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On the singularity of Quillen metrics

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Abstract. Let $\pi\colon X \to S$ be a holomorphic map from a compact Kähler manifold (X, g_X) to a compact Riemann surface S. Let Σ_π be the critical locus of π and let $\Delta = \pi(\Sigma_\pi)$ be the discriminant locus. Let (ξ, h_ξ) be a holomorphic Hermitian vector bundle on X. We determine the singularity of the Quillen metric on $\det R\pi_*\xi$ near Δ with respect to $g_X|_{TX/S}$ and h_ξ.

1. Introduction

Let X be a compact Kähler manifold of dimension $n+1$ with Kähler metric g_X, and let S be a compact Riemann surface. Let $\pi\colon X \to S$ be a surjective holomorphic map such that every connected component of X is mapped surjectively to S. Let $\Sigma_\pi := \{x \in X;\, d\pi(x) = 0\}$ be the critical locus of π. For $t \in S$, set $X_t := \pi^{-1}(t)$. The relative tangent bundle of $\pi\colon X \to S$ is the subbundle of $TX|_{X\setminus\Sigma_\pi}$ defined as $TX/S := \ker \pi_*|_{X\setminus\Sigma_\pi}$. Set

$$\Delta := \pi(\Sigma_\pi), \qquad S^o := S \setminus \Delta, \qquad X^o := X|_{S^o}, \qquad \pi^o := \pi|_{X^o}.$$

Then $\pi^o\colon X^o \to S^o$ is a holomorphic family of compact Kähler manifolds. Let $g_{X/S} := g_X|_{TX/S}$ be the Hermitian metric on TX/S induced from g_X.

Let $\xi \to X$ be a holomorphic vector bundle on X equipped with a Hermitian metric h_ξ. Let $\lambda(\xi) = \det R\pi_*\xi$ be the determinant of the cohomologies of ξ. By [5], [14], [15], $\lambda(\xi)|_{S^o}$ is equipped with the Quillen metric $\|\cdot\|^2_{\lambda(\xi),Q}$ with respect to the metrics $g_{X/S}$ and h_ξ.

Let $0 \in \Delta$ be an arbitrary critical value of π, and let $(\mathcal{U}, t)$ be a coordinate neighborhood of S centered at 0 with $\mathcal{U} \cap \Delta = \{0\}$. Set $\mathcal{U}^o := \mathcal{U} \setminus \{0\}$.

Let σ be a nowhere vanishing holomorphic section of $\lambda(\xi)$ on $\mathcal{U}$. Then $\log \|\sigma\|^2_{\lambda(\xi),Q}$ is a C^∞ function on $\mathcal{U}^o$ by [5]. The purpose of this article is to study the behavior of $\log \|\sigma(t)\|^2_{\lambda(\xi),Q}$ as $t \to 0$.

For a holomorphic vector bundle F over a complex manifold with zero-section Z, define the projective-space bundle $\mathbb{P}(F)$ as $\mathbb{P}(F) := (F \setminus Z)/\mathbb{C}^*$. The dual projective-space bundle $\mathbb{P}(F)^\vee$ is defined as $\mathbb{P}(F)^\vee := \mathbb{P}(F^\vee)$, where $F^\vee$ is the dual vector bundle of F.

Following Bismut [3], we consider the Gauss map $\mu\colon X \setminus \Sigma_\pi \to \mathbb{P}(TX)^\vee$ that assigns $x \in X \setminus \Sigma_\pi$ the hyperplane $\ker(\pi_*)_x \in \mathbb{P}(T_xX)^\vee$. Since μ extends to a meromorphic map $\mu\colon X \dashrightarrow \mathbb{P}(TX)^\vee$, there exists a resolution $q\colon (\widetilde{X}, E) \to (X, \Sigma_\pi)$ of the indeterminacy of μ such that $\widetilde{\mu} := \mu \circ q$ extends to a holomorphic map from $\widetilde{X}$ to $\mathbb{P}(TX)^\vee$ and such that E is a normal crossing divisor of $\widetilde{X}$. (For the scheme structure of E, see Sect. 3.) Let U be the universal hyperplane bundle of rank $n = \dim X/S$ over $\mathbb{P}(TX)^\vee$, and let $H := \mathcal{O}_{\mathbb{P}(TX)^\vee}(1)$.

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After Barlet [1], we define a subspace of $C^0(\mathcal{U})$ by

$$\mathcal{B}(\mathcal{U}) := C^\infty(\mathcal{U}) \oplus \bigoplus_{r\in\mathbb{Q}\cap(0,1]} \bigoplus_{k=0}^{n} |t|^{2r}(\log|t|)^k \cdot C^\infty(\mathcal{U}).$$

A function $\varphi(t) \in \mathcal{B}(\mathcal{U})$ has an asymptotic expansion at $0 \in \Delta$, i.e., there exist $r_1, \ldots, r_m \in \mathbb{Q}\cap(0,1]$ and $f_0, f_{l,k} \in C^\infty(\mathcal{U})$, $l = 1, \ldots, m$, $k = 0, \ldots, n$, such that

$$\varphi(t) = f_0(t) + \sum_{l=1}^{m}\sum_{k=0}^{n} |t|^{2r_l}(\log|t|)^k f_{l,k}(t).$$

In what follows, if $f(t), g(t) \in C^\infty(\mathcal{U}^o)$ satisfies $f(t) - g(t) \in \mathcal{B}(\mathcal{U})$, we write

$$f \equiv_{\mathcal{B}} g.$$

For a complex vector bundle F over a complex manifold, $c_i(F)$, $\mathrm{Td}(F)$, and $\mathrm{ch}(F)$ denote the i-th Chern class, the Todd genus, and the Chern character of F, respectively.

We can state the main result of this article, which generalizes [3, §5] and [16]:

Theorem 1.1. *The following identity holds:*

$$\log\|\sigma\|^2_{Q,\lambda(\xi)} \equiv_{\mathcal{B}} \left(\int_{E\cap q^{-1}(X_0)} \widetilde{\mu}^* \left\{\mathrm{Td}(U)\,\frac{\mathrm{Td}(H)-1}{c_1(H)}\right\} q^*\mathrm{ch}(\xi)\right) \log|t|^2.$$

By Theorem 1.1, $\|\cdot\|^2_{Q,\lambda(\xi)}$ extends to a singular Hermitian metric on $\lambda(\xi)$. Let π_* denote the integration along the fibers of π. As a consequence of Theorem 1.1 and the curvature formula for Quillen metrics [5], we get the following:

Corollary 1.2. *The $(1,1)$-form $\pi_*(\mathrm{Td}(TX/S, g_{X/S})\,\mathrm{ch}(\xi, h_\xi))^{(1,1)}$ lies in $L^p_{\mathrm{loc}}(S)$ for some $p > 1$, and the curvature current of $(\lambda(\xi), \|\cdot\|_{Q,\lambda(\xi)})$ is given by the following formula on $\mathcal{U}$:*

$$\begin{aligned} c_1(\lambda(\xi), \|\cdot\|_{Q,\lambda(\xi)}) &= \pi_*(\mathrm{Td}(TX/S, g_{X/S})\,\mathrm{ch}(\xi, h_\xi))^{(1,1)} \\ &\quad - \left(\int_{E\cap q^{-1}(X_0)} \widetilde{\mu}^* \left\{\mathrm{Td}(U)\,\frac{\mathrm{Td}(H)-1}{c_1(H)}\right\} q^*\mathrm{ch}(\xi)\right) \delta_0, \end{aligned}$$

where δ_0 denotes the Dirac δ-current supported at 0.

The proof of Theorem 1.1 is quite similar to that of Bismut in [3, §5], and we just follow his argument. There are essentially no new ideas except a systematic use of the Gauss maps for the family $\pi\colon X \to S$; in fact, the Gauss maps were already used by Bismut in [3].

The existence of an asymptotic expansion of the Quillen norm $\log\|\sigma\|^2_{Q,\lambda(\xi)}$ was first shown by Bismut-Bost[4, Sect. 13.(b)] when $\pi\colon X \to S$ is a family of curves and by the author [16] when Σ_π is isolated. In [9], Theorem 1.1 shall play an crucial role in the study of analytic torsion of Calabi-Yau threefolds.

Let $\mathbf{s}_\Delta$ be a section of $\mathcal{O}_S(\Delta)$ defining the reduced divisor Δ. Let $\|\cdot\|$ be a C^∞ Hermitian metric on $\mathcal{O}_S(\Delta)$. By Theorem 1.1,

$$\log\|\sigma(t)\|^2_{Q,\lambda(\xi)} - \left(\int_{E\cap q^{-1}(X_0)} \widetilde{\mu}^* \left\{\mathrm{Td}(U)\,\frac{\mathrm{Td}(H)-1}{c_1(H)}\right\} q^*\mathrm{ch}(\xi)\right) \log\|\mathbf{s}_\Delta(t)\|^2$$

has a finite limit as $t \to 0$. In Section 6, we shall compute this limit in terms of various secondary objects, which extends some results in [3, §5].

This article is organized as follows. In Sections 2 and 3, we explain the Gauss maps associated to the family $\pi\colon X \to S$ and their resolutions. In Sections 5 and 6, we prove the main theorem. In Sections 7 and 8, we verify the compatibility of Theorem 1.1 with the corresponding earlier results of Bismut [3] and the author [16]. In Sections 4 and 9, we prove some technical results. The problem treated in Section 9 seems to be related with the regularity problem of the star products of Green currents [8].

For a complex manifold, we set $d^c = \frac{1}{4\pi i}(\partial - \bar{\partial})$. Hence $dd^c = \frac{1}{2\pi i}\bar{\partial}\partial$. We keep the notation in Sect. 1 throughout this article.

2. **The Gauss maps**

Let Ω^1_X be the holomorphic cotangent bundle of X. Let $\Pi\colon \mathbb{P}(\Omega^1_X \otimes \pi^* TS) \to X$ be the projective-space bundle associated with $\Omega^1_X \otimes \pi^* TS$. Since $\dim S = 1$, we have $\mathbb{P}(\Omega^1_X \otimes \pi^* TS) = \mathbb{P}(\Omega^1_X)$. Let $\Pi^\vee\colon \mathbb{P}(TX)^\vee \to X$ be the dual projective-space bundle of $\mathbb{P}(TX)$, whose fiber $\mathbb{P}(T_xX)^\vee$ is the set of hyperplanes of T_xX passing through the zero vector of T_xX. We have the canonical isomorphisms

$$\mathbb{P}(\Omega^1_X \otimes \pi^* TS) = \mathbb{P}(\Omega^1_X) \cong \mathbb{P}(TX)^\vee.$$

Let $x \in X \setminus \Sigma_\pi$. Let t be a holomorphic local coordinate of S near $\pi(x) \in S$. We define the Gauss maps $\nu\colon X \setminus \Sigma_\pi \to \mathbb{P}(\Omega^1_X \otimes \pi^* TS)$ and $\mu\colon X \setminus \Sigma_\pi \to \mathbb{P}(TX)^\vee$ by

$$\nu(x) := [d\pi_x] = \left[\sum_{i=0}^{n} \frac{\partial (t \circ \pi)}{\partial z_i}(x)\, dz_i \otimes \frac{\partial}{\partial t}\right], \qquad \mu(x) := [T_x X_{\pi(x)}].$$

Under the canonical isomorphism $\mathbb{P}(\Omega^1_X \otimes \pi^* TS) \cong \mathbb{P}(TX)^\vee$, one has

$$\nu = \mu.$$

Let

$$L := \mathcal{O}_{\mathbb{P}(\Omega^1_X \otimes \pi^* TS)}(-1) \subset \Pi^*(\Omega^1_X \otimes \pi^* TS)$$

be the tautological line bundle over $\mathbb{P}(\Omega^1_X \otimes \pi^* TS)$, and set

$$Q := \Pi^*(\Omega^1_X \otimes \pi^* TS)/L.$$

We have the exact sequence of holomorphic vector bundles on $\mathbb{P}(\Omega^1_X \otimes \pi^* TS)$:

$$\mathcal{S}\colon 0 \longrightarrow L \longrightarrow \Pi^*(\Omega^1_X \otimes \pi^* TS) \longrightarrow Q \longrightarrow 0.$$

Let $H = \mathcal{O}_{\mathbb{P}(T_X)^\vee}(1)$, and let U be the universal hyperplane bundle of $(\Pi^\vee)^* TX$. Then the dual of $\mathcal{S}$ is given by

$$\mathcal{S}^\vee\colon 0 \longrightarrow U \longrightarrow (\Pi^\vee)^* TX \longrightarrow H \longrightarrow 0.$$

Since $T_x X_{\pi(x)} = \{v \in T_x X;\, d\pi_x(v) = 0\}$, we have on $X \setminus \Sigma_\pi$

$$TX/S = \mu^* U.$$

Let g_U be the Hermitian metric on U induced from $(\Pi^\vee)^* g_X$, and let g_H be the Hermitian metric on H induced from $(\Pi^\vee)^* g_X$ by the C^∞-isomorphism $H \cong U^\perp$. On $X \setminus \Sigma_\pi$, we have

$$(TX/S, g_{\mathcal{X}/S}) = \mu^*(U, g_U).$$

Let g_S be a Hermitian metric on S. Let $g_{\Omega^1_X}$ be the Hermitian metric on Ω^1_X induced from g_X. Let g_L be the Hermitian metric on L induced from the metric $\Pi^*(g_{\Omega^1_X} \otimes \pi^* g_S)$ by the inclusion $L \subset \Pi^*(\Omega^1_X \otimes \pi^* TS)$. Let g_Q be the Hermitian metric on Q induced from $\Pi^*(g_{\Omega^1_X} \otimes \pi^* g_S)$ by the C^∞-isomorphism $Q \cong L^\perp$.

Let $c_1(L, g_L)$ be the Chern form of (L, g_L). Since $d\pi$ is a nowhere vanishing holomorphic section of $\nu^*L|_{X\setminus\Sigma_\pi}$, we get the following equation on $X \setminus \Sigma_\pi$

$$-dd^c \log \|d\pi\|^2 = \nu^* c_1(L, g_L).$$

3. Resolution of the Gauss maps

Since Σ_π is a proper analytic subset of X, the maps $\nu\colon X\setminus\Sigma_\pi \to \mathbb{P}(\Omega^1_X \otimes \pi^*TS)$ and $\mu\colon X\setminus\Sigma_\pi \to \mathbb{P}(TX)^\vee$ extend to meromorphic maps $\nu\colon X \dashrightarrow \mathbb{P}(\Omega^1_X \otimes \pi^*TS)$ and $\mu\colon X \dashrightarrow \mathbb{P}(TX)^\vee$ by [13, Th. 4.5.3]. By Hironaka, there exists a compact Kähler manifold $\widetilde{X}$, a normal crossing divisor $E \subset \widetilde{X}$, a birational holomorphic map $q\colon \widetilde{X} \to X$, and holomorphic maps $\widetilde{\nu}\colon \widetilde{X} \to \mathbb{P}(\Omega^1_X \otimes \pi^*TS)$ and $\widetilde{\mu}\colon \widetilde{X} \to \mathbb{P}(TX)^\vee$ satisfying the following conditions:
(i) $q|_{\widetilde{X}\setminus q^{-1}(\Sigma_\pi)}\colon \widetilde{X}\setminus q^{-1}(\Sigma_\pi) \to X\setminus\Sigma_\pi$ is an isomorphism;
(ii) $q^{-1}(\Sigma_\pi) = E$;
(iii) $(\pi\circ q)^{-1}(b)$ is a normal crossing divisor of $\widetilde{X}$ for all $b \in \Delta$;
(iv) $\widetilde{\nu} = \nu\circ q$ and $\widetilde{\mu} = \mu\circ q$ on $\widetilde{X}\setminus E$.
Then $\widetilde{\nu} = \widetilde{\mu}$ under the canonical isomorphism $\mathbb{P}(\Omega^1_X \otimes \pi^*TS) \cong \mathbb{P}(TX)^\vee$. We set

$$\widetilde{\pi} := \pi\circ q$$

and $\widetilde{X}_s := \widetilde{\pi}^{-1}(s)$ for $s \in S$. Similarly, we set $E_b := E \cap \widetilde{X}_b$ for $b \in \Delta$. Sinec $E = q^{-1}(\Sigma_\pi) \subset \widetilde{\pi}^{-1}(\Delta)$, we have $E = \amalg_{b\in\Delta} E_b$.

Let $\mathcal{I}_{\Sigma_\pi}$ be the ideal sheaf of Σ_π. For every $p \in \Sigma_\pi$, the sheaf $\mathcal{I}_{\Sigma_\pi}$ has the following expression on a neighborhood of p:

$$\mathcal{I}_{\Sigma_\pi} = \mathcal{O}_X\left(\frac{\partial(t\circ\pi)}{\partial z_0}(z), \cdots, \frac{\partial(t\circ\pi)}{\partial z_n}(z)\right).$$

Define the ideal sheaf $\mathcal{I}_E$ of E as

$$\mathcal{I}_E = q^{-1}\mathcal{I}_{\Sigma_\pi}.$$

Denote by δ_E the $(1,1)$-current on $\widetilde{X}$ defined as the integration over E, i.e., $\delta_E(\psi) := \int_E \psi|_E$ for all C^∞ (n,n)-form on $\widetilde{X}$. Since $\widetilde{\nu}^*L = q^*\nu^*L$, $q^*d\pi$ extends to a holomorphic section of $\widetilde{\nu}^*L$ with zero divisor E by the definition of the ideal sheaf $\mathcal{I}_E$. By the Poincaré-Lelong formula, the following identity of currents on $\widetilde{X}$ holds

$$-dd^c(q^*\log\|d\pi\|^2) = \widetilde{\nu}^*c_1(L, g_L) - \delta_E.$$

4. Regularity of the direct image of differential forms

Recall that $(\mathcal{U}, t)$ is a coordinate neighborhood of S centered at the critical value $0 \in \Delta$. Set $D := \{(s,t) \in S\times\mathcal{U};\ s = t\}$. Then D is a divisor of $S\times\mathcal{U}$. Let $[D]$ be the line bundle on $S\times\mathcal{U}$ defined by the divisor D. Let $\mathbf{s}_D$ be a section of $[D]$ with zero divisor D. Let $B \subset S$ be a finite subset with $0 \in B$. By shrinking $\mathcal{U}$ if necessary, we may assume that $\mathcal{U}\cap B = \{0\}$. Let $\|\cdot\|_D$ be a C^∞ Hermitian metric on $[D]$ such that

$$\|\mathbf{s}_D(b,t)\|_D = 1, \qquad \forall\,(b,t) \in (B\setminus\{0\})\times\mathcal{U}. \tag{4.1}$$

We set $\mathbf{s}_t := \mathbf{s}_D|_{S\times\{t\}}$ and $\|\cdot\|_t := \|\cdot\|_D|_{S\times\{t\}}$ for $t \in \mathcal{U}$. Then $\mathrm{div}(\mathbf{s}_t) = \{t\}$ and $\|\mathbf{s}_t\|_t^2 \in C^\infty(S\times\mathcal{U})$.

Let V be a compact connected complex manifold with $\dim V = n+1$. Let $f\colon V \to S$ be a proper surjective holomorphic map. We set $V_t := f^{-1}(t)$ for $t \in S$.

Let $\overline{F} := (F, \|\cdot\|_F)$ be a holomorphic Hermitian line bundle on V, and let α be a holomorphic section of F with

$$\mathrm{div}(\alpha) \subset \sum_{b \in B} V_b.$$

Denote by f_* the integration along the fibers of f. In Section 4, we assume that φ is a ∂-closed and $\bar{\partial}$-closed C^∞ (n,n)-form on V.

Lemma 4.1. *There exists a Hörder continuous function η on $\mathcal{U}$ such that*

$$f_*\{(\log \|\alpha\|_F^2)\,\varphi\}^{(0,0)} - \left(\int_{\mathrm{div}(\alpha)\cap V_0} \varphi\right) \log \|\mathbf{s}_0\|_0^2 = \eta.$$

Proof. Since $\log \|\alpha\|_F^2\,\varphi$ is a locally integrable differential form on V, we have $f_*\{(\log \|\alpha\|^2)\,\varphi\}^{(0,0)} \in L^1_{\mathrm{loc}}(S) \cap C^\infty(S^o)$. Since dd^c commutes with f_* and since φ is d and d^c-closed, we get the following equation of currents on $\mathcal{U}$:

(4.2)
$$\begin{aligned} dd^c f_*\{(\log \|\alpha\|_F^2)\,\varphi\}^{(0,0)} &= [f_*\{dd^c((\log \|\alpha\|_F^2) \wedge \varphi)\}]^{(1,1)} \\ &= -[f_*\left\{(c_1(\overline{F}) - \delta_{\mathrm{div}(\alpha)}) \wedge \varphi\right\}]^{(1,1)} \\ &= \left(\int_{\mathrm{div}(\alpha)\cap V_0} \varphi\right) \delta_0 - [f_*\{c_1(\overline{F}) \wedge \varphi\}]^{(1,1)}. \end{aligned}$$

By Lemma 9.2 below, there exists $\psi \in \mathcal{B}(\mathcal{U})$ such that

$$[f_*\{c_1(\overline{F}) \wedge \varphi\}]^{(1,1)}(t) = \psi(t)\,\frac{dt \wedge d\bar{t}}{|t|^2}, \qquad \psi(0) = 0.$$

Since $\psi(0) = 0$, there exists $\nu \in \mathbb{Q} \cap (0,1]$ such that $\psi(t) \in \sum_{k \le n} |t|^{2\nu} (\log |t|)^k \cdot \mathcal{B}(\mathcal{U})$. Hence $|t|^{-2}\psi(t) \in L^p_{\mathrm{loc}}(\mathcal{U})$ for some $p > 1$. By the ellipticity of the Laplacian and the Sobolev embedding theorem, there exists a Hölder continuous function χ on $\mathcal{U}$ satisfying the following equation of currents on $\mathcal{U}$

$$[f_*\{c_1(\overline{F}) \wedge \varphi\}]^{(1,1)} = dd^c \chi.$$

This, together with (4.2) and the equation of currents $dd^c \log |t|^2 = \delta_0$ on $\mathcal{U}$, implies the assertion, because $\log \|\mathbf{s}_0\|_0^2 - \log |t|^2 \in C^\infty(\mathcal{U})$. $\square$

Lemma 4.2. *The following identity holds for all $t \in \mathcal{U}^o$:*

$$\begin{aligned} \int_{V_t} (\log \|\alpha\|_F^2)\,\varphi = \left(\int_{\mathrm{div}(\alpha)\cap V_0} \varphi\right) \log \|\mathbf{s}_t(0)\|_t^2 - \int_V (f^* \log \|\mathbf{s}_t\|_t^2)\, c_1(\overline{F}) \wedge \varphi \\ + \int_V (\log \|\alpha\|_F^2)\, f^* c_1([t], \|\cdot\|_t) \wedge \varphi. \end{aligned}$$

Proof. Since $V_t \cap \mathrm{div}(\alpha) = \emptyset$ for $t \in \mathcal{U}^o$, V_t meets $\mathrm{div}(\alpha)$ properly. Since φ is ∂ and $\bar{\partial}$-closed, we deduce from [11, Th. 2.2.2] the following identity by setting $X = W = V$, $Y = V_t$, $Z = \mathrm{div}(\alpha)$, and $g_Y = -f^* \log \|\mathbf{s}_t\|_t^2$, $g_Z = -\log \|\alpha\|_F^2$ in [11, Sect. 2.2.2]:

(4.3)
$$\int_{V_t} (\log \|\alpha\|_F^2)\, \varphi = \sum_{b\in B} \left(\int_{\mathrm{div}(\alpha)\cap V_b} \varphi\right) \log \|\mathbf{s}_t(b)\|_t^2 - \int_V (f^* \log \|\mathbf{s}_t\|_t^2)\, c_1(\overline{F}) \wedge \varphi$$
$$+ \int_V (\log \|\alpha\|_F^2)\, f^* c_1([t], \|\cdot\|_t) \wedge \varphi,$$

where we used the assumption $\mathrm{div}(\alpha) \subset \sum_{b\in B} V_b$. (See also [15, p.59, l.3-l.7].) Since $\|\mathbf{s}_t(b)\|_t = 1$ for $(b,t) \in (B \setminus \{0\}) \times \mathcal{U}$ by (4.1), the result follows from (4.3). $\square$

Lemma 4.3. *The following identity holds*
$$\lim_{t\to 0} \left\{ \int_{V_t} (\log \|\alpha\|_F^2)\, \varphi - \left(\int_{\mathrm{div}(\alpha)\cap V_0} \varphi\right) \log \|\mathbf{s}_0(t)\|_0^2 \right\} =$$
$$\int_V (\log \|\alpha\|_F^2)\, f^* c_1([0], \|\cdot\|_0) \wedge \varphi - \int_V (f^* \log \|\mathbf{s}_0\|_0^2)\, c_1(\overline{F}) \wedge \varphi.$$

Proof. By Lemma 4.2, we have

(4.4)
$$\int_{V_t} (\log \|\alpha\|_F^2)\, \varphi = \left(\int_{\mathrm{div}(\alpha)\cap V_0} \varphi\right) \log \|\mathbf{s}_0(t)\|_0^2 - \int_V (f^* \log \|\mathbf{s}_t\|_t^2)\, c_1(\overline{F}) \wedge \varphi$$
$$+ \int_V (\log \|\alpha\|_F^2)\, f^* c_1([t], \|\cdot\|_t) \wedge \varphi + \left(\int_{\mathrm{div}(\alpha)\cap V_0} \varphi\right) \log \frac{\|\mathbf{s}_t(0)\|_t^2}{\|\mathbf{s}_0(t)\|_0^2}.$$

Since $\lim_{s\to 0} \log(\|\mathbf{s}_t(0)\|_t^2 / \|\mathbf{s}_0(t)\|_0^2) = 0$, the assertion follows from (4.4). $\square$

Lemma 4.4. *The following identity of functions on $\mathcal{U}^o$ hold:*
$$f_*\{(\log \|\alpha\|_F^2)\, \varphi\}^{(0,0)} \equiv_{\mathcal{B}} \left(\int_{\mathrm{div}(\alpha)\cap V_0} \varphi\right) \log \|\mathbf{s}_0\|_0^2.$$

Proof. For $t \in \mathcal{U}^o$, set
$$I_1(t) := \int_V (f^* \log \|\mathbf{s}_t\|_t^2)\, c_1(\overline{F})\, \varphi, \qquad I_2(t) := \int_V (\log \|\alpha\|_F^2)\, f^* c_1([t], \|\cdot\|_t)\, \varphi.$$

By (4.4), it suffices to prove that $I_1 \in \mathcal{B}(\mathcal{U})$ and $I_2 \in \mathcal{B}(\mathcal{U})$.

Let $\{(W_\lambda, z_\lambda)\}_{\lambda\in\Lambda}$ be a system of local coordinates on V. Since V is compact, we may assume $\#\Lambda < +\infty$. For every $\lambda \in \Lambda$, there exist $F_\lambda \in \mathcal{O}(W_\lambda)$, $G_\lambda \in \mathcal{O}(W_\lambda)$, $A_\lambda \in C^\infty(W_\lambda)$, and $B_\lambda \in C^\infty(W_\lambda \times \mathcal{U})$ such that
$$\widetilde{\pi}^* \log \|\mathbf{s}_t\|_t^2|_{W_\lambda}(z_\lambda) = \log |F_\lambda(z_\lambda) - t|^2 + B_\lambda(z_\lambda, t),$$
$$\log \|\alpha\|_F^2|_{W_\lambda}(z_\lambda) = \log |G_\lambda(z_\lambda)|^2 + A_\lambda(z_\lambda).$$

Let $\{\varrho_\lambda\}_{\lambda\in\Lambda}$ be a partition of unity of V subject to the covering $\{W_\lambda\}_{\lambda\in\Lambda}$. We set $\chi_\lambda := \varrho_\lambda\, c_1(\overline{F})\, \varphi$. Then

(4.5)
$$I_1(t) = \sum_{\lambda\in\Lambda} \int_{W_\lambda} \log |F_\lambda(z_\lambda) - t|^2 \cdot \chi_\lambda(z_\lambda) + \sum_{\lambda\in\Lambda} \int_{W_\lambda} B_\lambda(z_\lambda, t)\, \chi_\lambda(z_\lambda).$$

Since the first term of the right hand side of (4.5) lies in $\mathcal{B}(\mathcal{U})$ by Theorem 9.1 below, we get $I_1 \in \mathcal{B}(\mathcal{U})$.

We set $\theta_\lambda := \varrho_\lambda\, \widetilde{\pi}^* c_1([t], \|\cdot\|_t)\, \varphi$. Then $\theta_\lambda(z_\lambda, t)$ is a C^∞ $(n+1, n+1)$-form on $W_\lambda \times \mathcal{U}$. Since

$$I_2(t) = \sum_{\lambda\in\Lambda} \int_{W_\lambda} \log|G_\lambda(z_\lambda)|^2 \cdot \theta_\lambda(z_\lambda, t) + \sum_{\lambda\in\Lambda} \int_{W_\lambda} A_\lambda(z_\lambda)\, \theta_\lambda(z_\lambda, t),$$

we get $I_2 \in C^\infty(\mathcal{U})$. This completes the proof. □

Corollary 4.5. *The following identity holds*

$$\lim_{t\to 0} \left\{ \int_{\widetilde{X}_t} q^*(\log\|d\pi\|^2)\, \varphi - \left(\int_{E_0} \varphi \right) \log\|\mathbf{s}_0(t)\|_0^2 \right\} =$$

$$\int_{\widetilde{X}} (q^* \log\|d\pi\|^2)\, \widetilde{\pi}^* c_1([0], \|\cdot\|_0) \wedge \varphi - \int_{\widetilde{X}} (\widetilde{\pi}^* \log\|\mathbf{s}_0\|_0^2)\, \widetilde{\nu}^* c_1(L, g_L) \wedge \varphi.$$

Proof. Setting $V = \widetilde{X}$, $f = \widetilde{\pi}$, $\overline{F} = \widetilde{\nu}^*(L, g_L)$ and $\alpha = q^*(d\pi)$ in Lemma 4.3, we get the result. □

Corollary 4.6. *The following identity of functions on $\mathcal{U}^o$ hold:*

$$\widetilde{\pi}_*(q^*(\log\|d\pi\|^2)\, \varphi)^{(0,0)} \equiv_{\mathcal{B}} \left(\int_{E_0} \varphi \right) \log\|\mathbf{s}_0\|_0^2.$$

Proof. Setting $V = \widetilde{X}$, $f = \widetilde{\pi}$, $\overline{F} = \widetilde{\nu}^*(L, g_L)$ and $\alpha = q^*(d\pi)$ in Lemma 4.4, we get the result. □

5. Behavior of the Quillen norm of the Knudsen-Mumford section

Let $\Gamma \subset X \times S$ be the graph of π, which is a smooth divisor on $X \times S$. Let $[\Gamma]$ be the holomorphic line bundle on $X \times S$ associated to Γ. Let $s_\Gamma \in H^0(X \times S, [\Gamma])$ be the canonical section of $[\Gamma]$, so that $\mathrm{div}(s_\Gamma) = \Gamma$. We identify X with Γ.

Let $i\colon \Gamma \hookrightarrow X \times S$ be the inclusion. Let $p_1\colon X \times S \to X$ and $p_2\colon X \times S \to S$ be the projections. On $X \times S$, we have the exact sequence of coherent sheaves,

(5.1)

$$0 \longrightarrow \mathcal{O}_{X\times S}([\Gamma]^{-1} \otimes p_1^*\xi) \xrightarrow{\otimes s_\Gamma} \mathcal{O}_{X\times S}(p_1^*\xi) \longrightarrow i_*\mathcal{O}_\Gamma(p_1^*\xi) \longrightarrow 0.$$

Let $\lambda(p_1^*\xi)$, $\lambda([\Gamma]^{-1}\otimes p_1^*\xi)$, $\lambda(\xi)$ be the determinants of the direct images $R(p_2)_* p_1^*\xi$, $R(p_2)_*([\Gamma]^{-1} \otimes p_1^*\xi)$, $R\pi_*\xi$, respectively. By definition [5], [12], [15],

$$\lambda(\xi) = \bigotimes_{q\geq 0} (\det R^q\pi_*\xi)^{(-1)^q}.$$

Under the isomorphism $p_1^*\xi|_\Gamma \cong \xi$ induced from the identification $p_1\colon \Gamma \to X$, the holomorphic line bundle on S

$$\lambda := \lambda\left([\Gamma]^{-1} \otimes p_1^*\xi\right) \otimes \lambda(p_1^*\xi)^{-1} \otimes \lambda(\xi)$$

carries the canonical nowhere vanishing holomorphic section σ_{KM} by [7], [12].

Let $\mathcal{V} \subset \mathcal{U}$ be a relatively compact neighborhood of $0 \in \Delta$, and set $\mathcal{V}^o := \mathcal{V} \setminus \{0\}$. On $\pi^{-1}(\mathcal{U})$, we identify π (resp. $d\pi$) with $t\circ\pi$ (resp. $d(t\circ\pi)$). Hence $\pi \in \mathcal{O}(\pi^{-1}(\mathcal{U}))$ and $d\pi \in H^0(\pi^{-1}(\mathcal{U}), \Omega_X^1)$ in what follows.

Let $h_{[\Gamma]}$ be a C^∞ Hermitian metric on $[\Gamma]$ with

(5.2)

$$h_{[\Gamma]}(s_\Gamma, s_\Gamma)(w, t) = \begin{cases} |\pi(w) - t|^2 & \text{if} \quad (w, t) \in \pi^{-1}(\mathcal{V}) \times \mathcal{V}, \\ 1 & \text{if} \quad (w, t) \in (X \setminus \pi^{-1}(\mathcal{U})) \times \mathcal{V}. \end{cases}$$

Let $h_{[\Gamma]^{-1}}$ be the metric on $[\Gamma]^{-1}$ induced from $h_{[\Gamma]}$.

Let $\|\cdot\|_{Q,\lambda(\xi)}$ be the Quillen metric on $\lambda(\xi)$ with respect to $g_{X/S}$, h_ξ. Let $\|\cdot\|_{Q,\lambda([\Gamma]^{-1}\otimes p_1^*\xi)}$ (resp. $\|\cdot\|_{Q,\lambda(p_1^*\xi)}$) be the Quillen metric on $\lambda([\Gamma]^{-1}\otimes p_1^*\xi)$ (resp. $\lambda(p_1^*\xi)$) with respect to g_X, $h_{[\Gamma]^{-1}}\otimes h_\xi$ (resp. g_X, h_ξ). Let $\|\cdot\|_{Q,\lambda}$ be the Quillen metric on λ defined as the tensor product of those on $\lambda([\Gamma]^{-1}\otimes p_1^*\xi)$, $\lambda(p_1^*\xi)^{-1}$, $\lambda(\xi)$.

For a complex manifold Y, $A^{p,q}(Y)$ denotes the vector space of C^∞ (p,q)-forms on Y. We set $\widetilde{A}(Y) := \bigoplus_{p\geq 0} A^{p,p}(Y)/\mathrm{Im}\,\partial + \mathrm{Im}\,\bar\partial$.

For a Hermitian vector bundle (F,h_F) over Y, $c_i(F,h_F), \mathrm{Td}(F,h_F), \mathrm{ch}(F,h_F) \in \bigoplus_{p\geq 0} A^{p,p}(Y)$ denote the i-th Chern form, the Todd form, and the Chern character form of (F,h_F) with respect to the holomorphic Hermitian connection, respectively. Let $\mathrm{R}(F)$ denote the R-genus of Gillet-Soulé [7, (0.4)], [15, p. 160].

Theorem 5.1. *The following identity of functions on $\mathcal{U}^o$ holds*

$$\log\|\sigma_{KM}\|^2_{Q,\lambda} \equiv_{\mathcal{B}} \left(\int_{E_0} \widetilde{\mu}^*\left\{\mathrm{Td}(U)\,\frac{\mathrm{Td}(H)-1}{c_1(H)}\right\} q^*\mathrm{ch}(\xi)\right)\log|t|^2.$$

Proof. We follow Bismut [3, Sect. 5]. (See also [17, Th. 6.3].)

(Step 1) Let $[X_t]$ be the holomorphic line bundle on X associated to the divisor X_t. Then $[X_t] = [\Gamma]|_{X_t}$. We define the canonical section s_t of $[X_t]$ by $s_t := s_\Gamma|_{X\times\{t\}} \in H^0(X,[X_t])$. Then $\mathrm{div}(s_t) = X_t$. Let $i_t\colon X_t \hookrightarrow X$ be the embedding, and set $\xi_t := \xi|_{X_t}$. By (5.1), we get the exact sequence of coherent sheaves on X,

$$0 \longrightarrow \mathcal{O}_X([X_t]^{-1}\otimes\xi) \xrightarrow{\otimes s_t} \mathcal{O}_X(\xi) \longrightarrow (i_t)_*\mathcal{O}_{X_t}(\xi) \longrightarrow 0. \tag{5.3}$$

Let $\lambda([X_t]^{-1}\otimes\xi)$ and $\lambda(\xi_t)$ be the determinants of the cohomology groups of $[X_t]^{-1}\otimes\xi$ and ξ_t, respectively. Then $\lambda_t = \lambda([X_t]^{-1}\otimes\xi)\otimes\lambda(\xi)^{-1}\otimes\lambda(\xi_t)$.

Set $h_{[X_t]} = h_{[\Gamma]}|_{X\times\{t\}}$ for $t\in\mathcal{V}$. Then $h_{[X_t]}$ is a Hermitian metric on $[X_t]$. Let $h^{-1}_{[X_t]}$ be the Hermitian metric on $[X_t]^{-1}$ induced from $h_{[X_t]}$.

Let $N_t = N_{X_t/X}$ (resp. $N_t^* = N^*_{X_t/X}$) be the normal (resp. conormal) bundle of X_t in X. Then $d\pi|_{X_t} \in H^0(X_t, N_t^*)$ generates N_t^* for $t\in\mathcal{U}^o$. Let $h_{N_t^*}$ be the Hermitian metric on N_t^* defined by

$$h_{N_t^*}(d\pi|_{X_t}, d\pi|_{X_t}) = 1. \tag{5.4}$$

Let h_{N_t} be the Hermitian metric on N_t induced from $h_{N_t^*}$. Then we have the identity $c_1(N_t,h_{N_t}) = 0$ for $t\in\mathcal{V}^o$.

For $(w,t)\in\pi^{-1}(\mathcal{U})\times\mathcal{U}$, set

$$\widetilde{s}_\Gamma(w,t) = \frac{s_\Gamma(w,t)}{\pi(w)-t}.$$

Since $\pi(w)-t$ is a holomorphic function on $\pi^{-1}(\mathcal{U})\times\mathcal{U}$ with divisor Γ, $\widetilde{s}_\Gamma$ is a nowhere vanishing holomorphic section of $[\Gamma]|_{\pi^{-1}(\mathcal{U})\times\mathcal{U}}$. Set $\widetilde{s}_{X_t} = \widetilde{s}_\Gamma|_{X_t\times\{t\}} \in H^0(X_t, [X_t]|_{X_t})$ and

$$ds_t|_{X_t} := d\pi\otimes\widetilde{s}_{X_t} \in H^0(X_t, N_t^*\otimes[X_t]|_{X_t}).$$

By (5.2), (5.4), the isomorphism

$$\otimes ds_t|_{X_t}\colon [X_t]^{-1}\otimes\xi|_{X_t} \ni v \to ds_t|_{X_t}(v) \in N_t^*\otimes\xi_t$$

gives an isometry of holomorphic Hermitian vector bundles

$$([X_t]^{-1}\otimes\xi, h_{[X_t]^{-1}}\otimes h_\xi)|_{X_t} \cong (N_t^*\otimes\xi_t, h_{N_t^*}\otimes h_\xi|_{X_t})$$

for all $t \in \mathcal{V}^o$. Hence the metrics $h_{[X_t]^{-1}} \otimes h_\xi$ and h_ξ verify assumption (A) of Bismut [2, Def.1.5] with respect to h_{N_t} and $h_\xi|_{X_t}$.

(Step 2) Associated to the exact sequence of holomorphic vector bundles on X_t,

$$\mathcal{E}_t \colon 0 \longrightarrow TX_t \longrightarrow TX|_{X_t} \longrightarrow N_t \longrightarrow 0,$$

one can define the Bott-Chern class $\widetilde{\mathrm{Td}}(\mathcal{E}_t; g_{X_t}, g_X, h_{N_t}) \in \widetilde{A}(X_t)$ by [5, I, f)], [10, I, Sect. 1], [15, Chap. IV, Sect. 3] such that

$$dd^c\widetilde{\mathrm{Td}}(\mathcal{E}_t; g_{X_t}, g_X, h_{N_t}) = \mathrm{Td}(TX_t, g_{X_t})\,\mathrm{Td}(N_t, h_{N_t}) - \mathrm{Td}(TX, g_X)|_{X_t}.$$

Notice that our $\widetilde{\mathrm{Td}}(\mathcal{E}_t;\, g_{X_t}, g_X, h_{N_t})$ and Bismut-Lebeau's $\widetilde{\mathrm{Td}}(TX_t, TX|_{X_t}, h_{N_t})$ are related as follows:

$$\widetilde{\mathrm{Td}}(\mathcal{E}_t;\, g_{X_t}, g_X, h_{N_t}) = -\widetilde{\mathrm{Td}}(TX_t, TX|_{X_t}, h_{N_t}).$$

Let Z be a general fiber of $\pi\colon X \to S$. By applying the embedding formula of Bismut-Lebeau [7, Th. 0.1] (see also [3, Th. 5.6]) to the embedding $i_t\colon X_t \hookrightarrow X$ and to the exact sequence (5.3), we get for all $t \in \mathcal{V}^o$:

(5.5)

$$\begin{aligned}\log\|\sigma_{KM}(t)\|^2_{Q,\lambda} = &\int_{X\times\{t\}} -\frac{\mathrm{Td}(TX, g_X)\,\mathrm{ch}(\xi, h_\xi)}{\mathrm{Td}([\Gamma], h_{[\Gamma]})}\log h_{[\Gamma]}(s_\Gamma, s_\Gamma)|_{X\times\{t\}}\\ &-\int_{X_t}\frac{\widetilde{\mathrm{Td}}(\mathcal{E}_t;\, g_{X_t}, g_X, h_{N_t})\,\mathrm{ch}(\xi, h_\xi)}{\mathrm{Td}(N_t, h_{N_t})}\\ &-\int_X \mathrm{Td}(TX)\,R(TX)\,\mathrm{ch}(\xi) + \int_Z \mathrm{Td}(TZ)\,R(TZ)\,\mathrm{ch}(\xi|_Z).\end{aligned}$$

Here we used the explicit formula for the Bott-Chern current [6, Rem. 3.5, especially (3.23), Th. 3.15, Th. 3.17] to get the first term of the right hand side of (5.5). Notice that the dual of our $\lambda(\xi)$ was defined as $\lambda(\xi)$ in [7].

By Theorem 9.1 below, the first term of the right hand side of (5.5) lies in $\mathcal{B}(\mathcal{U})$. Substituting $c_1(N_t, h_{N_t}) = 0$ into (5.5), we get

$$\log\|\sigma_{KM}(t)\|^2_{Q,\lambda} \equiv_{\mathcal{B}} \int_{X_t} -\widetilde{\mathrm{Td}}(\mathcal{E}_t;\, g_{X_t}, g_X, h_{N_t})\,\mathrm{ch}(\xi, h_\xi). \tag{5.6}$$

(Step 3) Let g_{N_t} be the Hermitian metric on N_t induced from g_X by the C^∞ isomorphism $N_t \cong (TX_t)^\perp$. Let $\widetilde{\mathrm{Td}}(N_t;\, h_{N_t}, g_{N_t}) \in \widetilde{A}(X_t)$ be the Bott-Chern class [5, I, e)], [10, Sect. 1.2.4], [15, Chap. IV, Sect. 3] such that

$$dd^c\widetilde{\mathrm{Td}}(N_t; h_{N_t}, g_{N_t}) = \mathrm{Td}(N_t, h_{N_t}) - \mathrm{Td}(N_t, g_{N_t}).$$

By [10, I, Prop. 1.3.2 and Prop. 1.3.4] (see also Lemma 5.3 below),

(5.7)

$$\widetilde{\mathrm{Td}}(\mathcal{E}_t;\, g_{X_t}, g_X, h_{N_t}) = \widetilde{\mathrm{Td}}(\mathcal{E}_t;\, g_{X_t}, g_X, g_{N_t}) + \mathrm{Td}(TX_t, g_{X_t})\,\widetilde{\mathrm{Td}}(N_t;\, h_{N_t}, g_{N_t}).$$

Since $c_1(N_t, h_{N_t}) = 0$ and $g_{N_t} = \|d\pi\|^{-2}\,h_{N_t}$, we deduce from [10, I, Prop. 1.3.1 and (1.2.5.1)] the identity

$$\begin{aligned}\widetilde{\mathrm{Td}}(N_t;\, h_{N_t}, g_{N_t}) &= \frac{1-\mathrm{Td}(dd^c\log\|d\pi\|^2)}{dd^c\log\|d\pi\|^2}\,\log\|d\pi\|^2\\ &= \nu^*\left\{\frac{1-\mathrm{Td}(-c_1(L, g_L))}{-c_1(L, g_L)}\right\}\log\|d\pi\|^2\bigg|_{X_t}.\end{aligned} \tag{5.8}$$

Substituting (5.8) and $(TX_t, g_{X_t}) = \mu^*(U, g_U)|_{X_t}$ into (5.7), we get

(5.9)
$$\begin{aligned}&\widetilde{\mathrm{Td}}(\mathcal{E}_t;\, g_{X_t}, g_X, h_{N_t}) =\\ &\widetilde{\mathrm{Td}}(\mathcal{E}_t;\, g_{X_t}, g_X, g_{N_t}) + \mu^*\mathrm{Td}(U, g_U)\,\nu^*\left\{\frac{1-\mathrm{Td}(-c_1(L,g_L))}{-c_1(L,g_L)}\right\}\log\|d\pi\|^2\bigg|_{X_t}.\end{aligned}$$

Since

$$\mathcal{E}_t = \mu^*\mathcal{S}^\vee|_{X_t},\quad g_{X_t} = \mu^* g_U|_{X_t},\quad g_X = \mu^*(\Pi^\vee)^* g_X|_{X_t},\quad g_{N_t} = \mu^* g_H|_{X_t},$$

we deduce from [10, I, Th. 1.2.2 (ii)] that

$$\widetilde{\mathrm{Td}}(\mathcal{E}_t;\, g_{X_t}, g_X, g_{N_t}) = \mu^*\widetilde{\mathrm{Td}}(\mathcal{S}^\vee;\, g_U, (\Pi^\vee)^* g_X, g_H)|_{X_t}. \tag{5.10}$$

Comparing (5.9) and (5.10), we get

(5.11)
$$\begin{aligned}\widetilde{\mathrm{Td}}(\mathcal{E}_t;\, g_{X_t}, g_X, h_{N_t}) &= \mu^*\widetilde{\mathrm{Td}}(\mathcal{S}^\vee;\, g_U, (\Pi^\vee)^* g_X, g_H)|_{X_t}\\ &\quad + \mu^*\mathrm{Td}(U, g_U)\,\nu^*\left\{\frac{1-\mathrm{Td}(-c_1(L,g_L))}{-c_1(L,g_L)}\right\}\log\|d\pi\|^2|_{X_t}.\end{aligned}$$

Substituting (5.11) into (5.6), we get

(5.12)
$$\begin{aligned}&\log\|\sigma_{KM}\|^2_{Q,\lambda}\\ &\equiv_{\mathcal{B}} -\pi_*\left[\mu^*\widetilde{\mathrm{Td}}(\mathcal{S}^\vee;\, g_U, (\Pi^\vee)^* g_X, g_H)\,\mathrm{ch}(\xi, h_\xi)\right]^{(0,0)}\\ &\quad -\pi_*\left[\mu^*\mathrm{Td}(U, g_U)\,\nu^*\left\{\frac{1-\mathrm{Td}(-c_1(L,g_L))}{-c_1(L,g_L)}\right\}\mathrm{ch}(\xi, h_\xi)\,\log\|d\pi\|^2\right]^{(0,0)}\\ &\equiv_{\mathcal{B}} -\widetilde{\pi}_*\left[\widetilde{\mu}^*\widetilde{\mathrm{Td}}(\mathcal{S}^\vee;\, g_U, (\Pi^\vee)^* g_X, g_H)\,q^*\mathrm{ch}(\xi, h_\xi)\right]^{(0,0)}\\ &\quad +\widetilde{\pi}_*\left[\widetilde{\mu}^*\mathrm{Td}(U, g_U)\,\widetilde{\nu}^*\left\{\frac{\mathrm{Td}(-c_1(L,g_L))-1}{-c_1(L,g_L)}\right\}q^*\mathrm{ch}(\xi, h_\xi)\,(q^*\log\|d\pi\|^2)\right]^{(0,0)}.\end{aligned}$$

Recall that for a C^∞ differential form φ on $\widetilde{X}$, one has $\widetilde{\pi}_*(\varphi)^{(0,0)} \in \mathcal{B}(\mathcal{U})$ by Barlet [1, Th. 4bis]. Since $q^*\mathrm{ch}(\xi, h_\xi)$ and

$$\widetilde{\mu}^*\widetilde{\mathrm{Td}}(\mathcal{S}^\vee;\, g_U, (\Pi^\vee)^* g_X, g_H),\quad \widetilde{\mu}^*\mathrm{Td}(U, g_U),\quad \widetilde{\nu}^*\{\frac{\mathrm{Td}(-c_1(L,g_L))-1}{-c_1(L,g_L)}\}$$

are C^∞ differential forms on $\widetilde{X}$, we deduce from (5.12), [1, Th. 4bis], and Corollary 4.6 that

(5.13)
$$\log\|\sigma_{KM}\|^2_{Q,\lambda} \equiv_{\mathcal{B}} \left(\int_{E_0}\widetilde{\mu}^*\left\{\mathrm{Td}(U)\,\frac{\mathrm{Td}(H)-1}{c_1(H)}\right\}q^*\mathrm{ch}(\xi)\right)\log|t|^2.$$

Here we used the identity $c_1(H) = -c_1(L) + (\Pi^\vee)^*\pi^* c_1(S)$ in $H^2(\mathbb{P}(TX)^\vee, \mathbb{Z})$ and the triviality of the line bundle $\widetilde{\mu}^*(\Pi^\vee)^*\pi^*(TS)|_{\widetilde{\pi}^{-1}(\mathcal{U})}$ to get (5.13). This completes the proof of Theorem 5.1. □

For simplicity, we set $\overline{L} := (L, g_L)$, $\overline{U} := (U, g_U)$, $\overline{\xi} := (\xi, h_\xi)$ in what follows.

Let $\widetilde{\mathrm{Td}}(\mathcal{S}^\vee;\, g_U, (\Pi^\vee)^* g_X, g_H)$ be the Bott-Chern secondary class associated with the Todd genus and the exact sequence of holomorphic vector bundles

$$\mathcal{S}^\vee : 0 \to U \to (\Pi^\vee)^* TX \to H \to 0$$

equipped with the Hermitian metrics g_U, $(\Pi^\vee)^* g_X$, g_H, such that

$$dd^c \widetilde{\mathrm{Td}}(\mathcal{S}^\vee;\, g_U, (\Pi^\vee)^* g_X, g_H) = \mathrm{Td}(U, g_U)\, \mathrm{Td}(H, g_H) - (\Pi^\vee)^* \mathrm{Td}(TX, g_X).$$

Recall that Z is a general fiber of $\pi\colon X \to S$.

Theorem 5.2. *The following identity holds*

$$\begin{aligned}
\lim_{t\to 0} &\left[\log \|\sigma_{KM}(t)\|^2_{Q,\lambda} - \left(\int_{E_0} \widetilde{\mu}^* \left\{\mathrm{Td}(U)\, \frac{\mathrm{Td}(H) - 1}{c_1(H)}\right\} q^* \mathrm{ch}(\xi)\right) \log \|\mathbf{s}_0(t)\|^2_0\right] = \\
&- \int_{X\times\{0\}} \frac{\mathrm{Td}(TX, g_X)\, \mathrm{ch}(\overline{\xi})}{\mathrm{Td}([\Gamma], h_{[\Gamma]})} \log \|s_\Gamma\|^2|_{X\times\{0\}} \\
&- \int_{\widetilde{X}_0} \widetilde{\mu}^* \widetilde{\mathrm{Td}}(\mathcal{S}^\vee;\, g_U, (\Pi^\vee)^* g_X, g_H)\, q^* \mathrm{ch}(\overline{\xi}) \\
&+ \int_{\widetilde{X}} (q^* \log \|d\pi\|^2)\, \widetilde{\pi}^* c_1([0], \|\cdot\|_0) \left[\widetilde{\mu}^* \mathrm{Td}(\overline{U})\, \widetilde{\nu}^* \left\{\frac{\mathrm{Td}(-c_1(\overline{L})) - 1}{-c_1(\overline{L})}\right\} q^* \mathrm{ch}(\overline{\xi})\right] \\
&- \int_{\widetilde{X}} (\widetilde{\pi}^* \log \|\mathbf{s}_0\|^2_0)\, \widetilde{\nu}^* c_1(\overline{L}) \left[\widetilde{\mu}^* \mathrm{Td}(\overline{U})\, \widetilde{\nu}^* \left\{\frac{\mathrm{Td}(-c_1(\overline{L})) - 1}{-c_1(\overline{L})}\right\} q^* \mathrm{ch}(\overline{\xi})\right] \\
&- \int_X \mathrm{Td}(TX)\, \mathrm{R}(TX)\, \mathrm{ch}(\xi) + \int_Z \mathrm{Td}(TZ)\, \mathrm{R}(TZ)\, \mathrm{ch}(\xi|_Z).
\end{aligned}$$

Proof. Define topological constants C_0 and C_1 by

$$C_0 := \int_{E_0} \widetilde{\mu}^* \left\{\mathrm{Td}(U)\, \frac{\mathrm{Td}(H) - 1}{c_1(H)}\right\} q^* \mathrm{ch}(\xi),$$

$$C_1 := -\int_X \mathrm{Td}(TX)\, \mathrm{R}(TX)\, \mathrm{ch}(\xi) + \int_Z \mathrm{Td}(TZ)\, \mathrm{R}(TZ)\, \mathrm{ch}(\xi|_Z).$$

Substituting (5.11) and $c_1(N_t, h_{N_t}) = 0$ into (5.5), we get for $t \in \mathcal{U}^o$

(5.14)

$$\begin{aligned}
\log \|\sigma_{KM}(t)\|^2_{Q,\lambda} &= -\int_{X\times\{t\}} \frac{\mathrm{Td}(TX, g_X)\, \mathrm{ch}(\overline{\xi})}{\mathrm{Td}([\Gamma], h_{[\Gamma]})} \log \|s_\Gamma\|^2|_{X\times\{t\}} \\
&\quad - \int_{X_t} \mu^* \widetilde{\mathrm{Td}}(\mathcal{S}^\vee;\, g_U, (\Pi^\vee)^* g_X, g_H)|_{X_t}\, \mathrm{ch}(\overline{\xi}) \\
&\quad - \int_{X_t} \mu^* \mathrm{Td}(\overline{U})\, \nu^* \left\{\frac{1 - \mathrm{Td}(-c_1(\overline{L}))}{-c_1(\overline{L})}\right\} \mathrm{ch}(\overline{\xi})\, \log \|d\pi\|^2 + C_1 \\
&= -\int_{X\times\{t\}} \frac{\mathrm{Td}(TX, g_X)\, \mathrm{ch}(\overline{\xi})}{\mathrm{Td}([\Gamma], h_{[\Gamma]})} \log \|s_\Gamma\|^2|_{X\times\{t\}} \\
&\quad - \int_{\widetilde{X}_t} \widetilde{\mu}^* \widetilde{\mathrm{Td}}(\mathcal{S}^\vee;\, g_U, (\Pi^\vee)^* g_X, g_H)|_{X_t}\, q^* \mathrm{ch}(\overline{\xi}) \\
&\quad + \int_{\widetilde{X}_t} \widetilde{\mu}^* \mathrm{Td}(\overline{U})\, \widetilde{\nu}^* \left\{\frac{\mathrm{Td}(-c_1(\overline{L})) - 1}{-c_1(\overline{L})}\right\} q^* \mathrm{ch}(\overline{\xi})\, q^*(\log \|d\pi\|^2) + C_1,
\end{aligned}$$

which yields that

(5.15)
$$
\begin{aligned}
&\log\|\sigma_{KM}(t)\|^2_{Q,\lambda} - C_0\,\log\|\mathbf{s}_0(t)\|^2_0 = \\
&-\int_{X\times\{t\}} \frac{\mathrm{Td}(TX,g_X)\,\mathrm{ch}(\overline{\xi})}{\mathrm{Td}([\Gamma],h_{[\Gamma]})}\log\|s_\Gamma\|^2 - \int_{\widetilde{X}_t}\widetilde{\mu}^*\widetilde{\mathrm{Td}}(\mathcal{S}^\vee;\,g_U,(\mathit{\Pi}^\vee)^*g_X,g_H)\,q^*\mathrm{ch}(\overline{\xi}) \\
&+\int_{\widetilde{X}_t}\left[\widetilde{\mu}^*\mathrm{Td}(\overline{U})\,\widetilde{\nu}^*\left\{\frac{\mathrm{Td}(-c_1(\overline{L}))-1}{-c_1(\overline{L})}\right\}q^*\mathrm{ch}(\overline{\xi})\right]q^*(\log\|d\pi\|^2) - C_0\,\log\|\mathbf{s}_0(t)\|^2_0 \\
&+C_1.
\end{aligned}
$$

By Corollary 4.5,

(5.16)
$$
\begin{aligned}
&\int_{\widetilde{X}_t}\left[\widetilde{\mu}^*\mathrm{Td}(\overline{U})\,\widetilde{\nu}^*\left\{\frac{\mathrm{Td}(-c_1(\overline{L}))-1}{-c_1(\overline{L})}\right\}q^*\mathrm{ch}(\overline{\xi})\right]q^*(\log\|d\pi\|^2) - C_0\,\log\|\mathbf{s}_0(t)\|^2_0 \\
&=\int_{\widetilde{X}}(q^*\log\|d\pi\|^2)\,\widetilde{\pi}^*c_1([0],\|\cdot\|_0)\left[\widetilde{\mu}^*\mathrm{Td}(\overline{U})\,\widetilde{\nu}^*\left\{\frac{\mathrm{Td}(-c_1(\overline{L}))-1}{-c_1(\overline{L})}\right\}q^*\mathrm{ch}(\overline{\xi})\right] \\
&\quad-\int_{\widetilde{X}}(\widetilde{\pi}^*\log\|\mathbf{s}_0\|^2_0)\,\widetilde{\nu}^*c_1(\overline{L})\,\left[\widetilde{\mu}^*\mathrm{Td}(\overline{U})\,\widetilde{\nu}^*\left\{\frac{\mathrm{Td}(-c_1(\overline{L}))-1}{-c_1(\overline{L})}\right\}q^*\mathrm{ch}(\overline{\xi})\right]+o(1).
\end{aligned}
$$

From (5.15) and (5.16), we get

(5.17)
$$
\begin{aligned}
&\lim_{t\to 0}\left[\log\|\sigma_{KM}(t)\|^2_{Q,\lambda} - C_0\,\log\|\mathbf{s}_0(t)\|^2_0\right] = \\
&-\int_{X\times\{0\}} \frac{\mathrm{Td}(TX,g_X)\,\mathrm{ch}(\overline{\xi})}{\mathrm{Td}([\Gamma],h_{[\Gamma]})}\log\|s_\Gamma\|^2|_{X\times\{0\}} \\
&-\int_{\widetilde{X}_0}\widetilde{\mu}^*\widetilde{\mathrm{Td}}(\mathcal{S}^\vee;\,g_U,(\mathit{\Pi}^\vee)^*g_X,g_H)\,q^*\mathrm{ch}(\overline{\xi}) \\
&+\int_{\widetilde{X}}(q^*\log\|d\pi\|^2)\,\widetilde{\pi}^*c_1([0],\|\cdot\|_0)\left[\widetilde{\mu}^*\mathrm{Td}(\overline{U})\,\widetilde{\nu}^*\left\{\frac{\mathrm{Td}(-c_1(\overline{L}))-1}{-c_1(\overline{L})}\right\}q^*\mathrm{ch}(\overline{\xi})\right] \\
&-\int_{\widetilde{X}}(\widetilde{\pi}^*\log\|\mathbf{s}_0\|^2_0)\,\widetilde{\nu}^*c_1(\overline{L})\,\left[\widetilde{\mu}^*\mathrm{Td}(\overline{U})\,\widetilde{\nu}^*\left\{\frac{\mathrm{Td}(-c_1(\overline{L}))-1}{-c_1(\overline{L})}\right\}q^*\mathrm{ch}(\overline{\xi})\right]+C_1.
\end{aligned}
$$

This completes the proof of Theorem 5.2. ☐

Lemma 5.3. *Let $\mathcal{E}\colon 0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$ be an exact sequence of holomorphic vector bundles over a complex manifold Y. Let h' and h be Hermitian metrics on E' and E, respectively. Let h'' and g'' be Hermitian metrics on E''. Then*
$$\widetilde{\mathrm{Td}}(\mathcal{E};\,h',h,h'') - \widetilde{\mathrm{Td}}(\mathcal{E};\,h',h,g'') = \mathrm{Td}(E',h')\,\widetilde{\mathrm{Td}}(E'';\,h'',g'').$$

Proof. Setting $\overline{L}_1 = (\mathcal{E},h',h,h'')$, $\overline{L}_2 = (\mathcal{E},h',h,g'')$, $\overline{L}_3 = 0$ in [10, I, Prop. 1.3.4], we get
$$\widetilde{\mathrm{Td}}(\mathcal{E};\,h',h,h'') - \widetilde{\mathrm{Td}}(\mathcal{E};\,h',h,g'') = \widetilde{\mathrm{Td}}(E'\oplus E'';\,h'\oplus h'',h'\oplus g'').$$

Since $\widetilde{\mathrm{Td}}(E'\oplus E'';\,h'\oplus h'',h'\oplus g'') = \mathrm{Td}(E',h')\,\widetilde{\mathrm{Td}}(E'';\,h'',g'')$ by [10, I, Prop. 1.3.2], we get the result. ☐

6. **The divergent term and the constant term**

Let α be a nowhere vanishing holomorphic section of $\lambda([\Gamma]^{-1}\otimes p_1^*\xi)^{-1}\otimes\lambda(p_1^*\xi)$ defined on $\mathcal{U}$.

Theorem 6.1. *Let σ be a nowhere vanishing holomorphic section of $\lambda(\xi)$ defined on $\mathcal{U}$. Then*

$$\log\|\sigma\|^2_{Q,\lambda(\xi)} \equiv_{\mathcal{B}} \left(\int_{E_0}\widetilde{\mu}^*\left\{\mathrm{Td}(U)\,\frac{\mathrm{Td}(H)-1}{c_1(H)}\right\} q^*\mathrm{ch}(\xi)\right)\log|t|^2.$$

Proof. There exists a nowhere vanishing holomorphic function $f(t)$ on $\mathcal{U}$ such that

$$\sigma(t) = f(t)\,\sigma_{KM}(t)\otimes\alpha(t).$$

Since $\log|f(t)|^2$ and $\log\|\alpha\|^2_{Q,\lambda([\Gamma]^{-1}\otimes p_1^*\xi)^{-1}\otimes\lambda(p_1^*\xi)}$ are C^∞ functions on $\mathcal{U}$, we deduce from Theorem 5.1 that

$$\begin{aligned}\log\|\sigma(t)\|^2_{Q,\lambda(\xi)} &= \log|f(t)|^2 + \log\|\sigma_{KM}(t)\|^2_{Q,\lambda} + \log\|\alpha(t)\|^2_{Q,\lambda([\Gamma]^{-1}\otimes p_1^*\xi)^{-1}\otimes\lambda(p_1^*\xi)}\\ &\equiv_{\mathcal{B}} \left(\int_{E_0}\widetilde{\mu}^*\left\{\mathrm{Td}(U)\,\frac{\mathrm{Td}(H)-1}{c_1(H)}\right\} q^*\mathrm{ch}(\xi)\right)\log|t|^2.\end{aligned}$$

This completes the proof of Theorem 6.1. □

Theorem 6.2. *The following identity holds:*

$$\begin{aligned}&\lim_{t\to0}\left[\log\|\sigma_{KM}\otimes\alpha\|^2_{Q,\lambda(\xi)}(t) - \left(\int_{E_0}\widetilde{\mu}^*\left\{\mathrm{Td}(U)\,\frac{\mathrm{Td}(H)-1}{c_1(H)}\right\} q^*\mathrm{ch}(\xi)\right)\log\|\mathbf{s}_0(t)\|_0^2\right]\\
&= \log\|\alpha(0)\|^2_Q - \int_{X\times\{0\}}\frac{\mathrm{Td}(TX,g_X)\,\mathrm{ch}(\overline{\xi})}{\mathrm{Td}([\Gamma],h_{[\Gamma]})}\log\|s_\Gamma\|^2|_{X\times\{0\}}\\
&\quad-\int_{\widetilde{X}_0}\widetilde{\mu}^*\widetilde{\mathrm{Td}}(\mathcal{S}^\vee;g_U,(\Pi^\vee)^*g_X,g_H)\,q^*\mathrm{ch}(\overline{\xi})\\
&\quad+\int_{\widetilde{X}}(q^*\log\|d\pi\|^2)\,\widetilde{\pi}^*c_1([0],\|\cdot\|_0)\left[\widetilde{\mu}^*\mathrm{Td}(\overline{U})\,\widetilde{\nu}^*\left\{\frac{\mathrm{Td}(-c_1(\overline{L}))-1}{-c_1(\overline{L})}\right\}q^*\mathrm{ch}(\overline{\xi})\right]\\
&\quad-\int_{\widetilde{X}}(\pi^*\log\|\mathbf{s}_0\|_0^2)\,\widetilde{\nu}^*c_1(\overline{L})\left[\widetilde{\mu}^*\mathrm{Td}(\overline{U})\,\widetilde{\nu}^*\left\{\frac{\mathrm{Td}(-c_1(\overline{L}))-1}{-c_1(\overline{L})}\right\}q^*\mathrm{ch}(\overline{\xi})\right]\\
&\quad-\int_X\mathrm{Td}(TX)\,\mathrm{R}(TX)\,\mathrm{ch}(\xi) + \int_Z\mathrm{Td}(TZ)\,\mathrm{R}(TZ)\,\mathrm{ch}(\xi|_Z).\end{aligned}$$

Proof. Since

$$\log\|\sigma_{KM}\otimes\alpha\|^2_{Q,\lambda(\xi)} = \log\|\sigma_{KM}\|^2_{Q,\lambda} + \log\|\alpha\|^2_{Q,\lambda([\Gamma]^{-1}\otimes p_1^*\xi)^{-1}\otimes\lambda(p_1^*\xi)},$$

the result follows from Theorem 5.2. □

7. **Critical points defined by a quadric polynomial of rank** 2

In this section, we assume that for every $x\in\Sigma_\pi\cap X_0$, there exists a system of coordinates $(z_0,\dots,z_n)$ centered at x such that

$$\pi(z) = z_0z_1.$$

Hence $\Sigma_\pi\subset X$ is a complex submanifold of codimension 2 defined locally by the equation $z_0=z_1=0$. Let $N_{\Sigma_\pi/X}$ be the normal bundle of Σ_π in X. In [3, Def. 5.1, Prop. 5.2], Bismut introduced the additive genus $\mathrm{E}(\cdot)$ associated with the generating function

$$\mathrm{E}(x) := \frac{\mathrm{Td}(x)\,\mathrm{Td}(-x)}{2x}\left(\frac{\mathrm{Td}^{-1}(x)-1}{x} - \frac{\mathrm{Td}^{-1}(-x)-1}{-x}\right),$$

where $\mathrm{Td}^{-1}(x) := (1-e^{-x})/x$.

The following result was proved by Bismut [3, Th. 5.9].

Theorem 7.1. *The following equation of functions on $\mathcal{U}^o$ holds:*

$$\log \|\sigma(t)\|^2_{\lambda(\xi),Q} \equiv_{\mathcal{B}} \frac{1}{2}\left(\int_{\Sigma_\pi\cap X_0} -\mathrm{Td}(T\Sigma_\pi)\,\mathrm{E}(N_{\Sigma_\pi/X})\,\mathrm{ch}(\xi)\right)\log|t|^2.$$

Remark 7.2. As mentioned before, the dual of our $\lambda(\xi)$ was defined as $\lambda(\xi)$ in [3, Th. 5.9], which explains the difference of the sign of the coefficient of $\log|t|^2$ in Theorem 7.1 with that of [3, Th. 5.9].

Proof. Let $q\colon \widetilde{X} \to X$ be the blowing-up along Σ_π with exceptional divisor

$$E = \mathbb{P}(N_{\Sigma_\pi/X}).$$

Then $\widetilde{\nu} = \nu\circ q$ extends to a holomorphic map from $\widetilde{X}$ to $\mathbb{P}(\Omega^1_X)$.

Since the Hessian of π is a non-degenerate symmetric bilinear form on $N_{\Sigma_\pi/X}$, we have $N_{\Sigma_\pi/X} \cong N^*_{\Sigma_\pi/X}$. Under the identification $\mathbb{P}(N_{\Sigma_\pi/X}) = \mathbb{P}(N^*_{\Sigma_\pi/X})$ induced from the Hessian of π, $\widetilde{\nu}$ is identified with the natural inclusion $\mathbb{P}(N^*_{\Sigma_\pi/X}) \hookrightarrow \mathbb{P}(\Omega^1_X|_{\Sigma_\pi})$, which yields that

$$\widetilde{\nu}^*L|_E = \mathcal{O}_{\mathbb{P}(N^*_{\Sigma_\pi/X})}(-1), \qquad \widetilde{\mu}^*H|_E = \mathcal{O}_{\mathbb{P}(N_{\Sigma_\pi/X})}(1). \tag{7.1}$$

Set $F := \mathcal{O}_{\mathbb{P}(N_{\Sigma_\pi/X})}(1)$.

By the exact sequence $\mathcal{S}^\vee$, we get

$$\mathrm{Td}(U) = \frac{\mathrm{Td}((\mathit{\Pi}^\vee)^*TX)}{\mathrm{Td}(H)}. \tag{7.2}$$

Since $\mathit{\Pi}^\vee\circ\widetilde{\mu} = q$, we deduce from the exact sequence of vector bundles on Σ_π

$$0 \longrightarrow T\Sigma_\pi \longrightarrow TX|_{\Sigma_\pi} \longrightarrow N_{\Sigma_\pi/X} \longrightarrow 0$$

the identity

$$\widetilde{\mu}^*\mathrm{Td}((\mathit{\Pi}^\vee)^*TX)|_E = q^*\left\{\mathrm{Td}(T\Sigma_\pi)\,\mathrm{Td}(N_{\Sigma_\pi/X})\right\}. \tag{7.3}$$

Substituting (7.3) into (7.2), we get

(7.4)
$$\widetilde{\mu}^*\mathrm{Td}(U)|_E = \frac{q^*\left\{\mathrm{Td}(T\Sigma_\pi)\,\mathrm{Td}(N_{\Sigma_\pi/X})\right\}}{\widetilde{\mu}^*\mathrm{Td}(H)|_E} = \frac{q^*\left\{\mathrm{Td}(T\Sigma_\pi)\,\mathrm{Td}(N_{\Sigma_\pi/X})\right\}}{\mathrm{Td}(F)},$$

where we used (7.1) to get the second equality.

Let p_* be the integration along the fibers of the projection $p\colon \mathbb{P}(N_{\Sigma_\pi/X}) \to \Sigma_\pi$. Since $q|_E = p$, we deduce from (7.1), (7.4) and the projection formula that

(7.5)
$$\begin{aligned}&\int_{E\cap X_0} \widetilde{\mu}^* \left\{ \mathrm{Td}(U)\, \frac{\mathrm{Td}(H)-1}{c_1(H)} \right\} q^*\mathrm{ch}(\xi)\\ &= \int_{\Sigma_\pi \cap X_0} \mathrm{Td}(T\Sigma_\pi)\,\mathrm{Td}(N_{\Sigma_\pi/X})\,\mathrm{ch}(\xi)\, p_* \left\{ \frac{1}{\mathrm{Td}(F)} \cdot \frac{\mathrm{Td}(F)-1}{c_1(F)} \right\}\\ &= \int_{\Sigma_\pi \cap X_0} \mathrm{Td}(T\Sigma_\pi)\,\mathrm{Td}(N_{\Sigma_\pi/X})\,\mathrm{ch}(\xi)\, p_* \left\{ \frac{1-\mathrm{Td}^{-1}(F)}{c_1(F)} \right\}.\end{aligned}$$

Since $N_{\Sigma_\pi/X} \cong N^*_{\Sigma_\pi/X}$, we have

$$c_1(N_{\Sigma_\pi/X}) = 0,$$

which, together with $\mathrm{rk}(N_{\Sigma_\pi/X}) = 2$, yields that

$$0 = c_1(F)^2 - p^*c_1(N_{\Sigma_\pi/X})\, c_1(F) + p^*c_2(N_{\Sigma_\pi/X}) = c_1(F)^2 + p^*c_2(N_{\Sigma_\pi/X}).$$

Since $p_*c_1(F) = 1$, this implies that for $m \geq 0$

$$p_*c_1(F)^m = \begin{cases} (-1)^k\, c_2(N_{\Sigma_\pi/X})^k & (m = 2k+1) \\ 0 & (m = 2k). \end{cases} \tag{7.6}$$

For a formal power series $f(x) = \sum_{j=0}^{\infty} a_j\, x^j \in \mathbb{C}[[x]]$, set

$$f_-(x) := \frac{f(x) - f(-x)}{2x} \in \mathbb{C}[[x]].$$

By (7.6), we get

$$p_*f(c_1(F)) = \sum_k a_{2k+1}\, p_*c_1(F)^{2k+1} = \sum_k (-1)^k a_{2k+1}\, c_2(N_{\Sigma_\pi/X})^k.$$

Let $f_-(N_{\Sigma_\pi/X})$ be the additive genus associated with $f_-(x) \in \mathbb{C}[[x]]$. Let x_1, x_2 be the Chern roots of $N_{\Sigma_\pi/X}$. Since $c_1(N_{\Sigma_\pi/X}) = x_1 + x_2 = 0$, we get

$$\begin{aligned} f_-(N_{\Sigma_\pi/X}) &= \frac{f(x_1) - f(-x_1)}{2x_1} + \frac{f(x_2) - f(-x_2)}{2x_2}\\ &= \sum_{k=0}^{\infty} a_{2k+1}\,(x_1^{2k} + x_2^{2k})\\ &= 2\sum_{k=0}^{\infty} a_{2k+1}(-x_1x_2)^k\\ &= 2\sum_{k=0}^{\infty} (-1)^k a_{2k+1}\, c_2(N_{\Sigma_\pi/X})^k = 2\, p_*f(c_1(F)). \end{aligned}$$

Setting $f(x) = (\mathrm{Td}^{-1}(x) - 1)/x$, we get

(7.7)
$$
\begin{aligned}
\mathrm{E}(N_{\Sigma_\pi/X}) &= \mathrm{Td}(x_1)\mathrm{Td}(x_2)\left\{\frac{f(x_1)-f(-x_1)}{2x_1} + \frac{f(x_2)-f(-x_2)}{2x_2}\right\} \\
&= 2\,\mathrm{Td}(N_{\Sigma_\pi/X})\,p_* f(c_1(F)) \\
&= -2\,\mathrm{Td}(N_{\Sigma_\pi/X})\,p_*\left(\frac{1-\mathrm{Td}^{-1}(F)}{c_1(F)}\right).
\end{aligned}
$$

By comparing (7.5) and (7.7), the desired formula follows from Theorem 6.1. □

8. Isolated critical points

In this section, we assume that $\mathrm{Sing}(X_0) = \Sigma_\pi \cap X_0$ consists of isolated points. Since Σ_π is discrete, we may identify $\mathbb{P}(\Omega^1_X)$ and $\mathbb{P}(TX)$ with the trivial projective-space bundle on a neighborhood of $\Sigma_\pi \cap X_0$ by fixing a system of coordinates near $\Sigma_\pi \cap X_0$. Under this trivialization, we consider the Gauss maps ν and μ only on a small neighborhood of $\Sigma_\pi \cap X_0$. Then we have the following expression on a neighborhood of each $p \in \Sigma_\pi \cap X_0$:

$$\mu(z) = \nu(z) = \left(\frac{\partial\pi}{\partial z_0}(z) : \cdots : \frac{\partial\pi}{\partial z_n}(z)\right).$$

For a formal power series $f(x) \in \mathbb{C}[[x]]$, let $f(x)|_{x^m}$ denote the coefficient of x^m. Let $\mu(\pi, p) \in \mathbb{N}$ be the Milnor number of the isolated critical point p of π. The following result was proved by the author [16, Main Th.].

Theorem 8.1. *The following identity of functions on $\mathcal{U}^o$ holds:*

$$\log\|\sigma\|^2_{\lambda(\xi),Q} \equiv_{\mathcal{B}} \frac{(-1)^n}{(n+2)!}\,\mathrm{rk}(\xi)\left(\sum_{p\in\mathrm{Sing}(X_0)}\mu(\pi,p)\right)\log|t|^2.$$

Proof. In Theorem 6.1, we can identify U (resp. L) with the universal hyperplane bundle (resp. tautological line bundle) on $\mathbb{P}^n$. Then $H = L^{-1}$. Set $x := c_1(H)$. Hence $\int_{\mathbb{P}^n} x^n = 1$. From the exact sequence $0 \to U \to \mathbb{C}^{n+1} \to H \to 0$, we get

$$\mathrm{Td}(U) = \mathrm{Td}^{-1}(x) = \frac{1-e^{-x}}{x}.$$

By substituting this and the equation $q^*\mathrm{ch}(\xi)|_{E\cap\widetilde{X}_0} = \mathrm{rk}(\xi)$ into the formula of Theorem 6.1, we get

(8.1)
$$
\begin{aligned}
&\int_{E_0} \widetilde{\mu}^*\mathrm{Td}(U)\,\widetilde{\nu}^*\left\{\frac{\mathrm{Td}(c_1(H))-1}{c_1(H)}\right\} q^*\mathrm{ch}(\xi) \\
&= \frac{1}{\mathrm{Td}(x)}\cdot\left.\frac{\mathrm{Td}(x)-1}{x}\right|_{x^n}\cdot\mathrm{rk}(\xi)\int_{E_0}\widetilde{\mu}^*c_1(H)^n \\
&= \left.\left\{\frac{1}{x} - \frac{1-e^{-x}}{x^2}\right\}\right|_{x^n}\cdot\mathrm{rk}(\xi)\int_{E_0}\widetilde{\mu}^*c_1(H)^n \\
&= \frac{(-1)^n}{(n+2)!}\,\mathrm{rk}(\xi)\int_{E_0}\widetilde{\mu}^*c_1(H)^n.
\end{aligned}
$$

Since

$$\begin{aligned}\widetilde{\pi}_*\left\{\widetilde{\mu}^*(-c_1(L,g_L))^n\, q^*(\log\|d\pi\|^2)\right\} &= \pi_*\left\{q^*(\log\|d\pi\|^2)\,(dd^c\log\|d\pi\|^2)^n\right\}\\ &= \sum_{p\in\mathrm{Sing}(X_0)}\mu(\pi,p)\,\log|t|^2+O(1)\end{aligned}$$

by [16, Th. 4.1], we get

$$\int_{E_0}\widetilde{\mu}^*c_1(H)^n=\sum_{p\in\mathrm{Sing}(X_0)}\mu(\pi,p) \tag{8.2}$$

by Corollary 4.6. The result follows from Theorem 6.1 and (8.1), (8.2). □

9. Some results on asymptotic expansion

Let $\mathcal{A}_{\mathbb{C}}$ (resp. $\mathcal{C}_{\mathbb{C}}$) be the sheaf of germs of C^∞ (resp. C^0) functions on $\mathbb{C}$. The stalk of $\mathcal{A}_{\mathbb{C}}$ (resp. $\mathcal{C}_{\mathbb{C}}$) at the origin is denoted by $\mathcal{A}_0$ (resp. $\mathcal{C}_0$). We define

$$\mathcal{B}_0:=\mathcal{A}_0\oplus\bigoplus_{r\in\mathbb{Q}\cap(0,1]}\bigoplus_{k=0}^{n}|t|^{2r}(\log|t|)^k\cdot\mathcal{A}_0\subset\mathcal{C}_0.$$

In this section, we prove the following

Theorem 9.1. *Let $\Omega\subset\mathbb{C}^n$ be a relatively compact domain. Let $F(z)$ be a holomorphic function on Ω with critical locus $\Sigma_F:=\{z\in\Omega;\,dF(z)=0\}$. Let $\chi(z)$ be a C^∞ (n,n)-form with compact support in Ω. Define a germ $\psi\in\mathcal{C}_0$ by*

$$\psi(t):=\int_\Omega\log|F(z)-t|^2\,\chi(z).$$

If $\Sigma_F\subset F^{-1}(0)$, then $\psi(t)\in\mathcal{B}_0$.

The continuity of similar integrals was studied by Bost-Gillet-Soulé [8, Sect. 1.5] in relation with the regularity of the star products of Green currents.

For the proof of Theorem 9.1, we prove some intermediary results.

Lemma 9.2. *Let Φ be a C^∞ (n,n)-form with compact support in Ω. Let $F_*(\Phi)$ be the locally integrable $(1,1)$-form on $\mathbb{C}$ defined as the integration of Φ along the fibers of $F\colon\Omega\to\mathbb{C}$. If $\Sigma_F\subset F^{-1}(0)$, then there exists a germ $A(t)\in\mathcal{B}_0$ such that*

$$F_*(\Phi)(t)=A(t)\,\frac{dt\wedge d\bar{t}}{|t|^2},\qquad A(0)=0$$

near $0\in\mathbb{C}$.

Proof. By Hironaka, there exists a proper holomorphic modification $\varpi\colon\widetilde{\Omega}\to\Omega$ such that
(i) $\varpi\colon\widetilde{\Omega}\setminus\varpi^{-1}(\Sigma_F)\to\Omega\setminus\Sigma_F$ is an isomorphism;
(ii) $(F\circ\varpi)^{-1}(\Sigma_F)$ is a normal crossing divisor of $\widetilde{\Omega}$.

Set $\widetilde{F}:=F\circ\varpi$. For any $z\in F^{-1}(0)$, there exist a system of coordinates $(U,(w_1,\dots,w_n))$ and integers $k_1,\dots,k_l\geq 1$, $l\leq n$, such that $\widetilde{F}(w)=w_1^{k_1}\cdots w_l^{k_l}$. Define a holomorphic $(n-1)$-form on U by

$$\tau:=\frac{1}{l}\sum_{i=1}^{l}\frac{1}{k_i}(-1)^{i-1}w_i\,dw_1\wedge\cdots\wedge dw_{i-1}\wedge dw_{i+1}\wedge\cdots\wedge dw_n.$$

Let ϱ_U be a C^∞ function with compact supported in U. Since $\varpi^*\Phi$ is a C^∞ (n,n)-form on $\widetilde{\Omega}$, there exists $h(w) \in C_0^\infty(U)$ such that

$$\varrho_U \varpi^*\Phi = h(w)\, dw_1 \wedge \cdots \wedge dw_n \wedge d\bar{w}_1 \wedge \cdots \wedge d\bar{w}_n.$$

We define a germ $B(t) \in \mathcal{C}_0$ by

$$B(t) := \int_{\widetilde{F}^{-1}(t)\cap U} h(w)\, \tau \wedge \bar{\tau}.$$

Then $B(t) \in \mathcal{B}_0$ by [1, p.166, Th. 4bis]. Since

$$\widetilde{F}^* \left(\frac{dt}{t}\right) \wedge \tau = dw_1 \wedge \cdots \wedge dw_n,$$

we get by the projection formula

(9.1)

$$\begin{aligned}\widetilde{F}_*(\varrho_U\, \varpi^*\Phi)(t) &= \widetilde{F}_*(h(w)\, dw_1 \wedge \cdots \wedge dw_n \wedge d\bar{w}_1 \wedge \cdots \wedge d\bar{w}_n)(t) \\ &= \frac{dt \wedge d\bar{t}}{|t|^2}\, \widetilde{F}_*\left(h(w)\, \tau \wedge \bar{\tau}\right) = B(t)\, \frac{dt \wedge d\bar{t}}{|t|^2}.\end{aligned}$$

For an $\epsilon > 0$ small enough, set $\Delta(\epsilon) := \{t \in \mathbb{C};\ |t| < \epsilon\}$. Since

$$\left| \int_{\Delta(\epsilon)} \widetilde{F}_*(\varrho_U\, \varpi^*\Phi) \right| = \left| \int_{\widetilde{F}^{-1}(\Delta(\epsilon))} \varrho_U\, \varpi^*\Phi \right| < \infty,$$

the $(1,1)$-form $B(t)\, dt \wedge d\bar{t}/|t|^2$ is locally integrable near the origin. Hence $B(0) = 0$.

Let $\{U_\beta\}_{\beta \in B}$ be a locally finite open covering of $\widetilde{\Omega}$ and let $\{\varrho_\beta\}_{\beta \in B}$ be a partition of unity subject to $\{U_\beta\}_{\beta \in B}$. By (9.1), there exists $B_\beta(t) \in \mathcal{B}_0$ for each $\beta \in B$ such that

$$\widetilde{F}_*(\varrho_\beta\, \varpi^*(\Phi)) = B_\beta(t)\, \frac{dt \wedge d\bar{t}}{|t|^2}, \qquad B_\beta(0) = 0.$$

There exist finitely many $\beta \in B$ with $B_\beta(t) \neq 0$ by the compactness of the support of $\varpi^*\Phi$. Since

$$F_*(\Phi) = \sum_{\beta \in B} \widetilde{F}_*(\varrho_\beta\, \varpi^*\Phi) = \Big(\sum_{\beta \in B} B_\beta(t)\Big)\, \frac{dt \wedge d\bar{t}}{|t|^2},$$

we get $A(t) = \sum_{\beta \in B} B_\beta(t) \in \mathcal{B}_0$ and $A(0) = 0$. □

We regard Ω as a domain in $(\mathbb{P}^1)^n$. Hence χ is a C^∞ (n,n)-form on $(\mathbb{P}^1)^n$. Let $z = (z_1, \ldots, z_n)$ be the inhomogeneous coordinates of $(\mathbb{P}^1)^n$. For $1 \leq i \leq n$, set

$$\omega_i := \frac{\sqrt{-1}\, dz_i \wedge d\bar{z}_i}{2\pi(1 + |z_i|^2)^2}.$$

Lemma 9.3. *Assume that* $F(z) = z_1^{\nu_1} \cdots z_n^{\nu_n}$, $\nu_1, \ldots, \nu_n \geq 0$ *and set*

$$\alpha := \int_{(\mathbb{P}^1)^n} \chi(z).$$

Then there exists $\eta(t) \in \mathcal{B}_0$ *such that*

$$\psi(t) = \alpha \int_{(\mathbb{P}^1)^n} \log |z_1^{\nu_1} \cdots z_n^{\nu_n} - t|^2\, \omega_1 \wedge \cdots \wedge \omega_n + \eta(t).$$

Proof. Let $((\zeta_1 : \xi_1), \dots, (\zeta_n : \xi_n))$ be the homogeneous coordinates of $(\mathbb{P}^1)^n$ such that $z_i = \zeta_i/\xi_i$. For $t \in \mathbb{C}$, set

$$Y_t := \{((\zeta_1 : \xi_1), \dots, (\zeta_n : \xi_n)) \in (\mathbb{P}^1)^n;\ \zeta_1^{\nu_1} \cdots \zeta_n^{\nu_n} - t\, \xi_1^{\nu_1} \cdots \xi_n^{\nu_n} = 0\},$$
$$D := \{((\zeta_1 : \xi_1), \dots, (\zeta_n : \xi_n)) \in (\mathbb{P}^1)^n;\ \xi_1^{\nu_1} \cdots \xi_n^{\nu_n} = 0\}.$$

Since

$$z_1^{\nu_1} \cdots z_n^{\nu_n} - t = \frac{\zeta_1^{\nu_1} \cdots \zeta_n^{\nu_n} - t\, \xi_1^{\nu_1} \cdots \xi_n^{\nu_n}}{\xi_1^{\nu_1} \cdots \xi_n^{\nu_n}}, \tag{9.2}$$

we get the following equation of currents on $(\mathbb{P}^1)^n$ by the Poincaré-Lelong formula:

$$dd^c \log |z_1^{\nu_1} \cdots z_n^{\nu_n} - t|^2 = \delta_{Y_t} - \delta_D. \tag{9.3}$$

Since $\chi(z)$ is cohomologous to $\alpha\, \omega_1 \wedge \cdots \wedge \omega_n$, there exists a C^∞ $(n-1, n-1)$-form γ on $(\mathbb{P}^1)^n$ by the dd^c-Poincaré lemma, such that

$$\chi(z) - \alpha\, \omega_1 \wedge \cdots \wedge \omega_n = dd^c \gamma.$$

Hence we get by (9.3)

(9.4)

$$\begin{aligned}
\psi(t) &= \alpha \int_{(\mathbb{P}^1)^n} \log |z_1^{\nu_1} \cdots z_n^{\nu_n} - t|^2\, \omega_1 \wedge \cdots \wedge \omega_n + \int_{(\mathbb{P}^1)^n} \log |z_1^{\nu_1} \cdots z_n^{\nu_n} - t|^2\, dd^c \gamma \\
&= \alpha \int_{(\mathbb{P}^1)^n} \log |z_1^{\nu_1} \cdots z_n^{\nu_n} - t|^2\, \omega_1 \wedge \cdots \wedge \omega_n + \int_{(\mathbb{P}^1)^n} dd^c (\log |z_1^{\nu_1} \cdots z_n^{\nu_n} - t|^2) \wedge \gamma \\
&= \alpha \int_{(\mathbb{P}^1)^n} \log |z_1^{\nu_1} \cdots z_n^{\nu_n} - t|^2\, \omega_1 \wedge \cdots \wedge \omega_n + \int_{Y_t} \gamma - \int_D \gamma.
\end{aligned}$$

For $t \in \mathbb{C}$, set

$$\eta(t) := \int_{Y_t} \gamma - \int_D \gamma. \tag{9.5}$$

Define a divisor of $(\mathbb{P}^1)^n \times \mathbb{C}$ by

$$Y := \{((\zeta_1 : \xi_1), \dots, (\zeta_n : \xi_n), t) \in (\mathbb{P}^1)^n \times \mathbb{C};\ \zeta_1^{\nu_1} \cdots \zeta_n^{\nu_n} - t \xi_1^{\nu_1} \cdots \xi_n^{\nu_n} = 0\}.$$

Let $\mathrm{pr}_1 \colon (\mathbb{P}^1)^n \times \mathbb{C} \to (\mathbb{P}^1)^n$ and $\mathrm{pr}_2 \colon (\mathbb{P}^1)^n \times \mathbb{C} \to \mathbb{C}$ be the projections. Then $Y_t = Y \cap \mathrm{pr}_2^{-1}(t)$. Let $P \colon \widetilde{Y} \to Y$ be the resolution of the singularities of Y. Then $\mathrm{pr}_2|_Y \circ P$ is a proper holomorphic function on the complex manifold $\widetilde{Y}$. Since $P^*(\mathrm{pr}_1)^*\gamma$ is a C^∞ $(n-1, n-1)$-form on $\widetilde{Y}$, we get

$$\eta(t) = \int_{(\mathrm{pr}_2|_Y \circ P)^{-1}(t)} P^*(\mathrm{pr}_1)^* \gamma - \int_D \gamma \in \mathcal{B}_0 \tag{9.6}$$

by [1, Th. 4bis]. The result follows from (9.4), (9.5), (9.6). $\square$

Define a germ $f \in \mathcal{C}_0$ by

$$f(t) := \int_{(\mathbb{P}^1)^n} \log |z_1^{\nu_1} \cdots z_n^{\nu_n} - t|^2\, \omega_1 \wedge \cdots \wedge \omega_n.$$

Lemma 9.4. *There exists a germ $g(t) \in \mathcal{B}_0$ such that*

$$dd^c f(t) = \frac{\sqrt{-1}}{4\pi}\, g(t)\, \frac{dt \wedge d\bar{t}}{|t|^2}, \qquad g(0) = 0.$$

Proof. We keep the notation in the proof of Lemma 9.3. Since the assertion is obvious when $\nu_1 = \cdots = \nu_n = 0$, we assume that $\nu_i > 0$ for some i. Since $z_1^{\nu_1} \cdots z_n^{\nu_n} - t$ is a meromorphic function on $(\mathbb{P}^1)^n \times \mathbb{C}$, we deduce from (9.2) and the Poincaré-Lelong formula the following equation of currents on $(\mathbb{P}^1)^n \times \mathbb{C}$:

$$dd^c \log |z_1^{\nu_1} \cdots z_n^{\nu_n} - t|^2 = \delta_Y - \delta_{D\times\mathbb{C}} = \delta_Y - \delta_{(\mathrm{pr}_1)^* D}. \tag{9.7}$$

Since

$$f = (\mathrm{pr}_2)_* \left\{ \log |z_1^{\nu_1} \cdots z_n^{\nu_n} - t|^2 \, (\mathrm{pr}_1)^*(\omega_1 \wedge \cdots \wedge \omega_n) \right\},$$

we get on $\mathbb{C} \setminus \{0\}$

(9.8)

$$\begin{aligned} dd^c f &= (\mathrm{pr}_2)_* \left\{ dd^c \log |z_1^{\nu_1} \cdots z_n^{\nu_n} - t|^2 \wedge (\mathrm{pr}_1)^*(\omega_1 \wedge \cdots \wedge \omega_n) \right\} \\ &= (\mathrm{pr}_2)_* \left\{ (\delta_Y - \delta_{(\mathrm{pr}_1)^* D}) \wedge (\mathrm{pr}_1)^*(\omega_1 \wedge \cdots \wedge \omega_n) \right\} \\ &= (\mathrm{pr}_2)_* \left\{ (\mathrm{pr}_1)^*(\omega_1 \wedge \cdots \wedge \omega_n)|_Y \right\} - (\mathrm{pr}_2)_* \left\{ (\mathrm{pr}_1)^*(\omega_1 \wedge \cdots \wedge \omega_n|_D) \right\} \\ &= (\mathrm{pr}_2|_Y)_* \left\{ (\mathrm{pr}_1)^*(\omega_1 \wedge \cdots \wedge \omega_n)|_Y \right\} \\ &= (\mathrm{pr}_2|_Y \circ P)_* \left\{ P^*(\mathrm{pr}_1)^*(\omega_1 \wedge \cdots \wedge \omega_n) \right\}, \end{aligned}$$

where the first equality follows from the commutativity $dd^c(\mathrm{pr}_2)_* = (\mathrm{pr}_2)_* dd^c$, the second equality follows from (9.7), and the fourth equality follows from the trivial identity $\omega_1 \wedge \cdots \wedge \omega_n|_D = 0$. Since $P^*(\mathrm{pr}_1)^*(\omega_1 \wedge \cdots \wedge \omega_n)$ is a C^∞ (n,n)-form on $\widetilde{Y}$ and since $\mathrm{pr}_2|_Y \circ P \colon \widetilde{Y} \to \mathbb{C}$ is a proper holomorphic map, the assertion follows from (9.8) and Lemma 9.2. □

Lemma 9.5. *The germ $f(t)$ is S^1-invariant, i.e., $f(t) = f(|t|)$.*

Proof. Without loss of generality, we may assume that $\nu_n > 0$. Since

$$\int_{\mathbb{P}^1} \log |A z_n^{\nu_n} + B|^2 \, \omega_n = \log(|A|^{2/\nu_n} + |B|^{2/\nu_n})$$

when $(A, B) \neq (0, 0)$, we get by Fubini's theorem

(9.9)

$$\begin{aligned} f(t) &= \int_{(\mathbb{P}^1)^n} \log |z_1^{\nu_1} \cdots z_n^{\nu_n} - t|^2 \, \omega_1 \wedge \cdots \wedge \omega_n \\ &= \int_{(\mathbb{P}^1)^{n-1}} \left(\int_{\mathbb{P}^1} \log |z_1^{\nu_1} \cdots z_n^{\nu_n} - t|^2 \, \omega_n \right) \omega_1 \wedge \cdots \wedge \omega_{n-1} \\ &= \int_{(\mathbb{P}^1)^{n-1}} \log \left(|z_1^{\nu_1} \cdots z_{n-1}^{\nu_{n-1}}|^{2/\nu_n} + |t|^{2/\nu_n} \right) \omega_1 \wedge \cdots \wedge \omega_{n-1}. \end{aligned}$$

The assertion follows from (9.9). □

Let (r, θ) be the polar coordinates of $\mathbb{C}$. Hence $t = r\, e^{i\theta}$.

Lemma 9.6. *Let $\lambda(t) \in C^\infty(\varDelta^*)$. Assume that $\lambda(t)$ is S^1-invariant, i.e., $\lambda(t) = \lambda(r)$. If $r\, \partial_r \lambda(t) \in \mathcal{B}_0$, then $\lambda(t) \in \mathcal{B}_0$.*

Proof. By the definition of $\mathcal{B}_0$, there exist a finite set $A \subset \mathbb{Q} \cap (0, 1]$ and germs $\mu_{\alpha,k}(t) \in \mathcal{A}_0$, $\alpha \in A$, $0 \le k \le n$ such that

$$r\, \partial_r \lambda(r) = \sum_{\alpha \in A} \sum_{k=0}^{n} r^{2\alpha} (\log r)^k \, \mu_{\alpha,k}(t). \tag{9.10}$$

We may assume that $\mu_{\alpha,k}(t) \in C^\infty(\Delta(2\epsilon))$ for some $\epsilon > 0$. Since the left hand side of (9.10) is S^1-invariant, we may assume that $\mu_{\alpha,k}(t) = \mu_{\alpha,k}(r)$ for all α and k after replacing $\mu_{\alpha,k}(t)$ by $\int_0^{2\pi} \mu_{\alpha,k}(e^{i\theta}t)\, d\theta/2\pi$. By (9.10), we get

$$\lambda(\epsilon) - \lambda(r) = \sum_{\alpha\in A}\sum_{k=0}^{n}\int_r^\epsilon u^{2\alpha-1}(\log u)^k\,\mu_{\alpha,k}(u)\,du. \tag{9.11}$$

By (9.11), we see that $\lambda(t) \in \mathcal{C}_0$ by setting

$$\lambda(0) := \lambda(\epsilon) - \sum_{\alpha\in A}\sum_{k=0}^{n}\int_0^\epsilon u^{2\alpha-1}(\log u)^k\,\mu_{\alpha,k}(u)\,du.$$

Since $\lambda(t) \in \mathcal{C}_0$, we get by (9.11)

$$\begin{aligned}
\lambda(r) &= \lambda(0) + \sum_{\alpha\in A}\sum_{k=0}^{n}\int_0^r u^{2\alpha-1}(\log u)^k\mu_{\alpha,k}(u)\,du \\
&= \lambda(0) + \sum_{\alpha\in A}\sum_{k=0}^{n} r^{2\alpha}\int_0^1 v^{2\alpha-1}(\log r + \log v)^k\mu_{\alpha,k}(vr)\,dv \\
&= \lambda(0) + \sum_{\alpha\in A}\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{k}{l} r^{2\alpha}(\log r)^l\int_0^1 v^{2\alpha-1}(\log v)^{k-l}\mu_{\alpha,k}(vt)\,dv,
\end{aligned}$$

which implies that $\lambda(t) \in \mathcal{B}_0$. □

Lemma 9.7. *If $F(z) = z_1^{\nu_1}\cdots z_n^{\nu_n}$, $\nu_1, \dots, \nu_n \geq 0$, then $\psi(t) \in \mathcal{B}_0$.*

Proof. By Lemma 9.3, it suffices to prove that $f \in \mathcal{B}_0$. Since $f(t) = f(r)$ by Lemma 9.5, we deduce from Lemma 9.4 the equation

$$\frac{1}{2\pi}\partial_t\partial_{\bar t}f(t) = \frac{1}{4\pi}\{f''(r) + r^{-1}f'(r)\} = \frac{g(t)}{4\pi r^2}.$$

Hence $g(t)$ is invariant under the rotation, i.e., $g(t) = g(r)$, and the following equation holds

$$(r\,\partial_r)^2 f(r) = g(r). \tag{9.12}$$

Since $g(t) \in \mathcal{B}_0$, we deduce from Lemma 9.6 and (9.12) that $r\,\partial_r f(r) \in \mathcal{B}_0$. By Lemma 9.6 again, we get $f(t) \in \mathcal{B}_0$. □

Proof of Theorem 9.1

We keep the notation in the proof of Lemma 9.2. There exists a system of coordinate neighborhoods $\{(U_\beta, w_\beta = (w_{1,\beta}, \dots, w_{n,\beta}))\}_{\beta\in B}$ of $\widetilde{\Omega}$ and integers $k_{1,\beta}, \dots, k_{n,\beta} \geq 0$ for each $\beta \in B$ such that $\widetilde{F}|_{U_\beta}(w_\beta) = w_{1,\beta}^{k_{1,\beta}}\cdots w_{n,\beta}^{k_{n,\beta}}$. Without loss of generality, we may assume that the covering $\{U_\beta\}_{\beta\in B}$ of $\widetilde{\Omega}$ is locally finite. Let $\{\varrho_\beta\}_{\beta\in B}$ be a partition of unity subject to the covering $\{U_\beta\}_{\beta\in B}$. Then $\chi_\beta := \varrho_\beta\,\varpi^*\chi$ is a C^∞ (n,n)-form with compact support in U_β. Since $\varpi^*\chi$ has a compact support in $\widetilde{\Omega}$, $\chi_\beta = 0$ except finitely many $\beta \in B$. By Lemma 9.7,

$$\psi_\beta(t) := \int_{U_\beta}\log|w_{1,\beta}^{k_{1,\beta}}\cdots w_{n,\beta}^{k_{n,\beta}} - t|^2\,\chi_\beta(w_\beta) \in \mathcal{B}_0. \tag{9.13}$$

Since

$$\psi(t) = \int_{\widetilde{\Omega}} \varpi^* \log |F - t|^2 \, \varpi^* \chi = \sum_{\beta \in B} \int_{U_\beta} \log \left| \widetilde{F}|_{U_\beta}(w_\beta) - t \right|^2 \varrho_\beta \varpi^* \chi = \sum_{\beta \in B} \psi_\beta(t),$$

we get $\psi(t) \in \mathcal{B}_0$ by (9.13). This completes the proof of Theorem 9.1. □

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