)$$

$$= -\frac{1}{4}\int_{a_k}\sum_{l,m=1}^{g}\left(B_{i,l}A_{j,m}\omega'_l\omega'_m + B_{i,l}\overline{A}_{j,m}\omega'_l\overline{\omega}'_m + \overline{B}_{i,l}A_{j,m}\overline{\omega}'_l\omega'_m + \overline{B}_{i,l}\overline{A}_{j,m}\overline{\omega}'_l\overline{\omega}'_m\right)$$

$$= -\frac{1}{2}\Re\left\{\sum_{l,m=1}^{g}\left(B_{i,l}A_{j,m}\int_{a_k}\omega'_l\omega'_m + B_{i,l}\overline{A}_{j,m}\int_{a_k}\omega'_l\overline{\omega}'_m\right)\right\}.$$

We use Lemma 3.5 in [6] and calculate

$$(g+1)^2\sum_{l,m=1}^{g} B_{i,l}A_{j,m}\int_{a_k}\omega'_l\omega'_m$$

$$= \sum_{l,m=1}^{g}\zeta^{-2il}(1+\zeta^l)\frac{\zeta^m(-1+\zeta^{-2jm})}{1-\zeta^m}\frac{1}{2}\zeta^{(l+m)(2k-1)}(1-2\zeta^m+\zeta^{l+m})$$

$$= \frac{1}{2}\sum_{m=1}^{g}\frac{1-\zeta^{2jm}}{1-\zeta^m}\zeta^{m(2k-2j)}\sum_{l=1}^{g}\zeta^{l(2k-2i-1)}(1+\zeta^l)(1-\zeta^m-\zeta^m(1-\zeta^l))$$

$$= \frac{1}{2}\sum_{m=1}^{g}\sum_{v=2k-2j}^{2k-1}\zeta^{mv}\left\{(1-\zeta^m)(t_{2k-2i-1}+t_{2k-2i})-\zeta^m(t_{2k-2i-1}-t_{2k-2i+1})\right\}$$

$$= \frac{1}{2}\sum_{m=1}^{g}\left\{\frac{1-\zeta^{2jm}}{1-\zeta^m}\zeta^{m(2k-2j)}(1-\zeta^m)(t_{2k-2i-1}+t_{2k-2i})-\sum_{v=2k-2j}^{2k-1}\zeta^{m(v+1)}(t_{2k-2i-1}-t_{2k-2i+1})\right\}$$

$$= \frac{1}{2}\left\{(t_{2k-2i-1}+t_{2k-2i})(t_{2k-2j}-t_{2k})-(t_{2k-2i-1}-t_{2k-2i+1})\sum_{v=2k-2j}^{2k-1}t_{v+1}\right\}$$

$$= \frac{1}{2}\left\{(t_{2k-2i-1}+t_{2k-2i})(t_{2k-2j}-t_{2k})-(t_{2k-2i-1}-t_{2k-2i+1})\sum_{v=2k-2j+1}^{2k}t_v\right\}.$$

Similarly, we have

$$(g+1)^2\sum_{l,m=1}^{g} B_{i,l}\overline{A}_{j,m}\int_{a_k}\omega'_l\overline{\omega}'_m$$

$$= \frac{1}{2}\left\{(t_{2k-2i-1}+t_{2k-2i})(t_{2k-2j}-t_{2k})-(t_{2k-2i-1}-t_{2k-2i+1})\sum_{v=2k-2j+1}^{2k}t_{-v}\right\}.$$

Therefore, we obtain the result

$$\int_{a_k}\alpha_i\beta_j$$

$$= \frac{-1}{2(g+1)^2}\frac{1}{2}\Re\left\{2(t_{2k-2i-1}+t_{2k-2i})(t_{2k-2j}-t_{2k})-2(t_{2k-2i-1}-t_{2k-2i+1})\sum_{\substack{v=2k-2j+1\\\text{even}}}^{2k}t_v\right\}$$

$$= \frac{-1}{2(g+1)^2} t_{2k-2i}(t_{2k-2j} - t_{2k}).$$

Similarly we compute the case (6). □

Using the symplectic basis $\{x_i, y_i\}_{i=1,2,\cdots,g} \subset H_1(C; \mathbb{Z})$ stated in §3, we choose a basis of K as follows:

$$\left\{ \begin{array}{lll} (1) & z_i \otimes z_j' & (i \neq j) \\ (2) & x_i \otimes y_i - x_1 \otimes y_1 & (i \neq 1) \\ (3) & x_i \otimes y_i + y_i \otimes x_i & (i = 1, 2, \ldots, g) \\ (4) & z_i \otimes z_i & (i = 1, 2, \ldots, g) \end{array} \right\},$$

where z_i denotes x_i or y_i and so on. By the definition of the pointed harmonic volume I_ν, we obtain

$$I_\nu((x_i \otimes y_i + y_i \otimes x_i) \otimes z_k'') \equiv 0 \mod \mathbb{Z} \quad \text{for any } i, k,$$

and

$$I_\nu(z_i \otimes z_i \otimes z_k'') \equiv \left\{ \begin{array}{ll} 1/2 \mod \mathbb{Z} & \text{if } z_i \otimes z_i \otimes z_k'' = x_i \otimes x_i \otimes y_i \text{ or } y_i \otimes y_i \otimes x_i, \\ 0 \mod \mathbb{Z} & \text{otherwise,} \end{array} \right.$$

for any hyperelliptic curve C. It is enough to consider the case (1) and (2). For the rest of this paper, we omit mod $\mathbb{Z}$, unless otherwise stated.

We compute the pointed harmonic volume of (C_0, Q_0). From Lemma 5.1, Lemma 3.8 in [6] and the equation $\int_{e_j} \eta = 0$ (Lemma 4.2 in [6]), it is to show

Proposition 5.2. *Case (1). If $i \neq k$ and $j \neq k$, then we have*

$$I_{Q_0}(z_i \otimes z_j' \otimes z_k'') \equiv 0.$$

If $i = k$ or $j = k$, then we have

$$\begin{array}{rcl} I_{Q_0}(x_i \otimes x_j \otimes y_i) & \equiv & \mu, \\ I_{Q_0}(x_i \otimes y_j \otimes y_i) & \equiv & \left\{ \begin{array}{ll} (g - j + 1)\mu & \text{if } i < j, \\ (2g - j + 2)\mu & \text{if } i > j, \end{array} \right. \\ I_{Q_0}(y_i \otimes x_j \otimes x_i) & \equiv & (2g + 1)\mu, \\ I_{Q_0}(y_i \otimes y_j \otimes x_i) & \equiv & \left\{ \begin{array}{ll} (g + j + 1)\mu & \text{if } i < j, \\ j\mu & \text{if } i > j, \end{array} \right. \end{array}$$

Case (2). If $i \neq k$ and $k \neq 1$, then we have

$$I_{Q_0}((x_i \otimes y_i - x_1 \otimes y_1) \otimes z_k'') \equiv 0.$$

If $i = k$ or $k = 1$, then we have

$$\begin{array}{rcl} I_{Q_0}((x_i \otimes y_i - x_1 \otimes y_1) \otimes x_i) & \equiv & (g + 2)\mu, \\ I_{Q_0}((x_i \otimes y_i - x_1 \otimes y_1) \otimes y_i) & \equiv & (2g - i + 2)\mu, \\ I_{Q_0}((x_i \otimes y_i - x_1 \otimes y_1) \otimes x_1) & \equiv & g\mu, \\ I_{Q_0}((x_i \otimes y_i - x_1 \otimes y_1) \otimes y_i) & \equiv & (g + 2)\mu. \end{array}$$

Here we denote $\mu = 1/(2g + 2)$.

Remark 5.3. From Remark 2.2, we do not need to compute $I_{Q_0}(x_j \otimes x_i \otimes y_i)$, $I_{Q_0}((y_i \otimes x_i - y_1 \otimes x_1) \otimes x_i)$ and so on.

We calculate the difference between I_ν and I_{Q_0}. For $h_1 \otimes h_2 \otimes h_3 \in K \otimes H$, we set $\Lambda_\nu(h_1 \otimes h_2 \otimes h_3) := I_\nu(h_1 \otimes h_2 \otimes h_3) - I_{Q_0}(h_1 \otimes h_2 \otimes h_3) \bmod \mathbb{Z}$. Let $\ell_\nu : [0,1] \to C_0$ be a path $t \mapsto (t\zeta^\nu, \sqrt{-1}\sqrt{1 - t^{2g+2}}) \in C_0$. It is clear that $\ell_\nu^{-1} \cdot e_j \cdot \ell_\nu$'s are loops in C_0 at the base point P_ν. From the equation (2.2) in [4], we have

Lemma 5.4.

$$\Lambda_\nu(h_1 \otimes h_2 \otimes h_3) \equiv (h_1, h_3) \int_{\ell_\nu} h_2 - (h_2, h_3) \int_{\ell_\nu} h_1 \quad \bmod \mathbb{Z}.$$

It is clear that

$$\int_{\ell_\nu} \alpha_i = \frac{1}{2(g+1)} \Re(t_{\nu-2i} + t_{\nu-2i+1}) \quad \text{and} \quad \int_{\ell_\nu} \beta_i = \frac{-1}{2(g+1)} \Re\left(\sum_{u=\nu-2i+1}^{\nu} t_u\right).$$

These equations and Lemma 5.4 give the following Lemma.

Lemma 5.5. *Case (1). If $i \neq k$ and $j \neq k$, then we have*

$$\Lambda_\nu(z_i \otimes z_j' \otimes z_k'') \equiv 0.$$

If $i = k$ or $j = k$, then we have

$$\Lambda_\nu(x_i \otimes x_j \otimes y_i) \equiv \begin{cases} g\mu & \text{if } \nu = 1 \text{ or } 2, \\ (2g+1)\mu & \text{if } \nu \neq 1 \text{ and } 2, \end{cases}$$

$$\Lambda_\nu(x_i \otimes y_j \otimes y_i) \equiv \begin{cases} j\mu & \text{if } \nu > 2i - 1, \\ (g+j+1)\mu & \text{if } \nu \leq 2i - 1, \end{cases}$$

$$\Lambda_\nu(y_i \otimes x_j \otimes x_i) \equiv \begin{cases} (g+2)\mu & \text{if } \nu = 1 \text{ or } 2, \\ \mu & \text{if } \nu \neq 1 \text{ and } 2, \end{cases}$$

$$\Lambda_\nu(y_i \otimes y_j \otimes x_i) \equiv \begin{cases} (2g-j+2)\mu & \text{if } \nu > 2i - 1, \\ (g-j+1)\mu & \text{if } \nu \leq 2i - 1. \end{cases}$$

Case (2). If $i \neq k$ and $k \neq 1$, then we have

$$\Lambda_\nu((x_i \otimes y_i - x_1 \otimes y_1) \otimes z_k'') \equiv 0.$$

If $i = k$ or $k = 1$, then we have

$$\Lambda_\nu((x_i \otimes y_i - x_1 \otimes y_1) \otimes x_i) \equiv \begin{cases} g\mu & \text{if } \nu = 2j - 1 \text{ or } 2j, \\ (2g+1)\mu & \text{if } \nu \neq 2j - 1 \text{ and } 2j, \end{cases}$$

$$\Lambda_\nu((x_i \otimes y_i - x_1 \otimes y_1) \otimes y_i) \equiv \begin{cases} i\mu & \text{if } \nu > 2i - 1, \\ (g+i+1)\mu & \text{if } \nu \leq 2i - 1, \end{cases}$$

$$\Lambda_\nu((x_i \otimes y_i - x_1 \otimes y_1) \otimes x_1) \equiv \begin{cases} (g+2)\mu & \text{if } \nu = 1 \text{ or } 2, \\ \mu & \text{if } \nu \neq 1 \text{ and } 2, \end{cases}$$

$$\Lambda_\nu((x_i \otimes y_i - x_1 \otimes y_1) \otimes y_1) \equiv \begin{cases} (2g+1)\mu & \text{if } \nu > 1, \\ g\mu & \text{if } \nu \leq 1. \end{cases}$$

By combining Proposition 5.2 and Lemma 5.5, we have the pointed harmonic volume I_ν of (C_0, P_ν).

Theorem 5.6. *Case (1). Elements of $K \otimes H$ at which the value of the pointed harmonic volumes I_ν are $1/2 \bmod \mathbb{Z}$ are given by*

$$\begin{array}{ll} x_i \otimes x_j \otimes y_i, \quad x_j \otimes x_i \otimes y_i & \text{if } \nu = 2j - 1 \text{ or } 2j, \\ x_i \otimes y_j \otimes y_i, \quad y_j \otimes x_i \otimes y_i & \text{if } (i < j, \nu > 2j - 1) \text{ or } (i > j, \nu \leq 2j - 1), \\ y_i \otimes x_j \otimes x_i, \quad x_j \otimes y_i \otimes x_i & \text{if } \nu = 2j - 1 \text{ or } 2j, \\ y_i \otimes y_j \otimes x_i, \quad y_j \otimes y_i \otimes x_i & \text{if } (i < j, \nu > 2j - 1) \text{ or } (i > j, \nu \leq 2j - 1). \end{array}$$

The values at the other elements are $0 \bmod \mathbb{Z}$.

Case (2). *Elements of $K \otimes H$ at which the value of the pointed harmonic volumes I_ν are $1/2 \bmod \mathbb{Z}$ are given by*

$$
\begin{array}{lll}
(x_i \otimes y_i - x_1 \otimes y_1) \otimes x_i, & (y_i \otimes x_i - y_1 \otimes x_1) \otimes x_i & \text{if } \nu \neq 2i-1 \text{ and } 2i, \\
(x_i \otimes y_i - x_1 \otimes y_1) \otimes y_i, & (y_i \otimes x_i - y_1 \otimes x_1) \otimes y_i & \text{if } \nu \leq 2i-1, \\
(x_i \otimes y_i - x_1 \otimes y_1) \otimes x_1, & (y_i \otimes x_i - y_1 \otimes x_1) \otimes x_1 & \text{if } \nu \neq 1 \text{ and } 2, \\
(x_i \otimes y_i - x_1 \otimes y_1) \otimes y_1, & (y_i \otimes x_i - y_1 \otimes x_1) \otimes y_1 & \text{if } \nu > 1.
\end{array}
$$

The values at the other elements are $0 \bmod \mathbb{Z}$.

From Proposition 4.1, this theorem can be extended to any hyperelliptic curve C with Weierstrass base points. But this extension is complicated. We reconsider Theorem 5.6 from a combinatorial viewpoint. We apply an element $A \in K \otimes H$ to the identification (3.1) in the group $(K \otimes H)_{\mathbb{Z}_2}$. Then we have $(A \bmod 2) = \sum_{p,q,r \neq \nu} A_{p,q,r} f_p \otimes f_q \otimes f_r$, where $A_{p,q,r} \in \mathbb{Z}_2 = \{0,1\}$. The notation $\sharp$ means the cardinality of a set. A counting function $\kappa_\nu \colon K \otimes H \to \frac{1}{2}\mathbb{Z}/\mathbb{Z} = \{0, 1/2\}$ is well-defined by

$$
\kappa_\nu(A) := \frac{1}{2}\Big(\sharp\{(p,q,r); A_{p,q,r} = 1, \sharp\{p,q,r\} = 2\}\Big) \bmod \mathbb{Z}.
$$

Here $\sharp\{p,q,r\} = 2$ means $p = q \neq r$ or $q = r \neq p$ or $r = p \neq q$. By the long but easy computation, we obtain the correspondence.

Corollary 5.7. *On the curve C_0, we have*

$$
I_\nu(A) \equiv \kappa_\nu(A) \bmod \mathbb{Z}.
$$

Example 5.8. (1) If $A = x_i \otimes x_j \otimes y_i$ ($i < j$ and $\nu = 2j - 1$), we have

$$
\begin{aligned}
\kappa(A) &= \kappa((f_{2i-1} + f_{2i}) \otimes f_{2j} \otimes (f_0 + f_1 + \cdots + f_{2i-1})) \\
&\equiv \kappa(f_{2i-1} \otimes f_{2j} \otimes f_{2i-1}) = 1/2.
\end{aligned}
$$

(2) If $A = x_i \otimes x_j \otimes y_i$ ($i > j$ and $2i < \nu$), we have

$$
\begin{aligned}
\kappa(A) &= \kappa((f_{2i-1} + f_{2i}) \otimes (f_{2j-1} + f_{2j}) \otimes (f_0 + f_1 + \cdots + f_{2i-1})) \\
&\equiv \kappa(f_{2i-1} \otimes f_{2j-1} \otimes f_{2j-1} + f_{2i-1} \otimes f_{2j-1} \otimes f_{2i-1} + f_{2i-1} \otimes f_{2j} \otimes f_{2j} \\
&\qquad + f_{2i-1} \otimes f_{2j} \otimes f_{2i-1} + f_{2i} \otimes f_{2j-1} \otimes f_{2j-1} + f_{2i} \otimes f_{2j} \otimes f_{2j}) \\
&= 1/2 + 1/2 + 1/2 + 1/2 + 1/2 + 1/2 \equiv 0.
\end{aligned}
$$

6. A COMBINATORIAL FORMULA OF I_ν

In this section, we compute the pointed harmonic volume $I_\nu = I_{P_\nu}$ of (C, P_ν) by another combinatorial way. Let S_{2g+1} be the $(2g+1)$-th symmetric group. Using the natural projection $\Delta_g^1 \to S_{2g+1}$, the group $H_{\mathbb{Z}_2}$ is naturally considered as a $\mathbb{Z}_2 S_{2g+1}$-module (Arnol'd, V. I. [1]). From the slight modification of Lemma 5.5 and Proposition 5.7 in [6], we have

Lemma 6.1.

$$
H^0(\Delta_g^1; (K \otimes H)^*) = H^0(S_{2g+1}; (H^{\otimes 3})^*) = \mathbb{Z}_2.
$$

Moreover the unique nontrivial element $\psi_\nu \in H^0(S_{2g+1}; (H^{\otimes 3})^*)$ is an S_{2g+1}-homomorphism $H^{\otimes 3} \to \mathbb{Z}_2$ defined by

$$
\psi_\nu(f_i \otimes f_j \otimes f_k) = \begin{cases} 1 & \text{if } \sharp\{i,j,k\} = 2, \\ 0 & \text{otherwise,} \end{cases}
$$

for any i, j, k except for ν.

From Lemma 6.1, we have

Theorem 6.2. *For $A \in K \otimes H$, we have*

$$I_\nu(A) \equiv \kappa_\nu(A) \bmod \mathbb{Z}.$$

Using the equation $f_i = \pi(e'_\nu) + \pi(e'_i)$, we obtain $A = \sum_{p,q,r} A'_{p,q,r} \pi(e'_p) \otimes \pi(e'_q) \otimes \pi(e'_r)$. Another counting function $\kappa'_\nu \colon K \otimes H \to \frac{1}{2}\mathbb{Z}/\mathbb{Z} = \{0, 1/2\}$ is defined by

$$\kappa'_\nu(A) := \frac{1}{2}\Big(\sharp\{(p,q,r); A'_{p,q,r} = 1, \sharp\{p,q,r\} = 2, \ p,q,r \neq \nu\}\Big) \bmod \mathbb{Z}.$$

Corollary 6.3.

$$I_\nu(A) \equiv \kappa'_\nu(A) \bmod \mathbb{Z}.$$

Proof. We use the notation $e(p,q,r) = \pi(e'_p) \otimes \pi(e'_q) \otimes \pi(e'_r)$ only here. The equation

$$\begin{aligned}
f_p \otimes f_q \otimes f_r =\,&e(p,q,r) + e(p,q,\nu) + e(p,\nu,r) + e(p,\nu,\nu) \\
&+ e(\nu,q,r) + e(p,q,\nu) + e(\nu,\nu,r) + e(\nu,\nu,\nu)
\end{aligned}$$

gives $\kappa_\nu(A) \equiv \kappa'_\nu(A)$. $\qquad\qquad\square$

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