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A boundary integral method for solving inverse heat conduction problem

by

X. Z. Jia and Y. B. Wang

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

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X. Z. Jia
021018038@fudan.edu.cn

Y. B. Wang
ybwang@fudan.edu.cn

School of Mathematical Sciences, Fudan University
Shanghai 200433, CHINA

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Abstract In this paper, a boundary integral method is used to solve an inverse heat conduction problem. An algorithm for the inverse problem of the one dimensional case is given by using the fundamental solution. Numerical results show that our algorithm is effective.

Keywords: ill-posed problem, boundary integral method;

1 Introduction

The inverse heat conduction problem (IHCP) arising in most thermal manufacturing processes has recently attracted much attention. The typical case is the determination of the heat flux on an inaccessible boundary through measurements on an accessible boundary or inside the domain. Only discrete data with noise at finite points are usually available for solving this problem ([10]). This problem is known to be extremely ill-posed ([1]): a small perturbation in the data may cause a dramatically large error in the solution.

This problem has been studied by many authors, for example: an approximate inverse method([14]), a residual-minimization lease squares method([9]), a boundary element method ([15]), a fundamental solution method([5], [13]), a wavelet and Fourier method([8]) and some other methods([3], [4], [7], [16], [17]). In this paper, we use the boundary integral method(BIM) to solve this problem. The BIM uses the prescribed initial and boundary data, together with the fundamental solution of a given differential equation defined in some domain Ω, and we construct integral equations on the boundary of Ω. In our case, the solution to the integral equation is a single layer potential. By the boundary integral equation one can obtain the unknown kernel, and the solution to the given problem will be obtained by integrating the product of the fundamental solution and the unknown kernel over the boundary.

The advantage of our approach is that the computation can be limited to the boundary, which reduces the problem from two dimensions to one dimension. As a result of the reduction, we may expect substantial savings in computer time and memory.

The work outlined below is based on the use of single layer potentials. Ammari and Kang used boundary integral method to solve inverse conductivity problem and related problems[2]. In [2], both the single layer potential and the double layer potential are used. In this article, our boundary integral method is based on the result of [11], which gives a representation formula for the heat conduction problem with Neumann boundary condition. The equation is assumed to be homogeneous.

As the trade-off we must execute integration for each point at which we want to know the solution. However, in many applications such as semiconductor fabrication, one only need to know the solution at a few points. In such cases, we expect our technique to be useful.

2 The Mathematical Problem

2.1 The direct problem

We consider the following problem:

$$u_t - a^2 u_{xx} = 0, \quad 0 \le x \le 1, \quad 0 \le t \le T, \tag{2.1}$$

$$u(x,0) = u_0(x), \quad 0 \le x \le 1, \tag{2.2}$$

$$\frac{\partial u}{\partial x}(0,t) = f(t), \quad 0 \le t \le T, \tag{2.3}$$

$$\frac{\partial u}{\partial x}(1,t) = g(t), \quad 0 \le t \le T, \tag{2.4}$$

where $a > 0$ is a constant and $u(x,t)$ is the temperature distribution.

Solving the equation with given $f(t), g(t)$ is called a direct problem.

From the theory of heat equation, we can see that for $f(t)$ and $g(t)$ in some function space there exists a unique solution ([6]).

2.2 The inverse problem

Let

$$u_t - a^2 u_{xx} = 0, \quad 0 \le x \le 1, \quad 0 \le t \le T, \tag{2.5}$$

$$u(x,0) = u_0(x), \quad 0 \le x \le 1, \tag{2.6}$$

$$\frac{\partial u}{\partial x}(1,t) = g(t), \quad 0 \le t \le T, \tag{2.7}$$

$$u(x^*,t) = \ell(t), \quad 0 \le t \le T, \tag{2.8}$$

where $x^* \in (0,1)$ is a fixed observation point.

The inverse problem is then to determine the value of $\frac{\partial u}{\partial x}(0,t) = f(t)$ and $u(x,t)$ from $g(t)$ and $\ell(t)$. In real applications, only the discrete values of $g(t)$ and $\ell(t)$ at some points are given. Therefore boundary conditions (2.7) and (2.8) will be changed to the following:

$$u(x^*,t_i) = \ell_i, \frac{\partial u}{\partial x}(1,t_i) = g(t_i) \quad i = 1,2,\cdots,N$$

where $0 < t_1 < t_2 < \cdots < t_N = T$.

This problem is seem to be an ill-posed problem([1]). In fact, if we know $\ell(t)$ and $g(t)$, then we can first solve the heat equation for $(x,t) \in (x^*,1) \times (0,T)$, and then we have to discuss the Cauchy boundary conditions problem for $(x,t) \in (0,x^*) \times (0,T)$. The second part is known to be ill-posed.

3 Boundary integral method

Here after [11], we explain a solution by the boundary integral equation. First we consider the case of homogeneous initial data, i.e., we assume that $u_0(x) \equiv 0$.

Let $G(x,t)$ be the fundamental solution of the heat conduction problem:

$$G(x,t) = \frac{H(t)}{2a\sqrt{\pi t}} exp(-x^2/4a^2 t)$$

where $H(t)$ is the Heaviside function.

For (2.1)-(2.4), we have the following formula according to the single layer potential(e.g. [11]):

$$u(x,t) = \int_0^t G(x, t-\tau)\psi_1(\tau)d\tau + \int_0^t G(x-1, t-\tau)\psi_2(\tau)d\tau. \tag{3.1}$$

Substitute (3.1) into (2.3) and (2.4), and we obtain

$$\frac{1}{2}\psi_1(t) - \int_0^t \frac{\partial G}{\partial x}(-1, t-\tau)\psi_2(\tau)d\tau = f(t) \tag{3.2}$$

$$\frac{1}{2}\psi_2(t) + \int_0^t \frac{\partial G}{\partial x}(1, t-\tau)\psi_1(\tau)d\tau = g(t) \tag{3.3}$$

We solve (3.2) and (3.3) with respect to $\psi_1(t)$ and $\psi_2(t)$, and then we can obtain $u(x,t)$. Thus the direct problem (2.1) - (2.4) is solved.

For inhomogeneous initial data, it can be seen that a function

$$v(x,t) = \int_0^1 G(x-y, t)u_0(y)dy$$

satisfies

$$v_t - a^2 v_{xx} = 0$$

and

$$\lim_{t \to 0+} v(x,t) = u_0(x).$$

Setting $\widetilde{u} = u - v$, we see

$$\begin{aligned}
&\widetilde{u}_t - a^2\widetilde{u}_{xx} = 0, \quad (x,t) \in (0,1) \times (0,T),\\
&\widetilde{u}(x,0) = 0, \quad x \in (0,1),\\
&\frac{\partial \widetilde{u}}{\partial x}(0,t) = f(t) - \frac{\partial v}{\partial x}(0,t), \quad 0 \le t \le T,\\
&\frac{\partial \widetilde{u}}{\partial x}(1,t) = g(t) - \frac{\partial v}{\partial x}(1,t), \quad 0 \le t \le T.
\end{aligned}$$

Since $\widetilde{u}$ is the solution of (2.1)-(2.4) with homogeneous initial data, it can be solved by (3.1)-(3.3). Thus we can solve the problem by the superpositions of v and the solution of homogenous initial data.

We will discretize (3.2) and (3.3) as follows:

Let $t_0 = 0$, $t_i = t_0 + ih$, $i = 1, 2, \cdots, N$ where N is an integer and $h = T/N$. Then we can discretize (3.2) and (3.3) as

$$\frac{1}{2}\psi_1(t_i) + h\sum_{j=1}^{i}\frac{\partial G}{\partial x}(-1, t_i - t_j)\psi_2(t_j) = f(t_i) \quad i = 1, 2, \cdots, N,$$

$$\frac{1}{2}\psi_2(t_i) + h\sum_{j=1}^{i}\frac{\partial G}{\partial x}(1, t_i - t_j)\psi_1(t_j) = g(t_i) \quad i = 1, 2, \cdots, N.$$

Thus the problem of solving integral equations (3.2) and (3.3) is changed to linear equations:

$$(\frac{1}{2}\mathbf{I} + \mathbf{G})\boldsymbol{x} \ = \boldsymbol{f}. \tag{3.4}$$

where we set

$$\mathbf{G} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}, \tag{3.5}$$

$$G_{11} = 0,\ G_{22} = 0,$$

$$G_{12}(i, j) = h\frac{\partial G}{\partial x}(-1, t_i - t_j),\ G_{21}(i, j) = h\frac{\partial G}{\partial x}(1, t_i - t_j), \text{if } j \le i,$$

$$G_{12}(i, j) = G_{21}(i, j) = 0, \text{if } j \ge i + 1.$$

$$\boldsymbol{f} = (\boldsymbol{f}_1, \boldsymbol{f}_2)^T,\ \boldsymbol{f}_1(i) = f(t_i),\ \boldsymbol{f}_2(i) = g(t_i),\ i = 1, 2, \cdots, N,$$

where T denotes the transpose of a vector.

By solving this linear equations, we can obtain $\boldsymbol{x}$, and then

$$\begin{aligned} \psi_1(i) &= \boldsymbol{x}(i), \quad i = 1, 2, \cdots, N, \\ \psi_2(i) &= \boldsymbol{x}(i + N), \quad i = 1, 2, \cdots, N. \end{aligned}$$

Thus the discrete values of $\psi_1(t)$ and $\psi_2(t)$ are obtained.

Next in (3.1), we set $x_0 = 0, x_i = x_0 + ih_x, i = 1, \cdots, M$ where M is an integer and $h_x = \frac{1}{M}, x_M = 1$. Thus we have

$$u(x_i, t_j) = \sum_{k=1}^{j} G(x_i, t_j - t_k) \cdot \psi_1(k) \cdot h + \sum_{k=1}^{j} G(x_i - 1, t_j - t_k) \cdot \psi_2(k) \cdot h$$
$$i = 0, 1, \cdots, M; j = 1, 2, \cdots, N.$$

4 Numerical scheme for the inverse problem

For the inverse problem, our numerical method is based on the boundary integral equation in section 3. As the measured data for inverse problem, the numerical data obtained by solving the direct problem can be used. The inverse problem is then to solve the following problem:

$$
\begin{aligned}
&u_t(x,t) - a^2 u_x x(x,t) = 0, \quad 0 \le x \le 1, \quad 0 \le t \le T, \\
&\frac{\partial u}{\partial x}(1,t) = g(t), \quad 0 \le t \le T, \\
&u(x,0) = 0, \quad 0 \le x \le 1, \\
&u(x^*, t_i) = \ell_i, i = 1, 2, \cdots, N.
\end{aligned}
$$

We use the following equation together with (3.3) to solve the above problem:

$$
u(x^*, t) = \int_0^t G(x^*, t-\tau)\psi_1(\tau)d\tau + \int_0^t G(x^*-1, t-\tau)\psi_2(\tau)d\tau. \tag{4.1}
$$

With (3.3) and (4.1), we have an equation of the first kind with respect to $\psi_1(t)$ and $\psi_2(t)$. Choose $x^* \in (0,1)$, and $u(x^*, t_i), i = 1, 2, \cdots, N$ are our discrete data. Similarly to (3.4), we can discretize the equation of the first kind.

Define

$$
\hat{\mathbf{G}} = \begin{pmatrix} \hat{G}_{11} & \hat{G}_{12} \\ \hat{G}_{21} & \frac{1}{2}\mathbf{I} \end{pmatrix}, \tag{4.2}
$$

$$
\begin{aligned}
\hat{G}_{11}(i,j) &= hG(x^*, t_i - t_j), \ \text{if } j \le i; \hat{G}_{11}(i,j) = 0, \ \text{if } j \ge i+1, \\
\hat{G}_{12}(i,j) &= hG(x^*-1, t_i - t_j), \ \text{if } j \le i; \hat{G}_{12}(i,j) = 0, \ \text{if } j \ge i+1, \\
\hat{G}_{21}(i,j) &= h\frac{\partial G}{\partial x}(-1, t_i - t_j), \ \text{if } j \le i; \hat{G}_{21}(i,j) = 0, \ \text{if } j \ge i+1.
\end{aligned}
$$

$$
\begin{aligned}
\hat{\boldsymbol{f}} &= (\hat{\boldsymbol{f}}_1, \hat{\boldsymbol{f}}_2)^T, \ \hat{\boldsymbol{f}}_1(i) = u(x^*, t_i), \ \hat{\boldsymbol{f}}_2(i) = g(t_i), \ i = 1, 2, \cdots, N, \\
\hat{\boldsymbol{x}} &= (\hat{\boldsymbol{x}}_1, \hat{\boldsymbol{x}}_2)^T, \ \hat{\boldsymbol{x}}_1(i) = \psi_1(t_i), \ \hat{\boldsymbol{x}}_2(i) = \psi_2(t_i), \ i = 1, 2, \cdots, N,
\end{aligned}
$$

Then we have

$$
\hat{\mathbf{G}}\hat{\boldsymbol{x}} = \hat{\boldsymbol{f}}
$$

After calculating ψ_1 and ψ_2 we substitute them into (3.2) and we have

$$
f(t_i) = \frac{1}{2}\psi_1(t_i) - h\sum_{j=1}^{i} \frac{\partial G}{\partial x}(-1, t_i - t_j)\psi_2(t_j), \quad i = 0, 1, \cdots, N.
$$

The approximation of $u(x,t)$ is also obtained:

$$
u(x_i, t_j) = \sum_{k=1}^{j} G(x_i, t_j - t_k)\psi_1(k)h + \sum_{k=1}^{j} G(x_i - 1, t_j - t_k)\psi_2(k)h
$$

$$
i = 0, 1, \cdots, M; j = 1, 2, \cdots, N.
$$

5 Numerical example

Example 1:

We consider the following direct problem:

$$\begin{aligned} & u_t - u_{xx} = 0, \quad 0 < x < 1, \quad 0 < t < 10, \\ & u(x,0) = 0, \\ & \frac{\partial u}{\partial x}(0,t) = 10sin(5t), \quad 0 \leq t \leq 10, \\ & \frac{\partial u}{\partial x}(1,t) = 1, \quad 0 \leq t \leq 10. \end{aligned}$$

We solve this problem by using boundary integral method which is presented in section 3. The parameter is chosen as $M = 50$, $N = 100$. A numerical result is shown in Figure 1:

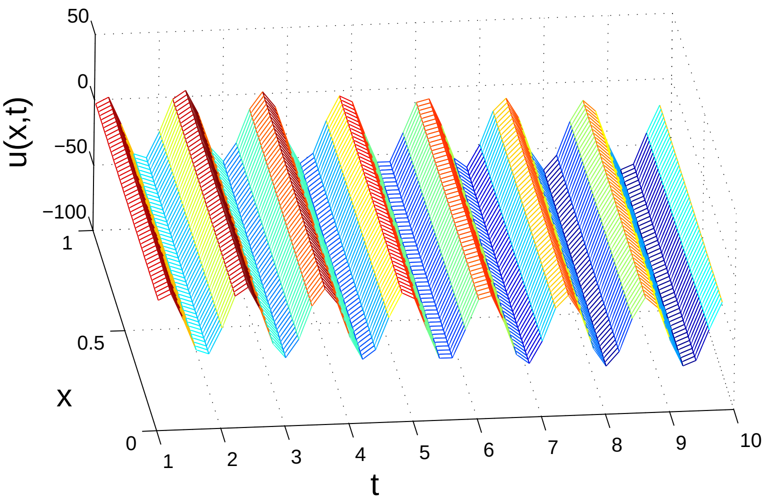

Figure 1: $u(x,t)$

Here x varies from 0 to 1 and t varies from 0 to 10.

Now we solve the inverse problem by values generated numerically by the direct problem. We choose $x^* = 0.95, M = 50, N = 100$. The condition number of the coefficient matrix $\hat{\boldsymbol{G}}$ is approximately 54.3253 and we can consider that the condition number is not large, so that we need not the regularization. If we have a large condition number, which means the resulting linear system is ill-posed, we can apply the Tikhonov regularization method([12]).

We denote the numerical result of the inverse problem as $f_*(t)$. If we do not introduce any random noises to data $u(x^*,t)$, which is generated by the direct problem, then the result is shown in Figure 2:

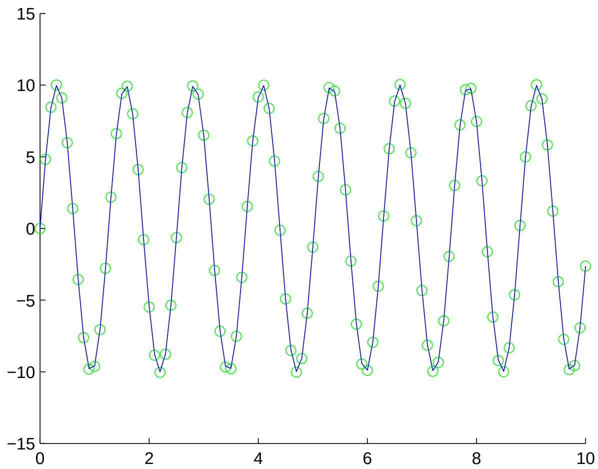

Figure 2: solid line–$f(t)$;   'o'–$f_*(t)$

Next, in order to test the ill-posedness, we add some random noises on the values of $u(x^*, t)$ whose level is at most 0.05. Then the result is shown in Figure 3:

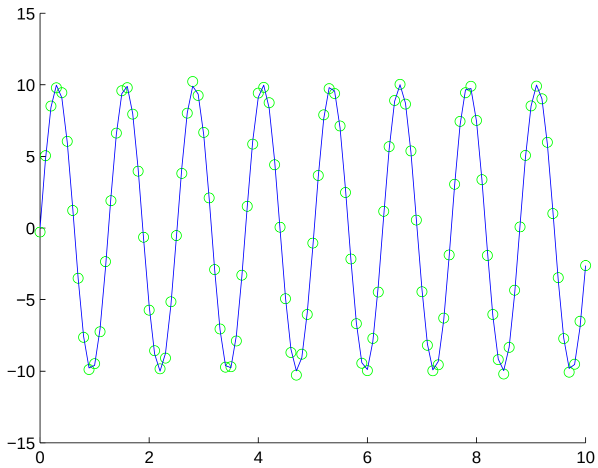

Figure 3: solid line–$f(t)$;   'o'–$f_*(t)$

The following two figures show the errors between $f(t)$ and $f_*(t)$. Figure 4a shows the error without random noise, and Figure 4b shows the error with random noise whose level is at most 0.05.

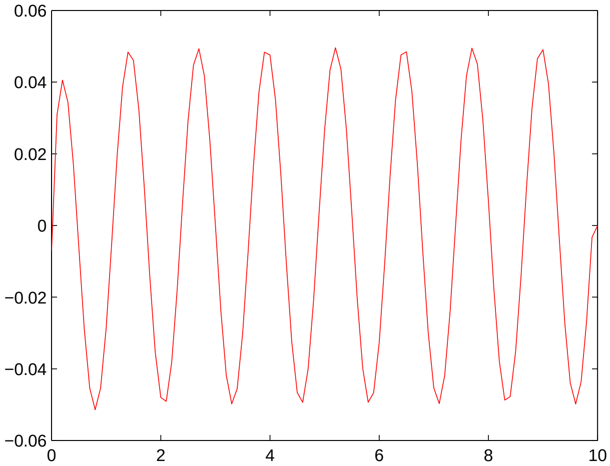

Figure 4a: $f(t) - f_*(t)$ without random noise

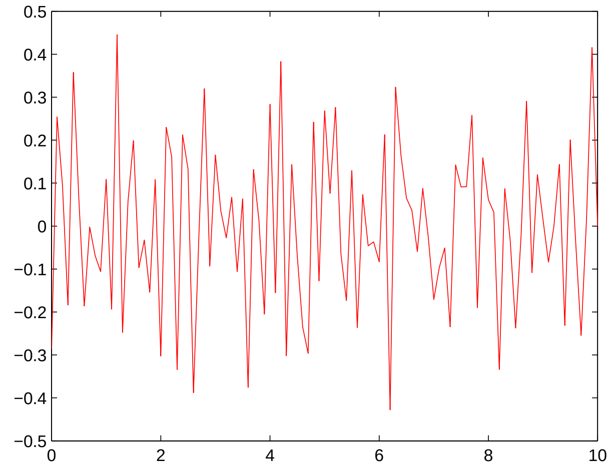

Figure 4b: $f(t) - f_*(t)$ with random noise

Example 2:

In this example we choose a piecewise continuous function $f(t)$ for problem (2.1)-(2.4). We choose $g(t) = 1$, and $f(t)$ is defined as follows:

$$f(t) = \begin{cases} 1, & 0 \leq t \leq 4, \\ -1, & 4 < t \leq 10. \end{cases}$$

First we use the boundary integral method to obtain the solution for the direct problem. Then we use these numerical values as measured data for the inverse problem. We choose $x^* = 0.5, M = 50, N = 50$. Then the condition number of the coefficient matrix $\hat{\boldsymbol{G}}$ is 48.8323. The numerical result for the inverse problem is denoted as $f_*(t)$.

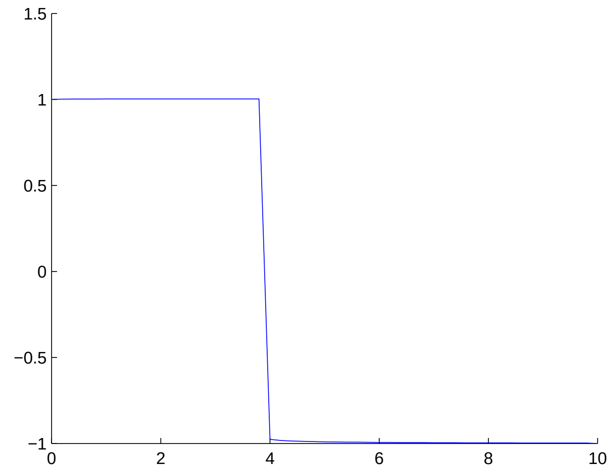

Figure 5a: $f_*(t)$

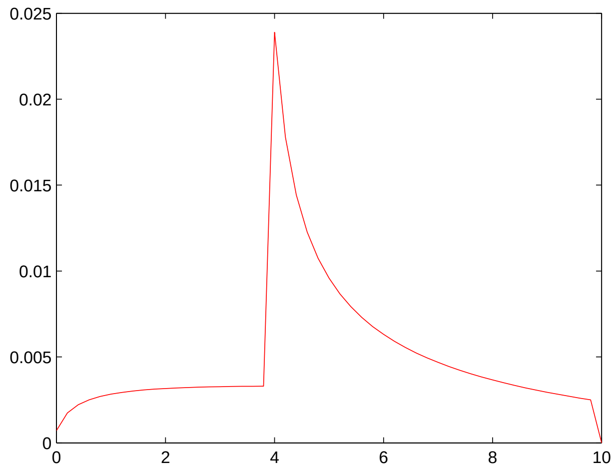

Figure 5: $f(t) - f_*(t)$

Figure 5a is the numerical result for $f(t)$, and the error between exact $f(t)$ and numerical result $f_*(t)$ is shown in Figure 5b.

From Figure 5, we can see that the error near the discontinuous point $t = 4$ is larger but still acceptable, and the error at other points are quite good.

6 Conclusion

In this paper, we discussed the boundary integral method for the one-dimensional inverse heat conduction problem. We presented an algorithm for the inverse problem. Numerical results show that this method is effective, even for noisy data and discontinuous functions. The algorithm for two-dimensional cases is to be considered.

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ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012