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Boundary rigity for Riemannian manifolds

by

A. Kh. Amirov

UNIVERSITY OF TOKYO

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES

KOMABA, TOKYO, JAPAN

A. Kh. Amirov
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Department of Mathematics, Zonguldak Karaelmas University

67100 Zonguldak Turkey, amirov@karaelmas.edu.tr

1. Introduction. In this paper the question of the uniqueness of the problem of restoring a Riemannian metric from the distances between the boundary points of the domain in the metric is investigated. This problem is of interest from the both viewpoints of mathematics and applications. For example, it appears in geophysics in connection with the study of distribution of the velocities of propagation of elastic waves inside the terrestial globe. ln the study of the problem of restoring a Riemannian metric a new type problem (problem 2) of integral geometry appears. It is known that the integral geometry problem is the mathematical base of tomography. This problem has many applications in various fields: the problem of the forecasting of earthquakes, diagnostics of plasma, problem of photometry, fiber optics and etc. (see [1,2,6,7]). The connection of the problem of integral geometry for differential forms with the inverse problems for equations of the hyperbolic type and kinetic equations are described in the works [2-7]. The basic results of this article were announced in the work [8] with one additional condition on metric g (see also [9]) .

2. Formulation of the problems and results.

Let D be a closed, bounded domain of variables $x = (x^1, ..., x^n)$ in the space R^n $(n > 1)$ with the boundary S of class C^5. A domain D is called convex with respect to a metric g, if any two points $x, y \in D$ can be joined by a unique geodesic $\Gamma(x, y)$ of this metric (see [10]). It is known (see also the proof of assertion a) of theorem 3) that if there exist a point $x^0 \in D$ such that any point $x \in D$ can be joined with x^0 by a unique geodesic ray of the metric g, then the metric g has a semigeodesic coordinates in D. So, if the domain D is convex with respect to a metric g, then for g, a semigeodesic system of coordinates can be introduced in the domain D. Moreover, in the semigeodesic system of coordinates x^i, the components of the metric $g = (g_{ij})$ satisfy the conditions: $g_{11} = 1$, $g_{1i} = 0$, $i = 2, ..., n$. Conversly, these conditions are sufficient for the system with coordinates x^i to be semigeodesic for the metric g in D (see [10]).

Let $\Gamma(x, y) = \xi(x, y, t) = \left\{\xi^1(x, y, t), ..., \xi^n(x, y, t)\right\}$ be a coordinate representation of the geodesic $\Gamma(x, y)$.

Problem 1. For each pair of the points $(x, y) \in \partial D \times \partial D$, let integral

$$\int_{\Gamma(x,y)} (\sum_{i,j=2}^{n} a_{ij}(\xi(x, y, t))\dot{\xi}^i(x, y, t)\, \dot{\xi}^j(x, y, t))\, dt,$$

be known, where t is the natural parameter, and the dot indicates the differentiation with respect to t: $\dot{\xi} = \frac{\partial \xi}{\partial t}$. Knowing these integrals determine functions $a_{ij}(x)$ in D $(i,\, j = 2..., n)$.

Theorem 1. Let D be convex with respect to metric $g \in C^6(D)$, functions $a_{ij}(x) \in C^5(R^n)$ be zero out of D and $g_{11} = 1$, $g_{1i} = 0$, $i = 2, ..., n$. Then problem 1 can have only one solution $(a_{ij}(x)) \in C^5(R^n)$.

lt is worthy to note, that if in the formulation of the problem 1, indexes i, j in the summation under the integral, run from 1 up to n, then the theorem 1 is not true (see [7]).

Let a function $h(x,'\xi) = (h^2(x,'\xi), ..., h^n(x,'\xi)) \in C^5(D \times (R^{n-1} \setminus \{0\}))$ be homogeneous of the first degree in $'\xi$: $h(x, l\, '\xi) = l\; h(x,'\xi), l > 0$; and the Jacobian $\frac{\partial\; h(x,'\xi)}{\partial\; '\xi} > 0$. Here for each $x \in D$ and for $2 \le k \le n$ the function $h^k(x,'\xi)$ depends on $'\xi = (\xi^2, ..., \xi^n)$ and $h^k(x, 0) = 0$.

Problem 2. For $(x, y) \in \partial D \times \partial D$, let the integral

$$\int_{\Gamma(x,y)} (\sum_{i,j=2}^{n} a_{ij}(\xi(x,y,t))\; h^i(\xi(x,y,t), '\dot{\xi}(x,y,t))\; h^j(\xi(x,y,t), '\dot{\xi}(x,y,t)))\; dt$$

be known.

Knowing these integrals determine functions $a_{ij}(x)$ in D $(i, j = 2, ..., n)$.

Theorem 2. Let the conditions of theorem 1 be satisfied and let the function $h(x,'\xi)$ satisfy the conditions formulated above. Then problem 2 can have only one solution $(a_{ij}(x)) \in C^5(R^n)$.

For the points $x, x_0 \in \partial D$ let us denote by $H_g(x, x_0)$ the distance between the points x, x_0 in metric g.

The function $H_g(x, x_0)$ determined on the set $\partial D \times \partial D$ is called the hodograph of the metric g.

Problem 3. Determine a metric g in D if the hodograph $H_g(x, x_0)$ is known.

It is easy to show the nonuniqueness of a solution of the problem 3. Indeed, let φ be the diffeomorphism of region D to itself of class C^1 which is identical on ∂D. It transforms each metric g_1 into $g_2 = \varphi^* g_1$ in the sense that for any vectors $\xi, \eta \in T_xD$, $\langle \xi, \eta \rangle_x^{(2)} = \langle \varphi_* \xi, \varphi_* \eta \rangle_{\varphi(x)}^{(1)}$, holds, where φ_* is the differential of map φ, $\langle ., . \rangle_x^{(i)}$ is the scalar product on T_xD is determined by the metric g_i $(i = 1, 2)$. These two metrics have different families of geodesic, but the same hodograph.

The following questions naturally arise:

1) when is a metric determined by its hodograph up to isometry and identicality on ∂D?

2) for what classes of metrics, does hodograph determine a metric uniquely?

Let us clarify the formulation of problem 3 as follows:

Problem 4. Let g_1, g_2 be two metrics which are convex in D. Does the existence of a diffeomorphism φ: $D \to D$ follow from the equality $H_1(x, x_0)$=$H_2(x, x_0)$, such that $\varphi \mid_{\partial D} = 1$, and $g_2 = \varphi^* g_1$. Here $H_k(x, x_0)$ is the hodograph of the g_k, $k = 1, 2$ and the equality $\varphi \mid_{\partial D} = 1$ means that mapping φ is identical on ∂D.

A positive answer to the question formulated in problem 4 is obtained only for a few class of metrics (see [11-19]). Below (in theorem 3) it is assumed that metric $g_1$and g_2 $(g_k = \left(g_{ij}^{(k)}(x)\right) \in C^6(D_{\varepsilon_0})$; $k = 1, 2)$ coincide on $D_{\varepsilon_0} \setminus D$,where D_{ε_0} is the ε_0- neigbourhood of $D, (\varepsilon_0 > 0)$ i.e. $D_{\varepsilon_0} = \{x \in R^n \;/\; d(x, D) < \varepsilon\}$, $d(x, D)$ is the euclidean distance between the point $x \in R^n$ and for a set D, we put $d(x, D) = \inf_{y \in D} |x - y|$. Let us note that the last condition is not, generally speaking, a restriction on metrics g_k, in D if their hodographs coincide. Indeed, it is proven in [13] that, if H_1 and H_2 coincide, then suitably chosen coordinates g_1 and g_2 will coincide in the space $C^2(\partial D)$. Consequently, it is possible to continue g_2 from boundary $\partial D \in C^3$ to $D_{\varepsilon_0} \setminus D$ by $g_2 = g_1$. Then the metrics g_1 and $\overline{g_2}$ will be from $C^2(D_{\varepsilon_0})$, have the same hodograph and coincide on $D_{\varepsilon_0} \setminus D$, where $\overline{g_2} = g_2$ when $x \in D$, and $\overline{g_2} = g_1$

when $x \in D_{\varepsilon_0} \setminus D$.

Theorem 3. Let D_{ε_0} be convex with respect to metrics $g_k \in C^6(D_{\varepsilon_0})$ $(k = 1, 2)$ and for $x, x_0 \in \partial D$, $H_1(x, x_0)$=$H_2(x, x_0)$. Then

a) there exists a diffeomorphism φ: $D \to D$ such that $\varphi \mid_{\partial D} = 1$, and $g_2 = \varphi^* g_1$, $\varphi \in C^5(D)$.

b) if $g_{11}^{(k)} = 1$, $g_{1i}^{(k)} = 0$, where $k = 1, 2$; $i = 2, ..., n$;then metrics g_1, g_2 coincide in D.

3. Auxiliary statements. Let us introduce functions

$$I(x,\xi) = \int_{\gamma(x,\xi)} b\left(z\left(x,\xi,t\right)\right) \dot{z}^i\left(x,\xi,t\right) \dot{z}^j\left(x,\xi,t\right) dt, \tag{1}$$

$$u(x,\xi) = \sum_{i,j=2}^{n} \int_{\gamma(x,\xi)} a_{ij}\left(z\left(x,\xi,t\right)\right) \dot{z}^i\left(x,\xi,t\right) \dot{z}^j\left(x,\xi,t\right) dt, \tag{2}$$

where $\gamma(x,\xi)$ is the ray of the metric $g = (g_{ij})$ starting from $x \in D$, in the direction ξ ; functions $b(x)$, $a_{ij}(x) \in C^5(R^n)$ are zero out of the D , in definition of $I(x,\xi)$ indeks i, j are fixed $2 \le i, j \le n$.

Let us investigate the smoothness of functions $I(x,\xi)$ and $u(x,\xi)$ and their Fourier transforms.

It is known that, $\gamma(x,\xi) = \left(z^1(x,\xi,t), ..., z^n(x,\xi,t)\right)$ is the solution of following system of differential equations

$$\frac{d^2}{dt^2} z^i = -\Gamma^i_{jk}(z) \dot{z}^j \dot{z}^k \ , \quad i = 1, 2, ..., n \tag{3}$$

with Cauchy data

$$z(0) = x, \ \ \dot{z}(0) = \xi, \tag{4}$$

where Γ^i_{jk} is the Christoffel symbols of the metric g, $z(x,\xi,t) = \left(z^1(x,\xi,t), ..., z^n(x,\xi,t)\right)$, $\dot{z}(x,\xi,t) = \left(\dot{z}^1(x,\xi,t), ..., \dot{z}^n(x,\xi,t)\right)$, $\dot{z}^i = \frac{d}{dt} z^i$, t is the natural parameter of the metric g.

It is easy to prove that, a solution of problem (3)-(4) has the following property :

$$z(x,\xi,t) = z(x,\nu,|\xi|\, t)\,; \quad \dot{z}(x,\xi,t) = |\xi|\, \dot{z}(x,\nu,|\xi|\, t)\,, \tag{5}$$

where $\nu = \frac{\xi}{|\xi|}$, $|\xi|^2 = \sum\limits_{i,j=1}^{n} g_{ij} \xi^i \xi^j$.

Let us recall that $\gamma(x,\xi)$ is a projection on the space $\left(z^1, ..., z^n\right)$ of the solution of the problem of Cauchy for the following system of differential equations

$$\begin{aligned} &\frac{d}{dt} z^i = \sum_{j=1}^{n} g^{ij}(z) p^j \\ &\frac{d}{dt} p^i = -\frac{1}{2} \sum_{j=1}^{n} p^i p^j \frac{\partial}{\partial z^i} g^{ij}(z) \end{aligned} \tag{6}$$

with the data

$$z^i(0) = x^i, \ \ p^i(0) = p^i_0, \quad i = 1, 2, ..., n \ , \tag{7}$$

where $\left(g^{ij}(x)\right)$ is the inverse of the matrix $(g_{ij}(x))$, $p^i_0 = \sum\limits_{j=1}^{n} g_{ij}(x) \xi^j$.

Let $'G$ denote the closed, bounded set of variables $'\xi$ =$(\xi^2, ..., \xi^n)$ in the set R^{n-1}, $0 \notin' G$ and let $G = \left\{\xi \in R^n \mid \xi = (\xi^1,' \xi),\ \xi^1 \in R^1,\ '\xi \in' G\right\}, \Omega = \left\{(x,\xi) | x \in D,\ \xi \in G\right\}.$

Note that by (2) differentiating $u(x,\xi)$ at the point x in the direction ξ and taking into account (3), (4) we have the following kinetic equation

$$\sum_{j=1}^{n} \xi^j u_{x^j} \ -\sum_{j,k,s=1}^{n} \Gamma^s_{jk} \xi^k \xi^j u_{\xi^s} = \sum_{j,k=2}^{n} a_{jk}(x) \xi^k \xi^j \ . \tag{8}$$

¿From the setting of the problem 1, from formulas (2), (5) and from the fact that $a_{jk}(x)$ is zero out of D it follows that the function $u(x,\xi)$ is known when $(x,\xi) \in \partial D \times R^n$ $('\xi \neq 0)$.

Since the uniqueness of solution of problem 1 is being investigated we have condition

$$for\ \ (x,\xi)\in\partial D\times R^n\ \ (\ \xi\neq 0)\ ,\ \ u(x,\xi)=0. \tag{9}$$

Lemma 1. Let D be convex with respect to the metric $g=(g_{ij}(x))\in C^6(D)$ and $g_{11}=1$, $g_{1i}=0$, $i=2...,n$. Then if $1\le s\le n$,

1) for $0\le|\beta|\le 4$, $D_\xi^\beta I$, $D_\xi^\beta I_{x^s}\in C(\Omega)$,

2) and for fixed $x\in D$, $\ \ {}'\xi\in{}'G\ \ ({}'\xi\neq 0)$,

a) if $|\beta|\le 2$, functions $D_\xi^\beta I$, $D_\xi^\beta I_{x^s}\in L_2\left(R^1_{\xi^1}\right)$,

b) if $|\beta|=3$, functions $D_\xi^\beta I$, $D_\xi^\beta I_{x^s}\in L_1\left(R^1_{\xi^1}\right)\cap L_2\left(R^1_{\xi^1}\right)$,

c) if $|\beta|=4$, functions $\xi^1 D_\xi^\beta I$, $\xi^1 D_\xi^\beta I_{x^s}\in L_1\left(R^1_{\xi^1}\right)\cap L_2\left(R^1_{\xi^1}\right)$,

where $\beta=(\beta_1,...,\beta_n)$, $D_\xi^\beta=D_{\xi^1}^{\beta_1}...D_{\xi^n}^{\beta_n}$, $|\beta|=\beta_1+...+\beta_n$, $D_{\xi^s}^{\beta_s}$ is the derivative in ξ^s of order $\beta_s\ge 0$.

Proof. Taking into account (5), let us rewrite the function $I(x,\xi)$ in the form

$$\begin{aligned} I(x,\xi)&=\int_0^\infty b\left(z\left(x,\xi,t\right)\right)\dot z^i\left(x,\xi,t\right)\dot z^j\left(x,\xi,t\right)dt\ = \\ &=\int_0^\infty b\left(z\left(x,\nu,|\xi|\,t\right)\right)|\xi|\,\dot z^i\left(x,\nu,|\xi|\,t\right)\,|\xi|\,\dot z^j\left(x,\nu,|\xi|\,t\right)dt= \\ &=\frac{1}{|\xi|}\int_0^\infty b\left(z\left(x,\nu,\tau\right)\right)|\xi|\,\dot z^i\left(x,\nu,\tau\right)|\xi|\,\dot z^j\left(x,\nu,\tau\right)d\tau \end{aligned} \tag{10}$$

where , $i,j\ge 2$.

Since the function $b(x)$ is finite in D, last integral in (10), actually, is taken on the finite interval $[0,d_0]$, where d_0- the diameter of D, in the metric g. Due to the condition $g_{ij}\in C^6(D)$, from the theory of ordinary differential equations it follows that $(z\left(x,\nu,t\right),\dot z\left(x,\nu,t\right))$ - solution of the problems (3)$-$ (4) - belong to the space $C^5\left(\Omega\left(d_0\right)\right)$, where $\Omega\left(d_0\right)=\{(x,\nu,t)\,|x\in D,\nu\in s^n,t\in[0,d_0]\}$, s^n is the unit sphere in R^n. Hence, taking into account the conditions $b(x)\in C^5\left(R^n\right)$, $\xi\in G$, ${}'\xi\neq 0$,and equality (10), from the theorem about the differentiation of integral on the parameter it follows that for $0\le|\beta|\le 4, 1\le s\le n$, $D_\xi^\beta I$, $D_\xi^\beta I_{x^s}\in C(\Omega)$.

In order to prove the assertion 2) of lemma 1, we investigate the behavior of the expression of the form $|\xi|\,\dot z^k\left(x,\nu,\tau\right)$ and its derivatives in ξ^j when $\xi^1\longrightarrow\infty$,where $j=1,2,...,n$, $k=2,...,n$.

Let $\xi^1=\frac{1}{\mu}$, then as $\xi^1\longrightarrow+\infty$, (i.e. when $\mu\longrightarrow+0$) the vector $\nu=\frac{\xi}{|\xi|}$, tends to $\nu^0=(1,0,...,0)\in R^n$. Therefore as known from the theory of ordinary differential equations ([20]), the unique solution of the problem (3) - (4), (it is uniform in $[0,d_0]$) tends to the solution $\left(z\left(x,\nu^0,t\right),\left(\dot z\left(x,\nu^0,t\right)\right)\right.$ when $\mu\longrightarrow+0$.Taking into account the facts that the metric $g=(g_{ij})$ is written down in the semigeodesic coordinates (i.e.$\Gamma^1_{1k}=\Gamma^k_{11}=0$, $k=1,2,...,n$) and the solution of problem (3) -(4) is unique, we have $z\left(x,\nu^0,t\right)=\left(z^1\left(x,\nu^0,t\right),...,z^n\left(x,\nu^0,t\right)\right)$, where $z^1\left(x,\nu^0,t\right)=x^1+t$, $z^k\left(x,\nu^0,t\right)=x^k$, $\dot z\left(x,\nu^0,t\right)=(1,0,\cdots,0)$, $\dot z^1\left(x,\nu^0,t\right)=1$, $\dot z^k\left(x,\nu^0,t\right)=0$, $k=2,3,\cdots,n$.

For $\xi^1>0\ \ (\xi^1=\frac{1}{\mu})$ we have :

$$|\xi|\,\dot z^k\left(x,\frac{\xi^1}{|\xi|},\cdots,\frac{\xi^n}{|\xi|},t\right)=\frac{1}{\mu}\left|{}'\xi\right|_\mu\dot z^k\left(x,\frac{1}{\left|{}'\xi\right|_\mu},\frac{\mu\xi^2}{\left|{}'\xi\right|_\mu},\cdots,\frac{\mu\xi^n}{\left|{}'\xi\right|_\mu},t\right), \tag{11}$$

where $\left|{}'\xi\right|_\mu=\left(1+\mu^2\sum\limits_2^n g_{ij}\xi^i\xi^j\right)^{1/2}$.

By the mean value theorem and smoothness of the function $\dot z^k\left(x,\nu,t\right)$ in set $\Omega\left(d_0\right)$ and by equality $\dot z^k\left(x,\nu^0,t\right)=0$, for $k\ge 2$, we have

$$\dot{z}^k\left(x, \frac{1}{|'\xi|_\mu}, \frac{\mu\xi^2}{|'\xi|_\mu}, \cdots, \frac{\mu\xi^n}{|'\xi|_\mu}, t\right) = \mu\ \dot{z}^k_{\mu_0}, 0 < \mu_0 < \mu \leq 1, \tag{12}$$

where $\dot{z}^k_{\mu_0}$are the derivatives in μ of the function $\dot{z}^k\left(x, \frac{1}{|'\xi|_\mu}, \frac{\mu\xi^2}{|'\xi|_\mu}, \cdots, \frac{\mu\xi^n}{|'\xi|_\mu}, t\right)$ at a point $\mu = \mu_0$.

Let us note that, since $\dot{z}^k(x,\nu,t) \in C^5(\Omega(d_0))$, the function $\dot{z}^k_\mu$is bounded on $\Omega(d_0)$ the set $\Omega(d_0)$ is closed and bounded. Therefore, taking into account the fact that when $\xi^1 \longrightarrow +\infty$, the vector $\nu = \frac{\xi}{|\xi|}$ tends to $\nu^0 = (1,0,...,0) \in R^n$, for $k \geqslant 2$ from (11), (12) in the set Ω, we have

$$\left|\ |\xi|\ \dot{z}^k(x,\nu,t)\right| \leq K_1\ , \tag{13}$$

where $K_1 > 0$ does not depend on $(x,\xi) \in (D \times G\ )$, but depends on the norm of the vector function $\dot{z}(x,\nu,t) = \left(\dot{z}^1(x,\nu,t), ..., \dot{z}^n(x,\nu,t)\right)$ in $C^1(\Omega(d_0))$ and on the diameter of $'G$. By analogous reasonings, as we proved inequality (13), when $\xi^1 \longrightarrow +\infty$, we can prove that it does occur also, in the case when $\xi^1 \longrightarrow -\infty$.

It is not difficult to verify that

$$\left(|\xi|\,\dot{z}^k\left(x, \frac{\xi}{|\xi|}, t\right)\right)'_{\xi^1} = \frac{\xi^1}{|\xi|^2}\left(|\xi|\,\dot{z}^k\left(x, \frac{\xi}{|\xi|}, t\right)\right) + |\xi|\left(-\sum_{j=1}^n \dot{z}^k_{\nu^j}\frac{\xi^j\xi^1}{|\xi|^3} + \frac{1}{|\xi|}\dot{z}^k_{\nu^1}\right),$$

$$\left(|\xi|\,\dot{z}^k\left(x, \frac{\xi}{|\xi|}, t\right)\right)'_{\xi^i} = \frac{1}{|\xi|^2}\sum_{j=2}^n g_{ij}\xi^j\left(|\xi|\,\dot{z}^k\left(x, \frac{\xi}{|\xi|}, t\right)\right) +$$
$$+|\xi|\left(-\sum_{s=1}^n \dot{z}^k_{\nu^s}\frac{1}{|\xi|^3}\xi^s\sum_{j=2}^n g_{ij}\xi^j + \frac{1}{|\xi|}\dot{z}^k_{\nu^i}\right), \quad \text{when} \quad 2 \leq i \leq n, \left(\ \nu = \frac{\xi}{|\xi|}\right). \tag{14}$$

The following equalities are true :

$$\left(|\xi|\,\dot{z}^k\left(x, \frac{\xi}{|\xi|}, t\right)\right)_{\xi^1\xi^1} = \frac{1}{|\xi|}\dot{z}^k - \frac{(\xi^1)^2}{|\xi|^3}\dot{z}^k - \sum_{j=2}^n \dot{z}^k_{\nu^j}\frac{\xi^j}{|\xi|^2} - 2\frac{\xi^1}{|\xi|^2}\dot{z}^k_{\nu^1} +$$
$$+2\xi^1\sum_{j=1}^n \dot{z}^k_{\nu^j}\frac{\xi^j\xi^1}{|\xi|^4} + \frac{\xi^1}{|\xi|^2}\left(\sum_{j=1}^n \dot{z}^k_{\nu^j}\frac{-\xi^j\xi^1}{|\xi|^2} + z^k_{\nu^1}\right) +$$
$$+\sum_{j=1}^n \frac{\xi^j\xi^1}{|\xi|^3}\left(\sum_{i=1}^n \dot{z}^k_{\nu^j\nu^i}\frac{\xi^i\xi^1}{|\xi|^2} - \dot{z}^k_{\nu^1\nu^i}\right) - \sum_{i=1}^n \dot{z}^k_{\nu^j\nu^i}\frac{\xi^i\xi^1}{|\xi|^3} + \frac{1}{|\xi|}\dot{z}^k_{\nu^1\nu^1},$$

$$\left(|\xi|\,\dot{z}^k\left(x, \frac{\xi}{|\xi|}, t\right)\right)_{\xi^1\xi^i} = -\frac{\xi^1}{|\xi|^3}\left(\sum_{j=2}^n g_{ij}\xi^j\right)\dot{z}^k - \left(\sum_{j=1}^n \dot{z}^k_{\nu^j}\frac{\xi^j\xi^1}{|\xi|^4}\right)\left(\sum_{j=2}^n g_{ij}\xi^j\right) + \dot{z}^k_{\nu^i}\frac{\xi^1}{|\xi|^2} +$$
$$+2\left(\sum_{j=1}^n \dot{z}^k_{\nu^j}\xi^j\xi^1\right)\left(\frac{1}{|\xi|^4}\sum_{j=2}^n g_{ij}\xi^j\right) -$$
$$-\dot{z}^k_{\nu^i}\frac{\xi^1}{|\xi|^2} + \sum_{j=1}^n \frac{\xi^i\xi^1}{|\xi|^3}\left(\sum_{m=1}^n \dot{z}^k_{\nu^j\nu^m}\frac{1}{|\xi|^2}\xi^m\sum_{s=2}^n g_{is}\xi^s - \dot{z}^k_{\nu^i\nu^j}\right) - \tag{15}$$
$$-\sum_{j=1}^n \dot{z}^k_{\nu^j\nu^1}\frac{1}{|\xi|^3}\xi^j\sum_{j=2}^n g_{ij}\xi^j + \frac{1}{|\xi|}\dot{z}^k_{\nu^i\nu^1}, \quad \text{when } 2 \leq i \leq n.$$

$$\left(|\xi|\,\dot{z}^k\left(x, \frac{\xi}{|\xi|}, t\right)\right)_{\xi^j\xi^i} = \frac{g_{ij}}{|\xi|}\dot{z}^k - \frac{1}{|\xi|^3}\left(\sum_{r=2}^n g_{jr}\xi^r\right)\left(\sum_{r=2}^n g_{ir}\xi^r\right)\dot{z}^k -$$
$$-\frac{1}{|\xi|^2}\left(\sum_{r=2}^n g_{jr}\xi^r\right)\left(\sum_{s=1}^n \dot{z}^k_{\nu^s}\frac{1}{|\xi|^2}\left(\xi^s\sum_{r=2}^n g_{ir}\xi^r\right) - \dot{z}^k_{\nu^i}\right) -$$

$$-\dot{z}^k_{\nu^i}\frac{1}{|\xi|^2}(\sum_{r=2}^{n} g_{ir}\xi^r) - \sum_{s=1}^{n}\dot{z}^k_{\nu^s}\frac{\xi^s g_{ij}}{|\xi|^2} + \frac{2}{|\xi|^4}\left(\sum_{r=2}^{n} g_{ir}\xi^r\right)\left(\sum_{s=1}^{n}\dot{z}^k_{\nu^s}\xi^s\left(\sum_{r=2}^{n} g_{jr}\xi^r\right)\right) +$$
$$+\sum_{s=1}^{n}\frac{1}{|\xi|^3}\xi^s(\sum_{r=2}^{n} g_{ir}\xi^r)\left(\sum_{m=1}^{n}\dot{z}^k_{\nu^s\nu^m}\frac{1}{|\xi|^2}\xi^m\sum_{r=2}^{n} g_{ir}\xi^r - \dot{z}^k_{\nu^i\nu^s}\right) -$$
$$-\frac{1}{|\xi|^3}\left(\sum_{r=2}^{n} g_{ir}\xi^r\right)\sum_{m=1}^{n}\xi^m\dot{z}^k_{\nu^j\nu^m} + \frac{1}{|\xi|}\dot{z}^k_{\nu^i\nu^j}, \text{ when } 2 \le i,\, j \le n.$$

Since $\dot{z}^k \in C^5(\Omega(d_0))$ and the set $'G$ is bounded, taking into account relations (13), (14), (15), it is easy to see that in Ω

$$\left||\xi|^{|\alpha|-1} D^\alpha_\xi\left(|\xi|\,\dot{z}^k\right)\right| \le K_2, \quad for \;\; 1 \le |\alpha| \le 4, \tag{16}$$

where $K_2 > 0$ does not depend on $(x,\xi) \in (D \times G\,)$. Moreover K_2 depends on the norm of the vector function $\dot{z}(x,\nu,t) = \left(\dot{z}^1(x,\nu,t),...,\dot{z}^n(x,\nu,t)\right)$ in the space $C^5(\Omega(d_0))$ and on the euclidean distance between the $'G$ and $0 \in R^{n-1}$ $(0 \notin' G)$ and on the euclidean diameter of the $'G$.

For $0 \le |\alpha| \le 4$, in the set Ω the following inequalities are also true :

$$\left||\xi|^{|\alpha|+1} D^\alpha_\xi\left(\frac{1}{|\xi|}\right)\right| \le K_3, \quad \left||\xi|^{|\alpha|} D^\alpha_\xi\left(b(z)\right)\right| \le K_4\,, \tag{17}$$

where K_3, K_4 depend on the same parametres as K_2 in (16) .

Using the theorem about the differentiation of integral on the parameter, from (10), taking into account boundedness of functions $|\xi|\,\dot{z}^k$, $\left(|\xi|\,\dot{z}^k\right)_{\xi^j}$ and by relations (16), (17), we obtain the proof of item 2) of lemma 1.

Corollary 1. Let conditions of lemma 1 be satisfied.Then

1) for $|'\beta| \le 2$, $D^{'\beta}_{'\xi}\,\widehat{I}$, $D^{'\beta}_{'\xi}\,\widehat{I}_{x^j} \in C\left(D \times \Delta^s_\eta \times' G\right) \cap L_2\left(R^1_\eta\right)$,

2) for $|'\beta| = 3$, $D^{'\beta}_{'\xi}\,\widehat{I}$, $D^{'\beta}_{'\xi}\,\widehat{I}_{x^j} \in C\left(D \times R^1_\eta \times' G\right) \cap L_2\left(R^1_\eta\right)$,

3) $\eta^r D^{'\beta}_{'\xi}\,\widehat{I}_\eta$, $\eta^r D^{'\beta}_{'\xi}\,\widehat{I}_{x^j\eta} \in C\left(D \times R^1_\eta \times' G\right) \cap L_2\left(R^1_\eta\right)$ if $r + |'\beta| = 4$, $0 \le r \le 4$

where $\widehat{I} = \widehat{I}(x,\eta,'\xi)$ is the Fourier transform in the variable ξ^1 of the function $I(x,\xi)$, η is dual to ξ^1 variable, $\Delta^s_\eta = \left\{\eta \in R^1_\eta |\; s\eta > 0\right\}$, $'\beta = (\beta_2,...,\beta_n)$, $D^{'\beta}_{'\xi} = D^{\beta_2}_{\xi^2}...D^{\beta_n}_{\xi^n}$, $|'\beta| = \beta_2 + ... + \beta_n$, $s = -1, 1$; $1 \le j \le n$.

Proof. From c) of lemma 1 it does follow that for fixed $x \in D$, $'\xi \in {}'G$ $('\xi \ne 0)$ and with $r + |'\beta| = 4$, $0 \le r \le 4$

$$\xi^1 D^r_{\xi^1}\left(D^{'\beta}_{'\xi} I\right),\;\; \xi^1 D^r_{\xi^1}\left(D^{'\beta}_{'\xi} I_{x^j}\right) \in L_1\left(R^1_{\xi^1}\right) \cap L_2\left(R^1_{\xi^1}\right).$$

Then taking into account 1) of lemma 1 and relations (13), (16), (17), from the theorem about the differentiation of integral on the parameter and from the properties of Fourier transform, we have:

$$\eta^r D^{'\beta}_{'\xi}\,\widehat{I}_\eta, \eta^r D^{'\beta}_{'\xi}\,\widehat{I}_{x^j\eta} \in C\left(D \times R^1_\eta \times' G\right) \cap L_2\left(R^1_\eta\right)\,, \tag{18}$$

where $r + |'\beta| = 4$, $0 \le r \le 4$ i.e. 3) of corollary 1 is true.

2) of corollary 1 is ensured by 1) and b) of the lemma 1.

By the assertions 1) and c) of lemma 1, we have that for $r + |'\beta| = 4$, $0 \le r \le 4, x \in D$, $'\xi \in {}'G$ $('\xi \ne 0)$

$$D^r_{\xi^1}\left(D^{'\beta}_{'\xi} I\right),\;\; D^r_{\xi^1}\left(D^{'\beta}_{'\xi} I_{x^j}\right) \in L_1\left(R^1_{\xi^1}\right) \cap L_2\left(R^1_{\xi^1}\right).$$

Last relations and 1) of lemma 1 show that

$$\eta^r D_{'\xi}^{'\beta}\,\widehat{I},\ \eta^r D_{'\xi}^{'\beta}\,\widehat{I}_{x^j} \in C\left(D\times R_\eta^1 \times' G\right)\cap L_2\left(R_\eta^1\right), \tag{19}$$

Consequently, for $|'\beta| \le 4$

$$D_{'\xi}^{'\beta}\,\widehat{I},\ D_{'\xi}^{'\beta}\,\widehat{I}_{x^j} \in C\left(D\times \Delta_\eta^s \times' G\right). \tag{20}$$

Assertion 1) of corollary 1 is ensured by assertion a) of lemma 1 and by (20).

Corollary 1 is proved.

Remark 1. As can be seen from equality (2), $u(x,\xi)$ is a sum of functions of the type $I(x,\xi)$, therefore, lemma 1 and Corollary 1 are true for $u(x,\xi)$ as well.

By remark 1 for each fixed $x\in D$, $'\xi \in {'G}$, $('\xi \neq 0)$, it is possible to apply the generalized Fourier transform in variable ξ^1 to equation (8).Then taking into account the fact that $\Gamma_{1k}^1 = \Gamma_{11}^k = 0$, $k=1,2,...,n$ (because of semigeodesicness of the system of coordinates x^i), for $\widehat{u} = \widehat{u}(x,\eta,'\xi)$ - Fourier's transform of the function $u(x,\xi)$ in variable ξ^1- we have :

$$i\widehat{u}_{x^1\eta} - 2i\sum_{j,k=2}^{n}\Gamma_{1k}^j\xi^k\,\widehat{u}_{\xi^j\eta} + \sum_{j=2}^{n}\xi^j\widehat{u}_{x^j} - i\sum_{j,k=2}^{n}\Gamma_{jk}^1\xi^k\xi^j\eta\,\widehat{u}-$$
$$-\sum_{j,k,s=2}^{n}\Gamma_{jk}^s\xi^k\xi^j\widehat{u}_{\xi^s} = 2\pi\delta(\eta)\sum_{k,j=2}^{n}a_{kj}(x)\,\xi^k\xi^j\,, \tag{21}$$

where i is the imaginary unit, $\delta(\eta)$ is the delta function of Dirac, $F(1){=}2\pi\delta(\eta)$, $F(1)$ is the generalized Fourier transform of the unit in the variable ξ^1.

Lemma 2. Let the conditions of lemma 1 be satisfied.Then for $0\le |'\beta| \le 2$, $1\le j\le n$, $s=-1,1$ and for fixed $(x,'\xi)\in D\times {'G}$, $('\xi\neq 0)$ $D_{'\xi}^{'\beta}\,\widehat{u}_\eta$, $D_{'\xi}^{'\beta}\,\widehat{u}_{\eta x^j} \in L_1\left(\Delta_\eta^s\right)\cap L_2\left(\Delta_\eta^s\right)$.

Proof. By the remark 1 (taking into account corollary 1), for $\eta>0$ $(\eta<0)$ functions $\widehat{u}$, $\widehat{u}_\eta$ are which continuously differentiable on $(x,'\xi)$ in the region $(D\times' G)$.Then from equation (21), it does follow that the function $\widehat{u}$ for $\eta>0$ $(\eta<0)$ satisfies the equation

$$i\widehat{u}_{x^1\eta} - 2i\sum_{j,k=2}^{n}\Gamma_{1k}^j\xi^k\,\widehat{u}_{\xi^j\eta} + \sum_{j=2}^{n}\xi^j\widehat{u}_{x^j} - i\sum_{j,k=2}^{n}\Gamma_{jk}^1\xi^k\xi^j\eta\,\widehat{u} - \sum_{j,k,s=2}^{n}\Gamma_{jk}^s\xi^k\xi^j\widehat{u}_{\xi^s} = 0 \tag{22}$$

in the classical sense.

Putting $\widehat{u} = p+iq$, from (22) for $\eta>0$ $(\eta<0)$ we have :

$$p_{x^1\eta} - 2\sum_{j,k=2}^{n}\Gamma_{1k}^j\xi^k p_{\xi^j\eta} = F_1 \tag{23}$$

,
$$q_{x^1\eta} - 2\sum_{j,k=2}^{n}\Gamma_{1k}^j\xi^k q_{\xi^j\eta} = F_2\ , \tag{24}$$

where $F_1 = \sum_{j,k=2}^{n}\Gamma_{jk}^1\xi^k\xi^j\eta p - \sum_{j=2}^{n}\xi^j q_{x^j} + \sum_{s,.j,k=2}^{n}\Gamma_{jk}^s\xi^k\xi^j q_{\xi^s}$,

$$F_2 = \sum_{j,k=2}^{n}\Gamma_{jk}^1\xi^k\xi^j\eta q + \sum_{j=2}^{n}\xi^j p_{x^j} - \sum_{s,j,k=2}^{n}\Gamma_{jk}^s\xi^k\xi^j p_{\xi^s}.$$

In this work the uniqueness of a solution of the problem 1 is investigated. Under the assumption of existence of the solution, in the region Ω there exists a solution. The solution $u(x,\xi)$ of the equation (8) (it means there exists the solution $\widehat{u} = p+iq$ of equation (21)) with the properties indicated in the remark 1 and satisfying the condition (see (9))

$$for\ \ (x,\xi)\in\partial D\times G\ ((x,'\xi)\in\partial D\times' G)\ ,\quad u=0\ (\ \widehat{u}(x,\eta,'\xi)=0\ ). \tag{25}$$

If we examine equation (24) as a differential equation for the function q_η then, as it follows from the theory of differential equations with partial derivatives of the 1st order, the following equalities holds :

$$\tfrac{d}{ds}x^1 = 1,\ \ \tfrac{d}{ds}\xi^k = -2\sum_{j=2}^{n}\Gamma_{1j}^k\,\xi^j,\ \ \tfrac{d}{ds}q_\eta = F_2, \quad k=2,3,...,n. \tag{26}$$

By remark 1, (see corollary 1) for $0 \le |'\beta| \le 2$, $j = 1, 2, ..., n$,

$$D'^{\beta}_{'\xi} F_2,\, D'^{\beta}_{'\xi} F_{2x_j} \in C\left(D \times \Delta^1_\eta \times' G\right) \cap L_2\left(\Delta^1_\eta\right). \tag{27}$$

¿From equalities (25), (26), it follows that

$$q_\eta(x, \eta,' \xi) = \int_{x_0^1}^{x^1} F_2\left(\tau,' x, \eta,' \zeta(\tau,' \xi)\right) d\tau, \tag{28}$$

where x_0^1- is the first component of boundary-point $(x_0^1,' x) \in \partial D$ of the region D (points $(x^1,' x) \in D$ possess the property $x^1 > x_0^1$). In equality (28), components of the vector $'\zeta(\tau,' \xi) = (\zeta^2(\tau,' \xi), ..., \zeta^n(\tau,' \xi))$ satisfy the system of the differential equations:

$$\tfrac{d}{d\tau}\zeta^k = -2\sum_{j=2}^{n} \Gamma^k_{1j}\, \zeta^j, \quad k = 2, 3, ..., n$$

Cauchy's conditions $'\zeta(x^1) = '\xi$.By the uniqueness of the solution of this Cauchy problem with the condition $'\zeta(x^1) = '\xi \neq 0$, it follows that $'\zeta(\tau,' \xi) \neq 0$, $\tau \in \left[x_0^1, x_0^1 + d_0\right]$, $x_0^1 \le x^1 \le x_0^1 + d_0$.

Since out of D the function $F_2\,(x, \eta,' \xi)$ is zero (this follows from the definition of the function $u(x, \xi)$ and (25)) and straight lines in R^n_x which are paralel to coordinate axis ox^1 are geodesics of metric (g_{ij}) the integral in (28) actually is taken on the finite interval $(x_0^1, x_0^1 + d_0)$, where d_0 is the diameter of the bounded domain D in the metric (g_{ij}).

¿From the relation (10), (16), (17), taking into account remark 1, we have that the integral $\int_{-\infty}^{+\infty} u^2(x, \xi)\, d\xi^1$ uniformly converges with respect to the parameters $(x,' \xi) \in (D \times' G)$ and is continuous in the set $(D \times' G)$. Then from the equality of Plansherel

$$2\pi \int_{-\infty}^{+\infty} u^2(x, \xi)\, d\xi^1 = \int_{-\infty}^{+\infty} \left|\widehat{u}(x, \eta,' \xi)\right|^2 d\eta$$

follows that the integrals $\int_0^{+\infty} q^2(x, \eta,' \xi) d\eta$, $\int_0^{+\infty} p^2(x, \eta,' \xi) d\eta$ are continuous in the set $(D \times' G)$. Note that for $|\beta| \le 3$, $1 \le j \le n$ from relations (10), (16), (17) taking into account remark 1, we have that the integrals $\int_{-\infty}^{+\infty} (D^\beta_\xi u(x, \xi))^2\, d\xi^1$, $\int_{-\infty}^{+\infty} (D^\beta_\xi u_{x^j}(x, \xi))^2\, d\xi^1$uniformly converge with respect to the parameters $(x,' \xi) \in (D \times' G)$ and are continuous in the set $(D \times' G)$.

Then by the analogous reasoning for the continuity of the integrals $\int_0^{+\infty} q^2(x, \eta,' \xi) d\eta$, $\int_0^{+\infty} p^2(x, \eta,' \xi) d\eta$, it may be proved that for $|'\beta| \le 3$, the integrals $\int_0^{+\infty} (D'^{\beta}_{'\xi} q(x, \eta,' \xi))^2 d\eta$, $\int_0^{+\infty} (D'^{\beta}_{'\xi} p(x, \eta,' \xi))^2 d\eta$, $\int_0^{+\infty} (D'^{\beta}_{'\xi} q_{x^j}(x, \eta,' \xi))^2 d\eta$, $\int_0^{+\infty} (D'^{\beta}_{'\xi} p_{x^j}(x, \eta,' \xi))^2 d\eta$ are continuous in the set $(D \times' G\,)$. Consequently, for $|'\beta| \le 2$, the integrals $\int_0^{+\infty} (D'^{\beta}_{'\xi} F_2(x, \eta,' \xi))^2 d\eta$, $\int_0^{+\infty} (D'^{\beta}_{'\xi} F_{2x^j}(x, \eta,' \xi))^2 d\eta$ are continuous in the set $(D \times' G)$.

From (28), we have

$$q_\eta^2 \le (x^1 - x_0^1) \int_{x_0^1}^{x^1} F_2^2\left(\tau,' x, \eta,' \zeta(\tau,' \xi)\right) d\tau. \tag{29}$$

On the other hand for $x_0^1 \le x^1 \le x_0^1 + d_0$ there is an integral $\int_{x_0^1}^{x^1} (\int_0^{+\infty} F_2^2\,(\tau,' x, \eta,' \zeta(\tau,' \xi))\, d\eta) d\tau$ and it is bounded by a number $M > 0$, which does not depend on $(x,' \xi)$ (since the function $\int_0^{+\infty} F_2^2\,(x, \eta,' \xi)\, d\eta$ is continuous with respect to the parameters $(x,' \xi) \in (D \times' G)$ and sets D, $'G$ are closed and bounded).

Then for each $N > 0$ from (29) by Fubini- Tonelli theorem, we have

$$\int_0^N q_\eta^2(x, \eta,' \xi) d\eta \le d_0 \int_0^N (\int_{x_0^1}^{x^1} F_2^2\,(\tau,' x, \eta,' \zeta(\tau,' \xi))\, d\tau) d\eta =$$
$$= d_0 \int_{x_0^1}^{x^1} (\int_0^N F_2^2\,(\tau,' x, \eta,' \zeta(\tau,' \xi))\, d\eta) d\tau \le d_0 M \quad .$$

For $|'\beta| \le 2$, from (28) by analogous reasonings, it may be proved that

$\int_0^N (D'^{\beta}_{'\xi} q_\eta(x,\eta,'\xi))^2 d\eta$, $\int_0^N (D'^{\beta}_{'\xi} q_{\eta x^j}(x,\eta,'\xi))^2 d\eta \le d_0 M_1$, (30)
where $M_1 > 0$ is the maximum on $(D \times' G\ )$ of the continuous functions
$\int_{x_0^1}^{x^1} (\int_0^{+\infty} (D'^{\beta}_{'\xi} F_2(\tau,\eta,'\zeta(\tau,'\xi)))^2 d\eta) d\tau$, and $\int_{x_0^1}^{x^1} (\int_0^{+\infty} (D'^{\beta}_{'\xi} F_{2x^j}(\tau,\eta,'\zeta(\tau,'\xi)))^2 d\eta) d\tau$.

Inequalities (30) show that for $|'\beta| \le 2$, $D'^{\beta}_{'\xi} q_\eta$, $D'^{\beta}_{'\xi} q_{\eta x^j} \in L_2\left(\Delta^1_\eta\right)$.Then from (18), taking into account remark 1, it follows that $D'^{\beta}_{'\xi} q_\eta(x,\eta,'\xi)$, $D'^{\beta}_{'\xi} q_{\eta x^j}(x,\eta,'\xi) \in L_1\left(\Delta^1_\eta\right) \cap C\left(D \times \Delta^1_\eta \times' G\right)$. By similar reasonings it may be proved that $D'^{\beta}_{'\xi} q_\eta(x,\eta,'\xi), D'^{\beta}_{'\xi} q_{\eta x^j}(x,\eta,'\xi) \in L_1\left(\Delta^{-1}_\eta\right) \cap C\left(D \times \Delta^{-1}_\eta \times' G\right)$.

Using equation (23), analogous to that as these were proved for the function $q(x,\eta,'\xi)$, it may be showed that for the function $p(x,\eta,'\xi)$ for $|'\beta| \le 2$, $s = -1, 1$ the following relations holds :

$$D'^{\beta}_{'\xi} p_\eta, D'^{\beta}_{'\xi} p_{\eta x^j} \in L_1\left(\Delta^s_\eta\right) \cap L_2\left(\Delta^s_\eta\right) \cap C\left(D \times \Delta^s_\eta \times' G\right).$$

Lemma 2 is proved.

Corollary 2. Let the conditions of lemma 2 be satisfied.Then
$D'^{\beta}_{'\xi}\ \widehat{u}$, $D'^{\beta}_{'\xi}\ \widehat{u}_{x^j} \in C\left(D \times \overline{\Delta}^s_\eta \times' G\right)$, where $|'\beta| \le 2$, $1 \le j \le n$; $s = -1, 1$; $\overline{\Delta}^s_\eta = \left\{\eta \in R^1_\eta |\ s\eta \ge 0\right\}$

Proof. According to the proof of lemma 2 for $|'\beta| \le 2$, $1 \le j \le n$

$$D'^{\beta}_{'\xi}\ \widehat{u}_\eta, D'^{\beta}_{'\xi}\ \widehat{u}_{\eta x^j} \in L_1\left(\Delta^s_\eta\right) \cap L_2\left(\Delta^s_\eta\right) \cap C\left(D \times \Delta^s_\eta \times' G\right).$$

Then the equality :

$$D'^{\beta}_{'\xi}\ q(x,\eta,'\xi) = -\int_\eta^\infty D'^{\beta}_{'\xi}\ q_\tau(x,\tau,'\xi) d\tau \tag{31}$$

holds, from which we have that for the point $(x,'\xi) \in (D \times' G\ )$

$$D'^{\beta}_{'\xi}\ q(x,+0,'\xi) = -\int_0^\infty D'^{\beta}_{'\xi}\ q_\tau(x,\tau,'\xi) d\tau\ . \tag{32}$$

Taking into account (30) - (32) we will obtain that

$$\left|D'^{\beta}_{'\xi} q(x,\eta,'\xi) - D'^{\beta}_{'\xi} q(x,+0,'\xi)\right|^2 = \left|\int_0^\eta D'^{\beta}_{'\xi} q_\tau(x,\tau,'\xi) d\tau\right|^2 \le$$
$$\le\ \eta \int_0^\eta (D'^{\beta}_{'\xi} q_\tau(x,\tau,'\xi))^2 d\tau\ \le\ \eta d_0 M_1$$

from which it follows that for $\eta \to +0$ function $D'^{\beta}_{'\xi} q(x,\eta,'\xi)$ tends to $D'^{\beta}_{'\xi} q(x,+0,'\xi)$ uniformly with respect to the parameters $(x,'\xi) \in (D \times' G)$. Consequently, $D'^{\beta}_{'\xi} q(x,+0,'\xi) \in C\left(D \times' G\right)$, since for $\eta > 0$ the function $D'^{\beta}_{'\xi} q(x,\eta,'\xi) \in C\left(D \times' G\right)$.

Analogously it may be proved that for $|'\beta| \le 2$, $1 \le j \le n$ the functions $D'^{\beta}_{'\xi} q_{x^j}(x,+0,'\xi)$, $D'^{\beta}_{'\xi} p(x,+0,'\xi)$, $D'^{\beta}_{'\xi} p_{x^j}(x,+0,'\xi)$, $D'^{\beta}_{'\xi} q(x,-0,'\xi)$, $D'^{\beta}_{'\xi} q_{x^j}(x,-0,'\xi)$, $D'^{\beta}_{'\xi} p(x,-0,'\xi)$, $D'^{\beta}_{'\xi} p_{x^j}(x,-0,'\xi)$ belong to the space $C\left(D \times' G\right)$.

Corollary 2 is proved.

For the known function $\overline{v}(\overline{x}) \in C^5_0(\widetilde{D})$, we introduce function $T\ ^v(\overline{x},\overline{\xi})$ of variables $\overline{x} = (x, x^{n+1}) = (x^1, ..., x^n, x^{n+1})$ and $\overline{\xi} = (\xi, \xi_{n+1}) = (\xi^1, ..., \xi^n, \xi^{n+1})$ by formula $T^v(\overline{x},\overline{\xi}) = \int_{\overline{\gamma}(\overline{x},\overline{\xi})} \left[\sum\limits_{i,j=2}^n a_{ij}\left(z\left(\overline{x},\overline{\xi},t\right)\right) \dot{z}^i\left(\overline{x},\overline{\xi},t\right) \dot{z}^j(\overline{x},\overline{\xi},t) + \overline{v}(\overline{z}\left(\overline{x},\overline{\xi},t\right))\right] dt$, where $\overline{\gamma}(\overline{x},\overline{\xi})$ is the ray of the metric $\overline{g}(\overline{x})$, which emerges from point $\overline{x} = (x, x_{n+1}) \in \widetilde{D}$ in the direction $\overline{\xi} = (\xi, \xi^{n+1}) \in \widetilde{G}$ with element of length $(ds)^2 = \sum\limits_{i,j=2}^n g_{ij}(x) dx^i dx^j + (dx^{n+1})^2$, and $\widetilde{D} = D \times (a_{n+1}, b_{n+1})$,

$\widetilde{G} = G \times (\frac{1}{4}, \frac{3}{4})$, $0 < a_{n+1} < b_{n+1} \leq d_0$. Here $\overline{\gamma}(\overline{x}, \overline{\xi})$ is solution of the problem of Cauchy (3) -(4), where $\overline{x} = (x, x^{n+1})$, $\overline{\xi} = (\xi, \xi^{n+1})$, $\overline{z} = \left(z\left(\overline{x}, \overline{\xi}, t\right), \, z^{n+1}\left(\overline{x}, \overline{\xi}, t\right)\right)$ is taken instead of n, x, ξ, z respectively.

Remark 2. Since functions $g_{ij}(x)$ $(1 \leq i, j \leq n)$ do not depend on x^{n+1}, the following equalities hold : $\Gamma^{n+1}_{ij} = \Gamma^k_{n+1j} = \Gamma^k_{in+1} = 0$ $(1 \leq i, j, k \leq n+1)$. Let us note that geodesic $\overline{\gamma}(\overline{x}, \overline{\xi})$ $(\overline{x} \in \widetilde{D}, \overline{\xi} \in S^{n+1})$ is determined from the problem (3) -(4) corresponding to the metric $\overline{g}(\overline{x})$ with the data $\overline{z}(0) = \overline{x}$, $\frac{d}{dt}\overline{z}(0) = \overline{\xi}$.

It is easy to see that the function $T^{\,v}(\overline{x}, \overline{\xi})$ satisfies the equation

$$\sum_{j=1}^{n+1} \xi^j T^v_{x^j} - \sum_{i,j,k=1}^{n} \Gamma^k_{ij} \xi^i \xi^j T^v_{\xi^k} = \overline{v}(\overline{x}) + \sum_{i,j=2}^{n} a_{ij}(x) \xi^i \xi^j \qquad (33)$$

Since the function $\overline{v}(\overline{x})$ is known and the function $u(x, \xi) = \sum_{i,j=2}^{n} \int_{\gamma(x,\xi)} a_{ij}\left(z\left(x, \xi, t\right)\right) \dot{z}^i\left(x, \xi, t\right) \dot{z}^j\left(x, \xi, t\right) dt$ is given on $\partial D \times G$, for $(\overline{x}, \overline{\xi}) \in \widetilde{\Gamma}$, it is possible to calculate the values of function $T^v(\overline{x}, \overline{\xi})$, where $\widetilde{\Gamma} = \partial D \times (a_{n+1}, b_{n+1}) \times \widetilde{G}$. Indeed, since functions $a_{ij}(x)$ $(2 \leq i, j \leq n)$ do not depend on x^{n+1}, $T^0_1(\overline{x}, \overline{\xi})$ $= \sum_{i,j=2}^{n} \int_{\overline{\gamma}(\overline{x},\overline{\xi})} a_{ij}\left(z\left(\overline{x}, \overline{\xi}, t\right)\right) \dot{z}^i\left(\overline{x}, \overline{\xi}, t\right) \dot{z}^j(\overline{x}, \overline{\xi}, t) dt$ $(\overline{x} \in \widetilde{D}, \overline{\xi} \in S^{n+1})$. Here integration is taken not on $\overline{\gamma}(\overline{x}, \overline{\xi})$, but along its projection to R^n_x, i.e. on $\gamma(x, \xi)$:
$T^0_1(\overline{x}, \overline{\xi}) = \sum_{i,j=2}^{n} \int_{\gamma(x,\xi)} a_{ij}\left(z\left(x, \xi, t\right)\right) \dot{z}^i\left(x, \xi, t\right) \dot{z}^j(x, \xi, t) dt$, where $\overline{\xi} \in S^{n+1}$, $\overline{\xi} = (\xi, \xi^{n+1})$, $\xi = (\xi^1, ..., \xi^n)$, $(\xi^1)^2 + \sum_{i,j=2}^{n} g_{ij}(x) \xi^i \xi^j + (\xi^{n+1})^2 = 1$. For each fixed (x^{n+1}, ξ^{n+1}) $(a_{n+1} \leq x^{n+1} \leq b_{n+1}$, $0 \leq \xi^{n+1} \leq \frac{3}{4})$ let us determine the vector $\nu = (\xi^1, ..., \xi^n)(1 - (\xi^{n+1})^2)^{-\frac{1}{2}} \in S^n$. By the condition of the problem 1, for $x \in \partial D$, $\nu \in S^n$ it is known the function $u(x, \nu) = \sum_{i,j=2}^{n} \int_{\gamma(x,\nu)} a_{ij}\left(z\left(x, \nu, t\right)\right) \dot{z}^i\left(x, \nu, t\right) \dot{z}^j\left(x, \nu, t\right) dt$; then for $x \in \partial D$, $\xi = \nu(1 - (\xi^{n+1})^2)^{\frac{1}{2}}$ and $\overline{\xi} \in S^{n+1}$, where $0 \leq \xi^{n+1} \leq \frac{3}{4}$, a function $T^0_1(\overline{x}, \overline{\xi}) = \sum_{i,j=2}^{n} \int_{\gamma(x,\xi)} a_{ij}\left(z\left(x, \xi, t\right)\right) \dot{z}^i\left(x, \xi, t\right) \dot{z}^j(x, \xi, t) dt$ can be calculated by formula (10), using values $u(x, \nu)$ on $(x, \nu) \in \partial D \times S^n$. Therefore, for $\overline{x} \in \partial D \times (a_{n+1}, b_{n+1})$ and for $\overline{\xi} \in R^{n+1}_0$, when $\frac{1}{4} \leq \xi^{n+1} \leq \frac{3}{4}$, the function $T^{\,0}_{\,1}(\overline{x}, \overline{\xi})$ (using it's values on S^{n+1}, when $0 \leq \xi^{n+1} \leq \frac{3}{4}$, $\overline{x} \in \partial D \times (a_{n+1}, b_{n+1})$) can be calculated by analogy of formula (10) for $T^0_1(\overline{x}, \overline{\xi})$. On the other hand calculation of the integral $T^0_2(\overline{x}, \overline{\xi}) = \int_{\overline{\gamma}(\overline{x},\overline{\xi})} \overline{v}(\overline{z}\left(\overline{x}, \overline{\xi}, t\right)) dt$ is not difficult.

So we can consider the following problem :

Problem 5. Determine a vector function $(a_{ij}(x))^n_2$ from equation (33) if $T^{\,v}$ is known on $\widetilde{\Gamma}$.

It is obvious that the uniqueness of solution to the problem 1 follows from the uniqueness of solution to the problem 5 in class $C^5_0(\widetilde{D})$.

¿From the reasonings connected with calculation of $T^0_1(\overline{x}, \overline{\xi})$ it is evident that, function $T^0_1(\overline{x}, \overline{\xi})$ depends on ξ^{n+1} complexly. In order to explain how $T^0_1(\overline{x}, \overline{\xi})$ depends on x^{n+1} let us note that $z^{n+1}(t) \equiv z^{n+1}\left(\overline{x}, \overline{\xi}, t\right) = x^{n+1} + t\xi^{n+1}$. Then $T^0_1(\overline{x}, \overline{\xi}) =$
$\sum_{i,j=2}^{n} \int_{\gamma(x,\overline{\xi})} a_{ij}\left(z\left(\overline{x}, \overline{\xi}, t\right)\right) \dot{z}^i\left(\overline{x}, \overline{\xi}, t\right) \dot{z}^j(\overline{x}, \overline{\xi}, t) dt = \sum_{i,j=2}^{n} \int_{\gamma(x,\overline{\xi})} a_{ij}\left(z\left(x, \nu, |\xi|\, t\right)\right) |\xi|\, \dot{z}^i\left(x, \nu, |\xi|\, t\right)$

$|\xi|\,\dot{z}^j(x,\nu,|\xi|\,t)dt = \sum\limits_{i,j=2}^{n} \int_{\gamma(x,\bar{\xi})} a_{ij}\left(z\left(x,\nu,\xi^{n+1}t(\frac{1}{\xi^{n+1}}-1)^{\frac{1}{2}}\right)\right)|\xi|\,\dot{z}^i\left(x,\nu,\xi^{n+1}t(\frac{1}{\xi^{n+1}}-1)^{\frac{1}{2}}\right)$
$|\xi|\,\dot{z}^j(x,\nu,\xi^{n+1}t(\frac{1}{\xi^{n+1}}-1)^{\frac{1}{2}})dt = \sum\limits_{i,j=2}^{n} \int_{\gamma(x,\bar{\xi})} a_{ij}\left(z\left(x,\nu,(z^{n+1}(t)-x^{n+1})(\frac{1}{\xi^{n+1}}-1)^{\frac{1}{2}}\right)\right)$
$|\xi|\,\dot{z}^i\left(x,\nu,(z^{n+1}(t)-x^{n+1})(\frac{1}{\xi^{n+1}}-1)^{\frac{1}{2}}\right)|\xi|\,\dot{z}^j(x,\nu,(z^{n+1}(t)-x^{n+1})(\frac{1}{\xi^{n+1}}-1)^{\frac{1}{2}})dt$.

It is clear that function $T^{\,0} = T^{\,v}_{\,2} - T^v_1 = T^0_1(\overline{x},\overline{\xi})$, satisfies the equation

$$\sum_{j=1}^{n+1}\xi^j T^0_{x^j} - \sum_{i,j,k=1}^{n}\Gamma^k_{ij}\xi^i\xi^j T^0_{\xi^k} = \sum_{i,j=2}^{n} a_{ij}(x)\xi^i\xi^j \tag{34}$$

and condition $T^{\,0} = 0$ on $\widetilde{\Gamma}$, where

$T^v_k = \int_{\overline{\gamma}(\overline{x},\overline{\xi})}\left[\sum\limits_{i,j=2}^{n} a^{(k)}_{ij}\left(z\left(\overline{x},\overline{\xi},t\right)\right)\dot{z}^i\left(\overline{x},\overline{\xi},t\right)\dot{z}^j(\overline{x},\overline{\xi},t) + \overline{v}(\overline{z}\left(\overline{x},\overline{\xi},t\right))\right]dt,\ a_{ij}(x) = a^{(2)}_{ij}(x) - a^{(1)}_{ij}(x),\ k = 1,2.$

Remark 3. The analysis of the proof of lemmas 1, 2 and corollaries 1, 2 shows that the function $T^{\,0}$ satisfies all the assertions of lemmas 1, 2 and corollaries 1,2 on set $\widetilde{D}\times\widetilde{G}$, where $'G$ contains the origin. Moreover all these assertions for function $T^{\,0}$ $(\widehat{T^0}, p_0, q_0)$ may be proved analoguosly as they are proved for the function u $(\widehat{u},\,p,q)$, where $\widehat{T^0} = p_0 + iq_0$ is the Fourier transform of the function $T^{\,0}$ with respect to ξ^1. We will not repeat here corresponding reasonings, let us simply note that in the proofs we substantially use the analog of formula (10) for the function T^0 and the analogs of the equations (23), (24) corresponding to equation (34).

Remark 4. Since the functions $a_{ij}(x)$ do not depend on x^{n+1}, taking into account remark 2, from the definition of $T^{\,0}$ and from equality $u(x,\xi) = 0$ on $\partial D\times R^n_0$ (see (9)) it follows that for $(\overline{x},\xi^{n+1}) \in \widetilde{D}\times(\frac{1}{4},\frac{3}{4})$, $T^{\,0}(\overline{x},\xi,\xi^{n+1}) + T^{\,0}(\overline{x},-\xi,\xi^{n+1}) = 0$. By remark 3, it is possible to apply the generalized Fourier transform in the variable ξ^1 to equation (34). Then for $\widehat{u} = \widehat{u}(\overline{x},\eta,'\xi,\xi^{n+1})$ which is the Fourier transform of $u(\overline{x},\overline{\xi}) = T^{\,0}(\overline{x},\overline{\xi})$, we have:

$$i\widehat{u}_{x^1\eta} - 2i\sum_{j,k=2}^{n}\Gamma^j_{1k}\xi^k\ \widehat{u}_{\xi^j\eta} + \sum_{j=2}^{n+1}\xi^j\widehat{u}_{x^j} - i\sum_{j,k=2}^{n}\Gamma^1_{jk}\xi^k\xi^j\eta\ \widehat{u} - \\ - \sum_{j,k,s=2}^{n}\Gamma^s_{jk}\xi^k\xi^j\widehat{u}_{\xi^s} = 2\pi\delta(\eta)\sum_{k,j=2}^{n} a_{kj}(x)\,\xi^k\xi^j \tag{35}$$

¿From the definition of the function $u(\overline{x},\overline{\xi})$ and from the fact that, for $'\xi = 0$ and $2 \le k \le n$, $\dot{z}^k\left(\overline{x},\xi^1,0,\xi^{n+1},t\right) = 0$ (see remark 2, taking into account the uniqueness of solution of problem (3) -(4)) it follows that $u(\overline{x},\xi^1,0,\xi^{n+1}) = u_{\xi^i}(\overline{x},\xi^1,0,\xi^{n+1}) = 0$ $(1 \le i \le n)$, then $u_{\xi^i\xi^1}(\overline{x},\xi^1,0,\xi^{n+1}) = u_{x^j\xi^i\xi^1}(\overline{x},\xi^1,0,\xi^{n+1}) = 0 (1 \le i \le n,\ 1 \le j \le n+1)$. Taking into account last equalities and analog of formula (10) for the function $u(\overline{x},\overline{\xi})$, by the same way as we proved 1) of lemma 1, we may prove that

$$u,\ u_{\xi^i}, u_{x^j\xi^i}, u_{\xi^1\xi^i}, u_{\xi^1\xi^i x^j} \to 0 \quad \text{in } L_2(R^1_{\xi^1}). \qquad as\ '\xi \to 0 \tag{36}$$

uniformly with respect to $(\overline{x},\xi^{n+1}) \in \widetilde{D}\times(\frac{1}{4},\frac{3}{4})$

Corollary 3. As $'\xi \to 0$, the functions $p, p_{x^j}, p_{\xi^i}, p_{\xi^i x^j}, \eta q$ tend to zero (uniformly with respect to $(\overline{x},\xi^{n+1})$) in space $L_1(R^1_\eta)$, where $\widehat{u} = \widehat{u}(\overline{x},\eta,'\xi,\xi^{n+1}) = p(\overline{x},\eta,'\xi,\xi^{n+1}) +\ iq(\overline{x},\eta,'\xi,\xi^{n+1})$, $(2 \le i \le n, 1 \le j \le n+1)$.

Proof. Since the Fourier transform is continuous in space $L_2(R^1_{\xi^1})$, from (36) it follows that as $'\xi \to 0$ functions $\widehat{u},\widehat{u}_{\xi^i},\eta\widehat{u},\eta^2\widehat{u},\eta\widehat{u}_{\xi^i},\eta\widehat{u}_{x^j},\widehat{u}_{x^j\xi^i},\eta^2\widehat{u}_{x^j},\eta\widehat{u}_{x^j\xi^i}$ tend to zero in space $L_2(R^1_\eta)$ $(2 \le$

$i \leq n, 1 \leq j \leq n{+}1$). Then from the inequalities $\int_1^\infty \left|p_{x^j\xi^i}\right| d\eta \leq \left(\int_1^\infty \eta^2 p^2_{x^j\xi^i} d\eta\right)^{\frac{1}{2}} \left(\int_1^\infty \frac{1}{\eta^2} d\eta\right)^{\frac{1}{2}}$, $\int_0^1 \left|p_{x^j\xi^i}\right| d\eta \leq \left(\int_0^1 p^2_{x^j\xi^i}\, d\eta\right)^{\frac{1}{2}}$ and from $\int_1^\infty \eta^2 p^2_{x^j\xi^i} d\eta \to 0$, $\int_0^1 p^2_{x^j\xi^i}\, d\eta \to 0$ it follows that, when $'\xi \to 0$, the function $p_{x^j\xi^i}$ tends to zero in space $L_1(0,\infty)$ uniformly with respect to $(\overline{x}, \xi^{n+1})$.

By the same arguments it can be proved that, when $'\xi \to 0$, the function $p_{x^j\xi^i}$ tends to zero in $L_1(-\infty, 0)$. The remaining assertions of corollary 3 may be proved analogously.

Now let us prove that, for $(\overline{x},' \xi, \xi^{n+1}) \in \widetilde{D} \times' G \times (\frac{1}{4}, \frac{3}{4})$ and $2 \leq i,j \leq n$ a function $\widehat{u}_{x^{n+1}\xi^i\xi^j}$ belongs to space $L_1(R^1_\eta)$. In fact, since $u_{x^{n+1}\xi^1\xi^i\xi^j} \in L_2(R^1_{\xi^1})$, we have $\eta \widehat{u}_{x^{n+1}\xi^i\xi^j} \in L_2(R^1_\eta)$. Then $\int_1^\infty \left|\widehat{u}_{x^{n+1}\xi^i\xi^j}\right| d\eta \leq \left(\int_1^\infty \eta^2 \left|\widehat{u}_{x^{n+1}\xi^i\xi^j}\right|^2 d\eta\right)^{\frac{1}{2}} \left(\int_1^\infty \frac{1}{\eta^2} d\eta\right)^{\frac{1}{2}}$, i.e. $\widehat{u}_{x^{n+1}\xi^i\xi^j} \in L_1(1,\infty)$. Analogously we have: $\widehat{u}_{x^{n+1}\xi^i\xi^j} \in L_1(-\infty,-1)$. On the other hand $u_{x^{n+1}\xi^i\xi^j} \in L_2(R^1_{\xi^1})$ and $\widehat{u}_{x^{n+1}\xi^i\xi^j} \in L_2(R^1_\eta)$, which means $\widehat{u}_{x^{n+1}\xi^i\xi^j} \in L_1(-1,1)$. Thus for $(\overline{x},' \xi, \xi^{n+1}) \in \widetilde{D} \times' G \times (\frac{1}{4}, \frac{3}{4})$, $\widehat{u}_{x^{n+1}\xi^i\xi^j} \in L_1(R^1_\eta)$, and

$$u_{x^{n+1}\xi^i\xi^j}(\overline{x}, 0,' \xi, \xi^{n+1}) = \int_{-\infty}^{\infty} p_{x^{n+1}\xi^i\xi^j}(\overline{x}, \eta,' \xi, \xi^{n+1}) d\eta$$

$$= \frac{\partial^2}{\partial \xi^i \partial \xi^j} \left(\int_{-\infty}^{\infty} p_{x^{n+1}}(\overline{x}, \eta,' \xi, \xi^{n+1}) d\eta\right). \tag{37}$$

Last equalities follow from the theorem of differentiability on parameter of integral. Since the functions $u_{x^{n+1}\xi^i\xi^j}(\overline{x}, \overline{\xi})$ are continuous (see remark 3 and lemma 1) as $'\xi \to 0$, for $(\overline{x}, \xi^{n+1}) \in \widetilde{D} \times (\frac{1}{4}, \frac{3}{4})$ we have:

$$u_{x^{n+1}\xi^i\xi^j}(\overline{x}, 0,' \xi, \xi^{n+1}) \to\ u_{x^{n+1}\xi^i\xi^j}(\overline{x}, 0, \xi^{n+1}). \tag{38}$$

In [21] it is proved

Theorem 3.1.3. Let the function $U(y)$, determined on the open set $Y \subset R$, belong to space $C^1(Y \,/\, \{y_0\})$ for some $y_0 \in Y$ and let the function $V(y)$, which coincides with $U\,'(y)$ for $y \neq y_0$, be integrable on some neighbourhood of the point y_0. Then there exists $U(y_0 \pm 0) = \lim\limits_{y \to y_0 \pm} U(y)$ and $U\,'(y) = V(y) + (U(y_0+0) - U(y_0-0))\delta_{y_0}$.

Let $\Gamma_k(x, x_0)$ be the ray of the metric $g_k(x) \in C^6(D)$ $(k = 1,2)$ connecting points $x \in D$ and $x_0 \in \partial D_{\varepsilon_0}$, $x \neq x_0$ and $\Gamma_2(x,x_0)$, $\Gamma_1(x,x_0)$ emanate from the point $x \in D$ at angles ξ and $\nu(\xi)$, respectively, where $\xi = (\xi^1, ..., \xi^n)$, $\nu = (\nu^1, ..., \nu^n)$, $\xi^i = g_2^{ij}(x) \left(\tau_2(x,x_0)\right)_{x^j}$, $\nu^k('\xi) = \sum\limits_{s=2}^{n} g_1^{ks}(\tau_1(x,x_0))_{x_s}$, $\tau_k(x,x_0)$ be the distance between points $x \in D$ and $x_0 \in \partial D_{\varepsilon_0}$ in the metric $g_k(x)$. In view of the convexity of the region D_{ε_0} with respect to the metric $g_k(x) \in C^6(D)$ $(k = 1,2)$, for each fixed $x \in D$ the functions $\xi = \xi(x,x_0)$ and $\nu = \nu(x,x_0)$ are invertible: $x_0 = x_0(\xi) \in C^5(S_1^{(2)})$, $x_0 = x_0(\xi) \in C^5(S_1^{(2)})$, where $S_1^{(k)}$ is the unit sphere of metric $g_k(x)$ centered at $x \in D$, $k = 1,2$. Consequently each vector $\xi \in S_1^{(2)}$ is assigned to vector $\nu \in S_1^{(1)}$ such that vectors ξ and ν correspond to the same $x_0 \in \partial D_{\varepsilon_0}$. lt is evident that a function $\nu = \nu(\xi)$ defined in this manner is invertible. In addition, let the Jacobian $\det \frac{\partial\, x_0}{\partial\, \nu} > 0$, $\det \frac{\partial\, x_0}{\partial\, \xi} > 0$ are positive, and so $\det \frac{\partial\, \nu}{\partial\, \xi} > 0$.

The equalities are true : $(\tau_1)_{x^1} = \nu^1$, for $2 \leq i \leq n$, $(\tau_1)_{x^i} = \sum\limits_{j=2}^{n} g_{ij}^{(1)} \nu^j$, and also $(\xi^1)^2 =$

$1-\sum\limits_{i,j=2}^{n} g_{ij}^{(2)}\xi^i\xi^j$ (sign of ξ^1 is determined by vector ξ). Recalling the invertibility of the function $\nu=\nu(\xi)$ on $S_1^{(2)}$, let us determine the function $f(\xi)=(f^1(\xi),...,f^n(\xi))$ as follows: $f^1(\xi)=(\tau_1)_{x^1}$, and for $2\le i\le n$, $f^i(\xi)=f^i('\xi)=\sum\limits_{j=2}^{n} g_2^{ij}(\tau_1)_{x^j}=\sum\limits_{k,j=2}^{n} g_2^{ij}g_{kj}^{(1)}\nu^k('\xi)$, where $\nu^k('\xi)=\sum\limits_{s=2}^{n} g_1^{ks}(\tau_1(x,x_0))_{x^s}$. Since $\det\frac{\partial\,\nu}{\partial\,\xi}>0$, then $\det\frac{\partial\,f}{\partial\,\xi}>0$.

Let

$$\eta^i\;=\;\sum_{j=1}^{n}\left(\frac{\partial\xi^i}{\partial z^j}+\sum_{k=1}^{n}\frac{\partial f^i}{\partial\xi^k}\;\frac{\partial\xi^k}{\partial z^j}\right)\zeta^j,\qquad 1\;\le\;i\;\le\;n,\tag{39}$$

where $\xi^i\;=\;\sum\limits_{j=1}^{n} g_2^{ij}\,(z)\,(\tau_2\,(z,x_0))_{z^j}$; $f^1\,(\xi)\;=\;(\tau_1\,(z,x_0))_{z^1}$,for $2\;\le\;s\;\le\;n$, $f^s\,(\xi)\;=\;f^s\,('\xi)\;=\;\sum\limits_{j=2}^{n} g_2^{sj}\,(z)\,(\tau_1\,(z,x_0))_{z^j}$, $\zeta^i=\xi^i\,(z,x_0)+f^i\,(\xi\,(z,x_0))\,.$

Let $x_0\in\partial D_{\varepsilon_0}$. Let us examine the Cauchy problem for the system

$$\tfrac{d}{dt}z^i\;=\;\zeta^i,\;\tfrac{d^2}{dt^2}z^i\;=\;\eta^i,\quad 1\;\le\;i\;\le\;n,\tag{40}$$

with the data

$$z\,(0)\;=\;x,\;\tfrac{d}{dt}z\,(0)\;=\;\dot{z}\,(0)\;=\;\zeta_0,\tag{41}$$

where $x\in D$, $\zeta_0=\left(\zeta_0^1,\cdots,\zeta_0^n\right)$, $\zeta_0^i=\sum\limits_{j=1}^{n} g_2^{ij}\,(x)\,(\tau_2\,(x,x_0)+\tau_1\,(x,x_0))_{x^j}$.

Under the assumptions of the theorem 3, the conditions of the theorem of existence and uniqueness of the solution to the Cauchy problem (40)-(41) hold. Hence a solution to problem (40)-(41) exists on a certain interval. Since the system (40) is $t-$independent, this solution $\gamma^+\,(x,\zeta_0)$ can be continued until the point $x_0\in\partial D_{\varepsilon_0}$. Moreover this solution is continuously dependent on the Cauchy data, i.e., $\gamma^+\,(x,\zeta_0)$ is five time differentiable with respect to x, ζ, t, (see [20]). It is not difficult to see that in the domain D_{ε_0}/D, $\gamma^+\,(x,\zeta_0)$ coincides with the ray of the metric $g_k=\left(g_{ij}^{(k)}\right)$, $k=1,2$.

Using these observations, we can prove that, at every fixed $x\in D$, the equation

$$F(\xi)\;\equiv\;\xi+f(\xi)\;=\;\zeta\tag{42}$$

can be solved uniquely on $S_1^{(2)}$: $\xi=F^{-1}(\zeta)$, where $\xi\in S_1^{(2)}$, $\zeta=F(\xi)$, $F^{-1}(\zeta)$ is five times continuously differentiable in its domain. In fact, by P_{12} let us denote 2- dimensional plane containing vectors $\xi_1,\xi_2\in S_1^{(2)}$ (i.e. the linear span of the vectors $\xi_1,\xi_2\in S_1^{(2)}$ passing through the point $x\in D$).

Let us construct the orthogonal system of coordinates on P_{12} (with the origin in $x\in D$) such that one of the coordinate axis coincides with vector ξ_1, and orientation of P_{12} is the same with orientation of $D\subset R^n$ $\left(\text{i.e. }\det\frac{\partial x_0}{\partial\xi}>0\right)$.

Let $\gamma^+(x,\xi)$ intersect $\partial D_{\varepsilon_0}$ at the point $z(\xi,P_{12})$, where $\xi\in S_1^{(2)}\cap P_{12}$, $\gamma^+(x,\xi)$ is the solution of Cauchy problem (40)-(41).

Let $C(P_{12})$ be a closed curve on $\partial D_{\varepsilon_0}$, consisting of points $z(\xi,P_{12}):C(P_{12})=\left\{z(\xi,P_{12});\;\xi\in S_1^{(2)}\cap P_{12}\right\}$, t is the length of the part of $C(P_{12})$ between points $z(\xi_1,P_{12})$ and $z(\xi,P_{12})$ in the metric g_2. Since the function $x_0=x_0(\xi)$ is invertible and the Jacobian $\det\frac{\partial x_0}{\partial\xi}>0$, the positive direction of movement of vector $\xi = \left(\xi^1(\theta), \ldots, \xi^n(\theta)\right)$ on unit circle of P_{12} centered at $x \in D$ (i.e. increasing of θ) corresponds to the positive direction of movement on $C(P_{12})$ of the point $z(\xi, P_{12})$ (i.e. to increasing of t). Then taking into account convexity of domain D_{ε_0} with respect to the metric g_2, we have that θ is an increasing function of t.

Since $\det \frac{\partial f}{\partial \xi} > 0$, positive direction of movement of vector ξ (i.e. increasing of θ and therefore, increasing of t) corresponds to the positive direction of movement of vector $f(\xi)$. Let $\xi_2 = \xi\,(t_0)\,,\ f\,(\xi_2) = f\,(\xi\,(t_0))\,,$ where $t_0 \in (0, d_e)\,,\ d_e$ is the length of curve $C(P_{12})$ in the metric g_2. Now we will prove that $\xi_1 \neq \xi_2$ implies $F\,(\xi_1) \neq F\,(\xi_2)$. Assume the contrary : $F\,(\xi_1) = F\,(\xi_2)$; then $\xi_2 = \xi_1 + f\,(\xi_1) - f\,(\xi_2)$ and from the continunity of the function $f(\xi)$ at the point $\xi_2 \neq 0$ it follows that as $t \to t_0 \ \ (t \in (0, d_e))$, $\xi(t) = \xi_1 + f\,(\xi_1) - f\,(\xi(t)) + O\,(t - t_0)\,.$ The last relation contradicts the assertion: " the positive direction of movement of vector $\xi(t)$ corresponds to the positive direction of movement of vector $f(\xi\,(t))$." Therefore $F\,(\xi_1) \neq F\,(\xi_2)\,,$ i.e. function $\zeta = F\,(\xi)$ is invertible on $S_1^{(2)}$. Obviously if we extend the function $f(\xi)$ by the formula $f(l\xi) = lf\,(\xi) \ \ \left(l > 0,\ \xi \in S_1^{(2)}\right)$ from the set $S_1^{(2)}$ over the set R_0^n (the functions $F\,(\xi)$ and $F^{-1}\,(\zeta)$ will then also be homogeneous functions), then equation (39) will be uniquely solvable in R_0^n, where R_0^n is R^n without origin. Here, the degrees of smoothness of the functions $\xi = F^{-1}\,(\zeta)$ and $\eta = f(\xi)$ coincide.

Moreover , if

$$\xi^i + f^i\,(\xi) = \zeta^i, \quad i = 2, \ldots, n \tag{43}$$

then expressing ξ^1 in terms of $\xi^2, \xi^3, \ldots, \xi^n$ in the equality $\|\xi\|_{g_2} = 1$ (the sign of ξ^1 is determined by the vector ξ) and substituting the result in (43), we have

$$\pi_i\,('\xi) = \xi^i + f^i\left(\xi^1\left(\xi^2, \ldots, \xi^n\right), \xi^2, \ldots, \xi^n\right) = \zeta^i, \quad i = 2, \ldots, n. \tag{44}$$

In the same way as in the case of equation (42) we can prove that in R_0^{n-1} system (44) has a unique solution $'\xi = \pi^{-1}\,('\zeta)\,,$ where $\pi^{-1}\,(\,'\zeta) \in C^5\left(R_0^{n-1}\right).$

For $\xi \in S_1^{(2)}$, the definition of the function $f(\xi)$ and equality (44) give

$$\nu_i^{(1)}\,(x, x_0) = f^i\,(\xi) = \zeta^i - \xi^i = \zeta^i - \pi_i^{-1}\,('\zeta)\,, \quad i = 2, \ldots, n. \tag{45}$$

Let $h(\zeta) = \left(h^2\,('\zeta), \ldots, h^n\,('\zeta)\right),\ \ h^i\,('\zeta) = \zeta^i - \pi_i^{-1}\,('\zeta)\,, \quad i = 2, \ldots, n,$ where $\zeta \in S_2^{(2)}$, $h(\zeta) \in S_1^{(2)}$. Equalities (45) yield $h('\zeta) =' f('\xi)$, where $'\xi = \pi^{-1}\,('\zeta)\,.$ Then from the uniqueness of the solution of the equation (44) and from the inequality $\det \frac{\partial' f}{\partial' \xi} > 0$ we get that the Jacobian $\det \frac{\partial' \xi}{\partial' \zeta} > 0$, therefore, $\det \frac{\partial h}{\partial' \zeta} = \det \frac{\partial' f}{\partial' \xi} \det \frac{\partial' \xi}{\partial' \zeta} > 0$.

Note that the conditions $g_\kappa^{1j} = 0;\ j = 2, 3, \ldots, n$ and $g_\kappa^{11} = 1$, the uniqueness of the ray $\gamma_\kappa\left(x, \nu^0\right)$ and relations (6), (7), we see that $\gamma_\kappa\left(x, \nu^0\right) = x + t\nu^0$, where $\kappa = 1, 2$ and $\nu^0 = (1, 0, \ldots, 0) \in R^n$. Therefore, we get $f\left(\nu^0\right) = \nu^0$ and $\zeta_0 = 2\nu^0$ (see (42)). Thus according to the theorem about continuous dependence of the solution to the Cauchy problem (which is the defining ray of the metric $g_\kappa^{(x)}$ for $\kappa = 1, 2$) on the initial data and the condition $f \in C^5\left(S_1^{(2)}\right)$, we obtain that $f\,(\xi)$ and ζ tend to ν_0 and $2\nu_0$, respectively, as $\xi \to \nu^0 \ \ \left(\xi \in S_1^{(2)}\right)$. Therefore, as $'\xi \to 0$, the functions $'f('\xi) = \left(f^2('\xi), \ldots, f^n('\xi)\right)$ and $'\zeta$ tend to zero; hence, by virtue of the smoothness of the function $\pi^{-1}\left('\zeta\right)$, we have $\pi^{-1}\left('\zeta\right)\to 0$, which means that the function $h\left(x,'\zeta\right)=\left(h^2\left(x,'\zeta\right),\ldots,h^n\left(x,'\zeta\right)\right)$ has the same property; namely,

$$h(x,0)=0, \qquad h\left(x,'\zeta\right)\to 0 \quad \text{as} \quad '\zeta\to 0 \tag{46}$$

Since the functions $f(\xi)$, $F(\xi)$, $F^{-1}(\zeta)$ are homogeneous of the first order, the function $h\left('\zeta\right)$ is also a homogeneous function of the first order.

Remark 5. Since $\gamma_k(x,\nu^0)=x+t\nu^0$ is the ray of the metric $g_k(x)$, $k=1,2$, then it is not difficult to see that it is possible to represent $f^1(\xi)$ - the first component of vector $f(\xi)$ - in the form $f^1(\xi)=\xi^1+\widetilde{f}('\xi)$, where $\widetilde{f}(0)=0$, $\xi\in S_1^{(2)}$.

Let $\xi=\overline{F}(\zeta)=\left(\overline{F}^1(\zeta),\cdots,\overline{F}^n(\zeta)\right)$, where $\overline{F}(\zeta)$ is the inverse function of $\zeta=F(\xi)$. Here for $2\le s\le n$, the function $\overline{F}^s(\zeta)=\overline{F}^s('\zeta)=\xi^s$ depends on $'\zeta$ and does not depend on ζ^1. Since the functions $\zeta=F(\xi)$, $\nu=f(\xi)$, $\xi=F^{-1}(\zeta)=\overline{F}(\zeta)$ are homogeneous of first degree in ξ and in ζ accordingly, for a constant $\alpha>0$ the equalities hold :

$$f(\alpha\xi)=\alpha f(\xi),\ \overline{F}(\alpha\zeta)=\alpha\overline{F}(\zeta),\ \tfrac{\partial}{\partial z^j}\overline{F}(\alpha\zeta)=\alpha\tfrac{\partial}{\partial z^j}\overline{F}(\zeta). \tag{47}$$

¿From the determination of ζ^i and from the equality (39) it follows that

$$\frac{d}{dt}\zeta^i=\eta^i=\sum_{j=1}^{n}\zeta^j\left(\frac{\partial}{\partial z^j}\overline{F}^i(\zeta)+\sum_{\kappa=1}^{n}\frac{\partial f^i}{\partial\xi^\kappa}\ \frac{\partial\overline{F}^\kappa(\zeta)}{\partial z^j}\right)=\sum_{j=1}^{n}\overline{a}_{ij}\zeta^j, \tag{48}$$

where $\overline{a}_{ij}=\frac{\partial\overline{F}^i(\zeta)}{\partial z^j}+\sum\limits_{\kappa=1}^{n}\frac{\partial f^i}{\partial\xi^\kappa}\ \frac{\partial\overline{F}^\kappa(\zeta)}{\partial z^j}$.

Since function $f(\xi)=\left(f^1(\xi),\cdots,f^n(\xi)\right)$ is homogeneous of first degree in ξ, it is not difficult to see that, function $\frac{\partial f^i}{\partial\xi^\kappa}$ is homogeneous of zero degree in ξ. Then taking into account equality (47) and (48) we have, that the function $\eta=\left(\eta^1,\cdots,\eta^n\right)$ is homogeneous of second degree in ζ. Consequently, problem (40)-(41) possesses the following property : if we substitute $\alpha\overline{t}$ for t and $\overline{\zeta}/\alpha$ for ζ, then the system (40) with the variables z, $\overline{\zeta}$, $\overline{t}$ will have the same form as the initial one, while in the data of Cauchy (41) for $\overline{\zeta}$ instead of ζ_0 there will be ζ_0/α.Therefore

$$z(x,\zeta_0,t)\ =\ z(x,\nu,|\zeta_0|\,t),\ \ \dot{z}(x,\zeta_0,t)\ =\ |\zeta_0|\,\dot{z}(x,\nu,|\zeta_0|\,t), \tag{49}$$

where $\nu=\frac{\zeta_0}{|\zeta_0|}$, $|\zeta_0|^2=\sum\limits_{i,j=1}^{n}g_{ij}^{(2)}(x)\,\zeta_0^i\zeta_0^j$.

Let us introduce the function:

$$u^+(x,\zeta)=\sum_{i,j=2}^{n}\ \int\limits_{\gamma^+(x,\zeta)}a_{ij}\left(z(x,\zeta,t)\right)h^i\left(z(x,\zeta,t),'\dot{z}(x,\zeta,t)\right)h^j\left(z(x,\zeta,t),'\dot{z}(x,\zeta,t)\right)dt, \tag{50}$$

where $\gamma^+(x,\zeta)=\left(z^1(x,\zeta,t),\cdots,z^n(x,\zeta,t)\right)$ is the solution of the problem (40)-(41) with $\zeta_0=\zeta$, the functions $a_{ij}(z)$, $h^i\left(z,'\dot{z}\right)$ possess the same properties with that in the formulation of theorem 2.

¿From (50) differentiating $u^+(x,\zeta)$ at point x in the direction ζ and taking into account (40), (41), (48), we have

$$\sum_{i=1}^{n}\zeta^i u^+_{x^i}+\sum_{i,j=1}^{n}\overline{a}_{ij}\zeta^j u^+_{\zeta^i}=\sum_{i,j=2}^{n}a_{ij}(x)\,h^i\left(x,'\zeta\right)h^j\left(x,'\zeta\right). \tag{51}$$

Problem 2'. Determine a vector function $(a_{ij}(x))_2^n$ in D from equation (51) if the condition

$$for\ (x,\zeta)\in\partial D\times R^n\ (\zeta\ne 0),\ \ u^+(x,\zeta)=0 \tag{52}$$

is given.

As in the study of problem 1, for the known function $\overline{v}(\overline{x})\in C_0^5(\widetilde{D})$, let us introduce a function $T_\nu\left(\overline{x},\overline{\zeta}\right)$ of variables $\overline{x}=(x,x^{n+1})=(x^1,...,x^n,x^{n+1})$ and $\overline{\zeta}=\left(\zeta,\zeta^{n+1}\right)$ by formula

$$T_\nu\left(\overline{x},\overline{\zeta}\right)=\int\limits_{\overline{\gamma}^+\left(\overline{x},\overline{\zeta}\right)}\Big(\sum_{i,j=2}^{n}a_{ij}\left(z\right)h^i\left(z,'\dot{z}\right)h^j\left(z,'\dot{z}\right)+\overline{v}(\overline{z})\Big)dt,$$

where $\overline{\gamma}^+\left(\overline{x},\overline{\zeta}\right)=\left(\gamma^+\left(\overline{x},\overline{\zeta}\right),\gamma^{n+1}\left(\overline{x},\overline{\zeta}\right)\right)$ is the solution of the following Cauchy problem for the system of equations

$$\frac{d}{dt}z^i=\zeta^i,\ \frac{d^2z^j}{dt^2}=\eta^j,\ \frac{d^2z^{n+1}}{dt^2}=0,\quad 1\le i\le n+1,\ 1\le j\le n \quad (53)$$

with the data

$$\overline{z}\left(0\right)=\left(z\left(0\right),z^{n+1}\left(0\right)\right)=\overline{x},\ \frac{d}{dt}\overline{z}\left(0\right)=\overline{\zeta}.$$

Here $\overline{x}\in\widetilde{D}$, $\overline{\zeta}\in\widetilde{G}$, $\widetilde{D}$ and $\widetilde{G}$ are the same sets we used in the determination of the function $T^\nu\left(\overline{x},\overline{\xi}\right)$.

lt is obvious that function $T_\nu\left(\overline{x},\overline{\zeta}\right)$ satisfies the equation

$$\sum_{j=1}^{n+1}\zeta^jT_{\nu x^j}+\sum_{i,j=1}^{n}\overline{a}_{ij}\zeta^jT_{\nu\zeta^i}=\overline{v}(\overline{x})+\sum_{i,j=2}^{n}a_{ij}\left(x\right)h^i\left(x,'\zeta\right)h^j\left(x,'\zeta\right)\quad (54)$$

The function $\overline{v}(\overline{x})$ is given and $u(x,\xi)=\sum\limits_{i,j=2}^{n}\int_{\gamma(x,\xi)}a_{ij}\left(z\left(x,\xi,t\right)\right)\dot{z}^i\left(x,\xi,t\right)\dot{z}^j\left(x,\xi,t\right)dt$

is known on $\partial D\times R^n(\zeta\neq0)$; therefore, as we calculated T^ν for $(\overline{x},\overline{\zeta})\in\widetilde{\Gamma}$, it is possible to calculate the values of the function $T_\nu\left(\overline{x},\overline{\zeta}\right)$ for $\widetilde{\Gamma}$, where $\widetilde{\Gamma}=\partial D\times(a_{n+1},b_{n+1})\times\widetilde{G}$.

Consequently, the uniqueness of solution of problem 2' in class $C_0^5\left(D\right)$ follows from the uniqueness of solution of the following problem :

Problem 5'. To determine a vector function $(a_{ij}(x))_2^n$ from equation (54) if T_ν is known on $\widetilde{\Gamma}$.

It is clear that the function $T^+=T_{2\nu}-T_{1\nu}=\sum\limits_{i,j=2}^{n}\int\limits_{\overline{\gamma}^+\left(\overline{x},\overline{\zeta}\right)}a_{ij}\left(z\right)h^i\left(z,'\dot{z}\right)h^j\left(z,'\dot{z}\right)dt$

satisfies the equation

$$\sum_{j=1}^{n+1}\zeta^jT^+_{x^j}+\sum_{i,j=1}^{n}\overline{a}_{ij}\zeta^jT^+_{\zeta^i}=\sum_{i,j=2}^{n}a_{ij}\left(x\right)h^i\left(x,'\zeta\right)h^j\left(x,'\zeta\right)\quad (55)$$

and $T^+=0$ on $\widetilde{\Gamma}$, where

$$T_{\kappa\nu}=\sum_{i,j=2}^{n}\int\limits_{\overline{\gamma}^+\left(\overline{x},\overline{\zeta}\right)}\Big(a_{ij}^{(\kappa)}\left(z\left(\overline{x},\overline{\zeta},t\right)\right)h^i\left(z\left(\overline{x},\overline{\zeta},t\right),'\dot{z}\left(\overline{x},\overline{\zeta},t\right)\right)h^j\left(z\left(\overline{x},\overline{\zeta},t\right),'\dot{z}\left(\overline{x},\overline{\zeta},t\right)\right)+$$

$$\overline{v}(\overline{z}\left(\overline{x},\overline{\zeta},t\right))\Big)dt,\ a_{ij}\left(x\right)=a_{ij}^{(2)}\left(x\right)-a_{ij}^{(1)}\left(x\right),\ \kappa=1,2.$$

Remark 6. The analysis of the proofs of lemma 1 and corollary 1, taking into account relationships (46), (49), shows that the function T^+ satisfies all the assertions of those given in lemmas 1, 2 and of their corollaries in set $\widetilde{D}\times\widetilde{G}$, where $'G$ contains the origin of R^{n-1}.

Remark 7. Since the functions $a_{ij}(x)$ do not depend on x^{n+1}, from the determination of function T^+ and from equality $u^+\left(x,\zeta\right)=0$ on $\partial D\times R_0^n$ (see (52)) it follows that (as for the function T_0 (see remark 4)) for $(\overline{x},\zeta^{n+1})\in\widetilde{D}\times\left(\frac{1}{4},\frac{3}{4}\right)$, $T^+\left(\overline{x},\zeta,\zeta^{n+1}\right)+T^+\left(\overline{x},-\zeta,\zeta^{n+1}\right)=0$.

Furthermore from the continuity of the function $T^+_{x^{n+1}\zeta^i\zeta^j}\left(\overline{x},\overline{\zeta}\right)$ (see remark 6 and lemma 1) for $(\overline{x},\zeta^{n+1})\in\widetilde{D}\times(\frac{1}{4},\frac{3}{4})$ as $'\zeta\to0$ we have:

$$T^+_{x^{n+1}\zeta^i\zeta^j}\left(\overline{x},0,'\zeta,\zeta^{n+1}\right)\to T^+_{x^{n+1}\zeta^i\zeta^j}\left(\overline{x},0,\zeta^{n+1}\right)=0\quad (56)$$

Differentiating the equality $(\xi^1)^2+\sum\limits_{i,j=2}^{n}g_{ij}^{(2)}\left(x\right)\xi^i\xi^j=1$ with respect to x^1 we have :

$\frac{\partial}{\partial x^1}(\xi^1)^2+ \frac{\partial}{\partial x^1}(\sum\limits_{i,j=2}^{n} g_{ij}^{(2)}(x)\,\xi^i\xi^j) = 0$. At the same time, in view of the independence of variables ζ^1 and x^1, from the equality $\zeta^1 = \xi^1 + f^1(\xi) = 2\xi^1 + \widetilde{f}\ ('\xi)$ (see remark 5), it follows that $2\frac{\partial}{\partial x^1}\xi^1 + \frac{\partial}{\partial x^1}\widetilde{f}\ ('\xi) = 0$, and so $\zeta^1\frac{\partial}{\partial x^1}\xi^1 = (2\xi^1 + \widetilde{f}('\xi))\frac{\partial}{\partial x^1}\xi^1 = \frac{\partial}{\partial x^1}(\xi^1)^2 - \frac{1}{2}\widetilde{f}\ ('\xi)\frac{\partial}{\partial x^1}\widetilde{f}('\xi) = -\frac{\partial}{\partial x^1}(\sum\limits_{i,j=2}^{n} g_{ij}^{(2)}(x)\,\xi^i\xi^j) - \frac{1}{2}\widetilde{f}\ ('\xi)\frac{\partial}{\partial x^1}\widetilde{f}\ ('\xi)$.

Since the function $'\xi$ ($'\xi = \pi^{-1}('\zeta)$) does not depend on ζ^1 from the above equalities, it follows that the functions $\frac{\partial}{\partial x^1}\xi^1$ and $\zeta^1\frac{\partial}{\partial x^1}\xi^1$ do not depend on ζ^1, and so $\frac{\partial}{\partial x^1}\xi^1 = 0$. Then taking into account that $\zeta^1 = 2\xi^1 + \widetilde{f}\ ('\xi)$ we have:

$$\frac{d}{dt}\,\zeta^1 = \eta^1 = 2(\,\zeta^1\frac{\partial}{\partial x^1}\xi^1 + \sum_{j=2}^{n}\zeta^j\frac{\partial}{\partial x^j}\xi^1) + \sum_{k=2}^{n}\frac{\partial}{\partial \xi^k}\widetilde{f}\sum_{j=1}^{n}\zeta^j\frac{\partial}{\partial x^j}\xi^k = 2\sum_{j=2}^{n}\zeta^j\frac{\partial}{\partial x^j}\xi^1 +$$

$$\zeta^1\sum_{k=2}^{n}\frac{\partial}{\partial \xi^k}\widetilde{f}\frac{\partial}{\partial x^1}\xi^k + \sum_{j,k=2}^{n}\zeta^j\frac{\partial}{\partial \xi^k}\widetilde{f}\frac{\partial}{\partial x^j}\xi^k = \zeta^1 e_{11} + \sum_{j=2}^{n} e_{1j}\,\zeta^j,$$

$e_{11} = \sum\limits_{k=2}^{n}\frac{\partial}{\partial \xi^k}\widetilde{f}\frac{\partial}{\partial x^1}\xi^k$, $e_{1j} = 2\frac{\partial}{\partial x^j}\xi^1 + \sum\limits_{k=2}^{n}\frac{\partial}{\partial \xi^k}\widetilde{f}\frac{\partial}{\partial x^j}\xi^k$.

Moreover for $2 \le i \le n$, we have: $\frac{d}{dt}\,\zeta^i = \eta^i = \overline{a}_{i1}\,\zeta^1 + \sum\limits_{j=2}^{n}\overline{a}_{ij}\zeta^j$, where for $1 \le j \le n$, $\overline{a}_{ij} = \frac{\partial\xi^i}{\partial z^j} + \sum\limits_{\kappa=2}^{n}\frac{\partial f^i}{\partial \xi^\kappa}\frac{\partial \xi^k}{\partial z^j}$.

Let us prove now that functions e_{1j} $(1 \le j \le n)$, $\overline{a}_{ij}$ $(2 \le i \le n, 1 \le j \le n)$, do not depend on ζ^1 .

Indeed, it suffices to note the following : 1) The function $'\xi$ does not depend on ζ^1 ($'\xi = \pi^{-1}('\zeta)$), 2) and the function $'f\,(\xi) = \left(f^2\,('\xi)\,,\cdots,f^n\,('\xi)\right)$ does not depend on ξ^1, i.e. $\frac{\partial f^\kappa}{\partial \xi^1} = 0$, $2 \le \kappa \le n$, 3) $\frac{\partial \xi^1}{\partial x^j} = (\tau_2\,(x,x_0))_{x^1x^j} = (\tau_2\,(x,x_0))_{x^jx^1}$. The function $((\tau_2\,(x,x_0))_{x^2}\,,...,(\tau_2\,(x,x_0))_{x^n})$ can be expressed by the vector $'\xi$ where, $2 \le j \le n$.

Let us rewrite now the equation (55) in the form:

$$\sum_{j=1}^{n+1}\zeta^j T^+_{x^j} + (\,\zeta^1 e_{11} + \sum_{j=2}^{n} e_{1j}\,\zeta^j)T^+_{\zeta^1} +$$

$$\sum_{k=2}^{n}(\overline{a}_{k1}\zeta^1 + \sum_{j=2}^{n}\overline{a}_{kj}\zeta^j)T^+_{\zeta^k} = \sum_{i,j=2}^{n} a_{ij}\,(x)\,h^i\,(x,'\zeta)\,h^j\,(x,'\zeta).$$

By remark 6, for each $\overline{x} \in \widetilde{D}$, $\overline{\zeta} \in' \widetilde{G}$, it is possible to apply the generalized Fourier transform in ζ^1 to the last equation. Then taking into account that the functions e_{1j} $(1 \le j \le n)$, $\overline{a}_{ij}$ $(2 \le i \le n, 1 \le j \le n)$ do not depend on ζ^1, for $\widehat{T}^+ = \widehat{T}^+\left(\overline{x},\eta,'\zeta,\zeta^{n+1}\right)$- Fourier's transform of the function $T^+\left(\overline{x},\overline{\zeta}\right)$ - we have:

$$i\widehat{T}^+_{x^1\eta} + \sum_{j=2}^{n+1}\zeta^j\widehat{T}^+_{x^j} - e_{11}\frac{\partial}{\partial \eta}(\eta\,\widehat{T}^+) + i\,\sum_{j=2}^{n} e_{1j}\zeta^j\eta\widehat{T}^+ + i\sum_{j=2}^{n}\overline{a}_{j1}\widehat{T}^+_{\eta\zeta^j}$$

$$+ \sum_{\kappa,j=2}^{n}\overline{a}_{\kappa j}\zeta^j\widehat{T}^+_{\zeta^\kappa} = 2\pi\delta\,(\eta)\,\sum_{\kappa,j=2}^{n} a_{\kappa j}h^\kappa\,('\zeta)\,h^j\,('\zeta)\,. \qquad (57)$$

Designating $\widehat{T}^+ = p^+ + iq^+$, from (57) for $\eta > 0$ $(\eta < 0)$ we have :

$$\frac{\partial}{\partial \eta}U_p^+ = F_1^+, \qquad \frac{\partial}{\partial \eta}U_q^+ = F_2^+\ , \qquad (58)$$

where $U_p^+ = p^+_{x^1} + \sum\limits_{j=2}^{n}\overline{a}_{j1}p^+_{\zeta^j} - e_{11}\eta q^+$, $U_q^+ = q^+_{x^1} + \sum\limits_{j=2}^{n}\overline{a}_{j1}\,q^+_{\zeta^j} + e_{11}\eta p^+$, $F_1^+ = -\sum\limits_{j=2}^{n+1}\zeta^j q^+_{x^j} -$

$\sum\limits_{\kappa,j=2}^{n}\overline{a}_{\kappa j}\zeta^j q^+_{\zeta^\kappa}\ -\eta p^+\sum\limits_{j=2}^{n}e_{1j}\zeta^j,\ F_2^+=\sum\limits_{j=2}^{n+1}\zeta^j p^+_{x^j}+\sum\limits_{\kappa,j=2}^{n}\overline{a}_{\kappa j}\zeta^j p^+_{\zeta^k}-\ \eta q^+\sum\limits_{j=2}^{n}e_{1j}\zeta^j.$

Then by analogous reasonings by which we proved the continuity of integrals $\int\limits_0^\infty\left(D'^{\beta}_{'\xi}F_2\left(x,\eta,'\zeta\right)\right)^2 d\eta,\ \int\limits_0^\infty\left(D'^{\beta}_{'\xi}F_{2x^i}\left(x,\eta,'\zeta\right)\right)^2 d\eta$ on $(x,'\xi)$ for $|'\beta|\le 3$ and by $D'^{\beta}_{'\xi}q_\eta,\ D'^{\beta}_{'\xi}q_{\eta x^i}\in L_1\left(\Delta^s_\eta\right)\cap C\left(D\times\Delta^s_\eta\times' G\right)$ in the proof of lemma 2, we can prove the continuity of the integrals $\int\limits_0^\infty\left(D'^{\beta}_{'\zeta}F_2^+\left(\bar{x},\eta,'\bar{\zeta}\right)\right)^2 d\eta\ ,\ \int\limits_0^\infty\left(D'^{\beta}_{'\zeta}F^+_{2x^i}\left(\bar{x},\eta,'\bar{\zeta}\right)\right)^2 d\eta$ in the set $\widetilde{D}\times'\widetilde{G}$ and $D'^{\beta}_{'\bar{\zeta}}U^+_q\in L_1\left(\Delta^s_\eta\right)\cap C\left(\widetilde{D}\times\Delta^s_\eta\times'\widetilde{G}\right)$, where $s=1,-1;\ |'\beta|\le 3,\ \ '\bar{\zeta}=\left('\zeta,\zeta^{n+1}\right)$.

Using these facts, (as corollary 2 was proved) we prove, that for $|'\beta|\le 3,\ s=-1,1$

$$D'^{\beta}_{'\bar{\zeta}}U^+_q\in C\left(\widetilde{D}\times\bar{\Delta}^s_\eta\times'\widetilde{G}\right),\ \text{where}\ \bar{\Delta}^s_\eta=\left\{s\eta\ge 0\mid\eta\in R^1_\eta\right\},\ (59)$$

$$D'^{\beta}_{'\bar{\zeta}}U^+_q\left(\bar{x},+0,'\bar{\zeta}\right)=\int\limits_0^\infty D'^{\beta}_{'\bar{\zeta}}F_2^+\left(\bar{x},\eta,'\bar{\zeta}\right)d\eta,$$

$$D'^{\beta}_{'\bar{\zeta}}U^+_q\left(x,-0,'\bar{\zeta}\right)=-\int\limits_{-\infty}^0 D'^{\beta}_{'\bar{\zeta}}F_2^+\left(x,\eta,'\bar{\zeta}\right)d\eta. \qquad (60)$$

¿From determination of the function $T^+\left(\overline{x},\overline{\zeta}\right)$ and from the fact that for $'\zeta=0$ and $2\le k\le n$, $\frac{d}{dt}z^k\left(x,\zeta^1,0\right)=0$, and also from equality $h^k\left(x,0\right)=0$ (taking into account the uniqueness of solution of the problem (40), (41)) it follows that $T^+\left(\bar{x},\zeta^1,0,\zeta^{n+1}\right)=T^+_{\zeta^i}\left(\bar{x},\zeta^1,0,\zeta^{n+1}\right)=0$ $(1\le i\le n)$ and $T^+_{\zeta^i\zeta^1}\left(\bar{x},\zeta^1,0,\zeta^{n+1}\right)=T^+_{x^j\zeta^i\zeta^1}\left(\bar{x},\zeta^1,0,\zeta^{n+1}\right)=0,\ (1\le i\le n,1\le j\le n+1)$. The last equalities and the analog of formula (10) for the function $T^+\left(\overline{x},\overline{\zeta}\right)$ show that by similar reasoning by which 1) of lemma 1 was proved, it is possible to prove that

$$\text{as}\ '\zeta\to 0,\ \ T^+,\ T^+_{\zeta^i},\ T^+_{x^j\zeta^i},\ T^+_{x^j\zeta^i\zeta^1}\to 0\quad\text{in}\quad L_2\left(R^1_{\zeta^1}\right). \qquad (61)$$

uniformly with respect to $(\overline{x},\zeta^{n+1})\in\widetilde{D}\times(\frac{1}{4},\frac{3}{4})$.

Using relationships (56), (61) by analogous reasonings by which corollary 3 proved we can prove

Corollary 4. Functions $p^+,\ p^+_{x^j},\ p^+_{x^j\zeta^i},\ p^+_{\zeta^i},\ p^+_{\zeta^i\zeta^j},\ \eta q^+,\eta q^+_{\zeta^i}$ as $'\zeta\to 0$ tend to zero (uniformly with respect to $(\overline{x},\zeta^{n+1})\in\widetilde{D}\times(\frac{1}{4},\frac{3}{4})$) in space $L_1\left(R^1_\eta\right)$, where $\widehat{T}^+=p^+\left(\bar{x},\eta,'\bar{\zeta}\right)+iq^+\left(\bar{x},\eta,'\bar{\zeta}\right),\ (2\le i\le n,1\le j\le n+1)$.

4. Proofs of the theorems.

Recalling that $\widehat{u}=p+iq$, from equation (35) we have:

$$\frac{\partial}{\partial\eta}\left(\ q_{x^1}-2\sum\limits_{j,k=2}^{n}\Gamma^j_{1k}\xi^k q_{\xi^j}\right)\ =\ -2\pi\delta(\eta)\sum\limits_{k,j=2}^{n}a_{jk}\left(x\right)\xi^k\xi^j+F_2, \qquad (62)$$

where $F_2=\sum\limits_{j,k=2}^{n}\Gamma^1_{jk}\xi^k\xi^j\eta q+\sum\limits_{j=2}^{n+1}\xi^j p_{x^j}-\sum\limits_{s,j,k=2}^{n}\Gamma^s_{jk}\xi^k\xi^j p_{\xi^s}$.

Lemma 2 shows that (taking into account the corollary 2 and remark 3) for fixed $(\overline{x},'\xi,\xi^{n+1})\in\widetilde{D}\times' G\times(a^0_{n+1},b^0_{n+1})$ the functions $U=q_{x^1}-2\sum\limits_{j,k=2}^{n}\Gamma^j_{1k}\xi^k q_{\xi^j}$ and $V=F_2$, for $y_0=0$ satisfy the conditions of theorem 3.1.3, if in the expressions for U and V variable y is replaced by η. Then from equality (62), for $'\xi_i \in' G$ we have :

$$\frac{\partial}{\partial\eta}\left(q_{x^1}(\overline{x},\eta,'\xi_i,\xi^{n+1}) - 2\varepsilon\sum_{j=2}^{n}\Gamma^j_{1i}(x)q_{\xi^j}(\overline{x},\eta,'\xi_i,\xi^{n+1})\right) =$$

$$-2\pi\delta(\eta)\,a_{ii}(x)\,\varepsilon^2 + F_{2i}(\overline{x},\eta,'\xi_i,\xi^{n+1}) \qquad (63)$$

where $'\xi_i = \varepsilon\ '\xi_i(1)$, $'\xi_i(1) = (\underbrace{0,\cdots,0}_{i-2},1,0,\cdots,0) \in R^{n-1}$, $\varepsilon \in R^1$, $'\xi = \left(\xi^2,\xi^3,\cdots,\xi^n\right)$,

$F_{2i}(\overline{x},\eta,'\xi_i,\xi^{n+1}) = \varepsilon^2\Gamma^1_{ii}(x)\eta q(\overline{x},\eta,'\xi_i,\xi^{n+1}) + \varepsilon\ p_{x^i}(\overline{x},\eta,'\xi_i,\xi^{n+1})$
$-\varepsilon^2\sum_{s=2}^{n}\Gamma^s_{ii}(x)p_{\xi^s}(\overline{x},\eta,'\xi_i,\xi^{n+1}) + \xi^{n+1}p_{x^{n+1}}(\overline{x},\eta,'\xi_i,\xi^{n+1})$, index i fixed $(2 \le i \le n)$.

In view of equality (63), according to the theorem 3.1.3, for $\varepsilon \in R^1$ we will obtain

$$U_{+q}(\overline{x},'\xi_i,\xi^{n+1}) - U_{-q}(\overline{x},'\xi_i,\xi^{n+1}) = -2\pi a_{ii}(x)\varepsilon^2, \qquad (64)$$

where $U_{\pm q}(\overline{x},'\xi_i,\xi^{n+1}) = q_{x^1}\left(\overline{x},\pm 0,'\xi_i,\xi^{n+1}\right) - 2\varepsilon\sum_{j=2}^{n}\Gamma^j_{1i}(x)q_{\xi^j}\left(\overline{x},\pm 0,'\xi_i,\xi^{n+1}\right)$.

On the other hand from (63), by corollaries 1,2, it is not difficult to obtain for $\varepsilon \in R^1$,

$$U_{+q}(\overline{x},'\xi_i,\xi^{n+1}) = -\int_0^\infty F_{2i}(\overline{x},\eta,'\xi_i,\xi^{n+1})d\eta, \quad U_{-q}(\overline{x},'\xi_i,\xi^{n+1}) = \int_{-\infty}^0 F_{2i}(\overline{x},\eta,'\xi_i,\xi^{n+1})d\eta.$$

Consequently, for $\varepsilon \in R^1$

$$U_{+q}(\overline{x},'\xi_i,\xi^{n+1}) - U_{-q}(\overline{x},'\xi_i,\xi^{n+1}) =$$

$$= \int_{-\infty}^{\infty}(\varepsilon^2\sum_{s=2}^{n}\Gamma^s_{ii}(x)p_{\xi^s}(\overline{x},\eta,'\xi_i,\xi^{n+1}) - \varepsilon^2\Gamma^1_{ii}(x)\eta q(\overline{x},\eta,'\xi_i,\xi^{n+1}) -$$

$$- \varepsilon p_{x^i}(\overline{x},\eta,'\xi_i,\xi^{n+1}) - \xi^{n+1}p_{x^{n+1}}(\overline{x},\eta,'\xi_i,\xi^{n+1}))d\eta. \qquad (65)$$

By corollary 3,

$\int_{-\infty}^{\infty} p_{x^i}(\overline{x},\eta,0,\xi^{n+1})d\eta = \int_{-\infty}^{\infty} p_{x^{n+1}}(\overline{x},\eta,0,\xi^{n+1})d\eta = \int_{-\infty}^{\infty} p_{\xi^i x^{n+1}}(\overline{x},\eta,0,\xi^{n+1})d\eta = 0$ and according to corollaries 1, 2, 3 (taking into account remark 3) we have, that $p_{x^i\xi^i}$, $p_{x^{n+1}\xi^i\xi^i} \in L_1\left(R^1_\eta\right) \cap L_2\left(R^1_\eta\right) \cap C\left(\widetilde{D} \times \Delta^s_\eta \times' G \times (a^0_{n+1}, b^0_{n+1})\right)$ $(s = -1, 1)$. Consequently, for each fixed $(\overline{x},\xi^{n+1}) \in \widetilde{D} \times (a^0_{n+1}, b^0_{n+1})$, from the mean value theorem it follows that

$$\int_{-\infty}^{\infty} p_{x^i}(\overline{x},\eta,'\xi_i,\xi^{n+1})d\eta = \varepsilon\int_{-\infty}^{\infty} p_{x^i\xi^i}(\overline{x},\eta,'\xi_i\theta_1,\xi^{n+1})d\eta,$$

$$\int_{-\infty}^{\infty} p_{x^{n+1}}(\overline{x},\eta,'\xi_i,\xi^{n+1})d\eta = \varepsilon^2\int_{-\infty}^{\infty} p_{x^{n+1}\xi^i\xi^i}(\overline{x},\eta,'\xi_i\theta_2,\xi^{n+1})d\eta, \qquad (66)$$

where $0 < \theta_1(\overline{x},'\xi_i,\xi^{n+1}),\ \theta_2(\overline{x},'\xi_i,\xi^{n+1}) < 1$.

Taking into account (66) in (65), we have :

$$U_{+q}(\overline{x},'\xi_i,\xi^{n+1}) - U_{-q}(\overline{x},'\xi_i,\xi^{n+1}) = q_i(\overline{x},'\xi_i,\xi^{n+1})\varepsilon^2 , \qquad (67)$$

where $q_i(\overline{x},'\xi_i,\xi^{n+1}) = \int_{-\infty}^{\infty}(\sum_{s=2}^{n}\Gamma^s_{ii}(x)p_{\xi^s}(\overline{x},\eta,'\xi_i,\xi^{n+1}) - \Gamma^1_{ii}(x)\eta q(\overline{x},\eta,'\xi_i,\xi^{n+1}) -$

$$- p_{x^i\xi^i}(\overline{x},\eta,'\xi_i\theta_1,\xi^{n+1}) - \xi^{n+1}p_{x^{n+1}\xi^i\xi^i}(\overline{x},\eta,'\xi_i\theta_2,\xi^{n+1}))d\eta.$$

Equalities (64) and (67) show that $-2\pi a_{ii}(x) = q_i(\overline{x},'\xi_i,\xi^{n+1})$, $(2 \le i \le n)$.

By remark 4, the function $u(\overline{x},\overline{\xi})$ $(u_{x^{n+1}\xi^i\xi^j}(\overline{x},\overline{\xi}))$ is odd with respect to ξ and from continuousness of $u_{x^{n+1}\xi^i\xi^j}(\overline{x},\overline{\xi})$ in $\widetilde{D} \times \widetilde{G}$ it follows that $u_{x^{n+1}\xi^i\xi^j}(\overline{x},0,\xi^{n+1}) = 0$. Then by corollary 3 and by relations (37), (38) we have: as $\varepsilon \to 0$, $q_i(\overline{x},'\xi_i,\xi^{n+1}) \to 0$, so $a_{ii}(x) = 0$, $(2 \le i \le n)$.

Let $'\xi_{ij}(1) = (\underbrace{0,\cdots,0}_{i-2},1,\underbrace{0,\cdots,0}_{j-i-1},1,\underbrace{0,\cdots,0}_{n-j}) \in R^{n-1}$, $i \neq j$, $'\xi_{ij} = \varepsilon\, '\xi_{ij}(1) \in' G$.

Taking into account equalities $a_{ii}(x) = 0$, $a_{ij}(x) = a_{ji}(x)$ from (44) for $'\xi_{ij}$ we have :

$$\frac{\partial}{\partial \eta}\left(q_{x^1}\left(\overline{x},\eta,'\xi_{ij},\xi^{n+1}\right) - 2\varepsilon \sum_{s=2}^{n}\sum_{\kappa=i,j}\Gamma^s_{1\kappa}(x)q_{\xi^s}\left(\overline{x},\eta,'\xi_{ij},\xi^{n+1}\right)\right) =$$

$$-4\pi\delta(\eta)\, a_{ij}(x)\varepsilon^2 + F_{2ij}\left(\overline{x},\eta,'\xi_{ij},\xi^{n+1}\right), \qquad (68)$$

where $F_{2ij}\left(\overline{x},\eta,'\xi_{ij},\xi^{n+1}\right) = \varepsilon^2 \sum_{s,\kappa=i,j}\Gamma^1_{s\kappa}(x)\eta q\left(\overline{x},\eta,'\xi_{ij},\xi^{n+1}\right) + \varepsilon\sum_{s=i,j}p_{x^s}\left(\overline{x},\eta,'\xi_{ij},\xi^{n+1}\right)$

$-\varepsilon^2 \sum_{s,\kappa=i,j}\sum_{r=2}^{n}\Gamma^r_{s\kappa}(x)p_{\xi^r}\left(\overline{x},\eta,'\xi_{ij},\xi^{n+1}\right) + \xi^{n+1}p_{x^{n+1}}(\overline{x},\eta,'\xi_{ij},\xi^{n+1})$, indexs i, j fixed ($2 \leq i$, $j \leq n$).

Now we repeat the same arguments which we used when we proved the equality $a_{ii}(x) = 0$.

By (68), taking into account the theorem 3.1.3, for $\varepsilon \in R^1$ we obtain

$$U_{+q}(\overline{x},'\xi_{ij},\xi^{n+1}) - U_{-q}(\overline{x},'\xi_{ij},\xi^{n+1}) = -4\pi a_{ij}(x)\varepsilon^2, \qquad (69)$$

where $U_{\pm q}(\overline{x},'\xi_{ij},\xi^{n+1}) = q_{x^1}\left(\overline{x},\pm 0,'\xi_{ij},\xi^{n+1}\right) - 2\varepsilon\sum_{s=2}^{n}\sum_{\kappa=i,j}\Gamma^s_{1\kappa}(x)q_{\xi^s}\left(\overline{x},\pm 0,'\xi_{ij},\xi^{n+1}\right)$

¿From (68), taking into account corollaries 1,2, for $\varepsilon \in R^1$, we have

$U_{+q}(\overline{x},'\xi_{ij},\xi^{n+1}) = -\int_0^\infty F_{2ij}\left(\overline{x},\eta,'\xi_{ij},\xi^{n+1}\right)d\eta, U_{-q}(\overline{x},'\xi_{ij},\xi^{n+1}) = \int_{-\infty}^0 F_{2ij}\left(\overline{x},\eta,'\xi_{ij},\xi^{n+1}\right)d\eta.$

Hence, for $\varepsilon \in R^1$

$$U_{+q}(\overline{x},'\xi_{ij},\xi^{n+1}) - U_{-q}(\overline{x},'\xi_{ij},\xi^{n+1}) =$$

$$= \int_{-\infty}^{\infty}\Big(\varepsilon^2\sum_{s,\kappa=i,j}\sum_{r=2}^{n}\Gamma^r_{s\kappa}(x)p_{\xi^r}\left(\overline{x},\eta,'\xi_{ij},\xi^{n+1}\right) - \varepsilon\sum_{s=i,j}p_{x^s}\left(\overline{x},\eta,'\xi_{ij},\xi^{n+1}\right) -$$

$$-\varepsilon^2\sum_{s,\kappa=i,j}\Gamma^1_{s\kappa}(x)\eta q\left(\overline{x},\eta,'\xi_{ij},\xi^{n+1}\right) - \xi^{n+1}p_{x^{n+1}}(\overline{x},\eta,'\xi_{ij},\xi^{n+1})\Big)d\eta. \qquad (70)$$

Recalling that $\int_{-\infty}^{\infty}p_{x^i}(\overline{x},\eta,0,\xi^{n+1})d\eta = \int_{-\infty}^{\infty}p_{x^{n+1}}(\overline{x},\eta,0,\xi^{n+1})d\eta = \int_{-\infty}^{\infty}p_{\xi^i x^{n+1}}(\overline{x},\eta,0,\xi^{n+1})d\eta =$ 0 by the mean value theorem, $\int_{-\infty}^{\infty}p_{x^s}(\overline{x},\eta,'\xi_{ij},\xi^{n+1})d\eta =$

$\varepsilon\sum_{\kappa=i,j}\int_{-\infty}^{\infty}p_{x^s\xi^k}(\overline{x},\eta,'\xi_{ij}\theta^s_1,\xi^{n+1})d\eta$, where $s = i,j$; $0 < \theta^s_1\left(\overline{x},'\xi_{ij},\xi^{n+1}\right) < 1$,

$\int_{-\infty}^{\infty}p_{x^{n+1}}(\overline{x},\eta,'\xi_{ij},\xi^{n+1})d\eta = \varepsilon^2\sum_{s,\kappa=i,j}\int_{-\infty}^{\infty}p_{x^{n+1}\xi^s\xi^k}(\overline{x},\eta,'\xi_{ij}\theta_2,\xi^{n+1})d\eta$, where $0 < \theta_2\left(\overline{x},'\xi_{ij},\xi^{n+1}\right) <$ 1. Taking into account last equalities in (70) we have :

$$U_{+q}(\overline{x},'\xi_{ij},\xi^{n+1}) - U_{-q}(\overline{x},'\xi_{ij},\xi^{n+1}) = q_{ij}(\overline{x},'\xi_{ij},\xi^{n+1})\varepsilon^2, \qquad (71)$$

where $q_{ij}(\overline{x},'\xi_{ij},\xi^{n+1}) = \int_{-\infty}^{\infty}\Big(\sum_{s,\kappa=i,j}\sum_{r=2}^{n}\Gamma^r_{s\kappa}(x)p_{\xi^r}\left(\overline{x},\eta,'\xi_{ij},\xi^{n+1}\right) - \sum_{s,\kappa=i,j}\Gamma^1_{s\kappa}(x)\eta q\left(\overline{x},\eta,'\xi_{ij},\xi^{n+1}\right) -$

$$-\sum_{s,\kappa=i,j}(p_{x^s\xi^k}(\overline{x},\eta,'\xi_{ij}\theta^s_1,\xi^{n+1}) - \xi^{n+1}p_{x^{n+1}\xi^s\xi^k}(\overline{x},\eta,'\xi_{ij}\theta_2,\xi^{n+1}))\Big)d\eta.$$

Equalities (69) and (71) show that, $-4\pi a_{ij}(x) = q_{ij}\left(x,'\xi_{ij},\xi^{n+1}\right)$, and since $\varepsilon \to 0$ and $q_{ij}(\overline{x},'\xi_{ij},\xi^{n+1}) \to 0$, it follows that, $a_{ij}(x) = 0$, $(2 \leq i,j \leq n)$.

Theorem 1 is proved.

The proof of theorem 2. Since the function $h(x,'\dot{z})$ is homogenuos in $'\dot{z}$ as in formula (10), let us rewrite the function

$$I_h(x,\xi) = \int_{\gamma(x,\xi)} b\,(z\,(x,\xi,t))\, h^i('\dot{z}\,(x,\xi,t))h^j('\dot{z}x,\xi,t))dt$$

in the form

$$I_h(x,\xi) = \frac{1}{|\xi|}\int_0^\infty b\,(z\,(x,\nu,\tau))\,|\xi|\; h^i('\dot{z}\,(x,\nu,\tau))\;|\xi|\; h^j('\dot{z}\,(x,\nu,\tau))d\tau. \quad (72)$$

Using the smoothness of the function $h(x,'\xi) \in C^5(D\times R_0^{n-1})$ and the condition $h(x,0) = (h^2(x,0),...,h^n(x,0)) = 0$ and the homogeneity of the function $h(x,'\xi)$ on $'\xi$, with the same arguments by which inequalities (13) and (16) were proved, the following inequalities

$$\left||\xi|\,h^k(x,'\,\dot{z}\,(x,\nu,\tau))\right| \le K_{1h},$$
$$\left||\xi|^{\alpha-1}\,D_\xi^\alpha(\;|\xi|\,h^k(x,'\,\dot{z}\,(x,\nu,\tau)))\right| \le K_{1h}, \quad for\;\; 1\le|\alpha|\le 4\;, \quad (73)$$

can be proved in the set Ω, where the number $K_{1h}>0$ does not depend on $(x,\xi)\in(D\times G)$. Moreover K_{1h} does differ from the numbers K_1, K_2 in (13), (16) by the fact that K_{1h} it depends also on the norm of the function $h(x,\dot{z})$ in the space $C^4(\Omega(d_0))$.

Relations (17), (72)-(73) and proof of lemma 1 (corollary 1) show that the following assertion is true :

Assertion 1. For the functions $I_h(x,\xi)$ $(\widehat{I}_h(x,\eta,'\xi))$ lemma 1 (corollary 1) is true, where $\widehat{I}_h = \widehat{I}_h(x,\eta,'\xi)$ is the Fourier transform of the function $I_h(x,\xi)$ in the variable ξ^1, η is the dual variable to ξ^1.

From assertion 1, it follows that for the function

$$u_h(x,\xi) = \sum_{i,j=2}^n \int_{\gamma(x,\xi)} a_{ij}\,(z\,(x,\xi,t))\,h^i(z\,(x,\xi,t)\,,\quad '\dot{z}\,(x,\xi,t))h^j(z\,(x,\xi,t)\,,'\,\dot{z}(x,\xi,t))dt$$

Lemma 1 is true. It is obvious that $u_h(x,\xi)$ satisfies equation (8), and $\widehat{u}_h(x,\eta,'\xi) = p_h(x,\eta,'\xi)+ iq_h(x,\eta,'\xi)$ is the Fourier transform in the variable ξ^1 of the solution $u_h(x,\xi)$ to equation (21) with the right sides $\sum\limits_{i,j=2}^n a_{ij}(x)h^i(x,'\xi)h^j(x,'\xi)$ and $2\pi\delta\,(\eta)\sum\limits_{k,j=2}^n a_{jk}\,(x)\,h^k(x,'\xi)h^j(x,'\xi)$ correspondingly. Then the functions p_h and q_h satisfy equations (23) and (24) respectively with the right sides of the form $F_{h1} = \sum\limits_{j,k=2}^n \Gamma_{jk}^1\xi^k\xi^j\eta p_h - \sum\limits_{j=2}^n \xi^j q_{hx^j} + \sum\limits_{s,.j,k=2}^n \Gamma_{jk}^s\xi^k\xi^j q_{h\xi^s}$, $F_{h2} = \sum\limits_{j,k=2}^n \Gamma_{jk}^1\xi^k\xi^j\eta q_h + \sum\limits_{j=2}^n \xi^j p_{hx^j} - \sum\limits_{s,j,k=2}^n \Gamma_{jk}^s\xi^k\xi^j p_{h\xi^s}$, respectively

Furthermore for the functions $u_h(x,\xi)$ and $\widehat{u}_h(x,\eta,'\xi)$ conditions (9) and (25) are true respectively . Therefore, analogously as lemma 2 and corollary 2 were proven for $\widehat{u}(x,\eta,'\xi)$, it may be proved that they are true for the function $\widehat{u}_h(x,\eta,'\xi)$.For each fixed $x\in D$, the mapping $'\eta =' f('\xi)$ it is one-to-one in R^{n-1}. (Let us recall that $'f(0)=0$); therefore for the right side $'\eta_i(1) = (\underbrace{0,\cdots,0}_{i-2},1,0,\cdots,0)\in R^{n-1}$, the equation $'f('\xi) = {}'\eta_i(1)$ has a unique solution $'\widetilde{\xi}_i(1) = \left(\xi_i^2(1),\cdots,\xi_i^n(1)\right)$ where index i $(2\le i\le n)$ is fixed. On the other hand, from equality (45), we get $h('\zeta) =' f('\xi)$, consequently, from the invertibility mapping $'\eta = h('\zeta)$ it follows that there is a unique vector $\zeta_i(1)$ such, that $h\,(\zeta_i(1)) = {}'f\left('\widetilde{\xi}_i(1)\right) = {}'\eta_i(1)$. Then from the homogeneity of the first degree of the functions $'f('\xi)$, $h('\zeta)$ with respect to $'\xi$, $'\zeta$ accordingly, it follows that $'f\left('\widetilde{\xi}_i\right) = h\,('\zeta_i)$ where $'\zeta_i = \varepsilon'\zeta_i(1) = \left(\varepsilon\zeta_i^2(1),\varepsilon\zeta_i^3(1),\cdots,\varepsilon\zeta_i^n(1)\right)$, $'\widetilde{\xi}_i =$

$\varepsilon'\widetilde{\xi}_i(1) = \left(\varepsilon\xi_i^2(1),, \cdots, \varepsilon\xi_i^n(1)\right) \equiv \left(\xi_i^2, \cdots, \xi_i^n\right), \varepsilon \geq 0.$

As in the study of problem 1, the auxiliary function T_h^0 of variables $(\overline{x}, \overline{\xi})$,

$T_h^0 = \sum\limits_{i,j=2}^{n} \int_{\overline{\gamma}(\overline{x},\overline{\xi})} a_{ij}\left(z\left(\overline{x}, \overline{\xi}, t\right)\right) h^i(z\left(\overline{x}, \overline{\xi}, t\right), '\dot{z}\left(\overline{x}, \overline{\xi}, t\right) h^j(z\left(\overline{x}, \overline{\xi}, t\right), '\dot{z}(\overline{x}, \overline{\xi}, t))dt,$ can be introduced.

If we denote $T_h^0(\overline{x}, \overline{\xi})$ again by $u_h(\overline{x}, \overline{\xi})$ and its Fourier transform in terms of the variable ξ^1 by $\widehat{u}_h(\overline{x}, \eta, '\xi, \xi^{n+1}) = p_h(\overline{x}, \eta, '\xi, \xi^{n+1})+ \ iq_h(\overline{x}, \eta, '\xi, \xi^{n+1})$, then taking into account that as $'\zeta \longrightarrow 0, 'h(x, '\zeta) \longrightarrow 0$ and $'h(x, 0) = 0$ (see (46)) with the arguments for corollary 3 for the functions $q(\overline{x}, \eta, '\xi, \xi^{n+1})$, $p(\overline{x}, \eta, '\xi, \xi^{n+1})$, the analogous assertions can be prove for the functions $q_h(\overline{x}, \eta, '\xi, \xi^{n+1})$, $p_h(\overline{x}, \eta, '\xi, \xi^{n+1})$ respectively. In this case the analog of equation (62) for the function $q_h(\overline{x}, \eta, '\xi, \xi^{n+1})$ takes the form:

$$\frac{\partial}{\partial\eta}(q_{hx^1}(\overline{x}, \eta, '\widetilde{\xi}_i, \xi^{n+1}) - 2\varepsilon \sum_{j,\kappa=2}^{n} \Gamma_{1\kappa}^j(x)\xi_i^\kappa(1)q_{h\xi_i^j}(\overline{x}, \eta, '\widetilde{\xi}_i, \xi^{n+1})) =$$
$$-2\pi\delta\left(\eta\right) a_{ii}(x)\ \varepsilon^2 + F_{2hi}(\overline{x}, \eta, '\widetilde{\xi}_i, \xi^{n+1}),$$

where $F_{2hi}(\overline{x}, \eta, '\widetilde{\xi}_i, \xi^{n+1}) = \varepsilon^2 \sum\limits_{j,\kappa=2}^{n} \Gamma_{j\kappa}^1(x)\xi_i^\kappa(1)\xi_i^j(1)\eta q_h(\overline{x}, \eta, '\widetilde{\xi}_i, \xi^{n+1})+$
$\varepsilon\sum\limits_{j=2}^{n}\xi_i^j(1)p_{hx^j}(\overline{x}, \eta, '\widetilde{\xi}_i, \xi^{n+1}) - \varepsilon^2 \sum\limits_{s,j,\kappa=2}^{n} \Gamma_{j\kappa}^s(x)\xi_i^\kappa(1)\xi_i^j(1)p_{h\xi_i^s}(\overline{x}, \eta, '\widetilde{\xi}_i, \xi^{n+1})+$
$\xi^{n+1}p_{hx^{n+1}}(\overline{x}, \eta, '\widetilde{\xi}_i, \xi^{n+1})$.

By the similar arguments, with the help of which equalities $a_{ii}(x) = q_i(\overline{x}, '\xi_i, \xi^{n+1})\ = 0$ were established in the proof of theorem 1, we can prove that the equalities $a_{ii}(x) = q_{hi}(\overline{x}, '\widetilde{\xi}_i, \xi^{n+1})$ $= 0$ are true, where

$$q_{hi}(\overline{x}, '\widetilde{\xi}_i, \xi^{n+1}) = \sum_{j,\kappa=2}^{n} \int\limits_{-\infty}^{\infty} (\xi_i^j(1)\xi_i^\kappa(1) \sum_{s=2}^{n}\Gamma_{j\kappa}^s(x)p_{h\xi_i^s}(\overline{x}, \eta, '\widetilde{\xi}_i, \xi^{n+1})-$$
$$-\Gamma_{j\kappa}^1(x)\xi_i^j(1)\xi_i^\kappa(1)\eta q_h(\overline{x}, \eta, '\widetilde{\xi}_i, \xi^{n+1}) - p_{hx^j\xi_i^\kappa}(\overline{x}, \eta, '\widetilde{\xi}_i\theta_j, \xi^{n+1})-$$
$$-\ \xi^{n+1}p_{hx^{n+1}\xi^j\xi^k}(\overline{x}, \eta, '\widetilde{\xi}_i\theta_{n+1}, \xi^{n+1}))d\eta, \quad 0 < \theta_s(\overline{x}, '\widetilde{\xi}_i, \xi^{n+1}) < 1,\ s = j, n+1.$$

For the proof of equality $\ a_{ij}(x) = 0\ (i \neq j)$, it is necessary to take the vectors $\eta_{ij}(1) =$ $(\underbrace{0, \cdots, 0}_{i-2}, 1, \underbrace{0, \cdots, 0}_{j-i-1}, 1, \underbrace{0, \cdots, 0}_{n-j}) \in R^{n-1}$, $'\widetilde{\xi}_{ij}(1)$, $'\zeta_{ij}(1)$, $'\widetilde{\xi}_{ij}$, $'\zeta_{ij}$ instead of the vectors $\eta_i(1), '\widetilde{\xi}_i(1), '\zeta_i(1), '\widetilde{\xi}_i, '\zeta_i$ respectively and (taking into account equality $a_{ii}(x) = 0$) to repeat arguments given above for the proof of equality $\ a_{ii}(x) = 0$ in the case when $\ '\xi =' \widetilde{\xi}_i$. Here $'\widetilde{\xi}_{ij}(1) = \left(\xi_{ij}^2(1), \cdots, \xi_{ij}^n(1)\right)$ $('\zeta_{ij}(1))$ is a unique solution of equation $\ 'f('\xi) = \eta_{ij}(1)$ $(h\left('\zeta_{ij}\right) = \eta_{ij}(1))$ and $'\widetilde{\xi}_{ij} = \varepsilon\ '\widetilde{\xi}_{ij}(1)$ $('\zeta_{ij} = \varepsilon'\zeta_{ij}(1))$, $\varepsilon \geq 0$.

Theorem 2 is proved.

Lemma 3. Let the conditions of theorem 2 be satisfied. Then problem 2' can have only one solution $(a_{ij}(x)) \in C^5(R^n)$.

Proof. As it was noted above, the uniqueness of the solution problem 2' follows from the uniqueness of solution of problem 5'. Consequently, for the proof of lemma 3 it suffices to show that if fulfills equation (55) and $T^+ = 0$ on $\widetilde{\Gamma}$, then $a_{ij}(x) \equiv 0$.

¿From equation (57) we have :

$$\frac{\partial}{\partial\eta}\left(U_q^+\right)=-2\pi\delta\left(\eta\right)\sum_{k,j=2}^{n}a_{kj}\left(x\right)h^k(x,'\zeta)h^j(x,'\zeta)+F_2^+ \ . \tag{74}$$

Let $'\nu_i(1)=(\underbrace{0,\cdots,0}_{i-2},1,0,\cdots,0)$, and let $'\xi_i(1)=\left(\xi_i^2(1),\cdots,\xi_i^n(1)\right)$ be a unique solution of equation $'f('\xi)='\nu_i(1)$, while $'\zeta_i(1)$ is a unique solution of equation $h('\zeta_i(1))$ $='\ f('\xi_i\,(1))$. From the homogeneity of the the first degree of the functions $f(\xi)$ and $h('\zeta)$ it follows that $'f('\xi_i)=h('\zeta_i)$, where $'\zeta_i=\varepsilon'\zeta_i(1)=\left(\varepsilon\zeta_i^2(1),\cdots,\varepsilon\zeta_i^n(1)\right)=\left(\zeta_i^2,\cdots,\zeta_i^n\right)$, $'\xi_i=\varepsilon\xi_i(1)=\ \left(\varepsilon\xi_i^2(1),,\cdots,\varepsilon\xi_i^n(1)\right)$, $\varepsilon>0$. Then for $'\zeta='\ \zeta_i$, equation (74) will take the form

$$\frac{\partial}{\partial\eta}\left(U_q^+\right)\ =\ -2\pi\delta\left(\eta\right)a_{ii}(x)\ \varepsilon^2+F_{2i}^+\ , \tag{75}$$

where $F_{2i}^+(\bar{x},\eta,'\zeta_i,\zeta^{n+1})=\varepsilon\sum\limits_{j=2}^{n}\left(p_{x^j}^+\ +\sum\limits_{\kappa=2}^{n}\overline{a}_{\kappa j}p_{\zeta^k}^+\ -\ \eta q^+e_{1j}\right)\zeta_i^j\left(1\right)+\zeta^{n+1}p_{x^{n+1}}$.

In view of equality (75), taking into account theorem 3.1.3 for $\varepsilon\in R_+^1$ we obtain:

$$U_q^+\left(\bar{x},+0,'\zeta_i,\zeta^{n+1}\right)-U_q^+\left(x,-0,'\zeta_i,\zeta^{n+1}\right)=-2\pi a_{ii}(x)\ \varepsilon^2 \tag{76}$$

On the other hand, from (75), taking into account (60), it is not difficult to obtain that for $\varepsilon\in R_+^1$,

$$U_q^+\left(\bar{x},+0,'\zeta_i,\zeta^{n+1}\right)-U_q^+\left(\bar{x},-0,'\zeta_i,\zeta^{n+1}\right)=$$
$$\varepsilon\sum_{j=2}^{n}\int_{-\infty}^{\infty}((p_{x^j}^+\ +\sum_{\kappa=2}^{n}\overline{a}_{\kappa j}p_{\zeta^k}^+\ -\eta q^+e_{1j})\zeta_i^j\left(1\right)+\zeta^{n+1}p_{x^{n+1}})d\eta. \tag{77}$$

By corollary 4 and remark 6

$$\int_{-\infty}^{\infty}p_{x^j}^+(\overline{x},\eta,0,\zeta^{n+1})d\eta\ =\ \int_{-\infty}^{\infty}p_{\zeta^k}^+\left(\overline{x},\eta,0,\zeta^{n+1}\right)d\eta\ =\ \int_{-\infty}^{\infty}p_{x^j\zeta^k}^+\left(\overline{x},\eta,0,\zeta^{n+1}\right)d\eta\ =$$
$$\int_{-\infty}^{\infty}\eta q_{\zeta^k}^+\left(\overline{x},\eta,0,\zeta^{n+1}\right)d\eta\ =\ \int_{-\infty}^{\infty}\eta q^+\left(\overline{x},\eta,0,\zeta^{n+1}\right)d\eta\ =\ \int_{-\infty}^{\infty}p_{\zeta^k\zeta^r}^+\left(\overline{x},\eta,0,\zeta^{n+1}\right)d\eta\ =\ 0, \tag{78}$$

and $p_{x^j}^+,p_{\zeta^k}^+,p_{x^j\zeta^k}^+,\eta q_{\zeta^k}^+,\eta q_{\zeta^k}^+,p_{\zeta^k\zeta^r}^+\ \in\ L_1\left(R_\eta^1\right)\cap L_2\left(R_\eta^1\right)\cap C\left(\widetilde{D}\times\Delta_\eta^s\times' G\times(\frac{1}{4},\frac{3}{4})\right)$, where $2\leq k,r\leq n,1\leq j\leq n+1,s=-1,1$. Consequently, from mean value theorem, for each fixed $(\bar{x},\zeta^{n+1})\in\widetilde{D}\times(\frac{1}{4},\frac{3}{4})$ we have :

$$\int_{-\infty}^{\infty}p_{x^j}^+(\overline{x},\eta,'\zeta_i,\zeta^{n+1})d\eta=\varepsilon\sum_{r=2}^{n}\ \int_{-\infty}^{\infty}p_{x^j\zeta^r}^+(\bar{x},\eta,'\zeta_i\theta_j^+,\zeta^{n+1})\zeta_i^r\left(1\right)d\eta,$$
$$\int_{-\infty}^{\infty}\eta q^+(\overline{x},\eta,'\zeta_i,\zeta^{n+1})d\eta=\ \varepsilon\sum_{r=2}^{n}\int_{-\infty}^{\infty}\eta q_{\zeta^r}^+(\overline{x},\eta,'\zeta_i\theta^+,\zeta^{n+1})\zeta_i^r\left(1\right)d\eta,$$
$$\int_{-\infty}^{\infty}p_{\zeta^k}^+(\overline{x},\eta,'\zeta_i,\zeta^{n+1})d\eta=\varepsilon\sum_{r=2}^{n}\ \int_{-\infty}^{\infty}p_{\zeta^k\zeta^r}^+(\overline{x},\eta,'\zeta_i\bar{\theta}_k^+,\zeta^{n+1})\zeta_i^r\left(1\right)d\eta, \tag{79}$$
$$\int_{-\infty}^{\infty}p_{x^{n+1}}(\overline{x},\eta,'\zeta_i,\xi^{n+1})d\eta=\ \varepsilon^2\sum_{r,j=2}^{n}\ \int_{-\infty}^{\infty}p_{x^{n+1}\zeta^r\zeta^j}(\overline{x},\eta,'\zeta_i\theta_{n+1}^+,\xi^{n+1})\zeta_i^r\left(1\right)\zeta_i^j\left(1\right)d\eta,$$

where $0<\theta_j^+(\overline{x},'\zeta_i,\zeta^{n+1})$, $\theta^+(\overline{x},'\zeta_i,\zeta^{n+1})$, $\bar{\theta}_k^+(\overline{x},'\zeta_i,\zeta^{n+1})$, $\theta_{n+1}^+(\overline{x},'\zeta_i,\zeta^{n+1})<1$.

Taking into account (79) in (77) we will obtain

$$U_q^+\left(\bar{x},+0,'\zeta_i,\zeta^{n+1}\right)-U_q^+\left(\bar{x},-0,'\zeta_i,\zeta^{n+1}\right)=q_i^+\left(\bar{x},'\zeta_i,\zeta^{n+1}\right)\varepsilon^2\ , \tag{80}$$

where $q_i^+\left(\bar{x},'\zeta_i,\zeta^{n+1}\right)=\sum\limits_{r,j=2}^{n}\int\limits_{-\infty}^{\infty}((p_{x^j\zeta^r}^+(\overline{x},\eta,'\zeta_i\theta_j^+,\zeta^{n+1})-\eta q_{\zeta^r}^+\left(\bar{x},\eta,'\zeta_i\theta^+,\zeta^{n+1}\right)e_{1j}\ +$

$\sum_{k=2}^{n} \bar{a}_{kj} p^{+}_{\zeta^k \zeta^r}(\overline{x}, \eta,' \zeta_i \bar{\theta}_k, \zeta^{n+1}))\zeta^r_i\,(1) + \zeta^{n+1}\ p_{x^{n+1}\zeta^r \zeta^j}(\overline{x}, \eta,' \zeta_i \theta^{+}_{n+1}, \zeta^{n+1})\zeta^r_i\,(1)\,\zeta^j_i\,(1))d\eta.$

Equalities (76) and (80) show that $-2\pi a_{ii}(x) = q^{+}_i\left(\bar{x},' \zeta_i, \zeta^{n+1}\right)$, $(2 \leq i \leq n)$. Then from corollary 4 and from (78) it follows that for $\varepsilon \to +0$, $q^{+}_i\left(\bar{x},' \zeta_i, \zeta^{n+1}\right) \to 0$. Therefore, $a_{ii}(x) = 0, (2 \leq i \leq n)$.

For the proof of equality $a_{ij}(x) = 0$ $(i \neq j)$ it is necessary to take vectors $'\nu_{ij}(1) = (\underbrace{0, \cdots, 0}_{i-2}, 1, \underbrace{0, \cdots, 0}_{j-i-1}, 1, \underbrace{0, \cdots, 0}_{n-j}) \in R^{n-1}$, $'\xi_{ij}(1)$, $'\zeta_{ij}(1)$, $'\xi_{ij}$, $'\zeta_{ij}$ instead of the vectors $'\nu_i(1)$, $'\xi_i(1),' \zeta_i(1)$, $'\xi_i,' \zeta_i$ respectively and (by taking into account equality $a_{ii}(x) = 0$) to repeat the reasoning given above for the proof of equality $a_{ii}(x) = 0$ in the case when $'\zeta =' \zeta_i$. In this case $'\xi_{ij}(1) = \left(\xi^2_{ij}(1), \cdots, \xi^n_{ij}(1)\right)$ $\left('\zeta_{ij}(1) = \zeta^2_{ij}(1), \cdots, \zeta^n_{ij}(1)\right)$ is a unique solution of equation $'f('\xi) =' \nu_{ij}(1)$ $(h\,('\zeta) =' f('\xi_{ij}(1)))$ and $'\xi_{ij} = \varepsilon\ '\xi_{ij}(1)$ $('\zeta_{ij} = \varepsilon\ '\zeta_{ij}(1))$, $\varepsilon > 0$.

Lemma 3 is proved.

Let us prove the assertion b) of theorem 3. Let $\tau_k(x, x_0)$ be the distance between the points $x_0 \in \partial D_{\varepsilon_0}$ and $x \in D$ in the metric $g_k = \left(g^{(k)}_{ij}(x)\right)$; $k = 1, 2$; $\zeta^i = \sum_{j=1}^{n} g^{(2)}_{ij}(x) p^j_0$, $p^i_0 = (\tau_1(x, x_0) + \tau_2(x, x_0))_{x^i}$, $i = 1, 2, ..., n$.

It is known that the function $\tau_k(x, x_0)$ satisfies the equation

$$\sum_{i,j=1}^{n} g^{ij}_k(x) \tau_{kx^i} \tau_{kx^j} = 1\ , \tag{81}$$

where $\left(g^{ij}_k(x)\right)$ is inverse of the matrix $\left(g^{(k)}_{ij}(x)\right)$.

Substracting equation (81) for $k = 1$ from the same equation for $k = 2$, and transforming the obtained equality correspondingly, for the functions $d(x, x_0) = \tau_2(x, x_0) - \tau_1(x, x_0)$ and $b^{ij}(x) = g^{ij}_2 - g^{ij}_1$, we have

$$\sum_{i,j=1}^{n} g^{ij}_2(x)\ p^i_0\ d_{x^j} + \sum_{i,j=2}^{n} b^{ij} \tau_{1x^i} \tau_{1x^j} = 0. \tag{82}$$

Since $g^{(k)}_{1i} = 0$ for $i = 2, 3, ..., n$ and $g^{(k)}_{11} = 1$ for $k = 1, 2$, we have $g^{1j}_k = 0$ for $j = 2, 3, ..., n$ and $g^{11}_k = 1$; therefore, $b^{1j} = 0$ for $j = 1, 2, ..., n$. lt can be seen that the expression $\sum_{i,j=1}^{n} g^{ij}_2(x)$ $p^i_0\ d_{x^j}$ is the derivative of $d(x, x_0)$ along $\gamma^{+}(x, \zeta)$. Integrating equality (82) along $\gamma^{+}(x, \zeta)$ and taking into account the fact that for the points $x, x_0 \in \partial D$, $d(x, x_0) = 0$, we obtain

$$d(x, x_0) = \int_{\gamma^{+}(x,\zeta)} \sum_{i,j=2}^{n} b^{ij}(z) \tau_{1x^i}(z, x_0) \tau_{1x^j}(z, x_0) dt = 0\ , \tag{83}$$

From $\tau_{1z^i}\,(z, x_0) = g^{(1)}_{ij}(z)\nu^j$, $f^i\,(\xi) = g^{ij}_2\,(z)\,g^{(1)}_{kj}(z)\nu^k$, $\nu^k = g^{ks}_1(z) g^{(2)}_{is}(z) f^i\,(\xi)$ and $h\,('\zeta)$ $=' f('\xi)$, we get

$$\sum_{i,j=2}^{n} b^{ij}(z) \tau_{1z^i}(z, x_0) \tau_{1z^j}(z, x_0) = \sum_{m,k=2}^{n} c_{mk}\,(z)\, h^m\,('\zeta)\, h^k\,('\zeta)\,, \tag{84}$$

where $c_{mk}\,(z) = \sum_{2}^{n} b^{ij}\,(z)\, g^{(1)}_{ir}(z) g^{(1)}_{js}(z) g^{rl}_1(z) g^{(2)}_{ml}(z) g^{st}_1(z) g^{(2)}_{kt}(z)$ and in the last sum indixes i, j, r, s, l, t changes from 2 till n.

Therefore, taking into account the homogeneity of the function $h\,(x,' \zeta)$ in $'\zeta$, for $x \in \partial D$, $\zeta \in R^n_0$ from (83), (84) we have :

$$\sum_{m,k=2}^{n}\int_{\gamma^{+}(x,\zeta)} c_{mk}\left(z\left(x,\zeta,t\right)\right)h^{m}(z\left(x,\zeta,t\right),'z\left(x,\zeta,t\right))h^{k}(z\left(x,\zeta,t\right),'z\left(x,\zeta,t\right))dt=0.(85)$$

lt was proved that the uniqueness of solution of problem 1 in the space $C_0^5(\widetilde{D})$ follows from the uniqueness of solution of the problem 5. By the same way, we can prove that the uniqueness of solution of the problem of determination of vector-function $(c_{mk}(x))$ from equalities (85), follows from the uniqueness of solution of the problem 2'. Then from lemma 3 we get : $c_{mk}(x)=0$, $2\le m,k\le n$, i.e. $b^{ij}(x)=0$, so $g_2^{ij}=g_1^{ij}$.

The assertion b) of the theorem 3 is proved.

Proof of the assertion a) of the theorem 3. In the region D for the metric g_k, $k=1,2$, we introduce semigeodesic coordinates as follows: Let us select any point $V_0\in D_{\varepsilon_0}/\overline{D}$ and let us consider the geodesics outgoing from it. We take the ends of the segments of constant length $s_k=r$ on the geodesics, outgoing from V_0.These ends form the hypersurface, which is called the geodesic hypersphere of radius r with center V_0 of the metric g_k. Let a number $r>0$ be so small that the hypersphere lies outside of $\overline{D}$. Let us examine a certain region on the hypersphere with the parameters $u_k^1,u_k^2,\ldots,u_k^{n-1}$. We will carry the geodesics to the same parameters, connecting center of hypersphere V_0 with the points of region D. We will characterize position of an arbitrary point L on the geodesic with arc length $s_k=V_0L$. Then it is obvious that in view of the condition on the metric g_k the variables $u_k^1,u_k^2,\ldots,u_k^{n-1},s_k$ form the semigeodesic coordinate system for the region D for the metric g_k (see [10]). Let D_k be the domain of semigeodesic coordinates of the metric g_k that we introduced for the region D, $k=1,2$.

We build a diffeomorphism φ as follows: let us assign to a point $x^{(2)}\in D$ with the semigeodesic coordinates $\left(u_2^1,u_2^2,\ldots,u_2^{n-1},s_2\right)\in D_2$ in the metric g_2, the point $x^{(1)}\in D$ with the semigeodesic coordinates $\left(u_1^1,u_1^2,\ldots,u_1^{n-1},s_1\right)\in D_1$ in the metric g_1 if the equalities $u_2^i=u_1^i$, $i=1,2,\ldots,n-1$, and $s_2=s_1$ hold. It is not difficult to see that $\varphi|_S=1$. Actually, taking into account equalities $H_1=H_2$ and $g_1=g_2$ outside D and condition $g_k\in C^2(D_{\varepsilon_0})$, it may be proved that, for the point $x\in S$ the rays $\Gamma_1(x,V_0)$, $\Gamma_2(x,V_0)$ outside of the region D coincide, where $\Gamma_k(x,V_0)$ is the geodesic of the metric g_k connecting points $x\in S$ and V_0, $k=1,2$. Here in construction of semigeodesic coordinates for the metrics g_1 and g_2 we take the same geodesic hypersphere with the center at the point V_0 which lies outside D. Then by the uniqueness of the ray $\Gamma_k(x,V_0)$ and the definition of the coordinates $u_1^1,u_1^2,\ldots,u_1^{n-1}$, the first $(n-1)$ components of the semigeodesic coordinates of the point $x\in S$ in the metrics g_1 and g_2 coincide, i.e $u_2^i=u_1^i$, $1,2,\ldots,n-1$. Moreover the equality of last components $(s_2=s_1)$ follows from the equality $H_1=H_2$ and from the fact that rays $\Gamma_1(x,V_0)$ and $\Gamma_2(x,V_0)$ outside of D coincide. Consequently, we have:

1) for $x\in S$, $\varphi(x)=x$,

2) the regions D_1 and D_2 coincide.

Then taking into account 2), the convexity of domain D with respect to g_k, $k=1,2$ and determination of φ, we obtain that φ transforms D to itself. According to the theorem about continuous differentiability, the dependence of the solution to the Cauchy problem (determining the ray of the metric $g_{(k)}\in C^6(D_{\varepsilon_0})$, $k=1,2$) on the initial data, $\partial D\in C^5$ and also from the determination of φ, we have that $\varphi\in C^5(D)$.

Now let us define the mapping $\Psi_k:D_k\to D$ $(k=1,2)$ as follows: it assigns to the point $\left(u_k^1,u_k^2,\ldots,u_k^{n-1},s_k\right)\in D_k$ the point $x^{(k)}\in D$.From determination of φ we have : $\varphi=\Psi_1\circ\Psi_2^{-1}$, and equality $H_1=H_2$ implies that $\widetilde{g}_1=\Psi_1^*g_1$ and $\widetilde{g}_2=\Psi_2^*g_2$ have the same hodograph. Then from assertion b) of theorem 3, we get $\widetilde{g}_1 = \widetilde{g}_2$, therefore, $g_2 = \varphi^* g_1$. Theorem 3 is proved.

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A.Kh. Amirov

Boundary Rigidity For Riemannian Manifolds

Abstract. In the work, the uniqueness of the solution of the problem of restoring the Riemannian metric by the distances between the pairs of the points of boundary of the region is investigated.The uniqueness of solution of the problem, up to the diffeomorphism identical on the boundary of the region is proved within a sufficiently wide class of the metrics.

Key words: hodograph, semigeodesic coordinates, inverse kinematic problem, integral geometry problem, special kinetic equation.

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The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

ADDRESS:
Graduate School of Mathematical Sciences, The University of Tokyo
3–8–1 Komaba Meguro-ku, Tokyo 153-8914, JAPAN
TEL +81-3-5465-7001 FAX +81-3-5465-7012