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THE LOCAL PROPERTY OF THE REGULARIZED SOLUTIONS IN NUMERICAL DIFFERENTIATION

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Abstract. Numerical differentiation is a typical ill-posed problem which can be treated by the Tikhonov regularization. In this paper, we prove that the L^2-norms of the second order derivatives of the regularized solutions blow up in any small interval I where the exact solution is not in $H^2(I)$.

1. Introduction

A numerical differentiation arises in many applications and engineering computations such as the determination of the underground water, the Dupire formulae in financial mathematics([15]), etc. One of the main difficulties for this problem is the ill-posedness, which means that the small errors of measurement may cause huge errors in computed derivatives ([4], [9], [12]). Therefore one needs regularizing techniques for reasonable computations. Several numerical algorithms have been proposed for overcoming the instabilities ([3], [4], [5], [6], [7], [8], [9], [10],[11], [13]). It has been shown that the Tikhonov regularization for treating the numerical differentiation problem is one of the effective methods.

In [14], the authors discussed the numerical differentiation by using the Tikhonov regularization method. It is shown that, if the exact solution is smooth, then the regularized solutions converge to the exact solution, and if the exact solution is not smooth, then the L^2-norms in the whole interval of the corresponding high order derivatives of the regularized solutions will blow up. From the numerical results, it can be seen that this kind of blow-up happens only near the irregular points. However to the authors' knowledge, this property has not been proved up to now.

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In this paper we discuss this local property of the regularized solution based on the work of [14]. We prove that, if the exact solution is not in $H^2(I)$ for an interval I, then the norms of the second order derivative of the regularized solutions in I blow up.

2. Formulation of the problem and theoretical result

Hereforth we set

$$g^{'} = \frac{dg}{dx}, \quad g^{''} = \frac{d^2g}{dx^2}$$

Suppose that $\Delta = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ is a uniform grid of $[0,1]$. Here $x_j = \frac{j}{n}$, $j = 0, 1, \cdots, n$, and we denote $h = \frac{1}{n}$. Let $y = y(x)$ be a continuous function defined on [0,1]. A noisy value of $y(x)$ at point x_j is given as y_j^δ which satisfies

$$|y(x_j) - y_j^\delta| < \delta, j = 0, 1, \cdots, n, \tag{2.1}$$

where δ is a given constant called the level of noise in the data. The numerical differentiation is then to approximate $y'(x)$ from the value of $y_j^\delta, j = 0, 1, \cdots, n$. Without loss of generality, we assume that there are no errors at the boundary for the sample data, i. e., $y_0^\delta = y(0)$, $y_n^\delta = y(1)$. Otherwise we can use a new function

$$Y(x) = y(x) + y_0^\delta - y(0) + (y_n^\delta - y(1) + y(0) - y_0^\delta)x.$$

It can be easily proved that $Y(0) = y_0^\delta$, $Y(1) = y_n^\delta$. Since

$$\left| y^{'}(x) - Y^{'}(x) \right| = \left| y_n^\delta - y(1) + y(0) - y_0^\delta \right| \le 2\delta$$

Therefore the approximation of $Y^{'}(x)$ is also an approximation of $y^{'}(x)$. ([9], [14]).

The **numerical differentiation** is to find a function f from the data $\{y_j^\delta\}_{j=0}^n$ such that $f^{'}$ approximates $y^{'}$. We will solve this problem by the Tikhonov regularization method as in [14]. The following spaces and norms will be used in this paper:

$$L^2(0,1) = \left\{ g \,|\, (\int_0^1 g^2(x)dx)^{1/2} < \infty \right\},$$

$$H^2(0,1) = \left\{ g \,|\, g \in L^2(0,1),\, g^{''} \in L^2(0,1) \right\},$$

$$C[0,1] = \{g \,|\, g \text{ is a continuous function on [0,1]}\},$$

$$\|g\|_{L^2(0,1)} = \left(\int_0^1 |g(x)|^2 dx\right)^{1/2},$$

$$\|g\|_{H^2(0,1)} = \left(\|g\|^2_{L^2(0,1)} + \|g^{''}\|^2_{L^2(0,1)}\right)^{\frac{1}{2}},$$

$$\|g\|_{C[0,1]} = \max_{x\in[0,1]} |g(x)|.$$

Define a cost functional by

$$\Phi(f) = \frac{1}{n}\sum_{i=1}^{n-1}(f(x_i) - y_i^\delta)^2 + \alpha\|f^{''}\|^2_{L^2(0,1)} \tag{2.2}$$

for all $f \in H^2(0,1)$ with $f(0) = y(0)$, $f(1) = y(1)$, where α is a regularization parameter.

Then we can prove (e.g. [14]):

Theorem 2.1. *Let $y \in C[0,1]$. There exists a unique minimizer $f_* = f_*(\delta, \alpha, h)$ of functional* (2.2)*.*

The minimizer f_* is called the regularized solution, and we have the following error estimate ([14]).

Theorem 2.2. *Suppose that $y \in H^2(0,1)$. Then we have the following error estimation for the regularized solution $f_*(\delta, \alpha, h)$:*

$$\|f_*^{'}(\delta, \alpha, h) - y^{'}\|_{L^2(0,1)} \le \left(2h + 4\alpha^{\frac{1}{4}} + \frac{h}{\pi}\right)\|y^{''}\|_{L^2(0,1)} + h\sqrt{\frac{\delta^2}{\alpha} + \frac{2\delta}{\alpha^{\frac{1}{4}}}}.$$

If we choose $\alpha = \delta^2$, then

$$\|f_*^{'}(\delta, \delta^2, h) - y^{'}\|_{L^2(0,1)} \le (2h + 4\sqrt{\delta} + \frac{h}{\pi})\|y^{''}\|_{L^2(0,1)} + h + 2\sqrt{\delta}.$$

It is practically important to detect subintervals where a state function f is not smooth. Thus we are more interested in an interval $(a,b) \subset (0,1)$ where $y^{''} \notin L^2(a,b)$. Therefore, it is important to know the behaviour of the regularized solution f_* in a subinterval as short as possible. In particular, if $(a,b) = (0,1)$, then the following theorem is proved in [14].

Theorem 2.3. *If $y \in C[0,1]\backslash H^2(0,1)$, and we choose the regularization parameter $\alpha = \delta^2$, then*

$$\|f_*^{''}(\delta, \delta^2, h)\|_{L^2(0,1)} \longrightarrow \infty, \qquad \text{as} \quad \delta, h \to 0. \tag{2.3}$$

Remark 2.4. The general study corresponding to Theorem 2.3 is found in [2].

This result indicates only that, if the exact solution is not in $H^2(0,1)$, then the L^2-norms of the second order derivatives of the regularized solutions on the whole interval $[0,1]$ blow up. In this paper we will give a localized version of this theorem. We want to know if the exact solution is not smooth on a small interval, namely, $y \in C[0,1] \setminus H^2(a,b)$, what will happen to $\|f_*^{''}\|_{L^2(a,b)}$? Here $(a,b) \subset [0,1]$.

We state our theoretical result on which our numerical method is based.

Theorem 2.5. *Suppose that $y \in C[0,1]$, $(a,b) \subset (0,1)$, and we choose the regularization parameter $\alpha = \delta^2$. If $y \notin H^2(a,b)$, then the regularized solution $f_*(\delta,\delta^2,h)$ satisfies*

$$\lim_{\delta,h\to 0} \|f_*^{''}(\delta,\delta^2,h)\|_{L^2(a,b)} = \infty.$$

In the proof of this theorem, we will use an interpolation inequality. (e.g., Theorem 4.14(p.75) in [1]):

Lemma 2.6. *Let $-\infty \le a < b \le \infty$, $1 \le p < \infty$, and $0 < \epsilon_0 < \infty$, f have the m-th order derivative $f^{(m)}$ in (a,b). There exists a constant $K > 0$ which depends on ϵ_0, p, m, and $b-a$ such that for every ϵ, $0 < \epsilon \le \epsilon_0$, $0 \le j < m$, we have*

$$(\int_a^b |f^{(j)}|^p dt)^{\frac{1}{p}} \le K\epsilon(\int_a^b |f^{(m)}|^p dt)^{\frac{1}{p}} + K\epsilon^{\frac{-j}{m-j}}(\int_a^b |f|^p dt)^{\frac{1}{p}}. \tag{2.4}$$

Proof of Theorem 2.5: Henceforth for simplicity, we set

$$f_*(\delta,n)(x) = f_*(\delta,\delta^2,h)(x)$$

where we recall $h = \frac{1}{n}$.

Assume contrarily that the conclusion of the theorem is not correct. This means that there exist two sequences $\{\delta_k\}, \{h_k\}$, $k = 1,2,\cdots$, such that

$$\lim_{k\to\infty} \delta_k = \lim_{k\to\infty} h_k = 0$$

and

$$\|f_*^{''}(\delta_k,n_k)\|_{L^2(a,b)} \le M, \qquad k = 1,2,\cdots \tag{2.5}$$

where M is a positive constant, $n_k = \frac{1}{h_k}$.

By $y \in C[0,1]$, we can take a sequence of functions $y_m \in H^2(0,1)$ satisfying

$$y_m(0) = y(0), \qquad y_m(1) = y(1),$$

and

$$\|y_m - y\|_{C[0,1]} \le \frac{1}{m}, \quad \sup_m \|y_m\|_{L^2(a,b)} < \infty. \tag{2.6}$$

In fact, y_m can be constructed for example by suitable interpolated polynomials.

Next we will prove

$$\sup_m \|y_m^{''}\|_{L^2(a,b)} = \infty. \tag{2.7}$$

Assume contrarily that $\sup_m \|y_m^{''}\|_{L^2(a,b)} < \infty$. Then from the definition of the norm, we know that $\sup_m \|y_m\|_{H^2(a,b)} < \infty$. By the reflexiveness of $H^2(a,b)$, there exists a subsequence $y_{m_k} \in H^2(a,b)$ and $\widetilde{y} \in H^2(a,b)$ so that $y_{m_k} \to \widetilde{y}$ weakly in $H^2(a,b)$. On the other hand, we see from (2.6) that $y_m \to y$ strongly in $L^2(a,b)$, so $\widetilde{y} = y$ and thus we have $y \in H^2(a,b)$. This is a contradiction. Thus the proof of (2.7) is complete.

Moreover, for each $k \in N$, we set

$$\ell(k) = \min\{j \in N; \delta_j \|y_k^{''}\|_{L^2(0,1)} < 1\}.$$

Such an $\ell(k)$ exists uniquely because $\delta_j \to 0$ as $j \to \infty$. For simplicity we denote $\delta_{\ell(k)}$ and $n_{\ell(k)}$ again by δ_k and n_k respectively. Hence

$$\delta_k \|y_k^{''}\|^2_{L^2(0,1)} < 1. \tag{2.8}$$

We set $f_{*k} = f_*(\delta_k, n_k)$. Since $\Phi(f_{*k}) \le \Phi(y_k)$, we have

$$\begin{aligned}
&\frac{1}{n_k}\sum_{j=1}^{n_k-1}\left(f_{*k}(x_j) - y_j^{\delta_k}\right)^2 \le \Phi(f_{*k}) \\
&\le \Phi(y_k) \le \frac{1}{n_k}\sum_{j=1}^{n_k-1}(y_k(x_j) - y_j^{\delta_k})^2 + \delta_k^2\|y_k^{''}\|^2_{L^2(0,1)} \\
&\le \frac{2}{n_k}\sum_{j=1}^{n_k-1}\left((y_k(x_j) - y(x_j))^2 + (y(x_j) - y_j^{\delta_k})^2\right) + \delta_k(\delta_k\|y_k^{''}\|^2_{L^2(0,1)}) \\
&\le \frac{2}{k^2} + 2\delta_k^2 + \delta_k.
\end{aligned} \tag{2.9}$$

At the last inequality, we need (2.8) and

$$|y_k(x_j) - y(x_j)| \le \frac{1}{k}, \quad 1 \le j \le n_k - 1,$$

by (2.6).

Suppose that $x_{i_0-1} \le a < x_{i_0}$ and $x_{j_0} < b \le x_{j_0+1}$. Then by (2.9), we have

$$\begin{aligned}\frac{1}{n_k}\sum_{j=i_0}^{j_0}\left(f_{*k}(x_j)-y_j^{\delta_k}\right)^2 &\le \frac{1}{n_k}\sum_{j=1}^{n_k-1}\left(f_{*k}(x_j)-y_j^{\delta_k}\right)^2\\ &\le \frac{2}{k^2}+2\delta_k^2+\delta_k.\end{aligned}$$

Moreover

$$\begin{aligned}|y_j^{\delta_k}|^2=(y_j^{\delta_k}-y(x_j)+y(x_j))^2 &\le 2((y_j^{\delta_k}-y(x_j))^2+y(x_j)^2)\\ &\le 2(\delta_k^2+\|y\|^2_{L^\infty(0,1)}).\end{aligned}$$

Therefore by (2.1) and (2.9)

$$\begin{aligned}\frac{1}{n_k}\sum_{j=i_0}^{j_0}f_{*k}^2(x_j) &\le \frac{2}{n_k}\sum_{j=i_0}^{j_0}\left((f_{*k}(x_j)-y_j^{\delta_k})^2+|y_j^{\delta_k}|^2\right)\\ \le\ &\frac{4}{k^2}+4\delta_k^2+2\delta_k+4(\delta_k^2+\|y\|^2_{L^\infty(0,1)})\le A.\end{aligned}\tag{2.10}$$

Here $A>0$ is a constant which is independent of $k\in N$. Moreover, denote

$$\eta_{i_0}=a,\qquad \eta_i=\frac{x_{i-1}+x_i}{2}, i=i_0+1,\cdots,j_0,\qquad \eta_{j_0+1}=b.$$

By the mean value theorem, we can choose $\xi_j\in(\eta_j,\eta_{j+1})$, and

$$\begin{aligned}\|f_{*k}\|^2_{L^2(a,b)} &= \int_{\eta_{i_0}}^{\eta_{j_0+1}}f_{*k}^2(x)dx=\sum_{j=i_0}^{j_0}\int_{\eta_j}^{\eta_{j+1}}f_{*k}^2(x)dx\le\frac{2}{n_k}\sum_{j=i_0}^{j_0}f_*^2(\xi_j)\\ &= \frac{2}{n_k}\sum_{j=i_0}^{j_0}(f_{*k}^2(\xi_j)-f_{*k}^2(x_j))+\frac{2}{n_k}\sum_{j=i_0}^{j_0}f_{*k}^2(x_j)\\ &\le \frac{2}{n_k}\int_a^b\left|2f_{*k}^{'}(x)f_{*k}(x)\right|dx+\frac{2}{n_k}\sum_{j=i_0}^{j_0}f_{*k}^2(x_j).\end{aligned}$$

Here we used

$$\begin{aligned}\left|\sum_{j=i_0}^{j_0}(f_{*k}^2(\xi_j)-f_{*k}^2(x_j))\right| &= \left|\sum_{j=i_0}^{j_0}\int_{x_j}^{\xi_j}\frac{d}{dx}(f_{*k}(x))^2dx\right|\\ \le \sum_{j=i_0}^{j_0}\int_{x_j}^{\xi_j}2|f_{*k}^{'}(x)f_{*k}(x)|dx &\le \int_{\eta_{i_0}}^{\eta_{j_0+1}}2|f_{*k}^{'}(x)f_{*k}(x)|dx.\end{aligned}$$

Hence (2.10) and the Schwarz inequality yield

$$\|f_{*k}\|^2_{L^2(a,b)}\le\frac{4}{n_k}\|f_{*k}\|_{L^2(a,b)}\|f_{*k}^{'}\|_{L^2(a,b)}+2A.\tag{2.11}$$

Choosing parameters as $p = 2, m = 2, j = 1, \epsilon = 1$ in Lemma 2.6, we have

$$\|f'_{*k}\|_{L^2(a,b)} \le K(\|f_{*k}\|_{L^2(a,b)} + \|f''_{*k}\|_{L^2(a,b)}). \tag{2.12}$$

Therefore, (2.11) yields

$$(1 - \frac{4K}{n_k})\|f_{*k}\|^2_{L^2(a,b)} \le \frac{4}{n_k}\|f_{*k}\|_{L^2(a,b)}\|f''_{*k}\|_{L^2(a,b)} + 2A.$$

Substituting

$$\|f_{*k}\|_{L^2(a,b)}\|f''_{*k}\|_{L^2(a,b)} \le \frac{1}{2}\|f_{*k}\|^2_{L^2(a,b)} + \frac{1}{2}\|f''_{*k}\|^2_{L^2(a,b)}$$

and choosing $k_0 \in N$ large, we have

$$\|f_{*k}\|_{L^2(a,b)} \le K_1\|f''_{*k}\|_{L^2(a,b)} + K_1\sqrt{A}$$

for all $k \ge k_0$. Here the constant $K_1 > 0$ is independent of k.

By using this inequality and (2.5), we have

$$\sup_{k \ge k_0} \|f_{*k}\|_{L^2(a,b)} < \infty.$$

Therefore, by (2.12) we have

$$\sup_k \|f_{*k}\|_{H^2(a,b)} < \infty, \quad k \ge k_0.$$

Since $H^2(a,b)$ is reflexive, there exist a subsequence f_{*k}, which is denoted by the same letter, and $\widetilde{f} \in H^2(a,b)$ such that

$$f_{*k} \to \widetilde{f} \quad \text{weakly in} \quad H^2(a,b).$$

Since the embedding from $H^2(a,b)$ to $C[a,b]$ is compact, we see that

$$f_{*k} \to \widetilde{f} \quad \text{strongly in} \quad C[a,b].$$

That is,

$$\lim_{k\to\infty} \|f_{*k} - \widetilde{f}\|_{C[a,b]} = 0. \tag{2.13}$$

By the definition of the integral, for any $\epsilon > 0$, there exists $k = k(\epsilon) \in N$ such that by (2.1) and (2.9) we obtain

$$
\begin{aligned}
& \|y - \widetilde{f}\|^2_{L^2(a,b)} \\
\leq \quad & \frac{1}{n_k} \sum_{j=i_0}^{j_0} (y(x_j) - \widetilde{f}(x_j))^2 + \epsilon \\
\leq \quad & \frac{3}{n_k} \sum_{j=i_0}^{j_0} \{(y(x_j) - y_j^{\delta_k})^2 + (y_j^{\delta_k} - f_{*k}(x_j))^2 + (f_{*k}(x_j) - \widetilde{f}(x_j))^2\} + \epsilon \\
\leq \quad & 3\delta_k^2 + \frac{6}{k^2} + 6\delta_k^2 + 3\delta_k + 3\|f_{*k} - \widetilde{f}\|_{C[a,b]} + \epsilon
\end{aligned}
$$

for any $k \geq k_1(\epsilon)$.

Hence letting $k \to \infty$, we have

$$\|y - \widetilde{f}\|_{L^2(a,b)} \leq \epsilon.$$

Since ϵ is arbitrary, we have

$$\|y - \widetilde{f}\|^2_{L^2(a,b)} = 0,$$

that is,

$$y(x) = \widetilde{f}(x) \qquad \text{for almost all } x \in [a,b].$$

Since $\widetilde{f} \in H^2(a,b)$, this implies that $y \in H^2(a,b)$, which contradicts the assumption $y \notin H^2(a,b)$. Thus the proof is complete.

3. Conclusion

In this paper, we established the behaviour near non-smooth points for the Tikhononv regularized solutions in the numerical differentiation. We proved that, if the exact solution has irregular points, then the norms of the second order derivatives of the regularized solutions at any small interval which contains the irregular points, will blow up. Unlike the result of [14], we proved that the blow-up will take place near the irregular points.

References

1. R. A. Adams, Sobolev Spaces. Pure and Applied Mathematics, Vol. 65, Academic Press, New York-London, 1975.
2. J. Cheng and M. Yamamoto, Regularity of the exact solution and the boundedness of the Tikhonov regularized solutions. Preprint 2001.

3. H. W. Engl, M. Hanke and A. Neubauer, Regularization of Inverse Problems. Kluwer Academic Publishers, Dordrecht Hardbound, 1996.
4. C. W. Groetsch, The Theory of Tikhonov Regularization for Fredholm Equations of the First Kind. Research Notes in Mathematics, 105. Pitman (Advanced Publishing Program), Boston, Mass.-London 1984.
5. C. W. Groetsch, Differentiation of approximately specified functions. Amer. Math. Monthly 98 (1991), pp. 847-850.
6. C. W. Groetsch, Lanczos' generalized derivative. Amer. Math. Montyly 105 (1998), No. 4, pp. 320-326.
7. C. W. Groetsch, Optimal order of accuracy in Vasin's method for differentiation of noisy functions. Journal of Optimization Theory and Applications 74 (1992), No. 2, pp373-378.
8. C. W. Groetsch and O. Scherzer, The optimal order of convergence for stable evaluation of differential operators. Electronic Journal of Differential Equations 1993 (1993), No. 4, pp. 1-10.
9. M. Hanke and O. Scherzer, Inverse problems light: numerical differentiation. Amer. Math. Monthly 108 (2001), No. 6, pp. 512-521.
10. D. A. Murio, Automatic numerical differentiation by discrete mollification, Compputers and Mathematics with Applications 13 (1987), pp. 381-386.
11. A. G. Ramm and A. B. Smirnova, On stable numerical differentiation. Math. Comp. 70 (2001), pp. 1131-1153.
12. A. N. Tikhonov and V. Y. Arsenin, Solutions of Ill-posed Problems. Winston and Sons, Washington, 1977.
13. V. V. Vasin, The stable evaluation of a derivative in the space $C(-\infty,\infty)$. USSR Computational Mathematics and Mathematical Physics 13 (1973), pp. 16-24.
14. Y. B. Wang, X. Z. Jia and J. Cheng, A numerical differentiation method and its application to reconstruction of discontinuity. Inverse Problems 18 (2002) No. 6, pp. 1461-1476.
15. B. B. Yin and Y. Z. Ye, Recovering the Local volatility in Black-Scholes model by numerical differentiation. Preprint 2005.

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