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Remarks on A Posteriori Error Estimation for Finite Element Solutions

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Abstract

We utilize the classical hypercircle method and the lowest-order Raviart-Thomas $H(\mathrm{div})$ element to obtain a posteriori error estimates of the P_1 finite element solutions for 2D Poisson's equation. A few other estimation methods are also discussed for comparison. We give some theoretical and numerical results to see the effectiveness of the methods.

Keywords: hypercircle method, error bound, P_1 triangular element, Raviart-Thomas triangular element

1 Introduction

The finite element method is now used as a representative numerical method for partial differential equations. Mathematical analysis of such a method have been also extensively performed, and the so-called "a priori" error estimation is now popular [3, 4, 5, 7, 8]. Moreover, "a posteriori" error estimation has also become available utilizing some information of the obtained finite element solutions, and can be used as a basis of adaptive computation [1, 3, 7, 8, 10, 12, 13]. In this paper, we will present some results on a special a posteriori estimation method.

As a model problem, we consider the 2D Poisson equation with the homogeneous Dirichlet boundary condition: Given f, find u that satisfies

$$-\Delta u = f \quad \text{in } \Omega\,, \quad u = 0 \quad \text{on } \partial\Omega\,, \tag{1}$$

where Ω is a bounded polygonal domain with boundary $\partial\Omega$, f a given function defined on Ω, and u an unknown function in Ω. In the finite element method (FEM), we usually use the following weak formulation of the above model problem: Given $f \in L_2(\Omega)$, find $u \in U := H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v)\ ;\ \forall v \in U\,, \tag{2}$$

where $(\cdot,\cdot)$ denotes the inner product of $L_2(\Omega)$ or $L_2(\Omega)^2$. Moreover, $L_2(\Omega)$ and $H_0^1(\Omega)$ are usual Sobolev spaces associated to Ω [5].

To solve the above problem by a typical P_1 FEM, we first consider a regular family of triangulations $\{\mathcal{T}^h\}_{h>0}$ of Ω, and then construct the P_1 (i.e., piecewise linear) finite element space U^h as a subspace of $U = H_0^1(\Omega)$ for each $\mathcal{T}^h$. Usually, h denotes the maximum edge length of all triangles in the triangulation. Finally, the finite element solution $u_h \in U^h$ is specified by

$$(\nabla u_h, \nabla v_h) = (f, v_h)\ ;\ \forall v_h \in U^h\,. \tag{3}$$

For the present u_h, we can obtain the following well-known a priori estimates:

$$\|\nabla u - \nabla u_h\| \le C_1 h^\delta \|u\|_{H^{1+\delta}(\Omega)} \le C_2 h^\delta \|f\|\,, \quad \|u - u_h\| \le C_2^2 h^{2\delta}\|f\|\,, \tag{4}$$

where $\|\cdot\|$ denotes the norm of $L_2(\Omega)$ or $L_2(\Omega)^2$, C_1 and C_2 are positive constants dependent on Ω and the family of triangulations only, δ is a constant such that $\frac{1}{2} < \delta \leq 1$ depending only on the maximum interior angle of Ω, and $H^{1+\delta}(\Omega)$ is the (fractional) Sobolev space. In particular, $\delta = 1$ when Ω is a convex polygonal domain. In this type of a priori estimation, the approximate solution u_h does not appear in the right-hand sides of the inequalities. Instead, some informations on u and/or f are used. Furthermore, we can also obtain similar a priori error estimates in some other norms. For quantitative purposes, the positive constants like C_1 and C_2 above should be evaluated beforehand, although such evaluation is not necessarily easy.

Another error estimation method developing rapidly is the so-called a posteriori method, where the approximate solution u_h is also used in the right-hand sides. Such a method is also used as basis of adaptive computation. There have been developed various methods in this category, and one of the most classical one is that based on the hypercircle method [11], which does not require any positive constants like C_1, C_2 for estimation in some special norms. However, it has been almost forgotten for a long time: in fact, its implementation in FEM is not easy from strict viewpoint. However, in some very special problems, we can apply such an idea after slightly relaxing the severe conditions required in the original hypercircle method. Such an approach was proposed by Destuynder and Métivet [6] utilizing the Raviart-Thomas $H(\mathrm{div})$-triangular element and the mixed FEM [4]. See also [2, 9] for related works.

In this paper, we will present some theoretical results on such an approach together with related methods and numerical results.

2 Hypercircle Method

Let us explain the essence of the hypercircle method for the model problem (1), which is, for a given $f \in L_2(\Omega)$, to find $u \in U = H_0^1(\Omega)$ such that $-\triangle u = f$. The Poisson differential equation can be decomposed into

$$p = \nabla u \ , \quad \mathrm{div}\, p + f = 0 \,. \tag{5}$$

Thus we naturally introduce the following affine set for the given $f \in L_2(\Omega)$:

$$H_f(\mathrm{div};\Omega) := \{q \in L_2(\Omega)^2 \,;\, \mathrm{div}\, q + f = 0\} \subset H(\mathrm{div};\Omega) := \{q \in L_2(\Omega)^2 \,;\, \mathrm{div}\, q \in L_2(\Omega)\}. \tag{6}$$

Clearly, $p = \nabla u$ belongs to this set, and we can easily obtain the Prager-Synge identity [11] :

$$\|\nabla u - \nabla v\|^2 + \|p - q\|^2 = \|\nabla v - q\|^2 \quad ; \forall v \in H_0^1(\Omega),\ \forall q \in H_f(\mathrm{div};\Omega)\,. \tag{7}$$

Essentially, this is the Pythagorean theorem based on the orthogonality condition $\nabla u - \nabla v \perp p - q$ in $L_2(\Omega)^2$, where the vertex of right angle is at $\nabla u = p$. Thus the three points $\nabla u = p$, ∇v and q lie on a hypercircle whose center and radius are respectively $\frac{\nabla v + q}{2}$ and $\|\nabla u - \frac{\nabla v + q}{2}\| = \frac{1}{2}\|\nabla v - q\|$.

The idea of the hypercircle method is very simple. If we take v as the finite element solution u_h, which is surely in U since $U^h \subset U$, we have the estimates

$$\|\nabla u - \nabla u_h\| \leq \|\nabla u_h - q\| \ , \quad \|\nabla u - q\| \leq \|\nabla u_h - q\| \ , \quad \|\nabla u - \frac{\nabla u_h + q}{2}\| = \frac{1}{2}\|\nabla u_h - q\| \ , \tag{8}$$

provided that we can find an appropriate q. If u_h and q are somehow obtained, these give a posteriori error estimates without any special (uncertain) positive constants. The best possible q is of course the one minimizing $\|\nabla u_h - q\|$, but this condition is equivalent to minimizing $\|\nabla u - q\|$ as may be seen from (7). Thus the best q is actually $p = \nabla u$, and hence independent of u_h.

In general, it is difficult to establish a systematic way of finding nice q. Above all, the condition $\mathrm{div}\, q + f = 0$ is hard to realize for general f. However, for some approximation of f, it can be achieved by using special finite elements such as the Raviart-Thomas ones, cf. [6]. We will explain such an approach in the subsequent section.

3 P_1 FEM and $H(\mathrm{div})$ Mixed FEM

As was already explained, we consider a regular family of triangulations $\{\mathcal{T}^h\}_{h>0}$ for a bounded polygonal domain Ω, where h is the maximum diameter of all triangles $K \in \mathcal{T}^h$. Then the popular P_1-finite element space U^h is defined by $U^h = \{v_h \in U\,;\, v_h|K \in P_1(K)\}$, where $P_1(K)$ is the space of linear polynomials on K. The P_1 finite element solution $u_h \in U^h$ is then defined by (3).

To implement the hypercircle method approximately, we also consider a mixed FEM based on the lowest order Raviart-Thomas $H(\mathrm{div})$ element. The mixed variation formulation related to (5) is, for a given $f \in L_2(\Omega)$, find $p \in V = H(\mathrm{div};\Omega)$ and $\lambda \in W = L_2(\Omega)$ such that

$$(p, q) + (\lambda, \mathrm{div}\, q) = 0 \quad (\forall q \in V)\,, \quad (\mathrm{div}\, p, \mu) = -(f, \mu) \quad (\forall \mu \in W)\,. \tag{9}$$

As is well known, the solution $\{p, \lambda\}$ exists uniquely with $p = \nabla u$ and $\lambda = u$ [4]. Moreover, p can be characterized by the minimization condition:

$$\|p\|^2 = \min_{q \in H_f(\mathrm{div};\Omega)} \|q\|^2\,, \tag{10}$$

while λ plays the role of the Lagrange multiplier for the constraint $q \in H_f(\mathrm{div};\Omega)$.

As finite element spaces for V and W, we introduce

$$V^h = H^h(\mathrm{div};\Omega) := \{q \in V\ ;\ q|K = (\alpha_K x + \beta_K, \alpha_K y + \gamma_K)\ (\forall K \in \mathcal{T}^h)\}\,, \tag{11}$$

$$W^h = \{\mu \in W\ ;\ \mu|K \in P_0(K)\ (\forall K \in \mathcal{T}^h)\}\,, \tag{12}$$

where V^h is the lowest-order Raviart-Thomas triangular finite element space, and $P_0(K)$ is the space of constant functions on K.

Now the mixed finite element scheme for (9) is, for a given $f \in L_2(\Omega)$, find $p_h \in V^h = H^h(\mathrm{div};\Omega)$ and $\lambda_h \in W^h$ such that

$$(p_h, q_h) + (\lambda_h, \mathrm{div}\, q_h) = 0 \quad (\forall q_h \in V^h)\,, \quad (\mathrm{div}\, p_h, \mu_h) = -(f, \mu_h) \quad (\forall \mu_h \in W^h)\,. \tag{13}$$

Mathematical properties related to the above, such as the inf-sup condition, existence, uniqueness and some a priori estimates etc. are well established and not repeated here, cf. [4].

To give a discrete analog of (10), define the orthogonal projection operator $Q_h : W = L_2(\Omega) \to W_h$. Then we find $\mathrm{div}\, V^h = W^h$, and hence the second relation in (13) is expressed by $\mathrm{div}\, p_h + Q_h f = 0$, instead of the condition $\mathrm{div}\, p_h + f = 0$ desired in the hypercircle method. Then defining

$$H_f^h(\mathrm{div};\Omega) = \{q_h \in H^h(\mathrm{div};\Omega); \mathrm{div}\, q_h + Q_h f = 0\}\,, \tag{14}$$

the approximate solution p_h to p is characterized by

$$\|p_h\|^2 = \min_{q_h \in H_f^h(\mathrm{div};\Omega)} \|q_h\|^2\,. \tag{15}$$

4 A Posteriori Error Estimation Using Mixed FEM

The mixed finite element solution $p_h \in H_f^h(\mathrm{div};\Omega)$ is not appropriate for the strict hypercircle method unless $Q_h f = f$. However, if we consider the exact solution $u^h \in U$ of (1) for $Q_h f$ in place of f, then ∇u^h, ∇u_h and $q_h \in H_f^h(\mathrm{div};\Omega)$ make a hypercircle in $L_2(\Omega)^2$, since $H_f^h(\mathrm{div};\Omega) \subset H_{Q_h f}(\mathrm{div};\Omega)$. From the Prager-Synge type equality

$$\|\nabla u_h - q_h\|^2 = \|\nabla u^h - \nabla u_h\|^2 + \|\nabla u^h - q_h\|^2\,, \tag{16}$$

together with the triangle inequalities, we have

$$\|\nabla u - \nabla u_h\| \le \|\nabla u - \nabla u^h\| + \|\nabla u^h - \nabla u_h\| \le \|\nabla u - \nabla u^h\| + \|\nabla u_h - q_h\|\,, \tag{17}$$

$$\|\nabla u - q_h\| \le \|\nabla u - \nabla u^h\| + \|\nabla u^h - q_h\| \le \|\nabla u - \nabla u^h\| + \|\nabla u_h - q_h\|\,. \tag{18}$$

These estimates are close to the former two in (8) when $\|\nabla u - \nabla u^h\|$ is sufficiently small. Now one possible strategy of choosing nice q_h is to minimize the last term $\|\nabla u_h - q_h\|$ above. But we have a very clear and simple answer to this problem as follows.

Theorem 1. *The minimum of $\|\nabla u_h - q_h\|$ for $q_h \in H_f^h(\mathrm{div};\Omega)$ is attained uniquely by the mixed finite element solution p_h.*

proof From (16), the desired minimum is attained by q_h minimizing $\|\nabla u^h - q_h\|$. Thanks to the orthogonality relation $(q_h - \nabla u^h, \nabla u^h) = -(Q_h f - Q_h f, u^h) = 0$, we find $\|q_h\|^2 = \|q_h - \nabla u^h\|^2 + \|\nabla u^h\|^2$. Thus the present minimization is equivalent to that of $\|q_h\|$, and is attained by the finite element solution p_h as may be seen from (15). □

The above result is meaningful since p_h can be computed independently of u_h, at the expense of costly finite element calculation. We can also show that $\|\nabla u - \nabla u^h\| = O(h^2)$ when f is smooth.

Proposition 1. *If $f \in H^1(\Omega)$, then $\|\nabla u - \nabla u^h\| \leq Ch^2\|\nabla f\|$, where C is a positive constant that can be taken common to the considered regular family of triangulations.*

proof From the definition of u and u^h, we find $(\nabla u - \nabla u^h, \nabla v) = (f - Q_h f, v) = (f - Q_h f, v - Q_h v)$ $(\forall v \in U)$. Using the well-known error estimates for the piecewise constant approximate functions [6] and the Schwarz inequality, we obtain $|(\nabla u - \nabla u^h, \nabla v)| \leq Ch\|\nabla f\| \cdot Ch\|\nabla v\|$, from which the desired estimation follows easily. □

From the present results, we have, for smooth f,

$$\max\{\|\nabla u - \nabla u_h\|\,,\, \|\nabla u - p_h\|\} \leq \|\nabla u_h - p_h\| + O(h^2)\,. \tag{19}$$

For smooth f, we can show the a priori estimates $\|\nabla u - \nabla u_h\| = O(h^\delta)$ and $\|\nabla u - p_h\| = O(h^\delta)$, so that the first term in the right-hand side of the above equation is of $O(h^\delta)$ $(\frac{1}{2} < \delta \leq 1)$. Thus this term is asymptotically dominant compared with the term of $O(h^2)$ as $h \to 0$.

Of course, the above estimates hold for general $q_h \in H_f^h(\mathrm{div};\Omega)$ other than p_h. The same conclusion is obtained in [6] by a slightly different approach. In [6], there is proposed a method to produce q_h with $\|\nabla u - q_h\| = O(h)$ by post-processing the obtained u_h when Ω is a convex polygonal domain. Moreover, some iteration methods are also given to improve the quality of q_h. Actually, these give p_h as limit if the iteration processes are convergent, and hence become essentially the same as the present method after sufficient iterations.

Strictly speaking, these methods are "quasi"-hypercircle ones since we must use $H_f^h(\mathrm{div};\Omega)$ for general f. Moreover, in real computations, there are many pollutants to general f caused by numerical methods such as numerical integration, interpolation and lumping. However, we can usually find appropriate methods to keep the induced errors as $O(h^2)$ for sufficiently smooth f.

5 Non-hypercircle Methods Using $H(\mathrm{div})$ Triangle

We call the preceding method as Method-1, which is based on the mixed finite element solution p_h. In this section, we will propose two other methods, which use $H(\mathrm{div})$ triangle but do not necessarily use $H_f^h(\mathrm{div};\Omega)$.

Define the error e of the finite element solution u_h by $e := u - u_h$. Moreover, Let $R_h : U \to U^h$ be the Ritz projection characterized by $(\nabla(R_h v - v), \nabla v^h) = 0\,;\, \forall v \in U, \forall v^h \in U^h$. Then the following formula for e is well known [1, 3, 7, 8] :

$$\|\nabla e\|^2 = (\mathrm{div}\, q_h + f, e - R_h e) + (q_h - \nabla u_h, \nabla e)\ ;\quad \forall q_h \in H^h(\mathrm{div};\Omega). \tag{20}$$

Applying the Nitsche trick, we have the estimate

$$\|\nabla e\| \leq C_2 h^\delta \|f + \mathrm{div}\, q_h\| + \|q_h - \nabla u_h\|\,, \tag{21}$$

where C_2 and δ can be taken to be the same as in (4). By choosing $q_h \in H^h(\mathrm{div};\Omega)$, we can obtain a posteriori error estimate. Notice that the right-hand side of (21) differs with q_h. If $q_h \in H^h_f(\mathrm{div};\Omega)$, $R_h e$ in (20) can be replaced with $Q_h f$, and then the first term in the right-hand side of (21) is evaluated as $Ch^2\|\nabla f\|$ when $f \in H^1(\Omega)$, thus giving essentially the same estimate as (19).

To give possible example of such q_h obtainable by post-processing u_h, let us make some definitions. Let K and K' be two triangles sharing an (internal) edge γ in $\mathcal{T}^h$. When γ is a portion of $\partial\Omega$, there is only one triangle K that has γ as an edge. For K, u_h^K and ν_K respectively denote the restriction of u_h to K and the unit outward normal to K on γ. For K', $u_h^{K'}$ and $\nu_{K'}$ can be defined similarly. Notice here that ν_K and $\nu_{K'}$ have orientations opposite to each other. Then we can define $p_h^{(1)} \in H^h(\mathrm{div};\Omega)$ by, for each edge γ,

$$p_h^{(1)} \cdot \nu_K = \frac{1}{2}\left(\frac{\partial u_h^K}{\partial \nu_K} + \frac{\partial u_h^{K'}}{\partial \nu_K}\right) \ (interior\ \gamma)\,, \quad p_h^{(1)} \cdot \nu_K = \frac{\partial u_h^K}{\partial \nu_K} \ (boundary\ \gamma)\,. \tag{22}$$

The present $p_h^{(1)}$ can be used as q_h in (20) and (21). We call the present approach Method-2.

Let us define the jump of $\frac{\partial u_h}{\partial \nu_K}$ for interior γ by

$$\left[\frac{\partial u_h}{\partial \nu_K}\right]\Bigg|_\gamma = \frac{\partial u_h^K}{\partial \nu_K}\Bigg|_\gamma - \frac{\partial u_h^{K'}}{\partial \nu_K}\Bigg|_\gamma = \frac{\partial u_h^{K'}}{\partial \nu_{K'}}\Bigg|_\gamma - \frac{\partial u_h^K}{\partial \nu_{K'}}\Bigg|_\gamma\,. \tag{23}$$

For boundary γ, $\left[\frac{\partial u_h}{\partial \nu_K}\right]\Big|_\gamma$ is specified as 0. Then we can give another expression for (20) [1, 3, 7]:

$$\|\nabla e\|^2 = (\mathrm{div}_h \nabla u_h + f, e - R_h e) - \tfrac{1}{2}\sum_K \int_{\partial K} (e - R_h e)\left[\frac{\partial u_h}{\partial \nu_K}\right] d\gamma\,, \tag{24}$$

where div_h is the element-wise divergence operator, ∂K is the boundary of $K \in \mathcal{T}^h$ composed of three edges, $[\frac{\partial u_h}{\partial \nu_K}]$ is of the form (22) when restricted to each edge, and $d\gamma$ is the infinitesimal line element. More precisely, for $q \in L_2(\Omega)$ with $q|K \in H(\mathrm{div};K)$ $(\forall K)$, $\mathrm{div}_h q$ is defined as a function in $L_2(\Omega)$ such that $(\mathrm{div}_h q)|K = \mathrm{div}(q|K)(\forall K)$. The above formula can be directly used to give a posteriori estimates in terms of $[\frac{\partial u_h}{\partial \nu_K}]$ on ∂K as in [6]. However, as such an approach sometimes gives too large error bounds, we further transform it by using $p_h^{(2)} \in L_2(\Omega)^2$ with $p_h^{(2)}|K \in H^h(\mathrm{div};K)$ $(\forall K)$ defined by

$$p_h^{(2)} \cdot \nu_K = -\frac{1}{2}\left[\frac{\partial u_h^K}{\partial \nu_K}\right] \quad (\partial K\ of\ \forall K). \tag{25}$$

Then, by using the Green formula and noting that $R_h e = 0$, (24) becomes

$$\|\nabla e\|^2 = (\mathrm{div}_h(\nabla u_h + p_h^{(2)}) + f, e - R_h e) + (p_h^{(2)}, \nabla e)\,. \tag{26}$$

As (21), we obtain

$$\|\nabla e\| \le C_2 h^\delta \|f + \mathrm{div}_h(\nabla u_h + p_h^{(2)})\| + \|p_h^{(2)}\|\,, \tag{27}$$

which is a posteriori error estimation for u_h. We call the present method as Method-3.

In general, Method-2 and -3 are based on different ideas. Method-2 can be viewed as an averaging or smoothing method [1, 8, 13], while Method-3 is a variant of an approach using the jump of the flux of u_h [1, 3, 7]. These methods can be actually different from each other in general situation. However, in the present case, they completely coincide with each other :

Theorem 2. *For the present U^h, V^h and W^h, Methods-2 and -3 are equivalent in the sense that $p_h^{(1)} = p_h^{(2)} + \nabla u_h$, i.e., the right-hand sides of (21) ($q_h = p_h^{(1)}$) and (27) coincide with each other.*

proof Clearly, $p_h^{(1)}$, $p_h^{(2)}$ and ∇u_h are the lowest-order Raviart-Thomas type functions in each K. Thus the equality can be shown by checking the degrees of freedom for each K, which are the normal components on the edges [4, 6]. From (22), (23) and (25), we can easily show that $p_h^{(1)} \cdot \nu_K = (p_h^{(2)} + \nabla u_h) \cdot \nu_K$ for each γ of K, and hence $p_h^{(1)} = p_h^{(2)} + \nabla u_h$ in K and also in Ω. □

As in [6], we can show that $\|\nabla u_h - p_h^{(1)}\| = O(h)$, and $\|f + \mathrm{div}\, p_h^{(1)}\| = O(1)$ when u is sufficiently smooth and the family of triangulations is quasi-uniform, although we omit the proof. In such cases, Method-2 (and -3) can give reasonable upper bounds to $\|\nabla e\|$. Moreover, especially for regular meshes, we can often observe $\|f + \mathrm{div}\, p_h^{(1)}\| = o(1)$ in numerical tests like in the subsequent section.

6 Numerical Experiments

We performed some numerical tests for the model problem (1). They include one and two dimensional (2-D) cases based on the P_1 elements, and 2-D one by the bilinear rectangular element. Here, we only show some typical results based on the 2-D P_1 element and the $H(\mathrm{div})$ one.

The test case to be shown is for the unit square $\Omega = (0,1) \times (0,1)$, and f is taken as $f(x,y) = 2\{x(1-x) + y(1-y)\}$. Then the exact solution is $u(x,y) = x(1-x)y(1-y)$. In actual computation, f is approximated by its P_1-interpolantation. The double and/or quadruple precision arithmetics are employed to retain sufficient accuracy, although no numerical verification is made.

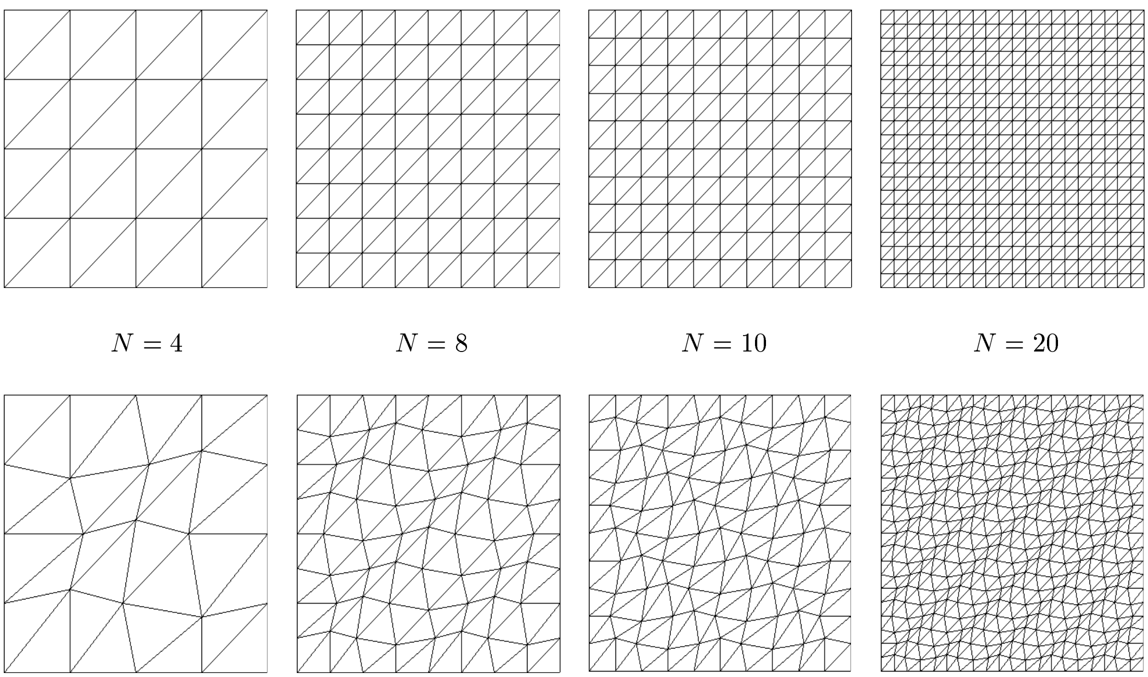

Figure 1: Triangulations (upper: uniform, lower: non-uniform)

Figure 1 depicts the triangulations used in the tests, where N denotes the number of division on each edge of Ω, and both uniform and non-uniform meshes are included.

Figures 2 and 3 show calculated errors by Method-1 and Method-2 (and -3) for the triangulations in Fig. 1. The results for $N = 40$ are also included. For Method-1, the exact errors $\|\nabla u - \nabla u_h\|$ and $\|\nabla u - p_h\|$ are plotted together with the bound $\|\nabla u - p_h\|$, where the higher order term in (19) is omitted. For Method-2, two terms in (21) are plotted with the total one. We employ $1/2$ as a tentative value of C_2, and δ is 1 since Ω is convex in the present case.

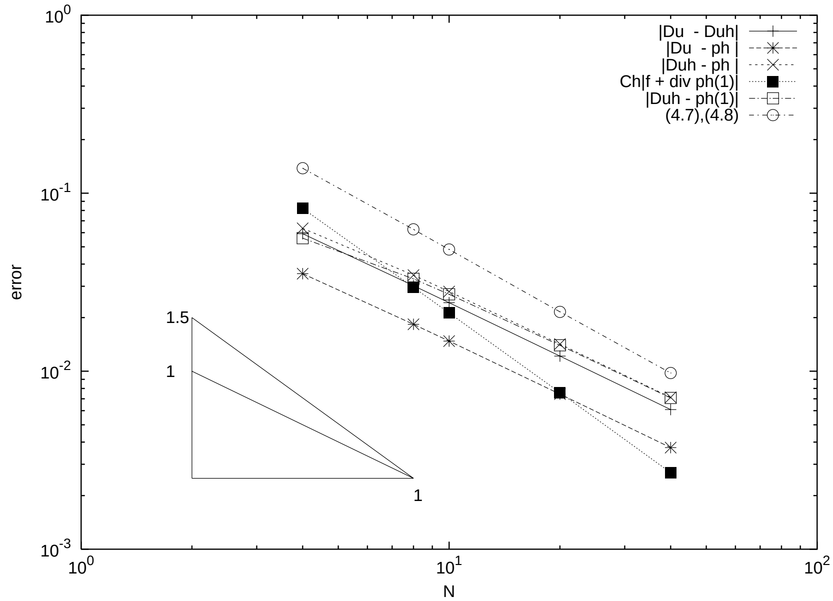

Figure 2: Calculated errors versus N : uniform triangulations

We can see that Method-1 gives a posteriori upper bounds $\|\nabla u_h - p_h\|$ of $O(h)$ since $h = O(1/N)$, which are about 1.2 times larger than $\|\nabla u - \nabla u_h\|$ in both uniform and non-uniform mesh cases. The present results are slightly superior or almost equal to those reported in [6]. We also checked the orthogonality condition $\nabla u - \nabla u_h \perp \nabla u - p_h$ numerically, since such orthogonality holds only asymptotically (i.e., $h \to 0$) in the present case. On the other hand, Methods-2 gives larger upper bound, though the 2nd term in (21) is often close to the bound by Method-1 and the first term $C_2 h\|f + \operatorname{div} p_h^{(1)}\|$ appears to decrease more rapidly than the second.

7 Concluding Remarks

We have given some results for a posteriori error estimation suggested by the hypercircle method. Such a method appears to be very effective when available, although its applicability is rather limited. Still it can play a role of model in the a posteriori estimation.

We are going to perform numerical tests for more singular problems, where δ in (1) is in the interval $1/2 < \delta < 1$. We should also consider some other boundary conditions, higher order elements, the 3-D Poisson equation, problems other than described by the Poisson equation, and error estimates in norms other than the energy norm. It is also important to develop simple post-processing methods based on the $H(\mathrm{div})$ elements.

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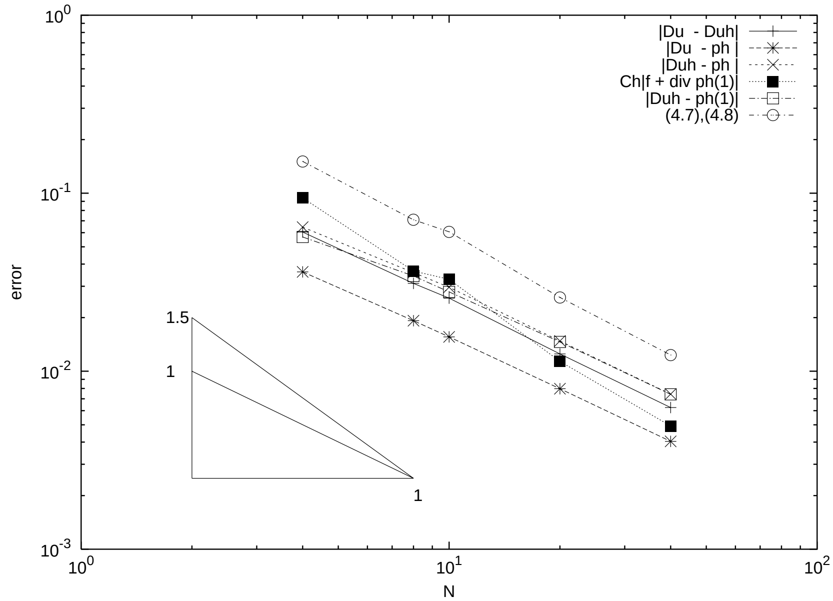

Figure 3: Calculated errors versus N : non-uniform triangulations

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