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INFINITESIMAL DERIVATIVE OF THE BOTT CLASS AND THE SCHWARZIAN DERIVATIVES

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Abstract. The Bott class is a cohomological invariant for transversely holomorphic foliations which can vary continuously as foliations are deformed. In this article, an infinitesimal derivative of the Bott class of transversely holomorphic foliations is defined by generalizing Heitsch's construction. It will be shown that the infinitesimal derivatives are expressed in terms of a kind of curvature tensors of the projective Schwarzian derivatives. Our formula is also valid for the Godbillon-Vey class of real foliations and it is a generalization of the Maszczyk formula for the Godbillon-Vey class of real codimension-one foliations. As an application, some properties of the Julia-Fatou decompositions due to Ghys, Gomez-Mont and Saludes will be discussed.

Introduction

The Bott class is a secondary characteristic class of transversely holomorphic foliations defined in a similar manner as the Godbillon-Vey class. One of the most important properties of these classes is that they can vary continuously if foliations are deformed. If an infinitesimal deformation is given, then one can naturally define the derivative of the Godbillon-Vey class with respect to it. An explicit construction was given by Heitsch [12], [13]. We call in this article such derivatives *infinitesimal derivatives.* An infinitesimal derivative of the Bott class is already defined by Heitsch if the complex normal bundle is trivial [13]. He has also constructed in the same paper the infinitesimal derivative of the imaginary part of the Bott class without assuming the triviality of normal bundles. When applications are considered, infinitesimal derivatives of the Bott class will be important in relationship

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with localizations of the Bott class and with the Julia sets in the sense of Ghys, Gomez-Mont and Saludes [11] when the complex codimension is equal to one, and in relationship to the Futaki invariant [9], [10] when the complex codimension is greater than one. In the both cases, it is necessary to generalize infinitesimal derivatives for arbitrary transversely holomorphic foliations. In the present paper, we will first give such a generalization and some applications concerning the Fatou-Julia decomposition will be discussed, while applications concerning Futaki invariants will be discussed elsewhere.

On the other hand, Maszczyk showed in [17] that infinitesimal derivatives of the Godbillon-Vey class of real codimension-one foliations can be written in terms of the Schwarzian derivative and cohomology classes representing infinitesimal deformations. His formula is also valid for the Bott class for transversely holomorphic foliations of complex codimension one. We will give in this article a version of the Maszczyk formula for arbitrary transversely holomorphic foliations. Instead of the classical Schwarzian derivative, multi-dimensional projective Schwarzian will appear when complex codimension is greater than one. Multi-dimensional Schwarzian derivatives and derived tensors which we will need are studied by several authors, e.g., [16], [19], [21], [18], [5], [20]. The Čech-de Rham cohomology is useful in the construction, because the Bott class is a cohomology class with coefficients in $\boldsymbol{C}/\boldsymbol{Z}$ so that the usual de Rham cohomology does not work very well. We will try to avoid the use of partitions of unity so far as possible, because it often makes difficult to calculations in examples. The Čech-de Rham cohomology is also useful for this purpose. Finally, we remark that our constructions are also valid for real foliations and that corresponding formulae for the Godbillon-Vey can be obtained.

This paper is organized as follows. In the first section, relevant definitions on transversely holomorphic foliations and basic tools treating the Bott class are recalled. In the second section, an infinitesimal derivative of the Bott class generalizing the previous ones will be introduced. In the third section, the projective Schwarzian derivatives will be introduced. In the forth section, a kind of Maszczyk formula will be shown. It will be also shown that the infinitesimal derivative is obtained as a result of composition of certain operators. In the fifth section, two kinds of residues are introduced. One is after Heitsch, and the other one is defined by using transverse projective structures. In the sixth section, complex codimension-one foliations are studied in relationship with the Julia-Fatou decomposition due to Ghys, Gomez-Mont and Saludes [11]. Finally in the seventh section, some examples are given.

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1. Relevant definitions

The Bott class is usually considered when the complex normal bundle of the foliation is trivial, and it is an element of $H^{2q+1}(M;\boldsymbol{C})$. Its deformations are studied by Heitsch in [13]. In this section, we will show that his construction can be adapted also in the case where the complex normal bundle is non-trivial so that the Bott class is an element of $H^{2q+1}(M;\boldsymbol{C}/\boldsymbol{Z})$.

We first briefly recall a construction in [2] of a representative of the Bott class in terms of the Čech-de Rham complex. In this paper, a manifold will always mean a smooth manifold without boundary. We begin with some relevant definitions.

Definition 1.1. A foliation $\mathcal{F}$ of a manifold is said to be transversely holomorphic if there is an open covering $\mathcal{U}=\{U_i\}$ of M with the following properties:

1) Each U_i is homeomorphic to $V_i \times D^{2q}$, where V_i is an open subset of $\boldsymbol{R}^p$ and D^{2q} is an open ball in $\boldsymbol{C}^q$ $(p+2q=\dim M)$.
2) The foliation restricted to U_i is given by $\{V_i \times \{z\}\}$, $z \in D^{2q}$.
3) Under the identification in 1), the transition function φ_{ji} from U_i to U_j is of the form $\varphi_{ji}(x,z)=(\psi_{ji}(x,z),\gamma_{ji}(z))$, where γ_{ji} is a local biholomorphic diffeomorphism.

Such an open covering $\mathcal{U}$ is called a foliation atlas. An open covering of M is said to be adapted if it is simple and a refinement of a foliation atlas for $\mathcal{F}$.

Definition 1.2. Let $\mathcal{F}$ be a transversely holomorphic foliation. Denote by $E=E(\mathcal{F})$ the complex subbundle $T_{\boldsymbol{C}}M=TM\otimes\boldsymbol{C}$ locally spanned by $\dfrac{\partial}{\partial x_k^i}$ and $\dfrac{\partial}{\partial \bar{z}_k^j}$, where $(x_k,z_k)=(x_k^1,\cdots,x_k^p,z_k^1,\cdots,z_k^q)$ is a local coordinate as in 1) of Definition 1.1. The complex normal bundle $Q(\mathcal{F})$ of $\mathcal{F}$ is by definition $T_{\boldsymbol{C}}M/E$. The line bundle $K_{\mathcal{F}}=\bigwedge^q Q(\mathcal{F})^*$ is called the canonical bundle, and $-K_{\mathcal{F}}=\bigwedge^q Q(\mathcal{F})$ is called the anti-canonical bundle.

Notation 1.3. We denote by $I_{(1)}(U)$ the ideal of $\boldsymbol{C}$-valued differential forms $\Omega^*(U)$ on U locally generated by $dz^1,\cdots,dz^q$, and set $I_{(k)}(U)=I_{(1)}(U)^k$. The sheaf on M formed by these ideals is denoted by $I_{(k)}$. The intersection $I_{(k)}(U)\cap\Omega^p(U)$ is denoted by $I_{(k)}^p(U)$ and the corresponding sheaf is denoted by $I_{(k)}^p$. We set $I_{(k,l)}^p=I_{(k)}^p/I_{(l)}^p$, namely, an element of $I_{(k,l)}^p(U)$ is a family $\{\omega_i\}$, where $\omega_i\in I_{(k)}^p$ is defined on an open subset V_i of U, where $\cup V_i=U$, such that $\omega_i=\omega_j \bmod I_{(l)}^p(V_i\cap V_j)$ if $V_i\cap V_j\neq\varnothing$. Finally, we set $I_{(k,l)}=\oplus_p I_{(k,l)}^p$.

Note that $I_{(k)} = \{0\}$ for $k > q$. If $p < l$, then $I^p_{(k,l)} = I^p_{(k)}$ because $I^p_{(l)} = \{0\}$. Note also that $E^* \cong I^1_{(0,1)}$, where the left hand side is regarded as the sheaf of germs of sections of E^*.

Notation 1.4. If $\mathcal{S}$ is a presheaf and $\mathcal{U}$ is an open covering of M, then denote by $\check{C}^r(\mathcal{U}; \mathcal{S})$ the Čech r-cochains valued in $\mathcal{S}$. When $\mathcal{U}$ is obvious, then it will be often omitted. Čech cochains are represented by adding or removing indices, for example, a cochain $\{\omega_i\}$ is usually denoted by ω and vice versa.

Definition 1.5. Denote by $\mathcal{A}^{p,q}(\mathcal{U}) = \check{C}^p(\mathcal{U}; \Omega^q)$ the space of the Čech-de Rham (p,q)-cochains. The modified Čech-de Rham complex is by definition the quotient $\mathcal{A}^{*,*}(\mathcal{U})/\check{C}^*(\mathcal{U}; \boldsymbol{Z})$ of the Čech-de Rham complex by the Čech complex with coefficients in $\boldsymbol{Z}$, where $\check{C}^p(\mathcal{U}; \boldsymbol{Z}) \subset \mathcal{A}^{p,0}(\mathcal{U})$. If $c \in \mathcal{A}^{p,q}(\mathcal{U})$ and $c' \in \mathcal{A}^{r,s}(\mathcal{U})$, then the product $c \cup c' \in \mathcal{A}^{p+r,q+s}(\mathcal{U})$ is defined by setting $(c \cup c')_{i_0 \cdots i_{p+r}} = (-1)^{qr} c_{i_0 \cdots i_p} \wedge c'_{i_p \cdots i_{p+r}}$.

Let $\mathcal{U} = \{U_i\}$ be an adapted covering, then $-K_{\mathcal{F}}$ is trivial when restricted to each U_i. Let e_i be a trivialization of $-K_{\mathcal{F}}|_{U_i}$, then there is a family $\{J_{ij}\}$ of non-zero functions such that $e_j = e_i J_{ij}$. (Under the notation in [2], $J_{ij} = \widetilde{\alpha}_{ij}^{-1}$.) Noticing that $\log J_{ij}$ is well-defined since the covering is adapted, set $\Theta = (2\pi\sqrt{-1})^{-1}\delta \log J$, where δ denotes the Čech differential. It is classical that Θ represents $c_1(Q(\mathcal{F}))$ in $\check{H}^2(M; \boldsymbol{Z})$. Let ∇_i be a Bott connection defined on U_i and let θ_i be its connection form with respect to e_i.

Definition 1.6. We set $\beta_{ij} = \theta_j - \theta_i - d \log J_{ij}$, then $\beta_{ij} \in I_{(1)}$. We call $\{\beta_{ij}\}$ the difference cochain of $\{\nabla_i\}$.

The Bott class is represented in terms of the following cochains in the modified Čech-de Rham complex:

Definition 1.7. Set $u_1(\nabla, e) = \frac{-1}{2\pi\sqrt{-1}}(\theta + \log J)$, $\bar{u}_1(\nabla, e) = \frac{1}{2\pi\sqrt{-1}}(\overline{\theta} + \overline{\log J})$, $v_1(\nabla, e) = \frac{-1}{2\pi\sqrt{-1}}(d\theta + \beta)$ and $\bar{v}_1(\nabla, e) = \frac{1}{2\pi\sqrt{-1}}(\overline{d\theta} + \overline{\beta})$. ∇ and e are omitted when they are clear.

The equations $\mathcal{D}u_1 = v_1 - \Theta$ and $\mathcal{D}\bar{u}_1 = \bar{v}_1 - \Theta$ hold.

The Bott class is a kind of so-called Cheeger-Chern-Simons classes. It is an element of $H^{2q+1}(M; \boldsymbol{C}/\boldsymbol{Z})$ characterized by certain properties. A representative of the Bott class can be given as follows.

Theorem 1.8 [2]**.** *Let $B_q(\nabla, e)$ be the cochain in the modified Čech-de Rham complex defined by the formula*

$$B_q(\nabla, e) = u_1 \cup v_1^q + \Theta \cup u_1 \cup v_1^{q-1} + \cdots + \Theta^q \cup u_1,$$

then $B_q(\nabla, e)$ is independent of the choice of $\mathcal{U}$, local trivializations e of $-K_{\mathcal{F}}$, the family of Bott connections ∇. This class is indeed the Bott class of $\mathcal{F}$ and denoted by $B_q(\mathcal{F})$.

Definition 1.9. Let $\{\mathcal{F}_s\}_{s\in S}$ be a family of transversely holomorphic foliations of a fixed codimension, of a fixed manifold M. Then $\{\mathcal{F}_s\}$ is said to be a continuous deformation of $\mathcal{F}_0$ if $\{\mathcal{F}_s\}$ is a continuous family as plane fields and the transverse holomorphic structures also vary continuously, where $0 \in S$ is the base point. If the family is in fact smooth and the transverse holomorphic structures vary smoothly, then it is said to be smooth.

In what follows, S is assumed to be $(-\epsilon, \epsilon)$, where ϵ is a sufficiently small positive number.

Given a smooth family $\{\mathcal{F}_s\}$ of transversely holomorphic foliations, set $-K_s = \bigwedge^q Q(\mathcal{F}_s)$, then we may assume that there is a family $\{e_{s,i}\}$ of local trivializations of $-K_s$ such that each $e_{s,i}$ is defined on U_i. We may assume that the function $J_{s,ij}$ defined by $e_{s,j} = e_{s,i}J_{s,ij}$ is independent of s, so $J_{s,ij}$ is denoted by J_{ij}. The cocycle $\Theta_s = (2\pi\sqrt{-1})^{-1}\delta \log J_s$ is also independent of s and denoted by Θ. Choose then a smooth family ∇_s of local Bott connections and let $\{\theta_{s,i}\}$ be its connection forms with respect to $\{e_{s,i}\}$. We denote by $\{\beta_{s,ij}\}$ the difference cochain of ∇_s, then by definition $\theta_{s,j} - \theta_{s,i} = d\log J_{ij} + \beta_{s,ij}$. Finally, for any cochain ω_s, we denote by $\dot{\omega}_s$ the partial derivative of ω_s with respect to s.

Under these choices of cochains, we have the following

Proposition 1.10. *Denote $u_1(\nabla_s, e_s)$ and $v_1(\nabla_s, e_s)$ by $u_1(s)$ and $v_1(s)$, respectively. Define $\bar{u}_1(s)$ and $\bar{v}_1(s)$ similarly, and set $\dot{u}_1(s) = \frac{-1}{2\pi\sqrt{-1}}\dot{\theta}_s$. Then, $\dfrac{\partial B_q(\mathcal{F}_s)}{\partial s}$ is naturally an element of $H^{2q+1}(M;\boldsymbol{C})$ and is represented by $\displaystyle\sum_{k=0}^{q} v_1(s)^k \cup \dot{u}_1(s) \cup v_1(s)^{q-k}$.*

Proof. First note that $\dot{u}_1(s)$ is indeed the partial derivative of $u_1(s)$ with respect to s. Moreover, if we set $\dot{v}_1(s) = \frac{-1}{2\pi\sqrt{-1}}(d\dot{\theta}_s + \dot{\beta}_s)$, then $D\dot{u}_1(s) = \dot{v}_1(s)$. We have the following equation, namely,

$$\begin{aligned}\frac{\partial}{\partial s}B_q(\nabla_s, e_s) =& \sum_{k=0}^{q} \Theta^k \cup \dot{u}_1(s) \cup v_1(s)^{q-k} \\ &+ \sum_{k=0}^{q-1}\sum_{l=0}^{q-k-1} \Theta^k \cup u_1(s) \cup v_1(s)^l \cup \dot{v}_1(s) \cup v_1(s)^{q-k-l-1}.\end{aligned}$$

For $k > 0$, set

$$\rho_k = \sum_{l=0}^{q-k} \Theta^{k-1} \cup u_1(s) \cup v_1(s)^l \cup \dot{u}_1(s) \cup v_1(s)^{q-k-l},$$

then

$$\begin{aligned}
D\rho_k =& \sum_{l=0}^{q-k} \Theta^{k-1} \cup (v_1(s) - \Theta) \cup v_1(s)^l \cup \dot{u}_1(s) \cup v_1(s)^{q-k-l} \\
& - \sum_{l=0}^{q-k} \Theta^{k-1} \cup u_1(s) \cup v_1(s)^l \cup \dot{v}_1(s) \cup v_1(s)^{q-k-l} \\
=& - \sum_{l=0}^{q-k} \Theta^{k} \cup v_1(s)^l \cup \dot{u}_1(s) \cup v_1(s)^{q-k-l} \\
& + \sum_{l=0}^{q-k} \Theta^{k-1} \cup v_1(s)^{l+1} \cup \dot{u}_1(s) \cup v_1(s)^{q-k-l} \\
& - \sum_{l=0}^{q-k} \Theta^{k-1} \cup u_1(s) \cup v_1(s)^l \cup \dot{v}_1(s) \cup v_1(s)^{q-k-l}.
\end{aligned}$$

By adding $D\rho_1 + \cdots + D\rho_q$ to $\dfrac{\partial}{\partial s} B_q(\nabla_s, e_s)$, the cocycle as in the statement is obtained. $\square$

Note that if each ∇_s is a global connection, then $\dfrac{\partial B_q(\mathcal{F}_s)}{\partial s}$ is represented by a globally well-defined $(2q+1)$-form $(-2\pi\sqrt{-1})^{-(q+1)}(q+1)\dot{\theta}_s \wedge (d\theta_s)^q$. This is the same formula as the derivative of the Bott class of transversely holomorphic foliations with trivial normal bundles given by Heitsch [13].

Finally we make some remarks on the imaginary part of the Bott class, which is an element of $H^{2q+1}(M; \boldsymbol{R})$. It can be represented without using the cocycle Θ as follows.

Theorem 1.11. *Let $\xi_q(\nabla, e)$ be the cocycle in the Čech-de Rham complex defined by the formula*

$$\xi_q(\nabla, e) = \frac{1}{2}\sqrt{-1} \sum_{k=0}^{q} \left(\bar{v}_1^k \cup (u_1 - \bar{u}_1) \cup v_1^q + v_1^k \cup (u_1 - \bar{u}_1) \cup \bar{v}_1^q \right),$$

then $\xi_q(\nabla, e)$ represents $\xi_q(\mathcal{F}) = \sqrt{-1}(B_q(\mathcal{F}) - \overline{B_q(\mathcal{F})})$ independent of the choice of ∇ and e.

Proof. We first show that $\sum_{k=0}^{q} \bar{v}_1^k \cup (u_1 - \bar{u}_1) \cup v_1^q$ is cohomologous to $B_q(\mathcal{F}) - \overline{B_q(\mathcal{F})}$. Define cochains $\alpha_{k,r}$ by $\alpha_{k,r} = \Theta^k \cup \bar{u}_1 \cup \bar{v}_1^r \cup (u_1 - \bar{u}_1) \cup v_1^{q-k-r-1}$ and set $\alpha_k = \sum_{r=0}^{q-k-1} \alpha_{k,r}$. Then

$$
\begin{aligned}
\mathcal{D}\alpha_k = & \sum_{r=1}^{q-k} \Theta^k \cup \bar{v}_1^r \cup (u_1 - \bar{u}_1) \cup v_1^{q-k-r} \\
& - \sum_{r=0}^{q-k-1} \Theta^{k+1} \cup \bar{v}_1^r \cup (u_1 - \bar{u}_1) \cup v_1^{q-k-r-1} \\
& - \Theta^k \cup \bar{u}_1 \cup v_1^{q-k} + \Theta^k \cup \bar{u}_1 \cup \bar{v}_1^{q-k}.
\end{aligned}
$$

It is easy to see that $\sum_{k=0}^{q} \bar{v}_1^k \cup (u_1 - \bar{u}_1) \cup v_1^q - \mathcal{D}(\alpha_0 + \cdots + \alpha_{q-1}) = B_q(\nabla, e) - \overline{B_q(\nabla, e)}$. It follows that $\xi_q(\nabla, e) - \mathcal{D}\sqrt{-1}(\alpha_0 + \cdots + \alpha_{q-1} - \overline{\alpha}_0 - \cdots - \overline{\alpha}_{q-1}) = 2\sqrt{-1}(B_q(\nabla, e) - \overline{B_q(\nabla, e)})$. This completes the proof. $\square$

A representative of $\dfrac{\partial}{\partial s}\xi_q(\mathcal{F}_s)$ can be obtained by Proposition 1.10 and Theorem 1.11. If $\log J$ is valued in $\sqrt{-1}\boldsymbol{R}$ and if $\beta = 0$, then $\widetilde{u}_1 = u_1 - \bar{u}_1$, v_1 and $\bar{v}_1$ are globally well-defined differential forms. In this case, the representative coincides with the Heitsch's one [13], and the formula in Theorem 1.11 becomes the standard definition of ξ_q. On the other hand, a representative of $\xi_q(\mathcal{F})$ is obtained by setting $\theta = 0$. The latter representative was used in [1] for studying local properties of this class. However, if we choose $\{e_{s,i}\}$ so that $\theta_{s,i} = 0$ for any s and i, then $J_{s,ij}$ depends on s in general. Thus corresponding formula for the derivative cannot be obtained.

2. Infinitesimal derivatives of the Bott class

We begin with the definition of infinitesimal deformations given in [12]. Let $\{\underline{e}_i = (\underline{e}_{i,1}, \cdots, \underline{e}_{i,q})\}$ be a family of local trivializations of $Q(\mathcal{F})$ and let $\{\underline{\omega}_i = {}^t(\underline{\omega}_i^1, \cdots, \underline{\omega}_i^q)\}$ be its dual. Let $\underline{A}_{ji} = ((\underline{a}_{ji})_l^k)$ be the matrix valued function such that $(\underline{e}_{i,1}, \cdots, \underline{e}_{i,q}) = (\underline{e}_{j,1}, \cdots, \underline{e}_{j,q})\underline{A}_{ji}$, then $\underline{A}_{ji}\underline{\omega}_i = \underline{\omega}_j$. Let $\underline{\nabla} = (\{\underline{\theta}_i\}, \{\underline{\beta}_{ij}\})$ be a pair of a family of local Bott connection forms and the difference cochain with respect to $\{\underline{e}_i\}$. That is, $\underline{\theta}_i$ is the connection form with respect to $\underline{e}_i$ of a Bott connection $\underline{\nabla}_i$ on U_i so that $\underline{\nabla}_i\underline{e}_i = (\underline{\nabla e}_{i,1}, \cdots, \underline{\nabla e}_{i,q}) = (\underline{e}_{i,1}, \cdots, \underline{e}_{i,q})\underline{\theta}_i$ and $\underline{\beta}_{ij} = \underline{A}_{ji}^{-1} d\underline{A}_{ji} + \underline{A}_{ji}^{-1}\underline{\theta}_j\underline{A}_{ji} - \underline{\theta}_i$. One has then $d\underline{\omega}_i + \underline{\theta}_i \wedge \underline{\omega}_i = 0$, $\underline{\beta}_{ji} = -\underline{A}_{ji}\underline{\beta}_{ij}\underline{A}_{ji}^{-1}$ and $\underline{\beta}_{ij} \in I_{(1)}(U_{ij})$, where $U_{ij} = U_i \cap U_j$.

In what follows, for a vector bundle V, the sheaf of germs of sections of V is also denoted by V by abuse of notation.

Definition 2.1. Set $E^s \otimes Q(\mathcal{F}) = \bigwedge^s E^* \otimes Q(\mathcal{F})$ and $\mathcal{E}^{t,s}(Q(\mathcal{F})) = \check{C}^t(E^s \otimes Q(\mathcal{F}))$. If an open covering $\mathcal{U}$ is specified, then $\mathcal{E}^{t,s}(Q(\mathcal{F}))$ is also denoted by $\mathcal{E}^{t,s}(\mathcal{U}; Q(\mathcal{F}))$. The complex $\mathcal{E}^{t,s}(Q(\mathcal{F}))$ is equipped with the differential $\delta + (-1)^t d_{\underline{\nabla}}$ where δ is the Čech differential and $d_{\underline{\nabla}}$ is defined below. The total complex is denoted by $\mathcal{E}^*(Q(\mathcal{F}))$.

The differential $d_{\underline{\nabla}}$ is defined as follows. Let $\mathcal{U} = \{U_i\}$ and let $s \in (E^s \otimes Q(\mathcal{F}))(U)$, where U is an open subset of M contained in U_i. Define then a mapping $d_{\underline{\nabla},i} : (E^s \otimes Q(\mathcal{F}))(U) \to (E^{s+1} \otimes Q(\mathcal{F}))(U)$ by setting

$$d_{\underline{\nabla},i}(s) = \underline{e}_i(d\varphi + \underline{\theta}_i \wedge \varphi),$$

where $\varphi = \underline{\omega}_i(s)$ and φ is considered as an s-form by arbitrarily extending.

Lemma 2.2. *$d_{\underline{\nabla},i}$ is independent of i so that $\{d_{\underline{\nabla},i}\}$ induces a well-defined mapping $d_{\underline{\nabla}} : E^s \otimes Q(\mathcal{F}) \to E^{s+1} \otimes Q(\mathcal{F})$.*

Proof. Let s be a section of $(E^s \otimes Q(\mathcal{F}))(U_i \cap U_j)$, then

$$d_{\underline{\nabla},j}(\underline{e}_j \underline{\omega}_j(s)) = d_{\underline{\nabla},i}(\underline{e}_i(\underline{\omega}_i(s)) + \underline{e}_i(\underline{\beta}_{ij} \wedge \underline{\omega}_i(s)).$$

The right hand side is equal to $d_{\underline{\nabla},i}(\underline{e}_i \underline{\omega}_i(s))$ as a section of $(E^{s+1} \otimes Q(\mathcal{F}))(U_i \cap U_j)$. $\square$

It is shown in [7] that $((E^* \otimes Q(\mathcal{F}))(M), d_{\underline{\nabla}})$ is a resolution of $\Theta_{\mathcal{F}}$ if $\underline{\nabla}$ is a global Bott connection. By using this fact it is easy to show that the cohomology of the total complex $(\mathcal{E}^*(Q(\mathcal{F})), \delta + (-1)^s d_{\underline{\nabla}})$ also coincides with $H^*(M; \Theta_{\mathcal{F}})$ even if $d_{\underline{\nabla}}$ is defined from a local Bott connection, because we consider smooth sections. The following type of deformations are useful.

Definition 2.3. Denote by $\underline{H}^*(M; \Theta_{\mathcal{F}})$ the cohomology of $((E^s \otimes Q(\mathcal{F}))(M), d_{\underline{\nabla}})$.

It is easy to see that the natural mapping $\underline{H}^p(M; \Theta_{\mathcal{F}}) \to H^p(M; \Theta_{\mathcal{F}})$ is injective if $p = 1$. Under our assumptions, $\underline{H}^p(M; \Theta_{\mathcal{F}})$ and $H^p(M; \Theta_{\mathcal{F}})$ are in fact isomorphic. However, we continue to distinguish them because there will appear a certain difference when defining infinitesimal derivatives without using partitions of unity (cf. Definitions 2.16 and 4.15).

Taking these observations into account, we introduce the following (cf. [13])

Definition 2.4. An element $\underline{\mu}$ of $H^1(M; \Theta_{\mathcal{F}})$ is called an infinitesimal deformation of $\mathcal{F}$. If $(\{\underline{\sigma}_i\}, \{\underline{s}_{ij}\}) \in \mathcal{E}^1(Q(\mathcal{F})) = \check{C}^0(E^* \otimes Q(\mathcal{F})) \oplus \check{C}^1(Q(\mathcal{F}))$ is a representative of $\underline{\mu}$, then the pair $(\{-\underline{\sigma}_i\}, \{-\underline{s}_{ij}\})$ is called the infinitesimal derivative of $\underline{\omega} = \{\underline{\omega}_i\}$.

It is shown in [12] that a smooth family $\{\mathcal{F}_s\}$ naturally determines an infinitesimal deformation as an element of $H^1(M;\Theta_{\mathcal{F}})$. We will briefly recall the construction after stating Theorem 2.17.

Noticing that $E^* \cong I^1_{(0,1)}$, the pair $(\{\underline{\sigma}_i\},\{\underline{s}_{ij}\})$ in the above definition should satisfy the following relations for some $\mathfrak{gl}(q;\boldsymbol{C})$-valued function $\underline{g}_{ij}$ on U_{ij} and some $\mathfrak{gl}(q;\boldsymbol{C})$-valued 1-form $\underline{\theta}'_i$ on U_i:

$$\underline{e}_i\left(d(\underline{\omega}_i(\underline{\sigma}_i)) + \underline{\theta}_i \wedge (\underline{\omega}_i(\underline{\sigma}_i))\right) = \underline{e}_i\underline{\theta}'_i \wedge \underline{\omega}_i, \tag{2.5.a}$$

$$(\underline{\sigma}_j - \underline{\sigma}_i) - \underline{e}_i\left(d(\underline{\omega}_i(\underline{s}_{ij})) + \underline{\theta}_i\underline{\omega}_i(\underline{s}_{ij})\right) = \underline{e}_i\underline{g}_{ij}\underline{\omega}_i, \tag{2.5.b}$$

$$(\underline{\delta} s)_{ijk} = 0, \tag{2.5.c}$$

where $\{\theta'_i\}$ and $\{g_{ij}\}$ are obtained by computing the left hand sides of the above equations after extending $\{\sigma_i\}$ to $Q(\mathcal{F})$-valued differential forms. Note that $g_{ij} \neq -g_{ji}$ in general.

Infinitesimal derivatives of Bott connections can be defined as follows if $\underline{\mu}$ is represented by an element of $\check{C}^0(E^* \otimes Q(\mathcal{F}))$. Note that cocycles in $\check{C}^0(E^* \otimes Q(\mathcal{F}))$ are elements of $(E^1 \otimes Q(\mathcal{F}))(M)$ closed under $d_{\underline{\nabla}}$.

Definition 2.6. Suppose that $\underline{\mu} \in \underline{H}^1(M;\Theta_{\mathcal{F}})$ and let $\sigma = \{\sigma_i\} \in \check{C}^0(E^* \otimes Q(\mathcal{F}))$ be a representative of μ. Then any pair $\underline{\nabla}' = (\{\underline{\theta}'_i\},\{g_{ij}\})$ which satisfies (2.5.a) and (2.5.b) is called an infinitesimal derivative of the Bott connection $\underline{\nabla} = (\{\underline{\theta}_i\},\{\underline{\beta}_{ij}\})$ with respect to σ.

The infinitesimal derivative of the Bott class is defined as follows.

Definition 2.7. Let $\underline{\mu} \in \underline{H}^1(M;\Theta_{\mathcal{F}})$ and let $\sigma \in (E^1 \otimes Q(\mathcal{F}))(M)$ be a representative. Set $\theta' = \operatorname{tr}\underline{\theta}'$, $\theta = \operatorname{tr}\underline{\theta}$, $\beta = \operatorname{tr}\underline{\beta}$ and $u'_1 = \frac{-1}{2\pi\sqrt{-1}}(\theta' + g)$. The cohomology class in $H^{2q+1}(M;\boldsymbol{C})$ represented by

$$\begin{aligned} D_\sigma B_q(\underline{\nabla},\underline{\nabla}') &= \sum_{k=0}^{q} v_1^k \cup u'_1 \cup v_1^{q-k} \\ &= (2\pi\sqrt{-1})^{-(q+1)} \sum_{k=0}^{q} (d\theta + \beta)^k \cup (\theta' + g) \cup (d\theta + \beta)^{q-k} \end{aligned}$$

is called the infinitesimal derivative of the Bott class with respect to $\underline{\mu}$ and denoted by $D_{\underline{\mu}} B_q(\mathcal{F})$.

We will later show in Theorems 2.15 and 2.17 that the infinitesimal derivative is well-defined and is independent of the choice of σ, ∇, ∇' and local trivializations.

The infinitesimal derivative of the Bott class can be reconstructed in terms of $-K_{\mathcal{F}}$ as follows. Let $\{e_i\}$ be a family of local trivializations of $-K_{\mathcal{F}}$, where e_i is defined on U_i. Let $\{J_{ij}\}$ a family of smooth functions such that $e_i = e_j J_{ji}$. A Bott connection on $Q(\mathcal{F})|_{U_i}$ naturally induces a connection on $-K_{\mathcal{F}}|_{U_i}$, which is also called a Bott connection. Then, a family of local Bott connections on $-K_{\mathcal{F}}$ is a pair $(\{\theta_i\}, \{\beta_{ij}\})$ satisfying $\theta_j - \theta_i = d\log J_{ij} + \beta_{ij}$, where θ_i is the connection form of a Bott connection on $-K_{\mathcal{F}}|_{U_i}$ with respect to e_i.

Recalling that $E^* \cong I^1_{(0,1)}$, we introduce the following

Definition 2.8. We denote $(E^* \otimes Q(\mathcal{F}))(U)$ also by $I^1_{(0,1)}(U; Q(\mathcal{F}))$, and set

$$I^*_{(q-1,q)}(U; -K_{\mathcal{F}}) = I^*_{(q-1,q)}(U) \otimes (-K_{\mathcal{F}}|_U) = \left(I^*_{(q-1)}(U)/I^*_{(q)}(U)\right) \otimes (-K_{\mathcal{F}}|_U).$$

Let $\varphi \in I^p_{(q-1,q)}(U_i \cap U_j; -K_{\mathcal{F}})$, then φ can be written as $\varphi = e_i \otimes \varphi_i$ on U_i, where $\varphi_i \in I^p_{(q-1,q)}(U_i)$. Set then $d_{\nabla,i}\varphi = e_i(d\varphi_i + \theta_i \wedge \varphi_i)$. Since $\beta_{ij} \in I_{(1)}(U_{ij})$, the equation

$$\begin{aligned} d_{\nabla,j}\varphi &= e_j(d\varphi_j + \theta_j \wedge \varphi_j) \\ &= e_i(d\varphi_i + \theta_i \wedge \varphi_i + \beta_{ij} \wedge \varphi_i) \\ &= d_{\nabla,i}\varphi \end{aligned}$$

holds. Hence $\{d_{\nabla,i}\}$ induces a globally well-defined map, which is denoted by d_∇. One has $d_\nabla \circ d_\nabla = 0$, indeed, the equalities

$$d_\nabla(d_\nabla(e_i\varphi_i)) = d_\nabla(e_i(d\varphi_i + \theta_i \wedge \varphi_i)) = e_i(d\theta_i \wedge \varphi_i)$$

hold on U_i. Since $\varphi_i \in I^p_{(q-1,q)}(U_i)$ and $d\theta_i \in I_{(1)}(U_i)$, $d\theta_i \wedge \varphi_i = 0$ in $I^p_{(q-1,q)}(U_i)$.

Definition 2.9. Set $\mathcal{K}^{r,s} = \check{C}^r(I^{s+q-1}_{(q-1,q)}(\mathcal{U}; -K_{\mathcal{F}}))$ and equip it with the differential $\delta + (-1)^r d_\nabla$. We denote by $\mathcal{K}^*$ the total complex and by $H^*(M; -K_{\mathcal{F}})$ the cohomology of $\mathcal{K}^*$.

In practice, $\mathcal{K}^0$, $\mathcal{K}^1$ and $\mathcal{K}^2$ are relevant. We have

$$\begin{aligned} \mathcal{K}^0 &= \check{C}^0(I^{q-1}(\mathcal{U}; -K_{\mathcal{F}})), \\ \mathcal{K}^1 &= \check{C}^0(I^q_{(q-1,q)}(\mathcal{U}; -K_{\mathcal{F}})) \oplus \check{C}^1(I^{q-1}(\mathcal{U}; -K_{\mathcal{F}})), \\ \mathcal{K}^2 &= \check{C}^0(I^{q+1}_{(q-1,q)}(\mathcal{U}; -K_{\mathcal{F}})) \oplus \check{C}^1(I^q_{(q-1,q)}(\mathcal{U}; -K_{\mathcal{F}})) \oplus \check{C}^2(I^{q-1}(\mathcal{U}; -K_{\mathcal{F}})). \end{aligned}$$

Let $\{\omega_i\}$ be the family of local trivializations of $K_{\mathcal{F}}$ dual to $\{e_i\}$. A version of infinitesimal deformations of $-K_{\mathcal{F}}$ is defined as follows.

Definition 2.10. An element μ of $H^1(M; -K_{\mathcal{F}})$ is called an infinitesimal deformation of $-K_{\mathcal{F}}$. If $(\{\sigma_i\}, \{s_{ij}\}) \in \mathcal{K}^1$ is a representative of μ, then the cocycle $(\{-\sigma_i\}, \{-s_{ij}\})$ is called the infinitesimal derivative of $\omega = \{\omega_i\}$ with respect to (σ, s).

If $(\{-\sigma_i\}, \{-s_{ij}\})$ is an infinitesimal derivative, then the following equations hold, namely,

$$e_i\bigl(d(\omega_i(\sigma_i)) + \theta_i \wedge (\omega_i(\sigma_i))\bigr) = e_i\theta'_i \wedge \omega_i, \tag{2.11.a}$$

$$(\sigma_j - \sigma_i) - e_i\bigl(d(\omega_i(s_{ij})) + \theta_i \wedge \omega_i(s_{ij})\bigr) = e_i g_{ij}\omega_i, \tag{2.11.b}$$

$$s_{jk} - s_{ik} + s_{ij} = 0. \tag{2.11.c}$$

Lemma 2.12. *Regard the complex $(I^{*+q-1}_{(q-1,q)}(M; -K_{\mathcal{F}}), d_\nabla)$ as a subcomplex of $(\mathcal{K}^*, \delta + (-1)^r d_\nabla)$ and denote by $\underline{H}^*(M; -K_{\mathcal{F}})$ its cohomology. Then, the natural mapping $\underline{H}^1(M; -K_{\mathcal{F}}) \to H^1(M; -K_{\mathcal{F}})$ is injective. It is an isomorphism if there is a partition of unity.*

The proof is easy and omitted.

When local trivializations and local connections of $Q(\mathcal{F})$ are given, we always consider local trivializations and local connections of $-K_{\mathcal{F}}$ in the following way. Let $\{\underline{e}_i\}$ be a family of local trivializations of $Q(\mathcal{F})$ and set $e_i = \underline{e}_{i,1} \wedge \cdots \wedge \underline{e}_{i,q}$, then $\{e_i\}$ is a family of local trivializations of $-K_{\mathcal{F}}$. Similarly, $\{\omega_i = \underline{\omega}_i^1 \wedge \cdots \wedge \underline{\omega}_i^q\}$ is a natural family of local trivializations of $K_{\mathcal{F}}$. Finally, $\{\theta_i = \operatorname{tr} \underline{\theta}_i\}$ is a family of local Bott connection forms with respect to $\{e_i\}$. They satisfy the equations $d\omega_i + \theta_i \wedge \omega_i = 0$ and $\theta_j - \theta_i = d\log J_{ij} + \beta_{ij}$, where $J_{ij} = \det \underline{A}_{ij}$ and $\beta_{ij} = \operatorname{tr} \underline{\beta}_{ij}$.

Lemma 2.13. *Let $\underline{\mu} \in H^1(M; \Theta_{\mathcal{F}})$ and let $\underline{m} = (\{\underline{\sigma}_i\}, \{\underline{s}_{ij}\}) \in \mathcal{E}^1(\mathcal{U}; Q(\mathcal{F}))$ be its representative. Set then*

$$r_0(\underline{m})_i = \sum_{k=1}^{q} \underline{\omega}_i^1 \wedge \cdots \wedge \underline{\omega}_i^{k-1} \wedge \underline{\omega}_i^k(\underline{\sigma}_i) \wedge \underline{\omega}_i^{k+1} \wedge \cdots \wedge \underline{\omega}_i^q \otimes e_i,$$

$$r_1(\underline{m})_{ij} = \sum_{k=1}^{q} (-1)^{k-1} \underline{\omega}_i^1 \wedge \cdots \wedge \underline{\omega}_i^{k-1} \wedge \underline{\omega}_i^k(\underline{s}_{ij}) \wedge \underline{\omega}_i^{k+1} \wedge \cdots \wedge \underline{\omega}_i^q \otimes e_i,$$

then $r = r_0 \oplus r_1$ induces an isomorphism from $H^1(M; \Theta_{\mathcal{F}})$ to $H^1(M; -K_{\mathcal{F}})$ under which $\underline{H}^1(M; \Theta_{\mathcal{F}})$ is mapped to $\underline{H}^1(M; -K_{\mathcal{F}})$. Moreover, if $\underline{m}$ satisfies (2.5.a), (2.5.b) *and* (2.5.c)*, then $r(\underline{m})$ satisfies* (2.11.a), (2.11.b) *and* (2.11.c) *with $\theta' = \operatorname{tr} \underline{\theta}'$ and $g = \operatorname{tr} \underline{g}$.*

The induced mapping on the cohomology is again denoted by r.

Proof. It is clear that the mapping r is well-defined at the cochain level. By (2.5.a), $d(\underline{\omega}_i(\underline{\sigma}_i))+\underline{\theta}_i\wedge(\underline{\omega}_i^k(\underline{\sigma}_i))=\underline{\theta}'_i\wedge\underline{\omega}_i$ for some $\mathfrak{gl}(q;\boldsymbol{C})$-valued 1-form $\underline{\theta}'_i$. Since $\theta=\operatorname{tr}\underline{\theta}$ and $\beta=\operatorname{tr}\underline{\beta}$, one has the following equations, namely,

$$
\begin{aligned}
& dr_0(\underline{m})_i+\theta_i\wedge r_0(\underline{m})_i \\
=&\sum_{k=1}^{q}\sum_{l\neq k}(-1)^{l-1}\underline{\omega}_i^1\wedge\cdots\wedge d\underline{\omega}_i^l\wedge\cdots\wedge\underline{\omega}_i^{k-1}\wedge\underline{\omega}_i^k(\underline{\sigma}_i)\wedge\underline{\omega}_i^{k+1}\wedge\cdots\wedge\underline{\omega}_i^q\otimes e_i \\
&+\theta_i\wedge\sum_{k=1}^{q}\underline{\omega}_i^1\wedge\cdots\wedge\underline{\omega}_i^{k-1}\wedge\underline{\omega}_i^k(\underline{\sigma}_i)\wedge\underline{\omega}_i^{k+1}\wedge\cdots\wedge\underline{\omega}_i^q\otimes e_i \\
&+\sum_{k=1}^{q}(-1)^{k-1}\underline{\omega}_i^1\wedge\cdots\wedge\underline{\omega}_i^{k-1}\wedge d(\underline{\omega}_i^k(\underline{\sigma}_i))\wedge\underline{\omega}_i^{k+1}\wedge\cdots\wedge\underline{\omega}_i^q\otimes e_i \\
=&-\sum_{k=1}^{q}\sum_{l\neq k}(\underline{\theta}_i)_l^l\wedge\underline{\omega}_i^1\wedge\cdots\wedge\underline{\omega}_i^{k-1}\wedge\underline{\omega}_i^k(\underline{\sigma}_i)\wedge\underline{\omega}_i^{k+1}\wedge\cdots\wedge\underline{\omega}_i^q\otimes e_i \\
&+\sum_{k=1}^{q}\sum_{l\neq k}(\underline{\theta}_i)_k^l\wedge\underline{\omega}_i^1\wedge\cdots\wedge\underline{\omega}_i^{l-1}\wedge\omega_i^k(\underline{\sigma}_i)\wedge\underline{\omega}_i^{l+1}\wedge\cdots\wedge\underline{\omega}_i^{k-1}\wedge\underline{\omega}_i^k\wedge\underline{\omega}_i^{k+1}\wedge\cdots\wedge\underline{\omega}_i^q\otimes e_i \\
&+\theta_i\wedge\sum_{k=1}^{q}\underline{\omega}_i^1\wedge\cdots\wedge\underline{\omega}_i^{k-1}\wedge\underline{\omega}_i^k(\underline{\sigma}_i)\wedge\underline{\omega}_i^{k+1}\wedge\cdots\wedge\underline{\omega}_i^q\otimes e_i \\
&+\sum_{l=1}^{q}(-1)^{l-1}\underline{\omega}_i^1\wedge\cdots\wedge\underline{\omega}_i^{l-1}\wedge d(\underline{\omega}_i^l(\underline{\sigma}_i))\wedge\underline{\omega}_i^{l+1}\wedge\cdots\wedge\underline{\omega}_i^q\otimes e_i \\
=&\sum_{l=1}^{q}\sum_{k=1}^{q}(-1)^{l-1}\underline{\omega}_i^1\wedge\cdots\wedge\underline{\omega}_i^{l-1}\wedge(\underline{\theta}'_i)_k^l\wedge\underline{\omega}_i^k\wedge\underline{\omega}_i^{l+1}\wedge\cdots\wedge\underline{\omega}_i^q\otimes e_i \\
=&(\operatorname{tr}\underline{\theta}')\wedge\omega_i\otimes e_i.
\end{aligned}
$$

On the other hand, by repeating similar calculations, one has the following equation on U_{ij};

$$
\begin{aligned}
& r_0(\underline{m})_j-r_0(\underline{m})_i \\
=&\sum_{k=1}^{q}\underline{\omega}_j^1\wedge\cdots\wedge\underline{\omega}_j^{k-1}\wedge\underline{\omega}_j^k(\underline{\sigma}_j)\wedge\underline{\omega}_j^{k+1}\wedge\cdots\wedge\underline{\omega}_j^q\otimes e_j, \\
&-\sum_{k=1}^{q}\underline{\omega}_i^1\wedge\cdots\wedge\underline{\omega}_i^{k-1}\wedge\underline{\omega}_i^k(\underline{\sigma}_i)\wedge\underline{\omega}_i^{k+1}\wedge\cdots\wedge\underline{\omega}_i^q\otimes e_i, \\
=&\sum_{k=1}^{q}\underline{\omega}_i^1\wedge\cdots\wedge\underline{\omega}_i^{k-1}\wedge\left(\underline{\omega}_i^k(\underline{\sigma}_j)-\underline{\omega}_i^k(\underline{\sigma}_i)\right)\wedge\underline{\omega}_i^{k+1}\wedge\cdots\wedge\underline{\omega}_i^q\otimes e_i,
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{q} \underline{\omega}_i^1 \wedge \cdots \wedge \underline{\omega}_i^{k-1} \wedge \big(d(\underline{\omega}_i^k(\underline{s}_{ij})) + \sum_{l=1}^{q} (\underline{\theta}_i)_l^k \wedge \underline{\omega}_i^l(\underline{s}_{ij})\big) \wedge \underline{\omega}_i^{k+1} \wedge \cdots \wedge \underline{\omega}_i^q \otimes e_i \\
&\quad + \sum_{k=1}^{q} \sum_{l=1}^{q} \underline{\omega}_i^1 \wedge \cdots \wedge \underline{\omega}_i^{k-1} \wedge (\underline{g}_i)_l^k \underline{\omega}_i^l \wedge \underline{\omega}_i^{k+1} \wedge \cdots \wedge \underline{\omega}_i^q \otimes e_i \\
&= (dr_1(\underline{m})_{ij} + \theta_i \wedge r_1(\underline{m})_{ij}) + (\operatorname{tr} \underline{g}_i)\, \omega_i \otimes e_i \\
&= d_\nabla r_1(\underline{m})_{ij} + (\operatorname{tr} \underline{g}_i)\, \omega_i \otimes e_i.
\end{aligned}
$$

Finally, it is easy to see that $\delta r_1(\underline{m})_{ijk} = 0$. Thus $r(\underline{m})$ is closed under $\delta + (-1)^r d_\nabla$ and the last part of the lemma is shown.

Assume that $\underline{m}$ is exact, then $\sigma_i = \underline{e}_i\,(df_i + \underline{\theta}_i f_i)$ and $s_{ij} = \underline{e}_j f_j - \underline{e}_i f_i$ for some collection $\{\underline{e}_i f_i\}$ of local sections of $Q(\mathcal{F})$. Setting

$$
\rho_i = \sum_{k=1}^{q} (-1)^{k-1} \underline{\omega}_i^1 \wedge \cdots \wedge \underline{\omega}_i^{k-1} \wedge\ f_i^k\ \wedge \underline{\omega}_i^{k+1} \wedge \cdots \wedge \underline{\omega}_i^q \otimes e_i,
$$

it is easy to verify that $d_\nabla \rho_i = r_0(\underline{m})_i$ and that $\rho_j - \rho_i = r_1(\underline{m})_{ij}$.

Conversely, let $m = (\{\sigma_i\}, \{s_{ij}\})$ be a cocycle in $\mathcal{K}^1$, then (2.11.a), (2.11.b) and (2.11.c) hold. Define 1-forms $\{\widetilde{\sigma}_i^k\}$ and functions $\{\widetilde{s}_{ij}^k\}$ by requiring $\underline{\omega}_i^k \wedge (\omega_i(\sigma_i)) = -\widetilde{\sigma}_i^k \wedge \omega_i$ and $\underline{\omega}_i^k \wedge \omega_i(s_{ij}) = \widetilde{s}_{ij}^k \wedge \omega_i$. Set $r_0'(m)_i = \sum_{k=1}^{q} \underline{e}_{i,k} \otimes \widetilde{\sigma}_i^k$ and $r_1'(m)_{ij} = \sum_{k=1}^{q} \underline{e}_{i,k} \otimes \widetilde{s}_{ij}^k$. It is clear that $r'(m) = r_0'(m) \oplus r_1'(m)$ is well-defined as an element of $\mathcal{K}^1$. Let $\widetilde{\sigma}_i = {}^t(\widetilde{\sigma}_i^1, \cdots, \widetilde{\sigma}_i^q)$ and $\underline{\omega}_i = {}^t(\underline{\omega}_i^1, \cdots, \underline{\omega}_i^q)$, then one has the following equations, namely,

$$
\begin{aligned}
d(\widetilde{\sigma}_i \wedge \omega_i) &= d\,(-\underline{\omega}_i \wedge \omega_i(\sigma_i)) \\
&= \underline{\theta}_i \wedge \underline{\omega}_i \wedge \omega_i(\sigma_i) + \underline{\omega}_i \wedge d(\omega_i(\sigma_i)) \\
&= -\underline{\theta}_i \wedge \widetilde{\sigma}_i \wedge \omega_i + \underline{\omega}_i \wedge d(\omega_i(\sigma_i)),
\end{aligned}
$$

while

$$
\begin{aligned}
d\,(\widetilde{\sigma}_i \wedge \omega_i) &= d\widetilde{\sigma}_i \wedge \omega_i - \widetilde{\sigma}_i \wedge d\omega_i \\
&= d\widetilde{\sigma}_i \wedge \omega_i + \widetilde{\sigma}_i \wedge \theta_i \wedge \omega_i \\
&= d\widetilde{\sigma}_i \wedge \omega_i - \underline{\omega}_i \wedge \theta_i \wedge \omega_i(\sigma_i).
\end{aligned}
$$

Hence

$$
(d\widetilde{\sigma}_i + \underline{\theta}_i \wedge \widetilde{\sigma}_i) \wedge \omega_i = \underline{\omega}_i \wedge (d(\omega_i(\sigma_i)) + \theta_i \wedge \omega_i(\sigma_i)) = 0.
$$

Similarly, by the equations

$$d(\widetilde{s}_{ij}\omega_i) = (d\widetilde{s}_{ij}) \wedge \omega_i - \widetilde{s}_{ij}\theta_i \wedge \omega_i = (d\widetilde{s}_{ij}) \wedge \omega_i + \underline{\omega}_i \wedge \theta_i \wedge \omega_i(s_{ij})$$

and

$$d(\widetilde{s}_{ij}\omega_i) = d(\underline{\omega}_i \wedge \omega_i(s_{ij})) = -\underline{\theta}_i \wedge \widetilde{s}_{ij}\omega_i - \underline{\omega}_i \wedge d(\omega_i(s_{ij})),$$

one sees that

$$\begin{aligned}
\underline{e}_i(d\widetilde{s}_{ij} + \underline{\theta}_i \wedge \widetilde{s}_{ij}) \wedge \omega_i &= -\underline{e}_i\underline{\omega}_i \wedge (d(\omega_i(s_{ij})) + \theta_i \wedge \omega_i(s_{ij})) \\
&= -\underline{e}_i\underline{\omega}_i \wedge (\omega_i(\sigma_j) - \omega_i(\sigma_i)) \\
&= -\underline{e}_j\underline{\omega}_j \wedge a_{ij}\omega_j(\sigma_j) - \underline{e}_i\widetilde{\sigma}_i\omega_i \\
&= \underline{e}_j\widetilde{\sigma}_j \wedge a_{ij}\omega_j - \underline{e}_i\widetilde{\sigma}_i \wedge \omega_i \\
&= (\underline{e}_j\widetilde{\sigma}_j - \underline{e}_i\widetilde{\sigma}_i) \wedge \omega_i.
\end{aligned}$$

It follows that $r'(m)$ is closed. Almost the same argument shows that r' descends to a mapping on the cohomology. Finally, it is clear from the construction that $\underline{H}^1(M;\Theta_{\mathcal{F}})$ is mapped to $\underline{H}^1(M;-K_{\mathcal{F}})$ under the mapping r. This completes the proof. $\square$

Infinitesimal derivatives of the Bott class is determined by infinitesimal deformations of $-K_{\mathcal{F}}$ as follows. We still assume that deformations are represented by elements of $I^q_{(q-1,q)}(M;-K_{\mathcal{F}})$.

Definition 2.14. Let $\mu \in \underline{H}^1(M;-K_{\mathcal{F}})$ and let $\sigma = \{\sigma_i\} \in I^q_{(q-1,q)}(M;-K_{\mathcal{F}})$ be a representative of μ. Then any pair $\nabla' = (\{\theta'_i\},\{g_{ij}\})$ which satisfies (2.11.a) and (2.11.b) is called an infinitesimal derivative of the Bott connection $\nabla = (\{\theta_i\},\{\beta_{ij}\})$ with respect to σ.

Theorem 2.15. *Let $\mu \in \underline{H}^1(M;-K_{\mathcal{F}})$ be an infinitesimal deformation. Let $\sigma = \{\sigma_i\} \in I^q_{(q-1,q)}(M;-K_{\mathcal{F}})$ be a representative and let $\nabla' = (\{\theta'_i\},\{g_{ij}\})$ be the infinitesimal derivative of ∇ with respect to σ. Set*

$$\begin{aligned}
D_\sigma B_q(\nabla,\nabla') &= \sum_{k=0}^{q} v_1^k \cup u'_1 \cup v_1^{q-k} \\
&= (2\pi\sqrt{-1})^{-(q+1)} \sum_{k=0}^{q} (d\theta+\beta)^k \cup (\theta'+g) \cup (d\theta+\beta)^{q-k},
\end{aligned}$$

where $u_1' = \frac{-1}{2\pi\sqrt{-1}}(\theta' + g)$, *then* $D_\sigma B_q(\nabla, \nabla')$ *represents a class in* $H^{2q+1}(M; \boldsymbol{C})$ *independent of the choice of cochains and connections.*

Proof. Claims are proved in steps. Denote by $\mathcal{D}$ the differential in the Čech-de Rham complex.

Claim 1. $D_\sigma B_q(\nabla, \nabla')$ is closed.

First of all,

$$\mathcal{D}(D_\sigma B_q(\nabla, \nabla')) = (2\pi\sqrt{-1})^{-(q+1)} \sum_{k=0}^{q} (d\theta + \beta)^k \cup \mathcal{D}(\theta' + g) \cup (d\theta + \beta)^{q-k}$$

since $\mathcal{D}(d\theta + \beta) = 0$. By (2.11.a), one has

$$d\theta_i \wedge (\omega_i(\sigma_i)) = d\theta'_i \wedge \omega_i. \tag{2.15.a}$$

One also has

$$\begin{aligned} e_j\theta'_j \wedge \omega_j &= e_j\big(d(\omega_j(\sigma_j)) + \theta_j \wedge (\omega_j(\sigma_j))\big) \\ &= e_i\big(d(\omega_i(\sigma_i)) + \theta_i \wedge (\omega_i(\sigma_i)) + \beta_{ij} \wedge (\omega_i(\sigma_i)) + dg_{ij} \wedge \omega_i\big) \\ &= e_i\theta'_i \wedge \omega_i + e_i\beta_{ij} \wedge (\omega_i(\sigma_i)) + e_i dg_{ij} \wedge \omega_i, \end{aligned}$$

namely,

$$(\delta\theta' - dg)_{ij} \wedge \omega_i = \beta_{ij} \wedge (\omega_i(\sigma)). \tag{2.15.b}$$

On the other hand, since $\sigma_i \in I^q_{(q-1,q)}(U_i; -K_\mathcal{F})$,

$$\underline{\omega}_i^m \wedge \omega_i(\sigma_i) = -\widetilde{\sigma}_i^m \wedge \omega_i \tag{2.15.c}$$

for some 1-form $\widetilde{\sigma}_i^m$ defined on U_i. We may assume that $\widetilde{\sigma}_i^m$ is well-defined modulo $I^1_{(1)}(U_i)$ and $\widetilde{\sigma}_j = \underline{A}_{ji}\widetilde{\sigma}_i$ modulo $I^1_{(1)}(U_i)$, where $\widetilde{\sigma}_i = {}^t(\widetilde{\sigma}_i^1, \cdots, \widetilde{\sigma}_i^q)$. Writing $d\theta_i = \sum_m \partial_m\theta_i \wedge \omega_i^m$ and $\beta_{ij} = \sum_m (b_{ij})_m \omega_i^m$, we set $\widetilde{\partial\theta}'_i = -\sum_m \partial_m\theta_i \wedge \tilde{\sigma}_i^m$ and $\tilde{\beta}'_{ij} = -\sum_m (b_{ij})_m \tilde{\sigma}_i^m$. Then by (2.15.a), (2.15.b) and (2.15.c),

$$\begin{aligned} \widetilde{\partial\theta}'_i \wedge \omega_i &= d\theta'_i \wedge \omega_i, \\ \tilde{\beta}'_{ij} \wedge \omega_i &= (\delta\theta' - dg)_{ij} \wedge \omega_i. \end{aligned}$$

Finally, $e_i(\delta g_{ijk})\omega_i = 0$ by (2.11.b) and by the assumption $s = \{s_{ij}\} = 0$. It follows that

$$(d\theta + \beta)^k \cup \mathcal{D}(\theta' + g) \cup (d\theta + \beta)^{q-k} = (d\theta + \beta)^k \cup (\widetilde{\partial\theta}' + \tilde{\beta}') \cup (d\theta + \beta)^{q-k}.$$

Let $\Phi = \{\varphi_0, \cdots, \varphi_q\}$ be an ordered sequence such that φ_k is either $d\theta$ or β, then

$$\sum_{k=0}^{q} \varphi_0 \cup \cdots \cup \varphi_{k-1} \cup \widetilde{\varphi}'_k \cup \varphi_{k+1} \cup \cdots \cup \varphi_q = 0.$$

Indeed, each φ_k in the above cochain appears in the form $\psi_k \wedge \omega_{l_k}$ for some ψ_k and l_k. This differential form is equal to $\psi_k \wedge \underline{A}_{l_k,i}\omega_i$. On the other hand, since $-\widetilde{\sigma}_j \wedge \omega_j = -\underline{A}_{ji}\widetilde{\sigma}_i \wedge a_{ji}\omega_i$, one has $\widetilde{\sigma}_j = \underline{A}_{ji}\widetilde{\sigma}_i$ modulo $I^1_{(1)}(U_{ij})$. It follows that the above sum is equal to

$$\frac{\partial}{\partial t} \sum_{k=0}^{q} \varphi_0(t) \cup \cdots \cup \varphi_q(t) \Big|_{t=0},$$

where $\varphi_k(t) = \psi_k A_{l_k,i} \wedge (\omega_i - t\widetilde{\sigma}_i)$. If we denote by $\mathcal{I}_i(t)$ the ideal generated by $\omega_i^m - t\widetilde{\sigma}_i^m$, $m = 1, \cdots, q$, then $\varphi_0(t) \cup \cdots \cup \varphi_q(t)$ belongs to $\mathcal{I}_i(t)^{q+1} = \{0\}$. Hence $D_\sigma B_q(\nabla, \nabla')$ is closed.

Claim 2. $D_\sigma B_q(\nabla, \nabla')$ is independent of the choice of ∇' once σ is fixed.

Let $(\{\tilde{\theta}'_i\}, \{\tilde{g}_{ij}\})$ be another choice of an infinitesimal derivative of ∇ with respect to σ, then $e_i(\tilde{\theta}'_i - \theta'_i) \wedge \omega_i = 0$ and $\tilde{g}_{ij} = g_{ij}$. Hence $(d\theta+\beta)^k \cup (\tilde{\theta}'+\tilde{g}) \cup (d\theta+\beta)^{q-k} = (d\theta+\beta)^k \cup (\theta'+g) \cup (d\theta+\beta)^{q-k}$ for each k.

Claim 3. The class $[D_\sigma B_q(\nabla, \nabla')]$ is independent of the choice of σ.

Let $\{\tilde{\sigma}_i\}$ be another representative of μ and choose an infinitesimal derivative $\widetilde{\nabla}' = (\{\tilde{\theta}'_i\}, \{\tilde{g}_{ij}\})$ of ∇ with respect to $\{\tilde{\sigma}_i\}$. Set $\psi = \tilde{\sigma} - \sigma$, then there is an element $\tau = \{\tau_i\} \in I^{q-1}_{(q-1,q)}(\mathcal{U}; -K_{\mathcal{F}})$ and a family $\{h_i\}$ of functions on U_i such that

$$e_i\omega_i(\psi_i) = e_i\big(d(\omega_i(\tau_i)) + \theta_i \wedge (\omega_i(\tau_i)) + h_i\omega_i\big), \tag{2.15.d}$$

$$\tau_j - \tau_i = 0. \tag{2.15.e}$$

Let now $\widetilde{\tau}_i^m$, $m = 1, \cdots, q$ be 1-forms such that $\underline{\omega}_i^m \wedge \omega_i(\tau) = -\widetilde{\tau}_i^m \wedge \omega_i$, and define $\widehat{\partial\theta}$ and $\widehat{\beta}_{ij}$ by setting $\widehat{\partial\theta}_i = -\sum_m \partial_m\theta_i \wedge \widetilde{\tau}_i^m$ and $\widehat{\beta}_{ij} = -\sum_m (b_{ij})_m \widetilde{\tau}_i^m$. Then by (2.15.d) and (2.15.e),

$$\begin{aligned}
e_i(\tilde{\theta}'_i - \theta'_i) \wedge \omega_i &= e_i\big(d(\omega_i(\psi_i)) + \theta_i \wedge (\omega_i(\psi_i))\big) \\
&= e_i(d\theta_i \wedge \omega_i(\tau) + dh_i \wedge \omega_i) \\
&= e_i(\widehat{\partial\theta}_i + dh_i) \wedge \omega_i, \\
e_j h_j \omega_j - e_i h_i \omega_i &= -\beta_{ij} \wedge \omega_i(\tau) + e_i(\tilde{g}_{ij} - g_{ij})\omega_i \\
&= (-\widehat{\beta}_{ij} + (\tilde{g}_{ij} - g_{ij})) \wedge \omega_i.
\end{aligned}$$

By repeating the same argument as in the proof of Claim 1, one obtains the equation

$$\sum_{k=0}^{q}(d\theta+\beta)^k \cup (\widehat{\partial\theta}+\widehat{\beta}) \cup (d\theta+\beta)^{q-k}=0.$$

Hence

$$\begin{aligned}
&(2\pi\sqrt{-1})^{q+1}\left(D_{\widetilde{\sigma}}B_q(\nabla,\widetilde{\nabla}')-D_{\sigma}B_q(\nabla,\nabla')\right)\\
=&\sum_{k=0}^{q}(d\theta+\beta)^k \cup (\tilde{\theta}'+\tilde{g}-\theta'-g) \cup (d\theta+\beta)^{q-k}\\
=&\sum_{k=0}^{q}(d\theta+\beta)^k \cup (\widehat{\partial\theta}+dh+\delta h+\widehat{\beta}) \cup (d\theta+\beta)^{q-k}\\
=&\sum_{k=0}^{q}(d\theta+\beta)^k \cup (\delta h+dh) \cup (d\theta+\beta)^{q-k}\\
=&\mathcal{D}\left(\sum_{k=0}^{q}(d\theta+\beta)^k \cup\, h\, \cup (d\theta+\beta)^{q-k}\right).
\end{aligned}$$

This completes the proof of Claim 3.

Claim 4. The class $[D_\sigma B_q(\nabla,\nabla')]$ is independent of the choice of ∇.

Let $\widetilde{\nabla}=(\{\varphi_i\},\{\rho_{ij}\})$ be another Bott connection and set $\psi_i=\varphi_i-\theta_i$, then $\psi_i\in I^1_{(1)}(U_i)$. Assume that $\{\sigma_i\}$ satisfies (2.11.a), (2.11.b) and (2.11.c), then $d(\omega_i(\sigma_i))+\varphi_i\wedge(\omega_i(\sigma_i))=\theta'_i\wedge\omega_i+\psi_i\wedge(\omega_i(\sigma_i))=\varphi'_i\wedge\omega_i$ for some 1-form φ'_i. Noticing that (2.11.b) for $\widetilde{\nabla}$ is the same as (2.11.b) for ∇ because $s=\{s_{ij}\}=0$, we may adopt $(\{\varphi'_i\},\{g_{ij}\})$ as an infinitesimal derivative of $\widetilde{\nabla}$. Denote $\{\psi_i\}$ by ψ, then $\mathcal{D}\psi=(d\varphi+\rho)-(d\theta+\beta)$. Setting $\psi'=\varphi'-\theta'$, one has $\psi_i\wedge(\omega_i(\sigma_i))=\psi'_i\wedge\omega_i$.

Define $\widetilde{\partial\theta}'$ and $\widetilde{\beta}'$ as in the proof of Claim 1 and define $\widetilde{\partial\varphi}'$ and $\widetilde{\rho}'$ in the same way, then by repeating the argument as above, one obtains

(2.15.f) $$\begin{aligned}&\psi'\cup(d\varphi+\rho)^q+\psi\cup(\widetilde{\partial\varphi}'+\widetilde{\rho}')\cup(d\varphi+\rho)^{q-1}+\cdots\\&+\psi\cup(d\varphi+\rho)^{q-1}\cup(\widetilde{\partial\varphi}'+\widetilde{\rho}')=0,\end{aligned}$$

(2.15.g) $$\begin{aligned}&(\widetilde{\partial\theta}'+\widetilde{\beta}')\cup(d\theta+\beta)^k\cup\psi\cup(d\varphi+\rho)^{q-k-1}\\
&+(d\theta+\beta)\cup(\widetilde{\partial\theta}'+\widetilde{\beta}')\cup(d\theta+\beta)^{k-1}\cup\psi\cup(d\varphi+\rho)^{q-k-1}\\
&+\cdots\\
&+(d\theta+\beta)^k\cup\psi'\cup(d\varphi+\rho)^{q-k-1}\\
&+(d\theta+\beta)^k\cup\psi\cup(\widetilde{\partial\varphi}'+\widetilde{\rho}')\cup(d\varphi+\rho)^{q-k-2}\\
&+\cdots\\
&+(d\theta+\beta)^k\cup\psi\cup(d\varphi+\rho)^{q-k-2}\cup(\widetilde{\partial\varphi}'+\widetilde{\rho}')=0.\end{aligned}$$

On the other hand,

$$(2.15.h)\quad (2\pi\sqrt{-1})^{q+1}\Big(D_\sigma B_q(\widetilde{\nabla},\widetilde{\nabla}')-D_\sigma B_q(\nabla,\nabla')\Big)$$

$$=\begin{pmatrix}\psi'\cup(d\varphi+\rho)^q+(\theta'+g)\cup\mathcal{D}\psi\cup(d\varphi+\rho)^{q-1}\\ +(\theta'+g)\cup(d\theta+\beta)\cup\mathcal{D}\psi\cup(d\varphi+\rho)^{q-2}\\ +\cdots+(\theta'+g)\cup(d\theta+\beta)^{q-1}\cup\mathcal{D}\psi\end{pmatrix}$$

$$+\begin{pmatrix}\mathcal{D}\psi\cup(\varphi'+g)\cup(d\varphi+\rho)^{q-1}+(d\theta+\beta)\cup\psi'\cup(d\varphi+\rho)^{q-1}\\ +(d\theta+\beta)\cup(\theta'+g)\cup\mathcal{D}\psi\cup(d\varphi+\rho)^{q-2}+\cdots\\ +(d\theta+\beta)\cup(\theta'+g)\cup(d\theta+\beta)^{q-2}\cup\mathcal{D}\psi\end{pmatrix}$$

$$+\cdots$$

$$+\mathcal{D}\psi\cup(d\varphi+\rho)^{q-1}\cup(\varphi'+g)+\cdots+(d\theta+\beta)^q\cup\psi'.$$

Since $\psi\in I^1_{(1)}(\mathcal{U})$, one has

$$(2.15.i)\quad \begin{aligned}&\mathcal{D}\big((d\theta+\beta)^m\cup(\theta'+g)\cup(d\theta+\beta)^k\cup\psi\cup(d\varphi+\rho)^l\big)\\ =&(d\theta+\beta)^m\cup(\widetilde{\partial\theta'}+\widetilde{\beta}')\cup(d\theta+\beta)^k\cup\psi\cup(d\varphi+\rho)^l\\ &-(d\theta+\beta)^m\cup(\theta'+g)\cup(d\theta+\beta)^k\cup\mathcal{D}\psi\cup(d\varphi+\rho)^l,\end{aligned}$$

and

$$(2.15.j)\quad \begin{aligned}&\mathcal{D}\big(-(d\theta+\beta)^m\cup\psi\cup(d\varphi+\rho)^k\cup(\varphi'+g)\cup(d\varphi+\rho)^l\big)\\ =&-(d\theta+\beta)^m\cup\mathcal{D}\psi\cup(d\varphi+\rho)^k\cup(\varphi'+g)\cup(d\varphi+\rho)^l\\ &+(d\theta+\beta)^m\cup\psi\cup(d\varphi+\rho)^k\cup(\widetilde{\partial\varphi'}+\widetilde{\rho}')\cup(d\varphi+\rho)^l,\end{aligned}$$

where $m+k+l=q-1$.

Adding (2.15.i) and (2.15.j) to the right hand side of (2.15.h) varying m,k,l and by using (2.15.f) and (2.15.g), one sees that $D_\sigma B_q(\widetilde{\nabla},\widetilde{\nabla}')-D_\sigma B_q(\nabla,\nabla')$ is exact.

Claim 5. $D_\sigma B_q(\nabla,\nabla')$ is independent of the choice of the family of local trivialization $\{e_i\}$.

Fix σ, $\nabla=(\{\theta_i\},\{\beta_{ij}\})$, and let $\{e'_i\}$ be another family of local trivializations, then we may assume that $e'_i=e_iu_i$ for some $\boldsymbol{C}^*$-valued function u_i. Hence $\omega'_i=u_i^{-1}\omega_i$ and $e'_j=u_ju_i^{-1}\alpha_{ij}e'_i$. It is easy to see that the connection form of ∇ with respect to $\{e'_i\}$ is $(\{\theta_i+u_i^{-1}du_i\},\{\beta_{ij}\})$. It follows the equation

$$\begin{aligned}&e'_i\big(d(\omega'_i(\sigma_i))+(\theta_i+u_i^{-1}du_i)\wedge(\omega'_i(\sigma_i))\big)\\ =&e_iu_i\big(-u_i^{-2}du_i\wedge\omega_i(\sigma_i)+u_i^{-1}d(\omega_i(\sigma_i))+u_i^{-1}\theta_i\wedge\omega_i(\sigma_i)+u_i^{-2}du_i\wedge\omega_i(\sigma_i)\big)\\ =&e_i\big(d(\omega_i(\sigma_i))+\theta_i\wedge(\omega_i(\sigma_i))\big)\\ =&-e_i\theta'_i\wedge\omega_i=-e'_i\theta'_i\wedge\omega'_i.\end{aligned}$$

One also has $\sigma_j - \sigma_i = e_i g_{ij} \omega_i = e'_i g_{ij} \omega'_i$. Hence we may still adopt $(\{\theta'_i\}, \{g_{ij}\})$ as an infinitesimal derivative of ∇. This completes the proof of Claim 5 and therefore of the theorem. $\square$

Definition 2.16. The infinitesimal derivative of the Bott class with respect to $\mu \in \underline{H}^1(M; -K_{\mathcal{F}})$ is the cohomology class in $H^{2q+1}(M; \boldsymbol{C})$ represented by $D_\sigma B_q(\nabla, \nabla')$ in Theorem 2.15 and denoted by $D_\mu B_q(\mathcal{F})$.

Definition 2.16 is compatible with Definition 2.7 as follows.

Theorem 2.17. *Let $\underline{\mu} \in \underline{H}^1(M; \Theta_{\mathcal{F}})$, then $D_{\underline{\mu}} B_q(\mathcal{F}) = D_{r(\underline{\mu})} B_q(\mathcal{F})$, where the left hand side is defined in Definition 2.7 and the right hand side is defined in Definition 2.16.*

Proof. This is an immediate consequence of Lemma 2.13 and Theorem 2.15. $\square$

In the rest of this section, some justification of Definition 2.16 will be given. We will first recall how a smooth family $\{\mathcal{F}_s\}$ of transversely holomorphic foliations induces an element of $H^1(M; \Theta_{\mathcal{F}})$ [12]. We may assume that there is a family $\{\underline{\omega}_s\}$ of local trivializations of $Q(\mathcal{F}_s)$ such that the transition functions $(\underline{A}_s)_{ji}$ are constant: $(\underline{\omega}_s)_j = \underline{A}_{ji} (\underline{\omega}_s)_i$. Choose a family $\{\underline{\nabla}_s\}$ of Bott connections so that the corresponding connection forms $(\{(\underline{\theta}_s)_i\}, \{(\underline{\beta}_s)_{ij}\})$ form a smooth family and the equation $d(\underline{\omega}_s)_i = -(\underline{\theta}_s)_i \wedge (\underline{\omega}_s)_i$ holds.

Fix a smooth family $T_{\boldsymbol{C}} M = E_s \oplus \nu_s$ of splittings and denote by π'_s the projection from $T_{\boldsymbol{C}} M$ to ν_s. Let $\pi_0 : T_{\boldsymbol{C}} M \to Q(\mathcal{F}_0)$ be the projection to the normal bundle. Define then a section σ of $E_0^* \otimes Q(\mathcal{F}_0)$ by setting $\sigma(X) = -\pi_0 \left(\left. \frac{\partial}{\partial s} \pi'_s(X) \right|_{s=0} \right)$. Note that σ is in fact a $Q(\mathcal{F}_0)$-valued 1-form so that $\sigma_j - \sigma_i = 0$, or equivalently, $g_{ij} = 0$. One can verify that σ is a cocycle by using the equation $d\dot{\underline{\omega}}_i + (\underline{\theta}_0)_i \wedge \dot{\underline{\omega}}_i = -\dot{\underline{\theta}}_i \wedge (\underline{\omega}_0)_i$, where $\dot{\underline{\omega}} = \left. \frac{\partial}{\partial s} \underline{\omega}_s \right|_{s=0}$ and $\dot{\underline{\theta}} = \left. \frac{\partial}{\partial s} \underline{\theta}_s \right|_{s=0}$. Thus $\{\mathcal{F}_s\}$ induces in fact an element of $\underline{H}^1(M; \Theta_{\mathcal{F}})$, hence also an element $\underline{H}^1(M; -K_{\mathcal{F}})$ by Lemma 2.13.

Theorem 2.18. *If $\mu \in \underline{H}^1(M; -K_{\mathcal{F}})$ is derived from a smooth family $\{\mathcal{F}_s\}$, then $D_\mu B_q(\mathcal{F}) = \left. \frac{\partial}{\partial s} B_q(\mathcal{F}_s) \right|_{s=0}$.*

Proof. Let $\underline{\mu}$ be the element of $\underline{H}^1(M; \Theta_{\mathcal{F}})$ determined by $\{\mathcal{F}_s\}$, then it suffices to show that $D_{r(\underline{\mu})} B_q(\mathcal{F}) = \left. \frac{\partial}{\partial s} B_q(\mathcal{F}_s) \right|_{s=0}$. One has then the following equation, namely,

$$\begin{aligned}
\sigma(X)|_{U_i} &= -\pi_0 \left(\frac{\partial}{\partial s} \sum_{k=1}^{q} \underline{\tilde{e}}(s)_{i,k} \underline{\omega}(s)_i^k(X) \Big|_{s=0} \right) \\
&= -\sum_{k=1}^{q} \underline{\tilde{e}}(0)_{i,k} \frac{\partial}{\partial s} \underline{\omega}(s)_i^k(X) \Big|_{s=0} \\
&= -\sum_{k=1}^{q} \underline{\tilde{e}}(0)_{i,k} \underline{\dot{\omega}}_i^k(X) \\
&= -\underline{e}(0)_i \underline{\dot{\omega}}_i(X).
\end{aligned}$$

Hence $\{\underline{\dot{\theta}}_i\}$ can be chosen as an infinitesimal derivative when calculating $D_{r(\underline{\mu})} B_q(\mathcal{F})$. Theorem now follows from Proposition 1.10. $\square$

The infinitesimal derivative of the Bott class constructed above is related with the previously constructed infinitesimal derivatives as follows.

Theorem 2.19. *Let* $\underline{\mu} \in \underline{H}^1(M; \Theta_{\mathcal{F}})$.

1) *Assume that* $-K_{\mathcal{F}}$ *is trivial, then* $D_{\underline{\mu}} B_q(\mathcal{F})$ *coincides with the infinitesimal derivative of the Bott class in* [13].
2) *Let* $D_{\underline{\mu}} \xi_q(\mathcal{F})$ *be the infinitesimal derivative of the imaginary part of the Bott class defined in* [13] (*and* [4]), *then* $D_{\underline{\mu}} \xi_q(\mathcal{F}) = -2\mathrm{Im}\, D_{\underline{\mu}} B_q(\mathcal{F})$.

Proof. These infinitesimal derivatives are constructed under the assumption that $\beta = 0$ and $g = 0$. Hence $D_{\underline{\mu}} B_q(\mathcal{F})$ is represented by a global $(2q+1)$-form $(-2\pi\sqrt{-1})^{-(q+1)} \theta' \wedge (d\theta)^q$. The claims are now obvious. $\square$

3. Schwarzian Derivatives

Notation 3.1. The natural coordinate of $\boldsymbol{C}^q$ will be usually denoted by $z = {}^t(z^1, \cdots, z^q)$. Holomorphic vectors of the form $X^1 \frac{\partial}{\partial z^1} + \cdots + X^q \frac{\partial}{\partial z^q}$ are usually abbreviated as $\frac{\partial}{\partial z} X$, where $\frac{\partial}{\partial z} = \begin{pmatrix} \frac{\partial}{\partial z^1} & \cdots & \frac{\partial}{\partial z^q} \end{pmatrix}$ and $X = {}^t(X^1 \ \cdots \ X^q)$. (The partial derivative of X by z is denoted by $\frac{\partial X}{\partial z}$.) Similarly, holomorphic 1-forms of the form $a_1 dz^1 + \cdot + a_q dz^q$ are denoted as $a\, dz = (a_1 \ \cdots \ a_q)\, {}^t(dz^1 \ \cdots dz^q)$. In what follows, tensors are usually represented in this way, namely, they will represented in matrices and the multiplications are considered under the usual multiplication laws together with the tensor or wedge products.

Notation 3.2. Sections of $\bigotimes^p Q(\mathcal{F}) \otimes \bigotimes^q Q(\mathcal{F})^*$ are said to be tensors of type $[p,q]$ or a $[p,q]$-tensor, in order to avoid confusions with Čech-de Rham cochains of bidegree (p,q). A $[0,q]$-tensor is also called simply a q-tensor.

Definition 3.3 ([16], [20], etc.)**.** Let γ be a biholomorphic local diffeomorphism of $\boldsymbol{C}^q$. Let $u = {}^t(u^1, \cdots, u^q)$ be the natural coordinate for the target so that $u = \gamma(z)$. Denote by γ^k be the k-th component of γ: $\gamma = {}^t(\gamma^1, \cdots, \gamma^q)$. The projective Schwarzian derivative Σ_γ of γ is a tensor of type $[1,2]$ given as follows:

$$\begin{aligned}\Sigma_\gamma = &\sum_{k,l,t,s} \frac{\partial z^l}{\partial u^k} \frac{\partial^2 \gamma^k}{\partial z^t \partial z^s} \frac{\partial}{\partial z^l} \otimes dz^t \otimes dz^s \\ &+ \sum_{l,t,s} \frac{-1}{q+1} \left(\frac{\partial \log J_\gamma}{\partial z^t} \delta_{l,s} \frac{\partial}{\partial z^l} \otimes dz^t \otimes dz^s + \frac{\partial \log J_\gamma}{\partial z^s} \delta_{l,t} \frac{\partial}{\partial z^l} \otimes dz^t \otimes dz^s \right),\end{aligned}$$

where $D\gamma$ denotes the differential of γ, $J_\gamma = \det D\gamma$ is the Jacobian and $\delta_{l,t}$ is the Kronecker delta. If $q > 1$, then let $\Sigma^l_{t,s}$ be the coefficient of $\dfrac{\partial}{\partial z^l} \otimes dz^t \otimes dz^s$ in Σ_γ and define a 2-tensor Λ_γ by the formula

$$\begin{aligned}\Lambda_\gamma = &\frac{-1}{q-1} \sum_{l=1}^q \left(\frac{\partial \Sigma^l_{t,s}}{\partial z^l} - \sum_{u=1}^q \Sigma^l_{t,u} \Sigma^u_{s,l} \right) dz^t \otimes dz^s \\ = &\sum_{t,s} \frac{-1}{q+1} \frac{\partial^2 \log J_\gamma}{\partial z^t \partial z^s} dz^t \otimes dz^s \\ &- \sum_{t,s} \frac{-1}{q+1} \frac{\partial \log J_\gamma}{\partial z^t} \frac{-1}{q+1} \frac{\partial \log J_\gamma}{\partial z^s} dz^t_i \otimes dz^s \\ &- \sum_{l,t,s} \frac{-1}{q+1} \frac{\partial \log J_\gamma}{\partial z^l} \frac{\partial z^l}{\partial u^k} \frac{\partial^2 \gamma^k}{\partial z^t \partial z^s} dz^t \otimes dz^s.\end{aligned}$$

If $q = 1$, then we define Λ_γ directly by the above formula because $\Sigma_\gamma = 0$, then

$$\Lambda_\gamma = -\frac{1}{2} \left(\frac{\gamma'''}{\gamma'} - \frac{3}{2} \left(\frac{\gamma''}{\gamma'} \right)^2 \right) dz \otimes dz,$$

where $\gamma'_{ji} = \dfrac{d\gamma_{ji}}{dz_i}$, $\gamma''_{ji} = \dfrac{d^2\gamma_{ji}}{dz_i^2}$ and $\gamma'''_{ji} = \dfrac{d^3\gamma_{ji}}{dz_i^3}$ by definition so that Λ_γ is the classical Schwarzian. Finally, the projective Schwarzian derivatives are also called the Schwarzian derivatives or the Schwarzians for short.

It is classical that γ is the restriction of a projective transformation if and only if $\Lambda_\gamma = 0$ if $q = 1$. When $q > 1$, then the following is a fundamental

Fact 3.4 ([18],[19] for 1) and 2), [8] and [18] for 3)).

1) *If $q > 1$, then γ is the restriction of a projective transformation if and only if $\Sigma_\gamma = 0$.*
2) $\Sigma^l_{t,s} = \Sigma^l_{t,s}$ *and* $\sum\limits_{l=1}^{q} \Sigma^l_{l,s} = 0$.
3) Λ_γ *can be seen as a kind of the curvature tensor for* Σ_γ.

The following Lemma is useful in succeeding calculations.

Lemma 3.5. *Set* $\partial \log J_\gamma = \begin{pmatrix} \dfrac{\partial \log J_\gamma}{\partial z^1} & \cdots & \dfrac{\partial \log J_\gamma}{\partial z^q} \end{pmatrix}$, *then*

$$\begin{aligned}
\Sigma_\gamma =& \frac{\partial}{\partial z} \otimes D\gamma^{-1} \cdot dD\gamma \otimes dz \\
&+ \sum_{k=1}^{q} \frac{-1}{q+1} \left(\frac{\partial}{\partial z^k} \otimes (\partial \log J_\gamma \cdot dz) \otimes dz^k + \frac{\partial}{\partial z^k} \otimes dz^k \otimes (\partial \log J_\gamma \cdot dz) \right), \\
\Lambda_\gamma =& \frac{-1}{q+1} d\partial \log J_\gamma \otimes dz - \frac{-1}{q+1} \partial \log J_\gamma D\gamma^{-1} \cdot dD\gamma \otimes dz \\
&- \frac{-1}{q+1} (\partial \log J_\gamma \cdot dz) \otimes \frac{-1}{q+1} (\partial \log J_\gamma \cdot dz).
\end{aligned}$$

Let X be a vector field on an open set U of $\boldsymbol{C}$, then denote by ι_X the interior product with X. For a p-form ω, set $\iota'_X \omega = (-1)^{p-1} \iota_X \omega$, or equivalently, $\iota'_X \omega = \omega(\,\cdot\,, \cdots, \,\cdot\,, X)$.

Definition 3.6. For a p-form ω, define a $Q(\mathcal{F})^*$-valued p-form $\langle \omega \,|\, \Sigma_\gamma \rangle$ be setting

$$\langle \omega \,|\, \Sigma_\gamma \rangle = \sum_{i,t,s} (\iota'_{\partial_i} \omega) \Sigma^i_{t,s} \wedge dz^t \otimes dz^s,$$

where $\iota'_{\partial_i} = \iota'_{\frac{\partial}{\partial z^i}}$. If in addition a $Q(\mathcal{F})$-valued 1-form $\sigma = \sum \dfrac{\partial}{\partial z^i} \otimes \sigma^i$ is given, set

$$\langle \omega \,|\, \Sigma_\gamma |\, \sigma \rangle = \sum_{i,t,s} (\iota'_{\partial_i} \omega) \Sigma^i_{t,s} \wedge dz^t \otimes \sigma^s,$$

which is also a $Q(\mathcal{F})^*$-valued p-form. We define a 2-tensor $\langle \Lambda_\gamma |\, \sigma \rangle$ and a p-form $\langle \omega \,|\, \sigma \rangle$ in a similar way. Note that we have

$$\begin{aligned}
&\langle \omega_1 \wedge \cdots \wedge \omega_r |\, \sigma \rangle \\
=& \langle \omega_1 | \sigma \rangle \wedge \omega_2 \wedge \cdots \wedge \omega_r + \omega_1 \wedge \langle \omega_2 | \sigma \rangle \wedge \omega_3 \wedge \cdots \wedge \omega_r + \cdots + \omega_1 \wedge \cdots \wedge \omega_{r-1} \wedge \langle \omega_r | \sigma \rangle
\end{aligned}$$

for differential forms $\omega_1, \cdots, \omega_r$.

By abuse of notation, the differential forms obtained by reduction are also denoted by the same symbols if there is no fear of confusions, e.g., $\langle \omega \,|\, \Sigma_\gamma \rangle$ will also stand for a $(p+1)$-form.

Lemma 3.7. *Let γ_{ji} and γ_{kj} be local biholomorphic diffeomorphisms and let z_i and z_j be the variables of γ_{ji} and γ_{kj}, respectively. Set $\gamma_{ki} = \gamma_{kj} \circ \gamma_{ji}$, and denote $\Sigma_{\gamma_{ab}}$ and $\Lambda_{\gamma_{ab}}$ by Σ_{ba} and Λ_{ba}, where $ab = ji$, kj or ki, then*

$$\begin{aligned}
&\gamma_{ji}^*\Lambda_{jk} - \Lambda_{ik} + \Lambda_{ij} = \frac{1}{q+1}\left\langle d\log J_{ij} \,\middle|\, \gamma_1^*\Sigma_{jk}\right\rangle,\\
&\gamma_{ji}^*\Sigma_{jk} - \Sigma_{ik} + \Sigma_{ij} = 0.
\end{aligned}$$

Proof. Denote $J_{\gamma_{ab}}$ simply by J_{ab}, where $ab = ji$, kj or ki, then we have by Lemma 3.5 the following equation, namely,

$$\begin{aligned}
(q+1)\Lambda_{ik} =&(q+1)\Lambda_{\gamma_{ki}}\\
=&-d\partial_i \log J_{ki} \otimes dz_i + \partial_i \log J_{ki} D\gamma_{ki}^{-1} dD\gamma_{ki} \otimes dz_i\\
&-\frac{1}{q+1}(d\log J_{ki}) \otimes (d\log J_{ki})\\
=&-d(\partial_j \log J_{kj} D\gamma_{ji} + \partial_i \log J_{ji}) \otimes dz_i\\
&+(\partial_j \log J_{kj} D\gamma_{ji} + \partial_i \log J_{ji}) D\gamma_{ji}^{-1} D\gamma_{kj}^{-1}(dD\gamma_{kj} D\gamma_{ji} + D\gamma_{kj} dD\gamma_{ji}) \otimes dz_i\\
&-\frac{1}{q+1}(d\log J_{kj} + d\log J_{ji}) \otimes (d\log J_{kj} + d\log J_{ji})\\
=&(q+1)\Lambda_{ij} + (q+1)\gamma_{ji}^*\Lambda_{jk}\\
&+\partial_j \log J_{ji} D\gamma_{kj}^{-1} dD\gamma_{kj} \otimes dz_j\\
&-\frac{1}{q+1}\left(d\log J_{kj} \otimes d\log J_{ji} + d\log J_{ji} \otimes d\log J_{kj}\right)\\
=&\left\langle d\log J_{ji} \middle| \, \Sigma_{jk}\right\rangle.
\end{aligned}$$

The equation for Σ can be shown in a parallel way. □

In what follows, pull-backs of the tensors are abbreviated, e.g., $\gamma_1^*\Lambda_{12}$ is simply denoted by Λ_{12}.

4. Relation between the infinitesimal derivative of the Bott class and the Schwarzian derivative

Let $\omega = \{\omega_i\}$ be a family of local trivializations of $-K_{\mathcal{F}} = \bigwedge^q Q^*(\mathcal{F})$ and let ∇ be a family of local Bott connections on $-K_{\mathcal{F}}$ induced by a family of Bott connections on $Q(\mathcal{F})$. For each i, let z_i be the local coordinate in the transversal direction and let $\{\gamma_{ji}\}$ be the transition functions in the transversal direction so that $z_j = \gamma_{ji}(z_i)$. Finally let μ be an element of $\underline{H}^1(M; -K_{\mathcal{F}})$, then μ can be regarded as an element of $\underline{H}^1(M; \Theta_{\mathcal{F}})$ by Lemma 2.13. Let $\sigma = \{\sigma_i\}$ be a representative of the latter element.

Definition 4.1. Set $\Sigma_{ij} = \Sigma_{\gamma_{ji}}$ and $\Lambda_{ij} = \Lambda_{\gamma_{ji}}$. Let $\{\theta_i\}$ be the family of local connection forms of ∇ with respect to $\left\{\frac{\partial}{\partial z_i^1} \wedge \cdots \wedge \frac{\partial}{\partial z_i^q}\right\}$. then we define Čech-de Rham $(1,2)$-cochains $L = \{L_{ij}\}$ and $S = \{S_{ij}\}$ by setting

$$\begin{aligned} L_{ij} &= \langle \Lambda_{ij} | \, \sigma_i \rangle \, , \\ S_{ij} &= \langle \theta_i | \, \Sigma_{ij} | \, \sigma_i \rangle \, , \end{aligned}$$

where the right hand sides are considered as 2-forms by reduction. More explicitly,

$$\begin{aligned} L_{ij} =& \frac{-1}{q+1}(d\partial_j \log J_{ji}) \wedge \sigma_i' - \frac{-1}{q+1}\partial_i \log J_{ji} D\gamma_{ji}^{-1} \cdot dD\gamma_{ji} \wedge \sigma_i' \\ & - \left(\frac{-1}{q+1} d\log J_{ji}\right) \wedge \left(\frac{-1}{q+1}\langle d\log J_{ji} | \sigma_i\rangle\right) \\ S_{ij} =& f_i D\gamma_{ji}^{-1} \cdot dD\gamma_{ji} \wedge \sigma_i' \\ & + \frac{-1}{q+1}\big((\partial_i \log J_{ji} \cdot dz_i) \wedge (f_i \cdot \sigma_i') + (f_i \cdot dz_i) \wedge (\partial_i \log J_{ji} \cdot \sigma_i')\big) \\ =& f_i D\gamma_{ji}^{-1} \cdot dD\gamma_{ji} \wedge \sigma_i' \\ & + \frac{-1}{q+1}\big(d\log J_{ji} \wedge \langle \theta_i | \sigma_i\rangle + \theta_i \wedge \langle d\log J_{ji} | \sigma_i\rangle\big), \end{aligned}$$

where $\partial_i \log J_{ji} = \left(\frac{\partial \log J_{ji}}{\partial z_i^1}, \cdots, \frac{\partial \log J_{ji}}{\partial z_i^q}\right)$, $\theta_i = f_i \cdot dz_i$, $J_{ji} = J_{\gamma_{ji}}$ and σ_i' is the 1-form such that $\sigma_i = \frac{\partial}{\partial z_i} \cdot \sigma_i'$.

In what follows, we adopt the following

Notation 4.2.

$$\begin{aligned} (\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l} &= d\log J_{i_0 i_1} \wedge d\log J_{i_1 i_2} \wedge \cdots \wedge d\log J_{i_{l-1} i_l}, \text{ and} \\ (d\log J)^l_{i_0\cdots i_l} &= (d\log J \cup d\log J \cup \cdots \cup d\log J)_{i_0\cdots i_l} \\ &= (-1)^{\frac{l(l-1)}{2}} d\log J_{i_0 i_1} \wedge \cdots \wedge d\log J_{i_{l-1} i_l} \\ &= (-1)^{\frac{l(l-1)}{2}} (\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}. \end{aligned}$$

A generalization of the Maszczyk formula [17] for arbitrary transversely holomorphic foliations is as follows.

Theorem 4.3. *Let $\mu \in \underline{H}^1(M; -K_{\mathcal{F}})$ be an infinitesimal derivative. Consider μ as an element of $\underline{H}^1(M; \Theta_{\mathcal{F}})$ and let $\sigma \in I^1_{(0,1)}(M; Q(\mathcal{F}))$ be a representative of μ,* *then the infinitesimal derivative of the Bott class $D_\mu B_q(\mathcal{F})$ is represented by the Čech-de Rham $(q,q+1)$-cocycle whose value on $U_{i_0\cdots i_q}$ is given by*

$$\begin{aligned}&\sum_{l=0}^{q-1}(-1)^{q-l-1}(\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge L_{i_l i_q}\wedge(\textstyle\bigwedge^{q-l-1}d\log J)_{i_{l+1}\cdots i_q}\\ &-\sum_{l=0}^{q-2}(-1)^{q-l-1}(\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge L_{i_{l+1} i_q}\wedge(\textstyle\bigwedge^{q-l-1}d\log J)_{i_{l+1}\cdots i_q}\end{aligned}$$

multiplied by $(-2\pi\sqrt{-1})^{-(q+1)}(q+1)^2(-1)^{\frac{q(q+1)}{2}}$. If $q=1$, then the infinitesimal Bott class is represented by the Čech-de Rham $(1,2)$-cocycle

$$-\frac{1}{2\pi^2}\left(\frac{\gamma'''}{\gamma'}-\frac{3}{2}\left(\frac{\gamma''^2}{\gamma'}\right)\right)dz\wedge\sigma.$$

Theorem 4.3 will be shown in steps. We compute firstly the derivatives of L and S.

Lemma 4.4. *The following equations hold modulo $I_{(q-1)}$ after reduction to differential forms:*

$$\begin{aligned}(\delta L)_{ijk}&=\frac{1}{q+1}\left\langle d\log J_{ij}\right|\Sigma_{jk}\left|\,\sigma_j\right\rangle,\\ (\delta S)_{ijk}&=\left\langle d\log J_{ij}\right|\Sigma_{jk}\left|\,\sigma_j\right\rangle,\\ (dL)_{ij}&=0,\\ (dS)_{ij}&=\langle d\theta_j|\,\Sigma_{ij}|\,\sigma_j\rangle.\end{aligned}$$

Proof. First, since $\sigma\in I^1_{(0,1)}(M;Q(\mathcal{F}))$, one has the following equations modulo $I_{(q-1)}$ by Lemma 3.7, namely,

$$\begin{aligned}(\delta L)_{ijk}=&\langle\Lambda_{ij}|\,\sigma_i\rangle-\langle\Lambda_{ik}|\,\sigma_i\rangle+\langle\Lambda_{jk}|\,\sigma_j\rangle\\ =&\langle(\delta\Lambda)_{ijk}|\,\sigma_j\rangle\\ =&\frac{1}{q+1}\langle d\log J_{ij}|\,\Sigma_{jk}|\,\sigma_j\rangle,\\ (\delta S)_{ijk}=&\langle\theta_i|\,\Sigma_{ij}|\,\sigma_i\rangle-\langle\theta_i|\,\Sigma_{ik}|\,\sigma_i\rangle+\langle\theta_j|\,\Sigma_{jk}|\,\sigma_j\rangle\\ =&\langle\theta_i|\,(\delta\Sigma)_{ijk}|\,\sigma_i\rangle+\langle(\delta\theta)_{ij}|\,\Sigma_{jk}|\,\sigma_j\rangle\\ =&\langle d\log J_{ij}|\,\Sigma_{jk}|\,\sigma_j\rangle.\end{aligned}$$

Second, from the fact that $d\sigma_k^i \wedge dz_i^1 \wedge \cdots \wedge dz_i^q = 0$ for any k, it follows easily that $(dL)_{ij} = 0$ modulo $I_{(q-1)}$. For the same reason, dS_{ij} modulo $I_{(q-1)}$ is calculated as follows:

$$\begin{aligned}
dS_{ij} &= df_i D\gamma_{ji}^{-1} \cdot dD\gamma_{ji} \wedge \sigma_i' + f_i d(D\gamma_{ji}^{-1}) \cdot dD\gamma_{ji} \wedge \sigma_i' \\
&\quad + \frac{-1}{q+1} \left(-d \log J_{ji} \wedge \langle d\theta_i |\, \sigma_i \rangle + d\theta_i \wedge \langle d \log J_{ji} |\, \sigma_i \rangle - \theta_i \wedge \langle d\partial_i \log J_{ji} |\, \sigma_i \rangle \right) \\
&= df_i D\gamma_{ji}^{-1} \cdot dD\gamma_{ji} \wedge \sigma_i' \\
&\quad + \frac{-1}{q+1} \left(-d \log J_{ji} \wedge \langle d\theta_i |\, \sigma_i \rangle + d\theta_i \wedge \langle d \log J_{ji} |\, \sigma_i \rangle \right) \\
&= \langle d\theta_j |\, \Sigma_{ij} |\, \sigma_j \rangle,
\end{aligned}$$

where the sign of the second term is due to the fact that df_j is a 1-form. $\square$

Definition 4.5. Let $c_0, \cdots, c_q$ be an ordered sequence of cochains such that c_0 and c_1 are cochains of degree 1 and the others are of degree 2. Set then

$$S(c_0, \cdots, c_q) = \sum_{\tau \in \mathfrak{S}_{q+1}} \epsilon\, c_{\tau(0)} \cup \cdots \cup c_{\tau(q)},$$

where $\epsilon = -1$ if $\tau(0) > \tau(1)$, otherwise $\epsilon = 1$. Even if c_0 or c_1 is of degree 2, we denote by $S(c_0, \cdots, c_q)$ the cochain obtained by the above formula, where ϵ is always set to be 1. By abuse of notation, repetition of cochains (of even degree) is represented by superscripts, e.g., $S(c_0, c_1, c_2^i, c_3^{q-i-1}) = S(c_0, c_1, c_2, \cdots, c_2, c_3, \cdots, c_3)$, where c_2 appears i-times and c_3 appears $(q-i-1)$-times.

Lemma 4.6. *Suppose that each c_i belongs to $I_{(1)}$, then*

$$S(\langle c_0 \,|\, \sigma \rangle, c_1, \cdots, c_q) + S(c_0, \langle c_1 \,|\, \sigma \rangle, c_2, \cdots, c_q) + \cdots + S(c_0, \cdots, c_{q-1}, \langle c_q \,|\, \sigma \rangle) = 0.$$

Proof. It is easy to see that the mapping such that

$$\varphi_0 \wedge \cdots \wedge \varphi_q \mapsto \langle \varphi_0 \,|\, \sigma \rangle \wedge \varphi_1 \wedge \cdots \wedge \varphi_q + \cdots + \varphi_0 \wedge \cdots \wedge \varphi_{q-1} \wedge \langle \varphi_q \,|\, \sigma \rangle$$

is well-defined. It follows that the left hand side of the formula in the claim is equal to the image of $S(c_0, \cdots, c_q)$ by this mapping. However, $S(c_0, \cdots, c_q) = 0$ because each c_i belongs to $I_{(1)}$. $\square$

Lemma 4.7. *$S\left((\theta'+g),(d\theta+\beta)^q\right)$ is cohomologous to $(-1)^q S\left((\theta'+g+\langle\theta\,|\,\sigma\rangle),(d\log J)^q\right)$.*

Proof. Taking the signature in Definition 4.5 into account, one sees that the following equations hold. First,

$$
\begin{aligned}
&\mathcal{D}S\left(\theta,(\theta'+g),(d\theta+\beta)^i,(d\log J)^{q-i-1}\right)\\
={}&S\left(d\theta+\beta+d\log J,(\theta'+g),(d\theta+\beta)^i,(d\log J)^{q-i-1}\right)\\
&+S\left(\theta,\langle d\theta+\beta\,|\,\sigma\rangle,(d\theta+\beta)^i,(d\log J)^{q-i-1}\right)\\
={}&S\left(d\theta+\beta,(\theta'+g),(d\theta+\beta)^i,(d\log J)^{q-i-1}\right)\\
&+S\left(d\log J,(\theta'+g),(d\theta+\beta)^i,(d\log J)^{q-i-1}\right)\\
&+S\left(\theta,\langle d\theta+\beta\,|\,\sigma\rangle,(d\theta+\beta)^i,(d\log J)^{q-i-1}\right)\\
={}&S\left((\theta'+g),(d\theta+\beta)^{i+1},(d\log J)^{q-i-1}\right)\\
&+S\left((\theta'+g),(d\theta+\beta)^i,(d\log J)^{q-i}\right)\\
&+S\left(\theta,\langle d\theta+\beta\,|\,\sigma\rangle,(d\theta+\beta)^i,(d\log J)^{q-i-1}\right),
\end{aligned}
\tag{4.7.a}
$$

and

$$
\begin{aligned}
&\mathcal{D}S\left(\theta,\langle\theta\,|\,\sigma\rangle,(d\theta+\beta)^j,(d\log J)^{q-j-1}\right)\\
={}&S\left(d\theta+\beta+d\log J,\langle\theta\,|\,\sigma\rangle,(d\theta+\beta)^j,(d\log J)^{q-j-1}\right)\\
&-S\left(\theta,\langle d\theta+\beta+d\log J\,|\,\sigma\rangle,(d\theta+\beta)^j,(d\log J)^{q-j-1}\right)\\
={}&S\left(\langle\theta\,|\,\sigma\rangle,(d\theta+\beta)^{j+1},(d\log J)^{q-j-1}\right)\\
&+S\left(\langle\theta\,|\,\sigma\rangle,(d\theta+\beta)^j,(d\log J)^{q-j}\right)\\
&-S\left(\theta,\langle d\theta+\beta\,|\,\sigma\rangle,(d\theta+\beta)^j,(d\log J)^{q-j-1}\right)\\
&-S\left(\theta,\langle d\log J\,|\,\sigma\rangle,(d\theta+\beta)^j,(d\log J)^{q-j-1}\right).
\end{aligned}
\tag{4.7.b}
$$

On the other hand, the following equation holds by Lemma 3.5, namely,

$$
\begin{aligned}
&S\left(\langle\theta\,|\,\sigma\rangle,(d\theta+\beta)^i,(d\log J)^{q-i}\right)\\
={}&-iS\left(\theta,\langle d\theta+\beta\,|\,\sigma\rangle,(d\theta+\beta)^{i-1},(d\log J)^{q-i}\right)\\
&-(q-i)S\left(\theta,\langle d\log J\,|\,\sigma\rangle,(d\theta+\beta)^i,(d\log J)^{q-i-1}\right).
\end{aligned}
\tag{4.7.c}
$$

Combining (4.7.b) and (4.7.c), one has

$$
\begin{aligned}
&\mathcal{D}S\left(\theta,\langle\theta\,|\,\sigma\rangle,(d\theta+\beta)^j,(d\log J)^{q-j-1}\right)\\
={}&-(j+2)S\left(\theta,\langle d\theta+\beta\,|\,\sigma\rangle,(d\theta+\beta)^j,(d\log J)^{q-j-1}\right)\\
&-(q-j-1)S\left(\theta,\langle d\log J\,|\,\sigma\rangle,(d\theta+\beta)^{j+1},(d\log J)^{q-j-2}\right)\\
&+S\left(\langle\theta\,|\,\sigma\rangle,(d\theta+\beta)^j,(d\log J)^{q-j}\right)\\
&-S\left(\theta,\langle d\log J\,|\,\sigma\rangle,(d\theta+\beta)^j,(d\log J)^{q-j-1}\right).
\end{aligned}
\tag{4.7.d}
$$

It follows from (4.7.a) and (4.7.d) that $S\left((\theta'+g),(d\theta+\beta)^q\right)$ is cohomologous to $s(j)$, where

$$\begin{aligned} s(j) &= (-1)^j S\left((\theta'+g),(d\theta+\beta)^{q-j},(d\log J)^j\right) \\ &\quad + (-1)^j \frac{j}{q+1} S\left(\langle\theta\,|\,\sigma\rangle,(d\theta+\beta)^{q-j},(d\log J)^j\right) \\ &\quad - (-1)^j \frac{j}{q+1} S\left(\theta,\langle d\log J\,|\,\sigma\rangle,(d\theta+\beta)^{q-j},(d\log J)^{j-1}\right). \end{aligned}$$

Indeed, the claim is obvious if $j=0$. One has from (4.7.a) the following equation modulo exact cochains:

$$\begin{aligned} s(j) &= (-1)^{j+1} S\left((\theta'+g),(d\theta+\beta)^{q-j-1},(d\log J)^{j+1}\right) \\ &\quad + (-1)^{j+1} S\left(\theta,\langle d\theta+\beta\,|\,\sigma\rangle,(d\theta+\beta)^{q-j-1},(d\log J)^j\right) \\ &\quad + (-1)^{j+1} \frac{j(q-j)}{q+1} S\left(\theta,\langle d\theta+\beta\,|\,\sigma\rangle,(d\theta+\beta)^{q-j-1},(d\log J)^j\right) \\ &\quad + (-1)^{j+1} \frac{j^2}{q+1} S\left(\theta,\langle d\log J\,|\,\sigma\rangle,(d\theta+\beta)^{q-j},(d\log J)^{j-1}\right) \\ &\quad - (-1)^j \frac{j}{q+1} S\left(\theta,\langle d\log J\,|\,\sigma\rangle,(d\theta+\beta)^{q-j},(d\log J)^{j-1}\right) \\ &= (-1)^{j+1} S\left((\theta'+g),(d\theta+\beta)^{q-j-1},(d\log J)^{j+1}\right) \\ &\quad + (-1)^{j+1} \frac{(j+1)(q-j+1)}{q+1} S\left(\theta,\langle d\theta+\beta\,|\,\sigma\rangle,(d\theta+\beta)^{q-j-1},(d\log J)^j\right) \\ &\quad + (-1)^{j+1} \frac{(j+1)j}{q+1} S\left(\theta,\langle d\log J\,|\,\sigma\rangle,(d\theta+\beta)^{q-j},(d\log J)^{j-1}\right). \end{aligned}$$

Then by (4.7.d), $s(j)$ is seen to be cohomologous to the cocycle

$$\begin{aligned} &(-1)^{j+1} S\left((\theta'+g),(d\theta+\beta)^{q-j-1},(d\log J)^{j+1}\right) \\ &\quad + (-1)^{j+1} \frac{j+1}{q+1} S\left(\langle\theta\,|\,\sigma\rangle,(d\theta+\beta)^{q-j-1},(d\log J)^{j+1}\right) \\ &\quad - (-1)^{j+1} \frac{j+1}{q+1} S\left(\theta,\langle d\log J\,|\,\sigma\rangle,(d\theta+\beta)^{q-j-1},(d\log J)^j\right) \\ &= s(j+1). \end{aligned}$$

Thus the claim is proved. Finally, by the equation

$$S(\theta,\langle d\log J\,|\,\sigma\rangle,(d\log J)^{q-1}) = -\frac{1}{q} S\left(\langle\theta\,|\,\sigma\rangle,(d\log J)^q\right)$$

we see that

$$s(q) = (-1)^q S\left((\theta'+g),(d\log J)^q\right) + (-1)^q S\left(\langle\theta\,|\,\sigma\rangle,(d\log J)^q\right).$$

Thus we are done. □

Definition 4.8. For a (q,q)-cochain φ and $0 \le k \le q$, define a family $\partial_{(k)}\varphi = \{(\partial_{(k)}\varphi)_{i_0\cdots i_q}\}$ of $Q(\mathcal{F})^*$-valued q-forms on $U_{i_0\cdots i_q}$ by setting

$$(\partial_{(k)}\varphi)_{i_0 i_1\cdots i_q} = \sum_{l=1}^{q} \frac{\partial \varphi_{i_0 i_1\cdots i_q}}{\partial z^l_{i_k}} \otimes dz^l_{i_k},$$

where $\dfrac{\partial}{\partial z^l_{i_k}} f dz^{l_1}_{i_k} \wedge \cdots \wedge dz^{l_q}_{i_k} = \dfrac{\partial f}{\partial z^l_{i_k}} dz^{l_1}_{i_k} \wedge \cdots \wedge dz^{l_q}_{i_k}$ by definition. Define then a $(q,q+1)$-cochain $\sigma_k(\varphi)$ by

$$\begin{aligned}\sigma_k(\varphi)_{i_0 i_1\cdots i_q} &= \langle (\partial_{(k)}\varphi)_{i_0 i_1\cdots i_q} \,|\, \sigma_{i_k}\rangle \\ &= \sum_{l=1}^{q} \frac{\partial \varphi_{i_0 i_1\cdots i_q}}{\partial z^l_{i_k}} \wedge \sigma^l_{i_k},\end{aligned}$$

where $\sigma_{i_k} = \sum\limits_{l=1}^{q} \dfrac{\partial}{\partial z^l_{i_k}} \sigma^l_{i_k}$. Finally, set

$$\sigma(\varphi) = \sum_{k=0}^{q} \sigma_k(\varphi).$$

Our calculations can be continued as follows.

Lemma 4.9. *$(d\log J)^k \cup (\theta' + g + \langle \theta \,|\, \sigma\rangle) \cup (d\log J)^{q-k}$ is cohomologous to $\sigma_k((d\log J)^q)$.*

Proof. First of all, note that $c_1 \cup \cdots \cup c_k = (-1)^{\frac{k(k-1)}{2}} c_1 \wedge \cdots \wedge c_k$ if each c_i is a $(1,1)$-cochain. Note also that we may assume that $\widetilde{\sigma}_{i_k} = \sigma^1_{i_k} \wedge dz^2_{i_k} \wedge \cdots \wedge dz^q_{i_k} + \cdots + dz^1_{i_k} \wedge \cdots \wedge dz^{q-1}_{i_k} \wedge \sigma^q_{i_k}$ by Lemma 2.13. Hence by the equation (2.11.b),

$$\begin{aligned}&(d\log J)^k \cup \theta' \cup (d\log J)^{q-k} \\ &= (-1)^{\frac{q(q+1)}{2}} \theta'_{i_k} \wedge \omega_{i_k} \det(\partial_{i_k} \log J_{i_0 i_1}, \cdots, \partial_{i_k} \log J_{i_{q-1} i_q}) \\ &= (-1)^{\frac{q(q+1)}{2}} (d\widetilde{\sigma}_{i_k} + \theta_{i_k} \wedge \widetilde{\sigma}_{i_k}) \det(\partial_{i_k} \log J_{i_0 i_1}, \cdots, \partial_{i_k} \log J_{i_{q-1} i_q}),\end{aligned}$$

where $\omega_{i_k} = dz^1_{i_k} \wedge \cdots \wedge dz^q_{i_k}$, and each $\partial_{i_k} \log J_{i_{l-1} i_l}$ is considered as a vector in row.

Noticing that $\theta_{i_k} \wedge \widetilde{\sigma}_{i_k} = -\langle \theta_{i_k} \,|\, \sigma_{i_k}\rangle \wedge \omega_{i_k}$, one has

$$\begin{aligned}&(-1)^{\frac{q(q+1)}{2}} (\theta_{i_k} \wedge \widetilde{\sigma}_{i_k}) \det(\partial_{i_k} \log J_{i_0 i_1}, \cdots, \partial_{i_k} \log J_{i_{q-1} i_q}) \\ &= -\left((d\log J)^k \cup \langle \theta \,|\, \sigma\rangle \cup (d\log J)^{q-k}\right)_{i_0\cdots i_q},\end{aligned}$$

where $\langle \theta \,|\, \sigma \rangle_i = \langle \theta_i | \, \sigma_i \rangle$. Similarly,

$$\begin{aligned}
&d\widetilde{\sigma}_{i_k} \det(\partial_{i_k} \log J_{i_0 i_1}, \cdots, \partial_{i_k} \log J_{i_{q-1} i_q}) \\
=&\langle d\log J_{i_0 i_1} | \, d\sigma_{i_k} \rangle \wedge (d\log J_{i_1 i_2}) \wedge \cdots \wedge (d\log J_{i_{q-1} i_q}) \\
&- (d\log J_{i_0 i_1}) \wedge \langle d\log J_{i_1 i_2} | \, d\sigma_{i_k} \rangle \wedge (d\log J_{i_2 i_3}) \wedge \cdots \wedge (d\log J_{i_{q-1} i_q}) \\
&+ \cdots \\
&+ (-1)^{q-1} (d\log J_{i_0 i_1}) \wedge \cdots \wedge (d\log J_{i_{q-2} i_{q-1}}) \wedge \langle d\log J_{i_{q-1} i_q} | \, d\sigma_{i_k} \rangle.
\end{aligned}$$

Let $\rho(k)$ be the (q,q)-cochain such that

$$\rho(k)_{i_0 \cdots i_q} = (-1)^{\frac{q(q-1)}{2}} \langle d\log J_{i_0 i_1} \wedge \cdots \wedge d\log J_{i_{q-1} i_q} | \, \sigma_{i_k} \rangle.$$

By the above equation, we see that

$$\begin{aligned}
&(-1)^{\frac{q(q+1)}{2}} (\mathcal{D}'' \rho(k))_{i_0 \cdots i_q} \\
=& d\widetilde{\sigma}_{i_k} \det(\partial_{i_k} \log J_{i_0 i_1}, \cdots, \partial_{i_k} \log J_{i_{q-1} i_q}) \\
&+ \langle d\partial_{i_k} \log J_{i_0 i_1} | \, \sigma_{i_k} \rangle \wedge (d\log J_{i_1 i_2}) \wedge \cdots \wedge (d\log J_{i_{q-1} i_q}) \\
&- (d\log J_{i_0 i_1}) \wedge \langle d\partial_{i_k} \log J_{i_1 i_2} | \, \sigma_{i_k} \rangle \wedge (d\log J_{i_2 i_3}) \wedge \cdots \wedge (d\log J_{i_{q-1} i_q}) \\
&+ \cdots \\
&+ (-1)^{q-1} (d\log J_{i_0 i_1}) \wedge \cdots \wedge (d\log J_{i_{q-2} i_{q-1}}) \wedge \langle d\partial \log J_{i_{q-1} i_q} | \, \sigma_{i_k} \rangle \\
=& d\widetilde{\sigma}_{i_k} \det(\partial_{i_k} \log J_{i_0 i_1}, \cdots, \partial_{i_k} \log J_{i_{q-1} i_q}) \\
&- (-1)^{\frac{q(q+1)}{2}} \sigma_{i_k} ((d\log J)^q)_{i_0 \cdots i_q}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(\mathcal{D}' \rho(k))_{i_0, \cdots, i_{q+1}} \\
=& \sum_{l=0}^{k} (-1)^l (-1)^{\frac{q(q-1)}{2}} \langle d\log J_{i_0 i_1} \wedge \cdots \wedge d\log J_{i_{l-1} i_{l+1}} \wedge \cdots \wedge d\log J_{i_q i_{q+1}} | \, \sigma_{i_{k+1}} \rangle \\
&+ \sum_{l=k+1}^{q} (-1)^l (-1)^{\frac{q(q-1)}{2}} \langle d\log J_{i_0 i_1} \wedge \cdots \wedge d\log J_{i_{l-1} i_{l+1}} \wedge \cdots \wedge d\log J_{i_q i_{q+1}} | \, \sigma_{i_k} \rangle \\
=&(-1)^{\frac{q(q-1)}{2}} (-1)^k \langle d\log J_{i_0 i_1} \wedge \cdots \wedge d\widehat{\log J_{i_k i_{k+1}}} \wedge \cdots \wedge d\log J_{i_q i_{q+1}} | \, \underline{g}_{i_k i_{k+1}} \omega_{i_k} \rangle \\
=&(-1)^{\frac{q(q-1)}{2}} (-1)^k d\log J_{i_0 i_1} \wedge \cdots \wedge d\log J_{i_{k-1} i_k} \cdot (\operatorname{tr} \underline{g}_{i_k i_{k+1}}) \cdot d\log J_{i_{k+1} i_{k+2}} \wedge \cdots \wedge d\log J_{i_q i_{q+1}} \\
=& \big((d\log J)^k \cup g \cup (d\log J)^{q-k} \big)_{i_0 i_1 \cdots i_{q+1}}.
\end{aligned}$$

Thus we are done. $\square$

Note that $\sigma((d\log J)^q)$ is the reduction of $\left(\sum_{k=0}^{q}\partial_{(k)}(d\log J)^q\right)\otimes\sigma$ to a differential form. We identify $(\bigwedge^l T^*M)\wedge(T^*M\otimes V)\wedge(\bigwedge^{q-l-1}T^*M)$ with $\bigwedge^q T^*M\otimes V$ for any vector bundle V, then the tensor $\sum_{k=0}^{q}\partial_{(k)}(d\log J)^q$ is calculated as follows;

Lemma 4.10.

$$\begin{aligned}
&(-1)^{\frac{q(q-1)}{2}}\sum_{k=0}^{q}\partial_{(k)}(d\log J)^q\\
=&(q+1)^2\sum_{l=0}^{q-1}(\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge\Lambda_{i_l i_q}\wedge(\textstyle\bigwedge^{q-l-1}d\log J)_{i_{l+1}\cdots i_q}\\
&-(q+1)^2\sum_{l=0}^{q-2}(\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge\Lambda_{i_{l+1} i_q}\wedge(\textstyle\bigwedge^{q-l-1}d\log J)_{i_{l+1}\cdots i_q},
\end{aligned}$$

where $(\bigwedge^l d\log J)_{i_0\cdots i_l}=d\log J_{i_0i_1}\wedge d\log J_{i_1i_2}\wedge\cdots\wedge d\log J_{i_{l-1}i_l}$.

Proof. Firstly,

$$D\gamma_{lk}^{-1}dD\gamma_{lk}\otimes dz_k=\sum_{m=k}^{l-1}D\gamma_{m,k}^{-1}D\gamma_{m+1,m}^{-1}dD\gamma_{m+1,m}\otimes dz_m.$$

Hence $d\partial_k\log J_{l,l+1}\otimes dz_k$ can be rewritten as follows if $k\neq l$.

Case 1. $k<l$:

$$\begin{aligned}
&d\partial_k\log J_{l,l+1}\otimes dz_k\\
=&d\partial_l\log J_{l,l+1}\otimes dz_l+\partial_l\log J_{l,l+1}D\gamma_{lk}D\gamma_{lk}^{-1}dD\gamma_{lk}\otimes dz_k\\
=&d\partial_l\log J_{l,l+1}\otimes dz_l+\sum_{m=k}^{l-1}\partial_m\log J_{l,l+1}D\gamma_{m+1,m}^{-1}dD\gamma_{m+1,m}\otimes dz_m.
\end{aligned}$$

Case 2. $k>l$:

$$\begin{aligned}
&d\partial_k\log J_{l,l+1}\otimes dz_k\\
=&d\partial_l\log J_{l,l+1}\otimes dz_l-\partial_l\log J_{l,l+1}D\gamma_{kl}^{-1}dD\gamma_{kl}\otimes dz_l\\
=&d\partial_l\log J_{l,l+1}\otimes dz_l-\sum_{m=l}^{k-1}\partial_m\log J_{l,l+1}D\gamma_{m+1,m}^{-1}dD\gamma_{m+1,m}\otimes dz_m.
\end{aligned}$$

Hence we have the following equations, namely,

$$
\begin{aligned}
&\sum_{\substack{0\le k\le q\\ 0\le l\le q-1}} (\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge d\partial_{i_k}\log J_{i_l i_{l+1}}\wedge(\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}\otimes dz_{i_k}\\
=&\sum_{l=0}^{q-1}\sum_{k=l+1}^{q} (\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge d\partial_{i_k}\log J_{i_l i_{l+1}}\wedge(\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}\otimes dz_{i_k}\\
&+\sum_{l=0}^{q-1}(\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge d\partial_{i_l}\log J_{i_l i_{l+1}}\wedge(\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}\otimes dz_{i_l}\\
&+\sum_{l=0}^{q-1}\sum_{k=0}^{l-1}(\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge d\partial_{i_k}\log J_{i_l i_{l+1}}\wedge(\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}\otimes dz_{i_k}\\
=&\sum_{l=0}^{q-1}\sum_{k=l+1}^{q} (\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge d\partial_{i_l}\log J_{i_l i_{l+1}}\wedge(\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}\otimes dz_{i_l}\\
&-\sum_{l=0}^{q-1}\sum_{k=l+1}^{q}\sum_{m=l}^{k-1}(\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge\partial_{i_m}\log J_{i_l i_{l+1}}D\gamma^{-1}_{i_{m+1}i_m}dD\gamma_{i_{m+1}i_m}\wedge(\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}\otimes dz_{i_m}\\
&+\sum_{l=0}^{q-1}(\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge d\partial_{i_l}\log J_{i_l i_{l+1}}\wedge(\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}\otimes dz_{i_l}\\
&+\sum_{l=0}^{q-1}\sum_{k=0}^{l-1}(\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge d\partial_{i_l}\log J_{i_l i_{l+1}}\wedge(\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}\otimes dz_{i_l}\\
&+\sum_{l=0}^{q-1}\sum_{k=0}^{l-1}\sum_{m=k}^{l-1}(\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge\partial_{i_m}\log J_{i_l i_{l+1}}D\gamma^{-1}_{i_{m+1}i_m}dD\gamma_{i_{m+1}i_m}\wedge(\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}\otimes dz_{i_m}\\
=&(q+1)\sum_{l=0}^{q-1}(\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge d\partial_{i_l}\log J_{i_l i_{l+1}}\wedge(\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}\otimes dz_{i_l}\\
&-\sum_{l=0}^{q-1}\sum_{m=l}^{q-1}(q-m)(\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge\partial_{i_m}\log J_{i_l i_{l+1}}D\gamma^{-1}_{i_{m+1}i_m}dD\gamma_{i_{m+1}i_m}\wedge(\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}\otimes dz_{i_m}\\
&+\sum_{l=0}^{q-1}\sum_{m=0}^{l-1}(m+1)(\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge\partial_{i_m}\log J_{i_l i_{l+1}}D\gamma^{-1}_{i_{m+1}i_m}dD\gamma_{i_{m+1}i_m}\wedge(\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}\otimes dz_{i_m}.
\end{aligned}
$$

On the other hand, the following equations hold. First,

$$
\begin{aligned}
&\sum_{l=0}^{q-1}(\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge(\partial_{i_m}\log J_{i_l i_{l+1}}D\gamma^{-1}_{i_{m+1}i_m}dD\gamma_{i_{m+1}i_m})\wedge(\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}\otimes dz_{i_m}\\
=&-d\log J_{i_0i_1}\wedge\cdots\wedge d\log J_{i_{q-1}i_q}\otimes d\log J_{i_mi_{m+1}}.
\end{aligned}
$$

Second, rewriting the formulae in Lemma 3.5, one has

$$\begin{aligned}\Lambda_{i_l i_{l+1}} =& -\frac{1}{q+1} d\partial_{i_l} \log J_{i_{l+1} i_l} \otimes dz_{i_l} - \frac{-1}{q+1} \partial_{i_l} \log J_{i_{l+1} i_l} D\gamma^{-1}_{i_{l+1} i_l} dD\gamma_{i_{l+1} i_l} \otimes dz_{i_l} \\ &- \frac{1}{(q+1)^2} d\log J_{i_{l+1} i_l} \otimes d\log J_{i_{l+1} i_l} \\ =& \frac{1}{q+1} d\partial_{i_l} \log J_{i_l i_{l+1}} \otimes dz_{i_l} + \frac{-1}{q+1} \partial_{i_l} \log J_{i_l i_{l+1}} D\gamma^{-1}_{i_{l+1} i_l} dD\gamma_{i_{l+1} i_l} \otimes dz_{i_l} \\ &- \frac{1}{(q+1)^2} d\log J_{i_l i_{l+1}} \otimes d\log J_{i_l i_{l+1}},\end{aligned}$$

and

$$\begin{aligned}\Sigma_{i_l i_{l+1}} =& \frac{\partial}{\partial z_{i_l}} \otimes D\gamma^{-1}_{i_{l+1} i_l} dD\gamma_{i_{l+1} i_l} \otimes dz_{i_l} \\ &- \frac{1}{q+1} \sum_{k=1}^{q} \left(\frac{\partial}{\partial z^k_{i_l}} \otimes d\log J_{i_{l+1} i_l} \otimes dz^k_{i_l} + \frac{\partial}{\partial z^k_{i_l}} \otimes dz^k_{i_l} \otimes d\log J_{i_{l+1} i_l} \right) \\ =& \frac{\partial}{\partial z_{i_l}} \otimes D\gamma^{-1}_{i_{l+1} i_l} dD\gamma_{i_{l+1} i_l} \otimes dz_{i_l} \\ &+ \frac{1}{q+1} \sum_{k=1}^{q} \left(\frac{\partial}{\partial z^k_{i_l}} \otimes d\log J_{i_l i_{l+1}} \otimes dz^k_{i_l} + \frac{\partial}{\partial z^k_{i_l}} \otimes dz^k_{i_l} \otimes d\log J_{i_l i_{l+1}} \right).\end{aligned}$$

It follows that

$$\begin{aligned}&(q+1)^2 \sum_{l=0}^{q-1} (\textstyle\bigwedge^l d\log J)_{i_0 \cdots i_l} \wedge \Lambda_{i_l i_{l+1}} \wedge (\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1} \cdots i_q} \\ =& (q+1) \sum_{l=0}^{q-1} (\textstyle\bigwedge^l d\log J)_{i_0 \cdots i_l} \wedge d\partial_{i_l} \log J_{i_l i_{l+1}} \wedge (\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1} \cdots i_q} \otimes dz_{i_l} \\ &- (q+1) \sum_{l=0}^{q-1} (\textstyle\bigwedge^l d\log J)_{i_0 \cdots i_l} \wedge \partial_{i_l} \log J_{i_l i_{l+1}} D\gamma^{-1}_{i_{l+1} i_l} dD\gamma_{i_{l+1} i_l} \wedge (\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1} \cdots i_q} \otimes dz_{i_l} \\ &- \sum_{l=0}^{q-1} (\textstyle\bigwedge^l d\log J)_{i_0 \cdots i_l} \wedge d\log J_{i_l i_{l+1}} \wedge (\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1} \cdots i_q} \otimes d\log J_{i_l i_{l+1}},\end{aligned}$$

and that

$$\begin{aligned}&\langle d\log J_{i_l i_{l+1}} | \Sigma_{i_m i_{m+1}} \rangle \\ =& \partial_{i_m} \log J_{i_l i_{l+1}} D\gamma^{-1}_{i_{m+1} i_m} dD\gamma_{i_{m+1} i_m} \otimes dz_{i_m} \\ &+ \frac{1}{q+1} \left(d\log J_{i_m i_{m+1}} \otimes d\log J_{i_l i_{l+1}} + d\log J_{i_l i_{l+1}} \otimes d\log J_{i_m i_{m+1}} \right).\end{aligned}$$

Combining these equations, one obtains the following equation, namely,

$$\sum_{\substack{0\le k\le q\\0\le l\le q-1}} (\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge d\partial_{i_k}\log J_{i_li_{l+1}}\wedge(\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}\otimes dz_{i_k}$$

$$=(q+1)^2\sum_{l=0}^{q-1}(\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge\Lambda_{i_li_{l+1}}\wedge(\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}$$

$$+(q+1)\sum_{l=0}^{q-1}(\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge\partial_{i_l}\log J_{i_li_{l+1}}D\gamma^{-1}_{i_{l+1}i_l}dD\gamma_{i_{l+1}i_l}\wedge(\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}\otimes dz_{i_l}$$

$$+\sum_{l=0}^{q-1} d\log J_{i_0i_1}\wedge\cdots\wedge d\log J_{i_{q-1}i_q}\otimes d\log J_{i_li_{l+1}}$$

$$-\sum_{l=0}^{q-1}\sum_{m=l}^{q-1}(q+1)(\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge\partial_{i_m}\log J_{i_li_{l+1}}D\gamma^{-1}_{i_{m+1}i_m}dD\gamma_{i_{m+1}i_m}\wedge(\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}\otimes dz_{i_m}$$

$$+\sum_{l=0}^{q-1}\sum_{m=0}^{q-1}(m+1)(\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge\partial_{i_m}\log J_{i_li_{l+1}}D\gamma^{-1}_{i_{m+1}i_m}dD\gamma_{i_{m+1}i_m}\wedge(\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}\otimes dz_{i_m}$$

$$=(q+1)^2\sum_{l=0}^{q-1}(\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge\Lambda_{i_li_{l+1}}\wedge(\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}$$

$$+\sum_{l=0}^{q-1} d\log J_{i_0i_1}\wedge\cdots\wedge d\log J_{i_{q-1}i_q}\otimes d\log J_{i_li_{l+1}}$$

$$-\sum_{m=1}^{q-1}\sum_{l=0}^{m-1}(q+1)(\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge\partial_{i_m}\log J_{i_li_{l+1}}D\gamma^{-1}_{i_{m+1}i_m}dD\gamma_{i_{m+1}i_m}\wedge(\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}\otimes dz_{i_m}$$

$$-\sum_{m=0}^{q-1}(m+1)d\log J_{i_0i_1}\wedge\cdots\wedge d\log J_{i_{q-1}i_q}\otimes d\log J_{i_mi_{m+1}}$$

$$=(q+1)^2\sum_{l=0}^{q-1}(\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge\Lambda_{i_li_{l+1}}\wedge(\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}$$

$$-\sum_{m=1}^{q-1}\sum_{l=0}^{m-1}(q+1)(\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge\langle d\log J_{i_li_{l+1}}|\Sigma_{i_mi_{m+1}}\rangle\wedge(\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}$$

$$+\sum_{m=1}^{q-1}\sum_{l=0}^{m-1}(\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge d\log J_{i_mi_{m+1}}\wedge(\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}\otimes d\log J_{i_li_{l+1}}$$

$$+\sum_{m=1}^{q-1}\sum_{l=0}^{m-1}(\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l}\wedge d\log J_{i_li_{l+1}}\wedge(\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}\otimes d\log J_{i_mi_{m+1}}$$

$$-\sum_{m=0}^{q-1} m\,d\log J_{i_0i_1}\wedge\cdots\wedge d\log J_{i_{q-1}i_q}\otimes d\log J_{i_mi_{m+1}}$$

$$= (q+1)^2 \sum_{l=0}^{q-1} (\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l} \wedge \Lambda_{i_l i_{l+1}} \wedge (\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}$$
$$- \sum_{l=0}^{q-2} \sum_{m=l+1}^{q-1} (q+1)(\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l} \wedge \langle d\log J_{i_l i_{l+1}} | \Sigma_{i_m i_{m+1}} \rangle \wedge (\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}$$
$$+ \sum_{m=1}^{q-1} m d\log J_{i_0 i_1} \wedge \cdots \wedge d\log J_{i_{q-1} i_q} \otimes d\log J_{i_m i_{m+1}}$$
$$- \sum_{m=0}^{q-1} m d\log J_{i_0 i_1} \wedge \cdots \wedge d\log J_{i_{q-1} i_q} \otimes d\log J_{i_m i_{m+1}}$$
$$= (q+1)^2 \sum_{l=0}^{q-1} (\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l} \wedge \Lambda_{i_l i_{l+1}} \wedge (\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}$$
$$- \sum_{l=0}^{q-2} (q+1)(\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l} \wedge \langle d\log J_{i_l i_{l+1}} | \Sigma_{i_{l+1} i_q} \rangle \wedge (\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}.$$

Finally by Lemma 3.7,

$$\langle d\log J_{i_l i_{l+1}} \,|\, \Sigma_{i_{l+1} i_q} \rangle = (q+1)(\Lambda_{i_{l+1} i_q} - \Lambda_{i_l i_q} + \Lambda_{i_l i_{l+1}}).$$

Lemma 4.10 follows from the last two equations. □

Proof of Theorem 4.3. First, by Lemmas 4.7 and 4.9, $D_\sigma B_q(\mathcal{F})$ is cohomologous to $-(2\pi\sqrt{-1})^{-(q+1)}\sigma((d\log J)^q) = -(2\pi\sqrt{-1})^{-(q+1)} \left\langle \sum_{k=0}^{q} \partial_{(k)}(d\log J)^q \;\middle|\; \sigma \right\rangle$. Second, by Lemma 4.10,

$$\left\langle \sum_{k=0}^{q} \partial_{(k)}(d\log J)^q \;\middle|\; \sigma \right\rangle_{i_0\cdots i_q}$$
$$= (-1)^{\frac{q(q-1)}{2}} (q+1)^2 \sum_{l=0}^{q-1} (-1)^{q-l-1} (\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l} \wedge L_{i_l i_q} \wedge (\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}$$
$$- (-1)^{\frac{q(q-1)}{2}} (q+1)^2 \sum_{l=0}^{q-2} (-1)^{q-l-1} (\textstyle\bigwedge^l d\log J)_{i_0\cdots i_l} \wedge L_{i_{l+1} i_q} \wedge (\textstyle\bigwedge^{q-l-1} d\log J)_{i_{l+1}\cdots i_q}.$$

Theorem 4.3 follows from these equations. □

In general, sections of vector bundles over M are said to be foliated if they are locally constant along the leaves and if they are transversely holomorphic. Let $\Gamma_{\mathcal{F}}(K_{\mathcal{F}})$ be the sheaf of germs of foliated sections of $K_{\mathcal{F}} = \bigwedge^q Q(\mathcal{F})^*$, and let $\Gamma_{\mathcal{F}}(K_{\mathcal{F}} \otimes Q(\mathcal{F})^*)$ be the sheaf of germs of foliated sections of $K_{\mathcal{F}} \otimes Q(\mathcal{F})^*$. Čech cochains with values in these sheaves are denoted respectively by $\check{C}^*_{\mathcal{F}}(\mathcal{U}; K_{\mathcal{F}})$ and $\check{C}^*_{\mathcal{F}}(\mathcal{U}; K_{\mathcal{F}} \otimes Q(\mathcal{F})^*)$, and the cohomology groups are denoted by $\check{H}^*_{\mathcal{F}}(M; K_{\mathcal{F}})$ and $\check{H}^*_{\mathcal{F}}(M; K_{\mathcal{F}} \otimes Q(\mathcal{F})^*)$.

Lemma 4.11. *The mapping*

$$\mathcal{S} = \sum_{k=0}^{q} \partial_{(k)} : \check{C}^*_{\mathcal{F}}(\mathcal{U}; K_{\mathcal{F}}) \to \check{C}^*_{\mathcal{F}}(\mathcal{U}; K_{\mathcal{F}} \otimes Q(\mathcal{F})^*)$$

given by

$$\mathcal{S}(\varphi)_{i_0\cdots i_p} = \sum_{k=0}^{q} \partial_{(k)}(\varphi)_{i_0\cdots i_p} = \sum_{k=0}^{q}\sum_{l=1}^{q} \frac{\partial \varphi_{i_0\cdots i_p}}{\partial z^l_{i_k}} \otimes dz^l_{i_k}$$

induces a homomorphism on the cohomology. We denote again by this homomorphism by $\mathcal{S}$.

The proof is straightforward and omitted. It is easy to verify that $\mathcal{S}$ is independent of the choice of a foliation chart.

Lemma 4.12. *There is a well-defined pairing*

$$(\,\cdot\,|\,\cdot\,) : \check{C}^q_{\mathcal{F}}(\mathcal{U}; K_{\mathcal{F}} \otimes Q(\mathcal{F})^*) \times \mathcal{E}^1(Q(\mathcal{F})) \to \mathcal{A}^{q,q+1}(\mathcal{U}) \oplus \mathcal{A}^{q+1,q}(\mathcal{U}) \subset \mathcal{A}^{2q+1}(\mathcal{U})$$

such that if $\eta = \{\eta_{i_0\cdots i_q}\}$ and $(a,b) = (\{a_i\},\{b_{ij}\})$, then

$$(\eta\,|(a,b))_{i_0\cdots i_q, i_0\cdots i_{q+1}} = \left\langle \eta_{i_0\cdots i_q} \middle| a_{i_q} \right\rangle \oplus (-1)^q \left\langle \eta_{i_0\cdots i_q} \middle| b_{i_q i_{q+1}} \right\rangle.$$

This pairing induces a pairing

$$\langle\,\cdot\,|\,\cdot\,\rangle : \check{H}^q_{\mathcal{F}}(M; K_{\mathcal{F}} \otimes Q(\mathcal{F})^*) \times H^1(M; \Theta_{\mathcal{F}}) \to H^{2q+1}(M; \boldsymbol{C}).$$

Proof. We adopt $\dfrac{\partial}{\partial z^1_i}, \cdots, \dfrac{\partial}{\partial z^q_i}$ as a local trivialization of $Q(\mathcal{F})$. Choose a family of local Bott connections and let $\{\theta_i\}$ be the family of local connection forms, then $\theta_i \in I^1_{(1)}(U_i)$ and $\theta_j - D\gamma_{ji}\theta_i D\gamma_{ji}^{-1} - dD\gamma_{ji}D\gamma_{ji}^{-1} \in I^1_{(1)}(U_{ij}; \mathrm{End}(Q(\mathcal{F}))$. Let μ be an element of $H^1(M;\Theta_{\mathcal{F}})$ represented by an element $(\{\alpha_i\},\{s_{ij}\})$ of $\mathcal{E}^1(Q(\mathcal{F}))$. Recall that each α_i and s_{ij} can be written as $\alpha_i = \dfrac{\partial}{\partial z_i}\alpha'_i$ for some $\boldsymbol{C}^q$-valued 1-form α'_i, and $s_{ij} = \dfrac{\partial}{\partial z_i} s'_{ij}$ for some $\boldsymbol{C}^q$-valued function s'_{ij}. The families $\{\alpha'_i\}$ and $\{s'_{ij}\}$ satisfy $d\alpha'_i + \theta_i \wedge \alpha'_i \in I_{(1)}(U_i)$, $(D\gamma_{ji}^{-1}\alpha'_j - \alpha'_i) - (ds'_{ij} + \theta_i s'_{ij}) \in I_{(1)}(U_{ij})$ and $s'_{ij} - s'_{ik} + D\gamma_{ji}s'_{jk} = 0$. Note that these conditions are consistent even though the connection is not necessarily globally well-defined. Let ρ be an element of $\check{H}^q_{\mathcal{F}}(M; K_{\mathcal{F}} \otimes Q(\mathcal{F})^*)$ represented by an element η of $\check{C}^q_{\mathcal{F}}(\mathcal{U}; K_{\mathcal{F}} \otimes Q(\mathcal{F})^*)$.

Claim 1. $(\eta\,|(\alpha,s))$ is a Čech-de Rham cocycle.

First, $d\langle \eta_{i_0\cdots i_q}\,|\,\alpha_{i_q}\rangle = 0$ because $d\alpha'_i \in I_{(1)}(U_i)$ and that η is foliated. Second, $(\delta\langle \eta\,|\,\alpha\rangle)_{i_0\cdots i_{q+1}} = (-1)^q\langle \eta_{i_0\cdots i_q}\,|\,ds_{i_q i_{q+1}}\rangle$, where $ds_{ij} = \dfrac{\partial}{\partial z_i} ds'_{ij}$. Since η is foliated, the right hand side is equal to $(-1)^{q+1}\mathcal{D}''\left\langle \eta_{i_0\cdots i_q} \,|\, s_{i_q i_{q+1}} \right\rangle$. Finally, $\delta\langle \eta | \, s\rangle = 0$ because $\delta\eta = 0$ and $\delta s = 0$. This completes the proof of the claim 1 and the first part of the lemma.

Claim 2. The cohomology class represented by $(\eta \,|(\alpha, s))$ depends only on the cohomology class of (α, s) once η is fixed.

Assume that there is an element $\{\beta_i\}$ of $\mathcal{E}^0(Q(\mathcal{F})) = \mathcal{A}^{0,0}(\mathcal{U})$ such that $\alpha'_i - (d\beta'_i + \theta_i\beta'_i) \in I^1_{(1)}(U_i)$ and that $s'_{ij} - (D\gamma_{ji}^{-1}\beta'_j - \beta'_i) \in I^0_{(1)}(U_{ij}) = \{0\}$, where $\beta_i = \dfrac{\partial}{\partial z_i}\beta'_i$. Then, $\mathcal{D}''\langle \eta \,|\, \beta_i\rangle = \left\langle \eta \;\middle|\; \dfrac{\partial}{\partial z_i} d\beta'_i \right\rangle = \left\langle \eta \;\middle|\; \dfrac{\partial}{\partial z_i}(d\beta'_i + \theta_i\beta'_i) \right\rangle = \langle \eta \,|\, \alpha_i\rangle$, where the index of η is omitted for simplicity. On the other hand, $\mathcal{D}'\langle \eta \,|\, \beta\rangle_{i_0\cdots i_{q+1}} = (-1)^q\langle \eta_{i_0\cdots i_q} \,|\, \beta_{i_{q+1}} - \beta_{i_q}\rangle = (-1)^q\langle \eta_{i_0\cdots i_q} \,|\, s_{i_q i_{q+1}}\rangle$. Thus $(\eta \,|(\alpha, s))$ is null-cohomologous.

Claim 3. The cohomology class represented by $(\eta \,|(\alpha, s))$ depends only on the cohomology class of η once (α, s) is fixed.

Suppose that $\eta = \delta\varphi$ for some $\varphi \in \check{C}^{q-1}_{\mathcal{F}}(\mathcal{U}; K_{\mathcal{F}} \otimes Q(\mathcal{F})^*)$. By repeating similar arguments as above, we see that $\mathcal{D}''\langle \varphi \,|\, \alpha\rangle = 0$, $\mathcal{D}'\langle \varphi \,|\, \alpha\rangle = \langle \eta \,|\, \alpha\rangle + (-1)^{q-1}\langle \varphi \,|\, ds\rangle$, $\mathcal{D}''\langle \varphi \,|s\rangle = \langle \varphi \,|\, ds\rangle$ and $\mathcal{D}'\langle \varphi \,|s\rangle = 0$, where indices are omitted for simplicity. Hence $\langle \eta \,|(\alpha, s)\rangle = \mathcal{D}(\langle \varphi \,|\, \alpha\rangle \oplus (-1)^q\langle \varphi \,|\, s\rangle)$. This completes the proof. $\square$

The Čech-de Rham (q, q)-cocycle $(d\log J)^q$ determines a class in $\check{H}^q_{\mathcal{F}}(M; K_{\mathcal{F}})$. This class is independent of the choice of a foliation chart and denoted by $[d\log J^q]$. Note that $d\log J$ determines a class $[d\log J]$ in $\check{H}^1_{\mathcal{F}}(M; Q(\mathcal{F})^*)$, and $[d\log J^q] = [d\log J]^q$.

Definition 4.13. Denote by $\mathcal{L}$ the cochain $\mathcal{S}(d\log J)^q$. The cohomology class in $\check{H}^q_{\mathcal{F}}(M; K_{\mathcal{F}} \otimes Q(\mathcal{F})^*)$ represented by $\mathcal{L}$ is denoted by $\mathcal{L}(\mathcal{F})$, that is, $\mathcal{L}(\mathcal{F}) = \mathcal{S}([d\log J]^q)$.

Corollary 4.14. *Under the assumption of Theorem 4.3, the cocycle $\sigma((d\log J)^q)$ is equal to $\langle \mathcal{L} \,|\, \sigma\rangle$. Hence $D_\mu B_q(\mathcal{F})$ is represented by $\langle \mathcal{L}(\mathcal{F}) | \, \mu\rangle$.*

From the viewpoint of calculations, Lemma 4.10 is the principal reason for which $D_\mu B_q(\mathcal{F})$ can be expressed in terms of the projective Schwarzian. It is naturally understood by a classical understanding of the Schwarzian derivative in terms of difference of Affine connections [16], [8] (cf. [18], [5], [20]). Indeed, the difference of the derivatives of $(d\log J)^q$ is calculated in defining $\mathcal{L}$. Here is an example of calculation of $\mathcal{L} = \mathcal{S}(d\log J)^q$ when $q = 1$. Note that $J_{ij} = \dfrac{\partial\gamma_{ij}}{\partial z_j} = \gamma'_{ij}$ and that $d\log J_{ij} = \dfrac{\gamma''_{ij}}{\gamma'_{ij}} dz_j = -\dfrac{\gamma''_{ji}}{\gamma'_{ji}} dz_i$. Then, we have the following equations as promised by Theorem 4.3, namely,

$$\begin{aligned}
&\mathcal{S}(d\log J)_{ij}\\
&= -\frac{\partial}{\partial z_i}\frac{\gamma_{ji}''}{\gamma_{ji}'}dz_i\otimes dz_i + \frac{\partial}{\partial z_j}\frac{\gamma_{ij}''}{\gamma_{ij}'}dz_j\otimes dz_j\\
&= -\left(\frac{\gamma_{ji}'''}{\gamma_{ji}'} - \left(\frac{\gamma_{ji}''}{\gamma_{ji}'}\right)^2\right)dz_i\otimes dz_i + \left(\frac{\gamma_{ij}'''}{\gamma_{ij}'} - \left(\frac{\gamma_{ij}''}{\gamma_{ij}'}\right)^2\right)dz_j\otimes dz_j\\
&= -2\left(\frac{\gamma_{ji}'''}{\gamma_{ji}'} - \frac{3}{2}\left(\frac{\gamma_{ji}''}{\gamma_{ji}'}\right)^2\right)dz_i\otimes dz_i.
\end{aligned}$$

Corollary 4.14, Lemmas 4.11 and 4.12 justify the following

Definition 4.15. Let $\mu \in H^1(M;-K_{\mathcal{F}}) \cong H^1(M;\Theta_{\mathcal{F}})$, then the infinitesimal derivative of the Bott class $D_\mu B_q(\mathcal{F})$ is by definition the Čech-de Rham cohomology class $\langle \mathcal{L}(\mathcal{F})|\,\mu\rangle$, where $\mathcal{L}(\mathcal{F}) = \mathcal{S}([d\log J]^q)$.

By Lemma 4.10, $\mathcal{L}(\mathcal{F})$ is the obstruction for $\mathcal{F}$ admitting a transverse projective structure if $q=1$. If $q>1$, it remains true that $\mathcal{L}(\mathcal{F})$ is closely related with the existence of transverse projective structures, however, it will be an obstruction for certain reduced structures. Indeed, the tensor Λ appears in the formulae instead of the Schwarzian derivative Σ. It is clear that $\Lambda = 0$ if $\Sigma = 0$ ($q>1$) but the converse is not true. Such a property of $\mathcal{L}(\mathcal{F})$ is reflected in infinitesimal derivatives as follows.

Definition 4.16. The Bott class of a transversely holomorphic foliation $\mathcal{F}$ is said to be infinitesimally rigid if $\langle \mathcal{L}(\mathcal{F})|\,\mu\rangle = 0$ for any $\mu \in H^1(M;\Theta_{\mathcal{F}})$.

Definition 4.17. A transversely holomorphic foliation $\mathcal{F}$ is said to be transversely complex projective on U if $\mathcal{F}$ admits a structure of a $(\mathrm{PSL}(q+1;\boldsymbol{C}), \boldsymbol{C}P^q)$-foliation on U. If $U = M$, then U is omitted. Here we always assume that the underlying transverse holomorphic structure coincides with the original one. A transverse complex projective structure is also called a transverse projective structure for short. If a transverse complex projective structure $\mathcal{P}$ is given on an open subset U, then a foliation atlas is said to be adapted to $\mathcal{P}$ if the atlas gives the structure $\mathcal{P}$ on U.

Corollary 4.18. *The Bott class of transversely projective foliations are infinitesimally rigid. Indeed, the Bott class of transversely holomorphic foliations are infinitesimally rigid if the tensor Λ is equal to zero.*

Note that there are transversely projective foliations with non-trivial Bott classes. In fact, there are transversely projective foliations with non-trivial Godbillon-Vey classes [4]. We will cite an example as Example 7.2.

Remark 4.19. The constructions are also valid for the Godbillon-Vey class of real foliations with obvious replacements. Especially, a formula of the same kind as Theorem 4.3 holds and a definition of the same kind as Definition 4.15 makes a sense. The codimension-one case is exactly the Maszczyk formula [17]. Theorem 4.3 and Definition 4.15 for real foliations are highly non-trivial, because it is well-known that the Godbillon-Vey class admits continuous deformations (due to Thurston, cf. [14]). However, it is known that the infinitesimal derivative of the Godbillon-Vey class always vanishes when restricted to transversely holomorphic foliations [4]. So these real versions make sense for foliations which do not admit transverse holomorphic structures.

5. Localization

We have obtained two expressions of the infinitesimal derivative as $D_\mu B_q(\mathcal{F})$ and $\langle \mathcal{L}(\mathcal{F}) | \, \mu \rangle$. These expressions lead to two kinds of localizations. We begin with some relevant definitions.

Definition 5.1. Let $\omega = \{\omega_{i_0,\cdots,i_p}\}$ be a Čech-de Rham (p,q)-cochain. Let I be the index set of the open covering $\mathcal{U} = \{U_i\}$ and set

$$I_\omega = \left\{ i \in I \,\middle|\, \exists (i_1, \cdots, i_p) \in I^p \text{ s.t. } \omega_{i,i_1,\cdots,i_p} \neq 0 \right\}.$$

The open set

$$\operatorname{supp} \omega = \bigcup_{i \in I_\omega} U_i.$$

is called the support of ω. If $\operatorname{supp}\omega$ is relatively compact, then ω is said to be of compact support.

Let ω be a globally defined differential form and denote by $s(\omega)$ the support of ω in the usual sense. If V is an open set containing $s(\omega)$, then taking refinements of coverings, we may assume that $s(\omega) \subset \operatorname{supp}\omega \subset V$.

We denote by H_c^* the cohomology of compactly supported cocycles. If U_1 and U_2 are open subsets of M such that $U \subset V$, then there is a natural mapping $H_c^*(U_1; R) \to H_c^*(U_2; R)$, where R is an arbitrary coefficient.

The localization of $D_\mu B_q(\mathcal{F})$ is defined by means of $\varGamma$-vector fields. The notion of $\varGamma$-vector fields and basic X-connections below is originally due to Heitsch [14]. In what follows, we take refinements of open coverings if necessary.

Definition 5.2 [2]**.** A vector field X defined on an open set O_X of M is said to be a Γ-vector field for $\mathcal{F}$ if $[E, X] \subset E$ on O_X. Set $Z_X = \{X \in E\} \cup (M \setminus O_X)$, then $\mathcal{F}$ and X form a transversely holomorphic foliation on the open set $M \setminus Z_X$. This foliation is denoted by $\mathcal{F}_X$.

Note that Z_X is saturated by leaves of $\mathcal{F}$ if O_X is saturated. Note also that if X is a Γ-vector field on O_X, then X is locally leafwise constant and transversely holomorphic as a section of $Q(\mathcal{F})$ on O_X.

Notation 5.3. Given a Γ-vector field X defined on O_X, denote by X_Q the foliated section of $Q(\mathcal{F})$ on O_X induced by X.

In what follows, for a Γ-vector field X, we denote by U_X an open neighborhood (which is not necessarily saturated) of Z_X and by V_X an open neighborhood of $M \setminus U_X$. The neighborhood U_X will be arbitrary small.

Definition 5.4. Let X be a Γ-vector field for $\mathcal{F}$ and let U_X and V_X be as above. A Bott connection $\nabla = \{\nabla_i\}$ of $-K_{\mathcal{F}}$ is said to be a basic X-connection for $\mathcal{F}$ supported off V_X if $(\nabla_i)_X s = \mathcal{L}_X s$ if $U_i \subset U$, where $\mathcal{L}_X$ denotes the Lie derivative with respect to X. Basic X-connections are usually denoted by ∇^X.

Note that basic X-connections depends only on X_Q but not on X itself. In other words, if X and X' are lifts to $T_{\boldsymbol{C}}M$ of the same foliated section of $Q(\mathcal{F})$, then $\nabla^X = \nabla^{X'}$.

Let $\nabla = \{\nabla_i\}$ be a basic X-connection, then it is easy to see the following properties:

1) The curvature form of ∇_i belongs to $I^2_{(1)}(\mathcal{F}_X)$ on V_X, where $I^2_{(1)}(\mathcal{F}_X)$ is the ideal $I_{(1)}$ appeared in Notation 1.3 but $\mathcal{F}$ is replaced with $\mathcal{F}_X$.
2) If $\nabla' = \{\nabla'_i\}$ be another basic X-connection, then $\nabla'_i - \nabla_i \in I^1_{(1)}(\mathcal{F}_X)$ on V_X.

It follows that the Bott vanishing for $\mathcal{F}_X$ can be applied on V_X when basic X-connections are used. If one admits to use a partition of unity, ∇^X may be assumed to be globally defined. Then it is a basic X-connection for $\mathcal{F}$ supported off V_X in the sense of Heitsch. Remark finally that once fixed an isomorphism $Q(\mathcal{F}) \cong \boldsymbol{C}X_Q \oplus Q(\mathcal{F}_X)$, a basic X-connection induces a Bott connection for $\mathcal{F}_X$ on V_X.

If W is an open subset of M, then elements of $H^1_c(W; \Theta_{\mathcal{F}}|_W)$ can be regarded as infinitesimal deformations of $\mathcal{F}$ whose support is compact and is contained in W.

Definition 5.5. Let X be a Γ-vector field for $\mathcal{F}$, and let U_X and V_X be as above. Let W be an open subset of M and let $\mu \in H^1_c(W; \Theta_{\mathcal{F}}|_W)$. Assume that μ admits a representative $\sigma \in I^q_{(q-1,q)}(W; -K_{\mathcal{F}})$ which is of compact support. Then, denote by $\operatorname{res} D_\mu B_q(\mathcal{F}, X)$ the element of $H^{2q+1}_c(U_X \cap W; \boldsymbol{C})$ represented by $D_\sigma B_q(\nabla^X, (\nabla^X)')$, where ∇^X is a basic X-connection supported off V_X, and $(\nabla^X)'$ is the infinitesimal derivative of ∇^X with respect to σ.

It is clear that $\operatorname{res} D_\mu B_q(\mathcal{F}, X)$ depends on X_Q but not on X itself so that $\operatorname{res} D_\mu B_q(\mathcal{F}, X)$ is also denoted by $\operatorname{res} D_\mu B(\mathcal{F}, X_Q)$.

Theorem 5.6. $\operatorname{res} D_\mu B_q(\mathcal{F}, X)$ *is well-defined and* $\iota_* \operatorname{res} D_\mu B_q(\mathcal{F}, X) = D_\mu B_q(\mathcal{F})$, *where* $\iota : U_X \cap W \to M$ *is the inclusion and* $\iota_* : H^{2q+1}_c(U_X \cap W; \boldsymbol{C}) \to H^{2q+1}(M; \boldsymbol{C})$ *is the natural mapping. Moreover, if* Z_X *is decomposed into connected components* $Z_1, \cdots, Z_r$*, then the residue is naturally decomposed into elements of* $H^{2q+1}(U_i \cap W; \boldsymbol{C})$ *as well, where* U_i*,* $i = 1, \ldots, r$*, are mutually disjoint open neighborhoods of* Z_i*.*

Proof. By the assumption, μ is represented by a cocycle compactly supported in W. It follows from (2.11.a) and (2.11.b) that the support of the infinitesimal derivative of any Bott connection is compact and contained in W when taken the wedge product with elements of $I_{(q)}(M)$. On the other hand, if basic-X connections are used in calculation, cochains such as $(d\theta + \beta)^q$ vanish on V_X thanks to the Bott vanishing for $\mathcal{F}_X$. It follows that the supports of the coboundaries constructed in Claims 3 and 4 in the proof of Theorem 2.15 are compact and contained in $U_X \cap W$. The last claim also follows from similar arguments. This complete the proof. $\square$

Let X be a Γ-vector field for $\mathcal{F}$ and let U_X and V_X be as above. Suppose that there is a trivialization e_{V_X} of $-K_{\mathcal{F}}|_{V_X}$ exists, then it is shown in [2] that the Bott class is naturally an element of $H^{2q+1}_c(U_X; \boldsymbol{C}/\boldsymbol{Z})$, which is called the residue of the Bott class. When residues are considered, a version of Theorem 2.18 holds under some additional conditions. Since the residue is defined as above, it is natural to consider a family of triples $\{(\mathcal{F}_s, X_s, e_s)\}$ with the following properties:

1) $\{\mathcal{F}_s\}$ is a smooth family of transversely holomorphic foliations with $\mathcal{F}_0 = \mathcal{F}$.
2) $\{X_s\}$ is a smooth family of Γ-vector fields with $X_0 = X$, that is, each X_s is a Γ-vector field for $\mathcal{F}_s$. We assume moreover that Z_{X_s} is independent of s, and denote Z_{X_s} by Z_X.
3) There is an open neighborhood U_X of Z_X and an open neighborhood V_X of $M \backslash U_X$ such that e_s restricted to V_X forms a smooth family of trivializations of $-K_{\mathcal{F}_s}|_{V_X}$. Note here that we may assume that $-K_{\mathcal{F}_s}$ is isomorphic by assuming that s is small. We denote e_0 simply by e.

Note that $\operatorname{res} B_q(\mathcal{F}_s, X_s, e_s)$ is well-defined as an element of $H^{2q+1}_c(U_X; \boldsymbol{C}/\boldsymbol{Z})$.

Theorem 5.7. *Let $\{(\mathcal{F}_s, X_s, e_s)\}$ be a smooth family of triples as above. Assume that e is foliated and that $\mathcal{L}_X e = 0$, where $\mathcal{L}_X$ denotes the Lie derivative with respect to X. Let $\mu \in H^1(M; -K_{\mathcal{F}})$ be the infinitesimal derivative induced from $\{\mathcal{F}_s\}$, then* $\operatorname{res} D_\mu B_q(\mathcal{F}, X) = \dfrac{\partial}{\partial s} \operatorname{res} B_q(\mathcal{F}_s, X_s, e_s)\Big|_{s=0}$.

Remark 5.8.

1) The assumption on e can be rephrased as 'e is foliated with respect to $\mathcal{F}_X$'.
2) As Example 7.3 shows, Theorem 5.7 fails if the assumption on e is dropped. Notice however that the left hand side is independent of e. The assumption is needed in order that Proposition 1.10 works in a compactly supported way.
3) If $q = 1$, then it is natural to choose X_Q as e_0.

Proof of Theorem 5.7. The proof is basically a repetition of the proof of Theorem 2.18. Under the assumptions, one can proceed in a parallel way and a cocycle of the same kind as the one given in the beginning of the proof of Proposition 1.10 can be obtained. Since there is a trivialization of $-K_{\mathcal{F}}$ on V_X, the cocycle Θ is zero on V_X. Moreover, since e is foliated with respect to $\mathcal{F}_X$, the cochain u_1 as well as v_1 belong to $I_{(1)}(\mathcal{F}_X)$. Hence the cochains ρ_k is equal to zero on V_X. Therefore, the both hand sides in the statement coincide as an element of $H_c^{2q+1}(M; \boldsymbol{C})$. $\square$

Another localization can be defined in terms of the cocycle $\mathcal{L}$ as follows. Recall that we do not need the assumption on representatives of μ (cf. Definition 4.15) and that the cocycle $\mathcal{L}$ strongly depends on the choice of foliation charts.

Theorem 5.9. *Let $\mathcal{F}$ be a transversely holomorphic foliation of M. Suppose that $\mathcal{F}$ admits on an open set V of M a transversal complex projective structure and fix such a structure $\mathcal{P}$. It is possible that $V = \varnothing$. Let U be an open neighborhood of $M \setminus V$. Finally let $\mu \in H_c^1(W; \Theta_{\mathcal{F}}|_W)$, where W is an open subset of M, and let σ be a representative of μ. Then $\langle \mathcal{L} \,|\, \sigma \rangle$ represents an element of $H_c^{2q+1}(U \cap W; \boldsymbol{C})$ which is independent of the choice of representatives, where foliation charts are always chosen to be adapted to $\mathcal{P}$ on V.*

Proof. By the choice of foliation charts, the support of $\mathcal{L}$ is contained in U. Hence the support of $\langle \mathcal{L} \,|\, \sigma \rangle$ is contained in $U \cap W$. If we choose another foliation chart adapted to $\mathcal{P}$ and denote by $\mathcal{L}'$ the resulting cocycle, it is clear that $\mathcal{L}$ and $\mathcal{L}'$ are cohomologous as cocycles supported on U. On the other hand, by Claim 2 in the proof of Lemma 4.12, $\langle \mathcal{L} \,|\, \sigma \rangle$ and $\langle \mathcal{L} \,|\, \sigma' \rangle$ represent the same cohomology class if σ and σ' are representatives of μ. $\square$

Definition 5.10. We denote by $\operatorname{res}\langle\mathcal{L}(\mathcal{F},\mathcal{P})|\,\mu\rangle$ the resulting element of $H_c^{2q+1}(U\cap W;\boldsymbol{C})$.

If a compactly supported infinitesimal derivative is given, then they coincide as follows.

Proposition 5.11. *Let W be an open subset of M and let $\mu\in H_c^1(W;-K_{\mathcal{F}}|_W)$. Then $\operatorname{res} D_\mu B_q(\mathcal{F},X)$ and $\operatorname{res}\langle\mathcal{L}(\mathcal{F},\mathcal{P})\,|\,\mu\rangle$ coincides as elements of $H_c^{2q+1}(W;\boldsymbol{C})$ for any Γ-vector field X and any transverse projective structure $\mathcal{P}$.*

Proof. Under the assumptions, the support of the coboundaries constructed in Lemmas 4.7 and 4.9 are compact and contained in W. □

6. Relation to the Fatou-Julia decomposition

If the complex codimension is equal to one, the localization in terms of $\operatorname{res}\langle\mathcal{L}(\mathcal{F},\mathcal{P})|\,\mu\rangle$ and the Fatou-Julia decomposition by Ghys, Gomez-Mont and Saludes [11] are related as follows. Note that in this case $\mathcal{L}$ is the classical Schwarzian (Definition 3.3). Let $\mathcal{B}_{\mathcal{F}}$ be the sheaf of germs of locally L^∞ foliated sections of $\overline{Q}(\mathcal{F})^*\otimes Q(\mathcal{F})$, where $\overline{Q}(\mathcal{F})$ denotes the complex conjugate of $Q(\mathcal{F})$, then $H^0(M;\mathcal{B}_{\mathcal{F}})$ is the space of locally L^∞ foliated sections of $\overline{Q}(\mathcal{F})^*\otimes Q(\mathcal{F})$. The space $H^0(M;\mathcal{B}_{\mathcal{F}})$ is a Banach space with the essential supremum norm and there is a natural mapping $\delta : H^0(M;\mathcal{B}_{\mathcal{F}})\to H^1(M;\Theta_{\mathcal{F}})$. It is natural to regard the image of δ as infinitesimal deformations preserving $\mathcal{F}_{\boldsymbol{R}}$, where $\mathcal{F}_{\boldsymbol{R}}$ denotes the real foliation obtained from $\mathcal{F}$ by forgetting the transverse holomorphic structure.

Lemma 6.1. *Let M be a closed manifold and let $\sigma\in H^0(M;\mathcal{B}_{\mathcal{F}})$. Set $\mu=\delta(\sigma)$, then $\langle\mathcal{L}\,|\,\sigma\rangle$ is well-defined as an integrable 3-form which is equal to $\langle\mathcal{L}(\mathcal{F})|\,\mu\rangle$ as an element of $\operatorname{Hom}_{\boldsymbol{C}}(H^{\dim M-3}(M;\boldsymbol{C}),\boldsymbol{C})\cong H^3(M;\boldsymbol{C})$.*

Proof. After extending σ as a section of $E^*\otimes Q(\mathcal{F})$ by requiring $\sigma|_{T\mathcal{F}}=0$, the lemma follows by approximating σ by differential forms of class C^∞ and by Lebesgue convergence theorem. □

More detailed information on $H^0(M;\mathcal{B}_{\mathcal{F}})$ was obtained in [11]. Let F be the Fatou set and F_l be its connected components, and let J be the Julia set and J_0 be the recurrent component and $J_1,\cdots,J_r$ be the ergodic components. Since measurable sections are considered, there is a decomposition

$$
\begin{aligned}
H^0(M;\mathcal{B}_{\mathcal{F}}) &= H^0(J;\mathcal{B}_{\mathcal{F}})\oplus H^0(F;\mathcal{B}_{\mathcal{F}}) \\
&= \bigoplus_{k=0}^{r} H^0(J_k;\mathcal{B}_{\mathcal{F}})\oplus\bigoplus_{l=1}^{s} H^0(F_l;\mathcal{B}_{\mathcal{F}}).
\end{aligned}
\tag{6.2}
$$

It is known that the mapping δ restricted to $H^0(J;\mathcal{B}_{\mathcal{F}})$ is injective [11, p.307]. Moreover $\delta|_{J_0}$ is by definition equal to zero and the image of $\delta|_{J_k}$, $k \neq 0$ is one-dimensional. Recalling that $H^0(M;\mathcal{B}_{\mathcal{F}})$ is a Banach space, choose a basis σ_k of unit length of $H^0(J_k;\mathcal{B}_{\mathcal{F}})$ for each $k > 0$. By choosing a section, we fix an isomorphism $H^1(M;\Theta_{\mathcal{F}}) \cong H^0(J;\mathcal{B}_{\mathcal{F}}) \oplus \mathcal{H}'_F \oplus \mathcal{H}_O$, where $H^0(J;\mathcal{B}_{\mathcal{F}}) \oplus \mathcal{H}'_F = \operatorname{Im}\delta$, and $\mathcal{H}_O \cong \operatorname{coker}\delta$.

Remark 6.3. Elements of $\mathcal{H}'_F$ correspond to infinitesimal deformations preserving $\mathcal{F}_{\boldsymbol{R}}$ but which cannot be induced by infinitesimal deformations supported on J. Elements of $\mathcal{H}_O$ correspond to infinitesimal deformations which cannot arise from deformations preserving $\mathcal{F}_{\boldsymbol{R}}$.

We normalize the volume of M to be 1 and denote by $|J_k|$ the volume of J_k. Note that $|J_k| > 0$ for $k > 0$. We propose the following

Definition 6.4. Let $\partial_{J_k} B_1(\mathcal{F})$, $k > 0$, be an element of $H^3(M;\boldsymbol{C})$ determined by $|J_k|\,\langle \mathcal{L} \,|\, \sigma_k\rangle$, and call it the infinitesimal derivative of the Bott class with respect to the ergodic component J_k.

It is easy to see that $\partial_{J_k} B_1(\mathcal{F})$ is independent of the choice of σ_k.

Corollary 6.5. *Let $\mu \in H^1(M;\Theta_{\mathcal{F}})$ and let $\mu = \mu_J + \mu'_F + \mu_O$ be the decomposition given by the isomorphism* (6.2). *Decompose further μ_J as $\sum_{k=1}^{r} a_k(|J_k|\,\sigma_k)$, then there is a measurable decomposition*

$$\langle \mathcal{L}(\mathcal{F})|\, \mu\rangle = \sum_{k=1}^{r} a_k \partial_{J_k} B_1(\mathcal{F}) + \langle \mathcal{L}(\mathcal{F})|\, \mu'_F\rangle + \langle \mathcal{L}(\mathcal{F})|\, \mu_O\rangle.$$

Although each Fatou component admits a transversal projective structure [11], $\langle \mathcal{L}(\mathcal{F})|\, \mu'_F\rangle$ need not vanish. Indeed, the transition functions from the Fatou set to the Julia set are not necessarily transversely projective. However, it is possible to assume that such phenomena occur only in an arbitrary small neighborhood of the Julia set. Recalling Theorem 5.9, we introduce the following

Definition 6.6. Let U be an arbitrary small neighborhood of the Julia set J. Fix a transverse projective structure on the union of Fatou components and denote it by $\mathcal{P}$. Let $\mu \in H^1(M;\Theta_{\mathcal{F}})$ and let σ be a representative of μ, then denote by $\operatorname{res}\langle \mathcal{L}(\mathcal{F},\mathcal{P})|\, \mu\rangle$ the element of $H^3_c(U;\boldsymbol{C})$ represented by $\langle \mathcal{L} \,|\, \sigma\rangle$.

The class $\operatorname{res}\langle \mathcal{L}(\mathcal{F},\mathcal{P})|\, \mu\rangle$ is independent of the choice of a foliation chart adapted to $\mathcal{P}$ and the representative σ. If J can be decomposed into connected components, then the residue admits a natural decomposition.

Remark 6.7. As Example 7.1 shows, the image of $\operatorname{res}\langle \mathcal{L}(\mathcal{F},\mathcal{P})|\,\mu\rangle$ in $H^3(M;\boldsymbol{C})$ and $\sum_{k=1}^{r} a_k\partial_{J_k}B_1(\mathcal{F})$ are distinct in general.

There are foliated sections (hence trivializations) of $Q(\mathcal{F})$ on most of the Fatou components. Indeed, the wandering Fatou components are locally trivial fibrations of which base spaces (or equivalently leaf spaces) are finite Riemann surfaces, while semi-wandering and dense components are G-Lie foliations [11], where G consists of projective transformations. The only exceptional case is that the Fatou component is a wandering component of which the base space is closed surface of genus $g \neq 1$. Let F' be the union of such wandering Fatou components and let U be an arbitrary small neighborhood of $J \cup F'$. Then one can always find a foliated trivialization X_Q of $Q(\mathcal{F})$ on a neighborhood of $M \setminus U$. Thus there is an element $\operatorname{res} D_\mu B_1(\mathcal{F}, X_Q)$ of $H^3_c(U;\boldsymbol{C})$, where the residue is defined by choosing a lift of X_Q as a Γ-vector field.

This residue and the residue given in Definition 6.6 can be related as follows. Let X be a Γ-vector field defined on O, then a transverse projective structure on O is naturally chosen because $T_{\boldsymbol{C}}O = E|_O \oplus \boldsymbol{C}X$. More precisely, let $\{O_i\}$ be an open covering of O by foliation charts, then there are projections $\pi_i : O_i \to \boldsymbol{C}$ which give the transverse holomorphic structure. Let X_i be the $(1,0)$-part of $\pi_{i*}X|_{O_i}$, then X_i is well-defined and holomorphic because X is a Γ-vector field. By integrating $2\operatorname{Re} X_i$, one obtains a foliation atlas $\{V_i\}$ with the following properties:

1) $\{V_i\}$ is a refinement of $\{O_i\}$.
2) The transversal direction of transition functions from V_i to V_j is the restriction of a translation in $\boldsymbol{C}$.
3) $X_i = \dfrac{\partial}{\partial z_i}$ on V_i, where z_i denotes the transverse coordinate on V_i.

Hence the atlas $\{V_i\}$ gives a transverse projective structure on O such that the connection form of the basic X-connection (note that it is unique because $T_{\boldsymbol{C}}M = E \oplus \boldsymbol{C}X$) is equal to zero with respect to the local trivializations $\{\frac{\partial}{\partial z_i}\}$ of $Q(\mathcal{F})$. Note that this family of local trivializations of $Q(\mathcal{F})$ is used in Section 4. It is clear that this transverse projective structure depends only on X_Q.

Definition 6.8. The transverse projective structure obtained as above is called the transverse projective structure associated with X and denoted by $\mathcal{P}_X$.

Recalling the previous definitions, we have then the following

Corollary 6.9. $\operatorname{res} D_\mu B_1(\mathcal{F}, X) = \operatorname{res}\langle \mathcal{L}(\mathcal{F},\mathcal{P}_X)|\,\mu\rangle$ *as elements of* $H^3_c(U_X;\boldsymbol{C})$.

The above Corollary 6.9 corresponds to the following version of Corollary 5.4 in [2], where a version of residues $\operatorname{res}^*_W B_1(\mathcal{F}, e)$ is defined by using transverse invariant Hermitian metric and trivialization of $Q(\mathcal{F})$ (Definition 5.1 in [2]). We give a proof because it is slightly stronger than the original one.

Proposition 6.10. *Let F', W and X as above. Then, there is a well-defined element* $\operatorname{res} B_1(\mathcal{F}, X, e)$ *of* $H^3_c(W; \boldsymbol{C})$, *where* $e = e_i$ *is a family of local trivializations of* $Q(\mathcal{F})$ *such that* $e_i = X$ *if* $U_i \subset M \setminus (J \cup F')$. *Moreover* $\operatorname{res} B_1(\mathcal{F}, X, e) = \operatorname{res}^*_W B_1(\mathcal{F}, e)$.

Proof. In [2], the claim is stated for $J \cup F_0$, where F_0 is the union of the wandering Fatou components. Since there is foliated trivializations of $Q(\mathcal{F})(= -K_{\mathcal{F}})$ on $F_0 \setminus F'$, the arguments in [2] remain valid even if F_0 is replaced with F'. □

7. Examples

We begin with a fundamental example.

Example 7.1. Let $X = \lambda_0 z^0 \dfrac{\partial}{\partial z^0} + \lambda_1 z^1 \dfrac{\partial}{\partial z^1}$ be a holomorphic vector field on $\boldsymbol{C}^2$, where (z^0, z^1) is the natural coordinate. Assume that $\lambda_0 \lambda_1 \neq 0$ and that $\lambda = \lambda_0/\lambda_1 \notin \boldsymbol{R}_{<0}$. Then, X naturally induces a transversely holomorphic foliation of S^3, which is denoted by $\mathcal{F}_\lambda$. If $\lambda = 1$, then $\mathcal{F}_1$ is formed by the orbits of the Hopf fibration, in particular $\mathcal{F}_1$ is transversely projective. The family $\{\mathcal{F}_\lambda\}$ is a smooth family of transversely holomorphic foliations and it is well-known that $B_1(\mathcal{F}_\lambda)$ is the natural image of $(\lambda + \lambda^{-1})[S^3]$, where $[S^3]$ is the generator of $H^3(S^3; \boldsymbol{Z})$. Let $Y = \nu z^1 \dfrac{\partial}{\partial z^1}$, then it induces a Γ-vector field for $\mathcal{F}_\lambda$. We denote the Γ-vector field again by Y. Let $\mu \in H^1(M; \mathcal{F}_\alpha)$ be the infinitesimal deformation induced by the family $\{\mathcal{F}_\lambda\}$, where $\mathcal{F}_\alpha$ is considered as the base point. Then Z_Y consists of two circles C_0 and C_1. Let U_i be a tubular neighborhood of C_i and identify $H^3_c(U_i; \boldsymbol{C})$ with $H^1(C_i; \boldsymbol{C})$ by integration along the fiber. The residue $\operatorname{res} D_\mu B_1(\mathcal{F}_\alpha, Y)$ is naturally decomposed into the sum of elements of $H^1(C_i; \boldsymbol{C})$, $i = 0, 1$. We denote these elements by $\operatorname{res}_{C_i} D_\mu B_1(\mathcal{F}_\alpha, Y)$, $i = 0, 1$, then by Theorem 5.7,

$$\begin{aligned}
\operatorname{res}_{C_0} D_\mu B_1(\mathcal{F}_\alpha, Y) &= [C_0] \in H^1(C_0; \boldsymbol{C}), \\
\operatorname{res}_{C_1} D_\mu B_1(\mathcal{F}_\alpha, Y) &= -\frac{1}{\alpha^2}[C_1] \in H^1(C_1; \boldsymbol{C}).
\end{aligned}$$

On the other hand, one has by Theorem 4.3 that

$$\langle \mathcal{L}(\mathcal{F}_\alpha) |\, \mu \rangle = \left(1 - \frac{1}{\alpha^2}\right)[S^3] \in H^3(S^3; \boldsymbol{C}).$$

Hence $\langle \mathcal{L}(\mathcal{F}_\alpha) | \, \mu \rangle = \operatorname{res}_{C_0} D_\mu B_1(\mathcal{F}_\alpha, Y) + \operatorname{res}_{C_1} D_\mu B_1(\mathcal{F}_\alpha, Y)$ in $H^3(S^3; \boldsymbol{C})$. If one adopts on a neighborhood of $S^3 \setminus (U_0 \cup U_1)$ the projective structure $\mathcal{P}_Y$ associated with Y, then $\operatorname{res} \langle L(\mathcal{F}_\alpha, \mathcal{P}_Y) | \, \mu \rangle$ is naturally decomposed into elements of $H^1(C_i; \boldsymbol{C})$, $i = 0, 1$. We denote these elements by $\operatorname{res}_i \langle L(\mathcal{F}_\alpha, \mathcal{P}_Y) | \, \mu \rangle$, $i = 0, 1$, then

$$\operatorname{res}_i \langle L(\mathcal{F}_\alpha, \mathcal{P}_Y) | \, \mu \rangle = \operatorname{res}_{C_i} D_\mu B_1(\mathcal{F}_\alpha, Y)$$

by Corollary 6.9.

If $\alpha = 1$, then $\operatorname{res}_{C_0} D_\mu B_1(\mathcal{F}_\alpha, Y) + \operatorname{res}_{C_1} D_\mu B_1(\mathcal{F}_\alpha, Y) = 0$ in $H^3(S^3; \boldsymbol{C})$. The foliation $\mathcal{F}_1$ is indeed the Hopf fibration so that $\mathcal{L}(\mathcal{F}_1) = 0$ in $\check{H}^1(S^3; Q^*(\mathcal{F}_1) \otimes Q^*(\mathcal{F}_1))$ and the infinitesimal derivative is always equal to zero. However, $\operatorname{res}_i \langle L(\mathcal{F}_1, \mathcal{P}_Y) | \, \mu \rangle$ can be non-trivial because the projective structure $\mathcal{P}_Y$ cannot be extended to the whole S^3.

If $\alpha \neq 1$, then $\mathcal{L}(\mathcal{F}_\alpha)$ is non-trivial so that $\mathcal{F}_\alpha$ cannot admit any transverse projective structures. Similar foliations can be constructed on S^{2q+1} from the vector field $\sum\limits_{i=0}^{q} \lambda_i z^i \dfrac{\partial}{\partial z^i}$ on $\boldsymbol{C}^{q+1}$, where $\lambda_i \neq 0$ for all i. Assuming that none of λ_i / λ_j is a negative real number, this vector field induces a foliation $\mathcal{F}_\lambda$ of S^{2q+1}, and it is known that $B_q(\mathcal{F}_\lambda) = \dfrac{(\lambda_0 + \cdots + \lambda_q)^q}{\lambda_0 \cdots \lambda_q} [S^{2q+1}]$. It follows from Theorem 2.18 that most of $\mathcal{F}_\lambda$ does not admit any transverse projective structures.

Coming back to complex codimension-one foliations, the localization given in Section 6 for foliations $\mathcal{F}_\alpha$ is as follows. If $\alpha = 1$, then the Julia set is empty. If $\alpha \neq 1$, then $J = C_0 \cup C_1$. In the both cases, J is of Lebesgue measure zero so that $\partial_J B_1(\mathcal{F}_\alpha) = 0$. This implies that $\langle \mathcal{L}(\mathcal{F}) | \, \mu'_F \rangle + \langle \mathcal{L}(\mathcal{F}) | \, \mu_O \rangle \neq 0$ in general. An example of μ_O is given in [3].

Example 7.2. Let $\mathcal{F}$ be a foliation given as follows. Let H be the subgroup of $\mathrm{SL}(q+1; \boldsymbol{C})$ defined by $H = \left\{ (a^i_j)_{0 \leq i,j \leq q} \,\middle|\, a^i_0 = 0 \text{ if } i > 0 \right\}$. Let $\mathit{\Gamma}$ be a discrete subgroup of $\mathrm{SL}(q+1; \boldsymbol{C})$ such that $M = \mathit{\Gamma} \backslash \mathrm{SL}(q+1; \boldsymbol{C}) / \mathrm{U}(q)$ is a closed manifold, where $\mathrm{U}(q)$ is considered as a subgroup of $\mathrm{SL}(q+1; \boldsymbol{C})$ by the mapping $A \in \mathrm{U}(q) \mapsto (\det A)^{-1} \oplus A \in \mathrm{SL}(q+1; \boldsymbol{C})$. Note that $\mathrm{U}(q) \subset H$. The cosets $\{gH\}_{g \in \mathrm{SL}(q+1; \boldsymbol{C})}$ naturally induce a transversely holomorphic foliation $\mathcal{F}$ of M, of complex codimension q, and it is known that the Godbillon-Vey class $\mathrm{GV}_{2q}(\mathcal{F})$ is non-trivial [4]. In this sense the dynamics of the foliation $\mathcal{F}$ is complicated. On the other hand, $\mathcal{F}$ is transversely projective since the transversal geometry of $\mathcal{F}$ is locally modeled on $\boldsymbol{C}P^q = \mathrm{SL}(q+1; \boldsymbol{C}) / H$. It follows that $\mathcal{L}(\mathcal{F})$ is equal to zero and the Bott class is infinitesimally rigid. Thus the rigidity of the Godbillon-Vey class of $\mathcal{F}$ is also derived from the rigidity of the Bott class, because

$\mathrm{GV}_{2q}(\mathcal{F}) = (\operatorname{Im} B_q(\mathcal{F}))\, c_1(Q(\mathcal{F}))^q$ up to multiplication of a constant determined by the codimension.

Example 7.3 [15]**.** Consider S^1 as $\boldsymbol{R}/\boldsymbol{Z}$ and let $\mathcal{F}_\lambda$ be the foliation of $S^1 \times \boldsymbol{C}^q$ induced by the vector field $\dfrac{\partial}{\partial t} + \sum\limits_{j=1}^{q} \lambda_j z^j \dfrac{\partial}{\partial z^j}$, there $(t, z^1, \cdots, z^q)$ denotes the standard coordinate. Let $Y_\delta = \sum\limits_{j=1}^{q} \delta_j z^j \dfrac{\partial}{\partial z^j}$, then Y_δ naturally induces a $\mathit{\Gamma}$-vector field for each λ. We adopt $e = \dfrac{\partial}{\partial z^1} \wedge \cdots \wedge \dfrac{\partial}{\partial z^q}$ as a trivialization of $Q(\mathcal{F}_\lambda)$, then $B_q(\mathcal{F}_\lambda, Y_\delta, e) = \dfrac{1}{2\pi\sqrt{-1}}(\lambda_1 + \cdots + \lambda_q)\dfrac{(\delta_1 + \cdots + \delta_q)^q}{\delta_1 \cdots \delta_q}[S^1]$, where $[S^1]$ denotes the natural generator of $H^1(S^1; \boldsymbol{Z})$. This implies that even if $\mathcal{F}_\lambda$ remains the same, the residue $D_\mu B_q(\mathcal{F}_\lambda, Y_\delta)$ can vary if $\mathit{\Gamma}$-vector fields are deformed. On the other hand, e is foliated with respect to $\mathcal{F}_{Y_\delta}$ if $\lambda_1 + \cdots + \lambda_q = \delta_1 + \cdots + \delta_q = 0$. If these conditions are fulfilled, then the derivative of $B_q(\mathcal{F}_\lambda, Y_\delta, e)$ is certainly well-defined and in fact equal to zero.

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