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Limit Theorem of a one dimensional Marokov Process to Sticky reflected Brownian Motion

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Abstract: We obtain a limit theorem of a 1-dimensional sticky reflected random walk X_n^λ with state space $[0, \infty)$. X_n^λ behaves like a normal random walk if it is away from the origin. Once X_n^λ reaches to 0, it stays 0 for a while and is repelled to the positive region. We consider a tightness of X_n^λ and a martingale problem for a discontinuous function.

1 Introduction

Let $\mu_W^\lambda, \lambda \in (0,1]$, be probability distributions on $\mathbf{R}$, $\mu_{Z+}^\lambda, \lambda \in (0,1]$, be probability distributions on $(0,\infty)$, and $p^\lambda \in (0,1]$, $\lambda \in (0,1]$. We assume the following.
$(A.1)$ There exist constants $a > 0$ and $K > 0$ such that

$$\sup_{\lambda \in (0,1]} \int_{\mathbf{R}} e^{a|x|} \mu_W^\lambda(dx) < \infty,$$

$$\left| \int_{\mathbf{R}} x \mu_W^\lambda(dx) \right| \le K\lambda^2, \ \int_{\mathbf{R}} x^4 \mu_W^\lambda(dx) + \int_0^\infty x^4 \mu_{Z+}^\lambda(dx) \le K\lambda^4, \ \lambda \in (0,1].$$

$(A.2)$ There exist constants $\sigma > 0, m_{Z+} > 0$ and $p > 0$ such that

$$\lim_{\lambda \to 0} \lambda^{-2} \int_{\mathbf{R}} x^2 \mu_W^\lambda(dx) = \sigma^2, \ \lim_{\lambda \to 0} \lambda^{-1} \int_0^\infty x \mu_{Z+}^\lambda(dx) = m_{Z+}, \ \lim_{\lambda \to 0} \lambda^{-1} p^\lambda = p.$$

For each $\lambda \in (0,1]$, let $\{W_n^\lambda\}_{n=1}^\infty, \{Z_n^\lambda\}_{n=1}^\infty$ be families of random variables defined on $(\Omega, \mathcal{F}, \mathbf{P})$ satisfying the following conditions.
(1) $W_n^\lambda, Z_n^\lambda, n = 1, 2, \cdots$, are independent,
(2) $W_n^\lambda, n = 1, 2, \cdots$, have the same probability law μ_W^λ,
(3) $Z_n^\lambda, n = 1, 2, \cdots$, have the same probability law, $Z_n^\lambda \ge 0$ $a.s.$,
$\mathbf{P}(Z_n^\lambda = 0) = 1 - p^\lambda$, and $\mathbf{P}(Z_n^\lambda \in dx | Z_n^\lambda > 0) = \mu_{Z+}^\lambda(dx), n = 1, 2, \cdots$.

Let $m_W(\lambda), v_W(\lambda,p), m_Z(\lambda), v(\lambda,p), m_{Z+}(\lambda)$ and $v_{Z+}(\lambda,p)$, $\lambda \in (0,1]$, $p \in (1,\infty)$, be given by the following.

$$m_W(\lambda) = \int_{\mathbf{R}} x\mu_W^\lambda(dx), \qquad v_W(\lambda,p) = \int_{\mathbf{R}} |x - m_W(\lambda)|^p \mu_W^\lambda(dx),$$

$$m_Z(\lambda) = p^\lambda \int_{(0,\infty)} x\mu_{Z+}^\lambda(dx), \qquad v(\lambda,p) = \int_{\mathbf{R}} |x|^p \mu_W^\lambda(dx) \vee \int_{(0,\infty)} |x|^p \mu_{Z+}^\lambda(dx),$$

$$m_{Z+}(\lambda) = \int_{(0,\infty)} x\mu_{Z+}^\lambda(dx), \quad \textit{and} \quad v_{Z+}(\lambda,p) = p^\lambda \int_{(0,\infty)} |x|^p \mu_{Z+}^\lambda(dx).$$

Let $\mathcal{F}_n = \mathcal{F}_n^\lambda = \sigma(W_m^\lambda, Z_m^\lambda; 0 \le m \le n)$. We define a family of random variables $\{X_n^\lambda(x)\}_{n=0}^\infty$ for each $x \in [0,\infty)$ and $\lambda \in (0,1]$, inductively by

$$X_0^\lambda(x) = x$$

$$X_{n+1}^\lambda(x) = \begin{cases} (X_n^\lambda(x) + W_{n+1}^\lambda) \vee 0, & X_n^\lambda(x) > 0, \\ Z_{n+1}^\lambda, & X_n^\lambda(x) = 0. \end{cases}$$

Then we see that $\{X_n^\lambda(x)\}_{n=0}^\infty$ is a Markov process. Let

$$S_n^\lambda(x) = x + \sum_{k=1}^n W_k^\lambda, \quad \lambda \in (0,1],$$

$$\tau(x) = \tau^\lambda(x) = \inf\{n \ge 0; X_n^\lambda(x) = 0\} = \inf\{n \ge 0; S_n^\lambda(x) \le 0\},$$

and

$$c(\lambda,\eta) = \mathbf{E}\left[e^{-\frac{\sqrt{2\eta}}{\sigma}W_1^\lambda}\right].$$

We assume following furthermore.

$(B.1)$ $m_W(\lambda) = 0$.

$(B.2)$ $$\overline{\lim}_{\eta\to 0}\ \overline{\lim}_{\lambda\to 0} \int_0^\infty \mathbf{E}\left[c(\lambda,\eta)^{-\tau(x)}\left(-\lambda^{-1}S_{\tau(x)}^\lambda(x)\right)\right]\mu_{Z+}^\lambda(dx)$$
$$= \underline{\lim}_{\eta\to 0}\ \underline{\lim}_{\lambda\to 0} \int_0^\infty \mathbf{E}\left[c(\lambda,\eta)^{-\tau(x)}\left(-\lambda^{-1}S_{\tau(x)}^\lambda(x)\right)\right]\mu_{Z+}^\lambda(dx) = \beta.$$

$(B.3)$ $$\lim_{\eta\to 0}\lim_{\lambda\to 0}\frac{1}{\lambda\sqrt{\eta}}\int_0^\infty \mathbf{E}\left[c(\lambda,\eta)^{-\tau(x)}\left(e^{-\frac{\sqrt{2\eta}}{\sigma}S_{\tau(x)}^\lambda(x)} - 1 + \frac{\sqrt{2\eta}}{\sigma}S_{\tau(x)}^\lambda(x)\right)\right]\mu_{Z+}^\lambda(dx) = 0.$$

Let us define stochastic processes $\{(\widetilde{X}_t^\lambda(x), \widetilde{S}_t^\lambda)\}_{t\in[0,\infty)}$ by

$$\widetilde{X}_t^\lambda(x) = X_{[\lambda^{-2}t]}^\lambda(x) + (\lambda^{-2}t - [\lambda^{-2}t])(X_{[\lambda^{-2}t]+1}^\lambda(x) - X_{[\lambda^{-2}t]}^\lambda(x)), \tag{1}$$

$$\widetilde{S}_t^\lambda = S_{[\lambda^{-2}t]}^\lambda(0) + (\lambda^{-2}t - [\lambda^{-2}t])(S_{[\lambda^{-2}t]+1}^\lambda(0) - S_{[\lambda^{-2}t]}^\lambda(0)), \tag{2}$$

and $\mathbf{Q}^\lambda$ be the probability measure induced by $\{(\widetilde{X}_t^\lambda(x), \sigma^{-1}\widetilde{S}_t^\lambda)\}_{t\in[0,\infty)}$ on $(\mathbf{C}([0,\infty);\mathbf{R}^2),\ \mathcal{B}(\mathbf{C}([0,\infty);\mathbf{R}^2)))$.

Let $\{W_t\}_{t\in[0,\infty)}$ be the Wiener process. Let X_t be a solution to the following stochastic differential equation.

$$X_t = x_0 + \sigma \int_0^t 1_{(0,\infty)}(X_s)dW_s + \delta \int_0^t 1_{\{0\}}(X_s)ds,$$

and

$$X_t \geq 0, \quad t \geq 0, \quad a.s..$$

This Stochastic Differential Equation has a unique solution called 1-dimensional sticky Brownian motion. (see Ikeda-Watanabe [4], p.222, Theorem 7.2.) Let $\mathbf{Q}$ be the probability law of $\{(X_t, W_t)\}_{t\in[0,\infty)}$ on $(\mathbf{C}([0,\infty);\mathbf{R}^2),\ \mathcal{B}(\mathbf{C}([0,\infty);\mathbf{R}^2)))$. Our main theorem is the following.

Theorem 1 $\mathbf{Q}^\lambda$ *converges weakly to* $\mathbf{Q}$ *as* $\lambda \downarrow 0$.

The limit theorem of a discrete process to this sticky Brownian motion has been studied Amir[1] and Harrison-Lemoine[6]. But both of them only consider special cases.

In this paper, we show the tightness of the distribution of $\{X_n^\lambda(x)\}_{n=0}^\infty$ in section 2. In section 3, we consider the sojourn time of $\{X_n^\lambda(x)\}_{n=0}^\infty$ at 0 which is important for the proof of our main theorem. In section 6, we solve martingale problem and prove our main theorem. In section 7, we show some examples and sufficient conditions of the assumption of the main theorem.

2 Tightness of $\{X_n^\lambda(x)\}$

In this section, we only assume (A.1) and (A.2). Let $\{M_n^\lambda(x)\}, \{Y_n^\lambda(x)\}, \{a_n^\lambda(x)\}$, $\{A_n^\lambda(x)\}$ be random variables given by

$$M_{n+1}^\lambda(x) = \sum_{k=0}^n 1_{(0,\infty)}(X_k^\lambda(x))(W_{k+1}^\lambda - m_W(\lambda)),$$

$$Y_{n+1}^\lambda(x) = \sum_{k=0}^n 1_{(0,\infty)}(X_k^\lambda(x))m_W(\lambda),$$

$$a_{n+1}^\lambda(x) = X_{n+1}^\lambda(x) - X_n^\lambda(x) - 1_{(0,\infty)}(X_n^\lambda(x))W_{n+1}^\lambda, \quad n = 0, 1, 2\cdots,$$

and

$$A_{n+1}^{\lambda}(x) = \sum_{k=1}^{n+1} a_k^{\lambda}(x).$$

Then we see that $X_n^{\lambda}(x) = x + M_n^{\lambda}(x) + (Y_n^{\lambda}(x) + A_n^{\lambda}(x))$.

Proposition 2 *For any $p \in (1, \infty)$, there exists a constant $C_p > 0$ such that*

$$\mathbf{E}[|X_n^{\lambda}(x) - X_m^{\lambda}(x)|^{2p}]$$
$$\leq C_p \left\{ (m_W(\lambda))^{2p} |n-m|^{2p} + v_W(\lambda, 2p)|n-m|^p + v(\lambda, 2p)|n-m| \right\},$$
$$n, m \in \mathbf{N}, m \leq n.$$

Proof. Step.1 First, we show the following claim.
Claim.

$$|X_n^{\lambda}(x) - X_m^{\lambda}(x)| \leq 3 \max_{m \leq l \leq n} \left| \sum_{k=m}^{l} (W_k^{\lambda} - m_W(\lambda)) \right| + 2(n-m)|m_W(\lambda)| + 2 \max_{m \leq k \leq n} |a_k^{\lambda}(x)|.$$

Let us prove Claim. In the case $\min_{m \leq k \leq n} X_k^{\lambda}(x) > 0$, we have

$$|X_n^{\lambda}(x) - X_m^{\lambda}(x)| \leq \left| \sum_{k=m+1}^{n} (W_k^{\lambda} - m_W(\lambda)) \right| + (n-m)|m_W(\lambda)|,$$

and so we have our assertion.

Suppose that $\min_{m \leq k \leq n} X_k^{\lambda}(x) = 0$. Let $r_1 = \max\{m \leq k \leq n; X_k^{\lambda}(x) = 0\}$ and $r_0 = \min\{m \leq k \leq n; X_k^{\lambda}(x) = 0\}$, we have

$$X_n^{\lambda}(x) - X_{r_1}^{\lambda}(x) = (M_n^{\lambda}(x) - M_{r_1}^{\lambda}(x)) + (Y_n^{\lambda}(x) - Y_{r_1}^{\lambda}) + a_{r_1+1}^{\lambda},$$

$$X_{r_0}^{\lambda}(x) - X_m^{\lambda}(x) = (M_{r_0}^{\lambda}(x) - M_m^{\lambda}(x)) + (Y_{r_0}^{\lambda} - Y_m^{\lambda}(x)) + a_{r_0}^{\lambda},$$

and

$$X_{r_1}^{\lambda} - X_{r_0}^{\lambda} = 0.$$

Note that

$$|M_{r_0}^{\lambda} - M_m^{\lambda}(x)| = \left| \sum_{k=m}^{r_0-1} (W_{k+1}^{\lambda} - m_W(\lambda)) \right| \leq \max_{m \leq l \leq n} \left| \sum_{k=m}^{l} (W_k^{\lambda} - m_W(\lambda)) \right|,$$

$$|M_n^\lambda(x) - M_{r_1}^\lambda| \le \max_{m+1\le l\le n} \left|\sum_{k=l}^n (W_k^\lambda - m_W(\lambda))\right|$$

$$\le \max_{m+1\le l\le n} \left|\sum_{k=m}^n (W_k^\lambda - m_W(\lambda)) - \sum_{k=m}^{l-1} (W_k^\lambda - m_W(\lambda))\right| \le 2\max_{m\le l\le n} \left|\sum_{k=m}^{l} (W_k^\lambda - m_W(\lambda))\right|,$$

$$|(Y_n^\lambda(x) - Y_{r_1}^\lambda) + (Y_{r_0}^\lambda - Y_m^\lambda(x))| \le \sum_{k=m}^n |m_W(\lambda)| \le 2(n-m)|m_W(\lambda)|,$$

and

$$|a_{r_1+1}^\lambda| + |a_{r_0}^\lambda| \le 2\max_{m\le k\le n} |a_k^\lambda(x)|.$$

So we have our Claim.

Step.2 Note that

$$\mathbf{E}\left[\left(\max_{m\le k\le n} |a_k^\lambda(x)|\right)^{2p}\right] \le \mathbf{E}\left[\sum_{k=m+1}^n \left(|W_k^\lambda| \vee |Z_k^\lambda|\right)^{2p}\right] \le |n-m|v(\lambda, 2p).$$

Using Claim, we have

$$\mathbf{E}\left[|X_n^\lambda(x) - X_m^\lambda(x)|^{2p}\right]^{\frac{1}{2p}}$$

$$\le 3\mathbf{E}\left[\max_{m\le l\le n}\left|\sum_{k=m}^l (W_k^\lambda - m_W(\lambda))\right|^{2p}\right]^{\frac{1}{2p}} + 2(n-m)|m_W(\lambda)| + 2(n-m)^{\frac{1}{2p}}\{v(\lambda, 2p)\}^{\frac{1}{2p}}.$$

Since $\sum_{k=m}^{l}(W_k^\lambda - m_W(\lambda))$, $l \ge m$, is a martingale, we see by Burkholder's inequality that there exists a constant C_p depending only on p such that

$$\mathbf{E}\left[\max_{m\le l\le n}\left|\sum_{k=m}^l (W_k^\lambda - m_W(\lambda))\right|^{2p}\right]^{\frac{1}{2p}} \le C_p\mathbf{E}\left[\left|\sum_{k=m}^n (W_k^\lambda - m_W(\lambda))^2\right|^p\right]^{\frac{1}{2p}}$$

$$\le C_p\mathbf{E}\left[(n-m+1)^{p-1}\sum_{k=m}^n (W_k^\lambda - m_W(\lambda))^{2p}\right]^{\frac{1}{2p}} \le C_p(n-m+1)^{\frac{1}{2}}(v_W(\lambda, 2p))^{\frac{1}{2p}}.$$

This completes the proof. ■

Proposition 3 *For any $p \in [2,\infty)$ there exists a constant C_p such that*

$$\mathbf{E}\left[\max_{0\le m<n\le 2^N, n-m\le K}|X_n^\lambda(x)-X_m^\lambda(x)|^{2p}\right]$$

$$\le C_p 2^N\{(m_W(\lambda))^{2p}K^{2p-1}+v_W(\lambda,2p)K^{p-1}+v(\lambda,2p)K\},\quad N\in\mathbf{N},\ 1\le K\le 2^N.$$

Proof. We define $n_{k'}, m_{k'},\ k'=1,2,\cdots,N'+1$, by

$$n_{k'}=\max\{k2^{N'-k'+1}; k\in\mathbf{N}, k2^{N'-k'+1}\le n\},$$

and

$$m_{k'}=\min\{k2^{N'-k'+1}; k\in\mathbf{N}, k2^{N'-k'+1}\ge m\}$$

for n,m satisfying $m\le n,\ n-m\le 2^{N'+1},\ 0\le N'\le N-1$. Then we have

$$|X_n^\lambda(x)-X_m^\lambda(x)|$$

$$\le\sum_{k'=1}^{N'}|X_{n_{k'+1}}^\lambda(x)-X_{n_{k'}}^\lambda(x)|+|X_{n_1}^\lambda(x)-X_{m_1}^\lambda(x)|+\sum_{k'=1}^{N'}|X_{m_{k'}}^\lambda(x)-X_{m_{k'+1}}^\lambda(x)|.$$

So we have

$$\max_{0\le m<n\le 2^N, n-m\le 2^{N'+1}}|X_n^\lambda(x)-X_m^\lambda(x)|\le 2\sum_{l=0}^{N'}\max_{1\le k\le 2^{N-l}}|X_{k2^l}^\lambda(x)-X_{(k-1)2^l}^\lambda(x)|.$$

Using Holder's inequality and Proposition 2, we have

$$\mathbf{E}\left[\max_{0\le m<n\le 2^N, n-m\le 2^{N'+1}}|X_n^\lambda(x)-X_m^\lambda(x)|^{2p}\right]$$

$$\le 2^{2p}\mathbf{E}\left[\left\{\sum_{l=0}^{N'}2^{\frac{l}{4p-2}}\right\}^{2p-1}\left\{\sum_{l=0}^{N'}2^{-\frac{l}{2}}\max_{1\le k\le 2^{N-l}}|X_{k2^l}^\lambda(x)-X_{(k-1)2^l}^\lambda(x)|^{2p}\right\}\right]$$

$$\le 4^p\left\{\frac{2^{\frac{N'+1}{4p-2}}-1}{2^{\frac{1}{4p-2}}-1}\right\}^{2p-1}\sum_{l=0}^{N'}2^{-\frac{l}{2}+N-l}C_p\left\{(m_W(\lambda))^{2p}2^{2pl}+v_W(\lambda,2p)2^{pl}+v(\lambda,2p)2^{2l}\right\}$$

$$\le 4^p\left\{\frac{2^{\frac{N'+1}{4p-2}}-1}{2^{\frac{1}{4p-2}}-1}\right\}^{2p-1}2^N C_p\Bigg\{(m_W(\lambda))^{2p}\frac{2^{\left(2p-\frac{3}{2}\right)(N'+1)}-1}{2^{2p-\frac{3}{2}}-1}$$

$$+v_W(\lambda,2p)\frac{2^{\left(p-\frac{3}{2}\right)(N'+1)}-1}{2^{(p-\frac{3}{2})}-1}+v(\lambda,2p)\frac{2^{\frac{1}{2}(N'+1)}-1}{2^{\frac{1}{2}}-1}\Bigg\}.$$

So there exists a constant C'_p such that

$$\mathbf{E}\left[\max_{0\le m<n\le 2^N, n-m\le 2^{N'+1}} |X_n^\lambda(x)-X_m^\lambda(x)|^{2p}\right]$$

$$\le C'_p 2^N\left\{(m_W(\lambda))^{2p}(2^{N'})^{2p-1}+v_W(\lambda,2p)(2^{N'})^{p-1}+v(\lambda,2p)2^{N'}\right\}$$

for any $N, N'\in\mathbf{N}$ satisfying $0\le N'\le N$. Therefore we have

$$\mathbf{E}\left[\max_{0\le m<n\le 2^N, n-m\le K} |X_n^\lambda(x)-X_m^\lambda(x)|^{2p}\right]$$

$$\le C'_p 2^N\{(m_W(\lambda))^{2p}K^{2p-1}+v_W(\lambda,2p)K^{p-1}+v(\lambda,2p)K\},$$

for any constant K satisfying $2^{N'}\le K\le 2^{N'+1}$. This completes the proof. ■

We define $(\widetilde{X}_t^\lambda, \widetilde{S}_t^\lambda)$, $t\in[0,T]$, by Equations (1) and (2) . Then we have the following.

Proposition 4 *For any* $T>0$,

$$\varlimsup_{\varepsilon\to 0}\varlimsup_{\lambda\to 0}\mathbf{E}\left[\sup_{0\le s<t\le T,\ t-s\le\varepsilon}|\widetilde{X}_t^\lambda-\widetilde{X}_s^\lambda|^4\right]=0,\qquad \varlimsup_{\lambda\to 0}\mathbf{E}\left[\sup_{0\le t\le T}|\widetilde{X}_t^\lambda|^4\right]<\infty,$$

$$\varlimsup_{\varepsilon\to 0}\varlimsup_{\lambda\to 0}\mathbf{E}\left[\sup_{0\le s<t\le T,\ t-s\le\varepsilon}|\widetilde{S}_t^\lambda-\widetilde{S}_s^\lambda|^4\right]=0,\quad \textit{and}\quad \varlimsup_{\lambda\to 0}\mathbf{E}\left[\sup_{0\le t\le T}|\widetilde{S}_t^\lambda|^4\right]<\infty.$$

Proof. We may assume that $\lambda<\sqrt{T}$. Let N be an integer satisfying $2^{N-1}\le\lambda^{-2}T\le 2^N-1$. If $0\le s<t\le T$, and $t-s\le\varepsilon$, then $0\le[\lambda^{-2}s]\le[\lambda^{-2}t]+1\le 2^N$, and $[\lambda^{-2}t]+1-[\lambda^{-2}s]\le\lambda^{-2}(t-s)+2\le 2^NT^{-1}\varepsilon+2$. Let $k=[\lambda^{-2}s]$, and $l=[\lambda^{-2}t]$. By Assumption (A.1) and Proposition 3, we see that there exist constants C_2 and K such that

$$\mathbf{E}\left[\sup_{0\le s<t\le T,\ t-s\le\varepsilon}|\widetilde{X}_t^\lambda-\widetilde{X}_s^\lambda|^4\right]$$

$$\le\mathbf{E}\left[\sup_{0\le k\le l+1\le 2^N,\ l+1-k\le 2^NT^{-1}\varepsilon+2}|X_l^\lambda(x)-X_k^\lambda(x)|^4\right]$$

$$\le C_2 2^N\left\{\ K^4\left(\frac{T}{2^{N-1}}\right)^4(2^NT^{-1}\varepsilon+2)^3+K^2\left(\frac{T}{2^{N-1}}\right)^2(2^NT^{-1}\varepsilon+2)\right.$$

$$+K\left(\frac{T}{2^{N-1}}\right)^2(2^N T^{-1}\varepsilon+2)\Bigg\}.$$

If $\lambda \to 0$, then $N \to \infty$. So we have

$$\overline{\lim_{\lambda\to 0}}\mathbf{E}\left[\sup_{0\le s<t\le T,\ t-s\le\varepsilon}|\widetilde{X}_t^\lambda-\widetilde{X}_s^\lambda|^4\right]\le C_2T\{K^4\varepsilon^3+2K\varepsilon\}.$$

Therefore we have

$$\overline{\lim_{\varepsilon\to 0}}\ \overline{\lim_{\lambda\to 0}}\mathbf{E}\left[\sup_{0\le s<t\le T,\ t-s\le\varepsilon}|\widetilde{X}_t^\lambda-\widetilde{X}_s^\lambda|^4\right]=0.$$

Furthermore, if $t=0$ and $\varepsilon=T$, we have

$$\overline{\lim_{\lambda\to 0}}\mathbf{E}\left[\sup_{0\le s\le T}|\widetilde{X}_s^\lambda|^4\right]<\infty.$$

By similar argument for $\{\widetilde{S}_t^\lambda\}$, we have our assertion. ■

We have the following by Proposition 4 and Billingsley [3], Theorem 7.3, p.82.

Corollary 5 *$\{\mathbf{Q}^{\lambda_n}; n\ge 0\}$ is tight as probability measures on $\mathbf{C}([0,\infty);\mathbf{R}^2)$, if $\lambda_n \downarrow 0,\ n\to\infty$.*

3 Sojourn time of $\{X_n^\lambda(x)\}$ at 0

From this Section, we assume Assumption (B.1)-(B.3) throughout this paper. Our main purpose in this section is to show the following Theorem.

Theorem 6 *For any $x>0$,*

$$\lim_{\eta\to 0}\lim_{\lambda\to 0}\lambda^2\sqrt{2\eta}\ \mathbf{E}\left[\sum_{n=0}^{\infty} c(\lambda,\eta)^{-n}1_{\{0\}}(X_n^\lambda(x))\right]=\frac{\sigma}{p(m_{Z+}+\beta)}.$$

We show some propositions before proving Theorem 6. Let a be the constant in Assumption (A.1). Let $c(\lambda,\eta)=\mathbf{E}\left[e^{-\frac{\sqrt{2\eta}}{\sigma}W_1^\lambda}\right]$. Note that $c(\lambda,\eta)\ge 1$ by Jensen's inequality. Since $\left|e^x-1-x-\dfrac{1}{2}x^2\right|\le\dfrac{1}{6}|x|^3e^{|x|}$, we have the following.

Proposition 7 *For any* $\eta \in \left(0, \frac{\sqrt{2}}{8}\sigma a\right)$,

$$\lim_{\lambda\to 0} \frac{c(\lambda,\eta)-1}{\lambda^2} = \eta \quad \textit{and} \quad \lim_{\lambda\to 0} c(\lambda,\eta)^{\lambda^{-2}} = e^{\eta}.$$

Let $\tau(x)$ and $\sigma(x)$ be stopping times given by

$$\tau(x) = \tau^{\lambda}(x) = \inf\{n \geq 0; X_n^{\lambda}(x) = 0\},$$

and

$$\sigma(x) = \sigma^{\lambda}(x) = \inf\{n > \tau(x); X_n^{\lambda}(x) > 0\}.$$

Then we have the following.

Proposition 8 *For any* $\lambda \in (0,1]$, $\eta \in \left(0, \frac{\sqrt{2}}{8}\sigma a\right)$ *and* $x > 0$,

$$\mathbf{E}\left[c(\lambda,\eta)^{-\tau(x)} e^{-\frac{\sqrt{2\eta}}{\sigma} S^{\lambda}_{\tau(x)}(x)}, \tau(x) < \infty\right] = e^{-\frac{\sqrt{2\eta}}{\sigma}x}.$$

Proof. We have

$$\mathbf{E}\left[e^{-\frac{\sqrt{2\eta}}{\sigma} S^{\lambda}_{n+1}(x)} | \mathcal{F}_n\right] = e^{-\frac{\sqrt{2\eta}}{\sigma} S^{\lambda}_{n}(x)} \mathbf{E}\left[e^{-\frac{\sqrt{2\eta}}{\sigma} W^{\lambda}_1}\right] = e^{-\frac{\sqrt{2\eta}}{\sigma} S^{\lambda}_{n}(x)} c(\lambda,\eta).$$

Let $M_n = c(\lambda,\eta)^{-n} e^{-\frac{\sqrt{2\eta}}{\sigma} S^{\lambda}_n(x)}$. Then M_n is a martingale, and so we have

$$\mathbf{E}[M_{n\wedge\tau(x)}] = e^{-\frac{\sqrt{2\eta}}{\sigma}x}.$$

By Fatou's Lemma, we have

$$\mathbf{E}\left[c(\lambda,\eta)^{-\tau(x)} e^{-\frac{\sqrt{2\eta}}{\sigma}} S^{\lambda}_{\tau(x)}, \tau(x) < \infty\right] \leq \liminf_{n\to\infty} \mathbf{E}[M_{n\wedge\tau(x)}] \leq e^{-\frac{\sqrt{2\eta}}{\sigma}x} \leq 1.$$

On the other hand, we have

$$M_{n\wedge\tau(x)} \leq 1_{\{\tau(x)<\infty\}} c(\lambda,\eta)^{-\tau(x)} e^{-\frac{\sqrt{2\eta}}{\sigma} S^{\lambda}_{\tau(x)}(x)} + 1.$$

Therefore by bounded convergence therorem, we have

$$\mathbf{E}\left[c(\lambda,\eta)^{-\tau(x)} e^{-\frac{\sqrt{2\eta}}{\sigma} S^{\lambda}_{\tau(x)}(x)}, \tau(x) < \infty\right] = \lim_{n\to\infty} \mathbf{E}[M_{n\wedge\tau(x)}] = e^{-\frac{\sqrt{2\eta}}{\sigma}x}. \qquad \blacksquare$$

Proposition 9 *Suppose that* $\{\nu^{\lambda,\alpha}\}_{\lambda\in(0,1],\alpha\in[0,1]}$ *is a family of distributions on* $[0,\infty)$ *satisfying* $\sup_{\lambda\in(0,1]}\sup_{\alpha\in[0,1]}\int_0^\infty |x|^2\nu^{\lambda,\alpha}(dx)<\infty$. *Then we have*

$$\overline{\lim_{\eta\to 0}}\ \overline{\lim_{\lambda\to 0}}\sup_{\alpha\in[0,1]}\frac{1}{\sqrt{\eta}}\int_0^\infty \mathbf{E}[1-c(\lambda,\eta)^{-\tau_1^\lambda(x)}]\nu^{\lambda,\alpha}(dx)\le \frac{2}{\sigma}\sup_{\lambda\in(0,1]}\sup_{\alpha\in[0,1]}\int_0^\infty x\ \nu^{\lambda,\alpha}(dx).$$

Proof. By Proposition 7, there exists a constant $\lambda_1(\eta)\in(0,1]$ for each $\eta\in\left(0,\dfrac{\sqrt{2}}{8}\sigma a\right)$ such that $1\le c(\lambda,\eta)^{\lambda^{-2}}\le e^{2\eta}$, for any $\lambda\in(0,\lambda_1(\eta)]$. We fix η for a while. Let $\alpha(\lambda)$ be a constant satisfying

$$\int_{[0,\infty)}\mathbf{E}[1-e^{-2\eta\lambda^2\tau_1^\lambda(x)}]\nu^{\lambda,\alpha(\lambda)}(dx)\ge\sup_{\alpha\in[0,1]}\int_{[0,\infty)}\mathbf{E}[1-e^{-2\eta\lambda^2\tau_1^\lambda(x)}]\nu^{\lambda,\alpha}(dx)-\lambda,$$

and let $\{\lambda_m\}$ be a sequence such that $\lambda_m\downarrow 0,\ m\to\infty$, and

$$\lim_{m\to\infty}\int_{[0,\infty)}\mathbf{E}[1-e^{-2\eta\lambda_m^2\tau_1^{\lambda_m}(x)}]\nu^{\lambda_m,\alpha(\lambda_m)}(dx)=\overline{\lim_{\lambda\to 0}}\sup_{\alpha\in[0,1]}\int_{[0,\infty)}\mathbf{E}[1-e^{-2\eta\lambda^2\tau_1^\lambda(x)}]\nu^{\lambda,\alpha}(dx).$$

Let ν_η^m denote $\nu^{\lambda_m,\alpha(\lambda_m)}$. Since $\sup_{\lambda\in(0,1]}\sup_{\alpha\in[0,1]}\int_0^\infty|x|^2\nu^{\lambda,\alpha}(dx)<\infty$, there exists a subsequence $\{\lambda_{m_l}\}$ of $\{\lambda_m\}$ such that $\nu_\eta^{m_l}$ converges weakly to a probability measure ν_η. Let $\xi^{\lambda,\alpha},\ \lambda\in(0,1],\alpha\in[0,1]$ be a random variable such that its distribution is $\nu^{\lambda,\alpha}$ and it is independent to $\{W_n^\lambda\}$. Let $\widetilde{S}_t^\lambda=\xi^{\lambda,\alpha}+S^\lambda_{[\lambda^{-2}t]}+(\lambda^{-2}t-[\lambda^{-2}t])W^\lambda_{[\lambda^{-2}t]+1}$, $\mathbf{P}^x$ be the distribution of Brownian motion whose variance is σ and its initial value is x and $\mathbf{E}^x$ denote the expectation under $\mathbf{P}^x$. Also, let $\tau(\varepsilon)=\tau(\varepsilon,w)=\inf\{t\ge 0; w(t)\le-\varepsilon\},\ w\in\mathbf{C}([0,\infty);\mathbf{R})$. Then we see by invariance principle that for any $t\ge 0$ and $\varepsilon>0$,

$$\begin{aligned}\underline{\lim}_{l\to\infty}\int_{[0,\infty)}\mathbf{P}\left((\lambda_{m_l})^2\tau_1^{\lambda_{m_l}}(x)\le t\right)\nu_\eta^{m_l}(dx)&\ge\underline{\lim}_{l\to\infty}\mathbf{P}\left(\min_{0\le s\le t}\widetilde{S}_s^{\lambda_{m_l}}<-\frac{\varepsilon}{2}\right)\\ &\ge\int_{[0,\infty)}\mathbf{P}^x\left(\min_{0\le s\le t}w(s)<-\frac{\varepsilon}{2}\right)\nu_\eta(dx)\ge\int_{[0,\infty)}\mathbf{P}^x\left(\tau(\varepsilon)\le t\right)\nu_\eta(dx).\end{aligned}$$

So we have

$$\begin{aligned}&\overline{\lim_{\lambda\to 0}}\sup_{\alpha\in[0,T]}\frac{1}{\sqrt{\eta}}\int_{[0,\infty)}\mathbf{E}[1-c(\lambda,\eta)^{-\tau_1^\lambda(x)}]\nu^{\lambda,\alpha}(dx)\\ &\le\overline{\lim_{l\to\infty}}\frac{1}{\sqrt{\eta}}\int_{[0,\infty)}\mathbf{E}[1-e^{-2\eta(\lambda_{m_l})^2\tau_1^{\lambda_{m_l}}(x)}]\nu_\eta^{m_l}(dx)\end{aligned}$$

$$= \overline{\lim_{l\to\infty}} \frac{1}{\sqrt{\eta}} \left\{ 1 - \int_{[0,\infty)} \int_{[0,\infty)} 2\eta e^{-2\eta t} \mathbf{P}\left((\lambda_{m_l})^2 \tau^{\lambda_{m_l}(\eta)}(x) \le t \right) \nu_\eta^{m_l}(dx)\ dt \right\}$$

$$\le \frac{1}{\sqrt{\eta}} \left\{ 1 - \int_{[0,\infty)} \mathbf{E}^x[e^{-2\eta\tau(\varepsilon)}]\nu_\eta(dx) \right\} = \frac{1}{\sqrt{\eta}} \int_{[0,\infty)} \left(1 - e^{-2\sqrt{\eta}\left(\frac{x+\varepsilon}{\sigma}\right)} \right) \nu_\eta(dx)$$

$$\le \frac{2}{\sigma} \sup_{\lambda\in[0,1]} \sup_{\alpha\in[0,T]} \int_{[0,\infty)} (x+\varepsilon)\ \nu^{\lambda,\alpha}(dx),$$

for any $\varepsilon > 0$ and $\eta \in \left(0, \frac{\sqrt{2}}{8}\sigma a\right)$. This implies our assertion. ■

Let $h(\lambda, \eta) = \sum_{n=0}^{\infty} c(\lambda,\eta)^{-n} p_\lambda (1-p_\lambda)^n = \frac{p_\lambda}{1 - c(\lambda,\eta)^{-1}(1-p_\lambda)}$. Then we have the following by Proposition 7.

Proposition 10 $\lim_{\lambda\to 0} \frac{\lambda(1-h(\lambda,\eta))}{1-c(\lambda,\eta)^{-1}} = \frac{1}{p}$ *for any* $\eta \in \left(0, \frac{\sqrt{2}}{8}\sigma a\right)$.

Proposition 11

$$\lim_{\eta\to 0} \lim_{\lambda\to 0} \frac{1}{\lambda\sqrt{\eta}} \left[1 - h(\lambda,\eta) \int_0^\infty \mathbf{E}[c(\lambda,\eta)^{-\tau(x)}] \mu_{Z+}^\lambda(dx) \right] = \frac{\sqrt{2}(m_{Z+} + \beta)}{\sigma}.$$

Proof. Note that $0 \le e^{-x} - 1 + x \le \frac{1}{2}x^2,\ x \ge 0$. Since $\overline{\lim_{\lambda\to 0}} \lambda^{-2} v_{Z+}(\lambda, 2) < \infty$ by Assumption (A.2), we have

$$\lim_{\eta\to 0} \lim_{\lambda\to 0} \frac{1}{\lambda\sqrt{\eta}} \int_0^\infty \left(e^{-\frac{\sqrt{2\eta}}{\sigma}x} - 1 + \frac{\sqrt{2\eta}}{\sigma}x \right) \mu_{Z+}^\lambda(dx) = 0. \tag{3}$$

By Proposition 8, we have

$$\frac{1}{\lambda\sqrt{\eta}} \left[1 - \int_0^\infty \mathbf{E}[c(\lambda,\eta)^{-\tau(x)}] \mu_{Z+}^\lambda(dx) \right]$$

$$= \frac{1}{\lambda\sqrt{\eta}} \int_0^\infty \mathbf{E}\left[c(\lambda,\eta)^{-\tau(x)} \left(e^{-\frac{\sqrt{2\eta}}{\sigma} S^\lambda_{\tau(x)}(x)} - 1 + \frac{\sqrt{2\eta}}{\sigma} S^\lambda_{\tau(x)}(x) \right) \right] \mu_{Z+}^\lambda(dx)$$

$$+\frac{\sqrt{2}}{\sigma}\left\{\int_0^\infty \mathbf{E}\left[c(\lambda,\eta)^{-\tau(x)}\left(-\lambda^{-1}S^\lambda_{\tau(x)}(x)\right)\right]\mu^\lambda_{Z+}(dx)+\lambda^{-1}\int_0^\infty x\mu^\lambda_{Z+}(dx)\right\}$$
$$-\frac{1}{\lambda\sqrt{\eta}}\int_0^\infty\left(e^{-\frac{\sqrt{2\eta}}{\sigma}x}-1+\frac{\sqrt{2\eta}}{\sigma}x\right)\mu^\lambda_{Z+}(dx).$$

By Assumptions (B.2), (B.3) and Equation (3), we have

$$\lim_{\eta\to 0}\lim_{\lambda\to 0}\frac{1}{\lambda\sqrt{\eta}}\left[1-\int_0^\infty \mathbf{E}[c(\lambda,\eta)^{-\tau(x)}]\mu^\lambda_{Z+}(dx)\right]=\frac{\sqrt{2}(m_{Z+}+\beta)}{\sigma}. \tag{4}$$

On the other hand, we have

$$\frac{1}{\lambda\sqrt{\eta}}\left[1-h(\lambda,\eta)\int_0^\infty \mathbf{E}[c(\lambda,\eta)^{-\tau(x)}]\mu^\lambda_{Z+}(dx)\right]$$
$$=\frac{1-h(\lambda,\eta)}{\lambda\sqrt{\eta}}+\frac{1}{\lambda\sqrt{\eta}}h(\lambda,\eta)\int_0^\infty \mathbf{E}[1-c(\lambda,\eta)^{-\tau(x)}]\mu^\lambda_{Z+}(dx).$$

Since $\lim_{\eta\to 0}\lim_{\lambda\to 0}\dfrac{1-h(\lambda,\eta)}{\lambda\sqrt{\eta}}=0$ by Propositions 7 and 10, we have our assertion from Equation (4). ■

Now let us prove Theorem 6. By the Storong Markov Property of $\{X^\lambda_n(x)\}$,

$$\mathbf{E}\left[c(\lambda,\eta)^{-\sigma(x)},X^\lambda_{\sigma(x)}(x)\in A\right]=\mathbf{E}\left[c(\lambda,\eta)^{-\tau(x)}\right]h(\lambda,\eta)\mu^\lambda_{Z+}(A).$$

So we have

$$\mathbf{E}\left[\sum_{n=0}^\infty c(\lambda,\eta)^{-n}1_{\{0\}}(X^\lambda_n(x))\right]=\frac{1}{1-c(\lambda,\eta)^{-1}}\mathbf{E}\left[c(\lambda,\eta)^{-\tau(x)}-c(\lambda,\eta)^{-\sigma(x)}\right]$$
$$+\mathbf{E}\left[c(\lambda,\eta)^{-\sigma(x)}\mathbf{E}\left[\sum_{n=0}^\infty c(\lambda,\eta)^{-n}1_{\{0\}}(X^\lambda_n(y))\right]\Bigg|_{y=X^\lambda_{\sigma(x)}(x)}\right]$$
$$=\mathbf{E}\left[c(\lambda,\eta)^{-\tau(x)}\right]\left\{\frac{1-h(\lambda,\eta)}{1-c(\lambda,\eta)^{-1}}\right.$$
$$\left.+h(\lambda,\eta)\int_0^\infty \mathbf{E}\left[\sum_{n=0}^\infty c(\lambda,\eta)^{-n}1_{\{0\}}(X^\lambda_n(y))\right]\mu^\lambda_{Z+}(dy)\right\}. \tag{5}$$

Integrating both sides by $\mu^\lambda_{Z+}(dy)$, we see that

$$\int_0^\infty \mathbf{E}\left[\sum_{n=0}^{\infty} c(\lambda,\eta)^{-n} 1_{(0)}(X^\lambda_n(y))\right] \mu^\lambda_{Z+}(dy)$$

$$= \int_0^\infty \frac{1-h(\lambda,\eta)}{1-c(\lambda,\eta)^{-1}} \frac{\mathbf{E}\left[c(\lambda,\eta)^{-\tau(y)}\right] \mu^\lambda_{Z+}(dy)}{1-h(\lambda,\eta)\int_0^\infty \mathbf{E}\left[c(\lambda,\eta)^{-\tau(y)}\right] \mu^\lambda_{Z+}(dy)}.$$

Sustituting this in Equation (5), we have

$$\lambda^2\sqrt{\eta}\,\mathbf{E}\left[\sum_{n=0}^{\infty} c(\lambda,\eta)^{-n} 1_{\{0\}}(X^\lambda_n(x))\right]$$

$$= \mathbf{E}[c(\lambda,\eta)^{-\tau(x)}] \frac{\lambda(1-h(\lambda,\eta))}{1-c(\lambda,\eta)^{-1}} \left[\lambda\sqrt{\eta} + \frac{\lambda\sqrt{\eta}\int_0^\infty \mathbf{E}[c(\lambda,\eta)^{-\tau(x)}]\mu^\lambda_{Z+}(dx)}{1-h(\lambda,\eta)\int_0^\infty \mathbf{E}[c(\lambda,\eta)^{-\tau(x)}]\mu^\lambda_{Z+}(dx)}\right].$$

Therefore we have our assertion by Propositions 9, 10 and 11. ∎

4 Preparations for Martingale Problem

Proposition 12 *There exists a constant C satisfying the following. For any $\varepsilon > 0$, there exists a $\lambda_\varepsilon > 0$ such that*

$$\mathbf{E}\left[\sum_{n=1}^{N-1} \lambda^2 1_{(0,\varepsilon)}(X^\lambda_n(x))\right] \le C\varepsilon \mathbf{E}\left[|X^\lambda_N(x)|\right], \quad \lambda \in (0,\lambda_\varepsilon],\ N \in \mathbf{N}.$$

In paticular, we have

$$\lim_{\varepsilon\to 0}\overline{\lim_{\lambda\to 0}}\mathbf{E}\left[\sum_{n=0}^{[\lambda^{-2}t]} \lambda^2 1_{(0,\varepsilon)}(X^\lambda_n(x))\right] = 0.$$

Proof. Step.1 First, we show the following.
Claim. There exists a constant $C > 0$ satisfying the following. For any $\varepsilon \in (0,1]$ there exists a constant $\lambda_\varepsilon \in (0,1]$ such that

$$\int_{[0,\varepsilon]} x^2\ \mu^\lambda_W(dx) \ge C\lambda^2, \quad \lambda \in (0,\lambda_\varepsilon].$$

By Assumption (A.2), there exists a constant $\lambda_0 \in (0,1]$ such that

$$\left\{\int_{\mathbf{R}} x^4 \mu_W^\lambda(dx)\right\}^{\frac{1}{3}} \left\{\int_{\mathbf{R}} |x| \mu_W^\lambda(dx)\right\}^{\frac{2}{3}} \geq \int_{\mathbf{R}} x^2 \mu_W^\lambda(dx) \geq \frac{\sigma^2}{2}\lambda^2, \quad \lambda \in (0, \lambda_0].$$

Let $K > 0$ be a constant in Assumption (A.1) and let $C = \dfrac{1}{2}\left(\dfrac{\sigma^2}{2}\right)^{\frac{3}{2}} K^{-\frac{1}{2}}$. Then we have

$$\int_{\mathbf{R}} |x| \mu_W^\lambda(dx) \geq 2C\lambda, \quad \lambda \in (0, \lambda_0].$$

Since $\displaystyle\int_{[0,\infty)} x \mu_W^\lambda(dx) = -\int_{(-\infty,0)} x \mu_W^\lambda(dx)$ by Assumption (B.1), we have

$$\int_{[0,\infty)} x \mu_W^\lambda(dx) \geq C\lambda.$$

So we have

$$\int_{[0,\varepsilon]} x^2 \mu_W^\lambda(dx) \geq \left(\int_{[0,\infty)} x \mu_W^\lambda(dx)\right)^2 - \int_{[\varepsilon,\infty)} x^2 \mu_W^\lambda(dx)$$

$$\geq C^2\lambda^2 - \frac{1}{\varepsilon^2}\int_{[0,\infty)} x^4 \mu_W^\lambda(dx) \geq C^2\lambda^2 - \frac{K}{\varepsilon^2}\lambda^4, \quad \lambda \in (0, \lambda_0].$$

Letting $\lambda_\varepsilon = \min\left\{\lambda_0, \dfrac{1}{2} C K^{-\frac{1}{2}} \varepsilon\right\}$, we have our Claim.

Step.2 Let

$$f(x) = \begin{cases} x^2, & (0 \leq x \leq 2\varepsilon), \\ 4\varepsilon x - 4\varepsilon^2, & (x \geq 2\varepsilon), \\ 0, & (x \leq 0). \end{cases}$$

$$g(x, y) = f(x + y) - f(x) - f'(x)y.$$

Then we have $g(x, y) \geq 0$, also we have $g(x, y) = y^2, \quad 0 \leq x \leq \varepsilon, \ 0 \leq y \leq \varepsilon$. Therefore we have

$$1_{\{X_n^\lambda(x)>0\}} \mathbf{E}\left[f(X_{n+1}^\lambda(x)) - f(X_n^\lambda(x)) \middle| \mathcal{F}_n\right]$$

$$\geq 1_{\{X_n^\lambda(x)>0\}} \mathbf{E}\left[f(X_n^\lambda(x) + W_{n+1}^\lambda) - f(X_n^\lambda(x)) - f'(X_n^\lambda(x)) W_{n+1}^\lambda \middle| \mathcal{F}_n\right]$$

$$\geq 1_{(0,\varepsilon)}(X_n^\lambda(x)) \int_{\mathbf{R}} g(X_n^\lambda(x), y) \mu_W^\lambda(dy) \geq 1_{(0,\varepsilon)}(X_n^\lambda(x)) \int_{[0,\varepsilon]} y^2 \mu_W^\lambda(dy)$$

$$\geq C\lambda^2 1_{(0,\varepsilon)}(X_n^\lambda(x)), \quad \lambda \in (0, \lambda_\varepsilon].$$

Note that $f(X_{n+1}^\lambda(x)) - f(X_n^\lambda(x)) = f(Z_{n+1}^\lambda) \geq 0$, if $X_n^\lambda(x) = 0$. So we have

$$\mathbf{E}\left[f(X_{n+1}^\lambda(x)) - f(X_n^\lambda(x)) \middle| \mathcal{F}_n \right] \geq C\lambda^2 1_{(0,\varepsilon)}(X_n^\lambda(x)), \quad \lambda \in (0, \lambda_\varepsilon].$$

Because $0 \leq f(x) \leq 4\varepsilon|x|$, we have

$$\mathbf{E}\left[\sum_{n=0}^{N-1} \lambda^2 1_{(0,\varepsilon)}(X_n^\lambda(x)) \right] \leq \frac{1}{C}\mathbf{E}\left[\sum_{n=0}^{N-1} f(X_{n+1}^\lambda(x)) - f(X_n^\lambda(x)) \right]$$

$$\leq \frac{1}{C}\mathbf{E}\left[f(X_N^\lambda(x)) - f(x) \right] \leq \frac{4}{C}\varepsilon \mathbf{E}\left[|X_N^\lambda(x)| \right], \quad \lambda \in (0, \lambda_\varepsilon].$$

This shows the first assertion. Then Proposition 4 implies our second assertion. ■

Let $\tau(x,t) = \inf\{n > [\lambda^{-2}t]; X_n^\lambda(x) = 0\}$. Then we have the following.

Proposition 13 $\lim_{\lambda \to 0} \sup_{s \in [0,t]} \mathbf{E}\left[c(\lambda,\eta)^{-\tau(0,s)} a_{\tau(0,s)}^\lambda \right] = 0$ *for any* $t > 0$.

Proof. If $[(n-1)\lambda^{-2}] \leq \tau(0,s) < [n\lambda^{-2}]$, then we have

$$c(\lambda,\eta)^{-\tau(0,s)} a_{\tau(0,s)} \leq c(\lambda,\eta)^{-[(n-1)\lambda^{-2}]} \max_{0 \leq m \leq [n\lambda^{-2}]} |W_m^\lambda|.$$

Let K be a constant in Assumption (A.1). Then we have

$$\mathbf{E}\left[\max_{0 \leq m \leq [\lambda^{-2}n]} |W_m^\lambda| \right] \leq \left\{ \mathbf{E}\left[\sum_{m=0}^{[\lambda^{-2}n]} |W_m^\lambda|^4 \right] \right\}^{\frac{1}{4}} \leq (\lambda^{-2}n + 1)^{\frac{1}{4}} \lambda K^{\frac{1}{4}}.$$

So we have

$$\sup_{k \in [0,t]} \mathbf{E}\left[c(\lambda,\eta)^{-\tau(0,s)} a_{\tau(0,s)}^\lambda \right] \leq \sum_{n=1}^{\infty} c(\lambda,\eta)^{-[(n-1)\lambda^{-2}]} (\lambda^{-2}n + 1)^{\frac{1}{4}} \lambda K^{\frac{1}{4}}.$$

Therefore we have our assertion from Propsition 7. ■

Proposition 14

$$\lim_{\eta\to 0}\lim_{\lambda\to 0}\sqrt{2\eta}\mathbf{E}\left[\sum_{n=0}^{\infty}c(\lambda,\eta)^{-n}1_{(0,\infty)}(X_n^\lambda(x))1_{\{0\}}(X_{n+1}^\lambda(x))a_{n+1}^\lambda(x)\right]=\frac{\sigma\beta}{m_{Z^\lambda+}+\beta}.$$

Proof. Let

$$g(\lambda,\eta)=\int_{\mathbf{R}}\mathbf{E}\left[\sum_{n=0}^{\infty}c(\lambda,\eta)^{-n}(1_{(0,\infty)}(X_n^\lambda(y)))1_{(0)}(X_{n+1}^\lambda(y))a_{n+1}^\lambda(y)\right]\mu_{Z+}^\lambda(dy).$$

By the Strong Markov Property of $\{X_n^\lambda(x)\}$, we have

$$\begin{aligned}&\mathbf{E}\left[\sum_{n=0}^{\infty}c(\lambda,\eta)^{-n}1_{(0,\infty)}(X_n^\lambda(x))1_{\{0\}}(X_{n+1}^\lambda(x))a_{n+1}^\lambda(x)\right]\\ &=\mathbf{E}[c(\lambda,\eta)^{-\tau(x)+1}a_{\tau(x)}^\lambda(x)]+\mathbf{E}\left[c(\lambda,\eta)^{-\sigma(x)}\mathbf{E}\left[\sum_{n=0}^{\infty}c(\lambda,\eta)^{-n}\right.\right.\\ &\qquad\left.\left.\times(1_{(0,\infty)}(X_n^\lambda(y)))1_{\{0\}}(X_{n+1}^\lambda(y))a_{n+1}^\lambda(y)\right]\Bigg|_{y=X_{\sigma(x)}^\lambda(x)}\right]\\ &=-\mathbf{E}[c(\lambda,\eta)^{-\tau(x)+1}S_{\tau(x)}^\lambda(x)]+\mathbf{E}[c(\lambda,\eta)^{-\tau(x)}]h(\lambda,\eta)g(\lambda,\eta).\end{aligned}\tag{6}$$

Integrating both sides by $\mu_{Z+}^\lambda(dx)$, we have

$$g(\lambda,\eta)=\frac{c(\lambda,\eta)\int_{\mathbf{R}}\mathbf{E}[c(\lambda,\eta)^{-\tau(y)}\left(-S_{\tau(y)}^\lambda\right)]\mu_{Z+}^\lambda(dy)}{1-h(\lambda,\eta)\int_{\mathbf{R}}\mathbf{E}[c(\lambda,\eta)^{-\tau(y)}]\mu_{Z+}^\lambda(dy)}.$$

Therefore, by Assumption (B.2) and Proposition 11, we have

$$\lim_{\eta\to 0}\lim_{\lambda\to 0}\sqrt{2\eta}g(\lambda,\eta)=\frac{\sigma\beta}{m_{Z^\lambda+}+\beta}.\tag{7}$$

Furthermore, by Proposition 13, we have

$$\lim_{\eta\to 0}\lim_{\lambda\to 0}\sqrt{2\eta}\mathbf{E}[c(\lambda,\eta)^{-\tau(x)+1}S_{\tau(x)}^\lambda(x)]=0.\tag{8}$$

So we have our assertion from Equations (6), (7) and (8). ■

Remind that $\delta=p(m_{Z+}+\beta)$. Then we have the following.

Proposition 15

$$\lim_{\eta\to 0}\lim_{\lambda\to 0}\sqrt{2\eta}\mathbf{E}\left[\sum_{n=0}^{\infty} c(\lambda,\eta)^{-n}\left\{X_{n+1}^{\lambda}(x)-X_n^{\lambda}(x)-\delta\lambda^2 1_{\{0\}}(X_n^{\lambda}(x))\right\}\right]=0.$$

Proof. Note that

$$\mathbf{E}\left[1_{\{0\}}(X_n^{\lambda}(x))1_{(0,\infty)}(X_{n+1}^{\lambda}(x))Z_{n+1}^{\lambda}\right]=\mathbf{E}\left[1_{\{0\}}(X_n^{\lambda}(x))\mathbf{E}[1_{(0,\infty)}(X_{n+1}^{\lambda}(x))Z_{n+1}^{\lambda}|\mathcal{F}_n]\right]$$

$$=p_{\lambda}m_Z(\lambda)\mathbf{E}\left[1_{\{0\}}(X_n^{\lambda}(x))\right]. \tag{9}$$

Since

$$\mathbf{E}\left[\sum_{n=0}^{\infty} c(\lambda,\eta)^{-n}(X_{n+1}^{\lambda}(x)-X_n^{\lambda}(x))\right]$$

$$=\mathbf{E}\left[\sum_{n=0}^{\infty} c(\lambda,\eta)^{-n}1_{\{0\}}(X_n^{\lambda}(x))1_{(0,\infty)}(X_{n+1}^{\lambda}(x))Z_{n+1}^{\lambda}\right]$$

$$+\mathbf{E}\left[\sum_{n=0}^{\infty} c(\lambda,\eta)^{-n}1_{(0,\infty)}(X_n^{\lambda}(x))\{W_{n+1}^{\lambda}+1_{\{0\}}(X_{n+1}^{\lambda}(x))a_{n+1}^{\lambda}(x)\}\right],$$

we have, by Equation (9), Theorem 6 and Proposition 14,

$$\lim_{\eta\to 0}\lim_{\lambda\to 0}\sqrt{2\eta}\mathbf{E}\left[\sum_{n=0}^{\infty} c(\lambda,\eta)^{-n}(X_{n+1}^{\lambda}(x)-X_n^{\lambda}(x))\right]=\sigma.$$

So we have our assertion by Theorem 6. ■

Proposition 16

$$\lim_{\eta\to 0}\lim_{\lambda\to 0}\sup_{s\in[0,t]}\left|\mathbf{E}\left[\sum_{n=0}^{[\lambda^{-2}s]} c(\lambda,\eta)^{-n}\left\{X_{n+1}^{\lambda}(0)-X_n^{\lambda}(0)-\delta\lambda^2 1_{\{0\}}(X_n^{\lambda}(0))\right\}\right]\right|=0.$$

Proof. Let

$$g(\lambda,\eta)=\sqrt{2\eta}\mathbf{E}\left[\sum_{n=0}^{\infty} c(\lambda,\eta)^{-n}\left\{X_{n+1}^{\lambda}(0)-X_n^{\lambda}(0)\right\}-\delta\sum_{n=0}^{\infty} c(\lambda,\eta)^{-n}\lambda^2 1_{\{0\}}(X_n^{\lambda}(0))\right]$$

Then we have, by the strong Markov property,

$$\left|\mathbf{E}\left[\sum_{n=0}^{\tau(0,s)-1} c(\lambda,\eta)^{-n}\{X_{n+1}^{\lambda}(0)-X_n^{\lambda}(0)-\delta\lambda^2 1_{\{0\}}(X_n^{\lambda}(0))\}\right]\right|$$
$$=\frac{1}{\sqrt{2\eta}}\left[1-\mathbf{E}[c(\lambda,\eta)^{-\tau(0,s)}]\right]|g(\lambda,\eta)|.$$

Let $\nu^{\lambda,s}$ be the distribution of $X_{[\lambda^{-2}s]}^{\lambda}(0)$. Then we have

$$\lim_{\eta\to0}\lim_{\lambda\to0}\sup_{s\in[0,t]}\frac{1}{\sqrt{2\eta}}\left[1-\mathbf{E}[c(\lambda,\eta)^{-\tau(0,s)}]\right]$$
$$=\lim_{\eta\to0}\lim_{\lambda\to0}\sup_{s\in[0,t]}\frac{1}{\sqrt{2\eta}}\left[1-c(\lambda,\eta)^{-[\lambda^{-2}s]}\int_{\mathbf{R}}\mathbf{E}\left[c(\lambda,\eta)^{-\tau_1^{\lambda}(x)}\right]\nu^{\lambda,s}(dx)\right]$$
$$\leq\frac{\sqrt{2}}{\sigma}\sup_{\lambda\in(0,1]}\sup_{s\in[0,t]}\mathbf{E}\left[|X_{[\lambda^{-2}s]}^{\lambda}(0)|\right]<\infty,$$

by Propositions 7 and 9. Because $\lim_{\eta\to0}\lim_{\lambda\to0}|g(\lambda,\eta)|=0$ by Proposition 15, we have

$$\lim_{\eta\to0}\lim_{\lambda\to0}\sup_{s\in[0,t]}\left|\mathbf{E}\left[\sum_{n=0}^{\tau(0,s)-1} c(\lambda,\eta)^{-n}\{X_{n+1}^{\lambda}(0)-X_n^{\lambda}(0)-\delta\lambda^2 1_{\{0\}}(X_n^{\lambda}(0))\}\right]\right|=0. \tag{10}$$

Let $M_n=\sum_{k=[\lambda^{-2}s]+2}^{n} c(\lambda,\eta)^{-k+1}W_k^{\lambda},\ n\geq[\lambda^{-2}s]+1$. Then M_n is a square-integrable martingale by Proposition 7 and Assumption (A.1). So, by Proposition 13, we have

$$\lim_{\lambda\to0}\sup_{s\in[0,t]}\left|\mathbf{E}\left[\sum_{n=[\lambda^{-2}s]+1}^{\tau(0,s)-1} c(\lambda,\eta)^{-n}\{X_{n+1}^{\lambda}(0)-X_n^{\lambda}(0)-\delta\lambda^2 1_{\{0\}}(X_n^{\lambda}(0))\}\right]\right|$$
$$=\lim_{\lambda\to0}\sup_{s\in[0,t]}\mathbf{E}\left[M_{\tau(0,s)}+c(\lambda,\eta)^{-\tau(0,s)+1}a_{\tau(0,s)}^{\lambda}\right]=0.$$

Therefore we have our assertion by this equation and Equation (10). ■

Corollary 17

$$\lim_{\eta\to0}\lim_{\lambda\to0}\sup_{s\in[0,t]}\left|\mathbf{E}\left[\sum_{n=0}^{[\lambda^{-2}s]}\left\{X_{n+1}^{\lambda}(0)-X_n^{\lambda}(0)-\delta\lambda^2 1_{\{0\}}(X_{n+1}^{\lambda}(0))\right\}\right]\right|=0.$$

Proof. Since

$$\sup_{s\in[0,t]}\left|\mathbf{E}\left[\sum_{n=0}^{[\lambda^{-2}s]}(1-c(\lambda,\eta)^{-n})\{X_{n+1}^{\lambda}(0)-X_n^{\lambda}(0)-\delta\lambda^2 1_{\{0\}}(X_{n+1}^{\lambda}(0))\}\right]\right|$$

$$\le \sup_{s\in[0,t]}\left\{(1-c(\lambda,\eta)^{-[\lambda^{-2}s]})\mathbf{E}\left[|X_{[\lambda^{-2}s]+1}^{\lambda}(0)|\right]\right.$$

$$\left.+\mathbf{E}\left[\sum_{n=1}^{[\lambda^{-2}s]}\left(c(\lambda,\eta)^{-n}-c(\lambda,\eta)^{n+1}\right)|X_n^{\lambda}(0)|\right]\right\}+(t+1)(1-c(\lambda,\eta)^{-[\lambda^{-2}t]})),$$

Then we have our assertion by Propositions 2, 4, 7 and 16. ■

Proposition 18 *Let* $\{G_n^{\lambda}\}$ *be an* $\mathcal{F}_n^{\lambda}-$*measurable random variable satisfying* $\sup_{\lambda\in(0,1]}\sup_{n,\omega}|G_n^{\lambda}(\omega)|<\infty$, *and let* $g\in\mathbf{C}_b^3(\mathbf{R}^2;\mathbf{R})$ *such that* $g(x,y)=0$, $(x,y)\in[0,\varepsilon]\times\mathbf{R}$, *for some constant* $\varepsilon>0$. *Then we have*

$$\lim_{\lambda\to 0}\mathbf{E}\left[\left\{\sum_{n=[\lambda^{-2}s]}^{[\lambda^{-2}t]}\left\{g(X_{n+1}^{\lambda}(x),S_{n+1}^{\lambda}(x))-g(X_n^{\lambda}(x),S_n^{\lambda}(x))\right.\right.\right.$$

$$\left.\left.\left.-\frac{\sigma^2}{2}\lambda^2\left(\frac{\partial^2}{\partial x^2}+2\frac{\partial^2}{\partial x\partial y}+\frac{\partial^2}{\partial y^2}\right)g(X_n^{\lambda}(x),S_n^{\lambda}(x))\right\}\right\}G_{[\lambda^{-2}s]}^{\lambda}\right]=0.$$

Proof. Let $M>0$ be a constant satisfying $\sup_{\lambda\in(0,1]}\sup_{n,\omega}|G_n^{\lambda}(\omega)|\le M$, and $\sup_{(x,y)\in\mathbf{R}^2}\max_{0\le l+m\le 3}\left|\frac{\partial^{l+m}}{\partial x^l\partial y^m}g(x,y)\right|\le M$. We define $I^{\lambda},\ I_k^{\lambda},\ k=1,2,\cdots,5$ as follows.

$$I^{\lambda}=\mathbf{E}\left[\left\{\sum_{n=[\lambda^{-2}s]}^{[\lambda^{-2}t]}\left\{g(X_{n+1}^{\lambda}(x),S_{n+1}^{\lambda}(x))-g(X_n^{\lambda}(x),S_n^{\lambda}(x))\right.\right.\right.$$

$$\left.\left.\left.-\frac{\sigma^2}{2}\lambda^2\left(\frac{\partial^2}{\partial x^2}+2\frac{\partial^2}{\partial x\partial y}+\frac{\partial^2}{\partial y^2}\right)g(X_n^{\lambda}(x),S_n^{\lambda}(x))\right\}\right\}G_{[\lambda^{-2}s]}^{\lambda}\right],$$

$$I_1^{\lambda}=\mathbf{E}\left[\left\{\sum_{n=[\lambda^{-2}s]}^{[\lambda^{-2}t]}\frac{\partial}{\partial x}g(X_n^{\lambda}(x),S_n^{\lambda}(x))(X_{n+1}^{\lambda}(x)-X_n^{\lambda}(x))\right\}G_{[\lambda^{-2}s]}^{\lambda}\right],$$

$$I_2^\lambda = \mathbf{E}\left[\left\{ \sum_{n=[\lambda^{-2}s]}^{[\lambda^{-2}t]} \frac{\partial}{\partial y} g(X_n^\lambda(x), S_n^\lambda(x))(S_{n+1}^\lambda(x) - S_n^\lambda(x)) \right\} G_{[\lambda^{-2}s]}^\lambda \right],$$

$$I_3^\lambda = \frac{1}{2}\mathbf{E}\left[\left\{ \sum_{n=[\lambda^{-2}s]}^{[\lambda^{-2}t]} \frac{\partial^2}{\partial x^2} g(X_n^\lambda(x), S_n^\lambda(x))\left\{(X_{n+1}^\lambda(x) - X_n^\lambda(x))^2 - \sigma^2\lambda^2\right\} \right\} G_{[\lambda^{-2}s]}^\lambda \right],$$

$$I_4^\lambda = \frac{1}{2}\mathbf{E}\left[\left\{ \sum_{n=[\lambda^{-2}s]}^{[\lambda^{-2}t]} \frac{\partial^2}{\partial y^2} g(X_n^\lambda(x), S_n^\lambda(x))\left\{(S_{n+1}^\lambda(x) - S_n^\lambda(x))^2 - \sigma^2\lambda^2\right\} \right\} G_{[\lambda^{-2}s]}^\lambda \right],$$

and

$$I_5^\lambda = \mathbf{E}\left[\left\{ \sum_{n=[\lambda^{-2}s]}^{[\lambda^{-2}t]} \frac{\partial^2}{\partial x \partial y} g(X_n^\lambda(x), S_n^\lambda(x))\left\{(X_{n+1}^\lambda(x) - X_n^\lambda(x))(S_{n+1}^\lambda(x) - S_n^\lambda(x)) - \sigma^2\lambda^2\right\} \right\} G_{[\lambda^{-2}s]}^\lambda \right].$$

First we will show $\lim_{\lambda\to 0} I_k^\lambda = 0$, $k = 1,2,3,4,5$. It is obvious that $I_2^\lambda = 0$ and $\lim_{\lambda\to 0} |I_4^\lambda| = 0$. Since $\frac{\partial}{\partial x} g(x,y) = 0$, $(x,y) \in [0,\varepsilon] \times \mathbf{R}$, we have

$$|I_1^\lambda| \le M^2 \mathbf{E}\left[\sum_{n=[\lambda^{-2}s]}^{[\lambda^{-2}t]} 1_{(\varepsilon,\infty)}(X_n^\lambda(x)) 1_{\{0\}}(X_{n+1}^\lambda(x)) a_{n+1}^\lambda(x) \right]$$

$$\le M^2 \sum_{n=[\lambda^{-2}s]}^{[\lambda^{-2}t]} \mathbf{E}[|W_{n+1}^\lambda|; W_{n+1}^\lambda < -\varepsilon] \le M^2 \sum_{n=[\lambda^{-2}s]}^{[\lambda^{-2}t]} \mathbf{E}\left[\frac{1}{\varepsilon^3}|W_{n+1}^\lambda|^4\right]$$

So we have $\lim_{\lambda\to 0} |I_1^\lambda| = 0$, by Assumption (A.1). Similarly we have

$$\lim_{\lambda\to 0} |I_3^\lambda| = 0,\ \lim_{\lambda\to 0} |I_5^\lambda| = 0.$$

Next we will show $\lim_{\lambda\to 0} \left| I^\lambda - \sum_{k=1}^{5} I_k^\lambda \right| = 0$. Since $\sup_{(x,y)\in\mathbf{R}^2} \max_{l+m=3} \left| \frac{\partial^3 g}{\partial x^l \partial y^m}(x,y) \right| \le M$, we have for any $(x_1,y_1),\ (x_2,y_2) \in \mathbf{R}^2$,

$$\Bigg| g(x_1,y_1) - g(x_2,y_2) - \frac{\partial}{\partial x} g(x_1,y_1)(x_1 - x_2) - \frac{\partial}{\partial y} g(x_1,y_1)(y_1 - y_2)$$

$$
-\frac{1}{2}\frac{\partial^2}{\partial x^2}g(x_1,y_1)(x_1-x_2)^2-\frac{1}{2}\frac{\partial^2}{\partial y^2}g(x_1,y_1)(y_1-y_2)^2
$$

$$
-\frac{\partial^2}{\partial x\partial y}g(x_1,y_1)(x_1-x_2)(y_1-y_2)\Bigg|\le \frac{4}{3}M\{|x_1-x_2|^3+|y_1-y_2|^3\}.
$$

Since $|X^\lambda_{n+1}(x)-X^\lambda_n(x)|\le |W^\lambda_{n+1}|\vee|Z^\lambda_{n+1}|$, we have

$$
\left|I^\lambda-\sum_{k=1}^5 I^\lambda_k\right|\le \frac{8}{3}M^2\sum_{n=[\lambda^{-2}s]}^{[\lambda^{-2}t]}\mathbf{E}[|W^\lambda_{n+1}|^3\vee|Z^\lambda_{n+1}|^3]\le 3M^2(t-s+1)\lambda\{\lambda^{-4}v(\lambda,4)\}^{\frac{3}{4}}.
$$

So we have $\lim_{\lambda\to 0}\left|I^\lambda-\sum_{k=1}^5 I^\lambda_k\right|=0$. This implies our assertion. ■

Corollary 19 *Let $\{G^\lambda_n\}$ be an $\mathcal{F}^\lambda_n-$measurable random variable satisfying $\sup_{\lambda\in(0,1]}\sup_{n,\omega}|G^\lambda_n(\omega)|<\infty$, and $g\in\mathbf{C}^3_b$ satisfy $g^{(k)}(0)=0,\ k=0,1,2$. Then we have*

$$
\lim_{\lambda\to 0}\mathbf{E}\left[\left\{\sum_{n=[\lambda^{-2}s]}^{[\lambda^{-2}t]}\left\{g(X^\lambda_{n+1}(x))-g(X^\lambda_n(x))-\frac{\sigma^2}{2}\lambda^2 g''(X^\lambda_n(x))\right\}\right\}G^\lambda_{[\lambda^{-2}s]}\right]=0.
$$

Proof. Let $M>0$ be a constant satisfying $\sup_{\lambda\in(0,1]}\sup_{n,\omega}|G^\lambda_n(\omega)|\le M$, and $|g^{(k)}|\le M,\ k=0,1,2,3$, and $\varphi\in C^\infty$ be a non-decreasing function such that

$$
\varphi(x)=\begin{cases}0, & x\in\left(-\infty,\frac{1}{2}\right],\\ 1, & x\in[1,\infty).\end{cases}
$$

Then there exists a constant $C\ge 1$ such that $|\varphi^{(k)}(x)|\le C,\quad k=1,2,3$.
Let $\varphi_\varepsilon(x)=\varphi(\varepsilon^{-1}x)$, and $g_\varepsilon(x)=\varphi_\varepsilon(x)g(x)$, for any $\varepsilon\in(0,1]$. Then $g_\varepsilon(x)=g(x)$ for $x\in[\varepsilon,\infty)$, and $g^{(k)}(x)=0,\ k=0,1,2,3$, for $x\in\left[0,\frac{\varepsilon}{2}\right]$. For $x\in[0,\varepsilon]$, we have $|g''(x)|=\left|\int_0^x g'''(y)dy\right|\le M\varepsilon,\ |g'(x)|\le M\varepsilon^2$, and $|g(x)|\le M\varepsilon^3$. Since $|\varphi^{(k)}_\varepsilon(x)|\le\frac{C}{\varepsilon^k},\ k=0,1,2,3$, we have $|g'_\varepsilon(x)|\le 2MC\varepsilon^2,\ |g''_\varepsilon(x)|\le 4MC\varepsilon$, and $|g'''_\varepsilon(x)|\le 8MC$.

Let

$$I^\lambda(\varepsilon) = \mathbf{E}\left[\left\{\sum_{n=[\lambda^{-2}s]}^{[\lambda^{-2}t]}\left\{g_\varepsilon(X^\lambda_{n+1}(x)) - g_\varepsilon(X^\lambda_n(x)) - \frac{\sigma^2}{2}\lambda^2 g''_\varepsilon(X^\lambda_n(x))\right\}\right\} G^\lambda_{[\lambda^{-2}s]}\right].$$

Then we have $\lim_{\lambda\to 0}|I^\lambda(\varepsilon)| = 0$, by Proposition 18. Let $h_\varepsilon(x) = (1-\varphi_\varepsilon(x))g(x)$, and

$$J^\lambda(\varepsilon) = \mathbf{E}\left[\left\{\sum_{n=[\lambda^{-2}s]}^{[\lambda^{-2}t]}\left\{h_\varepsilon(X^\lambda_{n+1}(x)) - h_\varepsilon(X^\lambda_n(x)) - \frac{\sigma^2}{2}\lambda^2 h''_\varepsilon(X^\lambda_n(x))\right\}\right\} G^\lambda_{[\lambda^{-2}s]}\right].$$

Since $h_\varepsilon(x) = h''_\varepsilon(x) = 0,\ \ |h''_\varepsilon(x)| \le 4MC\varepsilon$, and $|h_\varepsilon(x)| \le M\varepsilon^3$ for $x = 0,\ x \in [\varepsilon,\infty)$, we have

$$J^\lambda(\varepsilon) \le 2M\varepsilon^3 + 4M^2C\frac{\sigma^2}{2}\varepsilon(t-s+1).$$

Therefore we have $\lim_{\varepsilon\to 0}\lim_{\lambda\to 0} J^\lambda(\varepsilon) = 0$. This completes the proof. ■

5 Proof of Theorem 1

Fix $x_0 \ge 0$, and let

$$\widetilde{X}^\lambda_t = X^\lambda_{[\lambda^{-2}t]}(x_0) + (\lambda^{-2}t - [\lambda^{-2}t])(X^\lambda_{[\lambda^{-2}t]+1}(x_0) - X^\lambda_{[\lambda^{-2}t]}(x_0)),$$

$$\widetilde{S}^\lambda_t = S^\lambda_{[\lambda^{-2}t]}(x_0) + (\lambda^{-2}t - [\lambda^{-2}t])(S^\lambda_{[\lambda^{-2}t]+1}(x_0) - S^\lambda_{[\lambda^{-2}t]}(x_0)).$$

Let $\mathbf{Q}^\lambda$ be the distribution of $(\widetilde{X}^\lambda_t, \sigma^{-1}\widetilde{S}^\lambda_t)$. Suppose that $\{\lambda_m\}_{m=1}^\infty$ is a subsequence such that $\lambda_m \downarrow 0, m \to \infty$, and $\mathbf{Q}^{\lambda_m}$ converges weakly to a disutribution $\mathbf{Q}$ on $(\mathbf{C}([0,\infty);\mathbf{R}^2)$. Let $\mathbf{E}^{\mathbf{Q}}$ denote the expectation under $\mathbf{Q}$. Let

$$X_t = X_t(w) = \omega_1(t),\ W_t = W_t(w) = w_2(t),\quad w = (w_1, w_2) \in \mathbf{C}([0,\infty);\mathbf{R}^2),$$

and

$$\mathcal{G}_t = \sigma(X_s(w), W_s(w); 0 \le s \le t).$$

For $f \in \mathbf{C}^3(\mathbf{R})$, let

$$M^{[f]}_t = f(X_t) - f(X_0) - \frac{\sigma^2}{2}\int_0^t (1 - 1_{\{0\}}(X_u))f''(X_u)du - \delta\int_0^t 1_{\{0\}}(X_u)f'(X_u)du.$$

By Proposition 4, we have the following.

Proposition 20 *Let $g \in \mathbf{C}_b^1(\mathbf{R};\mathbf{R})$, and $G \in \mathbf{C}_b(\mathbf{R}^{2n};\mathbf{R})$. For any $0 \le s < t < \infty$, and the partition $0 \le s_1 < s_2 < \cdots < s_n \le s$, we have*

$$\mathbf{E}^{\mathbf{Q}}\left[\int_s^t g(X_u)du\ G((w(s_1),\cdots,w(s_n))\right]$$

$$= \lim_{m\to\infty} \mathbf{E}^{\mathbf{Q}^{\lambda_m}}\left[\sum_{n=[\lambda_m^{-2}s]}^{[\lambda_m^{-2}t]} \lambda_m^2 g(X_{\lambda_m^2 n}) G((w(s_1),\cdots,w(s_n))\right].$$

Proposition 21 *If $f(x) = x$, then $M^{[f]}$ is a $\mathcal{G}_t-$martingale.*

Proof. Let $G \in \mathbf{C}_b(\mathbf{R}^{2n};\mathbf{R})$, such that there exists a constant $M > 0$ satisfying $\sup_{x\in\mathbf{R}^{2n}} |G(x)| \le M$, and let $\varphi(x)$ be a smooth generalized monotone decreasing function satisfying following.

$$\varphi(x) = \begin{cases} 1, & 0 \le x \le \dfrac{1}{2}, \\ \\ 0, & x \ge 1. \end{cases}$$

$$0 \le \varphi(x) \le 1, \quad \sup_{x\in\mathbf{R}} |\varphi'(x)| \le 4.$$

We define $\varphi_\varepsilon(x) = \varphi\left(\varepsilon^{-1}x\right)$, for $\varepsilon \in (0,1]$. Since $f'(x) = 1$, and $f''(x) = 0$, we have

$$\begin{aligned} M_t^{[f]} - M_s^{[f]} &= X_t - X_s - \delta\int_s^t 1_{\{0\}}(X_u)du \\ &= X_t - X_s - \delta\int_s^t \varphi_\varepsilon(X_u)du + \delta\int_0^t \big(\varphi_\varepsilon(X_u) - 1_{\{0\}}(X_u)\big)\,du. \end{aligned}$$

For any partition $0 \le s_1 < s_2 < \cdots < s_n \le s$, we have

$$\mathbf{E}^{\mathbf{Q}}\left[\left\{X_t - X_s - \delta\int_s^t \varphi_\varepsilon(X_u)du\right\} G((w(s_1),\cdots,w(s_n))\right]$$

$$= \lim_{m\to\infty} \mathbf{E}^{\mathbf{Q}^{\lambda_m}}\left[\left\{\sum_{n=[\lambda_m^{-2}s]}^{[\lambda_m^{-2}t]} \{X_{\lambda_m^2(n+1)} - X_{\lambda_m^2 n} - \delta\lambda_m^2 1_{\{0\}}(X_{\lambda_m^2 n})\}\right\}\right.$$

$$-\delta\lambda_m^2 \sum_{n=[\lambda_m^{-2}s]}^{[\lambda_m^{-2}t]} \left(\varphi_\varepsilon(X_{\lambda_m^2 n}) - 1_{\{0\}}(X_{\lambda_m^2 n})\right) \Bigg\} G((w(s_1), \cdots, w(s_n)) \Bigg],$$

by Propositions 2 and 20. By Proposition 12, we have

$$\lim_{\varepsilon\to 0} \overline{\lim_{m\to\infty}} \mathbf{E}^{\mathbf{Q}^{\lambda_m}} \left[\left\{ \sum_{n=[\lambda_m^{-2}s]}^{[\lambda_m^{-2}t]} \lambda_m^2 \left(\varphi_\varepsilon(X_{\lambda_m^2 n}) - 1_{\{0\}}(X_{\lambda_m^2 n})\right) \right\} G((w(s_1), \cdots, w(s_n)) \right]$$

$$\leq \lim_{\varepsilon\to 0} \overline{\lim_{m\to\infty}} M\mathbf{E} \left[\sum_{n=[\lambda_m^{-2}s]}^{[\lambda_m^{-2}t]} \lambda_m^2 1_{(0,\varepsilon)}(\widetilde{X}_n^{\lambda_m}) \right] = 0.$$

So we have by Corollary 17

$$\lim_{\varepsilon\to 0} \mathbf{E}^{\mathbf{Q}} \left[\left\{ X_t - X_s - \delta \int_s^t \varphi_\varepsilon(X_u) du \right\} G((w(s_1), \cdots, w(s_n)) \right] = 0.$$

On the other hand, since $\varphi_\varepsilon(x) - 1_{\{0\}}(x) \to 0$, and $|\varphi_\varepsilon(x)| \leq 1$, we have by Bounded Convergence Theorem,

$$\lim_{\varepsilon\to 0} \left| \mathbf{E}^{\mathbf{Q}} \left[\left\{ \int_0^t \left(1_{\{0\}}(X_s) - \varphi_\varepsilon(X_s)\right) ds \right\} G((w(s_1), \cdots, w(s_n)) \right] \right| = 0.$$

This completes the proof. ■

Proposition 22 *If $f(x) \in C_b^3$ satisfies $f(0) = f'(0) = 0$, then $M^{[f]}$ is $\mathcal{G}_t-$martingale.*

Proof. Let $G \in \mathbf{C}_b(\mathbf{R}^{2n}; \mathbf{R})$, and let $M > 0$ be a constant satisfying $\sup_{x\in\mathbf{R}^{2n}} |G(x)| \leq M$, $\sup_{x\in\mathbf{R}} |f^{(k)}(x)| \leq M$, $k = 0, 1, 2, 3$. Also let $\varepsilon \in (0, 1]$, $a = 2\varepsilon^{-2}(f'''(\varepsilon)\varepsilon - f''(\varepsilon))$ and $b = \varepsilon^{-1}(2f''(\varepsilon) - f'''(\varepsilon)\varepsilon)$. Let

$$h_\varepsilon(x) = \begin{cases} ax + b, & x \in [0, \varepsilon], \\ f'''(x), & x \geq \varepsilon, \end{cases}$$

and

$$f_\varepsilon(x) = \int_0^x \int_0^y \int_0^z h_\varepsilon(w) dw dz dy.$$

Then $f_\varepsilon \in \mathbf{C}_b^3$, and $f_\varepsilon(0) = f'_\varepsilon(0) = f''_\varepsilon(0) = 0$, for any $\varepsilon \in (0,1]$. For any partition $0 \leq s_1 < s_2 < \cdots < s_n \leq s$, we have

$$\mathbf{E}^{\mathbf{Q}}\left[\left\{M_t^{[f_\varepsilon]} - M_s^{[f_\varepsilon]}\right\} G(w(s_1), \cdots, w(s_n))\right]$$

$$= \lim_{m\to\infty} \mathbf{E}^{\mathbf{Q}^{\lambda_m}}\left[\left\{f_\varepsilon(X_t) - f_\varepsilon(X_s) - \frac{\sigma}{2}\int_s^t f''_\varepsilon(X_u)du\right\} G(w(s_1), \cdots, w(s_n))\right]$$

$$= \lim_{m\to\infty} \mathbf{E}^{\mathbf{Q}^{\lambda_m}}\left[\left\{\sum_{n=[\lambda_m^2 s]}^{[\lambda_m^2 t]}\left\{f_\varepsilon(X_{\lambda_m^2(n+1)}) - f_\varepsilon(X_{\lambda_m^2 n}) - \frac{\sigma}{2}\lambda_m^2 f''_\varepsilon(X_{\lambda_m^2 n})\right\}\right\} G(w(s_1), \cdots, w(s_n))\right],$$

by Proposition 20. So $M^{[f_\varepsilon]}$ is a $\mathcal{G}_t-$martingale by Corollary 19.

Since $f''_\varepsilon(\varepsilon) = \displaystyle\int_0^\varepsilon h_\varepsilon(w)dw = \frac{1}{2}a\varepsilon^2 + b\varepsilon = f''(\varepsilon)$, and $f'''(x) = f'''_\varepsilon(x),\ x \in [\varepsilon, \infty)$, we have

$$f''(x) = f''_\varepsilon(x),\ x \in [\varepsilon, \infty).$$

So we have $\displaystyle\sup_{x\in[0,\varepsilon]} |f''(x) - f''_\varepsilon(x)| \leq M + \frac{1}{2}|a|\varepsilon^2 + |b|\varepsilon \leq 6M$. Since $f'(0) = f'_\varepsilon(0) = 0$, we have

$$\sup_{x\in\mathbf{R}} |f'(x) - f'_\varepsilon(x)| \leq 6M\varepsilon.$$

Therefore

$$\left|\mathbf{E}^{\mathbf{Q}}\left[\left\{(M_t^{[f]} - M_s^{[f]}) - (M_t^{[f_\varepsilon]} - M_s^{[f_\varepsilon]})\right\} G(w(s_1), \cdots, w(s_n))\right]\right|$$

$$= \left| \mathbf{E}^{\mathbf{Q}}\left[\left\{ (f(X_t) - f_\varepsilon(X_t)) - (f(X_s) - f_\varepsilon(X_s)) - \frac{\sigma}{2}\int_s^t 1_{(0,\infty)}(X_u)(f''(X_u) - f''_\varepsilon(X_u))du \right\} G(w(s_1), \cdots, w(s_n)) \right]\right|$$

$$\leq 6M^2\varepsilon\mathbf{E}^{\mathbf{Q}}\left[|X_t| + |X_s|\right] + 3M^2\sigma\mathbf{E}^{\mathbf{Q}}\left[\int_s^t 1_{(0,\varepsilon)}(X_u)du\right].$$

Since we have $\mathbf{E}^{\mathbf{Q}}\left[|X_t| + |X_s|\right] < \infty$ by Proposition 4, we have our assertion by taking $\varepsilon \to 0$. ■

The following is an easy consequence of Propositions 4 and 22.

Corollary 23 *If $f(x) \in C^3$ satisfies $f(0) = f'(0) = 0$, and $\max\{|f(x)|, |f''(x)|\} \le C(1+|x|^2)$, then $M^{[f]}$ is a $\mathcal{G}_t-$martingale.*

Let $g(x) = f(x) - f(0) - f'(0)x$, for $f(x) \in \mathbf{C}_b^3$, then we have $g(0) = g'(0) = 0$, $\max\{|g(x)|, |g''(x)|\} \le C(1+|x|^2)$, and $f(x) = f(0) + f'(0)x + g(x)$. So we have the following by Propositions 21 and 23.

Proposition 24 *For any $f(x) \in C_b^3$, $M^{[f]}$ is a $\mathcal{G}_t-$martingale.*

By Ikeda-Wtanabe [4], p.222, Theorems 7.1 and 7.2, we see that $M_t = X_t - x_0 - \delta \int_0^t 1_{\{0\}}(X_s)ds$ is a $\mathcal{G}_t$-martingale and

$$M_t = \int_0^t 1_{(0,\infty)}(X_s)dM_s, \quad \langle M \rangle_t = \sigma^2 \int_0^t 1_{(0,\infty)}(X_s)ds. \tag{11}$$

By Propositions 18 and 20, we have the following.

Proposition 25 *If $f(x,y) \in \mathbf{C}_b^3(\mathbf{R}^2;\mathbf{R})$ satisfies $f(x,y) = 0$, $(x,y) \in [0,\varepsilon] \times \mathbf{R}$, then $f(X_t, W_t) - \dfrac{\sigma^2}{2} \displaystyle\int_0^t \left(\frac{\partial^2}{\partial x^2} + 2\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2}\right) f(X_s, W_s)ds$ is a $\mathcal{G}_t-$martingale.*

Also, we have the following.

Proposition 26

$$M_t = \sigma \int_0^t 1_{(0,\infty)}(X_s)dW_s.$$

Proof. Let $N_t = M_t - \sigma \int_0^t 1_{(0,\infty)}(X_s)dW_s$. Then by Equation (11), we have

$$\langle N \rangle_t = \left\langle \int_0^\cdot 1_{(0,\infty)}(X_s)(dM_s - \sigma W_s) \right\rangle_t$$

$$= \int_0^t 1_{(0,\infty)}(X_s)(d\langle M\rangle_s - 2\sigma d\langle M, W\rangle_s + \sigma^2 d\langle W\rangle_s)$$

$$= 2\sigma \int_0^t 1_{(0,\infty)}(X_s)(\sigma ds - d\langle M, W\rangle_s).$$

Let $g \in \mathbf{C}_b^3(\mathbf{R}^2; \mathbf{R})$ such that $g(x, y) = 0,\ |x| < \varepsilon$, for some $\varepsilon > 0$. Then

$$g(X_t, S_t) - g(x, 0) - \frac{\sigma^2}{2} \int_0^t \left(\frac{\partial^2}{\partial x^2} + 2\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2} \right) g(X_s, W_s) ds$$

is a martingale by Proposition 18. On the other hand,

$$g(X_t, S_t) - g(x, 0) - \frac{1}{2} \int_0^t \frac{\partial^2}{\partial x^2} g(X_s, W_s) d\langle M\rangle_s + \int_0^t \frac{\partial^2}{\partial x \partial y} g(X_s, W_s) d\langle M, W\rangle_s$$

$$+ \frac{1}{2} \int_0^t \frac{\partial^2}{\partial y^2} g(X_s, W_s) d\langle W\rangle_s$$

is a local martingale by Ito's formula. Since $d\langle M\rangle_t = \sigma^2 1_{(0,\infty)}(X_t) dt$ by Equation (11) and $d\langle W\rangle_t = dt$,

$$A_t^{[g]} = \int_0^t \frac{\partial^2}{\partial x \partial y} g(X_s, W_s)(\sigma ds - d\langle M, W\rangle_s),$$

is a martingale. Furthermore the total variation of $\{A_t^{[g]}\}_{t\geq 0}$ is bounded. So we have $A_t^{[g]} = 0$. Let $h \in \mathbf{C}_b^3(\mathbf{R}^2; \mathbf{R})$ such that

$$h(x, y) = \begin{cases} xy, & x \in [2n^{-1}, n], y \in [-n, n], \\ 0, & x \in [0, n^{-1}]. \end{cases}$$

Then we have

$$\int_0^t 1_{(2n^{-1},n)}(X_s) 1_{(-n,n)}(W_s)(\sigma ds - d\langle M, W\rangle_s) = \int_0^t 1_{(2n^{-1},n)}(X_s) 1_{(-n,n)}(W_s) dA_s^{[h]} = 0.$$

Letting $n \to \infty$, we have

$$\int_0^t 1_{(0,\infty)}(X_s)(\sigma ds - d\langle M, W\rangle_s) = 0.$$

Therefore we have $N_t = 0$. This implies our assertion. ■

$\{W_t\}_{t\geq 0}$ is a Brownian Motion under $\mathbf{Q}$. By Proposition 26, $\{X_t\}$ satisfies

$$X_t = x_0 + \sigma \int_0^t 1_{(0,\infty)}(X_s) dW_s + \delta \int_0^t 1_{\{0\}}(X_s) ds,$$

and

$$X_t \geq 0, \quad t \geq 0, \quad a.s.$$

Suppose that $\{\lambda_m\}_{m\in\mathbf{N}}$ is a sequence such that $\lambda_m \downarrow 0$ and $\mathbf{Q}^{\lambda_m}$ converges weakly. Since the uniqueness of the strong solution of this Stochastic Differential Equation is guaranteed by Ikeda-Wtanabe [4], p.222, Theorems 7.1 and 7.2, $\mathbf{Q}^{\lambda}$ converges weakly to $\mathbf{Q}$, as $\lambda \downarrow 0$, by Corollary 5. This implies Theorem 1.

6 Example

In this section, we give some examples satisfying Assumptions (A.1)-(A.2) and (B.1)-(B.3), and we calculate δ.

Example 1 *Let μ_W be a probability distribution on $\mathbf{R}$, μ_{Z+} be a probability distribution on $(0,\infty)$. We assume that $W_n, Z_n,\ n = 1,2,3,\cdots$, are independent, $\{W_n\}_{n=0}^{\infty}$ have the same distribution μ_W, and $\{Z_n\}_{n=0}^{\infty}$ have the same distribution μ_{Z+}. If $W_n^{\lambda} = \lambda W_n,\ Z_n^{\lambda} = \lambda Z_n$, and $p^{\lambda} = p\lambda$. Then we have*

$$\delta = p\int_{(0,\infty)} \left\{x + \mathbf{E}[(-S^1_{\tau^1(x)}(x))]\right\} \mu_{Z+}(dx).$$

Example 2 *We assume that constants $\gamma > 0, C > 0$ satisfy $C < \gamma$, and that*

$$\mathbf{P}(W_n^{\lambda} < -x) = Ce^{-\gamma x}dx,\ x > 0.$$

Then we have

$$\delta = p(m_{Z+} + \gamma^{-1}),$$

because we have $\overline{\lim_{\eta\to 0}}\ \overline{\lim_{\lambda\to 0}}\mathbf{E}[c(\lambda,\eta)^{-\tau_1^{\lambda}(x)}(-\lambda^{-1}S^{\lambda}_{\tau_1^{\lambda}(x)}(x))] = \gamma^{-1}$ by the following equation

$$\frac{\displaystyle\int_{-\infty}^{-y}(-x-y)\mu_W^{\lambda}(dx)}{\displaystyle\int_{-\infty}^{-y}\mu_W^{\lambda}(dx)} = \frac{\displaystyle\int_{-\infty}^{0}(-x)\mu_W^{\lambda}(dx)}{\displaystyle\int_{-\infty}^{0}\mu_W^{\lambda}(dx)} = \frac{1}{\gamma}, \quad y > 0.$$

Example 3 *Suppose that $\mathbf{P}(\lambda^{-1}W_n^{\lambda} \in \mathbf{Z}) = 1$, and $\mathbf{P}(W_n^{\lambda} = -\lambda | W_n^{\lambda} < 0) = 1$. Since $\lambda^{-1}S^{\lambda}_{\tau_1^{\lambda}(x)} = -|[-x]| + x$, we see that*

$$\delta = p\int_{[0,\infty)} |[-x]|\mu_{Z+}^{\lambda}(dx).$$

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