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curves into algebraic varieties**

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Degeneracy of Holomorphic Curves into Algebraic Varieties *

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Abstract

Applying the Second Main Theorem of [20], we deal with the algebraic degeneracy of entire holomorphic curves $f : \mathbf{C} \rightarrow X$ from the complex plane \mathbf{C} into a complex algebraic normal variety X of positive log Kodaira dimension that admits a finite proper morphism to a semi-abelian variety. We will also discuss applications to the Kobayashi hyperbolicity problem.

1 Introduction and main result

Let $f : \mathbf{C} \rightarrow X$ be an entire holomorphic curve into a complex algebraic variety X which may be non-compact in general. We say that f is *algebraically degenerate* (resp. *algebraically nondegenerate*) if there is a (resp. no) proper algebraic subset of X containing the image $f(\mathbf{C})$. If X is compact, we denote by $\kappa(X)$ (resp. $q(X)$) the Kodaira dimension (resp. irregularity) of X ; in general, we write $\bar{\kappa}(X)$ (resp. $\bar{q}(X)$) for the log Kodaira dimension (resp. log irregularity) of X .

If $\bar{q}(X) > \dim X$, then every holomorphic curve $f : \mathbf{C} \rightarrow X$ is algebraically degenerate by log Bloch-Ochiai's Theorem (cf. Theorem 2.11). Here we would like to discuss the following problem:

Problem. If $\bar{q}(X) = \dim X$ and $\bar{\kappa}(X) > 0$, is every holomorphic curve $f : \mathbf{C} \rightarrow X$ algebraically degenerate?

With $\bar{q}(X) = \dim X$ the condition of $\bar{\kappa}(X) > 0$ prohibits X to be a semi-abelian variety, for which the Problem clearly fails to hold. By log Bloch-Ochiai's Theorem it is readily reduced to the case where the quasi-Albanese map $\pi : X \rightarrow A$ is dominant. If π is not proper, there is a counter-example to the Problem due to Dethloff-Lu [2] Proposition 7 a)

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such that $\dim X = \bar{q}(X) = 2$ and $\bar{\kappa}(X) = 1$. Thus it is necessary to assume that π is proper. In fact it is also sufficient:

Main Theorem *Let X be a complex algebraic variety and let $\pi : X \rightarrow A$ be a finite morphism onto a semi-abelian variety A . Let $f : \mathbf{C} \rightarrow X$ be an arbitrary entire holomorphic curve. If $\bar{\kappa}(X) > 0$, then f is algebraically degenerate.*

Moreover, the normalization of the Zariski closure of $f(\mathbf{C})$ is a semi-abelian variety which is a finite étale cover of a translate of a proper semi-abelian subvariety of A .

Corollary 1.1 *Let X be a complex algebraic variety whose quasi-Albanese map is a proper map. Assume that $\bar{\kappa}(X) > 0$ and $\bar{q}(X) \geq \dim X$. Then every entire holomorphic curve $f : \mathbf{C} \rightarrow X$ is algebraically degenerate.*

We will discuss in §4 applications for the Kobayashi hyperbolicity problem and the complements of divisors on $\mathbf{P}^n(\mathbf{C})$ in relation with those results which were obtained by C. Grant [5], Grauert [6], Dethloff-Schumacher-Wong [4], [3], Dethloff-Lu [2] §7, and [20].

In the surface case we provide in §6 a result towards the strong Green-Griffiths conjecture.

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2 Notation and preparation

In the present paper algebraic varieties and morphisms between them are defined over \mathbf{C} .

For a moment let X be a compact complex reduced space. We recall some notation in the value distribution theory of holomorphic curves (cf. [14], [19], [20]). Let $f : \mathbf{C} \rightarrow X$ be an entire holomorphic curve into X . Fixing a hermitian metric form ω on X , we define the order function $T_f(r)$ by

$$T_f(r) = T_f(r; \omega) = \int_1^r \frac{dt}{t} \int_{|z| < t} f^* \omega \quad (1 \leq r < \infty).$$

Let V be a complex subspace of X which may be reducible or non-reduced. As in [20] §2 we have

- (i) the proximity function $m_f(r; V)$ via a Weil function for V ,
- (ii) the curvature form $\omega_{V,f}$ along $f : \mathbf{C} \rightarrow X$, and the order function $T(r; \omega_{V,f})$ with respect to it,

(iii) the counting function $N(r; f^*V)$ and the truncated counting function $N_l(r; f^*V)$ to level l .

The following F.M.T. (First Main Theorem) holds (cf. [20] Theorem 2.9):

$$(2.1) \quad T(r; \omega_{V,f}) = N(r; f^*V) + m_f(r; V) + O(1),$$

provided that $f(\mathbf{C}) \not\subseteq \text{Supp } V$.

When V is a Cartier divisor D on X , following the notation of [14], we have the order function $T_f(r; L(D))$ with respect to the line bundle $L(D)$ determined by D and the proximity function $m_f(r, D)$ defined by a hermitian metric in $L(D)$. Then we have (cf. [20] §2)

$$\begin{aligned} T(r; \omega_{D,f}) &= T_f(r; L(D)) + O(1), \\ m_f(r; D) &= m_f(r, D) + O(1). \end{aligned}$$

If D is a big divisor on X and $f : \mathbf{C} \rightarrow X$ is algebraically nondegenerate, there is a constant $C > 0$ such that

$$(2.2) \quad C^{-1}T_f(r) \leq T_f(r; L(D)) \leq CT_f(r).$$

In this case we write

$$T_f(r) \sim T_f(r; L(D)).$$

Let $\lambda : X \rightarrow Y$ be a dominant rational mapping between compact algebraic varieties X and Y . Let $f : \mathbf{C} \rightarrow X$ be algebraically nondegenerate. Then there is a constant $C' > 0$ such that

$$(2.3) \quad T_{\lambda \circ f}(r) \leq C'T_f(r).$$

Furthermore, if $\dim X = \dim Y$, then (cf. [14] Lemma (6.1.5))

$$(2.4) \quad T_{\lambda \circ f}(r) \sim T_f(r)$$

Suppose that $\lambda : X \rightarrow Y$ is a morphism into a compact complex space Y , W is a complex subspace of Y and $V \subseteq \lambda^*W$ scheme-theoretically (i.e. the ideal sheaf defining λ^*W should be a subsheaf of the ideal sheaf defining V). Then by [20] Theorem 2.9

$$(2.5) \quad m_f(r; V) \leq m_{\lambda \circ f}(r; W)$$

for a holomorphic curve $f : \mathbf{C} \rightarrow X$.

We recall the Main Theorem of [20] in a form that we will use.

Theorem 2.6 ([20]) *Let A be a semi-abelian variety and let D be a reduced effective divisor on A . Then there exists an equivariant smooth compactification \bar{A} of A such that the following inequality holds for every algebraically nondegenerate holomorphic curve $f : \mathbf{C} \rightarrow A$*

$$(2.7) \quad m_f(r; \bar{D}) = S_f(r; L(\bar{D})),$$

$$(2.8) \quad T_f(r; L(\bar{D})) \leq N_1(r; f^*D) + \epsilon T_f(r; L(\bar{D}))|_{\epsilon}, \quad \forall \epsilon > 0.$$

Here as usual in the Nevanlinna theory, we use the notation:

(i) $\log^+ t = \log \max\{t, 1\}$ ($t \geq 0$),

(ii) $S_f(r; L(\bar{D})) = O(\log^+ r) + O(\log^+ T_f(r; L(\bar{D})))|_{\epsilon}$,

(iii) “ $|_{\epsilon}$ ” (resp. “ $|_{\epsilon}$ ”) stands for the inequality holds for $r > 1$ outside a Borel subset of finite Lebesgue measure (resp., where the Borel subset depends on $\epsilon > 0$).

Theorem 2.9 ([20]) *Let A be a semi-abelian variety and let V be an algebraic subvariety of A . Assume that $\text{codim}_A V \geq 2$. Then for every algebraically nondegenerate holomorphic curve $f : \mathbf{C} \rightarrow A$*

$$(2.10) \quad N(r; f^*V) \leq \epsilon T_f(r)|_{\epsilon}, \quad \forall \epsilon > 0.$$

The next is called log Bloch-Ochiai’s Theorem, which we will use in a reduction.

Theorem 2.11 ([11], [12], [15]) (i) *Let X be a connected compact Kähler manifold of dimension n and let U be a complement of a proper analytic subset of X . Assume that the log irregularity $\bar{q}(U) > n$. Then the image of an arbitrary holomorphic curve $f : \mathbf{C} \rightarrow U$ is contained in a proper analytic subset of X .*

(ii) *In particular, if U is a semi-abelian variety (resp. quasi-torus), the Zariski closure of the image $f(\mathbf{C})$ in X restricted to U is a translation of a semi-abelian subvariety (resp. quasi-subtorus) of U .*

We need the following result due to Kawamata.

Theorem 2.12 ([9] Theorem 27) *Let X be a normal algebraic variety. Let $\pi : X \rightarrow A$ be a finite morphism. Then $\bar{\kappa}(X) \geq 0$ and there are a semi-abelian subvariety B of A , finite étale Galois covers $\tilde{X} \rightarrow X$ and $\tilde{B} \rightarrow B$, and a normal algebraic variety Y such that*

(i) *there is a finite morphism from Y to the quotient A/B ,*

- (ii) \tilde{X} is a fiber bundle over Y with fiber \tilde{B} and with translations by \tilde{B} as structure group,
- (iii) $\bar{\kappa}(Y) = \dim Y = \bar{\kappa}(X)$.

In the special case where $\bar{\kappa}(X) = 0$ this result takes the following form:

Theorem 2.13 ([9] Theorem 26) *Let X be a normal algebraic variety, let A be a semi-abelian variety and let $\pi : X \rightarrow A$ be a surjective finite morphism. If $\bar{\kappa}(X) = 0$, then X is a semi-abelian variety and π is étale.*

3 The Compactification

Let X be a normal algebraic variety, let A be a semi-abelian variety and let $\pi : X \rightarrow A$ be a finite morphism. We need a good compactification.

Before constructing such a compactification we remark that given a finite morphism from a *normal* variety X to a smooth variety A we may desingularize X in order to get a generically finite morphism between smooth varieties. Conversely, given a generically finite morphism $p : \tilde{X} \rightarrow A$ between smooth varieties, the Stein factorization gives us a normal variety X together with a finite morphism from X to A and a proper connected morphism from \tilde{X} to X which is a desingularization for X .

3.1 Some toric geometry

Lemma 3.1 *Let A be a semi-abelian variety acting on a (possibly singular) projective variety X with an open orbit. Then there are only finitely many A -orbits in X .*

Proof. Let $\tilde{X} \rightarrow X$ be an equivariant desingularization. Then the number of A -orbits in \tilde{X} is finite ([20], Lemma 3.12). It follows that the number of A -orbits in X is finite as well. *Q.E.D.*

From toric geometry we know (see [21], §3):

Lemma 3.2 *Let Y be a normal toric variety with algebraic torus T . Then every point in Y admits a Zariski open invariant affine neighbourhood W .*

Lemma 3.3 *Let W be a Stein complex space on which a reductive complex Lie group H acts. Let V be a complex vector space, let $Z \subset W$ be an invariant closed analytic subset and let $f_0 : Z \rightarrow V$ be an H -invariant holomorphic map. Then f_0 extends to an H -invariant holomorphic map $f : W \rightarrow V$.*

Proof. Let $f_1 : W \rightarrow V$ be an arbitrary (i.e. not necessarily invariant) holomorphic extension (which exists, because W is Stein). Let K be a maximal compact subgroup of T with normalized Haar measure $d\mu$. Then we can define f by

$$f(x) = \int_K f_1(k \cdot x) d\mu(k)$$

Q.E.D.

Lemma 3.4 *Let W be an irreducible affine variety on which a reductive commutative complex Lie group $T = (\mathbf{C}^*)^g$ acts with finitely many orbits. Let S_i , $i = 1, 2$ be two T -orbits. Then either $S_1 = S_2$ or the isotropy Lie algebras differ.*

(Note that two points p, q in the same T -orbit S have the same isotropy group, because T is commutative. Therefore it makes sense to talk about the “isotropy Lie algebra of an orbit”.)

Proof. Let S_1, S_2 be two orbits with the same isotropy Lie algebra. Note that this implies $\dim(S_1) = \dim(S_2)$. Let Y denote the closure of $S_1 \cup S_2$ in W . Let H denote the connected component of the isotropy group for S_1 . By our assumption H acts trivially on Y . Now we choose a holomorphic embedding $i : Y \hookrightarrow \mathbf{C}^N$. Since H acts trivially on Y , this map i is H -invariant. Thus we can extend it to an H -invariant holomorphic map $F : W \rightarrow \mathbf{C}^N$ (using Lemma 3.3). We observe that

$$\dim(F(Y)) = \dim(i(S_1)) = \dim T - \dim H.$$

Since F is H -invariant,

$$\dim F(\Omega) \leq \dim T - \dim H$$

for every T -orbit $\Omega \subset W$. Recall that we assumed that the number of T -orbits is finite. Therefore the above inequality implies $\dim F(W) \leq \dim T - \dim H$. It follows that $\dim F(Y) = \dim F(W)$. Since W is irreducible and $Y \subset W$, it follows that $F(Y) = i(Y) \simeq Y$ is irreducible. Therefore $S_1 = S_2$. *Q.E.D.*

Lemma 3.5 *Let A be a semi-abelian variety acting effectively on an algebraic variety X . Let L be the maximal connected linear subgroup of A and $x \in X$. Then L contains the connected component of the isotropy group $A_x = \{a \in A : a \cdot x = x\}$.*

Proof. Let \mathcal{O}_x be the local ring at x and let m_x be its maximal ideal. Then A_x acts linearly on each vector space \mathcal{O}_x/m_x^k . Because the A -action is supposed to be effective, these actions can not be all trivial. Hence A_x is linear and its connected component is contained in L . *Q.E.D.*

Lemma 3.6 *Let A be a semi-abelian variety acting on a normal algebraic variety X with an open orbit. Let $\pi : \tilde{X} \rightarrow X$ be an equivariant desingularization and $a : \tilde{X} \rightarrow T$ the Albanese map. Then there exists an equivariant morphism $f : X \rightarrow T$ such that $a = f \circ \pi$.*

Proof. First we recall that there are only finitely many A -orbits in X and \tilde{X} (Lemma 3.1). Let L be the maximal linear connected subgroup of A . Let $p \in X$, let $\Omega = A(p)$ and let $\tilde{\Omega}$ be an irreducible component of $\pi^{-1}(\Omega)$. Since $\tilde{\Omega}$ is invariant and irreducible and contains only finitely many A -orbits, it contains a dense A -orbit. Furthermore observe that the fibers of π are irreducible, because X is normal. As a consequence we obtain that the fiber $\pi^{-1}(p)$ equals the closure of an orbit of the connected component of the isotropy group A_p^0 acting on $\tilde{\Omega}$. Since $A_p^0 \subset L$ (see Lemma 3.5), we see that all fibers of π are contained in closures of L -orbits in \tilde{X} . Being linear, L acts trivially on the abelian variety T . Therefore $a : \tilde{X} \rightarrow T$ is constant along the fibers of π . Since X is normal this implies that a fibers through π . *Q.E.D.*

Proposition 3.7 *Let A be a semi-abelian variety acting on a normal algebraic variety X with an open orbit. Then every point $x \in X$ admits a Zariski open neighbourhood W of x in X such that the following property holds:*

“Two points $y, z \in W$ are contained in the same A -orbit if and only if they have the same isotropy Lie algebra with respect to the A -action.”

Proof. We may assume that the A -action on X is effective. Let L be the maximal connected linear subgroup of A and $T = A/L$. Due to Lemma 3.6 there is a surjective equivariant morphism $\alpha : X \rightarrow T$ (which is the Albanese of a desingularization of X).

Now A -invariant subsets of X correspond to L -invariant subsets of a fiber F of this morphism $\alpha : X \rightarrow T$. Moreover, due to Lemma 3.5 every isotropy Lie algebra for the A -action is contained in the Lie algebra $Lie(L)$ and therefore coincides with the isotropy Lie algebra for the L -action. Therefore there is no loss in generality in assuming that $L = A$. Then X is a toric variety and every point $x \in X$ admits an invariant affine neighbourhood (Lemma 3.2). Due to Lemma 3.1 there are only finitely many orbits. Hence the statement follows from Lemma 3.4. *Q.E.D.*

Lemma 3.8 *Let G be a commutative algebraic group acting with a dense connected open orbit Ω on a variety X and acting transitively on a variety Y . Let $\pi : X \rightarrow Y$ be a surjective equivariant morphism. Then for every $y \in Y$ and every G -orbit $Z \subset X$ the intersection with the fiber $Z \cap \pi^{-1}(y)$ is connected.*

Proof. Fix $y \in Y$ and define $F = \pi^{-1}(y)$. Let H denote the isotropy group $H = \{g \in G : g(y) = y\}$. There is a one-to-one correspondence between G -orbits in X and H -orbits in F . $W = \Omega \cap F$ is a dense open H -orbit in F . Choose $p \in W$ and let $I = \{h \in H : h(p) = p\}$. Because H is commutative, the group I acts trivially on W . Since W is dense in F , the action of I on F is trivial. Let H^0 denote the connected component of H . Then $H^0 \cdot I = H$, because $W \simeq H/I$ is connected. Thus the fact that I acts trivially on F implies that the H -orbits on F coincide with the H^0 -orbits. In particular they are connected. Thus $Z \cap F$ which is an H -orbit must be connected. *Q.E.D.*

Lemma 3.9 *Let $\pi : X \rightarrow Y$ be a finite morphism from a normal variety X with singular locus S onto a smooth variety A . Let R denote the ramification divisor of the restriction of π to $X \setminus S$. Then S is contained in the closure of R .*

Proof. Let $p \in X \setminus \bar{R}$. Then there are small connected open Stein neighbourhoods V of p in $X \setminus \bar{R}$ and W of $\pi(p)$ in A such that π restricts to a finite morphism from V to W . We may assume that W is simply-connected. Because X is normal, the codimension of $\pi(S) \cap W$ is at least two. Hence $W \setminus \pi(S)$ is simply-connected. It follows that there is a section $\sigma : W \setminus \pi(S) \rightarrow V$ which (again because of $\text{codim} \pi(S) \geq 2$) extends to all of W . This yields a biholomorphic map between V and W . Since $W \subset A$ is smooth, we deduce $p \notin S$. *Q.E.D.*

3.2 Simple compactification

Proposition 3.10 *Let $\pi : X \rightarrow A$ be a finite surjective morphism between normal varieties and let $A \hookrightarrow \bar{A}$ be a normal compactification of A . Then there exists a unique normal compactification $X \hookrightarrow \bar{X}$ such that \bar{X} is normal and π extends to a finite morphism $\bar{\pi} : \bar{X} \rightarrow \bar{A}$.*

Proof. Let $\Gamma \subset X \times A$ be the graph of π . Choose a compactification $X \hookrightarrow \tilde{X}$ and let $\hat{\Gamma}$ be the closure of Γ in $\bar{A} \times \tilde{X}$. Then \bar{X} is obtained by first normalizing $\hat{\Gamma}$ and then taking the Stein factorization of the projection onto \bar{A} . It is easy to deduce unicity from the assumption of X being normal. *Q.E.D.*

3.3 A better compactification

Proposition 3.11 *Let π be a finite morphism from a normal algebraic variety X to a semi-abelian variety A . Let R denote the set of all non-singular points $p \in X$ at which*

π is ramified. Let $A \hookrightarrow \bar{A}$ be a smooth equivariant compactification and let ω be a log volume form on \bar{A} . Then there exists a compactification $X \hookrightarrow \bar{X}$ and a proper morphism $\bar{\pi} : \bar{X} \rightarrow \bar{A}$ such that:

- (i) $\bar{X} \setminus \bar{R}$ is smooth and $\bar{X} \setminus (X \cup \bar{R})$ is an s.n.c. (=“simple normal crossing”) divisor (where \bar{R} denotes the closure of R in \bar{X}),
- (ii) $\bar{\pi}|_X = \pi$,
- (iii) Let $\tilde{\omega} = \bar{\pi}^* \omega \in \Omega^d(\bar{X}; \log \partial X)$. Then $\tilde{\omega}$ has poles along all divisorial components of $\bar{X} \setminus (X \cup \bar{R})$.

Proof. Due to Lemma 3.9 the singular locus S of X is contained in the closure of R . We let D be a divisor on A containing $\pi(R)$. Then D contains $\pi(S)$, too.

(0) To prove the assertion we use the following strategy:

- We define a class of “admissible compactifications” of X .
- We show, using Lemma 3.10, that there exists an admissible compactification.
- For each admissible compactification we define an “indicator function” $\zeta : \partial A \rightarrow \mathbf{N}$ which measure the presence of singularities outside of \bar{R} .
- Using the theory of toroidal embeddings, we show that we can blow up admissible compactification in such a way that we stay inside the category of admissible compactification, but decrease the indicator function.
- We verify that after finitely many steps the indicator function vanishes and that then we have found a compactification as desired.

(1) A compactification $X \hookrightarrow X'$ is “admissible” if the following properties are satisfied:

- (i) X' is normal,
- (ii) the projection map $\pi : X \rightarrow A$ extends to a proper holomorphic map $\pi' : X' \rightarrow \bar{A}$ with $(\pi')^{-1}(A) = X$.
- (iii) For each point $p \in X' \setminus \bar{R}$ there is an open neighbourhood U of $\pi(p) = q$ in \bar{A} such that the connected component Ω of $(\pi')^{-1}(U) \setminus \bar{R}$ which contains p admits a biholomorphic map $\psi : \Omega \rightarrow W$ into an open subset W of a toric variety Z .

(iv) Let G denote the algebraic torus $(\mathbf{C}^*)^g$ acting on the toric variety Z . Then map $\pi \circ \psi^{-1} : W \rightarrow \bar{A}$ is (locally) equivariant for some holomorphic Lie group homomorphism with discrete fibers from G to A .

Condition (iii) could be rephrased by saying that $X \hookrightarrow X'$ should be locally a “toroidal embedding” in the sense of [10] except at \bar{R} .

(2) Using Lemma 3.10, we obtain a normal compactification $X \hookrightarrow X_1$ such that $\pi : X \rightarrow A$ extends to a finite morphism $\pi_0 : X_1 \rightarrow \bar{A}$. Let $p \in X' \setminus \bar{R}$, $q = (\pi')(p)$. Let U be an open neighbourhood of p in $X' \setminus \bar{R}$ and define $\Omega = (\pi')(U)$. Since ∂A is an s.n.c. divisor, we can shrink these open neighborhoods to obtain local coordinates z_i on Ω such that $z_i(q) = 0$ and such that ∂A is the zero locus of $z_1 \cdots z_k$ for some k . By shrinking Ω we may assume that Ω is biholomorphic to a polydisc with $\Omega \cap A \simeq (\Delta^*)^k \times \Delta^{n-k}$. Let $i : \Delta^* \rightarrow \mathbf{C}^*$ be the standard injection and $j : \Delta \rightarrow \mathbf{C}^*$ any open embedding (e.g. $j(z) = z + 2$). We obtain $\xi = (i^k, j^{n-k}) : \Omega \cap A \hookrightarrow (\mathbf{C}^*)^n = G$. Now $\pi_1(G) \simeq \mathbf{Z}^n$ and ξ induces an embedding of $\pi_1(\Omega \cap A) \simeq \mathbf{Z}^k$ into $\mathbf{Z}^n \simeq \pi_1(G) \simeq \mathbf{Z}^n$ as $\xi_*(\pi_1(\Omega \cap A)) = \mathbf{Z}^k \times \{0\}^{n-k}$. Thus for each subgroup Γ of fixed finite index d of $\pi_1(\Omega \cap A)$ we can choose a corresponding subgroup of the same finite index in $\pi_1(G) \simeq \mathbf{Z}^n$, namely $\xi_*(\Gamma) \times \mathbf{Z}^{n-k}$. Therefore the unramified covering $\pi^{-1}(\Omega \cap A) \rightarrow \Omega \cap A$ extends via ξ to an unramified covering $G_1 \rightarrow G$ of G . Now G_1 is again an algebraic group and as algebraic group isomorphic to $(\mathbf{C}^*)^n$. Since the procedure of normal compactification as in Proposition 3.10 is canonical, we can embed $\pi^{-1}(\Omega) \rightarrow \Omega$ into a finite morphism $\bar{G}_1 \rightarrow \bar{G} = (\mathbf{P}_1)^n$. Now \bar{G}_1 is a toric variety since the G_1 action on itself extends to the boundary $\partial G_1 \simeq \partial G$ via $G_1 \rightarrow G$.

This proves that $X \hookrightarrow X_1$ is admissible.

(3) We are looking for admissible compactifications which are smooth outside the closure of R . Thus given an admissible compactification $X \hookrightarrow \bar{X}$ with $\pi : \bar{X} \rightarrow \bar{A}$ we define our indicator function $\zeta : \partial A \rightarrow \mathbf{N}$ as follows: $\zeta(p)$ denotes the number of connected components of the fiber $\pi^{-1}(p)$ which intersect $\text{Sing}(\bar{X}) \setminus \bar{R}$. Evidently this function ζ vanishes iff $\bar{X} \setminus \bar{R}$ is smooth.

(4) We suppose given an admissible compactification \bar{X} with indicator function ζ . Since the level sets $\{z \in \partial A : \zeta(z) = c\}$ ($c \in \mathbf{N}$) of ζ are constructible sets, it makes sense to define $d_\zeta = \dim\{z : \zeta(z) \neq 0\}$ and to define a number n_ζ which is the generic value of ζ on $\{z : \zeta(z) \neq 0\}$. We choose a generic point $p \in \{z \in \partial A : \zeta(z) \neq 0\}$ and a point $q \in \pi^{-1}(p) \cap \text{Sing}(\bar{X}) \setminus \bar{R}$. Since the compactification is admissible, there is an isomorphism $\phi : U \rightarrow W$ where U is a neighbourhood of the connected component of $\pi^{-1}(p)$ containing q and W is an open neighbourhood of p in a toric variety Z . By the theory of toroidal embeddings (see e.g. [10]) there exists an equivariant desingularization

$\tilde{Z} \rightarrow Z$, and thus a desingularization of U . The problem is to extend this blow-up of U to a blow-up of \bar{X} . Because the blow-up of Z is equivariant and the number of G -orbits in Z is finite, it is given by a blow-up of invariant strata (=closures of G -orbits in Z). Now \bar{X} admits a natural stratification induced by the A -action on \bar{A} . In fact, there is a local A -action on $\bar{X} \setminus \bar{R}$ which gives this stratification. In order to extend the blow-up, we need to extend its center, and this means: Given a closed invariant subvariety $Q \subset Z$ we need to show that either $M \cap U = \phi^{-1}(Q)$ or $M \cap U = \emptyset$ for each stratum M of \bar{X} .¹ We prove this indirectly. So let us assume that this property fails. Then $\phi(M \cap U)$ intersects several G -orbits some of which are contained in Q and some of which are not. Since M is one stratum, $\pi(M)$ is one A -orbit in \bar{A} which implies that for each point in M we have the same isotropy Lie algebra. If we have chosen U sufficiently small, we can conclude from Proposition 3.7, that $\pi(M \cap U)$ is connected. Let $N = \pi^{-1}(\pi(M))$. Using Lemma 3.8 we may deduce that $M \cap U$ is connected, as desired.

(5) We recall that we introduced an indicator function ζ with associated numbers d_ζ and n_ζ . For a generic point $p \in \text{Sing}(\bar{X}) \setminus \bar{R}$ the above considerations show that there is an appropriate blow-up yielding an other admissible compactification which is smooth around (the preimage of) p . By the definition of d_ζ and n_ζ this means that given an admissible compactification we can always blow-up \bar{X} so that either d_ζ decreases or n_ζ decreases while d_ζ is kept fixed. Thus we can strictly decrease the value of $(d_\zeta, n_\zeta) \in \mathbf{N}^2$ where \mathbf{N}^2 is endowed with the lexicographic order. It follows that $(d_\zeta, n_\zeta) = (0, 0)$ after finitely many steps. But $(d_\zeta, n_\zeta) = (0, 0)$ implies the vanishing of $\zeta : \partial A \rightarrow \mathbf{N}$ which in turn implies that \bar{X} is smooth outside \bar{R} .

Thus we have established: There exists an admissible compactification $X \hookrightarrow X'$ such that X' is smooth outside the closure of R .

(6) By the definition of admissibility $X' \setminus \bar{R}$ is locally a toroidal embedding. This implies that $\partial X' \setminus \bar{R}$ is an s.n.c. divisor in $X' \setminus \bar{R}$. Furthermore: The logarithmic tangent bundle on a smooth toric variety is trivial. Hence conditions (iii) and (iv) of “admissibility” imply the statement about the poles of $\tilde{\omega}$. *Q.E.D.*

3.4 The best compactification

Theorem 3.12 *Let $\pi : X \rightarrow A$ be a finite morphism from a normal variety X onto a semi-abelian variety A . Let \bar{A} be a smooth equivariant compactification of A . Let D denote the critical locus, i.e. the closure of the set of all $\pi(z)$ where z is a smooth point*

¹In the language of [10], here we prove that the embedding of $X \setminus R$ into $\bar{X} \setminus \bar{R}$ is a toroidal embedding “without self-intersection”.

and $d\pi : T_z X \rightarrow T_{\pi(z)} A$ fails to have full rank. Then there exist

- (a) A desingularization $\tau : \tilde{X} \rightarrow X$ and a smooth compactification $j : \tilde{X} \hookrightarrow \hat{X}$ such that the boundary divisor $\partial\hat{X} = \hat{X} \setminus j(\tilde{X})$ has only simple normal crossings,
- (b) a proper holomorphic map $\psi : \hat{X} \rightarrow \bar{A}$ such that $\psi \circ j = \pi \circ \phi$ with $\psi^{-1}(A) = \tilde{X}$.
- (c) an effective divisor Θ on \hat{X} ,
- (d) a subvariety $\hat{S} \subset \hat{X}$

such that

- (i) Θ is linearly equivalent to the log canonical divisor $K_{\hat{X}} + \partial\hat{X}$ of \tilde{X} ,
- (ii) $\psi(\text{Supp } \Theta) \subset \bar{D}$,
- (iii) the image $\psi(\hat{S})$ has at least codimension two in A ,
- (iv) for every holomorphic curve $f : \Delta \rightarrow A$ from a disk in \mathbf{C} with lifting $F : \Delta \rightarrow \tilde{X}$ and for $z \in F^{-1}(\text{Supp } \Theta \setminus \hat{S})$ we have the following inequality of multiplicities:

$$(3.13) \quad \text{mult}_z F^* \Theta \leq \text{mult}_z f^* D - 1.$$

Proof. Let $\text{Sing}(X)$ be the singular locus of X and let R be the ramification divisor of π restricted to $X \setminus \text{Sing}(X)$. We apply Proposition 3.11 and obtain a first compactification X' of X and an extension $\pi' : X' \rightarrow \bar{A}$ of $\pi : X \rightarrow A$. We recall that $X' \setminus \bar{R}$ is smooth (assertion (i) of Proposition 3.11). Thus we can use Hironaka desingularization to desingularize X' without changing $X' \setminus \bar{R}$. We obtain a desingularization $\tau : \hat{X} \rightarrow X'$ which restricts to a desingularization $\tau_0 : \tilde{X} \rightarrow X$.

Set

$$\Theta = (\psi^* \omega) + \psi^{-1} \partial A.$$

Then the properties (i) and (ii) are satisfied.

Let S_1 denote the image of the set of singular points of X by π , let S_2 be the set of singularities of D , and let S_3 be the image of the set of singularities of $\text{Supp } \Theta|_{\tilde{X}}$ by ψ . Then $\text{codim } S_j \geq 2$ ($1 \leq j \leq 3$). Set $S = S_1 \cup S_2 \cup S_3$ and $\hat{S} = \psi^{-1}(\bar{S})$. Then $\text{codim } \hat{S} \geq 2$ and (iii) is satisfied.

Let $p \in \text{Supp } \Theta|_{\hat{X}} \setminus \hat{S}$. Then $q = \pi(p) \in D \setminus S$. Because of the construction there exist local coordinate systems (x_1, \dots, x_n) about p and (y_1, \dots, y_n) about q such that locally

$$y_1 = x_1^k \quad (k \in \mathbf{N}, k \geq 2), \quad y_j = x_j \quad (2 \leq j \leq n),$$

$$\Theta = (k-1)\{x_1 = 0\}, \quad D = \{y_1 = 0\}.$$

Set $F = (F_1, \dots, F_n)$ and $f = (f_1, \dots, f_n)$. Let ν denote the multiplicity of zero of F_1 at z . Note that $\text{mult}_z F^* \Theta = (k-1)\nu$ and $\text{mult}_z f^* D = k\nu$. Thus $(k-1)\nu \leq k\nu - 1$ and so (3.13) holds. *Q.E.D.*

Remark. Let the notation be as in Proposition 3.12. Let $\text{St}(D)$ be the stabilizer of D in A defined by the identity component of $\{a \in A; a + D = D\}$. Assume that X is of log general type. We claim

$$(3.14) \quad \text{St}(D) = \{0\}.$$

Suppose that it is not the case. Then $A \setminus D$ is also $\text{St}(D)$ -invariant. Then, setting $U = X \setminus \pi^{-1}D$, we have a sequence of surjective morphisms

$$U \xrightarrow{\pi|_U} A \setminus D \rightarrow (A/\text{St}(D)) \setminus (D/\text{St}(D))$$

induced from the restriction of π and the quotient map. Let $\nu : U \rightarrow (A/\text{St}(D)) \setminus (D/\text{St}(D))$ be the composed morphism. Then every connected component Z of the fibers of ν is a finite étale cover over $\text{St}(D)$. Since $\bar{\kappa}(Z) = 0$, $\bar{\kappa}(U) \leq \dim U - 1$ (Iitaka [8] Theorem 4). Then we have a contradiction, $\bar{\kappa}(X) \leq \dim X - 1$. By (3.14) \bar{D} is big on \bar{A} ([20] Proposition 3.9).

4 Proof of the Main Theorem

The following is an essential case.

Theorem 4.1 *Let X be a normal variety of log general type and $\pi : X \rightarrow A$ be a finite morphism onto a semi-abelian variety. Then every holomorphic curve $g : \mathbf{C} \rightarrow X$ is algebraically degenerate.*

Proof. We use the desingularization and compactification obtained by Theorem 3.12 and follow the notation there.

Assume that there exists an algebraically nondegenerate holomorphic curve $g : \mathbf{C} \rightarrow X$. Since g is algebraically nondegenerate and $\tilde{X} \rightarrow X$ is birational, we can lift g to an

algebraically nondegenerate entire curve $F : \mathbf{C} \rightarrow \tilde{X}$. Set $f = \pi \circ g$. We are going to deduce a contradictory estimate for the order function $T_F(r)$ of F .

Let \bar{A} be an equivariant compactification of A such that \bar{D} is in general position; that is, \bar{D} contains no A -orbit in \bar{A} .

Since \bar{D} is in general position in \bar{A} , it follows from (2.5), (2.7) and Theorem 3.12 (ii), (iii) that

$$(4.2) \quad m_F(r; \Theta) \leq m_f(r; \bar{D}) = S_f(r).$$

Since Θ is big, one infers from (2.4) that

$$(4.3) \quad T_f(r) \sim T_F(r).$$

Combining this with (4.2), one gets

$$(4.4) \quad m_F(r; \Theta) = S_F(r).$$

Theorem 3.12 (iv) implies

$$(4.5) \quad N(r; F^*\Theta) \leq N(r; F^*\hat{S}) + N(r; f^*D) - N_1(r; f^*D).$$

Now $\psi(\hat{S})$ is of codimension at least two in A . Therefore we can infer from Theorem 2.9 that

$$(4.6) \quad N(r; F^*\hat{S}) \leq N(r; f^*(\psi_*\hat{S})) \leq \epsilon T_f(r) \|\epsilon.$$

By virtue of (2.8) we have

$$\begin{aligned} N(r; f^*D) - N_1(r; f^*D) &\leq T_f(r; L(\bar{D})) - N_1(r; f^*D) \\ &\leq \epsilon T_f(r) \|\epsilon, \quad \forall \epsilon > 0. \end{aligned}$$

The combination of this with (4.3) yields

$$(4.7) \quad N(r; f^*D) - N_1(r; f^*D) \leq \epsilon T_F(r) \|\epsilon, \quad \forall \epsilon > 0.$$

Now one infers from (4.5)–(4.7) that

$$(4.8) \quad N(r; F^*\Theta) \leq \epsilon T_F(r) \|\epsilon, \quad \forall \epsilon > 0.$$

Note that $T_F(r) \sim T_F(r; L(\Theta)) \rightarrow \infty$ as $r \rightarrow \infty$. Henceforth, F.M.T. (2.1), (4.4) and (4.8) lead to a contradiction:

$$T_F(r) \leq \epsilon T_F(r) \|\epsilon, \quad \forall \epsilon > 0.$$

Q.E.D.

Remark. The case where X is compact was proved by Yamanoi [22] Corollary 3.1.14.

Proof of the Main Theorem. Let the notation be as in the Main Theorem. Assume that $\bar{\kappa}(X) > 0$ and $f : \mathbf{C} \rightarrow X$ is algebraically nondegenerate. By lifting f to the normalization of X we may assume further that X is normal.

We use Kawamata's Theorem 2.12 and the notation there. Since $\tilde{X} \rightarrow X$ is étale, we can lift f to a holomorphic curve $\tilde{f} : \mathbf{C} \rightarrow \tilde{X}$. By Theorem 4.1 the composed map of \tilde{f} with $\tilde{X} \rightarrow Y$ must be algebraically degenerate. This implies that f itself is algebraically degenerate, because $\dim Y = \bar{\kappa}(X) > 0$ due to our assumption; this is a contradiction.

Now let Z be the normalization of the Zariski closure of $f(\mathbf{C})$ in A . Then $\bar{\kappa}(Z) = 0$ by what we have just proved, and the last statement follows from Kawamata's Theorem 2.13. *Q.E.D.*

5 Applications

Here we give several applications. As a direct consequence of the Main Theorem we have the next:

Theorem 5.1 *Let A be a semi-abelian variety with smooth equivariant compactification $A \hookrightarrow \bar{A}$ and let \bar{X} be a projective variety with a finite morphism $\pi : \bar{X} \rightarrow \bar{A}$. Then $X = \pi^{-1}(A)$ is Kobayashi hyperbolic and hyperbolically embedded into \bar{X} unless there exists a semi-abelian subvariety $B \subset A$, a positive-dimensional orbit $B(p) \subset \bar{A}$, an étale cover $\rho : C \rightarrow B(p)$ from a semi-abelian variety C and a morphism $\tau : C \rightarrow \bar{X}$ such that $\rho = \pi \circ \tau$.*

Proof. By Brody-Green's Theorem either X is hyperbolic and hyperbolically embedded into \bar{X} or there is a non-constant holomorphic map from \mathbf{C} into one of the strata of the natural stratification on \bar{X} , the one induced by the stratification on \bar{A} which is given by the A -orbits. Using this, the statement follows from the Main Theorem. *Q.E.D.*

Theorem 5.2 *Let X be a normal algebraic variety which admits a finite morphism π onto a simple abelian variety A . Then either X is hyperbolic or X itself is an abelian variety.*

Proof. Since A is assumed to be simple, there does not exist any non-trivial (semi-) abelian subvariety. Hence every holomorphic map $f : \mathbf{C} \rightarrow X$ must be constant unless $\pi : X \rightarrow A$ is an étale cover by the Main Theorem. If every holomorphic map $f : \mathbf{C} \rightarrow X$

is constant, then X is hyperbolic by Brody's theorem. If π is étale, then X is an abelian variety, too. Q.E.D.

Remark. (i) The case of $\dim X = \kappa(X) = 2$ was proved by C. Grant [5].

(ii) If A is not simple, it is not sufficient to assume the Kobayashi hyperbolicity of the ramification locus R of π to obtain the Kobayashi hyperbolicity of X , as shown by the next proposition.

Proposition 5.3 *There exists an abelian surface A with a smooth ample hyperbolic curve $D \subset A$ and a smooth projective surface X with a finite covering $\pi : X \rightarrow A$ with ramification locus R such that $R = \pi^{-1}(D)$ and X is not hyperbolic.*

Proof. Let E be an elliptic curve with line bundles H and H' of degree 2 and 3 respectively. Then $\phi_H : E \rightarrow \mathbf{P}_1$ while $\phi_{H'}$ embeds E into \mathbf{P}_2 as a cubic curve C . Let $X = \mathbf{P}_1 \times C$. Furthermore let Z be the union of all $\{p\} \times C$ for points $p \in \mathbf{P}_1$ over which $\phi_H : E \rightarrow \mathbf{P}_1$ is ramified. Choose an even number d . Then using Bertini's theorem we obtain a hypersurface $L \subset X$ such that L is ample, smooth, with $L \cap Z$ being smooth too, and such that the bidegree is $(1, d)$. Let $\tau : E \times E \rightarrow X = \mathbf{P}_1 \times C$ be given as $\tau(x_1, x_2) = (\phi_H(x_1), \phi_{H'}(x_2))$. Then $D = \tau^*L$ is a smooth ample divisor (smoothness of D can be deduced from the conditions that both L and $L \cap Z$ are smooth) and by construction $L(D) = p_1^*L(H) \otimes p_2^*L(dH')$ where $p_i : E \times E \rightarrow E$ are the respective projections. Since D is smooth and ample in $E \times E$, it is a curve of genus larger than one and therefore hyperbolic. By construction there is a divisor D_0 such that $2D_0$ is linearly equivalent to D . Now, by the usual cyclic covering method there is a surface X with a two-to-one covering $\pi : X \rightarrow A = E \times E$ which is precisely ramified over D . More precisely, by taking squares fiber-wise there is a morphism from (the total space of) $L(D_0)$ to $L(D)$. We let σ denote a section of $L(D)$ whose zero-divisor is D and define $X = \{(x, t) \in L(D_0) : t^2 = \sigma(x)\}$. Now $d(t^2 - s(x)) = 2tdt - ds$ in local coordinates; hence X is smooth if D is smooth.

We claim that X is not hyperbolic. For each $q \in E$ let $E_q = E \times \{q\} \subset A$. Then $D \cap E_q$ is a divisor of degree 2 which is linearly equivalent to H . If in some neighborhood of q there are holomorphic functions a, b with values in E such that $\{a(p), b(p)\} = D \cap E_p$ for all p in this neighborhood, then $a+b$ is constant (because all $D \cap E_p$ are linearly equivalent to H). On the other hand a and b can not be constant. Hence $a-b$ is non-constant. If we now define an equivalence relation \sim on E via $z \sim -z$, we obtain a globally well-defined holomorphic non-constant map $E \rightarrow E/\sim$ locally given by $p \mapsto [a(p) - b(p)]$. As a non-constant holomorphic map between compact Riemann surfaces ($E/\sim \simeq \mathbf{P}_1$), this map

must be surjective. It follows that there exists a point $q \in E$ such that $D \cap E_q = 2\{r\}$ for some $r \in E_q$. Fix $F = E_q \subset A$ and define $F' = \pi^{-1}(F) \subset X$. Then $F' \rightarrow F$ is ramified at exactly one point. In local coordinates x for the base and t for the fiber of $L(D_0|_F)$ this implies $F' = \{(x, t) : t^2 = g(x)\}$ for some holomorphic function g vanishing at the chosen base point r of order 2. Then $g(x) = x^2 e^{h(x)}$ for some holomorphic function h and consequently F' locally decomposes into two irreducible components given by $t = \pm x e^{h(x)/2}$. For each of the two components the projection onto F is unramified.

Thus, if \hat{F} denotes the normalization of F' then the naturally induced map $\hat{F} \rightarrow F$ is an unramified covering. It follows that \hat{F} is an elliptic curve, and therefore it follows that X contains an image of an elliptic curve under a non-constant holomorphic map. In particular, X is not hyperbolic. *Q.E.D.*

Theorem 5.4 *Let $E_i, 1 \leq i \leq q$ be smooth hypersurfaces of the complex projective space $\mathbf{P}^n(\mathbf{C})$ of dimension n such that $E = \sum E_i$ is an s.n.c. divisor. Assume that*

- (i) $q \geq n + 1$.
- (ii) $\deg E \geq n + 2$.

Then every holomorphic curve $f : \mathbf{C} \rightarrow \mathbf{P}^n(\mathbf{C}) \setminus E$ is algebraically degenerate.

Proof. If $q \geq n + 2$, then this is immediate from Corollary 1.4. (iii) in [15].

Assume that $q = n + 1$. We observe that $K_{\mathbf{P}^n(\mathbf{C})} + E$ is ample, because $\deg(K_{\mathbf{P}^n(\mathbf{C})} + E) \geq (-n - 1) + n + 2 = 1$. Thus E is an s.n.c. divisor for which $K_{\mathbf{P}^n(\mathbf{C})} + E$ is ample, and therefore $\mathbf{P}^n(\mathbf{C}) \setminus E$ is of log general type.

Let d be the l.c.m. of $d_i = \deg E_i, 1 \leq i \leq n + 1$. Define $k_i = d/d_i$. Let $(x_0; \dots; x_n)$ be a homogeneous coordinate system of $\mathbf{P}^n(\mathbf{C})$. Let $P_i(x_0, \dots, x_n)$ be a homogeneous polynomial defining E_i .

Now we define a morphism $\bar{\pi}$ from $\mathbf{P}^n(\mathbf{C})$ to $\mathbf{P}^n(\mathbf{C})$ as follows:

$$\bar{\pi} : x \rightarrow (P_1^{k_1}(x); \dots; P_{n+1}^{k_{n+1}}(x))$$

Then $\bar{\pi}$ is a finite morphism which restricts to a finite morphism from $\mathbf{P}^n(\mathbf{C}) \setminus E$ to $A = \{(y_0; \dots; y_n) \in \mathbf{P}^n(\mathbf{C}) : y_0 \cdot y_1 \cdot \dots \cdot y_n \neq 0\}$.

Since A is biholomorphic to $(\mathbf{C}^*)^n$, we may now use our Main theorem and deduce that every holomorphic map $f : \mathbf{C} \rightarrow \mathbf{P}^n(\mathbf{C}) \setminus E$ is algebraically degenerate. *Q.E.D.*

Remark 5.5 (i) Cf. [20] Theorem 7.1, in which $E_i, 1 \leq i \leq n$ are hyperplanes and $\deg E_{n+1} \geq 2$.

- (ii) Grauert [6] dealt with the case where $n = 2$ and $E_i, 1 \leq i \leq 3$ are three smooth quadrics, and proved that $\mathbf{P}^2(\mathbf{C}) \setminus E$ is Kobayashi hyperbolic. The papers of Dethloff-Schumacher-P.M. Wong [4] and [3] followed it to make clear some part of the arguments in Grauert [6]. It was an essential step to prove the algebraic degeneracy of $f : \mathbf{C} \rightarrow \mathbf{P}^2(\mathbf{C}) \setminus \sum_{i=1}^3 E_i$.
- (iii) Dethloff-Lu [2] deals with the special case of $\dim X = 2$ and Brody curves $f : \mathbf{C} \rightarrow X$.

Examples. Let $D = \sum_{i=1}^3 D_i$ be the union of three quadrics of $\mathbf{P}^2(\mathbf{C})$ in sufficiently generic configuration in the sense of [3] Proposition 4.1. Then $X = \mathbf{P}^2(\mathbf{C}) \setminus D$ serves for an example of Theorem 5.1 (cf. Remark 5.5).

6 Strong Green-Griffiths conjecture

In [7] Green and Griffiths conjectured the following:

Conjecture 1 Let X be a projective complex variety of general type. Then every holomorphic map from \mathbf{C} to X is algebraically degenerate.

This can be strengthened as follows:

Conjecture 2 Let X be a projective complex variety of general type. Then there exists a closed subvariety $E \subsetneq X$ such that E contains the image $f(\mathbf{C})$ for every non-constant holomorphic map $f : \mathbf{C} \rightarrow X$.

Here we deal with this “strong Green-Griffiths’ conjecture” for surfaces.

Theorem 6.1 Let A be a semi-abelian surface, and let X be a smooth surface of log general type. Assume that there exists a proper finite morphism $\pi : X \rightarrow A$. Then there are only finitely many non-hyperbolic curves C on X ; moreover an arbitrary nonconstant holomorphic curve $f : \mathbf{C} \rightarrow X$ has the image contained in one of such C ’s.

Proof. The ramification locus R of π is the set of all points $x \in X$ for which the differential $d\pi_x : T_x X \rightarrow T_{\pi(x)} A$ is not surjective and we denote by R^* the set of all point $x \in X$ for which $d\pi_x$ has exactly rank one. Set $D = \pi(R)$. In the same way as in §3 (2) we see that $\text{St}(D)$ is trivial, since X is of log general type. We choose a smooth equivariant compactification $A \hookrightarrow \bar{A}$ in which D is in general position ([20] §3). Then we choose a

compactification \bar{X} of X such that $\pi : X \rightarrow A$ extends to a finite morphism from \bar{X} to \bar{A} (see Proposition 3.10).

Notice that there are only finitely many A -orbits in \bar{A} ([20] Lemma 3.12).

Now we observe the following: If E is a semi-abelian subvariety of A for which the fixed point set \bar{A}^E in \bar{A} is larger than \bar{A}^A , then E is the connected component of the isotropy group of A at a point in a one-dimensional A -orbit in \bar{A} . Since A is commutative, all points in the same orbit have the same isotropy group. Therefore the finiteness of the number of A -orbits implies that there are only finitely many semi-abelian subvarieties $E \subset A$ with $\bar{A}^E \neq \bar{A}^A$.

If C is a non-hyperbolic curve on X , then C is either an elliptic curve or \mathbf{C}^* , because a morphism from \mathbf{C} or \mathbf{P}_1 to a semi-abelian variety must be constant. Hence the image of C by π is necessarily an orbit in A of a semi-abelian subvariety of A . Thus it suffices to show that there exists only finitely many such semi-abelian subvarieties of A over which we can find a non-hyperbolic curve on X .

Thus we have to investigate semi-abelian subvarieties $E \subset A$ of dimension one with an orbit $E(q) \subset A$ such that $\pi^{-1}(E(q))$ contains a non-hyperbolic curve.

We may therefore assume that $\bar{A}^E = \bar{A}^A$.

Now E is either an elliptic curve or \mathbf{C}^* and in both cases $C \rightarrow E$ is an unramified covering. In particular $d\pi_x$ maps $T_x C$ surjectively on $T_{\pi(x)} E$ for every $x \in C$. Therefore $C \cap R \subset R^*$.

We claim that furthermore $\bar{C} \cap \bar{R} \subset C \cap R$. Indeed, if $C \neq \bar{C}$, then $\bar{C} = \mathbf{P}_1$ and $\pi(\bar{C} \setminus C) \subset \partial A$. Therefore $\bar{C} \cap R = C \cap R$. If $C = \bar{C}$, i.e., C is an elliptic curve, $\pi(\bar{C} \cap \partial X) \subset \bar{A}^E = \bar{A}^A$, because $\pi(\bar{C})$ is an E -orbit. Since D is generally positioned, we have $\pi(\bar{R}) \cap \bar{A}^A = \emptyset$. Hence $\bar{C} \cap \bar{R} \cap \partial X = \emptyset$. Together, these arguments yield

$$\bar{C} \cap \bar{R} \subset C \cap R \subset C \cap R^*$$

Next we define a ‘‘Gauss map’’ on \bar{R}^* : We set

$$\gamma : x \mapsto (\text{Image of } d\pi_x) \subset \mathbf{P}(\text{Lie}A)$$

It is readily verified that γ is a rational map, thus it extends to a morphism from the closure $R' = \bar{R}^*$ to $\mathbf{P}_1 \simeq \mathbf{P}(\text{Lie}A)$. Let R_1 denote the union of irreducible components of R' on which γ is locally constant. Then each irreducible component K of R_1 maps onto an orbit of a one-dimensional algebraic subgroup H of A in \bar{A} . The value of the Gauss map is evidently $\text{Lie}H$. But then $x \in K \cap E$ implies that $E = H$ and that there is an E -orbit inside D which is excluded. Therefore $C \cap R_1 = \emptyset$. Let R_2 denote the

union of all irreducible components of R' along which γ is nowhere locally constant. Let $x \in C \cap R_2$. A calculation in local coordinates shows that $\text{mult}_x(C, R) = \text{mult}_x\gamma + 1$. Therefore $\deg(C \cap R) \leq 2 \deg \gamma$.

In this way we obtain a universal bound for the degree of \bar{C} with respect to the big divisor \bar{R} . Therefore all such curves C are contained in a finite number of families. Now fix a one-dimensional semi-abelian subvariety E and consider all such curves \bar{C} for which $\pi(C)$ is an orbit of the fixed algebraic subgroup E of A . Then $\pi(C)$ is an E -orbit containing $\pi(R_2 \cap C)$. Therefore $\pi(C)$ is uniquely determined by $C \cap R_2$ provided the latter is not empty. Now $\{x \in R_2 : \gamma(x) = [\text{Lie}E]\}$ is finite and the number of curves with empty intersection with \bar{R} is finite as well, since \bar{R} is big. Thus we have finished the proof that there are only finitely many such curves.

The last assertion for entire holomorphic curves follows from the Main Theorem combined with the above obtained result. *Q.E.D.*

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UTMS

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- 2005–14 Yoshihiro Sawano and Hitoshi Tanaka: *Sharp maximal inequalities and commutators on Morrey spaces with non-doubling measures.*
- 2005–15 Atsushi Matsuo, Kiyokazu Nagatomo and Akihiro Tsuchiya: *Quasi-finite algebras graded by Hamiltonian and vertex operator algebras.*
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- 2005–18 Nariya Kawazumi: *Cohomological aspects of Magnus expansions.*
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The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

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