UTMS 2005–4 February 9, 2005

Completely integrable systems associated with classical root systems

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Abstract. We explicitly construct sufficient integrals of completely integrable quantum and classical systems associated with classical root systems, which include Calogero-Moser-Sutherland models, Inozemtsev models and Toda finite lattices with boundary conditions. We also discuss the classification of the completely integrable systems.

Contents

1. Introduction

A Schrödinger operator

$$P = -\frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2} + R(x) \tag{1.1}$$

with the potential function $R(x)$ of n variables $x = (x_1, \ldots, x_n)$ is called *completely integrable* if there exist n differential operators $P_1, \ldots, P_n$ such that

$$\begin{cases} [P_i, P_j] = 0 \qquad (1 \leq i < j \leq n), \\ P \in \mathbb{C}[P_1, \ldots, P_n], \\ P_1, \ldots, P_n \text{ are algebraically independent.} \end{cases} \tag{1.2}$$

In this note, we explicitly construct the integrals $P_1,\dots,P_n$ for completely integrable potential functions $R(x)$ of the form

$$R(x)=\sum_{1\le i<j\le n}\Big(u_{ij}^-(x_i-x_j)+u_{ij}^+(x_i+x_j)\Big)+\sum_{k=1}^n v_k(x_k) \tag{1.3}$$

appearing in other papers. The Schrödinger operators with these commuting differential operators treated in this paper include Calogero-Moser-Sutherland systems (cf. [Ca], [Mo], [Su], [OP2], [OP3]), Heckman-Opdam's hypergeometric systems (cf. [Se] for type A_{n-1}, [HO] in general) and some extensions of finite Toda lattices corresponding to (extended) Dynkin diagram for classical root systems (cf. [To], [Ko], [OP1], [vD], [Ru]).

Put $\partial_j=\frac{\partial}{\partial x_j}$ for simplicity. We denote by $\sigma(Q)$ the principal symbol of a differential operator of Q. For example, $\sigma(P)=-\frac12(\xi_1^2+\cdots+\xi_n^2)$.

We note that [Wa] proves that the potential function is of the form (1.3) if

$$\sigma(P_k)=\sum_{1\le j_1<\cdots<j_k\le n}\xi_{j_1}^2\cdots\xi_{j_k}^2\quad\text{for } k=1,\dots,n. \tag{1.4}$$

In this case $R(x)$ will be called to be of *type* B_n or of the *classical type*. Moreover when $R(x)$ is symmetric with respect to the coordinate $(x_1,\dots,x_n)$ and invariant under the coordinate transformation $(x_1,x_2,\dots,x_n)\mapsto(-x_1,x_2,\dots,x_n)$, then $R(x)$ is determined by [OS] for $n\ge 3$ and by [OO] for $n=2$ and P_k are given by [O1].

Classifications of the potential functions under certain conditions are given in [OOS], [Oc], [Ta], [Wa] and [O2] etc. We will present Conjecture 9.1 in §9 which claims that the potential functions given in this note exhaust those of the completely integrable systems satisfying (1.4).

If $v_k=0$ for $k=1,\dots,n$, we can expect $\sigma(P_k)=\sum_{1\le j_1<\cdots<j_k\le n}\xi_{j_1}^2\cdots\xi_{j_k}^2$ for $k=1,\dots,n-1$ and $\sigma(P_n)=\xi_1\xi_2\cdots\xi_n$ and the potential function is called to be of *type* D_n. If $v_k=0$ and $u_{ij}^+=0$ for $k=1,\dots,n$ and $1\le i<j\le n$, we can expect $P_1=\partial_1+\cdots+\partial_n$, $\sigma(P_k)=\sum_{1\le j_1<\cdots<j_k\le n}\xi_{j_1}\cdots\xi_{j_k}$ for $k=2,\dots,n$ and the potential function is called to be of *type* A_{n-1}.

The elliptic potential function of type A_{n-1} with

$$u_{ij}^-(t)=C\wp(t;2\omega_1,2\omega_2)+C',\ \ u_{ij}^+(t)=v_k(t)=0\quad (C,\,C'\in\mathbb{C}) \tag{1.5}$$

(cf. [OP3]) and that of type B_n with

$$\begin{cases}u_{ij}^-(t)=v_{ij}^+(t)=A\wp(t;2\omega_1,2\omega_2),\\ v_k(t)=\displaystyle\sum_{j=0}^3 C_j\wp(t+\omega_j;2\omega_1,\omega_2)-\frac{C}{2},\end{cases}\quad (A,\,C_i,\,C\in\mathbb{C}) \tag{1.6}$$

introduced by [I1] are most fundamental and their integrability and integrals of higher order are given by [O1] and [OOS]. Here $\wp(t;2\omega_1,2\omega_2)$ is the Weierstrass elliptic function whose fundamental periods are $2\omega_1$ and $2\omega_2$ and

$$\omega_0=0,\quad \omega_1+\omega_2+\omega_3=0. \tag{1.7}$$

Other potential functions are suitable limits of these elliptic potential functions. This fact is shown in [I2], [vD] and [Ru] etc., and we will construct other integrable systems by taking suitable analytic continuations of the integrals given in [O1]. The main purpose of this note is to give the explicit expression of the operators $P_1,\dots,P_n$ in (1.2) in this unified way. Such study of the systems of types A_{n-1}, B_2, B_n $(n\ge 3)$ and D_n are given in §3, §4, §5, §6, respectively.

Since our expression of P_k is natural, we can easily define their classical limits and get completely integrable Hamiltonians of dynamical systems together with their sufficient integrals. This is clarified in §7.

In §8 we examine the ordinary differential operators which are analogues of the Schrödinger operators studied in this note.

2. Notation and preliminary results

Let $\{e_1, \dots, e_n\}$ be the natural orthonormal base of the Euclidean space $\mathbb{R}^n$ with the inner product

$$\langle x, y\rangle = \sum_{j=1}^{n} x_j y_j \quad \text{for } x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n) \in \mathbb{R}^n. \tag{2.1}$$

Here $e_j = (\delta_{1j}, \dots, \delta_{nj}) \in \mathbb{R}^n$ with Kronecker's delta δ_{ij}.

Let $\alpha \in \mathbb{R}^n \setminus \{0\}$. The reflection w_α with respect to α is a linear transformation of $\mathbb{R}^n$ defined by $w_\alpha(x) = x - \dfrac{2\langle \alpha, x\rangle}{\langle \alpha, \alpha\rangle}\alpha$ for $x \in \mathbb{R}^n$. Furthermore we define a differential operator ∂_α by

$$(\partial_\alpha \varphi)(x) = \frac{d}{dt}\varphi(x + t\alpha)\Big|_{t=0} \tag{2.2}$$

and then $\partial_j = \partial_{e_j}$.

The root system $\Sigma(B_n)$ of type B_n is realized in $\mathbb{R}^n$ by

$$\begin{cases} \Sigma(A_{n-1})^+ = \{e_i - e_j;\ 1 \le i < j \le n\}, \\ \Sigma(D_n)^+ = \{e_i \pm e_j;\ 1 \le i < j \le n\}, \\ \Sigma(B_n)^+_S = \{e_k;\ 1 \le k \le n\}, \\ \Sigma(B_n)^+ = \Sigma(D_n)^+ \cup \Sigma(B_n)^+_S, \\ \Sigma(F) = \{\alpha, -\alpha;\ \alpha \in \Sigma(F)^+\} \quad \text{for } F = A_{n-1},\ D_n \text{ or } B_n. \end{cases} \tag{2.3}$$

The Weyl groups $W(B_n)$ of type B_n, $W(D_n)$ of type D_n and $W(A_{n-1})$ of type A_{n-1} are the groups generated by w_α for $\alpha \in \Sigma(B_n)$, $\Sigma(D_n)$ and $\Sigma(A_{n-1})$, respectively. The Weyl group $W(A_{n-1})$ is naturally identified with the permutation group $\mathfrak{S}_n$ of the set $\{1, \dots, n\}$ with n elements. Let ϵ be the group homomorphism of $W(B_n)$ defined by

$$\epsilon(w) = \begin{cases} 1 & \text{if } w \in W(D_n), \\ -1 & \text{if } w \in W(B_n) \setminus W(D_n). \end{cases} \tag{2.4}$$

The potential function (1.3) is of the form

$$R(x) = \sum_{\alpha \in \Sigma(D_n)^+} u_\alpha(\langle \alpha, x\rangle) + \sum_{\alpha \in \Sigma(B_n)^+_S} v_\beta(\langle \beta, x\rangle) \tag{2.5}$$

with functions u_α and v_β of one variable. For simplicity we will denote

$$\begin{cases} u_\alpha(x) = u_{-\alpha}(x) = u_\alpha(\langle \alpha, x\rangle) & \textit{for } \alpha \in \Sigma(D_n)^+, \\ v_\beta(x) = v_{-\beta}(x) = v_\beta(\langle \beta, x\rangle) & \textit{for } \beta \in \Sigma(B_n)^+_S, \\ u^\pm_{ij}(x) = u_{e_i \pm e_j}(x),\ v_k(x) = v_{e_k}(x). \end{cases} \tag{2.6}$$

Lemma 2.1. *For a bounded open subset U of $\mathbb{C}$, there exists an open neighborhood V of 0 in $\mathbb{C}$ such that the followings hold.*

i) *The function $\lambda \sinh^{-1} \lambda z$ is holomorphically extended to $(z, \lambda) \in (U \setminus \{0\}) \times V$ and the function equals $\frac{1}{z}$ when $\lambda = 0$.*

ii) *Suppose $\operatorname{Re} \lambda > 0$. Then the functions*

$$e^{2\lambda t} \sinh^{-2} \lambda(z \pm t) \quad \textit{and} \quad e^{4\lambda t}\bigl(\sinh^{-2} \lambda(z \pm t) - \cosh^{-2} \lambda(z \pm t)\bigr)$$

are holomorphically extended to $(z,q)\in U\times V$ *with* $q=e^{-2\lambda t}$ *and the functions equal* $4e^{\mp 2\lambda z}$ *and* $16e^{\mp 4\lambda z}$*, respectively, when* $q=0$*.*

Proof. The claims are clear from

$$\begin{aligned}\lambda^{-1}\sinh\lambda z&=z+\sum_{j=1}^{\infty}\frac{\lambda^{2j}z^{2j+1}}{(2j+1)!},\\ 4e^{-2\lambda t}\sinh^2\lambda(z\pm t)&=e^{\pm 2\lambda z}(1-e^{-2\lambda t}e^{\mp 2\lambda z})^2,\\ \sinh^{-2}\lambda z-\cosh^{-2}\lambda z&=4\sinh^{-2}2\lambda z.\end{aligned}\qquad\square$$

The elliptic functions $\wp$ and ζ of Weierstrass type are defined by

$$\wp(z)=\wp(z;2\omega_1,2\omega_2)=\frac{1}{z^2}+\sum_{\omega\neq 0}\Bigl(\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\Bigr),\tag{2.7}$$

$$\zeta(z)=\zeta(z;2\omega_1,2\omega_2)=\frac{1}{z}+\sum_{\omega\neq 0}\Bigl(\frac{1}{z-\omega}+\frac{1}{\omega}+\frac{z}{\omega^2}\Bigr)\tag{2.8}$$

where the sum ranges over all non-zero periods $2m_1\omega_1+2m_2\omega_2$ (m_1, $m_2\in\mathbb{Z}$) of $\wp$. The followings are some elementary properties of these functions (cf. [WW]).

$$\wp(z)=\wp(z+2\omega_1)=\wp(z+2\omega_2),\tag{2.9}$$

$$\zeta'(z)=-\wp(z),\tag{2.10}$$

$$\begin{aligned}(\wp')^2&=4\wp^3-g_2\wp-g_3=4(\wp-e_1)(\wp-e_2)(\wp-e_3),\\ e_\nu&=\wp(\omega_\nu)\text{ for }\nu=1,2,3,\ \omega_3=-\omega_1-\omega_2\ \textit{and}\ \omega_0=0,\end{aligned}\tag{2.11}$$

$$\wp(2z)=\frac{1}{4}\sum_{\nu=0}^{4}\wp(z+\omega_\nu)=\frac{(12\wp(z)^2-g_2)^2}{16\wp'(z)^2}-2\wp(z),\tag{2.12}$$

$$\wp(z;2\omega_2,2\omega_1)=\wp(z;2\omega_1,2\omega_2),\tag{2.13}$$

$$\wp(z+\omega_1;2\omega_1,2\omega_2)=e_1+\frac{(e_1-e_2)(e_1-e_3)}{\wp(z;2\omega_1,2\omega_2)-e_1},\tag{2.14}$$

$$\wp(z;\sqrt{-1}\lambda^{-1}\pi,\infty)=\lambda^2\sinh^{-2}\lambda z+\frac{1}{3}\lambda^2,\tag{2.15}$$

$$\wp(z;\infty,\infty)=z^{-2},\tag{2.16}$$

$$\wp(z;\omega_1,2\omega_2)=\wp(z;2\omega_1,2\omega_2)+\wp(z+\omega_1;2\omega_1,2\omega_2)-e_1,\tag{2.17}$$

$$\begin{vmatrix}\wp(z_1)&\wp'(z_1)&1\\ \wp(z_2)&\wp'(z_2)&1\\ \wp(z_3)&\wp'(z_3)&1\end{vmatrix}=0\quad\text{if}\quad z_1+z_2+z_3=0,\tag{2.18}$$

$$\begin{aligned}\wp(z;2\omega_1,2\omega_2)&=-\frac{\eta_1}{\omega_1}+\lambda^2\sinh^{-2}\lambda z+\sum_{n=1}^{\infty}\frac{8n\lambda^2e^{-4n\lambda\omega_2}}{1-e^{-4n\lambda\omega_2}}\cosh 2n\lambda z,\\ \eta_1&=\zeta(\omega_1;2\omega_1,2\omega_2)=\frac{\pi^2}{\omega_1}\Bigl(\frac{1}{12}-2\sum_{n=1}^{\infty}\frac{ne^{-4n\lambda\omega_2}}{1-e^{-4n\lambda\omega_2}}\Bigr),\\ \tau&=\frac{\omega_2}{\omega_1},\ q=e^{\pi i\tau}=e^{-2\lambda\omega_2}\quad\text{and}\quad\lambda=\frac{\pi}{2\sqrt{-1}\omega_1}.\end{aligned}\tag{2.19}$$

Here the sums in (2.19) converge if

$$2\operatorname{Im}\frac{\omega_2}{\omega_1}>\frac{|z|}{|\omega_1|}.\tag{2.20}$$

Let $0 \le k < 2m$. Then (2.19) means

$$\wp(z+\frac{k}{m}\omega_2;2\omega_1,2\omega_2) = -\frac{\eta_1}{\omega_1} + 4\lambda^2\Bigl(\frac{q^{\frac{k}{m}}e^{-2\lambda z}}{\bigl(1-e^{-2\lambda z}q^{\frac{k}{m}}\bigr)^2} + \sum_{n=1}^{\infty}\frac{q^{n(2-\frac{k}{m})}e^{2n\lambda z}}{1-q^{2n}}\Bigr),$$
$$-\frac{\eta_1}{\omega_1} = 4\lambda^2\Bigl(\frac{1}{12} - 2\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\Bigr).$$

Lemma 2.2. *Let k and m be integers satisfying $0 < k < 2m$. Put*

$$\begin{aligned}\wp_0(z;2\omega_1,2\omega_2) &= \wp(z;2\omega_1,2\omega_2) + \frac{\eta_1}{\omega_1},\\ \lambda = \frac{\pi}{2\sqrt{-1}\omega_1} \quad &\text{and} \quad t = q^{\frac{1}{m}} = e^{\frac{\pi i\omega_2}{m\omega_1}}.\end{aligned} \tag{2.21}$$

Then for any bounded open set U in $\mathbb{C}\times\mathbb{C}$, there exists a neighborhood of the origin V of $\mathbb{C}$ such that the following statements hold.

i) *$\wp_0(z;2\omega_1,2\omega_2) - \lambda^2\sinh^{-2}\lambda z$ and $\wp_0(z+\omega_1;2\omega_1,2\omega_2) + \lambda^2\cosh^{-2}\lambda z$ are holomorphic functions of $(z,\lambda,q)\in U\times V$ and vanish when $q=0$.*

ii) *$\wp_0(z+\frac{k}{m}\omega_2;2\omega_1,2\omega_2)$ is holomorphic for $(z,\lambda,t)\in U\times V$ and has zeros of order $\min\{k,2m-k\}$ along the hyperplane defined by $t=0$ and satisfies*

$$\begin{cases} t^{-k}\wp_0(z+\frac{k}{m}\omega_2;2\omega_1,2\omega_2)|_{t=0} = 4\lambda^2 e^{-2\lambda z} & (0<k<m),\\ t^{-k}\wp_0(z+\frac{k}{m}\omega_2;2\omega_1,2\omega_2)|_{t=0} = 8\lambda^2\cosh 2\lambda z & (k=m),\\ t^{k-2m}\wp_0(z+\frac{k}{m}\omega_2;2\omega_1,2\omega_2)|_{t=0} = 4\lambda^2 e^{2\lambda z} & (m<k<2m).\end{cases} \tag{2.22}$$

For our later convenience we list up some limiting formula discussed above. Fix ω_1 with $\sqrt{-1}\omega_1 > 0$ and let $\omega_2\in\mathbb{R}$ with $\omega_2>0$. Then $\lambda = \frac{\pi}{2\sqrt{-1}\omega_1} > 0$ and

$$\sinh^2\lambda(z+\omega_1) = -\cosh^2\lambda z,\ \ \cosh 2\lambda(z+\omega_1) = -\cosh 2\lambda z, \tag{2.23}$$

$$\lim_{\lambda\to 0}\lambda^2\sinh^{-2}\lambda z = \frac{1}{z^2}, \tag{2.24}$$

$$\lim_{R\to\pm\infty} e^{2\lambda|R|}\sinh^{-2}\lambda(z+R) = 4e^{\mp 2\lambda z}, \tag{2.25}$$

$$\lim_{\omega_2\to+\infty}\wp_0(z;2\omega_1,2\omega_2) = \lambda^2\sinh^{-2}\lambda z, \tag{2.26}$$

$$\lim_{\omega_2\to+\infty}\wp_0(z+\omega_1;2\omega_1,2\omega_2) = -\lambda^2\cosh^{-2}\lambda z, \tag{2.27}$$

$$\lim_{\omega_2\to\infty} e^{2r\lambda\omega_2}\wp_0(z+r\omega_2;2\omega_1,2\omega_2) = 4\lambda^2 e^{-2\lambda z} \quad \text{if } 0<r<1, \tag{2.28}$$

$$\lim_{\omega_2\to\infty} e^{2\lambda\omega_2}\wp_0(z+\omega_2;2\omega_1,2\omega_2) = 8\lambda^2\cosh 2\lambda z, \tag{2.29}$$

$$\lim_{\omega_2\to\infty} e^{2(2-r)\lambda\omega_2}\wp_0(z+r\omega_2;2\omega_1,2\omega_2) = 4\lambda^2 e^{2\lambda z} \quad \text{if } 1<r<2. \tag{2.30}$$

3. Type A_{n-1} ($n\ge 3$)

The completely integrable Schrödinger operator of type A_{n-1} is of the form

$$P = -\frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2} + \sum_{1\le i<j\le n} u_{ij}^-(x_i-x_j). \tag{3.1}$$

Denoting

$$u_{e_i-e_j}(x) = u_{e_j-e_i}(x) = u_{ij}^-(x_i-x_j), \tag{3.2}$$

we expect that the operators

$$
\begin{aligned}
P_k &= \sum_{0\le\nu\le\frac{k}{2}} \frac{1}{2^\nu\nu!(k-2\nu)!(n-k)!}\sum_{w\in\mathfrak{S}_n} \\
&\qquad u_{w(e_1-e_2)}u_{w(e_3-e_4)}\cdots u_{w(e_{2\nu-1}-e_{2\nu})}\partial_{w(e_{2\nu+1})}\cdots\partial_{w(e_k)} \\
&= \sum_{0\le\nu\le\frac{k}{2}}\sum u^-_{i_1i_2}\cdots u^-_{i_{2\nu-1}i_{2\nu}}\partial_{i_{2\nu+1}}\cdots\partial_{i_k}.
\end{aligned} \tag{3.3}
$$

satisfy

$$
[P_i,P_j]=0 \quad \text{for} \quad 1\le i<j\le n. \tag{3.4}
$$

Here we note that

$$
\begin{aligned}
P &= P_2 - \tfrac{1}{2}P_1^2, \\
P_1 &= \partial_1+\cdots+\partial_n, \\
P_2 &= \sum_{1\le i<j\le n}\partial_i\partial_j + \sum_{1\le i<j\le n} u^-_{ij}(x_i-x_j), \\
P_3 &= \sum_{1\le i<j<k\le n}\partial_i\partial_j\partial_k + \sum_{k=1}^{n}\sum_{\substack{1\le i<j\le n\\ i\ne k,\ j\ne k}} u^-_{ij}(x_i-x_j)\partial_k. \\
P_4 &= \sum_{1\le i<j<k<\ell\le n}\partial_i\partial_j\partial_k\partial_\ell + \sum_{1\le k<\ell\le n}\sum_{\substack{1\le i<j\le n\\ i\ne k,\ell,\ j\ne k,\ell}} u^-_{ij}\partial_k\partial_\ell \\
&\quad + \sum_{1\le i<j<k<\ell\le n}\Bigl(u^-_{ij}u^-_{k\ell}+u^-_{ik}u^-_{j\ell}+u^-_{i\ell}u^-_{jk}\Bigr) \\
&= \sum\partial_i\partial_j\partial_k\partial_\ell + \sum u^-_{ij}\partial_k\partial_\ell + \sum u^-_{ij}u^-_{k\ell}, \\
P_5 &= \sum\partial_i\partial_j\partial_k\partial_\ell\partial_m + \sum u^-_{ij}\partial_k\partial_\ell\partial_m + \sum u^-_{ij}u^-_{k\ell}\partial_m, \\
P_6 &= \sum\partial_i\partial_j\partial_k\partial_\ell\partial_m\partial_\nu + \sum u^-_{ij}\partial_k\partial_\ell\partial_m\partial_\nu + \sum u^-_{ij}u^-_{k\ell}\partial_m\partial_\nu + \sum u^-_{ij}u^-_{k\ell}u^-_{m\nu}, \\
&\cdots
\end{aligned}
$$

Since $W(A_{n-1})$ is naturally isomorphic to the permutation group $\mathfrak{S}_n$ of the set $\{1,\ldots,n\}$, we will identify them. The integrable potentials of type A_{n-1} which are invariant under the action of $\mathfrak{S}_n$ are determined by [OS] and [OOS] together with (3.4). They satisfy

$$
u_{e_i-e_j}(x) = u(\langle e_i-e_j,x\rangle) \tag{3.5}
$$

with an even function u and they are

(Ellip-A_{n-1}) *Elliptic potential of type* A_{n-1}:

$$
\begin{aligned}
u(t) &= C\wp_0(t;2\omega_1,2\omega_2), \\
R_E(A_{n-1};x_1,\ldots,x_n;C,2\omega_1,2\omega_2) &= C\sum_{1\le i<j\le n}\wp_0(x_i-x_j;2\omega_1,2\omega_2),
\end{aligned}
$$

(Trig-A_{n-1}) *Trigonometric potential of type* A_{n-1}:

$$
\begin{aligned}
u(t) &= C\sinh^{-2}\lambda t, \\
R_T(A_{n-1};x_1,\ldots,x_n;C,\lambda) &= C\sum_{1\le i<j\le n}\sinh^{-2}\lambda(x_i-x_j),
\end{aligned}
$$

(Rat-A_{n-1}) *Rational potential of type* A_{n-1}:

$$u(t) = \frac{C}{t^2},$$
$$R_R(A_{n-1}; x_1, \ldots, x_n; C) = \sum_{1\le i<j\le n} \frac{C}{(x_i - x_j)^2}.$$

Since

$$\lim_{\omega_2\to\infty} R_E(A_{n-1}; x; \frac{C}{\lambda^2}, 2\omega_1, 2\omega_2) = R_T(A_{n-1}; x; C, \lambda),$$
$$u(t) = \lim_{\omega_2\to\infty} \frac{C}{\lambda^2}\wp_0(t; 2\omega_1, 2\omega_2) = C\sinh^{-2}\lambda t, \tag{3.6}$$

the integrability (3.4) for (Trig-A_{n-1}) follows from that for (Ellip-A_{n-1}) by the analytic continuation of P_j with respect to q (cf. Lemma 2.2 i)).

The integrability for (Rat-A_{n-1}) is similarly clear from Lemma 2.1 with

$$\lim_{\lambda\to 0} R_T(A_{n-1}; x; \lambda^2 C, \lambda) = R_R(A_{n-1}; x; C),$$
$$u(t) = \lim_{\lambda\to 0} \lambda^2 C\sinh^{-2}\lambda t. \tag{3.7}$$

Other integrable potentials of type A_{n-1} are

(Toda-$A^{(1)}_{n-1}$) *Toda potential of type* $A^{(1)}_{n-1}$:

$$R_L(A^{(1)}_{n-1}; x; C, \lambda) = \sum_{i=1}^{n-1} Ce^{\lambda(x_i - x_{i+1})} + Ce^{\lambda(x_n - x_1)},$$

(Toda-A_{n-1}) *Toda potential of type* A_{n-1}:

$$R_L(A_{n-1}; x; C, \lambda) = \sum_{i=1}^{n-1} Ce^{\lambda(x_i - x_{i+1})}.$$

The integrability (3.4) for these potentials follows from Lemma 2.1, 2.2 and

$$\begin{aligned}
\lim_{\omega_2\to\infty} R_E(A_{n-1}; &x_1 - \frac{2\omega_2}{n}, \ldots, x_k - \frac{2k\omega_2}{n}, \ldots, x_n - 2\omega_2; \frac{e^{\frac{4}{n}\lambda\omega_2}}{4\lambda^2}C, 2\omega_1, 2\omega_2) \\
&= R_L(A^{(1)}_{n-1}; x; C, -2\lambda), \\
u_{e_i - e_j}(x) &= \lim_{\omega_2\to\infty} \frac{e^{\frac{4}{n}\lambda\omega_2}}{4\lambda^2} C\wp_0\big(x_i - x_j + \frac{2(j-i)\omega_2}{n}; 2\omega_1, 2\omega_2\big) \\
&= \begin{cases} Ce^{-2\lambda(x_i - x_{i+1})} & \text{if } 1 < j = i+1 \le n, \\ Ce^{-2\lambda(x_n - x_1)} & \text{if } i = 1 \text{ and } j = n, \\ 0 & \text{if } 1 \le i < j \le n \text{ and } j - i \ne 1,\ n-1 \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\lim_{R\to\infty} R_T(A_{n-1}; &x_1 - R, \ldots, x_n - nR; \frac{e^{2\lambda R}}{4}C, \lambda) = R_L(A_{n-1}; x; C, -2\lambda), \\
u_{e_i - e_j}(x) &= \lim_{R\to\infty} \frac{e^{2\lambda R}}{4} C\sinh^{-2}\lambda\big(x_i - x_j + (j-i)R\big) \\
&= \begin{cases} Ce^{-2\lambda(x_i - x_j)} & \text{if } 1 < j = i+1 \le n, \\ 0 & \text{if } 1 \le i < j \le n \text{ and } j \ne i+1, \end{cases}
\end{aligned}$$

respectively, if $\operatorname{Re}\lambda > 0$. Thus we have the following theorem

Theorem 3.1 (A_{n-1})**.** *The Schrödinger operators with the potential functions* (Ellip-A_{n-1}), (Trig-A_{n-1}), (Rat-A_{n-1}), (Toda-$A_{n-1}^{(1)}$) *and* (Toda-A_{n-1}) *are completely integrable and their integrals are given by* (3.3) *with* $u_{e_i-e_j}(x)$ *in the above.*

Remark 3.2. i) In [OS, Theorem 5.2] the complete integrability for (Ellip-A_{n-1}) is proved as follows. The equations $[P_1,P_k]=[P_2,P_k]=0$ for $k=1,\dots,n$ are obtained by direct calculations. Then the relation $[P_2,[P_i,P_j]]=0$ and periodicity and symmetry of P_k imply $[P_i,P_j]=0$ (cf. [OS, Lemma 3.5]).

ii) If $\operatorname{Re}\lambda>0$, we also have

$$\lim_{R\to\infty} R_L(A_{n-1}^{(1)};x_k+kR;Ce^{-2\lambda R};-2\lambda)=R_L(A_{n-1};x;C,-2\lambda). \tag{3.8}$$

iii) Note that

$$R_L(A_{n-1};x_k+R_k;C,\lambda)=\sum_{i=1}^{n-1}Ce^{\lambda(R_i-R_{i+1})}\cdot e^{\lambda(x_i-x_{i+1})}.$$

Hence $e^{\lambda(x_1-x_2)}-e^{\lambda(x_2-x_3)}$ gives the potential function of a completely integrable system of type A_{n-1} with $n=3$ but the potential function

$$\lim_{\lambda\to0}\lambda^{-1}\bigl(e^{\lambda(x_1-x_2)}-e^{\lambda(x_2-x_3)}\bigr)=x_1-2x_2+x_3 \tag{3.9}$$

is not so because it doesn't satisfy (9.2) in Remark 9.3.

The limit of the parameters of the integrable potential function should be taken care of the integrability.

iv) Let $P_n(t)$ denote the differential operator P_n in (3.3) defined by replacing u_{ij}^- by $\tilde u_{ij}^-=u_{ij}^-+t$ with a constant $t\in\mathbb{C}$. Then

$$P_n(t)=\sum_{0\le k\le\frac n2}\frac{(2k)!}{k!2^k}P_{n-2k}t^k\qquad\text{with}\quad P_0=1, \tag{3.10}$$

$$[P_n(s),P_n(t)]=0\qquad\text{for}\quad s,\,t\in\mathbb{C}. \tag{3.11}$$

In fact, the term $u_{12}^-u_{34}^-\cdots u_{2j-1,2j}^-\partial_{2j+1}\partial_{2j+2}\cdots\partial_{n-2k}$ only appears in the coefficient of t^k in the right hand side of (3.10) and it is originated to the term

$$\tilde u_{i_{n-2k+1}i_{n-2k+2}}^-\cdots\tilde u_{i_{n-1}i_n}^-\tilde u_{12}^-\cdots\tilde u_{2j-1,2j}^-\partial_{2j+1}\cdots\partial_{n-2k}$$

of $P_n(t)$, where the number of the possibilities of these $\tilde u_{i_{n-2k+1}i_{n-2k+2}}\cdots\tilde u_{i_{n-1}i_n}^-$ equals $\frac{(2k)!}{2^kk!}$ because $\{i_{n-2k+1},i_{n-2k+2},\dots,i_n\}=\{n-2k+1,\dots,n\}$.

v) Since

$$P_{k-1}=(n-k+1)[P_k,x_1+\cdots+x_n]\qquad\text{for}\quad k=2,\dots.n, \tag{3.12}$$

$[P_k,P_2]=0$ implies $[P_{k-1},P_2]$ by the Jacobi identity. Here we note that

$$[u_{12}^-u_{34}^-\cdots u_{2j-1,2j}^-\partial_{2j+1}\cdots\partial_{k-1}\partial_\nu,x_\nu]=u_{12}^-u_{34}^-\cdots u_{2j-1,2j}^-\partial_{2j+1}\cdots\partial_{k-1}$$

for $\nu=k,k+1,\dots,n$.

In the following diagram we show the relations among integrable potentials of type A_{n-1} by taking limits.

Hierarchy of Integrable Potentials of Type A_{n-1} ($n\ge3$)

$$\begin{array}{ccccc} & & \text{Toda-}A_{n-1}^{(1)} & \to & \text{Toda-}A_{n-1}\\ & \nearrow & & \nearrow & \\ \text{Ellip-}A_{n-1} & \to & \text{Trig-}A_{n-1} & \to & \text{Rat-}A_{n-1}\end{array}$$

4. Type B_2

In this section we study the following commuting differential operators.

$$\begin{cases} P = -\frac{1}{2}\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) + R(x,y),\\ P_2 = \frac{\partial^4}{\partial x^2 \partial y^2} + S \quad \text{with } \operatorname{ord} S < 4,\\ [P, P_2] = 0. \end{cases} \tag{4.1}$$

First we review the arguments given in [OO] and [Oc]. Since P is self-adjoint, we may assume P_2 is also self-adjoint by replacing P_2 by its self-adjoint part if necessary. Here for $A = \sum a_{ij}(x,y)\frac{\partial^{i+j}}{\partial x^i \partial y^j}$ we define ${}^tA = \sum (-1)^{i+j}\frac{\partial^{i+j}}{\partial x^i \partial y^j} a_{ij}(x,y)$ and A is called self-adjoint if ${}^tA = A$. Then we may assume

$$\begin{aligned} R(x,y) &= u^+(x+y) + u^-(x-y) + v(x) + w(y),\\ P_2 &= \Big(\frac{\partial^2}{\partial x \partial y} + u^-(x-y) - u^+(x+y)\Big)^2 - 2w(y)\frac{\partial^2}{\partial x^2} - 2v(x)\frac{\partial^2}{\partial y^2}\\ &\quad + 4v(x)w(y) + T(x,y), \end{aligned} \tag{4.2}$$

and the function $T(x,y)$ satisfies

$$\begin{aligned} \frac{\partial T(x,y)}{\partial x} &= 2\big(u^+(x+y) - u^-(x-y)\big)\frac{\partial w(y)}{\partial y} + 4w(y)\frac{\partial}{\partial y}\big(u^+(x+y) - u^-(x-y)\big),\\ \frac{\partial T(x,y)}{\partial y} &= 2\big(u^+(x+y) - u^-(x-y)\big)\frac{\partial v(x)}{\partial x} + 4v(x)\frac{\partial}{\partial x}\big(u^+(x+y) - u^-(x-y)\big). \end{aligned} \tag{4.3}$$

Conversely, if a function $T(x,y)$ satisfies (4.3) for suitable functions $u^\pm(t)$, $v(t)$ and $w(t)$, then (4.1) is valid for $R(x,y)$ and P_2 defined by (4.2).

Remark 4.1. If $w(x) = 0$, then $T(x,y)$ does not depend on x.

Since $T(x,y)$ is determined by $(u^-, u^+; v, w)$ up to the difference of constants, we will write $T(u^-, u^+; v, w)$ for the corresponding $T(x,y)$ which is an element of the space of meromorphic functions of (x,y) modulo constant functions and define $Q(u^-, u^+; v, w)$ by

$$T(u^-, u^+; v, w) = 2\big(u^-(x-y) + u^+(x+y)\big)\big(v(x) + w(y)\big) - 4Q(u^-, u^+; v, w). \tag{4.4}$$

The following lemma is a direct consequence of (4.3) and this definition of Q.

Lemma 4.2. *For C, C_i, $C'_j \in \mathbb{C}$ we have*

$$\begin{aligned} T\big(u^-(t) + C, u^+(t) + C; v(t), w(t))\big) &= T\big(u^-(t), u^+(t); v(t), w(t)\big),\\ Q\big(u^-(t), u^+(t); v(t) + C, w(t) + C\big) &= Q\big(u^-(t), u^+(t); v(t), w(t)\big),\\ Q\big(u^-(Ct), u^+(Ct); v(Ct), w(Ct)\big) &= Q\big(u^-(t), u^-(t); v(t), w(t)\big)|_{x\to Cx,\, y\to Cy},\\ Q\Big(\sum_{i=1}^2 A_i u_i^-, \sum_{i=1}^2 A_i u_i^+; \sum_{j=1}^2 C_j v_j, \sum_{j=1}^2 C_j w_j\Big) &= \sum_{i=1}^2\sum_{j=1}^2 A_i C_j Q(u_i^-, u_i^+; v_j, w_j). \end{aligned}$$

For simplicity we will use the notation

$$\begin{aligned} Q(u^-, u^+; v) = Q(u^-, u^+; v, v), \quad Q(u; v, w) = Q(u, u; v, w),\\ Q(u; v) = Q(u, u; v, v). \end{aligned} \tag{4.5}$$

The same convention will be also used for $T(u^-, u^+; v, w)$. The integrable potentials of type B_2 in this note are classified into three kinds. The potentials in the first kinds are the unified integrable potentials which are in the same form as those of type B_n with $n \geq 3$, which we call *normal integrable potentials* of type B_2.

The integrable potentials of type B_2 admit a special transformation called *dual* which does not exists in B_n with $n \ge 3$. Hence there are normal potentials and their dual in the invariant integrable potentials of type B_2. Because of this duality, there exist another kind of invariant integrable potentials of type B_2, which we call *special integrable potentials* of type B_2.

In this section we present $\bigl(R(x,y),T(x,y)\bigr)$ as suitable limits of elliptic functions as in the previous section since it helps to study the potentials of type B_n in §5. But we can also easily check (4.3) by calculations (cf. Remark 4.6).

4.1. **Normal Case.** In this subsection we study the integrable systems (4.1) with (4.2) which have natural extension to type B_n for $n \ge 3$ and have the form

$$\begin{aligned} &u^-(t) = Au_0^-(t), && u^+(t) = Au_0^+(t),\\ &v(t) = \sum_{j=0}^{3} C_j v_j(t), && w(t) = \sum_{j=0}^{3} C_j w_j(t) \end{aligned} \tag{4.6}$$

with any A, C_0, C_1, C_2, $C_3 \in \mathbb{C}$.

Theorem 4.3 (B_2: Normal Case)**.** *The operators P and P_2 defined by the following pairs of $R(x,y)$ and $T(x,y)$ satisfy (4.1) and (4.2).*

(Ellip-B_2): $\bigl(\langle\wp(t)\rangle; \langle\wp(t),\wp(t+\omega_1),\wp(t+\omega_2),\wp(t+\omega_3)\rangle\bigr)$

$$\begin{aligned} R(x,y) &= A\bigl(\wp(x-y)+\wp(x+y)\bigr) + \sum_{j=0}^{3} C_j\bigl(\wp(x+\omega_j)+\wp(y+\omega_j)\bigr),\\ T(x,y) &= 2\bigl(u^-(x-y)+u^+(x+y)\bigr)\bigl(v(x)+w(y)\bigr)\\ &\quad - 4A\sum_{j=0}^{3} C_j\wp(x+\omega_j)\cdot\wp(y+\omega_j). \end{aligned}$$

(Trig-B_2): $\bigl(\langle\sinh^{-2}\lambda t\rangle; \langle\sinh^{-2}\lambda t, \sinh^{-2}2\lambda t, \sinh^2\lambda t, \sinh^2 2\lambda t\rangle\bigr)$

$$\begin{aligned} R(x,y) &= A\bigl(\sinh^{-2}\lambda(x-y)+\sinh^{-2}\lambda(x+y)\bigr)\\ &\quad + C_0\bigl(\sinh^{-2}\lambda x+\sinh^{-2}\lambda y\bigr) + C_1\bigl(\cosh^{-2}\lambda x+\cosh^{-2}\lambda y\bigr)\\ &\quad + C_2\bigl(\sinh^2\lambda x+\sinh^2\lambda y\bigr) + \tfrac14 C_3\bigl(\sinh^2 2\lambda x+\sinh^2 2\lambda y\bigr),\\ T(x,y) &= 2\bigl(u^-(x-y)+u^+(x+y)\bigr)\bigl(v(x)+w(y)\bigr)\\ &\quad - 4A\bigl(C_0\sinh^{-2}\lambda x\cdot\sinh^{-2}\lambda y - C_1\cosh^{-2}\lambda x\cdot\cosh^{-2}\lambda y\\ &\qquad + C_3(\sinh^2\lambda x+\sinh^2\lambda y+2\sinh^2\lambda x\cdot\sinh^2\lambda y)\bigr). \end{aligned}$$

(Rat-B_2): $\bigl(\langle t^{-2}\rangle; \langle t^{-2}, t^2, t^4, t^6\rangle\bigr)$

$$\begin{aligned} R(x,y) &= \frac{A}{(x-y)^2} + \frac{A}{(x+y)^2}\\ &\quad + C_0\bigl(x^{-2}+y^{-2}\bigr) + C_1\bigl(x^2+y^2\bigr) + C_2\bigl(x^4+y^4\bigr) + C_3\bigl(x^6+y^6\bigr),\\ T(x,y) &= 2\bigl(u^-(x-y)+u^+(x+y)\bigr)\bigl(v(x)+w(y)\bigr)\\ &\quad - 4A\Bigl(\frac{C_0}{x^2y^2} + C_2(x^2+y^2) + C_3(x^4+y^4+3x^2y^2)\Bigr)\\ &= 8A\frac{2C_0+2C_1x^2y^2+C_2x^2y^2(x^2+y^2)+2C_3x^4y^4}{(x^2-y^2)^2}. \end{aligned}$$

(Toda-$D_2^{(1)}$-bry): $\big(\langle\cosh 2\lambda t\rangle; \langle\sinh^{-2}\lambda t, \sinh^{-2} 2\lambda t\rangle, \langle\sinh^{-2}\lambda t, \sinh^{-2} 2\lambda t\rangle\big)$

$$\begin{aligned}R(x,y) &= A\cosh 2\lambda(x-y) + A\cosh 2\lambda(x+y)\\ &\quad + C_0\sinh^{-2}\lambda x + C_1\sinh^{-2} 2\lambda x + C_2\sinh^{-2}\lambda y + C_3\sinh^{-2} 2\lambda y,\\ T(x,y) &= 8A(C_0\cosh 2\lambda y + C_2\cosh 2\lambda x).\end{aligned}$$

(Toda-$B_2^{(1)}$-bry): $\big(\langle e^{-2\lambda t}\rangle; \langle e^{2\lambda t}, e^{4\lambda t}\rangle, \langle\sinh^{-2}\lambda t, \sinh^{-2} 2\lambda t\rangle\big)$

$$\begin{aligned}R(x,y) &= Ae^{-2\lambda(x-y)} + Ae^{-2\lambda(x+y)}\\ &\quad + C_0e^{2\lambda x} + C_1e^{4\lambda x} + C_2\sinh^{-2}\lambda y + C_3\sinh^{-2} 2\lambda y,\\ T(x,y) &= 4A\big(C_0\cosh 2\lambda y + 2C_2e^{-2\lambda x}\big).\end{aligned}$$

(Trig-$A_1^{(1)}$-bry): $\big(\langle\sinh^{-2}\lambda t\rangle, 0; \langle e^{-2\lambda t}, e^{-4\lambda t}, e^{2\lambda t}, e^{4\lambda t}\rangle\big)$

$$\begin{aligned}R(x,y) &= A\sinh^{-2}\lambda(x-y) + C_0\big(e^{-2\lambda x} + e^{-2\lambda y}\big) + C_1\big(e^{-4\lambda x} + e^{-4\lambda y}\big)\\ &\quad + C_2\big(e^{2\lambda x} + e^{2\lambda y}\big) + C_3\big(e^{4\lambda x} + e^{4\lambda y}\big),\\ T(x,y) &= 2A\sinh^{-2}\lambda(x-y)\Big(C_0\big(e^{-2\lambda x} + e^{-2\lambda y}\big) + 2C_1e^{-2\lambda(x+y)}\\ &\qquad + C_2\big(e^{2\lambda x} + e^{2\lambda y}\big) + 2C_3e^{2\lambda(x+y)}\Big).\end{aligned}$$

(Toda-$C_2^{(1)}$): $\big(\langle e^{-2\lambda t}\rangle, 0; \langle e^{2\lambda t}, e^{4\lambda t}\rangle, \langle e^{-2\lambda t}, e^{-4\lambda t}\rangle\big)$

$$\begin{aligned}R(x,y) &= Ae^{-2\lambda(x-y)} + C_0e^{2\lambda x} + C_1^{4\lambda x} + C_2e^{-2\lambda y} + C_3^{-4\lambda y},\\ T(x,y) &= 2A\big(C_0e^{2\lambda y} + C_2e^{-2\lambda x}\big).\end{aligned}$$

(Rat-A_1-bry): $\big(\langle t^{-2}\rangle, 0; \langle t, t^2, t^3, t^4\rangle\big)$

$$\begin{aligned}R(x,y) &= \frac{A}{(x-y)^2} + C_0(x+y) + C_1(x^2+y^2) + C_2(x^3+y^3) + C_3(x^4+y^4),\\ T(x,y) &= \frac{2A}{(x-y)^2}\big(C_0(x+y) + C_1(x^2+y^2) + C_2xy(x+y) + 2C_3x^2y^2\big).\end{aligned}$$

Remark 4.4. For example, $\big(\langle t^{-2}\rangle, 0; \langle t, t^2, t^3, t^4\rangle\big)$ in the above (Rat-A_1-bry) means

$$u^-(t) = At^{-2},\ u^+(t) = 0,\ v(t) = w(t) = C_0t + C_1t^2 + C_2t^3 + C_3t^4$$

with using the similar convention as in (4.5).

Remark 4.5. All the invariant integrable potentials of type B_2 together with P_2 are determined by [OOS] and [OO]. They are (Ellip-B_2), (Trig-B_2) and (Rat-B_2) which have the following unified expression of the invariant potentials given by [OS, Lemma 7.3], where the periods $2\omega_1$ and $2\omega_2$ may be infinite.

$$\begin{aligned}R(x,y) &= A\wp(x-y) + A\wp(x+y)\\ &\quad + \frac{C_4\wp(x)^4 + C_3\wp(x)^3 + C_2\wp(x)^2 + C_1\wp(x) + C_0}{\wp'(x)^2}\\ &\quad + \frac{C_4\wp(y)^4 + C_3\wp(y)^3 + C_2\wp(y)^2 + C_1\wp(y) + C_0}{\wp'(y)^2},\\ T(x,y) &= 4A\big(\wp(x) - \wp(y)\big)^{-2}\Big(C_4\wp(x)^2\wp(y)^2 + \frac{C_3}{2}\wp(x)^2\wp(y) +\\ &\quad + \frac{C_3}{2}\wp(x)\wp(y)^2 + C_2\wp(x)\wp(y) + \frac{C_1}{2}\wp(x) + \frac{C_1}{2}\wp(y) + C_0\Big),\end{aligned}$$

When $\omega_1=\omega_2=\infty$, $\wp(t)=t^{-2}$ and $\bigl(\wp(x)-\wp(y)\bigr)^{-2}=x^4y^4(x^2-y^2)^{-2}$ and

$$\begin{aligned}R(x,y)&=\frac{A}{(x-y)^2}+\frac{A}{(x+y)^2}\\&\quad+\tfrac14(C_4x^{-2}+C_3+C_2x^2+C_1x^4+C_0x^6)\\&\quad+\tfrac14(C_4y^{-2}+C_3+C_2y^2+C_1y^4+C_0y^6),\\T(x,y)&=2A(x^2-y^2)^{-2}\bigl(2C_4+C_3(x^2+y^2)+2C_2x^2y^2+C_1x^2y^2(x^2+y^2)\\&\quad+2C_0x^4y^4\bigr).\end{aligned}$$

We review these cases discussed in [OS] and [OO]. Owing to the identity

$$\begin{aligned}&v'(u^--u^+)+2v((u^-)'-(u^+)')+\partial_y\bigl((v+w)(u^-+u^+)-2vw\bigr)\\&=\begin{vmatrix}v&v'&1\\w&-w'&1\\u^-&-(u^-)'&1\end{vmatrix}+\begin{vmatrix}v&-v'&1\\w&-w'&1\\u^+&(u^+)'&1\end{vmatrix}\end{aligned}$$

and (2.18), the right hand side of the above equals zero and we have (4.3) when

$u^-=C\wp(x-y)+C',\ u^+=C\wp(x+y)+C',\ v=C\wp(x)+C'$ and $w=C\wp(y)+C'$

with $T(x,y)=2(u^-+u^+)(v+w)-4vw$. Hence $Q\bigl(\wp(t);\wp(t)\bigr)=\wp(x)\wp(y)$. Using the transformations $(x,y)\mapsto(x+\omega_j,y+\omega_j)$, we have
(Ellip-B_2)

$$Q\bigl(\wp(t);\wp(t+\omega_j)\bigr)=\wp(x+\omega_j)\cdot\wp(y+\omega_j)\quad\text{for}\quad j=0,1,2,3. \tag{4.7}$$

because the functions $\wp(x\pm y)$ do not change under this transformations (cf. (2.9)).

Here we note that $\wp$ may be replaced by $\wp_0$.

By the limit under $\omega_2\to\infty$, we have the following (Trig-B_2) from (Ellip-B_2) as was shown in the proof of [O1, Proposition 6.1].
(Trig-B_2)

$$Q\bigl(\sinh^{-2}\lambda t;\sinh^{-2}\lambda t\bigr)=\sinh^{-2}\lambda x\cdot\sinh^{-2}\lambda y, \tag{4.8}$$

$$Q\bigl(\sinh^{-2}\lambda t;\cosh^{-2}\lambda t\bigr)=-\cosh^{-2}\lambda x\cdot\cosh^{-2}\lambda y, \tag{4.9}$$

$$Q\bigl(\sinh^{-2}\lambda t;\sinh^{2}\lambda t\bigr)=0, \tag{4.10}$$

$$Q\bigl(\sinh^{-2}\lambda t;\tfrac14\sinh^{2}2\lambda t\bigr)=\sinh^2\lambda x+\sinh^2\lambda y+2\sinh^2\lambda x\cdot\sinh^2\lambda y. \tag{4.11}$$

The equations (4.8), (4.9) and (4.10) correspond to (2.26), (2.27) and (2.29), respectively. Moreover (2.17) should be noted and (4.11) corresponds to (2.29) with replacing (ω_1,λ) by $(\frac12\omega_1,2\lambda)$.

By the limit under $\lambda\to0$, we have the following (Rat-B_2) from (Trig-B_2) as was shown in the proof of [O1, Proposition 6.3]. Here we note (2.25) and

$$\begin{aligned}&\cosh^{-2}\lambda t\cdot\sinh^2\lambda t=1-\cosh^{-2}\lambda t,\\&\cosh^{-2}\lambda t\cdot\sinh^4\lambda t=-1+\cosh^{-2}\lambda t+\sinh^2\lambda t,\\&\cosh^{-2}\lambda t\cdot\sinh^6\lambda t=1-\cosh^{-2}\lambda t-2\sinh^2\lambda t+\tfrac14\sinh^2 2\lambda t,\\&\lim_{\lambda\to0}\lambda^{-2j}\cosh^{-2}\lambda t\cdot\sinh^{2j}\lambda t=t^{2j}\quad\text{for}\quad j=1,2,3,\\&\frac{1}{(x-y)^2}+\frac{1}{(x+y)^2}=\frac{2(x^2+y^2)}{(x^2-y^2)^2}.\end{aligned}$$

(Rat-B_2):

$$Q(t^{-2};t^{-2})=x^{-2}y^{-2},$$

$$
\begin{aligned}
&T(t^{-2};t^{-2}) = 4(x^{-2}+y^{-2})\big((x-y)^{-2}+(x+y)^{-2}\big) - 4x^{-2}y^{-2},\\
&\quad = \frac{4(x^2+y^2)^2 - 4(x^2-y^2)^2}{x^2y^2(x^2-y^2)^2} = \frac{16}{(x^2-y^2)^2},\\
&Q(t^{-2};t^2) = 0,\\
&T(t^{-2};t^2) = \frac{4(x^2+y^2)^2}{(x^2-y^2)^2} = \frac{16x^2y^2}{(x^2-y^2)^2} + 4,\\
&Q(t^{-2};t^4) = x^2+y^2,\\
&T(t^{-2},t^4) = \frac{4(x^2+y^2)(x^4+y^4)}{(x^2-y^2)^2} - 4(x^2+y^2) = \frac{8x^2y^2(x^2+y^2)}{(x^2-y^2)^2},\\
&Q(t^{-2};t^6) = x^4+y^4+3x^2y^2,\\
&T(t^{-2},t^6) = \frac{4(x^2+y^2)(x^6+y^6)}{(x^2-y^2)^2} - 4(x^4+3x^2y^2+y^4) = \frac{16x^4y^4}{(x^2-y^2)^2}.
\end{aligned}
$$

This expression of $T(x,y)$ for (Rat-B_2) is also given in Remark 4.5. Note that we ignore the difference of constant functions for Q and T.

Proof. (Toda-$D_2^{(1)}$-bry) $\leftarrow$ (Ellip-B_2): Replacing (x,y) by $(x+\omega_2,y)$, we have

$$
\begin{aligned}
&Q(\cosh 2\lambda t; 0, \sinh^{-2}\lambda t)\\
&\quad = \lim_{\omega_2\to\infty} Q\big(\frac{e^{2\lambda\omega_2}}{8\lambda^2}\wp_0(t+\omega_2); \frac{1}{\lambda^2}\wp_0(t+\omega_2), \frac{1}{\lambda^2}\wp_0(t)\big)\\
&\quad = \lim_{\omega_2\to\infty} \frac{e^{2\lambda\omega_2}}{8\lambda^4}\wp_0(x+\omega_2)\wp_0(y) = \cosh 2\lambda x\cdot\sinh^{-2}\lambda y,\\
&Q(\cosh 2\lambda t; 0, \cosh^{-2}\lambda t)\\
&\quad = \lim_{\omega_2\to\infty} Q\big(\frac{e^{2\lambda\omega_2}}{8\lambda^2}\wp_0(t+\omega_2); -\frac{1}{\lambda^2}\wp_0(t+\omega_1+\omega_2); -\frac{1}{\lambda_2}\wp_0(t+\omega_1)\big)\\
&\quad = -\lim_{\omega_2\to\infty} \frac{e^{2\lambda\omega_2}}{8\lambda^4}\wp_0(x+\omega_1+\omega_2)\wp_0(y+\omega_1) = -\cosh 2\lambda x\cdot\cosh^{-2}\lambda y,\\
&Q(\cosh 2\lambda t; \sinh^{-2}\lambda t, 0) = \sinh^{-2}\lambda x\cdot\cosh^2\lambda y,\\
&Q(\cosh 2\lambda t; \cosh^{-2}\lambda t, 0) = -\cosh^{-2}\lambda x\cdot\sinh^2\lambda y.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&T(\cosh 2\lambda t; 0, \sinh^{-2}\lambda t) = 2\big(\cosh 2\lambda(x+y) + \cosh 2\lambda(x-y)\big)\cdot\sinh^{-2}\lambda y\\
&\qquad - 4\cosh 2\lambda x\cdot\sinh^{-2}\lambda y = 8\cosh 2\lambda x,\\
&T(\cosh 2\lambda t; 0, \cosh^{-2}\lambda t) = 2\big(\cosh 2\lambda(x+y) + \cosh 2\lambda(x-y)\big)\cdot\cosh^{-2}\lambda y\\
&\qquad + 4\cosh 2\lambda x\cdot\cosh^{-2}\lambda y = 8\cosh 2\lambda x,\\
&T(\cosh 2\lambda t; 0, \sinh^{-2}2\lambda t) = T(\cosh 2\lambda t; \sinh^{-2}2\lambda t, 0) = 0,\\
&T(\cosh 2\lambda t; \sinh^{-2}\lambda t, 0) = 8\cosh 2\lambda y.
\end{aligned}
$$

(Toda-$B_2^{(1)}$-bry) $\leftarrow$ (Toda-$D_2^{(1)}$-bry): Replacing (x,y) by $(x-R,y)$, we have

$$\begin{aligned}
&T\bigl(e^{-2\lambda t}; 0, \sinh^{-2}\lambda t\bigr) = \lim_{R\to\infty} T\bigl(2e^{-2\lambda R}\cosh 2\lambda(t-R); 0, \sinh^{-2}\lambda t\bigr)\\
&\quad = \lim_{R\to\infty} 16e^{-2\lambda R}\cosh 2\lambda(x-R) = 8e^{-2\lambda x},\\
&T\bigl(e^{-2\lambda t}; 0, \sinh^{-2} 2\lambda t\bigr) = \lim_{R\to\infty} T\bigl(2e^{-2\lambda R}\cosh 2\lambda(t-R); 0, \sinh^{-2} 2\lambda t\bigr) = 0,\\
&T\bigl(e^{-2\lambda t}; e^{2\lambda t}, 0\bigr) = \lim_{R\to\infty} T\bigl(2e^{-2\lambda R}\cosh 2\lambda(t-R); \tfrac14 e^{2\lambda R}\sinh^{-2}\lambda(t-R), 0\bigr)\\
&\quad = 4\cosh 2\lambda y,\\
&T\bigl(e^{-2\lambda t}; e^{4\lambda t}, 0\bigr) = \lim_{R\to\infty} T\bigl(2e^{-2\lambda R}\cosh 2\lambda(t-R); \tfrac14 e^{4\lambda R}\sinh^{-2} 2\lambda(t-R), 0\bigr) = 0.
\end{aligned}$$

(Trig-$C_2^{(1)}$) $\leftarrow$ (Trig-$B_2^{(1)}$-bry): Replacing (x,y) by $(x+R,y+R)$,

$$\begin{aligned}
&T(e^{-2\lambda t}, 0; e^{2\lambda t}, 0) = \lim_{R\to\infty} T(e^{-2\lambda t}, e^{-2\lambda(t+2R)}; e^{-2\lambda R}e^{2\lambda(t+R)}, 0)\\
&\quad = \lim_{R\to\infty} e^{-2\lambda R}\cdot 4\cosh 2\lambda(y+R) = 2e^{2\lambda y},\\
&T(e^{-2\lambda t}, 0; e^{4\lambda t}, 0) = \lim_{R\to\infty} T(e^{-2\lambda t}, e^{-2\lambda(t+2R)}; e^{-4\lambda R}e^{4\lambda(t+R)}, 0) = 0,
\end{aligned}$$

By the transformation $(x,y,\lambda)\mapsto(y,x,-\lambda)$, we have

$$T(e^{-2\lambda t}, 0; 0, e^{-2\lambda t}) = 2e^{-2\lambda x},\quad T(e^{-2\lambda t}, 0; 0, e^{-4\lambda t}) = 0.$$

(Trig-$A_1^{(1)}$-bry) $\leftarrow$ (Trig-B_2): Replacing (x,y) by $(x+R,y+R)$,

$$\begin{aligned}
&Q\bigl(\sinh^{-2}\lambda t, 0; e^{2\lambda t}\bigr)\\
&\quad = \lim_{R\to\infty} Q\bigl(\sinh^{-2}\lambda t, \sinh^{-2}\lambda(t+2R); \tfrac14 e^{2\lambda R}\sinh^{-2}\lambda(t+R)\bigr)\\
&\quad = \lim_{R\to\infty} \tfrac14 e^{2\lambda R}\cdot\sinh^{-2}\lambda(x+R)\cdot\sinh^{-2}\lambda(y+R) = 0,\\
&T\bigl(\sinh^{-2}\lambda t, 0; e^{2\lambda t}\bigr) = 2\bigl(e^{2\lambda x}+e^{2\lambda y}\bigr)\sinh^{-2}\lambda(x-y),\\
&Q\bigl(\sinh^{-2}\lambda t, 0; e^{4\lambda t}\bigr)\\
&\quad = \lim_{R\to\infty} Q\bigl(\sinh^{-2}\lambda t, \sinh^{-2}\lambda(t+2R); 4e^{-4\lambda R}\sinh^2 2\lambda(t+R)\bigr)\\
&\quad = 16\lim_{R\to\infty} e^{-4\lambda R}\Bigl(\sinh^2\lambda(x+R)+\sinh^2\lambda(y+R)\\
&\qquad + 2\sinh^2\lambda(x+R)\cdot\sinh^2\lambda(y+R)\Bigr) = 2e^{2\lambda(x+y)},\\
&T(\sinh^{-2}\lambda t, 0; e^{4\lambda t}\bigr) = 2\sinh^{-2}\lambda(x-y)\bigl(e^{4\lambda x}+e^{4\lambda y}\bigr) - 8e^{2\lambda(x+y)}\\
&\quad = 2\sinh^{-2}\lambda(x-y)\bigl(e^{4\lambda x}+e^{4\lambda y} - e^{2\lambda(x+y)}(e^{\lambda(x-y)}-e^{-\lambda(x-y)})^2\bigr)\\
&\quad = 4e^{2\lambda(x+y)}\sinh^{-2}\lambda(x-y).
\end{aligned}$$

(Rat-A_1-bry) $\leftarrow$ (Trig-A_1-bry): Taking the limit $\lambda\to 0$,

$$\begin{aligned}
&Q(t^{-2}, 0; t) = \lim_{\lambda\to 0} Q\bigl(\lambda^2\sinh^{-2}\lambda t, 0; \tfrac{1}{2\lambda}(e^{2\lambda t}-1)\bigr) = 0,\\
&T(t^{-2}, 0; t) = \frac{2(x+y)}{(x-y)^2},\\
&Q(t^{-2}, 0; t^2) = \lim_{\lambda\to 0} Q\bigl(\lambda^2\sinh^{-2}\lambda t, 0; \tfrac{1}{4\lambda^2}(e^{2\lambda t}+e^{-2\lambda t}-2)\bigr) = 0,\\
&T(t^{-2}, 0; t^2) = 2\frac{x^2+y^2}{(x-y)^2},
\end{aligned}$$

$$
\begin{aligned}
Q(t^{-2},0;t^3) &= \lim_{\lambda\to 0} Q\big(\lambda^2\sinh^{-2}\lambda t, 0; \tfrac{1}{8\lambda^3}(e^{4\lambda t}-3e^{2\lambda t}-e^{-2\lambda t}+3)\big)\\
&= \lim_{\lambda\to 0} \tfrac{2}{8\lambda}\big(e^{2\lambda(x+y)}-1\big) = \tfrac{1}{2}(x+y),\\
T(t^{-2},0;t^3) &= 2\frac{x^3+y^3}{(x-y)^2} - 2(x+y) = 2\frac{xy(x+y)}{(x-y)^2},\\
Q(t^{-2},0;t^4) &= \lim_{\lambda\to 0} Q\big(\lambda^2\sinh^{-2}\lambda t, 0; \tfrac{1}{16\lambda^4}(e^{4\lambda t}+e^{-4\lambda t}-4e^{2\lambda t}-4e^{-2\lambda t}+6)\big)\\
&= \lim_{\lambda\to 0} \tfrac{2}{16\lambda^2}\big(e^{2\lambda(x+y)}+e^{-2\lambda(x+y)}-2\big) = \tfrac{1}{2}(x+y)^2,\\
T(t^{-2},0;t^4) &= 2\frac{x^4+y^4}{(x-y)^2} - 2(x+y)^2 = \frac{4x^2y^2}{(x-y)^2}.
\end{aligned}
$$

Thus we have completed the proof of Theorem 4.3. □

Remark 4.6. Theorem 4.3 can easily checked by direct calculations. For example, Remark 4.1 and the equations

$$
\begin{aligned}
&2\big(\varepsilon e^{-2\lambda(x+y)} - e^{-2\lambda(x-y)}\big)\big(e^{2\lambda x}\big)' + 4e^{2\lambda x}\frac{\partial}{\partial x}\big(\varepsilon e^{-2\lambda(x+y)} - e^{-2\lambda(x-y)}\big)\\
&\quad = 4\lambda\big(\varepsilon e^{-2\lambda y} - e^{2\lambda y}\big) - 8\lambda\big(\varepsilon e^{-2\lambda y} - e^{2\lambda y}\big) = \frac{\partial}{\partial y}\Big(2(\varepsilon e^{-2\lambda y} + e^{2\lambda y})\Big),\\
&2\big(\varepsilon e^{-2\lambda(x+y)} - e^{-2\lambda(x-y)}\big)\big(e^{4\lambda x}\big)' + 4e^{4\lambda x}\frac{\partial}{\partial x}\big(\varepsilon e^{-2\lambda(x+y)} - e^{-2\lambda(x-y)}\big) = 0
\end{aligned}
$$

with $\varepsilon = 1$ give $T(e^{-2\lambda t}; e^{2\lambda t}, 0)$ and $T(e^{-2\lambda t}; e^{4\lambda t}, 0)$ for (Trig-B_2-bry). Moreover $T(e^{-2\lambda t}, 0; e^{2\lambda t}, 0)$ and $T(e^{-2\lambda t}, 0; e^{4\lambda t}, 0)$ for (Toda-$C_2^{(1)}$) also follow from these equations with $\varepsilon = 0$.

4.2. **Special Case.** In this subsection we study the integrable systems (4.1) with (4.2) which are of the form

$$
\begin{aligned}
&R(x,y) = u^-(x-y) + u^+(x+y) + v(x) + w(y),\\
&u^-(t) = \sum_{j=0}^{1} A_j u_0^-(t), \qquad u^+(t) = \sum_{j=0}^{1} A_j u_0^+(t),\\
&v(t) = \sum_{j=0}^{1} C_j v_j(t), \qquad w(t) = \sum_{j=0}^{1} C_j w_j(t)
\end{aligned}
\tag{4.12}
$$

with A_0, A_1, C_0, $C_1 \in \mathbb{C}$.

Theorem 4.7 (B_2 : Special Case)**.** *The operators P and P_2 defined by the following pairs of $R(x,y)$ and $T(x,y)$ satisfy* (4.1) *and* (4.2)*.*

(Ellip-B_2-S): $\big(\langle\wp(t;2\omega_1,2\omega_2), \wp(t;\omega_1,2\omega_2)\rangle; \langle\wp(t;\omega_1,2\omega_2), \wp(t;\omega_1,\omega_2)\rangle\big)$

$$
\begin{aligned}
R(x,y) &= A_0\wp(x-y;2\omega_1,2\omega_2) + A_0\wp(x+y;2\omega_1,2\omega_2)\\
&\quad + A_1\wp(x-y;\omega_1,2\omega_2) + A_1\wp(x+y;\omega_1,2\omega_2)\\
&\quad + C_0\wp(x;\omega_1,2\omega_2) + C_0\wp(y;\omega_1,2\omega_2)\\
&\quad + C_1\wp(x;\omega_1,\omega_2) + C_1\wp(y;\omega_1,\omega_2),
\end{aligned}
$$

$$\begin{aligned}
T(x,y) &= 2\big(u^-(x-y)+u^+(x+y)\big)\big(v(x)+w(y)\big)\\
&\quad - 4A_0C_0\sum_{j=0}^{1}\wp(x+\omega_j;2\omega_1,2\omega_2)\cdot\wp(y+\omega_j;2\omega_1,2\omega_2)\\
&\quad - 4A_0C_1\sum_{j=0}^{3}\wp(x+\omega_j;2\omega_1,2\omega_2)\cdot\wp(y+\omega_j;2\omega_1,2\omega_2)\\
&\quad - 4A_1C_0\wp(x;\omega_1,2\omega_2)\cdot\wp(y;\omega_1,2\omega_2)\\
&\quad - 4A_1C_1\sum_{j=0}^{1}\wp(x+\omega_{2j};\omega_1,2\omega_2)\cdot\wp(y+\omega_{2j};\omega_1,2\omega_2),
\end{aligned}$$

(Trig-B_2-S): $\big(\langle\sinh^{-2}\lambda t,\sinh^{-2}2\lambda t\rangle;\langle\sinh^{-2}2\lambda t,\sinh^{2}2\lambda t\rangle\big)$

$$\begin{aligned}
R(x,y) &= A_0\sinh^{-2}\lambda(x+y)+A_0\sinh^{-2}\lambda(x-y)\\
&\quad + A_1\sinh^{-2}2\lambda(x+y)+A_1\sinh^{-2}2\lambda(x-y)\\
&\quad + C_0\sinh^{-2}2\lambda x+C_0\sinh^{-2}2\lambda y+C_1\sinh^{2}2\lambda x+C_1\sinh^{2}2\lambda y,\\
T(x,y) &= 2\big(u^-(x-y)+u^+(x+y)\big)\big(v(x)+w(y)\big)\\
&\quad - A_0C_0\big(\sinh^{-2}\lambda x\cdot\sinh^{-2}\lambda y+\cosh^{-2}\lambda x\cdot\cosh^{-2}\lambda y\big)\\
&\quad - 4A_0C_1\big(\sinh^{2}\lambda x+\sinh^{2}\lambda y+2\sinh^{2}\lambda x\cdot\sinh^{2}\lambda y\big)\\
&\quad - 4A_1C_0\sinh^{-2}2\lambda x\cdot\sinh^{-2}2\lambda y,
\end{aligned}$$

(Rat-B_2-S): $\big(\langle t^{-2},t^{2}\rangle;\langle t^{-2},t^{2}\rangle\big)$

$$\begin{aligned}
R(x,y) &= \frac{A_0}{(x-y)^2}+\frac{A_0}{(x+y)^2}+A_1(x-y)^2+A_1(x+y)^2\\
&\quad + \frac{C_0}{x^2}+\frac{C_0}{y^2}+C_1x^2+C_1y^2,\\
T(x,y) &= \frac{16A_0C_0+16A_0C_1x^2y^2}{(x^2-y^2)^2}+16A_1C_1x^2y^2,
\end{aligned}$$

(Toda-$D_2^{(1)}$-S-bry): $\big(\langle\cosh 2\lambda,\cosh 4\lambda t\rangle;\langle\sinh^{-2}2\lambda t\rangle,\langle\sinh^{-2}2\lambda t\rangle\big)$

$$\begin{aligned}
R(x,y) &= A_0\cosh 2\lambda(x-y)+A_0\cosh 2\lambda(x+y)\\
&\quad + A_1\cosh 4\lambda(x-y)+A_1\cosh 4\lambda(x+y)\\
&\quad + C_0\sinh^{-2}2\lambda x+C_1\sinh^{-2}2\lambda y,\\
T(x,y) &= 8A_1\big(C_0\cosh 4\lambda y+C_1\cosh 4\lambda x\big),
\end{aligned}$$

(Toda-$B_2^{(1)}$-S-bry): $\big(\langle e^{-2\lambda t},e^{-4\lambda t}\rangle;\langle e^{4\lambda t}\rangle,\langle\sinh^{-2}2\lambda t\rangle\big)$

$$\begin{aligned}
R(x,y) &= A_0e^{-2\lambda(x-y)}+A_0e^{-2\lambda(x+y)}+A_1e^{-4\lambda(x-y)}+A_1e^{-4\lambda(x+y)}\\
&\quad + C_0e^{4\lambda x}+C_1\sinh^{-2}2\lambda y,\\
T(x,y) &= 4A_1\big(C_0\cosh 4\lambda y+2C_1e^{-4\lambda x}\big).
\end{aligned}$$

(Toda-$C_2^{(1)}$-S): $\big(\langle e^{-2\lambda t},e^{-4\lambda t}\rangle,0;\langle e^{4\lambda t}\rangle,\langle e^{-4\lambda t}\rangle\big)$

$$\begin{aligned}
R(x,y) &= A_0e^{-2\lambda(x-y)}+A_1e^{-4\lambda(x-y)}+C_0e^{4\lambda x}+C_1e^{-4\lambda y},\\
T(x,y) &= 2A_1\big(C_0e^{4\lambda y}+C_1e^{-4\lambda x}\big).
\end{aligned}$$

(Trig-A_1-S-bry): $\big(\langle\sinh^{-2}\lambda t, \sinh^{-2}2\lambda t\rangle, 0; \langle e^{-4\lambda t}, e^{4\lambda t}\rangle\big)$

$$\begin{aligned}R(x,y) &= A_0\sinh^{-2}\lambda(x-y) + A_1\sinh^{-2}2\lambda(x-y)\\&\quad + C_0e^{-4\lambda x} + C_0e^{-4\lambda y} + C_1e^{4\lambda x} + C_1e^{4\lambda y},\\T(x,y) &= 2A_1\sinh^{-2}2\lambda(x-y)\big(C_0e^{-4\lambda x} + C_0e^{-4\lambda y} + C_1e^{4\lambda x} + C_1e^{4\lambda y}\big)\\&\quad + 4A_0\sinh^{-2}\lambda(x-y)\big(C_0e^{-2\lambda(x+y)} + C_1e^{2\lambda(x+y)}\big).\end{aligned}$$

Proof. (Ellip-B_2-S): We have the following from (4.7), Lemma 4.2 and (2.17).

$$\begin{aligned}&Q(\wp(t;\omega_1,2\omega_2);\wp(t;\omega_1;2\omega_2)) = \wp(t;\omega_1,2\omega_2)\cdot\wp(t;\omega_1;2\omega_2),\\&Q(\wp(t;2\omega_1,2\omega_2);\wp(t;\omega_1,2\omega_2))\\&\quad = Q(\wp(t;2\omega_1,2\omega_2);\wp(t;2\omega_1,2\omega_2)+\wp(t+\omega_1;2\omega_1,2\omega_2))\\&\quad = \wp(x;2\omega_1,2\omega_2)\cdot\wp(y;2\omega_1,2\omega_2)+\wp(x+\omega_1;2\omega_1,2\omega_2)\cdot\wp(y+\omega_1;2\omega_1,2\omega_2),\\&Q(\wp(t;\omega_1,2\omega_2);\wp(t;\omega_1,\omega_2))\\&\quad = \wp(x;\omega_1,2\omega_2)\cdot\wp(y;\omega_1,2\omega_2)+\wp(x+\omega_2;\omega_1,2\omega_2)\cdot\wp(y+\omega_2;\omega_1,2\omega_2),\\&Q(\wp(t;2\omega_1,2\omega_2);\wp(t;\omega_1,\omega_2)) = \sum_{j=0}^{3}\wp(x+\omega_j;2\omega_1,2\omega_2)\cdot\wp(y+\omega_j;2\omega_1,2\omega_2).\end{aligned}$$

(Rat-B_2) is given in [OS, (7.13)] but it is easy to check (4.3) or prove the result as a limit of (Trig-B_2-S). Moreover (Trig-B_2-S), (Toda-$D_2^{(1)}$-S-bry), (Toda-$C_2^{(1)}$-S) and (Trig-A_1-S-bry) are obtained from the corresponding normal cases together with Lemma 4.2. For example, Q for (Trig-B_2-S) is given by (4.8), (4.10) and

$$\begin{aligned}Q\big(\sinh^{-2}\lambda t;\sinh^{-2}2\lambda t\big) &= Q\big(\sinh^{-2}\lambda t;\tfrac14\sinh^{-2}\lambda t-\tfrac14\cosh^{-2}\lambda t\big)\\&= \tfrac14(\sinh^{-2}\lambda t\sinh^{-2}\lambda y+\cosh^{-2}\lambda x\cosh^{-2}\lambda y),\\Q\big(\sinh^{-2}2\lambda t;\sinh^{-2}2\lambda t\big) &= \sinh^{-2}2\lambda x\cdot\sinh^{-2}2\lambda y,\\Q\big(\sinh^{-2}2\lambda t;\sinh^{2}2\lambda t\big) &= 0.\end{aligned}\tag{4.13}$$

Thus we easily get Theorem 4.7 from Theorem 4.3. □

4.3. **Duality.**

Definition 4.8 (Duality in B_2)**.** Under the coordinate transformation

$$(x,y)\mapsto(X,Y)=\left(\frac{x+y}{\sqrt2},\frac{x-y}{\sqrt2}\right)\tag{4.14}$$

the pair (P, P^2-P_2) also satisfies (4.1), which we call the *duality* of the commuting differential operators of type B_2.

Denoting $\partial_x=\frac{\partial}{\partial x}$, $\partial_y=\frac{\partial}{\partial y}$ and put

$$L=P^2-P_2-(\tfrac12\partial_x^2-\tfrac12\partial_y^2+w-v)^2+u^-\cdot(\partial_x+\partial_y)^2+u^+\cdot(\partial_x-\partial_y)^2.$$

Then the order of L is at most 2 and the second order term of L equals

$$\begin{aligned}&-(u^++u^-+v+w)(\partial_x^2+\partial_y^2)-2(u^--u^+)\partial_x\partial_y+2w\partial_x^2+2v\partial_y^2\\&\qquad-(w-v)(\partial_x^2-\partial_y^2)+u^-(\partial_x+\partial_y)^2+u^+(\partial_x-\partial_y)^2=0.\end{aligned}$$

Since L is self-adjoint, L is of order at most 0 and the 0-th order term of L equals

$$\begin{aligned}&-\tfrac12(\partial_x^2+\partial_y^2)(u^++u^-+v+w)+(u^++u^-+v+w)^2-4vw-T-\partial_x\partial_y(u^--u^+)\\&\qquad-\tfrac12(\partial_x^2-\partial_y^2)(w-v)=(u^++u^-+v+w)^2-4vw-T\end{aligned}$$

and therefore we have the following proposition.

Proposition 4.9. i) *By the duality in Definition 4.8 the pair* $\big(R(x,y),T(x,y)\big)$ *changes into* $\big(\tilde R(x,y),\tilde T(x,y)\big)$ *with*

$$\begin{cases}\tilde R(x,y)=v\Big(\dfrac{x+y}{\sqrt2}\Big)+w\Big(\dfrac{x-y}{\sqrt2}\Big)+u^+\big(\sqrt2x\big)+u^-\big(\sqrt2y\big),\\ \tilde T(x,y)=\tilde R(x,y)^2-4v\Big(\dfrac{x+y}{\sqrt2}\Big)w\Big(\dfrac{x-y}{\sqrt2}\Big)-T\Big(\dfrac{x+y}{\sqrt2},\dfrac{x-y}{\sqrt2}\Big).\end{cases}\tag{4.15}$$

ii) *Combining the duality with the scaling map* $R(x,y)\mapsto c^{-2}R(cx,cy)$*, the following pair* $\big(R^d(x,y),T^d(x,y)\big)$ *defines commuting differential operators if so is* $\big(R(x,y),T(x,y)\big)$*. This* $R^d(x,y)$ *is also called the dual of* $R(x,y)$*.*

$$\begin{cases}R^d(x,y)=v(x+y)+w(x-y)+u^+(2x)+u^-(2y),\\ T^d(x,y)=R^d(x,y)^2-4v(x+y)w(x-y)-T(x+y,x-y).\end{cases}\tag{4.16}$$

Remark 4.10. i) We list up the systems of type B_2 given in §4.1 and §4.2:

Name	$\big(u^-(t),\ u^+(t);\ v(t),\ w(t)\big)$
(Ellip-B_2)	$\big(\langle\wp(t)\rangle;\ \langle\wp(t),\ \wp(t+\omega_1),\ \wp(t+\omega_2),\ \wp(t+\omega_3)\rangle\big)$,
(Ellip-B_2-S)	$\big(\langle\wp(t;2\omega_1;2\omega_2),\ \wp(t;\omega_1;2\omega_2)\rangle;\ \langle\wp(t;\omega_1;2\omega_2),\ \wp(t;\omega_1;\omega_2)\rangle\big)$,
(Trig-B_2)	$\big(\langle\sinh^{-2}\lambda t\rangle;\ \langle\sinh^{-2}\lambda t,\ \sinh^{-2}2\lambda t,\ \sinh^2\lambda t,\ \sinh^2 2\lambda t\rangle\big)$,
(Trig-B_2-S)	$\big(\langle\sinh^{-2}\lambda t,\ \sinh^{-2}2\lambda\rangle,\ \langle\sinh^{-2}2\lambda t,\ \sinh^2 2\lambda t\rangle\big)$,
(Rat-B_2)	$\big(\langle t^{-2}\rangle;\ \langle t^{-2},\ t^2,\ t^4,\ t^6\rangle\big)$,
(Rat-B_2-S)	$\big(\langle t^{-2},\ t^2\rangle;\ \langle t^{-2},\ t^2\rangle\big)$,
(Toda-$D_2^{(1)}$-bry)	$\big(\langle\cosh 2\lambda t\rangle;\ \langle\sinh^{-2}\lambda t,\ \sinh^{-2}2\lambda t\rangle,\ \langle\sinh^{-2}\lambda t,\ \sinh^{-2}2\lambda t\rangle\big)$,
(Toda-$D_2^{(1)}$-S-bry)	$\big(\langle\cosh\lambda t,\ \cosh 2\lambda t\rangle;\ \langle\sinh^{-2}\lambda t\rangle,\ \langle\sinh^{-2}\lambda t\rangle\big)$,
(Toda-$B_2^{(1)}$-bry)	$\big(\langle e^{-2\lambda t}\rangle;\ \langle e^{2\lambda t},\ e^{4\lambda t}\rangle,\ \langle\sinh^{-2}\lambda t,\ \sinh^{-2}2\lambda t\rangle\big)$,
(Toda-$B_2^{(1)}$-S-bry)	$\big(\langle e^{-\lambda t},\ e^{-2\lambda t}\rangle;\ \langle e^{2\lambda t}\rangle,\ \langle\sinh^{-2}\lambda t\rangle\big)$,
(Toda-$C_2^{(1)}$)	$\big(\langle e^{-2\lambda t}\rangle,\ 0;\ \langle e^{2\lambda t},\ e^{4\lambda t}\rangle,\ \langle e^{-2\lambda t},\ e^{-4\lambda t}\rangle\big)$,
(Toda-$C_2^{(1)}$-S)	$\big(\langle e^{-2\lambda t},\ e^{-4\lambda t}\rangle,\ 0;\ \langle e^{4\lambda t}\rangle,\ \langle e^{-4\lambda t}\rangle\big)$,
(Trig-A_1-bry)	$\big(\langle\sinh^{-2}\lambda t\rangle,\ 0;\ \langle e^{-2\lambda t},e^{-4\lambda t},e^{2\lambda t},e^{4\lambda t}\rangle\big)$,
(Trig-A_1-S-bry)	$\big(\langle\sinh^{-2}\lambda t,\ \sinh^{-2}2\lambda t\rangle,\ 0;\ \langle e^{-4\lambda t},\ e^{4\lambda t}\rangle\big)$.

ii) The dual is indicated by superfix ^{d}. For example, the dual of (Ellip-B_2) is denoted by (Ellip^d-B_2) whose potential function is

$$R(x,y)=A\wp(2x)+A\wp(2y)+\sum_{j=0}^{3}C_j\big(\wp(x-y+\omega_j)+\wp(x+y+\omega_j)\big)$$

and the dual of (Toda-$C_2^{(1)}$) is

(Toda^d-$C_2^{(1)}$) $\qquad\big(\langle e^{-2\lambda t},\ e^{-4\lambda t}\rangle,\ \langle e^{2\lambda t},\ e^{4\lambda t}\rangle;\ 0,\ \langle e^{-4\lambda t}\rangle\big)$

since the dual of $\big(u^-(t),u^+(t);v(t),w(t)\big)$ is $\big(w(t),v(t);u^+(2t),u^-(2t)\big)$. Similarly

(Ellip^d-B_2) $\qquad\big(\langle\wp(t;\omega_1,2\omega_2),\wp(t;\omega_1,\omega_2)\rangle\,;\,\langle\wp(2t;2\omega_1,2\omega_2),\wp(2t;\omega_1,2\omega_2)\rangle\big)$.

Since $4\wp(2t;2\omega_1,2\omega_2)=\wp(t;\omega_1,\omega_2)$ etc., (Ellip^d-B_2) coincides with (Ellip^d-B_2) by replacing (ω_1,ω_2) by $(2\omega_2,\omega_1)$.

Then we have the following diagrams and their duals, where the arrows with double lines represent specializations of parameters. For example, "Trig-$B_2 \overset{5:3}{\Rightarrow}$ Trig-BC_2-reg" means that 2 parameters out of 5 in the potential function (Trig-B_2) are specialized to get the potential function (Trig-BC_2-reg). See Definition 5.14 for their naming.

Hierarchy of Normal Integrable Potentials of type B_2

$$
\begin{array}{ccccccc}
 & & \text{Trig-}BC_2\text{-reg} & \rightarrow & \text{Toda-}D_2\text{-bry} & & \\
 & & \Uparrow 5:3 & & \Uparrow 5:3 & & \\
 & & \text{Trig-}B_2 & \rightarrow & \text{Toda-}B_2^{(1)}\text{-bry} & \overset{5:3}{\Rightarrow} & \text{Toda-}B_2^{(1)} \\
 & \nearrow & & \nearrow & \downarrow & & \\
\text{Ellip-}B_2 & \rightarrow & \text{Toda-}D_2^{(1)}\text{-bry} & & \text{Toda-}C_2^{(1)} & \overset{5:3}{\Rightarrow} & \text{Toda-}BC_2 \\
 & \searrow & & & & & \\
 & & \text{Trig-}B_2 & \rightarrow & \text{Trig-}A_1\text{-bry} & \overset{5:3}{\Rightarrow} & \text{Trig-}A_1\text{-bry-reg} \\
 & & & \searrow & & \searrow & \\
 & & & & \text{Rat-}B_2 & \rightarrow & \text{Rat-}A_1\text{-bry}
\end{array}
$$

Hierarchy of Special Integrable Potentials of type B_2

$$
\begin{array}{ccccccc}
 & & \text{Trig}^{(d)}\text{-}B_2\text{-S-reg} & \rightarrow & \text{Toda}^{(d)}\text{-}D_2\text{-S-bry} & & \\
 & & \Uparrow 4:3 & & \Uparrow 4:3 & & \\
 & & \text{Trig}^{(d)}\text{-}B_2\text{-S} & \rightarrow & \text{Toda}^{(d)}\text{-}B_2^{(1)}\text{-S-bry} & \overset{4:3}{\Rightarrow} & \text{Toda}^{(d)}\text{-}B_2^{(1)}\text{-S} \\
 & \nearrow & & \nearrow & \downarrow & & \\
\text{Ellip-}B_2\text{-S} & \rightarrow & \text{Toda}^{(d)}\text{-}D_2^{(1)}\text{-S-bry} & & \text{Toda}^{(d)}\text{-}C_2^{(1)}\text{-S} & \overset{4:3}{\Rightarrow} & \text{Toda}^{(d)}\text{-}B_2\text{-S} \\
 & \searrow & & & & & \\
 & & \text{Trig}^{(d)}\text{-}B_2\text{-S} & \rightarrow & \text{Trig}^{(d)}\text{-}A_1\text{-S-bry} & \overset{4:3}{\Rightarrow} & \text{Trig}^{(d)}\text{-}A_1\text{-S-bry-reg} \\
 & & & \searrow & & & \\
 & & & & \text{Rat-}B_2\text{-S} & &
\end{array}
$$

Definition 4.11. We define some potential functions as specializations.

(Trig-B_2-S-reg): *Trigonometric special potential of type B_2 with regular boundary* is (Trig-B_2-S) with $C_1 = 0$.

(Toda-D_2-S-bry): *Toda special potential of type D_2 with boundary* is (Toda-$B_2^{(1)}$-S-bry) with $C_0 = 0$.

(Toda-$B_2^{(1)}$-S): *Toda special potential of type $B_2^{(1)}$* is (Toda-$B_2^{(1)}$-S-bry) with $C_1 = 0$.

(Toda-B_2-S): *Toda special potential of type B_2* is (Toda-$C_2^{(1)}$-S-bry) with $C_0 = 0$.

(Trig-A_1-S-bry-reg): *Toda special potential of type A_1 with regular boundary* is (Trig-A_1-S-bry) with $C_1 = 0$.

Remark 4.12. We have some equivalences as follows.

$$(\text{Ellip-}B_2\text{-S}) = (\text{Ellip}^d\text{-}B_2\text{-S}), \tag{4.17}$$

$$(\text{Rat-}B_2\text{-S}) = (\text{Rat}^d\text{-}B_2\text{-S}), \tag{4.18}$$

$$(\text{Trig-}BC_2\text{-reg}) = (\text{Trig}^d\text{-}B_2\text{-S-reg}), \tag{4.19}$$

$$(\text{Trig-}A_1\text{-bry-reg}) = (\text{Toda}^d\text{-}D_2\text{-S-bry}), \tag{4.20}$$

$$(\text{Toda-}D_2\text{-bry}) = (\text{Trig}^d\text{-}A_1\text{-S-bry-reg}), \tag{4.21}$$

$$(\text{Toda-}B_2^{(1)}) = (\text{Toda}^d\text{-}B_2^{(1)}\text{-S}), \tag{4.22}$$

$$(\text{Toda-}BC_2) = (\text{Toda}^d\text{-}B_2\text{-S}). \tag{4.23}$$

5. Type B_n $(n \geq 3)$

In this section we construct integrals of the completely integrable systems of type B_n appearing in the following diagram (cf. [vD]):

Hierarchy of Integrable Potentials with 5 parameters $(n \geq 2)$

$$\begin{array}{ccccccc}
 & & \text{Toda-}D_n^{(1)}\text{-bry} & \to & \text{Toda-}B_n^{(1)}\text{-bry} & \to & \text{Toda-}C_n^{(1)} \\
 & \nearrow & & \nearrow & & & \\
\text{Ellip-}B_n & \to & \text{Trig-}B_n & \to & \text{Rat-}B_n & & \\
 & & & \searrow & & \searrow & \\
 & & & & \text{Trig-}A_{n-1}\text{-bry} & \to & \text{Rat-}A_{n-1}\text{-bry}
\end{array}$$

Definition 5.1. The potential functions $R(x)$ of (1.1) are as follows:
Here A, C_0, C_1, C_2 and C_3 are any complex numbers.

(Ellip-B_n) *Elliptic potential of type* B_n:

$$\sum_{1\le i<j\le n} A\big(\wp(x_i - x_j; 2\omega_1, 2\omega_2) + \wp(x_i + x_j; 2\omega_1, 2\omega_2)\big)$$
$$+ \sum_{k=1}^{n}\sum_{j=0}^{3} C_j \wp(x_k + \omega_j; 2\omega_1, 2\omega_2),$$

(Trig-B_n) *Trigonometric potential of type* B_n:

$$\sum_{1\le i<j\le n} A\big(\sinh^{-2}\lambda(x_i - x_j) + \sinh^{-2}\lambda(x_i + x_j)\big)$$
$$+ \sum_{k=1}^{n}\Big(C_0 \sinh^{-2}\lambda x_k + C_1\cosh^{-2}\lambda x_k + C_2\sinh^2\lambda x_k + \frac{C_3}{4}\sinh^2 2\lambda x_k\Big),$$

(Rat-B_n) *Rational potential of type* B_n:

$$\sum_{1\le i<j\le n}\Big(\frac{A}{(x_i - x_j)^2} + \frac{A}{(x_i + x_j)^2}\Big) + \sum_{k=1}^{n}\big(C_0 x_k^{-2} + C_1 x_k^2 + C_2 x_k^4 + C_3 x_k^6\big),$$

(Trig-A_{n-1}-bry) *Trigonometric potential of type* A_{n-1} *with boundary*:

$$\sum_{1\le i<j\le n} A\sinh^{-2}\lambda(x_i - x_j)$$
$$+ \sum_{k=1}^{n}\big(C_0 e^{-2\lambda x_k} + C_1 e^{-4\lambda x_k} + C_2 e^{2\lambda x_k} + C_3 e^{4\lambda x_k}\big),$$

(Toda-$B_n^{(1)}$-bry) *Toda potential of type* $B_n^{(1)}$ *with boundary*:

$$\sum_{i=1}^{n-1} A\big(e^{-2\lambda(x_i - x_{i+1})} + e^{-2\lambda(x_{n-1}+x_n)}\big)$$
$$+ C_0 e^{2\lambda x_1} + C_1 e^{4\lambda x_1} + C_2\sinh^{-2}\lambda x_n + C_3\sinh^{-2} 2\lambda x_n,$$

(Toda-$C_n^{(1)}$) *Toda potential of type* $C_n^{(1)}$:

$$\sum_{i=1}^{n-1} A e^{-2\lambda(x_i - x_{i+1})} + C_0 e^{2\lambda x_1} + C_1 e^{4\lambda x_1} + C_2 e^{-2\lambda x_n} + C_3 e^{-4\lambda x_n},$$

(Toda-$D_n^{(1)}$-bry) *Toda potential of type $D_n^{(1)}$ with boundary*:

$$\sum_{i=1}^{n-1} A\big(e^{-2\lambda(x_i-x_{i+1})}+e^{-2\lambda(x_{n-1}+x_n)}+e^{2\lambda(x_1+x_2)}\big)$$
$$+C_0\sinh^{-2}\lambda x_1+C_1\sinh^{-2}2\lambda x_1+C_2\sinh^{-2}\lambda x_n+C_3\sinh^{-2}2\lambda x_n,$$

(Rat-A_{n-1}-bry) *Rational potential of type A_{n-1} with boundary*:

$$\sum_{1\le i<j\le n}\frac{A}{(x_i-x_j)^2}+\sum_{k=1}^{n}\big(C_0x_k+C_1x_k^2+C_2x_k^3+C_3x_k^4\big).$$

Remark 5.2. In these cases the Schrödinger operator P is of the form

$$P=-\frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}+R(x),$$

$$R(x)=\sum_{1\le i<j\le n}\Big(u_{e_i-e_j}(x)+u_{e_i+e_j}(x)\Big)+\sum_{j=0}^{3}C_jv^j(x), \tag{5.1}$$

$$v^j(x)=\sum_{k=1}^{n}v^j_{e_k}(x).$$

Here

$$\partial_a u_\alpha(x)=\partial_b v^j_\beta(x)=0\qquad\text{if }\ a,\,b\in\mathbb{R}^n\text{ satisfy }\langle a,\alpha\rangle=\langle b,\beta\rangle=0.$$

Definition 5.3. Let $u_\alpha(x)$ and $T_I^o(v^j)$ are functions given for $\alpha\in\Sigma(D_n)$ and subsets $I=\{i_1,\dots,i_k\}\subset\{1,\dots,n\}$ such that

$$u_\alpha(x)=u_{-\alpha}(x)\ \text{ and }\ \partial_y u_\alpha=0\ \text{ for } y\in\mathbb{R}^n \text{ with } \langle\alpha,y\rangle=0. \tag{5.2}$$

Define a differential operator

$$P_n(C)=\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\sum_{w\in\mathfrak{S}_n}\Big(q_{\{w(1),\dots,w(k)\}}(C)\cdot\Delta^2_{\{w(k+1),\dots,w(n)\}}\Big) \tag{5.3}$$

by

$$\Delta_{\{i_1,\dots,i_k\}}=\sum_{0\le j\le[\frac{k}{2}]}\frac{1}{2^kj!(k-2j)!}\sum_{w\in W(B_k)}\varepsilon(w)w\Big(u_{e_{i_1}-e_{i_2}}(x)\cdot u_{e_{i_3}-e_{i_4}}(x)\cdots u_{e_{i_{2j-1}}-e_{i_{2j}}}(x)\cdot\partial_{i_{2j+1}}\partial_{i_{2j+2}}\cdots\partial_{i_k}\Big), \tag{5.4}$$

$$q_{\{i_1,\dots,i_k\}}(C)=\sum_{\nu=1}^{k}\sum_{I_1\amalg\cdots\amalg I_\nu=\{i_1,\dots,i_k\}}T_{I_1}\cdots T_{I_\nu}, \tag{5.5}$$

$$T_I=(-1)^{\#I-1}\Big(CS_I^o-\sum_{j=0}^{3}C_jT_I^o(v^j)\Big), \tag{5.6}$$

where

$$S^o_{\{i_1,i_2,\dots,i_k\}}=\tfrac{1}{2}\sum_{w\in W(B_k)}w\big(u_{e_{i_1}-e_{i_2}}(x)u_{e_{i_2}-e_{i_3}}(x)\cdots u_{e_{i_{k-1}}-e_{i_k}}(x)\big), \tag{5.7}$$

$$S^o_\emptyset=0,\ S^o_{\{k\}}=1,\ S^o_{\{i,j\}}=2u_{e_i-e_j}(x)+2u_{e_i+e_j}(x),$$

$$T^o_\emptyset(v^j)=0,\ T^o_{\{k\}}(v^j)=2v^j_{e_k}(x)\qquad\text{for }\ 1\le k\le n, \tag{5.8}$$

$$q_\varnothing=1,\ q_{\{i\}}=T_{\{i\}},\ q_{\{i_1i_2\}}=T_{\{i_1\}}T_{\{i_2\}}+T_{\{i_1,i_2\}},\dots \tag{5.9}$$

In the above, we identify $W(B_k)$ with the reflection group generated by $w_{e_{i_k}}$ and $w_{e_{i_\nu}-e_{i_{\nu+1}}}$ $(\nu=1,\dots,k-1)$.

Replacing ∂_j by ξ_j for $j=1,\dots,n$ in the definition of $\Delta_{\{i_1,\dots,i_k\}}$ and $P_n(C)$, we define functions $\bar\Delta_{\{i_1,\dots,i_k\}}$ and $\bar P_n(C)$ of (x,ξ), respectively.

We will define $u_\alpha(x)$ and $T^o_I(v^j)$ so that

$$[P_n(C),P_n(C')]=0 \qquad \text{for } \ C,\ C'\in\mathbb{C}. \tag{5.10}$$

Then putting

$$q^o_I=q_I\big|_{C=0}, \tag{5.11}$$

$$P_n=P_n(0)=\sum_{k=0}^n\frac{1}{k!(n-k)!}\sum_{w\in\mathfrak{S}_n}\Bigl(q^o_{\{w(1),\dots,w(k)\}}\Delta^2_{\{w(k+1),\dots,w(n)\}}\Bigr), \tag{5.12}$$

$$P_{n-j}=\sum_{i=j}^n\sum_{k=i}^n\frac{(-1)^{i-j}}{i!(k-i)!(n-k)!}\sum_{w\in\mathfrak{S}_n}\sum_{I_1\amalg\cdots\amalg I_j=w(\{1,\dots,i\})} S^o_{I_1}\cdots S^o_{I_j}q^o_{w(\{i+1,\dots,k\})}\Delta^2_{w(\{k+1,\dots,n\})}\Bigr), \tag{5.13}$$

we have $P_n(C)=\sum_{j=0}^n C^jP_{n-j}$ and (1.4) and

$$[P_i,P_j]=0 \quad \text{for } \ 1\le i<j\le n,$$

$$-\frac{P_1}{2}=-\frac12\sum_{j=1}^n\frac{\partial^2}{\partial x_j^2}+\sum_{1\le i<j\le n}\Bigl(u_{e_i-e_j}(x)+u_{e_i+e_j}(x)\Bigr)+\sum_{j=0}^3C_jv^j(x). \tag{5.14}$$

Remark 5.4. i) When $n=2$, $T(x,y)$ in the last section corresponds to T_{12}, namely

$$T(x_1,x_2)=T_{12}\big|_{C=0}.$$

ii) Put

$$U(x)=\sum_{1\le i<j\le n}\bigl(u^-_{ij}(x_i-x_j)+u^+_{ij}(x_i+x_j)\bigr) \quad\text{and}\quad V(x)=\sum_{k=1}^n v_k(x_k)$$

in (1.3) and let $T_I(U;V)$ be the corresponding T_I given by (5.6). Then [O1, Remark 4.3] says

$$T_I(c_0U;c_1V+c_2W)=c_0^{\#I-1}c_1T_I(U;V)+c_0^{\#I-1}c_2T_I(U;W) \quad \text{for } \ c_i\in\mathbb{C}. \tag{5.15}$$

iii) The definition (5.6) may be replace by

$$T_I=(-1)^{\#I-1}\Bigl(C\sum_{\nu=1}^{\#I}\sum_{I_1\amalg\cdots\amalg I_\nu}(-2\lambda)^{\nu-1}(\nu-1)!\cdot S^o_{I_1}\cdots S^o_{I_\nu}-\sum_{j=0}^3C_jT^o_I(v^j)\Bigr) \tag{5.16}$$

because λ can be any complex number in [O1, Lemma 5.2 ii)] when $v=C$. Note that we fixed $\lambda=1$ in [O1]. Combining (5.16) and (5.15) we may put

$$T_I=(-1)^{\#I-1}\Bigl(CS^o_I+c'\sum_{\nu\ge2}c^{\nu-1}\sum_{I_1\amalg\cdots\amalg I_\nu}S^o_{I_1}\cdots S^o_{I_\nu}-\sum_{j=0}^3C_jT^o_I(v^j)\Bigr)$$

for any $c,\ c'\in\mathbb{C}$ and hence

$$T_I=(-1)^{\#I-1}\Bigl(CS^o_I+\sum_{\nu\ge2}c_\nu\sum_{I_1\amalg\cdots\amalg I_\nu}S^o_{I_1}\cdots S^o_{I_\nu}-\sum_{j=0}^3C_jT^o_I(v^j)\Bigr) \tag{5.17}$$

for any $c_2,c_3,\ldots\in\mathbb{C}$.

Theorem 5.5 (Ellip-B_n, [O1, Theorem 7.2]). *Put*

$$\begin{cases} u_{e_i\pm e_j}(x) = A\wp_0(x_i \pm x_j; 2\omega_1, 2\omega_2) & \text{for } 1 \le i < j \le n, \\ v^j_{e_k}(x) = \wp_0(x_k + \omega_j; 2\omega_1, 2\omega_2) & \text{for } 1 \le k \le n \text{ and } 0 \le j \le 3 \end{cases} \tag{5.18}$$

and

$$T^o_I(v^j) = \sum_{\nu=1}^{\#I} \sum_{I_1 \amalg \cdots \amalg I_\nu = I} (-A)^{\nu-1}(\nu-1)! \cdot S_{I_1}(v^j) \cdots S_{I_\nu}(v^j), \tag{5.19}$$

$$\begin{aligned} S_{\{i_1,\ldots,i_k\}}(v^j) = \sum_{w \in W(B_k)} & v^j_{w(e_{i_1})}(x) u_{w(e_{i_1}-e_{i_2})}(x) u_{w(e_{i_2}-e_{i_3})}(x) \cdots \\ & \cdots u_{w(e_{i_{k-1}}-e_{i_k})}(x). \end{aligned} \tag{5.20}$$

Then (5.10) *holds.*

Example 5.6. Put $v^j_k = v^j_{e_k}$, $\tilde{v}_k = \sum_{j=0}^3 C_j v^j_k$, and $w^{\pm}_{ij} = u^-_{ij} \pm u^+_{ij}$. Then

$$\begin{aligned}
\Delta_{\{1\}} &= \partial_1, \\
\Delta_{\{1,2\}} &= \partial_1\partial_2 + u^-_{12} - u^+_{12} = \partial_1\partial_2 + w^-_{12}, \\
\Delta_{\{1,2,3,4\}} &= \partial_1\partial_2\partial_2 + w^-_{12}\partial_3 + w^-_{23}\partial_1 + w^-_{13}\partial_2, \\
\Delta_{\{1,2,3,4\}} &= \partial_1\partial_2\partial_3 + w^-_{34}\partial_1\partial_2 + w^-_{24}\partial_1\partial_3 + w^-_{23}\partial_1\partial_4 + w^-_{14}\partial_2\partial_3 + w^-_{13}\partial_2\partial_4 \\
&\quad + w^-_{12}\partial_3\partial_4 + w^-_{12}w^-_{34} + w^-_{13}w^-_{24} + w^-_{14}w^-_{23}, \\
S_1(v^j) &= 2v^j_1, \\
S_{\{1,2\}}(v^j) &= 2v^j_1u^-_{12} + 2v^j_1u^+_{12} + 2v^j_2u^-_{12} + 2v^j_2u^+_{12} = 2(v^j_1 + v^j_2)(u^-_{12} + u^+_{12}), \\
S_{\{1,2,3\}}(v^j) &= 2v^j_1u^-_{12}u^-_{23} + 2v^j_1u^-_{12}u^+_{23} + 2v^j_1u^+_{12}u^-_{23} + 2v^j_1u^+_{12}u^+_{23} + \cdots \\
&= 2(v^j_1 + v^j_3)w^+_{12}w^+_{23} + 2(v^j_1 + v^j_2)w^+_{23}w^+_{13} + 2(v^j_2 + v^j_3)w^+_{12}w^+_{13}, \\
T_{\{1\}} &= CS_{\{1\}} - \sum_{j=0}^3 C_j T^o_{\{1\}}(v^j) = C - 2\tilde{v}_1, \\
T_{\{1,2\}} &= -CS^o_{\{1,2\}} + \sum_{j=0}^3 C_j T^o_{\{1,2\}}(v^j), \\
q_{\{1\}} &= T_{\{1\}}, \\
q_{\{1,2\}} &= T_{\{1\}}T_{\{2\}} + T_{\{1,2\}}, \\
q_{\{1,2,3\}} &= T_{\{1\}}T_{\{2\}}T_{\{3\}} + T_{\{1,2\}}T_{\{3\}} + T_{\{1,3\}}T_{\{2\}} + T_{\{1\}}T_{\{2,3\}} + T_{\{1,2,3\}}.
\end{aligned}$$

If $T^o(v^j)$ and $S_{i_1,\ldots,i_k}(v^j)$ are given by (5.19) and (5.20), then

$$\begin{aligned}
T^o_{\{1\}}(v^j) &= 2v^j_1, \\
T^o_{\{1,2\}}(v^j) &= S_{\{1,2\}}(v^j) - AT^o_{\{1\}}T^o_{\{2\}} = 2(v^j_1 + v^j_2)w^+_{12} - 4Av^j_1v^j_2, \\
T^o_{\{1,2,3\}}(v^j) &= S_{\{1,2,3\}}(v^j) - 2A\big(v^j_1S_{\{2,3\}}(v^j) + v^j_2S_{\{1,3\}}(v^j) + v^j_3S_{\{1,2\}}(v^j)\big) \\
&\quad + 16A^2v^j_1v^j_2v^j_3.
\end{aligned}$$

In particular, if $n = 2$, then

$$\begin{aligned}
P_2(C) &= \Delta^2_{\{1,2\}} + q_{\{1\}}\Delta^2_{\{2\}} + q_{\{2\}}\Delta^2_{\{1\}} + q_{\{1,2\}} \\
&= \big(\partial_1\partial_2 + u^-_{12} - u^+_{12}\big)^2 + T_{\{1\}}\partial_2^2 + T_{\{2\}}\partial_1^2 \\
&\quad + T_{\{1\}}T_{\{2\}} - CS^o_{\{1,2\}} + \sum_{j=0}^3 C_jT^o_{\{1,2\}}(v^j)
\end{aligned}$$

$$\begin{aligned}
&= \left(\partial_1\partial_2 + u_{12}^- - u_{12}^+\right)^2 + (C - 2\tilde v_1)\partial_2^2 + (C - 2\tilde v_2)\partial_1^2\\
&\quad + (C - 2\tilde v_1)(C - 2\tilde v_2) - 2C(u_{12}^- + u_{12}^+)\\
&\quad + 2(\tilde v_1 + \tilde v_2)(u_{12}^- + u_{12}^+) - 4A\sum_{j=0}^{3} C_j v_1^j v_2^j\\
&= C^2 - 2P\cdot C + P_2
\end{aligned}$$

with

$$\begin{aligned}
P &= -\tfrac12(\partial_1^2 + \partial_2^2) + \tilde v_1 + \tilde v_2 + u_{12}^- + u_{12}^+,\\
P_2 &= \left(\partial_1\partial_2 + u_{12}^- - u_{12}^+\right)^2 - 2\tilde v_1\partial_2^2 - 2\tilde v_2\partial_1^2\\
&\quad + 4\tilde v_1\tilde v_2 + 2(\tilde v_1 + \tilde v_2)(u_{12}^- + u_{12}^+) - 4A\sum_{j=0}^{3} C_j v_1^j v_2^j.
\end{aligned}$$

In general

$$\begin{aligned}
P_1 &= \sum_{k=1}^{n}\Big(\Delta_{\{k\}}^2 + q_{\{k\}}^o\Big) - \sum_{1\le i<j\le n} S_{\{i,j\}}^o \qquad (5.21)\\
&= \sum_{k=1}^{n}\Big(\partial_k^2 - 2\tilde v_k^j\Big) - 2\sum_{1\le i<j\le n} w_{ij}^+,
\end{aligned}$$

$$\begin{aligned}
P_2 &= \sum_{1\le i<j\le n}\Delta_{\{i,j\}}^2 + \sum_{\substack{1\le i\le n\\ 1\le j\le n,\, i\ne j}}\sum_{j=1}^{n} q_{\{i\}}^o\Delta_{\{j\}}^2 + \sum_{1\le i<j\le n} q_{\{i,j\}}^o \qquad (5.22)\\
&\quad - \sum_{\substack{1\le i<j\\ 1\le k\le n,\, k\ne i,\, j}} S_{\{i,j\}}^o\big(\Delta_{\{k\}}^2 + q_{\{k\}}^o\big)\\
&\quad + \sum_{\substack{1\le i<j\le n\\ i<k<\ell\le n\\ j\ne k,\,\ell}} S_{\{i,j\}}^o S_{\{k,\ell\}}^o + \sum_{1\le i<j<k\le n} S_{\{i,j,k\}}^o,\\
&= \sum\Big(\partial_i\partial_j + w_{ij}^-\Big)^2 - \sum 2\tilde v_i\partial_j^2 + \sum 4\tilde v_i\tilde v_j + \sum C_j T_{\{k,\ell\}}^o(v^j) \qquad (5.23)\\
&\quad - 2\sum w_{ij}^+\big(\partial_k^2 - 2\tilde v_k\big) + 4\sum w_{ij}^+ w_{k\ell}^+ + 2\sum w_{ij}^+ w_{jk}^+.
\end{aligned}$$

Here if (5.19) and (5.20) are valid, then

$$T_{\{k,\ell\}}^o(v^j) = 2(v_k^j + v_\ell^j)w_{k\ell}^+ - 4Av_k^j v_\ell^j. \qquad (5.24)$$

The commuting operator P_3 of the 6-th order is

$$\begin{aligned}
P_3 &= \sum\Delta_{\{i,j,k\}}^2 + \sum q_{\{k\}}^o\Delta_{\{i,j\}}^2 + \sum q_{\{i,j,k\}}^o \qquad (5.25)\\
&\quad - \sum S_{\{k,\ell\}}^o\Delta_{\{i,j\}}^2 - \sum S_{\{k,\ell\}}^o q_{\{i\}}^o\Delta_{\{j\}}^2 - \sum S_{\{k,\ell\}}^o q_{\{i,j\}}^o\\
&\quad + \sum S_{\{i,j,k,\ell,m\}}^o + \sum S_{\{i,j,k,\ell\}}^o S_{\{\mu,\nu\}}^o\\
&\quad + \sum S_{\{i,j,k\}}^o S_{\{\ell,\mu,\nu\}}^o + \sum S_{\{i,j,k\}}^o S_{\{\ell,m\}}^o S_{\{\mu,\nu\}}^o\\
&\quad + \sum S_{\{i_1,i_2\}}^o S_{\{j_1,j_2\}}^o S_{\{k_1,k_2\}}^o S_{\{\ell_1,\ell_2\}}^o.
\end{aligned}$$

In Theorem 5.5 the Schrödinger operator is

$$P = -\frac{1}{2}\sum_{k=0}^{n}\frac{\partial^2}{\partial x_k^2} + A\sum_{1\le i<j\le n}\big(\wp(x_i-x_j)+\wp(x_i+x_j)\big) + \sum_{j=0}^{3} C_j \sum_{k=1}^{n}\wp(x_k+\omega_j)$$

and the operator P_2 satisfying $[P, P_2] = 0$ is given by (5.23) and (5.24) with

$$\tilde v_k = \sum_{\nu=0}^{3} C_\nu \wp(x_k+\omega_\nu),\ v_k^j = \wp(x_k+\omega_j),\ w_{ij}^{\pm} = A\big(\wp(x_i-x_j)\pm\wp(x_i+x_j)\big).$$

Theorem 5.7 (Toda-$D_n^{(1)}$-bry)**.** *For*

$$(5.26)\qquad \begin{cases} u_{e_i-e_j}(x) = \begin{cases} Ae^{-2\lambda(x_i-x_{i+1})} & (j=i+1)\\ 0 & (|j-i|>1)\end{cases}\\ u_{e_i+e_j}(x) = \begin{cases} Ae^{2\lambda(x_1+x_2)} & (i+j=3)\\ Ae^{-2\lambda(x_{n-1}+x_n)} & (i+j=2n-1)\\ 0 & (i+j\notin\{3,2n-1\})\end{cases}\\ v_k^0(x) = \delta_{1k}\sinh^{-2}\lambda x_1,\quad v_k^1(x)=\delta_{1k}\sinh^{-2}2\lambda x_1,\\ v_k^2(x) = \delta_{nk}\sinh^{-2}\lambda x_n,\quad v_k^3(x)=\delta_{nk}\sinh^{-2}2\lambda x_n,\end{cases}$$

we have integrals P_j by (5.12), (5.4), (5.5), (5.6) *and*

(5.27)

$$\begin{aligned} S^o_{\{k\}} &= 1 \quad \textit{for } 1\le k\le n,\\ S^o_I &= 0 \quad \textit{if } I\ne\{k,k+1,\ldots,\ell\} \quad \textit{for } 1\le k<\ell\le n,\\ S^o_{\{k,k+1,\ldots,\ell\}} &= 2A^{\ell-k+1}\big(e^{-2\lambda(x_k-x_\ell)}+\delta_{1k}e^{2\lambda(x_1+x_\ell)}+\delta_{\ell n}e^{-2\lambda(x_k+x_n)}\big),\\ T^0_{\{k\}}(v^j) &= 2v_k^j(x) \quad \textit{for}\ \ 0\le j\le 3,\ k=1,\ldots,n,\\ T^0_I(v^0) &= 0 \quad \textit{if } I\ne\{1,\ldots,k\} \textit{ for } k=1,\ldots,n,\\ T^0_I(v^2) &= 0 \quad \textit{if } I\ne\{k,\ldots,n\} \textit{ for } k=1,\ldots,n,\\ T^0_I(v^1) &= T^0_I(v^3) = 0 \quad \textit{if } \#I>1,\\ T^0_{\{1,\ldots,k\}}(v^0) &= 8A^{k-1}(e^{2\lambda x_k}+\delta_{kn}e^{-2\lambda x_n}) \quad \textit{for } k\ge 2,\\ T^0_{\{n-k+1,\ldots,n\}}(v^2) &= 8A^{k-1}\big(e^{-2\lambda x_{n-k+1}}+\delta_{kn}e^{2\lambda x_1}\big) \quad \textit{for } k\ge 2.\end{aligned}$$

Proof. Put

$$\begin{aligned} \tilde x &= (x_1-\tfrac{1-1}{n-1}\omega_2,\ldots,x_k-\tfrac{k-1}{n-1}\omega_2,\ldots,x_n-\tfrac{n-1}{n-1}\omega_2),\\ \tilde u_{e_i\mp e_j}(\tilde x) &= A\frac{e^{\frac{2\lambda\omega_2}{n-1}}}{4\lambda^2}\wp_0\big(x_i-\tfrac{i-1}{n-1}\omega_2\mp(x_j-\tfrac{j-1}{n-1}\omega_2);2\omega_1,2\omega_2\big),\\ \tilde v_k^j(\tilde x) &= \frac{(-1)^j}{\lambda^2}\wp_0(x_k-\tfrac{k-1}{n-1}\omega_2+\omega_j;2\omega_1,2\omega_2) \quad \text{for}\ \ 0\le j\le 3,\ 1\le k\le n.\end{aligned}$$

When $\omega_2\to\infty$, $\tilde u_{e_i\mp e_j}(\tilde x)$ and $\tilde v_k^\ell$ $(\ell=0,1,2,3)$ converge to $u_{e_i\mp e_j}(x)$ in (5.26) and

$$\begin{cases} v_k^0(x) = \delta_{1k}\sinh^{-2}\lambda x_1,\quad v_k^1(x) = \delta_{1k}\cosh^{-2}\lambda x_1,\\ v_k^2(x) = \delta_{nk}\sinh^{-2}\lambda x_n,\quad v_k^3(x) = \delta_{nk}\cosh^{-2}\lambda x_n,\end{cases}$$

respectively. Under the notation in Theorem 5.5, let $\tilde S_I(\tilde v^\ell)$ and $\tilde T^o_I(\tilde v^\ell)$ be the functions defined in the same way as $S_I(v^\ell)$ and $T^o_I(v^\ell)$, respectively, where $(u_{e_i\mp e_j}(x),$

$v_k^\ell(x)$) are replaced by $(\tilde u_{e_i\mp e_j}(x),\ \tilde v_k^\ell(\tilde x))$. Then by taking the limits for $\omega_2\to\infty$ $\tilde T_I^o(\tilde v^\ell)$ converge the following $\bar T_I^o(v^\ell)$.

$$\begin{aligned}\bar T_I^0(v^0)=\bar T_I^0(v^1)=0 \quad &\text{if } I\neq\{1,\dots,k\} \text{ for } k=1,\dots,n,\\ \bar T_I^0(v^2)=\bar T_I^0(v^3)=0 \quad &\text{if } I\neq\{k,\dots,n\} \text{ for } k=1,\dots,n.\end{aligned}$$

If $k\geq 2$, then

$$\begin{aligned}\bar T^0_{\{1,\dots,k\}}(v^0)&=\lim_{\omega_2\to\infty}\sum_{\nu=1}^{\#I}\sum_{I_1\amalg\cdots\amalg I_\nu=I}\Bigl(-A\frac{e^{\frac{2\lambda\omega_2}{n-1}}}{4\lambda^2}\Bigr)^{\nu-1}(\nu-1)!\cdot\tilde S_{I_1}(\tilde v^0)\cdots\tilde S_{I_\nu}(\tilde v^0)\\ &=\lim_{\omega_2\to\infty}\tilde S_{\{1,\dots,k\}}(\tilde v^0)-\lim_{\omega_2\to\infty}\tilde S_{\{1\}}(\tilde v^0)\cdot A\frac{e^{\frac{2\lambda\omega_2}{n-1}}}{4\lambda^2}\tilde S_{\{2,\dots,k\}}(\tilde v^0)\\ &=2A^{k-1}\sinh^{-2}\lambda x_1\bigl(e^{-2\lambda(x_1-x_k)}+e^{2\lambda(x_1+x_k)}+\delta_{kn}e^{2\lambda(x_1-x_n)}\\ &\quad+\delta_{kn}e^{-2\lambda(x_1+x_n)}\bigr)-2\sinh^{-2}\lambda x_1\cdot 2A^{k-1}\bigl(e^{2\lambda x_k}+\delta_{kn}e^{-2\lambda x_n}\bigr)\\ &=8A^{k-1}\bigl(e^{2\lambda x_k}+\delta_{kn}e^{-2\lambda x_n}\bigr).\\ \bar T^0_{\{1,\dots,k\}}(v^1)&=2A^{k-1}\cosh^{-2}\lambda x_1\bigl(e^{-2\lambda(x_1-x_k)}+e^{2\lambda(x_1+x_k)}+\delta_{kn}e^{2\lambda(x_1-x_n)}\\ &\quad+\delta_{kn}e^{-2\lambda(x_1+x_n)}\bigr)+4A^{k-1}\cosh^{-2}\lambda x_1\bigl(e^{2\lambda x_k}+\delta_{kn}e^{-2\lambda x_n}\bigr)\\ &=8A^{k-1}\bigl(e^{2\lambda x_k}+\delta_{kn}e^{-2\lambda x_n}\bigr),\\ \bar T^0_{\{n-k+1,\dots,n\}}(v^2)&=8A^{k-1}\bigl(e^{-2\lambda x_{n-k+1}}+\delta_{kn}e^{2\lambda x_1}\bigr),\\ \bar T^0_{\{n-k+1,\dots,n\}}(v^3)&=8A^{k-1}\bigl(e^{-2\lambda x_{n-k+1}}+\delta_{kn}e^{2\lambda x_1}\bigr).\end{aligned}$$

Replacing v^1 and v^3 by $\frac14(v^0-v^1)$ and $\frac14(v^2-v^3)$, respectively, we have the theorem by the analytic continuation given in Lemma 2.2. □

Suitable limits of the functions in Theorem 5.5 give the following theorem.

Theorem 5.8 (Trig-B_n, [O1, Proposition 6.1]). *For complex numbers λ, C, $C_0,\dots,$ C_3 and A with $\lambda\neq 0$, we have (5.14) by putting*

$$\begin{cases}u_{e_i\pm e_j}(x)=A\sinh^{-2}\lambda(x_i\pm x_k),\\ v^0_{e_k}(x)=\sinh^{-2}\lambda x_k, \qquad v^1_{e_k}(x)=\cosh^{-2}\lambda x_k,\\ v^2_{e_k}(x)=\sinh^2\lambda x_k, \qquad v^3_{e_k}(x)=\frac14\sinh^2 2\lambda x_k\end{cases}\tag{5.28}$$

and

$$\begin{aligned}T_I&=(-1)^{\#I-1}\Bigl(CS_I^o-C_0T_I^o(v^0)-C_1T_I^o(v^1)-C_2S_I(v^2)-C_3S_I(v^3)\\ &\quad+2C_3\sum_{I_1\amalg I_2=I}\bigl(S_{I_1}(v^2)\cdot S_{I_2}(v^2)+S_{I_1}(v^2)\cdot S^o_{I_2}+S^o_{I_1}\cdot S_{I_2}(v^2)\bigr)\Bigr),\\ T_I^o(v^0)&=\sum_{\nu=1}^{\#I}\sum_{I_1\amalg\cdots\amalg I_\nu=I}(-A)^{\nu-1}(\nu-1)!\cdot S_{I_1}(v^0)\cdots S_{I_\nu}(v^0),\\ T_I^o(v^1)&=\sum_{\nu=1}^{\#I}\sum_{I_1\amalg\cdots\amalg I_\nu=I}A^{\nu-1}(\nu-1)!\cdot S_{I_1}(v^1)\cdots S_{I_\nu}(v^1).\end{aligned}$$

Theorem 5.9 (Trig-A_{n-1}-bry). *For*

$$\begin{cases}u_{e_i-e_j}(x)=A\sinh^{-2}\lambda(x_i-x_j), \qquad u_{e_i+e_j}(x)=0,\\ v^0_{e_k}(x)=e^{-2\lambda x_k},\ v^1_{e_k}(x)=e^{-4\lambda x_k},\ v^2_{e_k}(x)=e^{2\lambda x_k},\ v^3_{e_k}(x)=e^{4\lambda x_k}\end{cases}\tag{5.29}$$

we have (5.14) *by putting*

$$T_I = (-1)^{\#I-1}\Big(CS_I^o - \sum_{j=0}^{3} C_j S_I(v^j) + 2\sum_{I_1 \amalg I_2 = I}\big(C_1 S_{I_1}(v^0)S_{I_2}(v^0) + C_3 S_{I_1}(v^2)S_{I_2}(v^2)\big)\Big). \tag{5.30}$$

Proof. Putting

$$\begin{aligned}
\tilde u_{e_i\pm e_j} &= A\sinh^{-2}\lambda\big((x_i+R)\pm(x_j+R)\big),\\
\tilde v_k^0 &= \tfrac14 e^{2\lambda R}\sinh^{-2}\lambda(x_k+R),\\
\tilde v_k^1 &= \tfrac14 e^{4\lambda R}\sinh^{-2}2\lambda(x_k+R)\\
&= \tfrac1{16} e^{4\lambda R}\big(\sinh^{-2}2\lambda(x_k+R)-\cosh^{-2}2\lambda(x_k+R)\big),\\
\tilde v_k^2 &= 4e^{-2\lambda R}\sinh^2\lambda(x_k+R),\\
\tilde v_k^3 &= 4e^{-4\lambda R}\sinh^2 2\lambda(x_k+R),\\
\tilde x &= (x_1+R, x_2+R,\dots,x_n+R)
\end{aligned}$$

under the notation in Theorem 5.8, we have

$$\begin{aligned}
(\bar u_{e_i-e_j}, \bar u_{e_i+e_j}, \bar v_k^0, \bar v_k^1, \bar v_k^2, \bar v_k^3) &:= \lim_{R\to\infty}(\tilde u_{e_i-e_j}, \tilde u_{e_i+e_j}, \tilde v_k^0, \tilde v_k^1, \tilde v_k^2, \tilde v_k^3)\\
&= (A\sinh^{-2}\lambda(x_i-x_k), 0, e^{-2\lambda x_k}, e^{-4\lambda x_k}e^{2\lambda x_k}, e^{4\lambda x_k}),\\
\lim_{R\to\infty}\tfrac14 e^{2\lambda R}T_I^o(v^0)(\tilde x) &= \bar S_I(\bar v^0),\\
\lim_{R\to\infty}\tfrac1{16} e^{4\lambda R}\big(T_I^o(v^0)(\tilde x) - T_I^o(v^1)(\tilde x)\big) &= \bar S_I(\bar v^1) - 2\sum_{I_1\amalg I_2=I}\bar S_{I_1}(\bar v^0)\bar S_{I_2}(\bar v^0),\\
\lim_{R\to\infty} 4e^{-2\lambda R}T_I^o(v^2)(\tilde x) &= \bar S_I(\bar v^2),\\
\lim_{R\to\infty} 4e^{-2\lambda R}T_I^o(v^3)(\tilde x) &= \bar S_I(\bar v^3) - 2\sum_{I_1\amalg I_2=I}\bar S_{I_1}(\bar v^2)\bar S_{I_2}(\bar v^2).
\end{aligned}$$

Here $\bar S_I(\bar v^\ell)$ are defined by (5.20) with $u_{e_i\pm e_j}$ and $v_{e_k}^\ell$ replaced by $\bar u_{e_i\pm e_j}$ and $\bar v_{e_k}^\ell$, respectively. Then the theorem is clear. □

Theorem 5.10 (Toda-$B_n^{(1)}$-bry)**.** *For the potential function defined by*

$$\begin{cases}
u_{e_i-e_j}(x) = \begin{cases} Ae^{-2\lambda(x_i-x_{i+1})} & \text{if } j=i+1,\\ 0 & \text{if } 1\le i<i+1<j\le n,\end{cases}\\
u_{e_i+e_j}(x) = \begin{cases} Ae^{-2\lambda(x_{n-1}+x_n)} & \text{if } i=n-1,\ j=n,\\ 0 & \text{if } 1\le i<j\le n \text{ and } i\ne n-1,\end{cases}\\
v_k^0(x) = \delta_{k1}e^{2\lambda x_1}, \qquad v_k^1(x) = \delta_{k1}e^{4\lambda x_1},\\
v_k^2(x) = \delta_{kn}\sinh^{-2}\lambda x_n, \qquad v_k^3(x) = \delta_{kn}\sinh^{-2}2\lambda x_n,
\end{cases} \tag{5.31}$$

we have (5.14) *with*

$$
\begin{aligned}
S^o_{\{k\}} &= 1 \quad \textit{for } 1 \le k \le n,\\
S^o_I &= 0 \quad \textit{if } I \ne \{k, k+1, \dots, \ell\} \textit{ for } 1 \le k < \ell \le n,\\
S^o_{\{k,k+1,\dots,\ell\}} &= 2A^{\ell-k+1}\bigl(e^{-2\lambda(x_k - x_\ell)} + \delta_{\ell n} e^{-2\lambda(x_k + x_n)}\bigr),\\
T^0_{\{k\}}(v^j) &= 2v^j_k(x) \quad \textit{for } \ 0 \le j \le 3,\ k = 1, \dots, n,\\
T^0_I(v^0) &= 0 \quad \textit{if } I \ne \{1, \dots, k\} \textit{ for } k = 1, \dots, n,\\
T^0_I(v^2) &= 0 \quad \textit{if } I \ne \{k, \dots, n\} \textit{ for } k = 1, \dots, n,\\
T^0_I(v^1) &= T^0_I(v^3) = 0 \quad \textit{if } \#I > 1,\\
T^0_{\{1,\dots,k\}}(v^0) &= 2A^{k-1}(e^{2\lambda x_k} + \delta_{kn} e^{-2\lambda x_n}) \quad \textit{for } k \ge 2,\\
T^0_{\{n-k+1,\dots,n\}}(v^2) &= 8A^{k-1} e^{-2\lambda x_{n-k+1}} \quad \textit{for } k \ge 2.
\end{aligned}
\tag{5.32}
$$

Proof. Suppose $\operatorname{Re}\lambda > 0$. In (Toda-$D^{(1)}_n$-bry) put

$$
\begin{aligned}
\tilde x &= \bigl(x_1 - (n-1)R, \dots, x_k - (n-k)R, \dots, x_n - (n-n)R\bigr),\\
\tilde u_{e_i - e_j} &= \begin{cases} Ae^{-2\lambda R} \cdot e^{-2\lambda(x_i - (n-i)R - x_{i+1} + (n-i-1)R)} & (j = i+1)\\ 0 & (|j-i| > 1)\end{cases}\\
\tilde u_{e_i + e_j} &= \begin{cases} Ae^{-2\lambda R} \cdot e^{2\lambda(x_1 - (n-1)R + x_2 - (n-2)R)} & (i+j = 3)\\ Ae^{-2\lambda R} \cdot e^{-2\lambda(x_{n-1} - R + x_n)} & (i+j = 2n-1)\\ 0 & (i+j \notin \{3, 2n-1\})\end{cases}\\
\tilde v^0_k &= \delta_{1k} \frac{e^{2\lambda(n-1)R}}{4} \sinh^{-2} \lambda\bigl(x_1 - (n-1)R\bigr),\\
\tilde v^1_k &= \delta_{1k} \frac{e^{4\lambda(n-1)R}}{4} \sinh^{-2} 2\lambda\bigl(x_1 - (n-1)R\bigr),\\
\tilde v^2_k &= \delta_{nk} \sinh^{-2} \lambda x_n,\\
\tilde v^3_k &= \delta_{nk} \sinh^{-2} 2\lambda x_n
\end{aligned}
$$

and we have (5.31) by the limit $R \to \infty$. Moreover for $k \ge 2$, it follows from Theorem 5.7 and (5.15) that

$$
\begin{aligned}
\tilde T_{\{1,\dots,k\}}(\tilde v^1) &= \tilde T_{\{1,\dots,k\}}(\tilde v^3) = 0,\\
\tilde T_{\{1,\dots,k\}}(\tilde v^0) &= \tfrac14 e^{2\lambda(n-1)R} \cdot \bigl(Ae^{-2\lambda R}\bigr)^{k-1} \cdot \bigl(8e^{2\lambda(x_k - (n-k)R)} + 8\delta_{kn} e^{-2\lambda x_n}\bigr)\\
&= 2A^{k-1}\bigl(e^{2\lambda x_k} + \delta_{kn} e^{-2\lambda x_n}\bigr),\\
\tilde T_{\{n-k+1,\dots,n\}}(\tilde v^2) &= \bigl(Ae^{-2\lambda R}\bigr)^{k-1}\bigl(8e^{-2\lambda(x_{n-k+1} - (k-1)R)} + 8\delta_{1k} e^{2\lambda(x_1 - (n-1)R)}\bigr)\\
&= 8A^{k-1} e^{-2\lambda x_{n-k+1}},
\end{aligned}
$$

which implies the theorem. □

Theorem 5.11 (Toda-$C^{(1)}_n$)**.** *For the potential function defined by*

$$
\begin{cases}
u_{e_i - e_j}(x) = \begin{cases} Ae^{-2\lambda(x_i - x_{i+1})} & \textit{if } \ j = i+1,\\ 0 & \textit{if } \ 1 \le i < i+1 < j \le n,\end{cases}\\
u_{e_i + e_j}(x) = 0 \quad \textit{for } \ 1 \le i < j \le n,\\
v^0_k(x) = \delta_{k1} e^{2\lambda x_1}, \qquad v^1_k(x) = \delta_{k1} e^{4\lambda x_1},\\
v^2_k(x) = \delta_{kn} e^{-2\lambda x_n}, \qquad v^3_k(x) = \delta_{kn} e^{-4\lambda x_n},
\end{cases}
\tag{5.33}
$$

we have (5.14) *with*

$$
\begin{aligned}
S^o_{\{k\}} &= 1 \quad \textit{for } 1 \le k \le n,\\
S^o_I &= 0 \quad \textit{if } I \ne \{k, k+1, \ldots, \ell\} \textit{ for } 1 \le k < \ell \le n,\\
S^o_{\{k,k+1,\ldots,\ell\}} &= 2A^{\ell-k+1} e^{-2\lambda(x_k - x_\ell)},\\
T^0_{\{k\}}(v^j) &= 2v^j_k(x) \quad \textit{for } \ 0 \le j \le 3,\ k = 1, \ldots, n,\\
T^0_I(v^0) &= 0 \quad \textit{if } I \ne \{1, \ldots, k\} \textit{ for } k = 1, \ldots, n,\\
T^0_I(v^2) &= 0 \quad \textit{if } I \ne \{k, \ldots, n\} \textit{ for } k = 1, \ldots, n,\\
T^0_I(v^1) &= T^0_I(v^3) = 0 \quad \textit{if } \#I > 1,\\
T^0_{\{1,\ldots,k\}}(v^0) &= 2A^{k-1} e^{2\lambda x_k} \quad \textit{for } k \ge 2,\\
T^0_{\{n-k+1,\ldots,n\}}(v^2) &= 2A^{k-1} e^{-2\lambda x_{n-k+1}} \quad \textit{for } k \ge 2.
\end{aligned} \tag{5.34}
$$

Proof. Substituting x_k by $x_k + R$ for $k = 1, \ldots, n$ and multiplying v^0_k, v^1_k, v^2_k and v^3_k by $e^{-2\lambda R}$, $e^{-4\lambda R}$, $\frac{1}{4}e^{2\lambda R}$ and $\frac{1}{4}e^{4\lambda R}$, respectively, we have the claim from Theorem 5.10. □

Theorem 5.12 (Rat-A_{n-1}-bry)**.** *We have* (5.14) *if*

$$
\begin{cases}
u_{e_i - e_j}(x) = \dfrac{A}{(x_i - x_j)^2}, \quad u_{e_i + e_j}(x) = 0,\\
v^j_k(x) = x^{j+1}_k,
\end{cases} \tag{5.35}
$$

$$
\begin{aligned}
T_I = (-1)^{\#I-1}\Big(&CS^o_I - \sum_{j=0}^{3} C_j S_I(v^j)\\
&+ \sum_{I_1 \amalg I_2 = I} C_1\big(S_{I_1}(v^0)\cdot S^o_{I_2} + S^o_{I_1}\cdot S_{I_1}(v^0)\big)\\
&+ \sum_{I_1 \amalg I_2 = I} C_3\big(S_{I_1}(v^1)\cdot S^o_{I_2} + S_{I_1}(v^0)\cdot S_{I_2}(v^0) + S^o_{I_1}\cdot S_{I_2}(v^1)\big)\Big).
\end{aligned} \tag{5.36}
$$

Proof. Put

$$
\begin{aligned}
\tilde u_{e_i - e_j} &= \lambda^2 \sinh^{-2}\lambda(x_i - x_j),\\
\tilde u_{e_i + e_j} &= 0,\\
\tilde v^0_k &= \tfrac{1}{2\lambda}\big(e^{2\lambda x_k} - 1\big),\\
\tilde v^1_k &= \tfrac{1}{4\lambda^2}\big(e^{2\lambda x_k} + e^{-2\lambda x_k} - 2\big),\\
\tilde v^2_k &= \tfrac{1}{8\lambda^3}\big(e^{4\lambda x_k} - 3e^{2\lambda x_k} - e^{-2\lambda x_k} + 3\big),\\
\tilde v^3_k &= \tfrac{1}{16\lambda^4}\big(e^{4\lambda x_k} + e^{-4\lambda x_k} - 4e^{2\lambda x_k} - 4e^{-2\lambda x_k} + 6\big).
\end{aligned}
$$

Then taking $\lambda \to 0$ we have the required potential function.

Owing to (Trig-A_{n-1}-bry) and Remark 5.4, we have

$$
\begin{aligned}
&\lim_{\lambda\to 0} \tilde S_I(\textstyle\sum \tilde v^j_k) = \bar S_I(\sum x^{j+1}_k),\\
&\lim_{\lambda\to 0} \lambda^2 \cdot \tfrac{1}{8\lambda^3} \cdot \Big(\tilde S_{I_1}(\textstyle\sum e^{2\lambda x_k})\tilde S_{I_2}(\sum e^{2\lambda x_k}) - 4\tilde S^o_{I_1}\tilde S^o_{I_2}\Big)\\
&\quad = \tfrac{1}{2}\Big(\bar S_{I_1}(\textstyle\sum x_k)\cdot \bar S^o_{I_2} + \bar S^o_{I_1}\cdot \bar S_{I_2}(\sum x_k)\Big),
\end{aligned}
$$

$$\lim_{\lambda\to 0}\lambda^2\cdot\tfrac{1}{16\lambda^4}\cdot\Big(\tilde S_{I_1}(\sum e^{2\lambda x_k})\tilde S_{I_2}(\sum e^{2\lambda x_k})+\tilde S_{I_1}(\sum e^{-2\lambda x_k})\tilde S_{I_2}(\sum e^{-2\lambda x_k})$$
$$-8\tilde S^o_{I_1}\tilde S^o_{I_2}\Big)$$
$$=\tfrac12\Big(\bar S_{I_1}(\sum x_k^2)\cdot\bar S^o_{I_2}+\bar S_{I_1}(\sum x_k)\cdot\bar S_{I_2}(\sum x_k)+\bar S^o_{I_1}\cdot\bar S_{I_2}(\sum x_k)\Big)$$

and thus the theorem. □

Suitable limits of the functions in Theorem 5.8 give the following theorem.

Theorem 5.13 (Rat-B_n, [O1, Proposition 6.3])**.** *Put*

$$\begin{cases} u_{e_i-e_j}(x)=\dfrac{A}{(x_i-x_j)^2}, & u_{e_i+e_j}(x)=\dfrac{A}{(x_i+x_j)^2}\\ v_k^0(x)=x_k^{-2},\quad v_k^1(x)=x_k^2,\quad v_k^2(x)=x_k^4,\quad v_k^3(x)=x_k^6. \end{cases}\tag{5.37}$$

Then (5.14) *holds with*

$$T_I=(-1)^{\#I-1}\Big(CS^o_I-C_0T^o_I(v^0)-C_1S_I(v^1)$$
$$-C_2S_I(v^2)+2C_2\sum_{I_1\amalg I_2=I}\Big(S_{I_1}(v^1)\cdot S^o_{I_2}+S^o_{I_1}\cdot S_{I_2}(v^1)\Big)$$
$$-2C_3S^o_I(v^3)+C_3\sum_{I_1\amalg I_2=I}\Big(S_{I_1}(v^1)\cdot S_{I_2}(v^1)+2S_{I_1}(v^2)\cdot S^o_{I_2}+2S^o_{I_1}\cdot S_{I_2}(v^2)\Big)$$
$$-24C_3\sum_{I_1\amalg I_2\amalg I_3=I}\Big(S_{I_1}(v^1)\cdot S^o_{I_2}\cdot S^o_{I_3}+S^o_{I_1}\cdot S_{I_2}(v^1)\cdot S^o_{I_3}+S^o_{I_1}\cdot S^o_{I_2}\cdot S_{I_3}(v^1)\Big),$$
$$T^o_I(v^0)=\sum_{\nu=1}^{\#I}\sum_{I_1\amalg\cdots\amalg I_\nu=I}(-A)^{\nu-1}(\nu-1)!\cdot S_{I_1}(v^0)\cdots S_{I_\nu}(v^0).$$

Definition 5.14. We define some potential functions as specializations of potential functions in Definition 5.1.

(Trig-A_{n-1}-bry-reg): *Trigonometric potential of type A_{n-1} with regular boundary* is (Trig-A_{n-1}-bry) with $C_2=C_3=0$.
(Trig-A_{n-1}): *Trigonometric potential of type A_{n-1}* is (Trig-A_{n-1}-bry) with $C_0=C_1=C_2=C_3=0$.
(Trig-BC_n-reg): *Trigonometric potential of type BC_n with regular boundary* is (Trig-B_n) with $C_2=C_3=0$.
(Toda-D_n-bry): *Toda potential of type D_n with boundary* is (Toda-$B_n^{(1)}$-bry) with $C_0=C_1=0$.
(Toda-$B_n^{(1)}$): *Toda potential of type $B_n^{(1)}$* is (Toda-$B_n^{(1)}$-bry) with $C_2=C_3=0$.
(Toda-$D_n^{(1)}$): *Toda potential of type $D_n^{(1)}$* is (Toda-$D_n^{(1)}$-bry) with $C_0=C_1=C_2=C_3=0$.
(Toda-A_{n-1}): *Toda potential of type A_{n-1}* is (Toda-$C_n^{(1)}$) with $C_0=C_1=C_2=C_3=0$.
(Toda-BC_n): *Toda potential of type B_n* is (Toda-$C_n^{(1)}$) with $C_0=C_1=0$.
(Ellip-D_n): *Elliptic potential of type D_n* is (Ellip-B_n) with $C_0=C_1=C_2=C_3=0$.
(Trig-D_n): *Trigonometric potential of type D_n* is (Trig-B_n) with $C_0=C_1=C_2=C_3=0$.
(Rat-D_n): *Rational potential of type D_n* is (Rat-B_n) with $C_0=C_1=C_2=C_3=0$.
(Toda-D_n): *Toda potential of type D_n* is (Toda-$B_n^{(1)}$-bry) with $C_0=C_1=C_2=C_3=0$.
(Rat-B_n-2): *Rational potential of type B_n-2* is (Rat-B_n) with $C_2=C_3=0$.

(Rat-A_{n-1}-bry2): *Rational potential of type* A_{n-1} *with* 2*-boundary* is (Rat-A_{n-1}-bry) with $C_2 = C_3 = 0$. In this case, we may assume $C_0 = 0$ or $C_1 = 0$ by the transformation $x_k \mapsto x_k + c$ $(k = 1, \dots, n)$ with a suitable $c \in \mathbb{C}$.

Then we have the following diagrams for $n \geq 3$. Note that we don't write all the arrows in the diagrams (ex. (Toda-D_n-bry) $\to$ (Toda-BC_n)).

Hierarchy of Elliptic-Trigonometric-Rational Integrable Potentials

$$\begin{array}{ccccccc}
 & & \text{Rat-}B_n\text{-}2 & & \text{Ellip-}D_n & & \\
 & & \Uparrow_{5:3} & & \downarrow & & \\
 & & \text{Rat-}B_n & \nwarrow & \text{Trig-}D_n & \to & \text{Rat-}D_n \\
 & & \uparrow & & \Uparrow_{3:1} & & \\
\text{Ellip-}B_n & \to & \text{Trig-}B_n & \overset{5:3}{\Rightarrow} & \text{Trig-}BC_n\text{-reg} & & \text{Ellip-}A_{n-1} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \text{Trig-}A_{n-1}\text{-bry} & \overset{5:3}{\Rightarrow} & \text{Trig-}A_{n-1}\text{-bry-reg} & \overset{3:1}{\Rightarrow} & \text{Trig-}A_{n-1} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \text{Rat-}A_{n-1}\text{-bry} & \overset{5:3}{\Rightarrow} & \text{Rat-}A_{n-1}\text{-bry2} & \overset{3:1}{\Rightarrow} & \text{Rat-}A_{n-1}
\end{array}$$

Hierarchy of Toda Integrable Potentials

$$\begin{array}{ccccccc}
 & & \text{Trig-}BC_n\text{-reg} & \to & \text{Toda-}D_n\text{-bry} & \overset{3:1}{\Rightarrow} & \text{Toda-}D_n \\
 & & \Uparrow_{5:3} & & \Uparrow_{5:3} & & \Uparrow_{3:1} \\
 & & \text{Trig-}B_n & \to & \text{Toda-}B_n^{(1)}\text{-bry} & \overset{5:3}{\Rightarrow} & \text{Toda-}B_n^{(1)} \\
 & \nearrow & & \nearrow & & \searrow & \\
\text{Ellip-}B_n & \to & \text{Toda-}D_n^{(1)}\text{-bry} & \overset{5:1}{\Rightarrow} & \text{Toda-}D_n^{(1)} & & \text{Toda-}C_n^{(1)} \\
 & & & \nearrow & & & \Downarrow_{5:3} \\
 & & \text{Ellip-}D_n & & \text{Trig-}A_{n-1} & & \text{Toda-}BC_n \\
 & & & \nearrow & & \searrow & \Downarrow_{3:1} \\
 & & \text{Ellip-}A_{n-1} & \to & \text{Toda-}A_{n-1}^{(1)} & \to & \text{Toda-}A_{n-1}
\end{array}$$

6. Type D_n $(n \geq 3)$

Theorem 6.1 (Type D_n)**.** *The Schrödinger operators* (Ellip-D_n)*,* (Trig-D_n)*,* (Rat-D_n)*,* (Toda-$D_n^{(1)}$)*,* (Toda-D_n) *are in the commutative algebra of differential operators generated by* $P_1, P_2, \dots, P_{n-1}$ *and* $\Delta_{\{1,\dots,n\}}$ *which are the corresponding operators for* (Ellip-B_n)*,* (Trig-B_n)*,* (Rat-B_n)*,* (Toda-$D_n^{(1)}$-bry)*,* (Toda-D_n-bry) *with* $C_0 = C_1 = C_2 = C_3 = 0$*, respectively.*

Proof. This theorem is proved by [O1] in the cases (Ellip-D_n), (Trig-D_n), (Rat-D_n). Other two cases have been defined by suitable analytic continuation and therefore the claim is clear. □

Remark 6.2. In the above theorem we have $P_n = \Delta^2_{\{1,\dots,n\}}$ because $q_I^o = 0$ if $I \neq \emptyset$. Then $[P_j, P_n] = 0$ implies $[P_j, \Delta_{\{1,\dots,n\}}] = 0$.

Hierarchy of Integrable Potentials of Type D_n $(n \geq 3)$

$$\begin{array}{ccccc}
 & & \text{Toda-}D_n^{(1)} & \to & \text{Toda-}D_n \\
 & \nearrow & & \nearrow & \\
\text{Ellip-}D_n & \to & \text{Trig-}D_n & \to & \text{Rat-}D_n
\end{array}$$

7. Classical limits

For functions $f(\xi,x)$ and $g(\xi,x)$ of $(\xi,x)=(\xi_1,\ldots,\xi_n,x_1,\ldots,x_n)$, we define their Poisson bracket by

$$\{f,g\}=\sum_{k=1}^{n}\Bigl(\frac{\partial f}{\partial\xi_k}\frac{\partial g}{\partial x_k}-\frac{\partial g}{\partial\xi_k}\frac{\partial f}{\partial x_k}\Bigr). \tag{7.1}$$

Theorem 7.1. *Put*

$$\bar P(\xi,x)=-\frac12\sum_{k=1}^{n}\xi_k^2+R(x). \tag{7.2}$$

Then for the integrable potential function $R(x)$ given in this note, the functions $\bar P_k(\xi,x)$ and $\bar\Delta_{\{1,\ldots,n\}}(\xi,x)$ of (ξ,x) defined by replacing ∂_k by ξ_k in the definitions of P_k and $\Delta_{\{1,\ldots,n\}}$ in §3, §4 and §5 satisfy

$$\bigl\{\bar P_i(\xi,x),\bar P_j(\xi,x)\bigr\}=\bigl\{\bar P(\xi,x),\bar P_k(\xi,x)\bigr\}=0\quad\text{for } 1\le i<j\le n \text{ and } 1\le k\le n.$$

Hence $\bar P(\xi,x)$ are Hamiltonians of completely integrable dynamical systems.

Moreover if the potential function $R(x)$ is of type D_n, then

$$\bigl\{\bar\Delta_{\{1,\ldots,n\}}(\xi,x),\bar P_k(\xi,x)\bigr\}=\bigl\{\bar\Delta_{\{1,\ldots,n\}}(\xi,x),\bar P(\xi,x)\bigr\}=0\quad\text{for } 1\le k\le n.$$

Proof. If $R(x)$ is a potential function of (Ellip-A_{n-1}), (Ellip-B_n) or (Ellip-D_n), the claim is proved in [OS] and [O1]. Since the claim keeps valid under suitable holomorphic continuations with respect to the parameters which are given in the former sections, we have the theorem. □

8. Analogue for one variable

Putting $n=1$ for the Schrödinger operator P of type A_n in §3 or of type B_n in §5, we examine the ordinary differential equation $Pu=Cu$ with $C\in\mathbb{C}$ (cf. [WW, §10·6]). We will write the operators $Q=P-C$.

(Ellip-B_1) The Heun equation ([OS, §8], [WW, pp.576])

$$-\frac12\frac{d^2}{dt^2}+\sum_{j=0}^{3}C_j\wp(t+\omega_j)-C.$$

(Ellip-A_1) The Lamé equation

$$-\frac12\frac{d^2}{dt^2}+A\wp(t)-C.$$

(Trig-BC_1-reg) The Gauss hypergeometric equation

$$-\frac12\frac{d^2}{dt^2}+\frac{C_0}{\sinh^2\lambda t}+\frac{C_1}{\sinh^2 2\lambda t}-C.$$

(Trig-A_1) The Legendre equation

$$-\frac12\frac{d^2}{dt^2}+\frac{C_0}{\sinh^2\lambda t}-C.$$

(Trig-B_1) with $C_0=C_1=C_3=0$. The (Modified) Mathieu equation

$$-\frac12\frac{d^2}{dt^2}+C_2\cosh 2\lambda t-C.$$

(Rat-B_1-2) Equation of the paraboloid of revolution

$$-\frac12\frac{d^2}{dt^2}+\frac{C_0}{t^2}+C_1t^2-C.$$

This is the Weber equation if $C_0 = 0$. Putting $s = t^2$, the above equation is reduced to the Whittaker equation:

$$-\frac{1}{2}\frac{d^2}{ds^2} + \frac{C_0'}{s^2} + \frac{C_1'}{s} - C'.$$

(Rat-A_0-bry2) with $C_2 = C_3 = 0$:

$$-\frac{1}{2}\frac{d^2}{dt^2} + C_0 t + C_1 t^2 - C.$$

If $C_1 \neq 0$, this is transformed into the Weber equation under the coordinate $s = t + \frac{C_2}{2C_1}$. If $C_1 = 0$, this is the Stokes equation which is reduced to the Bessel equation. In particular the Airy equation corresponds to $C = C_1 = 0$.

$$\text{(Toda-}BC_1\text{)} \qquad -\frac{1}{2}\frac{d^2}{dt^2} + C_0 e^{-2t} + C_1 e^{-4t} - C,$$

which is transformed into (Rat-B_1-2) by putting $s = e^{-t}$. In particular

$$\text{(Toda-}A_1\text{)} \qquad -\frac{1}{2}\frac{d^2}{dt^2} + C_0 e^{-2t} - C$$

is reduced to the Bessel equation.

(Rat-A_1) the Bessel equation

$$-\frac{1}{2}\frac{d^2}{dt^2} + \frac{C_0}{t^2} - C.$$

In fact, the equation $-\frac{u''}{2} + \frac{C_0 u}{t^2} = Cu$ is equivalent to

$$\Big(\frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} - \frac{2C_0 + \frac{1}{4}}{t^2} + 2C\Big)t^{-\frac{1}{2}}u = 0$$

since $t^{-\frac{1}{2}} \circ \frac{d^2}{dt^2} \circ t^{\frac{1}{2}} = \frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} - \frac{1}{4t^2}$. Hence if $C \neq 0$, the function $v = t^{-\frac{1}{2}}u$ satisfies the following Bessel equation with $s = \sqrt{-2C}t$:

$$\frac{d^2 v}{ds^2} + \frac{1}{s}\frac{dv}{ds} - \Big(1 - \frac{C_0 + \frac{1}{8}}{Cs^2}\Big)v = 0.$$

Hierarchy of ordinary differential equations

$$\begin{array}{ccccccccc}
\text{Heun} & \overset{5:2}{\Rrightarrow} & \text{Lamé} & \longrightarrow & \text{Legendre} & & & & \\
 & \searrow & & & \Uparrow_{3:2} & \searrow & & & \\
\text{Mathieu} & \overset{2:5}{\Lleftarrow} & \text{Trig-}B_1 & \overset{5:3}{\Rrightarrow} & \text{Gauss} & & \text{Bessel (Stokes)} & \overset{2:1}{\rightarrow} & \text{Airy} \\
 & & \downarrow_{5:4} & \searrow & & \searrow & \Uparrow_{3:2} & & \uparrow_{2:1} \\
 & & \text{Rat-}A_0\text{-bry} & & \text{Rat-}B_1 & \overset{5:3}{\Rrightarrow} & \text{Whittaker} & \overset{3:2}{\rightarrow} & \text{Weber}
\end{array}$$

9. A classification

In this section we assume that P is the Schrödinger operator (1.1) which admits commuting differential operators (1.2) satisfying (1.3).

Conjecture 9.1. Under a suitable affine transformation of the coordinate $x \in \mathbb{C}^n$ which keeps the algebra $\mathbb{C}[\sum_{k=1}^n \partial_k^2, \sum_{k=1}^n \partial_k^4, \ldots, \sum_{k=1}^n \partial_k^{2n}]$ invariant, P is transformed into an integrable Schrödinger operator studied in this note or in general a *direct sum* of such operators and/or trivial operators

$$(A_1) \qquad \frac{d^2}{dx^2} + v(x)$$

with arbitrary functions $v(x)$ of one variable.

Here the direct sum of the two operators $P_j(x, \partial_x) = \sum_{\alpha \in \{0,1,\ldots\}^{n_j}} a_\alpha(x)\partial_x^\alpha$ of $x \in \mathbb{C}^{n_j}$ for $j = 1, 2$ means the operator $P_1(x, \partial_x) + P_2(y, \partial_y)$ of $(x, y) \in \mathbb{C}^{n_1+n_2}$.

Remark 9.2. The condition

$$\text{there exists } P_2 \text{ such that } [P,P_2]=0 \text{ and } \sigma(P_2)=\sum_{1\le i<j\le n}\xi_i^2\xi_j^2 \tag{9.1}$$

may be sufficient to assure the claim of the conjecture.

Remark 9.3 (Type A_2). If $n=2$ and there exists P_3 satisfying

$$\sigma(P_3)=\xi_1\xi_2+\xi_2\xi_3+\xi_3\xi_1 \text{ and } [P,P_3]=[\partial_1+\partial_2+\partial_3,P]=[\partial_1+\partial_2+\partial_3,P_3]=0,$$

then Conjecture 9.1 is true.

In fact this case is reduced to solve the equation

$$\begin{vmatrix} u(x) & u'(x) & 1\\ v(y) & v'(y) & 1\\ w(z) & w'(z) & 1\end{vmatrix}=0 \quad \text{for} \quad x+y+z=0 \tag{9.2}$$

for three unknown functions $u(t)$, $v(t)$ and $w(t)$, which is solved by [BP] and [BB]. Here $u(t)=u_{e_1-e_2}(t)$, $v(t)=u_{e_2-e_3}(t)$ and $w(t)=u_{e_1-e_3}(-t)$.

9.1. Pairwise interactions and meromorphy.

Theorem 9.4 ([Wa]). *The potential function $R(x)$ is of the form*

$$R(x)=\sum_{\alpha\in\Sigma(B_n)^+}u_\alpha(\langle\alpha,x\rangle) \tag{9.3}$$

with functions $u_\alpha(t)$ of one variable.

Remark 9.5. The condition (9.1) assures

$$R(x)=\sum_{\alpha\in\Sigma(B_n)^+}u_\alpha(\langle\alpha,x\rangle)+\sum_{1\le i<j<k\le n}C_{ijk}x_ix_jx_k$$

with $C_{ijk}\in\mathbb{C}$ and thus the above theorem is proved in the invariant case (cf. §9.2) by [OS] or in the case of Type B_2 by [Oc] or in the case of Type A_{n-1}. This theorem is proved in [Wa] by using $[P,P_2]=[P,P_3]=0$.

Definition 9.6. By the expression (9.3), put

$$S=\{\alpha\in\Sigma(B_n)^+\,;\,u'_\alpha\ne 0\} \tag{9.4}$$

and let $W(S)$ be the Weyl group generated by $\{w_\alpha\,;\,\alpha\in S\}$ and moreover put $\bar S=W(S)S$.

Theorem 9.7 ([Oc] for Type B_2, [Wa] in general). *If the root system $\bar S$ has no irreducible component of rank one, then (9.1) assures that any function $u_\alpha(t)$ extends to a meromorphic function on $\mathbb{C}$.*

Remark 9.8 ([OO, (6.4)-(6.5)], [Wa, §3]). The condition (9.1) is equivalent to

$$S_{ij}=S_{ji} \qquad (1\le i<j\le n) \tag{9.5}$$

with

$$
\begin{aligned}
S^{ij} = &\Big(\partial_i^2 v_i(x_i) + \sum_{\nu\in I(i,j)} \partial_i^2\big(u_{i\nu}^+(x_i+x_\nu) + u_{i\nu}^+(x_i-x_\nu)\big)\Big)\\
&\cdot\Big(u_{ij}^+(x_i+x_j) - u_{ij}^-(x_i-x_j)\Big)\\
&+3\Big(\partial_i v_i(x_i) + \sum_{\nu\in I(i,j)} \partial_i\big(u_{i\nu}^+(x_i+x_\nu) + u_{i\nu}^-(x_i-x_\nu)\big)\Big)\\
&\cdot\Big(\partial_i u_{ij}^+(x_i+x_j) - \partial_i u_{ij}^-(x_i-x_j)\Big)\\
&+2\Big(v_i(x_i) + \sum_{\nu\in I(i,j)} \big(u_{i\nu}^+(x_i+x_\nu) + u_{i\nu}^-(x_i-x_\nu)\big)\Big)\\
&\cdot\Big(\partial_i^2 u_{ij}^+(x_i+x_j) - \partial_i^2 u_{ij}^-(x_i-x_j)\Big)\\
&+\sum_{\nu\in I(i,j)}\Big(\partial_i^2 u_{i\nu}^+(x_i+x_\nu) - \partial_i^2 u_{i\nu}^-(x_i-x_\nu)\Big)\Big(u_{j\nu}^+(x_j+x_\nu) - u_{j\nu}^-(x_j-x_\nu)\Big).
\end{aligned}
$$

Here $I(i,j) = \{1,2,\dots,n\}\setminus\{i,j\}$.

Lemma 9.9. *Suppose P satisfies (9.1). Let S_0 be a subset of $\bar S$ such that*

$$S_0 \subset \sum_{i=1}^m \mathbb{R}e_i \quad \textit{and} \quad \bar S\setminus S_0 \subset \sum_{i=m+1}^n \mathbb{R}e_i$$

with a suitable m. Then the Schrödinger operator

$$P' = -\frac{1}{2}\sum_{i=1}^m \partial_i^2 + \sum_{\alpha\in S_0\cap S} u(\langle\alpha,x\rangle)$$

on $\mathbb{R}^m$ admits a differential operator P_2' on $\mathbb{R}^m$ satisfying $[P',P_2']=0$ and $\sigma(P_2') = \sum_{1\le i<j\le n}^m \xi_i^2\xi_j^2$, that is, the corresponding condition (9.1) for P'.

Proof. This lemma clearly follows from the equivalent condition (9.5) given in Remark 9.8 □

9.2. **Invariant case.**

Theorem 9.10 ([OOS], [OS], [OO], [O1])**.** *Assume that P in (1.1) is invariant under the Weyl group $W = W(A_{n-1})$, $W(B_n)$ or $W(D_n)$ with $n\ge 3$ or $W = W(B_2)$ with $n = 2$. Suppose*

$$P_1 = \partial_1+\partial_2+\cdots+\partial_n \qquad \textit{if}\;\; W = W(A_{n-1}),$$

$$\sigma(P_k) = \begin{cases} \displaystyle\sum_{1\le j_1<j_2<\cdots<j_k\le n} \xi_{j_1}\xi_{j_2}\cdots\xi_{j_k} & \textit{if}\;\; W=W(A_{n-1}) \;\;\textit{and}\;\; 1\le k\le n,\\ \displaystyle\sum_{1\le j_1<j_2<\cdots<j_k\le n} \xi_{j_1}^2\xi_{j_2}^2\cdots\xi_{j_k}^2 & \textit{if}\;\; W=W(B_n) \;\;\textit{and}\;\; 1\le k\le n,\\ \displaystyle\sum_{1\le j_1<j_2<\cdots<j_k\le n} \xi_{j_1}^2\xi_{j_2}^2\cdots\xi_{j_k}^2 & \textit{if}\;\; W=W(D_n) \;\;\textit{and}\;\; 1\le k< n,\end{cases}$$

$$\sigma(P_n) = \xi_1\xi_2\cdots\xi_n \qquad \textit{if}\;\; W = W(D_n).$$

If P is not a direct sum of trivial operators (A_1), then P is (Ellip-F) *or* (Trig-F) *or* (Rat-F) *with $F = A_{n-1}$ or B_n or D_n.*

Remark 9.11. The condition

$$\begin{cases}[P,P_1]=[P,P_3]=0 & \text{if}\;\; W=W(A_{n-1}),\\ [P,P_2]=0 & \text{if}\;\; W=W(B_n)\text{ or }W=W(D_n)\end{cases} \tag{9.6}$$

is sufficient for the proof of Theorem 9.2 with (9.3).

9.3. **Enough singularities.** Put $\Xi = \{\alpha \in \Sigma(B_n)\,;\, u_\alpha(t) \text{ is not entire.}\}$

Theorem 9.12. i) ([Oc]) *Suppose $n = 2$ and let $\bar S$ is of type B_2. If $\#\Xi \ge 2$, then Conjecture 9.1 is true.*

ii) ([Wa]) *If $\bar S$ is of type A_{n-1} or of type B_n and moreover the reflections w_α for $\alpha \in \Xi$ generate $W(A_{n-1})$ or $W(B_n)$, respectively, then Conjecture 9.1 is true.*

This theorem follows from the following key Lemma.

Lemma 9.13 ([Oc], [Ta], [Wa])**.** *Suppose there exist α and β in S such that $\alpha \ne \beta$, $\langle \alpha, \beta\rangle \ne 0$ and $u_\alpha(t)$ has a singularity at $t = t_0$. Then $u_\alpha(t - t_0)$ is an even function with a pole of order two at the origin and*

$$\begin{cases} u_{w_\alpha(\beta)}(t - 2t_0 \frac{\langle\alpha,\gamma\rangle}{\langle\alpha,\alpha\rangle}) = u_\beta(t) & \text{if } \ w_\alpha(\beta) \in \Sigma(B_n)^+, \\ u_{-w_\alpha(\beta)}(-t + 2t_0 \frac{\langle\alpha,\gamma\rangle}{\langle\alpha,\alpha\rangle}) = u_\beta(t) & \text{if } \ -w_\alpha(\beta) \in \Sigma(B_n)^+. \end{cases} \tag{9.7}$$

Corollary 9.14. *Assume the assumption in Lemma 9.13.*

i) *If $u_\alpha(t)$ has another singularity at $t_1 \ne t_0$, then*

$$u_\gamma\big(t + 2(t_1 - t_0)\tfrac{\langle\alpha,\gamma\rangle}{\langle\alpha,\alpha\rangle}\big) = u_\gamma(t) \quad \text{for } \ \gamma \in S. \tag{9.8}$$

ii) *Assume that u_α has poles at 0, t_0 and t_1 such that t_0 and t_1 are linearly independent over $\mathbb{R}$. Then $u_\beta(t)$ is a doubly periodic function and therefore $u_\beta(t)$ has poles and hence $u_\alpha(t)$ is also a doubly periodic function. We may moreover assume that u_β has a pole at 0 by a pararell transformation of the variable x.*

Case I: *Suppose $\alpha = e_i - e_j$, $\beta = e_j - e_k$ with $1 \le i < j < k \le n$.*

$$u_{e_i-e_j}(t) = u_{e_j-e_k}(t) = u_{e_i-e_k}(t) = C\wp(t; 2\omega_1, 2\omega_2) + C'. \tag{9.9}$$

with suitable C, $C' \in \mathbb{C}$, which corresponds to (Ellip-A_2).

Case II: *Suppose $\alpha = e_i - e_j$ and $\beta = e_k$ with $1 \le i < j \le n$. Then $\big(u_{e_i-e_j}(t), u_{e_i+e_j}(x), u_{e_i}(t), u_{e_j}(t)\big)$ is* (Ellip-B_2), (Ellip-B_2-S) *or* (Ellip^d-B_2).

iii) If $\bar S$ is of type A_{n-1} or B_n or D_n and one of $u_\alpha(t)$ is a doubly periodic function with poles, then P transforms into (Ellip-A_{n-1}) or (Ellip-B_n) or (Ellip-D_n) under a suitable parallel transformation on $\mathbb{C}^n$.

Proof. i) is a direct consequence of Lemma 9.13 and iii) follows from ii). We have only to show ii).

Case I: It follows from (9.7) that $u_\alpha(t) = u_\beta(t) = u_{e_i-e_k}(t)$ and they are even functions. Let

$$\Gamma_{2\omega_1,2\omega_2} = \{2m_1\omega_1 + 2m_2\omega_2\,;\, m_1, m_2 \in \mathbb{Z}\}$$

be the set of poles of u_α. Then (9.8) implies $u_\beta(t + 2\omega_1) = u_\beta(t + 2\omega_2) = u_\beta(t)$. Since $2\omega_1$ and $2\omega_2$ are periods of $\wp(t)$ and there exists only one double pole in the fundamental domain defined by these periods, we have the claim.

Case II: It follows from (9.7) that $u_{e_i-e_j}(t) = u_{e_i+e_j}(t)$ and $u_{e_i}(t) = u_{e_k}(t)$ and they are even functions. Let $\Gamma_{2\omega_1,2\omega_2}$ be the poles of $u_{e_i-e_j}(t)$. Then (9.8) means $u_{e_i}(t + 2\omega_1) = u_{e_i}(t + 2\omega_2) = u_{e_i}(t)$. Considering the poles of $u_{e_i-e_k}(t)$ with (9.8), we have four possibilities of poles of u_{e_i}:

$$\begin{cases} \text{(Case II-0): } \Gamma_{2\omega_1,2\omega_2}, \\ \text{(Case II-1): } \Gamma_{2\omega_1,2\omega_2} \cup (\omega_1 + \Gamma_{2\omega_1,2\omega_2}), \\ \text{(Case II-2): } \Gamma_{2\omega_1,2\omega_2} \cup (\omega_2 + \Gamma_{2\omega_1,2\omega_2}), \\ \text{(Case II-3): } \Gamma_{2\omega_1,2\omega_2} \cup (\omega_1 + \Gamma_{2\omega_1,2\omega_2}) \cup (\omega_2 + \Gamma_{2\omega_1,2\omega_2}). \end{cases}$$

Here we note that (Case II-1) changes into (Case II-2) if we exchange ω_1 and ω_2. Then we have

$$\begin{cases}\text{(Case II-0): } u_{e_i-e_j}(t+4\omega_1)=u_{e_i-e_j}(t+4\omega_2)=u(t),\\ \text{(Case II-2): } u_{e_i-e_j}(t+4\omega_1)=u_{e_i-e_j}(t+2\omega_2)=u(t),\\ \text{(Case II-3): } u_{e_i-e_j}(t+2\omega_1)=u_{e_i-e_j}(t+2\omega_2)=u(t).\end{cases}$$

Thus (Case II-0), (Case II-2) and (Case II-3) are reduced to (Ellip^d-B_2), (Ellip-B_2-S) and (Ellip-B_2), respectively. □

Let $\mathcal{H}$ be a finite set of mutually nonparallel vectors in $\mathbb{R}^n$ and suppose

$$P=-\frac{1}{2}\sum_{j=1}^{n}\partial_j^2+R(x),\quad R(x)=\sum_{\alpha\in\mathcal{H}}C_\alpha\frac{\langle\alpha,\alpha\rangle}{\langle\alpha,x\rangle^2}+\tilde{R}(x). \tag{9.10}$$

Here C_α are nonzero complex numbers and $\tilde{R}(x)$ is real analytic at the origin. We assume that $\mathcal{H}$ is *irreducible*, namely,

$$\mathbb{R}^n=\sum_{\alpha\in\mathcal{H}}\mathbb{R}\alpha,$$

$$\emptyset\neq\forall\mathcal{H}'\subsetneq\mathcal{H}\Rightarrow\exists\alpha\in\mathcal{H}'\text{ and }\exists\beta\in\mathcal{H}\setminus\mathcal{H}'\text{ with }\langle\alpha,\beta\rangle\neq0.$$

Definition 9.15. The potential function $R(x)$ of a Schrödinger operator is *reducible* if $R(x)$ and $\mathbb{R}^n$ is decomposed as $R(x)=R_1(x)+R_2(x)$ and $\mathbb{R}^n=V_1\oplus V_2$ such that

$$0\subsetneq V_1\subsetneq\mathbb{R}^n,\ V_2=V_1^\perp,\ \partial_{v_2}R_1(x)=\partial_{v_1}R_2(x)=0\quad\text{for }\ \forall v_2\in V_2\text{ and }\forall v_1\in V_1.$$

If $R(x)$ is not reducible, $R(x)$ is called to be irreducible.

Theorem 9.16 ([Ta]). *Suppose $n\geq2$ and there exists a differential operator Q with $[P,Q]=0$ whose principal symbol does not depend on x and is not a polynomial of $\sum_{i=0}^n\xi_i^2$. Put $W=\{w_\alpha;\alpha\in\mathcal{H}\}$. If*

$$2C_\alpha\neq k(k+1)\quad\text{for }\ k\in\mathbb{Z}\text{ and }\alpha\in\mathcal{H}, \tag{9.11}$$

then W is a finite reflection group and $\sigma(Q)$ is W-invariant.

9.4. **Periodic potentials.** The following theorem is a little generalization of the result in [O2].

Theorem 9.17. *Assume $R(x)$ is of the form (9.3) with meromorphic functions $u_\alpha(t)$ on $\mathbb{C}$ and*

$$R\big(x+\frac{2\pi\sqrt{-1}\alpha}{\langle\alpha,\alpha\rangle}\big)=R(x)\qquad\text{for }\ \alpha\in\Sigma(B_n) \tag{9.12}$$

and moreover assume that

$$\left\{\begin{array}{l}\text{the root system }\bar{S}\text{ does not contain an irreducible component of}\\ \text{type }B_2\text{ or even if }\bar{S}\text{ contains an irreducible component }\bar{S}_2=\\ \{\pm e_i\pm e_j,\pm e_i,\pm e_j\}\text{ of type }B_2\text{, the origin }s=0\text{ is not an isolated}\\ \text{essential singularity of }u_\alpha(\log s)\text{ for }\alpha\in\bar{S}_2\cap\Sigma(D_n)^+.\end{array}\right. \tag{9.13}$$

Then (9.1) implies that P is transformed into an integrable Schrödinger operator classified in §3, 4, 5 or a direct sum of such operators and/or the trivial operators.

Remark 9.18. The assumption (9.12) in Theorem 9.17 implies that $u_\alpha(\log s)$ is a meromorphic function on $\mathbb{C}\setminus\{0\}$ for any $\alpha\in\Sigma(B_n)^+$.

Lemma 9.19. *Assume $n=2$, $\bar{S}$ is of type B_2 and $u_\alpha(\log s)$ are holomorphic for $\alpha\in\Sigma(B_2)^+$ and $0<|s|\ll1$. If the origin is at most a pole of $u_\beta(\log s)$ for $\beta\in\Sigma(D_2)^+$, the origin is also at most a pole of $u_\alpha(\log s)$ for $\alpha\in\Sigma(B_2)^+$.*

Proof. Use the notation as in (4.2). Put

$$u^-(\log s) = U_0^- + \sum_{\nu=r}^{\infty} \nu U_\nu^- s^\nu, \qquad u^+(\log s) = U_0^+ + \sum_{\nu=m}^{\infty} \nu U_\nu^+ s^\nu,$$
$$v(\log s) = V_0 + \sum_{\nu=-\infty}^{\infty} \nu V_\nu s^\nu, \qquad w(\log s) = W_0 + \sum_{\nu=-\infty}^{\infty} \nu W_\nu s^\nu.$$

with U_ν^-, U_ν^+, V_ν, $W_\nu \in \mathbb{C}$, $rm \neq 0$ and $(U_r^-, U_m^+) \neq 0$. Then as is shown in [O2] the condition for the existence of $T(x,y)$ in (4.3) is equals to

$$pq(2p-q)(p-q)(V_{2p-q}U_{q-p}^+ + V_q U_{p-q}^- + W_{q-2p}U_p^+ - W_q U_p^-) = 0 \text{ for } p, q \in \mathbb{Z}. \tag{9.14}$$

Hence if $p < r$ and $p < m$,

$$p(p-k)(p+k)k(V_{p+k}U_{-k}^+ + V_{p-k}U_k^-) = 0 \quad \text{for } k \in \mathbb{Z}.$$

Case $U_r^- \neq 0$: Put $k = r$. Suppose q is negative with a sufficiently large absolute value. Then $V_q = \bigl(-\frac{U_{-r}^+}{U_r^-}\bigr)V_{q+2r}$, which implies $V_q = 0$ since $\sum_{\nu=-\infty}^{\infty} \nu V_\nu s^\nu$ converges for $0 < |s| \ll 1$.

Suppose q is negative with a sufficiently large absolute value compared to p. Then by the relation $W_{q-2p}U_p^+ - W_q U_p^- = 0$ we similarly conclude $W_q = 0$.

Case $U_m^+ \neq 0$: Putting $k = -m$, we have the same conclusion as above in the same way. □

Proof of Theorem 9.17. Lemma 9.9 assures that we may assume $\bar{S}$ is an irreducible root system. We may moreover assume that the rank of $\bar{S}$ is greater than one.

Suppose that there exists $\gamma \in S$ such that the origin is not a removable singularity nor an isolated singularity of $u_\gamma(\log s)$. Then $u_\gamma(t)$ is a doubly periodic function with poles. Owing to Corollary 9.14, $\sum_{\alpha \in S_0} u_\alpha(\langle \alpha, x\rangle)$ is reduced to the potential function of (Ellip-A_{n-1}) or (Ellip-B_n) or (Ellip-D_n).

Thus we may assume that the origin is a removable singularity or an isolated singularity of $u_\alpha(\log s)$ for any $\alpha \in S$.

Let $\alpha, \beta \in S \cap \Sigma(D_n)$ with $\alpha \neq \beta$ and $\langle \alpha, \beta \rangle \neq 0$. Put $\gamma = w_\alpha \beta$ or $\gamma = -w_\alpha \beta$ so that $\gamma \in \Sigma(D_n)^+$. Then [O2] shows that $u(t) = u_\alpha(t)$, $v(t) = u_\beta(t)$ and $w(t) = u_\gamma(-t)$ satisfy (9.2). Then Remark 9.3 says that the origin is at most a pole of $u(\log s)$, $v(\log s)$ and $w(\log s)$.

Let $\alpha \in S \cap \Sigma(D_n)$ and $\beta \in S \setminus \Sigma(D_n)$ with $\langle \alpha, \beta \rangle \neq 0$. Let W be the reflection group generated by w_α and w_β and put $S^o = W\{\alpha, \beta\} \cap \Sigma(B_n)$. Then [O2] shows that

$$R(x) = \sum_{\gamma \in S^o} u_\gamma(\langle \gamma, x \rangle)$$

defines an integrable potential function of type B_2. Hence Lemma 9.19 assures that the origin is at most a pole of $u_\alpha(\log s)$ for $\alpha \in S^o$.

Since S is irreducible, the origin is at most a pole of $u_\alpha(\log s)$ for $\alpha \in S$. Then Conjecture 9.1 follows from [O2]. □

Remark 9.20. The integrable systems classified in this note which satisfy the assumption of Theorem 9.17 under a suitable coordinate system are (Ellip-$*$) and (Trig-$*$) and (Toda-$*$). But (Rat-$*$) does not satisfy it.

9.5. **Uniqueness.** We give some remarks on the operator which commutes with the Schrödinger operator P.

Remark 9.21 ([OS, Lemma 3.1 ii)]). If a differential operators Q and Q' satisfy $[Q,Q']=0$, $\sigma(Q')=\sum_{j=1}^n \xi_j^N$ and $\mathrm{ord}(Q)\le N-2$, then Q has a constant principal symbol, that is, $\sigma(Q)$ does not depend on x.

Hence if there exist differential operators $Q_1,\dots,Q_n$ with constant principal symbols such that $\sigma(Q_1),\dots,\sigma(Q_n)$ are algebraically independent and moreover they satisfy $[Q_i,Q_j]=0$ for $1\le i<j\le n$, then any operator Q satisfying $[Q,Q_j]=0$ for $j=1,\dots,n$ has a constant principal symbol. In particular, if a differential operator Q satisfies $[Q,P_k]=0$ for P_k in (1.2) and (1.4) with $k=1,\dots,n$, then $\sigma(Q)$ does not depend on x.

Remark 9.22. Assume that a differential operator Q commutes with a Schrödinger operator P and moreover assume that there exist linearly independent vectors $c_j\in\mathbb{C}^n$ for $j=1,\dots,n$ such that the operators are invariant under the parallel transformations $x\mapsto x+c_j$ for $j=1,\dots,n$. Then $\sigma(Q)$ does not depend on x (cf. [OS, Lemma 3.1 i)]).

Furthermore assume that P is of type (Ellip-F) or (Trig-F) or (Rat-F) with $F=A_{n-1}$ or B_n or D_n. If the condition (9.11) holds or Q is $W(F)$-invariant, it follows from Theorem 9.16 or [O1, Proposition 3.6] that Q is in the ring $\mathbb{C}[P_1,\dots,P_n]$ generated by the $W(F)$-invariant commuting differential operators. If the condition (8.11) is not valid, $\sigma(Q)$ is not necessarily $W(F)$-invariant (cf. [CV], [VSC]).

Remark 9.23 ([OS, Theorem 3.2]). Let P be the Schrödinger operator in Theorem 9.10. Under the notation in Theorem 9.10 suppose P_k are W-invariant for $1\le k\le n$. Then the ring $\mathbb{C}[P_1,\dots,P_n]$ is uniquely determined by P and Q, where $Q=P_3$ if $W=W(A_{n-1})$ and $Q=P_2$ if $W=W(B_n)$ or $W(D_n)$.

Remark 9.24. If $P_c=-\frac12\sum_{j=1}^n\partial_j^2+cR(x)$ be a Schrödinger operator with a parameter $c\in\mathbb{C}$ such that P_c admits a commuting differential operator Q_c of order four for any $c\in\mathbb{C}$, then the operator P_c may be a system stated in Conjecture 9.1.

The following example does not satisfy this condition nor the condition (9.11). It does not admit commuting differential operators (1.2) satisfying (1.3) if $m\neq 0$.

Example 9.25. It is shown in [CFV] that the Schrödinger operator

$$P=-\frac12\sum_{j=1}^n\frac{\partial^2}{\partial x_j^2}+\sum_{1\le i<j<n}\frac{m(m+1)}{(x_i-x_j)^2}+\sum_{i=1}^{n-1}\frac{m+1}{(x_i-\sqrt{m}x_n)^2}\tag{9.15}$$

is completely integrable for any m and *algebraically integrable* if m is an integer.

The following example shows that the Shrödinger operator P does not necessarily determine the commuting system $\mathbb{C}[P_1,\dots,P_n]$.

Example 9.26. Let α, β, γ and λ be complex numbers. Put $(A_0,A_1,C_0,C_1)=(\alpha,\frac{\gamma}{2}-\frac{\lambda}{2},\beta,\lambda)$ for (Rat-B_2-S) in Theorem 4.7 (cf. [OS, Remark 3.7]). Then the Schrödinger operator

$$P_{\alpha,\beta,\gamma}=-\frac12\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big)+(x^2+y^2)\Big(\frac{2\alpha}{(x^2-y^2)^2}+\frac{\beta}{x^2y^2}+\gamma\Big)\tag{9.16}$$

commutes with

$$
(9.17)\quad \begin{aligned}
Q_{\alpha,\beta,\gamma,\lambda} = \Big(&\frac{\partial^2}{\partial x\partial y} + \frac{4\alpha xy}{(x^2-y^2)^2} - 2(\gamma-\lambda)xy\Big)^2 \\
&- 2\Big(\frac{\beta}{y^2}+\lambda y^2\Big)\frac{\partial^2}{\partial x^2} - 2\Big(\frac{\beta}{x^2}+\lambda x^2\Big)\frac{\partial^2}{\partial y^2} \\
&+ 4\Big(\frac{\beta}{x^2}+\lambda x^2\Big)\Big(\frac{\beta}{y^2}+\lambda y^2\Big) \\
&+ \frac{16\alpha\lambda x^2y^2+16\alpha\beta}{(x^2-y^2)^2} + 8\lambda(\gamma-\lambda)x^2y^2
\end{aligned}
$$

for any $\lambda\in\mathbb{C}$. Note that $[Q_{\alpha,\beta,\gamma,\lambda}, Q_{\alpha,\beta,\gamma,\lambda'}]\neq 0$ if $\lambda\neq\lambda'$ and these operators are $W(B_2)$-invariant. The half of the coefficient of the term λ of $Q_{\alpha,\beta,\gamma,\lambda}$ considered as a polynomial function of λ equals

$$
S_{\alpha,\beta,\gamma} = -\Big(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\Big)^2 + 2\alpha\Big(\frac{xy}{(x-y)^2} - \frac{xy}{(x+y)^2}\Big) + 2\beta\Big(\frac{y^2}{x^2}+\frac{x^2}{y^2}\Big) + 4\gamma x^2y^2.
$$

In particular, $P = -\frac{1}{2}(\partial_x^2+\partial_y^2) + \gamma(x^2+y^2)$ commutes with $\partial_x\partial_y - 2\gamma xy$ and $x\partial_y - y\partial_x$.

Note that if $R(x)$ is a polynomial function on $\mathbb{C}^n$, the condition $[-\frac{1}{2}\sum_{j=1}^n \partial_j^2 + R(x), Q] = 0$ for a differential operator Q implies that the coefficients of Q are polynomial functions (cf. [OS, Lemma 3.4]).

9.6. **Regular singularities.**

Definition 9.27 ([KO]). Put $\vartheta_k = t_k\frac{\partial}{\partial t_k}$ and $Y_k = \{t=(t_1,\ldots,t_n)\in\mathbb{C}^n\,;\, t_k=0\}$. Then a differential operator Q of the variable t is said to have *regular singularities* along the set of walls $\{Y_1,\ldots,Y_n\}$ if

$$
(9.18)\qquad Q = q(\vartheta_1,\ldots,\vartheta_n) + \sum_{k=1}^n t_kQ_k(t,\vartheta).
$$

Here q is a polynomial of n variables and Q_k are differential operators with the form

$$
Q_k(t,\vartheta) = \sum a_\alpha(t)\vartheta_1^{\alpha_1}\cdots\vartheta_n^{\alpha_n}
$$

and $a_\alpha(t)$ are analytic at $t=0$. In this case we define

$$
(9.19)\qquad \sigma_*(Q) = q(\xi_1,\ldots,\xi_n)
$$

and $\sigma_*(Q)$ is called the *indicial polynomial* of Q.

Theorem 9.28. *Let $R(t)$ be a holomorphic function defined on a neighborhood of the origin of $\mathbb{C}^n$. Let Q_1 and Q_2 be differential operators of t which have regular singularities along the set of walls $\{Y_1,\ldots,Y_n\}$. Suppose $\sigma_*(Q_1)=\sigma_*(Q_2)$ and $[Q_1,P]=[Q_2,P]=0$ with the Schrödinger operator*

$$
(9.20)\qquad P = -\frac{1}{2}\Big(\vartheta_n^2 + \sum_{k=1}^{n-1}(\vartheta_{j+1}-\vartheta_j)^2\Big) + R(t).
$$

Then $Q_1=Q_2$.

Proof. Put $t_j = e^{-(x_j-x_{j+1})}$ for $j=1,\ldots,n-1$ and $t_n = e^{-x_n}$. Then $\partial_j = \vartheta_{j+1}-\vartheta_j$ for $j=1,\ldots,n-1$ and $\partial_n = -\vartheta_n$. Under the coordinate system $x=(x_1,\ldots,x_n)$ Remark 9.22 says that Q_1-Q_2 has a constant principal symbol, which implies $Q_1=Q_2$ because $\sigma_*(Q_1-Q_2)=0$. □

The following corollary is a direct consequence of this theorem.

Corollary 9.29. *Put* $t_j = e^{-\lambda(x_j - x_{j+1})}$ *for* $j = 1, \ldots, n$ *and* $t_n = e^{-\lambda x_n}$. *Suppose* P *is the Schrödinger operator of type* (Trig-A_n), (Trig-A_n-bry-reg), (Trig-BC_n-reg), (Trig-D_n), (Toda-A_n), (Toda-BC_n) *or* (Toda-D_n).

i) P *and* P_k *for* $k = 1, \ldots, n$ *have regular singularities along the set of walls* $\{Y_1, \ldots, Y_n\}$.

ii) *Let* Q *be a differential operator which has regular singularities along the set of walls* $\{Y_1, \ldots, Y_n\}$ *and satisfies* $[Q, P] = 0$. *If* $\sigma_*(Q) = \sigma_*(\tilde{Q})$ *for an operator* $\tilde{Q} \in \mathbb{C}[P_1, \ldots, P_n]$, *then* $Q = \tilde{Q}$.

Hierarchy starting from (Trig-BC_n-reg)

$$\begin{array}{ccccc}
 & & & & \text{Toda-}A_{n-1} \\
 & & & \nearrow & \\
\text{Rat-}D_n & & \text{Trig-}A_{n-1} & \rightarrow & \text{Rat-}A_{n-1} \\
\Uparrow_{3:1} & & \Uparrow_{3:1} & & \Uparrow_{3:1} \\
\text{Rat-}B_n\text{-}2 & & \text{Trig-}A_{n-1}\text{-bry-reg} & \rightarrow & \text{Rat-}A_{n-1}\text{-bry2} \\
\uparrow & \nearrow & & & \\
\text{Trig-}BC_n\text{-reg} & \rightarrow & \text{Toda-}D_n\text{-bry} & \rightarrow & \text{Toda-}BC_n \\
\Downarrow_{3:1} & & \Downarrow_{3:1} & & \Downarrow_{3:1} \\
\text{Trig-}D_n & \rightarrow & \text{Toda-}D_n & \rightarrow & \text{Toda-}A_{n-1} \\
\downarrow & \searrow & & & \\
\text{Trig-}A_{n-1} & & \text{Rat-}D_n & &
\end{array}$$

9.7. **Other forms.** If a Schrödinger operator P is in the commutative algebra $\mathbb{D} = \mathbb{C}[P_1, \ldots, P_n]$, then the differential operator $\tilde{P} := \psi(x)^{-1} P \circ \psi(x)$ with a function $\psi(x)$ is in the commutative algebra $\tilde{\mathbb{D}} = \mathbb{C}[\psi(x)^{-1}P_1 \circ \psi(x), \ldots, \psi(x)^{-1}P_n \circ \psi(x)]$ of differential operators. If

$$\tilde{P} = -\frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2}{\partial x_j^2} + \sum_{j=1}^{n} a_j(x)\frac{\partial}{\partial x_j} + \tilde{R}(x), \tag{9.21}$$

then

$$\frac{\partial \psi(x)}{\partial x_j} = a_j(x) \quad \text{for} \quad j = 1, \ldots, n. \tag{9.22}$$

Conversely, if a function $\psi(x)$ satisfies (9.22) for a differential operator $\tilde{P}$ of the form (9.21), then $P = \psi(x)\tilde{P} \circ \psi(x)^{-1}$ is of the form (1.1), which we have studied in this note.

If $\psi(x)$ is a function satisfying

$$\frac{1}{2\psi(x)}\sum_{j=1}^{n} \frac{\partial^2 \psi}{\partial x_j^2}(x) = R(x), \tag{9.23}$$

then

$$\tilde{P} = \psi(x)^{-1}\Big(-\frac{1}{2}\sum_{j=1}^{n}\partial_j^2 + R(x)\Big)\circ\psi(x) = -\frac{1}{2}\sum_{j=1}^{n}\partial_j^2 - \psi(x)^{-1}\sum_{j=1}^{n}\frac{\partial\psi}{\partial x_j}(x)\partial_j. \tag{9.24}$$

Note that

$$\begin{aligned}
e^{-\phi(x)}\frac{\partial e^{\phi(x)}}{\partial x_j} &= \frac{\partial \phi(x)}{\partial x_j}, \\
e^{-\phi(x)}\sum_{j=1}^{n}\frac{\partial^2 e^{\phi(x)}}{\partial x_j^2} &= \sum_{j=1}^{n}\frac{\partial^2\phi(x)}{\partial x_j^2} + \sum_{j=1}^{n}\Big(\frac{\partial\phi(x)}{\partial x_j}\Big)^2,
\end{aligned} \tag{9.25}$$

Putting

$$\phi(x) = m \sum_{1\le i<j\le n} \log\sinh\lambda(x_i - x_j),$$

we have

$$\begin{aligned}
\frac{\partial\phi(x)}{\partial x_k} &= \lambda m \sum_{1\le i\le n,\, i\ne k} \coth\lambda(x_k - x_i),\\
\sum_{j=1}^n \frac{\partial^2\phi(x)}{\partial x_j^2} + \sum_{k=1}^n \Big(\frac{\partial\phi(x)}{\partial x_k}\Big)^2 &= -2\lambda^2 m \sum_{1\le i<j\le n} \sinh^{-2}\lambda(x_i - x_j)\\
&\quad + 2\lambda^2 m^2 \sum_{1\le i<j\le n} \coth^2\lambda(x_i - x_j) + \lambda^2 m^2 \frac{n(n-1)(n-2)}{3}\\
&= 2\lambda^2 m(m-1) \sum_{1\le i<j\le n} \sinh^{-2}\lambda(x_i - x_j) + \lambda^2 m^2 \frac{n(n^2-1)}{3}
\end{aligned}$$

since

$$\coth\alpha\cdot\coth\beta + \coth\beta\cdot\coth\gamma + \coth\gamma\cdot\coth\alpha = -1 \quad \text{if} \quad \alpha+\beta+\gamma = 0.$$

Hence

$$\tilde P = -\frac12 \sum_{j=1}^n \partial_j^2 - m \sum_{1\le i<j\le n} \lambda\coth\lambda(x_i - x_j)(\partial_i - \partial_j), \tag{9.26}$$

$$\psi(x) = \prod_{1\le i<j\le n} \lambda^m \sinh^m\lambda(x_i - x_j), \tag{9.27}$$

$$\psi(x)\circ\tilde P\circ\psi^{-1}(x) = -\frac12\sum_{j=1}^n \partial_j^2 + \sum_{1\le i<j\le n} \frac{m(m-1)\lambda^2}{\sinh^2\lambda(x_i - x_j)} + \frac{m^2 n(n^2-1)\lambda^2}{6}$$

and $\tilde P$ is transformed into the Schrödinger operator of type (Trig-A_{n-1}).

Now we put

$$\begin{aligned}
\phi(x) = m_0 &\sum_{1\le i<j\le n} \big(\log\sinh\lambda(x_i - x_j) + \log\sinh\lambda(x_i + x_j)\big)\\
&+ m_1 \sum_{1\le k\le n} \log\sinh\lambda x_k + m_2 \sum_{1\le k\le n} \log\sinh 2\lambda x_k
\end{aligned}$$

and we have

$$\begin{aligned}
&\frac{\partial\phi(x)}{\partial x_k} = \lambda m_0 \sum_{1\le i\le n,\, i\ne k} \big(\coth\lambda(x_k + x_i) + \coth\lambda(x_k - x_i)\big)\\
&\quad + \lambda m_1 \coth\lambda x_k + 2\lambda m_2 \coth 2\lambda x_k,\\
&\coth\lambda x_k\cdot\coth 2\lambda x_k = 1 + \tfrac12\sinh^{-2}\lambda x_k,\\
&\sum_{\{i,j,k\}=I} \Big(2\coth\lambda(x_k + x_i)\cdot\coth\lambda(x_k - x_i) + 2\coth\lambda(x_k + x_i)\cdot\coth\lambda(x_k - x_j)\\
&\quad + \coth\lambda(x_k + x_i)\cdot\coth\lambda(x_k + x_j) + \coth\lambda(x_k - x_i)\cdot\coth\lambda(x_k - x_j)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\{i,j,k\}=I} \Big(\coth\lambda(x_k+x_i)\cdot\coth\lambda(x_k+x_j) \\
&\quad + \coth\lambda(x_i-x_j)\cdot\coth\lambda(x_i+x_k) + \coth\lambda(x_j-x_i)\cdot\coth\lambda(x_j+x_k)\Big) \\
&\quad + \sum_{\{i,j,k\}=I} \Big(\coth\lambda(x_k-x_i)\cdot\coth\lambda(x_k-x_j)\Big) \\
&\quad = 8 \qquad \text{for} \quad I\subset\{1,\dots,n\} \text{ with } \#I=3,
\end{aligned}
$$

$$
\coth\lambda(x_k+x_i)+\coth\lambda(x_k-x_i) = \frac{\sinh 2\lambda x_k}{\sinh\lambda(x_k+x_i)\cdot\sinh\lambda(x_k-x_i)},
$$

$$
\frac{\cosh 2\lambda x_k - \cosh 2\lambda x_i}{\sinh\lambda(x_k+x_i)\cdot\sinh\lambda(x_k-x_i)} = \frac{2\cosh^2\lambda x_k - 2\cosh^2\lambda x_i}{\sinh\lambda(x_k+x_i)\cdot\sinh\lambda(x_k-x_i)} = 2,
$$

$$
\begin{aligned}
\sum_{k=1}^n \Big(\frac{\partial\phi(x)}{\partial x_k}\Big)^2 &= 2\lambda^2 m_0^2 \sum_{1\le i<j\le n} \big(\coth^2\lambda(x_i-x_j)+\coth^2\lambda(x_i+x_j)\big) \\
&\quad + \lambda^2 m_1^2 \sum_{k=1}^n \coth^2\lambda x_k + 4\lambda^2 m_2^2 \sum_{k=1}^n \coth^2 2\lambda x_k + 2\lambda^2 m_1 m_2 \sum_{k=1}^n \sinh^{-2}\lambda x_k \\
&\quad + \frac{4\lambda^2 m_0^2 n(n-1)(n-2)}{3} + 2\lambda^2 m_0(m_1+2m_2)n(n-1) + 4\lambda^2 m_1 m_2 n,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{j=1}^n \frac{\partial^2\phi(x)}{\partial x_j^2} + \sum_{k=1}^n \Big(\frac{\partial\phi(x)}{\partial x_k}\Big)^2 \\
&\quad = 2\lambda^2 m_0(m_0-1) \sum_{1\le i<j\le n} \big(\sinh^{-2}\lambda(x_i-x_j)+\sinh^{-2}\lambda(x_i+x_j)\big) \\
&\qquad + \lambda^2 m_1(m_1+2m_2-1)\sum_{k=1}^n \sinh^{-2}\lambda x_k + 4\lambda^2 m_2(m_2-1)\sum_{k=1}^n \sinh^{-2} 2\lambda x_k \\
&\qquad + \lambda^2\big((\tfrac{2}{3}m_0(2n-1)+2m_1+4m_2)m_0(n-1)+(m_1+2m_2)^2\big)n.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\tilde P = -\frac12\sum_{j=1}^n \partial_j^2 - \sum_{k=1}^n \lambda\Big(&\sum_{1\le i<j\le n} m_0\big(\coth\lambda(x_i-x_k)+\coth\lambda(x_i+x_k)\big) \\
&+ m_1\coth\lambda x_k + 2m_2\coth 2\lambda x_k\Big)\partial_k,
\end{aligned}
\tag{9.28}
$$

$$
\begin{aligned}
\psi(x) = &\prod_{1\le i<j\le n} \Big(\sinh^{m_0}\lambda(x_i-x_j)\cdot\sinh^{m_0}\lambda(x_i+x_j)\Big) \\
&\cdot \prod_{k=1}^n \sinh^{m_1}\lambda x_k \cdot \prod_{k=1}^n \sinh^{m_2} 2\lambda x_k
\end{aligned}
\tag{9.29}
$$

and $\tilde P$ is transformed into the Schrödinger operator of type (Trig-BC_n-reg):

$$
\begin{aligned}
&\psi(x)\circ\tilde P\circ\psi^{-1}(x) \\
&\quad = -\frac12\sum_{j=1}^n \partial_j^2 + m_0(m_0-1)\sum_{1\le i<j\le n}\Big(\frac{\lambda^2}{\sinh^2\lambda(x_i-x_j)}+\frac{\lambda^2}{\sinh^2\lambda(x_i+x_j)}\Big) \\
&\qquad + \sum_{k=1}^n \frac{m_1(m_1+2m_2-1)\lambda^2}{2\sinh^2\lambda x_k} + \sum_{k=1}^n \frac{2m_2(m_2-1)\lambda^2}{\sinh^2 2\lambda x_k} \\
&\qquad + \lambda^2\Big(\frac{m_0^2}{3}(2n-1)(n-1)+m_0(m_1+2m_2)(n-1)+\frac{(m_1^2+2m_2)^2}{2}\Big)n.
\end{aligned}
$$

Remark 9.30. As is shown in [He, Theorem 5.24 in Ch. II], the operator (9.26) or (9.28) gives the radial part of the differential equation satisfied by the zonal spherical function of a Riemannian symmetric space G/K of the non-compact type which corresponds to the Laplace-Beltrami operator on G/K. Here G is a real connected semisimple Lie group with a finite center, K is a maximal compact subgroup of G and $2m$, $2m_0$, $2m_1$ and $2m_2$ correspond to the multiplicities of the roots of the restricted root system for G.

Similarly the following operator $\tilde{P}$ is for the K-fixed Whittaker vector v on $G = GL(n,\mathbb{R})$.

$$\tilde{P} = -\frac{1}{2}\sum_{j=1}^{n}\partial_j^2 + \sum_{j=1}^{n}\Big(\frac{n+1}{4} - \frac{j}{2}\Big)\partial_j - \sum_{j=1}^{n-1} e^{2(x_j - x_{j+1})}, \tag{9.30}$$

$$\psi(x) = e^{\sum_{j=1}^{n}(\frac{j}{2} - \frac{n+1}{4})x_j}, \tag{9.31}$$

$$\psi(x)\circ\tilde{P}\circ\psi^{-1}(x) = -\frac{1}{2}\sum_{j=1}^{n}\partial_j^2 - \sum_{j=1}^{n-1} e^{2(x_j - x_{j+1})} + \frac{n(n^2-1)}{48}.$$

Namely v is a simultaneous eigenfunction of the invariant differential operators on G/K and satisfies $v(nx) = \chi(n)v(x)$ with $n \in N$ and $x \in G/K$. Here $G = KAN$ is an Iwasawa decomposition of G and χ is a nonsingular character of the nilpotent Lie group N. Then $v|_A$ is a simultaneous eigenfunction of the commuting algebra of differential operators determined by $\tilde{P}$.

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The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

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