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Confluence from Siegel-Whittaker functions to Whittaker functions on $Sp(2, R)$

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CONFLUENCE FROM SIEGEL-WHITTAKER FUNCTIONS TO WHITTAKER FUNCTIONS ON $Sp(2,\mathbf{R})$

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ABSTRACT. We discuss a confluence from Siegel-Whittaker functions to Whittaker functions on $Sp(2,\mathbf{R})$ by using their explicit formulae. In our proof, we use expansion theorems of the good Whittaker functions by the secondary Whittaker functions.

Introduction

Let π be an irreducible admissible representation of a real semisimple Lie group G. Given a pair (R,η) of a closed subgroup R of G and an irreducible unitary representation η of R, we can consider the intertwining space $\mathrm{Hom}_G(\pi,\mathrm{Ind}_R^G(\eta))$ or $\mathrm{Hom}_{(\mathfrak{g}_{\mathbf{C}},K)}(\pi,\mathrm{Ind}_R^G(\eta))$ with values in the smoothly induced G-module $\mathrm{Ind}_R^G(\eta)$. Here K is a maximal compact subgroup of G and $\mathfrak{g}_{\mathbf{C}}$ the complexification of the Lie algebra of G. If we take a non-zero element I in this space, i.e., an intertwining operator I between π and $\mathrm{Ind}_R^G(\eta)$, then its image $\mathrm{Im}(I)$ in $\mathrm{Ind}_R^G(\eta)$ is a generalized spherical model or realization of π. Let us take a closed subgroup A_R of the split component of a maximally $\mathbf{R}$-split Cartan subgroup of G. Then we can compute the holonomic system of A_R-radial part of elements of $U(\mathfrak{g}_{\mathbf{C}})$ which annihilate the τ-isotypic component $\mathrm{Im}(I)[\tau]$ of $\mathrm{Im}(I)$ for a good choice of K-type τ of π (usually with multiplicity one), when G has a double coset decomposition $G = RA_RK$ and when the space $\mathrm{Hom}_{(\mathfrak{g}_{\mathbf{C}},K)}(\pi,\mathrm{Ind}_R^G(\eta))$ is of finite dimension. For example, when $G = Sp(2,\mathbf{R})$, we gave explicit formulae of these holonomic systems and their solutions (i.e., generalized spherical functions on G) for various representations π and pairs (R,η), under our motive for the study of automorphic forms (*cf.* [12], [13], [14], [16], [17], [22], [23], [26]).

Once one had two different models of the same π with respecet to the different pairs (R_1,η_1) and (R_0,η_0), one naturally wants to know the relation between these two models. Heuristically speaking, we consider "deformation" of pairs (R_t,η_t) of spherical subgroups R_t of G and representations η_t of R_t, such that each pair (R_t,η_t) is an inner twist of (R_1,η_1) for $t \neq 0$ and gives the "contracted" pair (R_0,η_0) in the limit $t \to 0$. This is a variation of contraction or deformation of Lie groups and their representations developed, for example, by Dooley [3].

We discuss in this paper the confluence of two models of a generalized principal series representation π of $G = Sp(2,\mathbf{R})$ induced from the Jacobi maximal parabolic subgroup P_J of G associated with the long root. In our setting, (R_0,η_0) is the pair of a maximal unipotent subgroup R_0 and a non-degenerate character η_0 of R_0, and hence the associated spherical model is a (usual) Whittaker model of π (*cf.* [18], [30], [19]). And the pair (R_1,η_1) leads to a Siegel-Whittaker model of π, which is investigated by Miyazaki [21] and Ishii [16], for example. The rigorous development of the idea on contraction or deformation requires sometimes a heavy preparation compared with the results so obtained. To avoid this high cost, here we discuss only the A_R-radial part of the spherical functions involved in the family (R_t,η_t) in a direct and computational way. The original heuristics is described in section 5 as a kind of afterthought.

Our main results are in section 4, which assert confluences of the holonomic systems for the spherical functions (Theorem 4.1), the secondary spherical functions (Theorem 4.2), and the spherical functions with good growth conditions (Theorem 4.3). Here the secondary spherical functions are the power series solutions at the regular singularities of the holonomic system,

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which are originally considered by Harish-Chandra [7] in order to investigate the asymptotics of the matrix coefficients and are generalized by Heckman-Opdam [11], [28]. Moreover, it is known that these functions play a fundamental role to construct the Poincaré series as in the papers of Miatello-Wallach [20] and Oda-Tsuzuki [27]. In order to prove the confluence of the good spherical functions, we use that of the secondary functions together with the expansion theorem of the good functions by the secondary functions (Theorems 2.4 and 3.4). We expect that our results are useful for the study of automorphic forms, e.g., computing the archimedean local zeta functions (*cf.* [25]).

The contents of this paper is organized as follows. We prepare some basic notation in section 1, including the definitions of Whittaker functions, Siegel-Whittaker functions, and P_J-principal series representations. In sections 2 and 3, we discuss the explicit formulae of these spherical functions and also derive the expansion theorem of the good spherical functions. Section 4 contains our main results. In section 5, we explain our results heuristically from the points of view of deformation and contractions. Section 6 is for some miscellaneous remarks.

Convention For $a \in \mathbf{C}$ and $n \in \mathbf{Z}$, $(a)_n = \Gamma(a+n)/\Gamma(a)$ the Pochhammer symbol. For complex numbers a_i $(1 \le i \le r)$ and b_j $(1 \le j \le s)$, set

$$\Gamma[\, a_1, \dots, a_r\,] = \prod_{i=1}^{r} \Gamma(a_i), \quad \Gamma\left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \right] = \prod_{i=1}^{r} \Gamma(a_i) \Big/ \prod_{i=1}^{s} \Gamma(b_i).$$

1. Preliminaries

1.1. **Basic notions.** Let G be the real symplectic group of degree two:

$$G = Sp(2, \mathbf{R}) = \left\{ g \in SL(4, \mathbf{R}) \,\middle|\, {}^t g J_2 g = J_2 = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} \right\},$$

with 1_2 the unit matrix of degree two.

Fix a maximal compact subgroup K of G by

$$K = \left\{ k(A, B) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in G \,\middle|\, A, B \in M(2, \mathbf{R}) \right\}.$$

It is isomorphic to the unitary group $U(2)$ via the homomorphism

$$K \ni k(A, B) \longmapsto A + \sqrt{-1} B \in U(2).$$

Then the set of irreducible representations of K is parameterized by $\{(\lambda_1, \lambda_2) \in \mathbf{Z} \oplus \mathbf{Z} \mid \lambda_1 \ge \lambda_2\}$ and we denote by $\tau_{(\lambda_1, \lambda_2)} = \mathrm{Sym}^{\lambda_1 - \lambda_2} \otimes \det^{\lambda_2}$ the representation corresponding to (λ_1, λ_2).

We define two spherical subgroups R_i of G and their representationes. The first one is a maximal unipotent radical of G given by

$$R_1 = \left\{ n(n_0, n_1, n_2, n_3) = \left(\begin{array}{cc|cc} 1 & n_0 & & \\ & 1 & & \\ \hline & & 1 & \\ & & -n_0 & 1 \end{array} \right) \left(\begin{array}{cc|cc} 1 & & n_1 & n_2 \\ & 1 & n_2 & n_3 \\ \hline & & 1 & \\ & & & 1 \end{array} \right) \,\middle|\, n_i \in \mathbf{R} \right\}.$$

Any unitary character η_1 of R_1 can be written as

$$\eta_1(n(n_0, n_1, n_2, n_3)) = \exp(2\pi\sqrt{-1}(c_0 n_0 + c_3 n_3))$$

with some $c_0, c_3 \in \mathbf{R}$. In this paper we assume η_1 is non-degenerate, that is, $c_0 c_3 \ne 0$. Taking a maximal split torus A of G by

$$A = \{a(a_1, a_2) = \mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \mid a_i > 0\},$$

we have the Iwasawa decomposition $G = R_1 A K$.

The second spherical subgroup R_2 is defined as follows. Let $P_S = L_S \ltimes N_S$ be the Siegel parabolic subgroup with the Levi part L_s and the abelian unipotent radical N_S given by

$$L_S = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \mid A \in GL(2, \mathbf{R}) \right\},$$
$$N_S = \{ n(0, n_1, n_2, n_3) \mid n_1, n_2, n_3 \in \mathbf{R} \}.$$

Fix a non-degenerate unitary character ξ of N_S by

$$\xi(n(0, n_1, n_2, n_3)) = \exp(2\pi\sqrt{-1}\,\mathrm{Tr}(H_\xi T))$$

with $T = \begin{pmatrix} n_1 & n_2 \\ n_2 & n_3 \end{pmatrix}$, $H_\xi = \begin{pmatrix} h_1 & h_3/2 \\ h_3/2 & h_2 \end{pmatrix} \in M(2, \mathbf{R})$ and $\det H_\xi \neq 0$. Consider the action of L_S on N_S by conjugation and the induced action on the character group $\widehat{N_S}$. Define $SO(\xi)$ to be the identity component of the subgroup of L_S which stabilize ξ:

$$SO(\xi) := \mathrm{Stab}_{L_S}(\xi)^\circ = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \mid {}^tAH_\xi A = H_\xi \right\}.$$

Then $SO(\xi)$ is isomorphic to $SO(2)$ if $\det H_\xi > 0$ and to $SO_o(1,1)$ if $\det H_\xi < 0$. In this paper we treat the case that ξ is a 'definite' character, that is, $\det H_\xi > 0$. So we may assume $h_1, h_2 > 0$ and $h_3 = 0$ without loss of generality. We sometimes identify the element of $SO(\xi)$ with its upper left 2×2 component. Fix a unitary character χ_{m_0} ($m_0 \in \mathbf{Z}$) of $SO(\xi) \cong SO(2)$ by

$$\chi_{m_0}\left(\begin{pmatrix} \sqrt{h_1} & \\ & \sqrt{h_2} \end{pmatrix}^{-1} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \sqrt{h_1} & \\ & \sqrt{h_2} \end{pmatrix} \right) = \exp(\sqrt{-1}\, m_0 \theta).$$

We define $R_2 = SO(\xi) \ltimes N_S$ and $\eta_2 = \chi_{m_0} \boxtimes \xi$. Note that we also have the decomposition $G = R_2AK$.

1.2. **Spherical functions.** For the pair (R_i, η_i) defined as above, consider the space $C^\infty_{\eta_i}(R_i\backslash G)$ of complex valued C^∞ functions f on G satisfying

$$f(rg) = \eta_i(r) f(g) \quad \text{for all } (r, g) \in R_i \times G.$$

By the right translation, $C^\infty_{\eta_i}(R_i\backslash G)$ is a smooth G-module and we denote by the same symbol its underlying $(\mathfrak{g}_\mathbf{C}, K)$-module ($\mathfrak{g}_\mathbf{C}$ is the complexification of the Lie algebra of G). For an irreducible admissible representation (π, H_π) of G and the subspace $H_{\pi,K}$ of K-finite vectors, the intertwining space

$$\mathcal{I}_{\eta_i,\pi} = \mathrm{Hom}_{(\mathfrak{g}_\mathbf{C},K)}(H_{\pi,K}, C^\infty_{\eta_i}(R_i\backslash G))$$

between the $(\mathfrak{g}_\mathbf{C}, K)$-modules is called the space of *algebraic Whittaker functionals* for $i = 1$, or *algebraic Siegel-Whittaker functionals* for $i = 2$. For a finite-dimensional K-module (τ, V_τ), denote by $C^\infty_{\eta_i,\tau}(R_i\backslash G/K)$ the space

$$\{\phi : G \to V_\tau,\ C^\infty \mid \phi(rgk) = \eta_i(r)\tau(k^{-1})\phi(g), \quad \text{for all } (r, g, k) \in R_i \times G \times K\}.$$

Let (τ^*, V_{τ^*}) be a K-type of π and $\iota : V_{\tau^*} \to H_\pi$ be an injection. Here τ^* means the contragredient representation of τ. Then for $\Phi \in \mathcal{I}_{\eta_i,\pi}$, we can find an element ϕ_ι in

$$C^\infty_{\eta_i,\tau}(R_i\backslash G/K) = C^\infty_{\eta_i}(R_i\backslash G) \otimes V_{\tau^*} \cong \mathrm{Hom}_K(V_{\tau^*}, C^\infty_{\eta_i}(R_i\backslash G))$$

via $\Phi(\iota(v^*))(g) = \langle v^*, \phi_\iota(g)\rangle$ with $\langle\ ,\ \rangle$ the canonical paring on $V_{\tau^*} \times V_\tau$.

Since there is the decomposition $G = R_iAK$, our (generalized) spherical function ϕ_ι is determined by its restriction $\phi_\iota|_A$ to A, which we call the *radial part* of ϕ_ι. For a subspace X of $C^\infty_{\eta_i,\tau}(R_i\backslash G/K)$, we denote $X|_A = \{\phi|_A \in C^\infty(A) \mid \phi \in X\}$.

Let us define two spaces $\mathrm{Wh}(\pi, \eta_1, \tau)$ and $\mathrm{SW}(\pi, \eta_2, \tau)$ of spherical functions and their subspaces $\mathrm{Wh}(\pi, \eta_1, \tau)^{\mathrm{mod}}$ and $\mathrm{SW}(\pi, \eta_2, \tau)^{\mathrm{rap}}$ as follows:

$$\mathrm{Wh}(\pi, \eta_1, \tau) = \bigcup_{\iota \in \mathrm{Hom}_K(\tau^*, \pi)} \{\phi_\iota \mid \Phi \in \mathcal{I}_{\eta_1,\pi}\},$$

$$\mathrm{Wh}(\pi,\eta_1,\tau)^{\mathrm{mod}} = \{\phi_\iota \in \mathrm{Wh}(\pi,\eta_1,\tau) \mid \phi_\iota|_A \text{ is of moderate growth as } a_1, a_2 \to \infty\},$$

$$\mathrm{SW}(\pi,\eta_2,\tau) = \bigcup_{\iota\in \mathrm{Hom}_K(\tau^*,\pi)} \{\phi_\iota \mid \Phi \in \mathcal{I}_{\eta_2,\pi}\},$$

and

$$\mathrm{SW}(\pi,\eta_2,\tau)^{\mathrm{rap}} = \{\phi_\iota \in \mathrm{SW}(\pi,\eta_2,\tau) \mid \phi_\iota|_A \text{ decays rapidly as } a_1, a_2 \to \infty\}.$$

We call an element in $\mathrm{Wh}(\pi,\eta_1,\tau)$ (resp. $\mathrm{SW}(\pi,\eta_2,\tau)$) a *Whittaker function* (resp. *Siegel-Whittaker function*) for (π,η_i,τ).

As we shall see in the next two sections, radial parts of spherical functions satisfy certain holonomic systems of regular singular type. We call the power series solutions at the regular singularities of the systems *secondary spherical functions*, and the elements of $\mathrm{Wh}(\pi,\eta_1,\tau)^{\mathrm{mod}}$ and $\mathrm{SW}(\pi,\eta_2,\tau)^{\mathrm{rap}}$ *good spherical functions.*

1.3. **P_J-principal series representations.** In this section we recall the generalized principal series representations of G associated with the Jacobi maximal parabolic subgroup P_J of G corresponding to the long root. A Langlands decomposition $P_J = M_J A_J N_J$ is given by

$$M_J = \left\{ \left(\begin{array}{cc|cc} \epsilon & & & \\ & a & & b \\ \hline & & \epsilon & \\ & c & & d \end{array}\right) \,\middle|\, \epsilon\in\{\pm1\},\ \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL(2,\mathbf{R}) \right\},$$
$$A_J = \{a(a_1,1) = \mathrm{diag}(a_1,1,a_1^{-1},1) \in A \mid a_1 > 0\},$$
$$N_J = \{n(n_0,n_1,n_2,0) \in N = R_1 \mid n_i \in \mathbf{R}\}.$$

A discrete series representation (σ, V_σ) of the semisimple part $M_J \cong \{\pm1\}\times SL(2,\mathbf{R})$ of P_J is of the form $\sigma = \varepsilon \boxtimes D_k^{\pm}$ $(k \geq 2)$, where $\varepsilon : \{\pm1\} \to \mathbf{C}^*$ is a character and D_k^+ (resp. D_k^-) is the discrete series representation of $SL(2,\mathbf{R})$ with Blattner parameter k (resp. $-k$). For $\nu \in \mathbf{C}$, define a quasi-character $\exp(\nu)$ of A_J by $\exp(\nu)(a(a_1,1)) = a_1^\nu$. We call an induced representation

$$I(P_J;\sigma,\nu) = C^\infty\text{-}\mathrm{Ind}_{P_J}^G(\sigma\otimes \exp(\nu+1)\otimes 1_{N_J})$$

the *P_J-principal series representation* of G.

The K-types of $I(P_J;\sigma,\nu)$ is fully described in [23, Proposition 2.1] and [13, Proposition 2.3]. In particular, if $\pi = I(P_J;\varepsilon\boxtimes D_k^+,\nu)$ with $\varepsilon(\mathrm{diag}(-1,1,-1,1)) = (-1)^k$ (*even P_J-principal series*), then the corner K-type $\tau^* = \tau_{(k,k)}$ occurs in π with multiplicity one.

2. Whittaker functions

2.1. **Basic results.** Let $\pi = I(P_J;\varepsilon\boxtimes D_k^+,\nu)$ be an irreducible even P_J-principal series representation of G with $\varepsilon(\mathrm{diag}(-1,1,-1,1)) = (-1)^k$, and $\tau^* = \tau_{(k,k)}$ is the corner K-type of π. We first prepare some basic facts on the Whittaker functions for (π,η_1,τ). Throughout this section we use a coordinate $x = (x_1,x_2)$ on A defined by

$$x_1 = \left(\pi c_0 \frac{a_1}{a_2}\right)^2, \quad x_2 = 4\pi c_3 a_2^2.$$

By combining the results of Kostant ([18, §6]), Wallach ([30, Theorem 8.8]), Matumoto ([19, Theorem 6.2.1]) and Miyazaki and Oda ([23, Proposition 7.1, Theorem 8.1]), we obtain

Proposition 2.1. *Let π and τ be as above. Then we have the following:*
(i) We have $\dim \mathcal{I}_{\eta_1,\pi} = \dim \mathrm{Wh}(\pi,\eta_1,\tau) = 4$*, and a function*

$$\phi_W(a) = a_1^{k+1} a_2^{k+1} \exp(-2\pi c_3 a_2^2) h_W(a)$$

on A is in the space $\mathrm{Wh}(\pi,\eta_1,\tau)|_A$ *if and only if* $h_W(a)=h_W(x)$ *is a smooth solution of the following holonomic system of rank* 4:

$$\Big\{\partial_{x_1}\Big(-\partial_{x_1}+\partial_{x_2}+\frac{1}{2}\Big)+x_1\Big\}h_W(x)=0, \tag{2.1}$$

$$\Big\{\Big(\partial_{x_2}+\frac{k+\nu}{2}\Big)\Big(\partial_{x_2}+\frac{k-\nu}{2}\Big)-x_2\Big(-\partial_{x_1}+\partial_{x_2}+\frac{1}{2}\Big)\Big\}h_W(x)=0, \tag{2.2}$$

where $\partial_{x_i}=x_i(\partial/\partial x_i)$ *(*$i=1,2$*) is the Euler operator with respect to* x_i*.*

(ii) $\dim \mathrm{Wh}(\pi,\eta_1,\tau)^{\mathrm{mod}}\le 1$*. Moreover this inequality is an equality if and only if* $c_3>0$*.*

Remark 1. Since [23] treated the case $\sigma=\varepsilon\boxtimes D_k^-$, we need a minor change by using the explicit formulas of 'shift operators' ([22, Proposition 8.3]).

2.2. **Explicit formulas of secondary Whittaker functions.** In this section we determine the space of smooth solutions of the holonomic system in Proposition 2.1, therefore the space of the Whittaker functions $\mathrm{Wh}(\pi,\eta_1,\tau)$, explicitly. Set

$$h_W(a)=h_W(x)=\sum_{m,n\ge 0}a_{m,n}x_1^{\sigma_1+m}x_2^{\sigma_2+n}$$

with $a_{0,0}\neq 0$. Then we have the following difference equations for $\{a_{m,n}\}$:

$$(\sigma_1+m)\Big\{-(\sigma_1+m)+(\sigma_2+n)+\frac{1}{2}\Big\}a_{m,n}+a_{m-1,n}=0, \tag{2.3}$$

$$\Big(\sigma_2+n+\frac{k+\nu}{2}\Big)\Big(\sigma_2+n+\frac{k-\nu}{2}\Big)a_{m,n}+\Big\{(\sigma_1+m)-(\sigma_2+n)+\frac{1}{2}\Big\}a_{m,n-1}=0. \tag{2.4}$$

Here we promise $a_{m,n}=0$ if $m<0$ or $n<0$. By putting $m=n=0$ in (2.3) and (2.4), we can find the characteristic indices

$$(\sigma_1,\sigma_2)=\Big(0,\ \frac{-k\pm\nu}{2}\Big),\Big(\frac{-k\pm\nu+1}{2},\ \frac{-k\pm\nu}{2}\Big).$$

If ν is not an integer, we can determine the coefficients $a_{m,n}$ inductively for each case and thus obtain

Proposition 2.2. ($cf.$ [29, Proposition 2.1]) *For* $\nu\notin\boldsymbol{Z}$*, define the functions* $h_W^i(\nu;a)=h_W^i(\nu;x)$ *on* A *by*

$$h_W^1(\nu;x)=\sum_{m,n\ge 0}c^1_{m,n}x_1^m x_2^{n+(-k+\nu)/2},$$
$$h_W^2(\nu;x)=\sum_{m,n\ge 0}c^2_{m,n}x_1^{m+n+(-k+\nu+1)/2}x_2^{n+(-k+\nu)/2},$$

with

$$c^1_{m,n}=\Gamma\begin{bmatrix}-n-\nu, & -m+n+\frac{-k+\nu+1}{2}\\ -\nu, & \frac{-k+\nu+1}{2}\end{bmatrix}\frac{(-1)^{m+n}}{m!\,n!},$$
$$c^2_{m,n}=\Gamma\begin{bmatrix}-n-\nu, & -m-n+\frac{k-\nu-1}{2}\\ -\nu, & \frac{k-\nu-1}{2}\end{bmatrix}\frac{(-1)^{m+n}}{m!\,n!}.$$

Then the power series $h_W^i(\nu;x)$ *converges for any* $x\in\mathbf{C}^2$ *and the set* $\{h_W^i(\varepsilon\nu;x)\mid i=1,2,\ \varepsilon\in\{\pm1\}\}$ *forms a basis of the space of solutions of the system in Proposition 2.1.*

2.3. **Explicit formulas of good Whittaker functions.** When $c_3<0$, Proposition 2.1 tells us that there is no non-zero moderate growth Whittaker function. Therefore let us assume $c_3>0$ in the following discussion. The integral expression for the Whittaker functions of moderate growth was obtained by Miyazaki and Oda.

Proposition 2.3. ([23, Theorem 8.1]) *Let π and τ be as before. Define*

$$g_W(a) = g_W(x) := x_2^{-1/2} \int_0^\infty t^{-k+1/2} W_{0,\nu}(t) \exp\Big(-\frac{t^2}{16x_2} - \frac{16x_1x_2}{t^2}\Big)\frac{dt}{t},$$

with $W_{\kappa,\mu}$ the classical Whittaker function. Then the function

$$\phi_W(a) = a_1^{k+1} a_2^{k+1} \exp(-2\pi c_3 a_2^2) g_W(a)$$

gives a non-zero element in $\mathrm{Wh}(\pi, \eta_1, \tau)^{\mathrm{mod}}|_A$ which is unique up to constant multiple.

2.4. **Expansion theorem for Whittaker functions.** Now we express the moderate growth Whittaker function g_W as a linear combination of h_W^i.

Theorem 2.4. *For $\nu \notin \boldsymbol{Z}$, let $h_W^i(\nu; a)$ and $g_W(a)$ be the function defined in Proposition 2.2 and 2.3, respectively. Then*

$$g_W(a) = c_W \sum_{\varepsilon \in \{\pm 1\}} \Big(\Gamma\Big[-\varepsilon\nu,\ \frac{-k+\varepsilon\nu+1}{2}\Big] h_W^1(\varepsilon\nu; a) + \Gamma\Big[-\varepsilon\nu,\ \frac{k-\varepsilon\nu-1}{2}\Big] h_W^2(\varepsilon\nu; a)\Big)$$

with $c_W = 2^{1-2k}\pi^{-1/2}$.

Proof. Since

$$\begin{aligned} W_{0,\nu}(t) &= \Big(\frac{t}{\pi}\Big)^{1/2} K_\nu\Big(\frac{t}{2}\Big) \\ &= -\frac{(\pi t)^{1/2}}{2\sin \pi\nu}\Big(I_\nu\Big(\frac{t}{2}\Big) - I_{-\nu}\Big(\frac{t}{2}\Big)\Big) \\ &= \frac{1}{2}\Big(\frac{t}{\pi}\Big)^{1/2}\Big\{\sum_{n\ge 0}\frac{(-1)^n\Gamma(-\nu-n)}{n!}\Big(\frac{t}{4}\Big)^{2n+\nu} + \sum_{n\ge 0}\frac{(-1)^n\Gamma(\nu-n)}{n!}\Big(\frac{t}{4}\Big)^{2n-\nu}\Big\} \end{aligned} \tag{2.5}$$

(K_ν and I_ν are modified Bessel functions, see [5, 7.2.2]), we have

$$g_W(x) = g_W(\nu; x) + g_W(-\nu; x)$$

with

$$g_W(\nu; x) = \frac{x_2^{-1/2}}{2\pi^{1/2}} \int_0^\infty \sum_{n \ge 0} \frac{(-1)^n \Gamma(-\nu-n)}{n!}\Big(\frac{t}{4}\Big)^{2n+\nu} t^{-k+1}\exp\Big(-\frac{t^2}{16x_2} - \frac{16x_1x_2}{t^2}\Big)\frac{dt}{t}.$$

If we substitute $t = 4\sqrt{\frac{x_1x_2}{r}}$ and change the order of integration and infinite sum then

$$\begin{aligned} g_W(\nu; x) = \frac{x_2^{-1/2}}{2^{2k-1}\pi^{1/2}} &\sum_{n\ge 0} \frac{(-1)^n\Gamma(-\nu-n)}{n!}(x_1x_2)^{n+(-k+\nu+1)/2} \\ &\cdot \int_0^\infty r^{-n+(k-\nu-1)/2}\exp\Big(-\frac{x_1}{r} - r\Big)\frac{dr}{2r}. \end{aligned} \tag{2.6}$$

Here, the estimate

$$\begin{aligned} \int_0^\infty \Big| r^{-n+(k-\nu-1)/2}\exp\Big(-\frac{x_1}{r} - r\Big)\Big| \frac{dr}{2r} &\le \int_0^\infty r^{n-(k-\mathrm{Re}(\nu)-1)/2}\exp(-rx_1)\frac{dr}{2r} \\ &= x_1^{-n+(k-\mathrm{Re}(\nu)-1)/2}\,\Gamma\Big(n - \frac{k-\mathrm{Re}(\nu)-1}{2}\Big), \end{aligned}$$

which is valid except for the finite number of n (satisfying $(k - \mathrm{Re}(\nu) - 1)/2 > n$), ensures the change of order. By using the formula

$$\int_0^\infty t^{-\mu}\exp\Big(-t - \frac{z^2}{4t}\Big)\frac{dt}{t} = 2^{1+\mu}z^{-\mu}K_\mu(z),$$

([5, 7.12 (23)]) and the relation (2.5) again, the integral in (2.6) becomes

$$x_1^{-n/2+(k-\nu-1)/4} K_{n+(-k+\nu+1)/2}(2\sqrt{x_1})$$

$$= \sum_{m\ge 0} \frac{(-1)^m \Gamma(n + \frac{-k+\nu+1}{2} - m)}{m!} x_1^{m-n+(k-\nu-1)/2} + \sum_{m\ge 0} \frac{(-1)^m \Gamma(-n + \frac{k-\nu-1}{2} - m)}{m!} x_1^m.$$

Therefore we arrive at

$$g_W(\nu;x) = c_W\Big(\Gamma\Big[-\nu,\ \frac{-k+\nu+1}{2}\Big] h_W^1(\nu;x) + \Gamma\Big[-\nu,\ \frac{k-\nu-1}{2}\Big] h_W^2(\nu;x)\Big)$$

and complete the proof. □

3. Siegel-Whittaker functions

3.1. **Basic results.** Miyazaki ([21]) studied the Siegel-Whittaker functions for P_J-principal series and obtained the multiplicity one property and the explicit integral representation for rapidly decreasing function. As in the previous section, we introduce the coordinate $y = (y_1, y_2)$ on A by

$$y_1 = \frac{h_1 a_1^2}{h_2 a_2^2}, \quad y_2 = 4\pi h_2 a_2^2.$$

We remark on a compatibility condition. For a non-zero element ϕ of $C^\infty_{\eta_i, \tau_{(-k,-k)}}(R_i \backslash G/K)$, we have

$$\phi(a) = \phi(mam^{-1}) = (\chi_{m_0} \boxtimes \xi)(m)\tau_{(-k,-k)}(m)\phi(a),$$

where $a \in A$ and $m \in SO(\xi) \cap Z_K(A) = \{\pm 1_4\}$. If we take $m = -1_4$, $(\chi_{m_0} \boxtimes \xi)(m) = \chi_{m_0}(m) = \exp(\pi\sqrt{-1}m_0)$ and $\tau_{(-k,-k)}(m) = 1$ imply that m_0 is an even integer.

Proposition 3.1. ([21, Proposition 7.2]) *Let π and τ be as in §2.1. Then we have the following:*
(i) We have $\dim \mathcal{I}_{\eta_2,\pi} = \dim \mathrm{SW}(\pi, \eta_2, \tau) \le 4$ *and a function*

$$\phi_{SW}(a) = a_1^{k+1} a_2^{k+1} \exp(-2\pi(h_1 a_1^2 + h_2 a_2^2)) h_{SW}(a)$$

is in the space $\mathrm{SW}(\pi, \eta_2, \tau)|_A$ *if and only if* $h_{SW}(a) = h_{SW}(y)$ *is a smooth solution of following system:*

$$\text{(3.1)}\quad \Big\{\partial_{y_1}\Big(-\partial_{y_1} + \partial_{y_2} + \frac{1}{2}\Big) + \frac{y_1}{y_1 - 1}\Big(-\partial_{y_1} + \frac{1}{2}\partial_{y_2}\Big) + \frac{m_0^2}{4}\frac{y_1}{(y_1-1)^2}\Big\} h_{SW}(y) = 0,$$

$$\text{(3.2)}\quad \Big\{\Big(\partial_{y_2} + \frac{k+\nu}{2}\Big)\Big(\partial_{y_2} + \frac{k-\nu}{2}\Big) - y_1 y_2\Big(\partial_{y_1} + \frac{1}{2}\Big) - y_2\Big(-\partial_{y_1} + \partial_{y_2} + \frac{1}{2}\Big)\Big\} h_{SW}(y) = 0,$$

with $\partial_{y_i} = y_i(\partial/\partial y_i)$.
(ii) $\dim \mathrm{SW}(\pi, \eta_2, \tau)^{\mathrm{rap}} \le 1$.

Remark 2. The above system has singularities along the three divisors $y_1 = 0$, $y_1 = 1$ and $y_2 = 0$, and they are regular singularities.

3.2. **Explicit formulas of secondary Siegel-Whittaker functions.** We consider the power series solution of the system in Proposition 3.1 around $(y_1, y_2) = (0,0)$. In the notation in [16], this is the solution at Q_∞.

Proposition 3.2. *For $\nu \notin \boldsymbol{Z}$, set $h^i_{SW}(\nu; a) = h^i_{SW}(\nu; y)$ by*

$$h^1_{SW}(\nu;y) = (1-y_1)^{|m_0|/2} \sum_{m,n\ge 0} c^1_{m,n}\, \Gamma\begin{bmatrix} m - n + \frac{k+|m_0|-\nu}{2}, & m + \frac{|m_0|+1}{2} \\ -n + \frac{k+|m_0|-\nu}{2}, & \frac{|m_0|+1}{2} \end{bmatrix} y_1^m y_2^{n+(-k+\nu)/2},$$

$$h^2_{SW}(\nu;y) = (1-y_1)^{|m_0|/2} \sum_{m,n\ge 0} c^2_{m,n}\, \Gamma\begin{bmatrix} m + n + \frac{-k+|m_0|+\nu}{2} + 1, & m + \frac{|m_0|+1}{2} & \frac{k+|m_0|-\nu}{2} \\ \frac{-k+|m_0|+\nu}{2} + 1, & \frac{|m_0|+1}{2} & n + \frac{k+|m_0|-\nu}{2} \end{bmatrix}$$
$$\cdot\, y_1^{m+n+(-k+\nu+1)/2} y_2^{n+(-k+\nu)/2}.$$

Here $c^1_{m,n}$ and $c^2_{m,n}$ are the coefficients defined in Proposition 2.2. Then the power series $h^i_{SW}(\nu;y)$ converges $|y_1|<1$ and $y_2\in\mathbf{C}$ and the set $\{h^i_{SW}(\varepsilon\nu;y)\mid i=1,2,\,\varepsilon\{\pm1\}\}$ forms a basis of the space of solutions of the system in Proposition 3.1.

Proof. If we put $h_{SW}(y)=(1-y_1)^{|m_0|/2}y_2^{|m_0|/2}\tilde{h}_{SW}(y)$, (3.1) and (3.2) are transformed into

$$\Big[\partial_{y_1}\Big(\partial_{y_1}-\partial_{y_2}-\frac{|m_0|+1}{2}\Big)-y_1(\partial_{y_1}-\partial_{y_2})\Big(\partial_{y_1}+\frac{|m_0|+1}{2}\Big)\Big]\tilde{h}_{SW}(y)=0, \tag{3.3}$$

and

$$\begin{aligned}&\Big[\Big(\partial_{y_2}+\frac{k+|m_0|+\nu}{2}\Big)\Big(\partial_{y_2}+\frac{k+|m_0|-\nu}{2}\Big)\\&+y_2\Big(\partial_{y_1}-\partial_{y_2}-\frac{|m_0|+1}{2}\Big)-y_1y_2\Big(\partial_{y_1}+\frac{|m_0|+1}{2}\Big)\Big]\tilde{h}_{SW}(y)=0.\end{aligned} \tag{3.4}$$

Set $\tilde{h}_{SW}(y)=\sum_{m,n\geq 0}b_{m,n}y_1^{\tau_1+m}y_2^{\tau_2+n}$ with $b_{0,0}\neq 0$. Then we can find the recurrence relations

$$\begin{aligned}&(\tau_1+m)\Big\{-(\tau_1+m)+(\tau_2+n)+\frac{|m_0|+1}{2}\Big\}b_{m,n}\\&+\{(\tau_1+m-1)-(\tau_2+n)\}\Big\{(\tau_1+m-1)+\frac{|m_0|+1}{2}\Big\}b_{m-1,n}=0,\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}&\Big\{(\tau_2+n)+\frac{k+|m_0|+\nu}{2}\Big\}\Big\{(\tau_2+n)+\frac{k+|m_0|-\nu}{2}\Big\}b_{m,n}\\&+\Big\{(\tau_1+m)-(\tau_2+n-1)-\frac{|m_0|+1}{2}\Big\}b_{m,n-1}\\&-\Big\{(\tau_1+m-1)+\frac{|m_0|+1}{2}\Big\}b_{m-1,n-1}=0\end{aligned} \tag{3.6}$$

from (3.3) and (3.4) respectively. Here we promise $b_{m,n}=0$ for $m<0$ or $n<0$. Let $m=n=0$ in (3.5) and (3.6). Then we get

$$(\tau_1,\tau_2)=\Big(0,\,\frac{-k-|m_0|}{2}\pm\frac{\nu}{2}\Big),\ \Big(\frac{-k+1}{2}\pm\frac{\nu}{2},\,\frac{-k-|m_0|}{2}\pm\frac{\nu}{2}\Big).$$

For each case, we can solve the recurrence relations (3.5) and (3.6). □

3.3. **Explicit formulas of good Siegel-Whittaker functions.** The integral representation of the unique element in $\mathrm{SW}(\pi,\eta_2,\tau)^{\mathrm{rap}}|_A$ is given by Miyazaki ([21, Theorem 7.5]). For our purpose, however, we need another integral expression for this function. Inspired by the work of Debiard and Gaveau ([1],[2]), we obtain the following Euler type integral. See also Iida ([15]) and Gon ([6]).

Proposition 3.3. *Define*

$$\begin{aligned}g_{SW}(a)=g_{SW}(y)&:=(1-y_1)^{|m_0|/2}y_2^{|m_0|/2}\\&\cdot\int_0^1 t^{(|m_0|-1)/2}(1-t)^{(|m_0|-1)/2}F\Big(\frac{y_2}{2}\{1-t(1-y_1)\}\Big)dt,\end{aligned}$$

with

$$F(z)=e^z(2z)^{(-k-|m_0|-1)/2}W_{(k-|m_0|-1)/2,\nu/2}(2z).$$

Then the function

$$\phi_{SW}(a)=a_1^{k+1}a_2^{k+1}\exp(-2\pi(h_1a_1^2+h_2a_2^2))g_{SW}(a)$$

gives a non-zero element in $\mathrm{SW}(\pi,\eta_2,\tau)^{\mathrm{rap}}|_A$ which is unique up to constant multiple.

Proof. See [6, 8.4]. □

3.4. **Expansion theorem for Siegel-Whittaker functions.**

Theorem 3.4. *For $\nu \notin \boldsymbol{Z}$, let $h^i_{SW}(\nu;a)$ and $g_{SW}(a)$ be the function defined in Proposition 3.2 and 3.3, respectively. Then*

$$g_{SW}(a) = c_{SW} \sum_{\varepsilon\in\{\pm1\}} \left(\Gamma\left[-\varepsilon\nu,\ \frac{-k+\varepsilon\nu+1}{2} \right] h^1_{SW}(\varepsilon\nu;a) \right.$$
$$\left. + \Gamma\left[\begin{matrix} -\varepsilon\nu,\ \frac{k-\varepsilon\nu-1}{2},\ \frac{-k+|m_0|+\varepsilon\nu}{2}+1 \\ \frac{k+|m_0|-\varepsilon\nu}{2} \end{matrix} \right] h^2_{SW}(\varepsilon\nu;a) \right)$$

with

$$c_{SW} = \Gamma\left[\begin{matrix} \frac{|m_0|+1}{2} \\ \frac{-k+|m_0|-\nu}{2}+1,\ \frac{-k+|m_0|+\nu}{2}+1 \end{matrix} \right].$$

Proof. In the same way as the case of Whittaker functions, we start from the relation

$$W_{\kappa,\mu}(z) = \Gamma\left[\begin{matrix} -2\mu \\ \frac{1}{2}-\mu-k \end{matrix} \right] M_{\kappa,\mu}(z) + \Gamma\left[\begin{matrix} 2\mu \\ \frac{1}{2}+\mu-k \end{matrix} \right] M_{\kappa,-\mu}(z)$$

with

$$M_{\kappa,\mu}(z) = z^{\mu+1/2}e^{-z/2}{}_1F_1\left(\begin{matrix} \frac{1}{2}+\mu-k \\ 2\mu+1 \end{matrix} \,\middle|\, z \right) \qquad (2\mu \notin \mathbf{Z})$$

is the classical Whittaker function. Then we obtain

$$g_{SW}(y) = g_{SW}(\nu;y) + g_{SW}(-\nu;y)$$

with

$$\begin{aligned} g_{SW}(\nu;y) = {} & \Gamma\left[\begin{matrix} -\nu, & \nu+1 \\ \frac{-k+|m_0|-\nu}{2}+1, & \frac{-k+|m_0|+\nu}{2}+1 \end{matrix} \right] (1-y_1)^{|m_0|/2} y_2^{|m_0|/2} \\ & \cdot \sum_{n\ge0} \frac{1}{n!} \Gamma\left[\begin{matrix} n+\frac{-k+|m_0|+\nu}{2}+1 \\ n+\nu+1 \end{matrix} \right] \\ & \cdot \int_0^1 t^{(|m_0|-1)/2}(1-t)^{(|m_0|-1)/2}\{y_2(1-t(1-y_1))\}^{n+(-k-|m_0|+\nu)/2}dt. \end{aligned} \tag{3.7}$$

Here the change of the order of integration and infinite sum is justified because the integral in (3.7) is bounded by

$$y_2^{n+(-k-|m_0|+\mathrm{Re}(\nu))/2} \int_0^1 t^{(|m_0|-1)/2}(1-t)^{(|m_0|-1)/2}dt.$$

except for the finite number of n. The last integral in (3.7) is Euler integral representation for the hypergeometric function ${}_2F_1$. Then we have

$$\begin{aligned} g_{SW}(\nu;y) = {} & c_{SW}\,\Gamma\left[\begin{matrix} \frac{|m_0|+1}{2} \\ |m_0|+1 \end{matrix} \right] (1-y_1)^{|m_0|/2} \\ & \cdot \sum_{n\ge0} \frac{(-1)^n}{n!} \Gamma\left[-n-\nu,\ n+\frac{-k+|m_0|+\nu}{2}+1 \right] \\ & \cdot {}_2F_1\left(\begin{matrix} -n+\frac{k+|m_0|-\nu}{2},\ \frac{|m_0|+1}{2} \\ |m_0|+1 \end{matrix} \,\middle|\, 1-y_1 \right) y_2^{n+(-k+\nu)/2}. \end{aligned}$$

Here we used

$$\Gamma\left[\begin{matrix} -\nu,\ \nu+1 \\ n+\nu+1 \end{matrix} \right] = (-1)^n\Gamma(-n-\nu).$$

Finally, we apply the formula

$$
{}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\,\middle|\, 1-z\right) = \Gamma\left[\begin{matrix} c,\ c-a-b \\ c-a,\ c-b \end{matrix}\right] {}_2F_1\left(\begin{matrix} a,\ b \\ a+b-c+1 \end{matrix}\,\middle|\, z\right)
$$
$$
+\Gamma\left[\begin{matrix} c,\ a+b-c \\ a,\ b \end{matrix}\right] z^{c-a-b}\,{}_2F_1\left(\begin{matrix} c-a,\ c-b \\ c-a-b+1 \end{matrix}\,\middle|\, z\right),
$$

([4, 2.10.(1)]) to obtain

$$
\begin{aligned}
g_{SW}(\nu;y) &= c_{SW}(1-y_1)^{|m_0|/2} \\
&\cdot \sum_{n\geq 0} \frac{(-1)^n}{n!}\Gamma\Big[-n-\nu,\ n+\frac{-k+|m_0|+\nu}{2}+1\Big] y_2^{n+(-k+\nu)/2} \\
&\cdot \left\{ \Gamma\left[\begin{matrix} n+\frac{-k+\nu+1}{2} \\ n+\frac{-k+|m_0|+\nu}{2}+1 \end{matrix}\right] {}_2F_1\left(\begin{matrix} -n+\frac{k+|m_0|-\nu}{2},\ \frac{|m_0|+1}{2} \\ -n+\frac{k-\nu+1}{2} \end{matrix}\,\middle|\, y_1\right) \right. \\
&\left. + \Gamma\left[\begin{matrix} -n+\frac{k-\nu-1}{2} \\ n+\frac{k+|m_0|-\nu}{2} \end{matrix}\right] {}_2F_1\left(\begin{matrix} n+\frac{-k+|m_0|+\nu}{2}+1,\ \frac{|m_0|+1}{2} \\ n+\frac{-k+\nu+3}{2} \end{matrix}\,\middle|\, y_1\right) y_1^{n+(-k+\nu+1)/2} \right\}.
\end{aligned}
$$

By expressing ${}_2F_1(y_1)$ as a power series, we complete the proof. □

4. Confluences

In this section we state our main results.

4.1. **Confluence of the differential equations.**

Theorem 4.1. *If we substitute*

$$
h_1 = t^2 c_3, \quad h_2 = c_3, \quad m_0 = \frac{2\pi c_0}{t} \tag{4.1}
$$

in the system in Proposition 3.1 and take the limit $t \to 0$, then we obtain the system in Proposition 2.1.

Proof. After the substitution,

$$
\partial_{y_i} = \partial_{x_i}
$$

is immediate and $\lim_{t\to 0}$ leads

$$
y_1 \to 0
$$

and

$$
\frac{m_0^2 y_1}{4(y_1-1)^2} = \frac{1}{4}\Big(\frac{2\pi c_0}{t}\Big)^2 \frac{(ta_1/a_2)^2}{\{(ta_1/a_2)^2-1\}^2} \to \Big(\pi c_0 \frac{a_1}{a_2}\Big)^2 = x_1.
$$

□

4.2. **Confluence of the secondary spherical functions.**

Theorem 4.2. *For $\nu \notin \boldsymbol{Z}$, define the functions $h^i_{SW}(\nu,t;a)$ $(i=1,2)$ by substituting (4.1) in $h^i_{SW}(\nu;a)$. Then*

$$
\lim_{t\to 0} h^1_{SW}(\nu,t;a) = h^1_W(\nu;a),
$$
$$
\lim_{t\to 0} t^{k-\nu-1} h^2_{SW}(\nu,t;a) = h^2_W(\nu;a).
$$

Proof. By definition,

$$
h^1_{SW}(\nu,t;a) = \Big(1 - t^2\frac{a_1^2}{a_2^2}\Big)^{\pi|c_0|/t} \sum_{m,n\geq 0} d^1_{m,n}(t)\Big(\frac{a_1}{a_2}\Big)^{2m}(4\pi c_3 a_2^2)^{n+(-k+\nu)/2}
$$

with

$$d^1_{m,n}(t) = c^1_{m,n}\,\Gamma\begin{bmatrix} m-n+\frac{k-\nu}{2}+\frac{\pi|c_0|}{t}, & m+\frac{1}{2}+\frac{\pi|c_0|}{t} \\ -n+\frac{k-\nu}{2}+\frac{\pi|c_0|}{t}, & \frac{1}{2}+\frac{\pi|c_0|}{t} \end{bmatrix}\cdot t^{2m}.$$

Since

$$t^m\,\Gamma\begin{bmatrix} m-n+\frac{k-\nu}{2}+\frac{\pi|c_0|}{t} \\ -n+\frac{k-\nu}{2}+\frac{\pi|c_0|}{t} \end{bmatrix} \to (\pi c_0)^m, \quad t^m\,\Gamma\begin{bmatrix} m+\frac{1}{2}+\frac{\pi|c_0|}{t} \\ \frac{1}{2}+\frac{\pi|c_0|}{t} \end{bmatrix} \to (\pi c_0)^m$$

and

$$\left(1-t^2\frac{a_1^2}{a_2^2}\right)^{\pi|c_0|/t} \to 1$$

as $t\to 0$, our claim follows. The proof for $h^2_{SW}(\nu,t;a)$ can be done in the same way. □

4.3. **Confluence of the spherical functions.**

Theorem 4.3. *Define the function $g_{SW}(t;a)$ by substituting (4.1) in $g_{SW}(a)$. Then*

$$\lim_{t\to 0}\frac{g_{SW}(t;a)}{c_{SW}} = \frac{g_W(a)}{c_W}.$$

Proof. Let $\nu\notin\mathbf{Z}$. Combined with the expansion theorems (Theorem 2.4 and Theorem 3.7) and the confluence of power series (Theorem 4.2), our task is reduced to show

$$\lim_{t\to 0} t^{-k+\varepsilon\nu+1}\,\Gamma\begin{bmatrix} \frac{-k+\varepsilon\nu}{2}+\frac{\pi|c_0|}{t}+1 \\ \frac{k-\varepsilon\nu}{2}+\frac{\pi|c_0|}{t} \end{bmatrix} = (\pi|c_0|)^{-k+\varepsilon\nu+1}.$$

Since

$$t^{-k+\varepsilon\nu+1}\,\Gamma\begin{bmatrix} \frac{-k+\varepsilon\nu}{2}+\frac{\pi|c_0|}{t}+1 \\ \frac{k-\varepsilon\nu}{2}+\frac{\pi|c_0|}{t} \end{bmatrix} = t^{\varepsilon\nu}\,\Gamma\begin{bmatrix} \frac{k+\varepsilon\nu}{2}+\frac{\pi|c_0|}{t}+1 \\ \frac{k-\varepsilon\nu}{2}+\frac{\pi|c_0|}{t} \end{bmatrix}\cdot\frac{t^{-k+1}}{(\frac{-k+\varepsilon\nu}{2}+\frac{\pi|c_0|}{t})_{k-1}},$$

the asymptotic formula $\Gamma(z+\alpha)/\Gamma(z+\beta) = z^{\alpha-\beta}(1+O(z^{-1}))$ for large $|z|$ ([4, 1.18 (4)]) implies the result. By an analytic continuation, we can extend the assertion for all $\nu\in\mathbf{C}$. □

5. Deformation from (R_2,η_2) to (R_1,η_1) and the confluence

In this section we explain the main results in the previous section from the points of view of deformations and contractions of Lie groups (cf. [3]). This is to supply a heuristic background for the computations in the previous sections.

5.1. **From $SO(2)$ to N_0.** We first consider the deformation of two subgroups of $SL(2,\mathbf{R})$:

$$SO(2) = \left\{\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\,\middle|\,\theta\in\mathbf{R}\right\},$$

$$N_0 = \left\{\begin{pmatrix}1 & c \\ 0 & 1\end{pmatrix}\,\middle|\,c\in\mathbf{R}\right\}.$$

Under the usual action of $SL(2,\mathbf{R})$ to the upper half plane $\mathfrak{h} = \{z\in\mathbf{C}\mid \mathrm{Im}(z)>0\}$, $SO(2)$ is the stabilizer subgroup of $\sqrt{-1}$ and N_0 fixes $\sqrt{-1}\infty$. Set $z_t = \sqrt{-1}/t$ for $t>0$. Then $\lim_{t\to\infty} z_t = \sqrt{-1}\infty$ and the stabilizer subgroup $\mathrm{Stab}_{SL(2,\mathbf{R})}(z_t)$ of z_t in $SL(2,\mathbf{R})$ is

$$\begin{aligned}\mathrm{Stab}_{SL(2,\mathbf{R})}(z_t) &= \left\{r_\theta(t) = \begin{pmatrix}\cos\theta & \sin\theta/t \\ -t\sin\theta & \cos\theta\end{pmatrix}\,\middle|\,\theta\in\mathbf{R}\right\} \\ &= \begin{pmatrix}t^{-1/2} & 0 \\ 0 & t^{1/2}\end{pmatrix} SO(2) \begin{pmatrix}t^{1/2} & 0 \\ 0 & t^{-1/2}\end{pmatrix}.\end{aligned}$$

For our purpose, we have to move $\theta = \theta(t)$ such as $\sin\theta(t)/t \to c$ as $t \to 0$. Let $\theta(t) = ct$. Then

$$r_{\theta(t)}(t) = \begin{pmatrix} \cos\theta(t) & \sin\theta(t)/t \\ -t\sin\theta(t) & \cos\theta(t) \end{pmatrix} \to \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \tag{5.1}$$

as desired.

5.2. **From R_2 to R_1.** Let ξ_t $(t \neq 0)$ be the definite character of N_S associated with $H_{\xi_t} = \begin{pmatrix} ht^2 & 0 \\ 0 & h \end{pmatrix}$ $(h > 0)$:

$$\xi_t(n(0, n_1, n_2, n_3)) = \exp\bigl(2\pi\sqrt{-1}(ht^2 n_1 + hn_3)\bigr).$$

Then the stabilizer subgroup $SO(\xi_t)$ is identified with $\{r_\theta(t) \mid t > 0\}$. Therefore if we take $\theta = \theta(t) = tn_0$ as in the previous subsection, (5.1) implies

$$\lim_{t\to 0} SO(\xi_t) = \{n(n_0, 0, 0, 0) \mid n_0 \in \mathbf{R}\} \tag{5.2}$$

in L_S makes up $R_1 = N$ together with N_S, as we expected.

5.3. **From η_2 to η_1.** Define the character $\chi_{m_0(t)}$ of $SO(\xi_t)$ by

$$\chi_{m_0(t)}(r_\theta(t)) = \exp\sqrt{-1}(m_0(t)\theta).$$

and put $\eta_{2,t} = \chi_{m_0(t)} \boxtimes \xi_t$.

$$\eta_{2,t}(r_{\theta(t)}(t) \cdot n(0, n_1, n_2, n_3)) = \exp 2\pi\sqrt{-1}(ht^2 n_1 + hn_3) \cdot \exp\sqrt{-1}(m_0(t)\theta(t)).$$

Since $\theta(t) = n_0 t$, we should take

$$m_0(t) = \frac{2\pi c_0}{t}.$$

Then the right hand side goes to $\exp 2\pi\sqrt{-1}(c_3 n_3 + c_0 n_0)$ (after the replacement $h = c_3$), and thus combined with (5.2), we obtain

$$\lim_{t\to 0} \eta_{2,t}(r_{\theta(t)}(t) \cdot n(0, n_1, n_2, n_3)) = \eta_1(n(n_0, n_1, n_2, n_3)).$$

Remark 3. Our result should be regarded as the investigation of the intertwining spaces:

$$\mathrm{Hom}_{(\mathfrak{g}_{\mathbf{C}}, K)}(H_{\pi,K}, C^\infty_{\eta_{2,t}}(R_t\backslash G))$$

with

$$R_t = SO(\xi_t) \ltimes N_s \quad (t > 0).$$

6. Further comments

We only treat the even P_J-principal series, however, we also have the same results for the odd case, that is, $\varepsilon(\mathrm{diag}(-1, 1, -1, 1)) = -(-1)^k$.

In the case of the principal series (induced from minimal parabolic subgroup of G), the holonomic systems of rank 8 for the radial part of Whittaker functions (resp. Siegel-Whittaker functions) are obtained in [22] (resp. [21], [16]) and we can prove the same assertion as Theorem 4.1. However, explicit formulas for secondary spherical functions are known only for Whittaker functions ([17]), we can not say any more.

The other kinds of spherical functions on $Sp(2, \mathbf{R})$ are studied by Moriyama ([24]) and by Hirano ([12], [13], [14]). The spherical subgroup of [24] is $SL(2, \mathbf{C})$ and of [12], [13] and [14] is $SL(2, \mathbf{R}) \ltimes H_3$, with H_3 the 3-dimensional Heisenberg group. We hope that similar results hold between the two spherical functions.

We finally remark on the case of the special unitary group $SU(2, 2)$, which has the same restricted root system as $Sp(2, \mathbf{R})$. (Siegel-) Whittaker functions on $SU(2, 2)$ are studied by Hayata and Oda ([10], [8], [9]) and by Gon ([6]). Since their differential equations are compatible to those of $Sp(2, \mathbf{R})$, analogous argument seems to be possible.

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