UTMS 2004–9 February 27, 2004

Functional inequalities and an application for parabolic stochastic partial differential equations containing rotation

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Functional Inequalities and an Application for Parabolic Stochastic Partial Differential Equations Containing Rotation

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Abstract: The main purpose of this paper is to establish a gradient estimate and a parabolic Harnack inequality for the non-symmetric transition semigroup with respect to the Gibbs measure on a path space. This semigroup is related to a diffusion process which is represented by the solution of a certain parabolic stochastic partial differential equation(=SPDE, in abbreviation) containing rotation. We also discuss the relationship between the Gibbs measure and stationary measures of our dynamics. For the proof of our functional inequalities, we formulate a suitable domain of the infinitesimal generator for the semigroup. As an application of our results, we study a certain lower estimate on the transition probability for our dynamics.

1 Introduction

In this paper, we consider a dynamics of unbounded continuous spins on $\mathbb{R}$ containing rotation. This dynamics is described by the following parabolic SPDE which is called the time dependent Ginzburg-Landau type SPDE:

$$\begin{cases} dX_t(x) = \dfrac{1}{2}\big\{\Delta_x X_t(x) - \nabla U(X_t(x))\big\}dt + BX_t(x)dt + dW_t(x), & x \in \mathbb{R},\ t>0, \\[2ex] X_0(x) = w(x), \end{cases} \tag{1.1}$$

where $U(z): \mathbb{R}^d \to \mathbb{R}$, $B \in \mathbb{R}^d \otimes \mathbb{R}^d$, $\Delta_x = d^2/dx^2$, $\nabla = (\partial/\partial z_i)_{i=1}^d$ and $W_t(x)$ is a white noise process. Throughout of this paper, we also use the notation $b(z) := -\frac{1}{2}\nabla U(z) +$

*This work was supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists and by Grant-in-Aid for Scientific research 15-03706, Japan Society for the Promotion of Science, Japan.

$Bz,\ z \in \mathbb{R}^d$. Such equations as the SPDE (1.1) often appear in statistical mechanics to represent dynamic phenomena approaching to equilibrium. In the case of $B = O$, the SPDE (1.1) describes a diffusion process associated with $P(\phi)_1$-model which has its origin in Parisi and Wu's stochastic quantization model. On the other hand, Funaki [4] discussed the SPDE (1.1) as an equation describing a random motion of an elastic string.

The main purpose of this paper is to discuss some functional inequalities and an application. Especially, we establish a gradient estimate (cf. Theorem 5.1) and a parabolic Harnack inequality (cf. Theorem 6.1) for the transition semigroup $\{P_t\}$ associated with the SPDE (1.1). This semigroup is non-symmetric with respect to a Gibbs measure on the path space $C(\mathbb{R}, \mathbb{R}^d)$. In the former paper Kawabi [9], we established these inequalities for the transition semigroup in the case of $B = O$. Needless to say, the semigroup is symmetric with respect to the Gibbs measure.

In this paper, we assume the following conditions on the matrix B and the potential function U. In physical view, the condition (B) means that $\{Bz\}_{z\in\mathbb{R}^d}$ is a magnetic field.

(U1) U is a radial symmetric function of $C^2(\mathbb{R}^d, \mathbb{R})$.

(U2) There exists a constant $K_1 \in \mathbb{R}$ such that $\nabla^2 U(z) \geq -K_1$ holds for any $z \in \mathbb{R}^d$.

(U3) There exist $K_2 > 0$ and $p > 0$ such that $|\nabla U(z)| \leq K_2(1 + |z|^p)$ holds for any $z \in \mathbb{R}^d$.

(U4) $\lim_{|z|\to\infty} U(z) = \infty$.

(B) $B^* = -B$.

As examples of U satisfying above conditions, we are interested in a square potential and a double-well potential. Those are , $U(z) = a|z|^2$ and $U(z) = a(|z|^4 - |z|^2)$, $a > 0$, respectively. We can also give a simple example of B in the case of $d = 2$. It is $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ which generates the rotation matrix $e^{tB} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$.

Now we explain our framework. First, we give a precise meaning of the solution to the SPDE (1.1). When we discuss the existence and the uniqueness of solution of the SPDE (1.1), we have to introduce suitable function spaces to control the growth of $X_t(x)$ as $|x| \to \infty$. We introduce Hilbert spaces $L^2_\lambda(\mathbb{R}, \mathbb{R}^d) := L^2(\mathbb{R}, e^{-2\lambda\chi(x)}dx), \lambda > 0$ where $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ is a positive symmetric convex function satisfying $\chi(x) = |x|$ for $|x| \geq 1$. $L^2_\lambda(\mathbb{R}, \mathbb{R}^d)$ has an inner product defined by

$$(X, Y)_\lambda := \int_{\mathbb{R}} \big(X(x), Y(x)\big)_{\mathbb{R}^d} e^{-2\lambda\chi(x)} dx, \quad X, Y \in L^2_\lambda(\mathbb{R}, \mathbb{R}^d).$$

The corresponding norms are denoted by $\|\cdot\|_\lambda$. In this paper, we fix $\bar{\lambda} > 0$ and denote $E := L^2_{\bar{\lambda}}(\mathbb{R}, \mathbb{R}^d)$ and $H := L^2(\mathbb{R}, \mathbb{R}^d)$.

We also define a suitable subspace of $C(\mathbb{R},\mathbb{R}^d)$. For functions of $C(\mathbb{R},\mathbb{R}^d)$, we define

$$|||X|||_\lambda := \sup_{x\in\mathbb{R}} |X(x)|e^{-\lambda\chi(x)} \quad \text{for } \lambda > 0.$$

Let

$$\mathcal{C} := \bigcap_{\lambda>0} \{X(\cdot) \in C(\mathbb{R},\mathbb{R}^d) \mid |||X|||_\lambda < \infty\}.$$

$\mathcal{C}$ becomes a Fréchet space with the system of norms $|||\cdot|||_\lambda$. We easily see that the dense inclusion $\mathcal{C} \subset E \cap C(\mathbb{R},\mathbb{R}^d)$ holds with respect to the topology of E. We regard these spaces as the state spaces of our dynamics.

We denote by $C_b(E,\mathbb{R})$ the set of bounded continuous functions on E and $\langle u, v\rangle$ is defined by $\int_\mathbb{R}(u(x), v(x))_{\mathbb{R}^d}dx$ if the integral is absolutely converging. We say a function $F : E \longrightarrow \mathbb{R}$ is in class $\mathcal{FC}_b^\infty$ if there exist a function $f := f(\alpha_1, \cdots, \alpha_n) \in C_b^\infty(\mathbb{R}^n)$, $n = 1, 2, \cdots$ and $\{\phi_k\}_{k=1}^n \subset C_0^\infty(\mathbb{R},\mathbb{R}^d)$ satisfying

$$F(w) \equiv f(\langle w, \phi_1\rangle, \cdots, \langle w, \phi_n\rangle).$$

Let $(\Theta, \mathcal{F}, P)$ be a probability space. We define a white noise process (H-cylindrical Brownian motion) $W := \{W_t\}_{t\geq 0}$ on this probability space. Here we call that a family of random linear functionals W on H is a white noise process if the linear functional $\langle W_t, \phi\rangle$ is a one-dimensional Brownian motion multiplied by $\|\phi\|_H$ for every $\phi \in H$ and $\langle W_0, \phi\rangle = 0$ holds. Here we also denote $\langle W_t, \phi\rangle$ by $\int_\mathbb{R}(W_t(x), \phi(x))_{\mathbb{R}^d}dx$. In this paper, we consider a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ by the Brownian filtration $\mathcal{F}_t := \sigma(W_s; s \leq t) \vee \mathcal{N}$, where $\mathcal{N}$ is the family of P-null sets.

Following Iwata [8] and Shiga [18], we call that $\mathcal{C}$-valued $\{\mathcal{F}_t\}$-adapted continuous stochastic process $X := \{X_t(x)\}_{t\geq 0}$ is a mild solution of (1.1) with the initial data $w \in \mathcal{C}$ if there exists a $\{\mathcal{F}_t\}$-white noise process $W = \{W_t\}$ and X satisfies the stochastic integral equation

$$\begin{aligned} X_t(x) &= \int_\mathbb{R} G_t(x,y)w(y)dy + \int_0^t\int_\mathbb{R} G_{t-s}(x,y)b(X_s(y))dsdy \\ &\quad + \int_0^t\int_\mathbb{R} G_{t-s}(x,y)dW_s(y)dy, \quad x \in \mathbb{R},\ t > 0 \end{aligned} \tag{1.2}$$

for P-almost surely. Here we denote the heat kernel by $G_t(x,y) := \frac{1}{\sqrt{2\pi t}}\exp\Big\{-\frac{(x-y)^2}{2t}\Big\}$.

We also give the notion of the weak form solution. It is a solution of the following stochastic integral equation:

$$\langle X_t, \phi\rangle = \langle w, \phi\rangle + \frac{1}{2}\int_0^t \langle X_s, \Delta_x\phi\rangle ds + \int_0^t \Big\langle b(X_s(\cdot)), \phi\Big\rangle ds + \langle W_t, \phi\rangle \tag{1.3}$$

for every $t > 0$ and $\phi \in C_0^\infty(\mathbb{R},\mathbb{R}^d)$ P-almost surely.

It is known that two definitions of the SPDE (1.1) are mutually equivalent. Moreover the SPDE (1.1) has a solution living in $C([0,\infty),\mathcal{C})$ for the initial data $w \in \mathcal{C}$ and the pathwise uniqueness of solutions holds under slightly weaker conditions than (U1) and (U2). See Theorem 3.2, Theorem 5.1 and Theorem 5.2 in [8] and Theorem 2.1 in [18] for the details.

In the sequel, we denote by P_w, $w \in \mathcal{C}$ the probability measure on $C([0,\infty),E)$ induced by X and $\mathbb{M} := (X, \{P_w\}_{w\in\mathcal{C}})$. Moreover we denote by $Y := \{Y_t(x)\}_{t\geq 0}$ the solution of the SPDE

$$dY_t(x) = \frac{1}{2}\big\{\Delta_x Y_t(x) - \nabla U(Y_t(x))\big\}dt + dW_t(x), \quad x \in \mathbb{R},\ t > 0, \tag{1.4}$$

with the initial data $w \in \mathcal{C}$ and $P_w^{(0)}$, $w \in \mathcal{C}$ by the probability measure on $C([0,\infty),E)$ induced by Y and $\mathbb{M}^{(0)} := (Y, \{P_w^{(0)}\}_{w\in\mathcal{C}})$.

We define the transition semigroup $\{P_t\}_{t\geq 0}$ of the dynamics $\mathbb{M}$ by

$$P_t F(w) := \int_E F(y) P_w(X_t \in dy), \quad F \in \mathcal{FC}_b^\infty,\ w \in \mathcal{C}. \tag{1.5}$$

We also define the transition semigroup $\{P_t^{(0)}\}_{t\geq 0}$ of the dynamics $\mathbb{M}^{(0)}$ as above.

The organization of this paper is as follows: In Section 2, we prepare a simple lemma about the stochastic flow for our dynamics $\mathbb{M}$. Moreover, we state a fundamental property for the transition semigroup $\{P_t\}$. In Section 3, we introduce Gibbs measures and stationary measures of our dynamics $\mathbb{M}$. Here we also discuss the relationship between $\{P_t\}$ and $\{P_t^{(0)}\}$. By using this relationship, we prove that a Gibbs measure is a stationary measure of $\mathbb{M}$. In Section 4, we formulate a suitable domain for the infinitesimal generator of the semigroup $\{P_t\}$ by adopting a stochastic approach. In infinite dimensional settings, it is very difficult to find a good domain $\mathcal{D}(\mathcal{L})$ which has both the ring property and the stability under the operation $\{P_t\}$. However we insist that it is not difficult to construct such a domain $\mathcal{D}(\mathcal{L})$ if we handle diffusion processes which are represented by the solution of some stochastic equations (cf. Theorem 4.4). In Theorem 4.4, we also discuss the relationship between $\mathcal{D}(\mathcal{L})$ and $\mathcal{D}(\mathcal{E})$, where $\mathcal{D}(\mathcal{E})$ is the domain of the symmetric Dirichlet form related to the diffusion process $\mathbb{M}^{(0)}$. Here the Littlewood-Paley-Stein inequality plays a significant role. In Section 5, we establish a gradient estimate for $\{P_t\}$. To prove this inequality, the key lemma in Section 2 is used effectively. In Section 6, we establish a parabolic Harnack inequality for $\{P_t\}$. To prove this inequality, various results discussed in Section 4 and the gradient estimate play fundamental roles. We also discuss the some smoothing property of $\{P_t\}$. Finally in Section 7, we give an application of the parabolic Harnack inequality. This is the lower bound on the small time asymptotics of the transition probability for our dynamics $\mathbb{M}$. At present, we do not have the upper bound. This will be discussed in separate papers.

Acknowledgement. This paper is a part of my Ph.D dissertation [10] under the direction of Professor Shigeo Kusuoka. I would like to express my sincere gratitude to him for invaluable advice and constant encouragement. I also thank Professor Shigeki Aida for fruitful discussion and useful comments during the preparation of this paper.

2 Key Estimate for the Stochastic Flow

In this section, we prepare a key estimate for the stochastic flow of the solution of the SPDE (1.1). This estimate plays a significant role in this paper.

Lemma 2.1 *Let X^w and $X^{w'}$ be the solutions of the SPDE* (1.1) *with the initial conditions $X_0^w = w \in \mathcal{C}$ and $X_0^{w'} = w' \in \mathcal{C}$, respectively. Then for every $\lambda > 0$,*

$$\left\|X_t^w - X_t^{w'}\right\|_\lambda \leq e^{\frac{(K_1+2\lambda^2)t}{2}} \left\|w - w'\right\|_\lambda \tag{2.1}$$

holds for P-almost surely. Moreover, for every $h \in H \cap \mathcal{C}$, we have the following estimate for P-almost surely.

$$\|X_t^{w+h} - X_t^w\|_H \leq e^{\frac{K_1 t}{2}} \|h\|_H. \tag{2.2}$$

Proof. The proof of this lemma goes similarly as Lemma 2.1 in [9]. So we only outline the proof. We realize X^w and $X^{w'}$ on the same probability space as solutions of (1.1) with the same cylindrical Brownian motion. Here we set $Z^{w,w'} := X^w - X^{w'}$. By (1.2), Y satisfies the following integral equation:

$$Z_t^{w,w'}(x) = \int_{\mathbb{R}} G_t(x,y)h(y)dy + \int_0^t \int_{\mathbb{R}} G_{t-s}(x,y)\Psi(s,y)dsdy,$$

where $\Psi(s,y) := b(X_s^w(y)) - b(X_s^{w'}(y))$. This expression leads us to the semi-linear heat equation

$$\begin{cases} \dfrac{\partial}{\partial t} Z_t^{w,w'}(x) = \dfrac{1}{2}\Delta_x Z_t^{w,w'}(x) + \Psi(t,x), \quad x \in \mathbb{R},\ t > 0, \\ \\ Z_0^{w,w'}(x) = w(x) - w'(x). \end{cases}$$

Here we take $\lambda \in (0, \bar{\lambda}]$, multiply both sides by $2Z_t^{w,w'}(x)e^{-2\lambda\chi(x)}$ and integrate over $(0,t) \times \mathbb{R}$. We remark that the convexity of χ implies $\|\nabla\chi\|_{L^\infty} \leq 1$. Then by applying integration by parts, we obtain

$$\begin{aligned} &\int_{\mathbb{R}} |Z_t^{w,w'}(x)|^2 e^{-2\lambda\chi(x)} dx \\ \leq\ & \int_{\mathbb{R}} |w(x) - w'(x)|^2 e^{-2\lambda\chi(x)} dx + 2\lambda^2 \int_0^t \int_{\mathbb{R}} |Z_s^{w,w'}(x)|^2 e^{-2\lambda\chi(x)} dsdx \\ & + \int_0^t \int_{\mathbb{R}} \left(\Psi(s,x), Z_s^{w,w'}(x)\right)_{\mathbb{R}^d} e^{-2\lambda\chi(x)} dsdx. \end{aligned} \tag{2.3}$$

Needless to say, by the lack of the regularity for $Z^{w,w'}$, above computations are formal, however we can use the mollifier technique to justify (2.3) holds. See Lemma 2.1 in [9] for the details.

Here we note the condition (B) implies

$$\big(Bz - Bz', z - z'\big)_{\mathbb{R}^d} = 0, \;\; z, z' \in \mathbb{R}^d. \tag{2.4}$$

Hence the condition (U1), (2.3) and (2.4) lead us to the following estimate:

$$\begin{aligned}
&\int_{\mathbb{R}} |Z_t^{w,w'}(x)|^2 e^{-2\lambda\chi(x)} dx \\
&\leq \int_{\mathbb{R}} |w(x) - w'(x)|^2 e^{-2\lambda\chi(x)} dx + (K_1 + 2\lambda^2) \int_0^t \int_{\mathbb{R}} |Z_s^{w,w'}(x)|^2 e^{-2\lambda\chi(x)} ds dx.
\end{aligned}$$

By using Gronwall's lemma, we obtain

$$\int_{\mathbb{R}} |Z_t^{w,w'}(x)|^2 e^{-2\lambda\chi(x)} dx \leq e^{(K_1+2\lambda^2)t} \int_{\mathbb{R}} |w(x) - w'(x)|^2 e^{-2\lambda\chi(x)} dx. \tag{2.5}$$

This completes the proof of (2.1). For the assertion (2.2), we complete the proof by letting $\lambda \downarrow 0$. ∎

Before closing this section, we present a certain continuity for the transition semigroup. As a consequence of this lemma, we can see

Corollary 2.2 *For $F \in \mathcal{F}C_b^\infty$ and $t \geq 0$, P_tF and $P_t^{(0)}F$ can be extended functions of $C_b(E, \mathbb{R})$. (Throughout of this paper, we also denote them P_tF and $P_t^{(0)}F$, respectively.)*

Proof. The proof of this lemma is quite similar as the proof of Corollary 2.2 in [9]. So we also outline the proof for P_tF.

By (2.1), we have the following estimate for every $w, w' \in \mathcal{C} \subset E$:

$$|P_tF(w) - P_tF(w')| \leq K(\bar{\lambda}, F) e^{(\frac{K_1+2\bar{\lambda}^2}{2}t)} \cdot \|w - w'\|_E, \tag{2.6}$$

where $K(\bar{\lambda}, F)$ is a positive constant defined by $\|\nabla f\|_{L^\infty(\mathbb{R}^n)} \cdot \Big\{ \sum_{i=1}^n \big(\int_{\mathbb{R}} |\phi_i(x)|^2 e^{2\bar{\lambda}\chi(x)} dx \big) \Big\}^{1/2}$ and $\langle X_t^w, \phi \rangle$ is denoted by $(\langle X_t^w, \phi_1 \rangle, \cdots, \langle X_t^w, \phi_n \rangle)$ for simplicity. This estimate means that P_tF is uniformly continuous on $\mathcal{C}$. Finally by recalling $\mathcal{C} \subset E$ is a dense inclusion, we can completes the proof. ∎

3 The Relationship between a Gibbs Measure and Stationary Measures on a Path Space

In this section, we discuss the relationship between a Gibbs measure on the path space $\mathcal{C}$ and stationary measures for a non-symmetric diffusion process described by the SPDE

(1.1). Roughly speaking, we prove that a Gibbs measure keeps the invariance for our dynamics under the rotation. In what follows, we denote $\mathcal{P}(\mathcal{C})$ and $\mathcal{P}(E)$ the class of all probability measures on the space $\mathcal{C}$ and E, respectively. Moreover we denote by $\mathcal{B}_r$ and $\mathcal{B}_r^*$ the σ-field generated by $\mathcal{C}|_{[-r,r]}$ and $\mathcal{C}|_{\mathbb{R}\setminus(-r,r)}$, respectively.

3.1 Preliminary Facts and Results

In this subsection, we prepare some terminologies on Gibbs measures and stationary measures for our dynamics to state results.

Firstly, we introduce a Gibbs measure. Consider a Schrödinger operator $H := -\frac{1}{2}\Delta + U$ on $L^2(\mathbb{R}^d, dz)$, where Δ is the d-dimensional Laplacian. Then the condition (U4) assures that H has purely discrete spectrum and a complete set of eigenfunctions. We denote $\kappa > 0$ by the minimal eigenvalue and Ω by the corresponding eigenfunction with $\|\Omega\|_{L^2} = 1$. We define $\mu(A)$ for $A \in \mathcal{B}_r$, $r > 0$ by

$$\mu(A) := e^{2r\kappa} \int_{\mathbb{R}^d \times \mathbb{R}^d} \Omega(z)\Omega(z')p(2r, z, z')\mathbb{E}_{-r,r}^{z,z'}\Big[\exp\big(- \int_{-r}^{r} U(w(x))dx\big); A\Big] dzdz', \quad (3.1)$$

where $p(t, x, y) := \left(\frac{1}{\sqrt{2\pi t}}\right)^d \exp\left\{ - \frac{|x-y|^2}{2t} \right\}$ and $\mathbb{E}_{-r,r}^{z,z'}[\cdot]$ is the expectation with respect to the path measure of Brownian bridge such that $w(-r) = z, w(r) = z'$.

Then we can easily check that μ is well-defined as an element of $\mathcal{P}(\mathcal{C})$. Since the inclusion map of $\mathcal{C}$ into E is continuous, we can also regard $\mu \in \mathcal{P}(E)$ by identifying it with its image measure under the inclusion map.

By applying the Feynman-Kac formula, it is not difficult to see that μ satisfies the following DLR-equation for every $r \in \mathbb{N}$ and μ-a.e. $\xi \in \mathcal{C}$:

$$\mu(dw|\mathcal{B}_r^*)(\xi) \;=\; Z_{r,\xi}^{-1} \exp\big(- \int_{-r}^{r} U(w(x))dx\big)\mathcal{W}_{r,\xi}(dw), \quad (3.2)$$

where $\mathcal{W}_{r,\xi}$ is the path measure of the Brownian bridge on $[-r, r]$ with a boundary condition $w(-r) = \xi(-r), w(r) = \xi(r)$ and $Z_{r,\xi} := \mathbb{E}^{\mathcal{W}_{r,\xi}}\Big[\exp\big(- \int_{-r}^{r} U(w(x))dx\big)\Big]$ is the normalization constant. See Proposition 2.7 in Iwata [7] for details. Although generally there exist another μ's satisfying (3.2), in this paper we only consider the Gibbs measure μ which has been constructed in (3.1).

From the expression (3.1), we easily see that μ is shift invariant and

$$\int_E \Big\{ \int_{\mathbb{R}} |w(x)|^{2m} e^{-2\lambda\chi(x)} dx\Big\}\mu(dw) \le \frac{1}{\lambda}\int_{\mathbb{R}^d} |z|^{2m}\Omega(z)^2 dz < \infty. \quad (3.3)$$

holds for any integer m and $\lambda > 0$.

Moreover we have to mention the $C_0^\infty(\mathbb{R}, \mathbb{R}^d)$-quasi-invariance of the Gibbs measure μ:

$$\int_E F(w + h)\mu(dw) = \int_E F(w)e^{\Phi(h,w)}\mu(dw), \quad h \in C_0^\infty(\mathbb{R}, \mathbb{R}^d), F \in B_b(E, \mathbb{R}), \quad (3.4)$$

where $\Phi(h,w)$ is defined by

$$\Phi(h,w)=\int_{\mathbb{R}}\Big\{U\big(w(x)\big)-U\big(w(x)-h(x)\big)-\frac{1}{2}|h'(x)|^2-(w(x),\Delta_x h(x))_{\mathbb{R}^d}\Big\}dx. \quad (3.5)$$

For details the reader is referred to Funaki [5] and [7]. This property will be used in the sequel of this paper.

Next we recall the notion of the stationary measure. We call that $\mu\in\mathcal{P}(E)$ is a stationary measure of the SPDE (1.1) if it satisfies

$$\int_E P_tF(w)\mu(dw)=\int_E F(w)\mu(dw)$$

for every $t>0$ and $F\in\mathcal{FC}_b^\infty$. We denote by $\mathcal{S}(b)$ the family of tempered stationary measures. Here we say a probability measure $\mu\in\mathcal{P}(E)$ is tempered if $\mathbb{E}^\mu\big[\|w\|_\lambda^2\big]<\infty$ holds for all $\lambda>0$.

The following theorem is our main result in this section.

Theorem 3.1 *Under the conditions* (U1)-(U4) *and* (B)*, The Gibbs measure μ belongs to $\mathcal{S}(b)$.*

We also present the following theorem as a by-product of Theorem 3.1. We assume the following condition which is stronger than the condition (U2).

(U5) U is strictly convex, i.e., there exists a constant $K_3>0$ such that $\nabla^2U(z)\ge K_3$ holds for any $z\in\mathbb{R}^d$.

Theorem 3.2 *Under the conditions* (U1), (U3)-(U5) *and* (B)*, The Gibbs measure μ is the unique element of $\mathcal{S}(b)$.*

3.2 The Relationship between $\{P_t\}$ and $\{P_t^{(0)}\}$

In this subsection, we study a relationship between our dynamics $\mathbb{M}=\big(X,P_w\big)$ and $\mathbb{M}^{(0)}=\big(Y,P_w^{(0)}\big)$. It is known that the Gibbs measure μ is $\{P_t^{(0)}\}$-reversible, i.e.,

$$\int_E P_t^{(0)}F(w)G(w)\mu(dw)=\int_E P_t^{(0)}G(w)F(w)\mu(dw)$$

holds for every $t>0$ and $F,G\in\mathcal{FC}_b^\infty$.

Especially, we discuss the relationship between the semigroups $\{P_t\}$ and $\{P_t^{(0)}\}$. This relationship will plays a important role in the proof of Theorem 3.1.

At the beginning, we prepare the following semigroup $\{Q_t\}_{t\ge0}$ as follows:

$$Q_tF(w):=F\big(R_tw\big),\quad F\in C_b(E,\mathbb{R}),\ w\in E, \quad (3.6)$$

where $R_t:E\to E$ is defined by $(R_tw)(\cdot):=e^{tB}(w(\cdot))$. Then we have the following theorem.

Theorem 3.3 (1) *For any* $F \in \mathcal{F}\mathcal{C}_b^\infty$ *and* $s, t \geq 0$,

$$P_t^{(0)} Q_s F(w) = Q_s P_t^{(0)} F(w), \quad w \in E. \tag{3.7}$$

(2) *For any* $F \in \mathcal{F}\mathcal{C}_b^\infty$ *and* $t \geq 0$,

$$P_t F(w) = P_t^{(0)} Q_t F(w) = Q_t P_t^{(0)} F(w), \quad w \in E. \tag{3.8}$$

For the proof of this theorem, we prepare the following lemma.

Lemma 3.4 (1) *Let* $W := \{W_t\}_{t\geq 0}$ *be a white noise process and* $\{\phi_i\}_{i=1}^\infty$ *be a C.O.N.S. of* H. *Then there exists a sequence of independent one-dimensional Brownian motions* $\{\beta_i\}_{i=1}^\infty$ *and a Hilbert space* $\mathcal{H}$ *such that the inclusion* $H \subset \mathcal{H}$ *is a Hilbert-Schmidt operator and the expansion*

$$W_t = \sum_{i=1}^\infty \beta_i(t)\phi_i, \ t \geq 0 \tag{3.9}$$

holds. Here we regard the right hand of (3.9) *as a* $\mathcal{H}$*-valued continuous square integrable* $\{\mathcal{F}_t\}$*-martingale.*
(2) *For* $\{\beta_i\}_{i=1}^\infty$ *and* $\{\phi_i\}_{i=1}^\infty$ *denoted above, we define a* $\mathcal{H}$*-valued stochastic process* $\hat{W} := \{\hat{W}\}_{t\geq 0}$ *by*

$$\hat{W}_t := \sum_{i=1}^\infty \int_0^t (R_s\phi_i) d\beta_i(s), \quad t \geq 0. \tag{3.10}$$

Then it is also a white noise process. Here we regard (3.10) *as the assertion* (1).

Proof. The assertion (1) is well-known. See Da Prato-Zabczyk's book [3] for the detail. We show the assertion (2). By recalling the condition (B), we easily have

$$\begin{aligned} \Big\langle \langle \hat{W}_\cdot, \phi\rangle, \langle \hat{W}_\cdot, \psi\rangle \Big\rangle_t &= \int_0^t \sum_{i=1}^\infty \big(R_s\phi_i, \phi\big)_H \big(R_s\phi_i, \psi\big)_H ds \\ &= \int_0^t \big(R_s\phi, R_s\psi\big)_H ds = t(\phi, \psi)_H \end{aligned} \tag{3.11}$$

and $\{\hat{W}_t\}_{t\geq 0}$ is a martingale. Hence by Levy's characterization, this is also a white noise process. ∎

Proof of Theorem 3.3. At the beginning, we introduce the heat semigroup $\{G_t\}_{t\geq 0} : \mathcal{C} \to \mathcal{C}$ defined by

$$G_t w(x) := \int_{\mathbb{R}} G_t(x, y) w(y) dy, \quad w \in \mathcal{C}, \ x \in \mathbb{R}. \tag{3.12}$$

(1) For the solution of (1.4) with the initial data $w \in \mathcal{C}$ and $s \geq 0$, we consider a stochastic process $\tilde{Y} := \{R_s Y_t\}_{t\geq 0}$. By recalling (1.2) in the case of $B = O$, this process satisfies the following stochastic integral equation.

$$\begin{aligned}
\tilde{Y}_t(x) &= G_t(R_s w)(x) - \frac{1}{2}\int_0^t G_{t-\tau}\big(R_s \nabla U(Y_\tau(\cdot))\big)(x)d\tau \\
&\quad + \sum_{i=1}^{\infty}\int_0^t G_{t-\tau}(R_s\phi_i)(x)d\beta_i(\tau) \\
&= G_t(R_s w)(x) - \frac{1}{2}\int_0^t G_{t-\tau}\big(\nabla U(R_s Y_\tau(\cdot))\big)(x)d\tau \\
&\quad + \sum_{i=1}^{\infty}\int_0^t G_{t-\tau}(R_s\phi_i)(x)d\beta_i(\tau) \\
&= G_t(R_s w)(x) - \frac{1}{2}\int_0^t G_{t-\tau}\big(\nabla U(\tilde{Y}_\tau(\cdot))\big)(x)d\tau \\
&\quad + \sum_{i=1}^{\infty}\int_0^t G_{t-\tau}(R_s\phi_i)(x)d\beta_i(\tau) \\
&= \int_{\mathbb{R}} G_t(x,y)(e^{sB}w)(y)dy - \frac{1}{2}\int_0^t\int_{\mathbb{R}} G_{t-\tau}(x,y)\nabla U(\tilde{Y}_\tau(y))d\tau dy \\
&\quad + \int_0^t\int_{\mathbb{R}} G_{t-\tau}(x,y)d\tilde{W}_\tau(y)dy, \qquad (3.13)
\end{aligned}$$

where we used the conditions (B) and (U1) for the second line and $\tilde{W}$ is another white noise process defined by

$$\tilde{W}_t := \sum_{i=1}^{\infty}\beta_i(t)(R_s\phi_i), \quad t \geq 0. \tag{3.14}$$

Then (3.13) means that $\tilde{P}^{(0)}_w$ is equal to $P^{(0)}_{R_s w}$ for every $w \in \mathcal{C}$. Here $\tilde{P}^{(0)}_w, w \in \mathcal{C}$ is the probability measure on $C([0,\infty), E)$ induced by $\tilde{Y}$.

Hence we have

$$\begin{aligned}
P^{(0)}_t Q_s F(w) &= \mathbb{E}[Q_s F(Y^w_t)] \\
&= \mathbb{E}[F(\tilde{Y}^w_t)] \\
&= \mathbb{E}[F(Y^{R_s w}_t)] \\
&= P^{(0)}_t F(R_s w) = Q_s P^{(0)}_t F(w), \quad w \in \mathcal{C}. \qquad (3.15)
\end{aligned}$$

Therefore we have (3.7) by combining (3.15) and Corollary 2.2.

(2) For the solution of (1.4) with the initial data $w \in \mathcal{C}$, we consider a stochastic process $\tilde{X} := \{R_t Y_t\}_{t\geq 0}$. We are going to look for the stochastic integral equation of which $\tilde{X}$ is a solution.

Since Y is the mild solution of (1.4), we have

$$\begin{aligned}
\tilde{X}_t(x) &= G_t(R_t w)(x) - \frac{1}{2}\int_0^t G_{t-\tau}\big(R_t \nabla U(Y_\tau(\cdot))\big)(x)d\tau \\
&\quad + \sum_{i=1}^{\infty}\int_0^t G_{t-\tau}(R_t\phi_i)(x)d\beta_i(\tau) \\
&= G_t(R_t w)(x) - \frac{1}{2}\int_0^t G_{t-\tau}\Big\{R_{t-\tau}\nabla U(R_\tau Y_\tau(\cdot))\Big\}(x)d\tau \\
&\quad + \sum_{i=1}^{\infty}\int_0^t G_{t-\tau}(R_t\phi_i)(x)d\beta_i(\tau) \\
&= G_t(R_t w)(x) - \frac{1}{2}\int_0^t R_{t-\tau}\Big\{G_{t-\tau}\Big(\nabla U(\tilde{X}_\tau(\cdot))\Big)\Big\}(x)d\tau \\
&\quad + \sum_{i=1}^{\infty}\int_0^t G_{t-\tau}\big\{R_{t-\tau}(R_\tau\phi_i)\big\}(x)d\beta_i(\tau) \\
&=: S_t^{(1)}(x;w) - \frac{1}{2}S_t^{(2)}(x;\tilde{X}) + S_t^{(3)}(x;W),
\end{aligned} \tag{3.16}$$

where we used the conditions (B) and (U1) for the second line.

To expand the right hand side of (3.16), we prepare the following $\mathbb{R}^d \otimes \mathbb{R}^d$-valued equality:

$$e^{(t-\tau)B} = I_{\mathbb{R}^d} + e^{-\tau B}\int_\tau^t Be^{sB}ds, \quad 0 \le \tau \le t. \tag{3.17}$$

By recalling (3.17) and the semigroup property for $\{G_t\}$, we have the following expansion on the term $S_t^{(1)}(x;w)$:

$$\begin{aligned}
S_t^{(1)}(x;w) &= G_t\Big(\int_0^t B(R_s w)(\cdot)ds + w\Big)(x) \\
&= G_t w(x) + \int_0^t G_t\big(B(R_s w)(\cdot)\big)(x)ds \\
&= G_t w(x) + \int_0^t G_{t-s}\big\{BR_s\big((G_s w)(\cdot)\big)\big\}(x)ds \\
&= G_t w(x) + \int_0^t G_{t-s}\Big\{B\big(S_s^{(1)}(\cdot;w)\big)\Big\}(x)ds.
\end{aligned} \tag{3.18}$$

Next we proceed to the expansion on the term $S_t^{(2)}(x;\tilde{X})$. By using Fubini's theorem, the semigroup property for $\{G_t\}$ and (3.17), we have

$$\begin{aligned}
&S_t^{(2)}(x;\tilde{X}) \\
&= \int_0^t G_{t-\tau}\big\{\nabla U(\tilde{X}_\tau(\cdot))\big\}(x)d\tau \\
&\quad + \int_0^t G_{t-\tau}\Big\{\Big(e^{-\tau B}\int_\tau^t Be^{sB}ds\Big)\cdot\nabla U(\tilde{X}_\tau(\cdot))\Big\}(x)d\tau
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t G_{t-s}\{\nabla U(\tilde{X}_s(\cdot))\}(x)ds \\
&\quad + \int_0^t ds \int_0^s G_{t-\tau}\Big(Be^{(s-\tau)B} \cdot \nabla U(\tilde{X}_\tau(\cdot))\Big)(x)d\tau \\
&= \int_0^t G_{t-s}\{\nabla U(\tilde{X}_s(\cdot))\}(x)ds \\
&\quad + \int_0^t G_{t-s}\Big[B\Big\{\int_0^s R_{s-\tau}\Big(G_{s-\tau}\big(\nabla U(\tilde{X}_s(\cdot))\big)\Big)d\tau\Big\}\Big](x)ds \\
&= \int_0^t G_{t-s}\{\nabla U(\tilde{X}_s(\cdot))\}(x)ds + \int_0^t G_{t-s}\Big\{B\Big(S_s^{(2)}(\cdot;\tilde{X})\Big)\Big\}(x)ds, \qquad (3.19)
\end{aligned}
$$

Next we proceed to the expansion on the term $S_t^{(3)}(x;W)$. By using stochastic Fubini's theorem, the semigroup property for $\{G_t\}$ and (3.17), we have

$$
\begin{aligned}
&S_t^{(3)}(x;W) \\
&= \sum_{i=1}^\infty \int_0^t G_{t-\tau}(R_\tau\phi_i)(x)d\beta_i(\tau) \\
&\quad + \sum_{i=1}^\infty \int_0^t G_{t-\tau}\Big\{\Big(e^{-\tau B}\int_\tau^t Be^{sB}ds\Big)\cdot(R_\tau\phi_i)\Big\}(x)d\beta_i(\tau) \\
&= \sum_{i=1}^\infty \int_0^t G_{t-s}(R_s\phi_i)(x)d\beta_i(s) \\
&\quad + \sum_{i=1}^\infty \int_0^t ds \int_0^s G_{t-\tau}\Big(Be^{(s-\tau)B}(R_\tau\phi_i)\Big)(x)d\beta_i(\tau) \\
&= \sum_{i=1}^\infty \int_0^t G_{t-s}(R_s\phi_i)(x)d\beta_i(s) \\
&\quad + \sum_{i=1}^\infty \int_0^t G_{t-s}\Big[B\Big\{\int_0^s G_{s-\tau}\Big(R_{s-\tau}\big(R_\tau\phi_i\big)\Big)(\cdot)d\beta_i(\tau)\Big\}\Big](x)ds \\
&= \int_0^t\int_{\mathbb{R}} G_{t-s}(x,y)d\hat{W}_s(y)dy + \int_0^t G_{t-s}\Big\{B\big(S_s^{(3)}(\cdot;W)\big)\Big\}(x)ds, \qquad (3.20)
\end{aligned}
$$

where we used (3.10) in Lemma 3.4 for the fourth line.

Finally we combine (3.16), (3.18), (3.19) and (3.20). Then we have

$$
\begin{aligned}
\tilde{X}_t(x) &= \Big(G_t w(x) + \int_0^t G_{t-s}\Big\{B\big(S_s^{(1)}(\cdot;w)\big)\Big\}(x)ds\Big) \\
&\quad -\frac{1}{2}\Big(\int_0^t G_{t-s}\{\nabla U(\tilde{X}_s(\cdot))\}(x)ds + \int_0^t G_{t-s}\Big\{B\Big(S_s^{(2)}(\cdot;\tilde{X})\Big)\Big\}(x)ds\Big) \\
&\quad +\Big(\int_0^t\int_{\mathbb{R}} G_{t-s}(x,y)d\hat{W}_s(y)dy + \int_0^t G_{t-s}\Big\{B\big(S_s^{(3)}(\cdot;W)\big)\Big\}(x)ds\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= G_t w(x) - \frac{1}{2}\int_0^t G_{t-s}\big\{\nabla U(\tilde{X}_s(\cdot))\big\}(x)ds \\
&\quad + \int_0^t G_{t-s}\Big\{B\Big(S_s^{(1)}(\cdot; w) - \frac{1}{2}S_s^{(2)}(\cdot;\tilde{X}) + S_s^{(3)}(\cdot; W)\Big)\Big\}(x)ds \\
&\quad + \int_0^t\int_{\mathbb{R}} G_{t-s}(x,y)d\hat{W}_s(y)dy \\
&= \int_{\mathbb{R}} G_t(x,y)w(y)dy + \int_0^t\int_{\mathbb{R}} G_{t-s}(x,y)b(\tilde{X}_s(y))dsdy \\
&\quad + \int_0^t\int_{\mathbb{R}} G_{t-s}(x,y)d\hat{W}_s(y)dy, \quad x\in\mathbb{R},\ t>0. \qquad (3.21)
\end{aligned}
$$

This means the stochastic process $\tilde{X}$ is also a mild solution of the SPDE (1.1) with the initial data $w\in\mathcal{C}$. Therefore the uniqueness implies that P_w is equal to $\tilde{P}_w$ for every $w\in\mathcal{C}$. Here $\tilde{P}_w$ is the probability measure on $C([0,\infty),E)$ induced by $\tilde{X}$.

Hence we have

$$
\begin{aligned}
P_t^{(0)}Q_tF(w) &= \mathbb{E}[Q_tF(Y_t^w)] \\
&= \mathbb{E}[F(\tilde{X}_t^w)] \\
&= \mathbb{E}[F(X_t^w)] = P_tF(w), \quad w\in\mathcal{C}. \qquad (3.22)
\end{aligned}
$$

Moreover by putting $s=t$ in (3.7), we complete the proof. ∎

3.3 Proof of Theorem 3.1 and Theorem 3.2

As a preparation, we present the following lemma.

Lemma 3.5 *The Gibbs measure μ is $\{Q_t\}$-invariant, i.e.,*

$$
\int_E Q_tF(w)\mu(dw) = \int_E F(w)\mu(dw) \qquad (3.23)
$$

holds for every $t\geq 0$ and $F\in\mathcal{FC}_b^\infty$.

Proof. We take $F\in\mathcal{FC}_b^\infty$. By taking a sufficient large number r such that $r\geq \max\{\mathrm{supp}(\phi_i); 1\leq i\leq n\}$, we have the following expressions:

$$
\begin{aligned}
F(w) &= f\Big(\int_{-r}^r (w(x),\phi_1(x))_{\mathbb{R}^d}dx,\cdots,\int_{-r}^r (w(x),\phi_n(x))_{\mathbb{R}^d}dx\Big), \\
Q_tF(w) &= f\Big(\int_{-r}^r (e^{tB}w(x),\phi_1(x))_{\mathbb{R}^d}dx,\cdots,\int_{-r}^r (e^{tB}w(x),\phi_n(x))_{\mathbb{R}^d}dx\Big).
\end{aligned}
$$

Firstly we consider the finite volume Gibbs measure $\mu_{r,0}$ defined by

$$
\mu_{r,0}(dw) := Z_{r,0}^{-1}\exp\big(-\int_{-r}^r U(w(x))dx\big)\mathcal{W}_{r,0}(dw),
$$

where $\mathcal{W}_{r,0}$ is the path measure of the Brownian bridge on $[-r,r]$ with a boundary condition $w(-r)=w(r)=0$ and $Z_{r,0}$ is the normalization constant.

Here we consider a transformation $R_t^{(r)}: C([-r,r],\mathbb{R}^d)\to C([-r,r],\mathbb{R}^d)$ defined by $(R_t^{(r)}w)(x):=e^{tB}(w(x)), x\in[-r,r]$. Then by recalling that the potential function U is radial symmetric and $\mathcal{W}_{r,0}(dw)$ is invariant under the operation $R_t^{(r)}$, we have

$$\int_{C([-r,r],\mathbb{R}^d)} Q_tF(w)\mu_{r,0}(dw)=\int_{C([-r,r],\mathbb{R}^d)} F(w)\mu_{r,0}(dw). \tag{3.24}$$

Next we define the extension of $\mu_{r,0}$ to the probability measure $\tilde{\mu}_{r,0}$ on $\mathcal{C}$ as $\tilde{\mu}_{r,0}(A)=\mu_{r,0}(A)$ for $A\in\mathcal{B}_r$ and $\tilde{\mu}_{r,0}\big(w(x)\equiv 0 \text{ for } x\in\mathbb{R}\setminus(-r,r)\big)=1$. We also recall the probability measure $\tilde{\mu}_{r,0}$ converges weakly to μ as $r\to\infty$ on the space $\mathcal{C}$. See Proposition 3.2 in Funaki [4]. Hence by recalling (3.24), we have

$$\begin{aligned}\int_E Q_tF(w)\mu(dw) &= \lim_{r\to\infty}\int_{\mathcal{C}} Q_tF(w)\tilde{\mu}_{r,0}(dw)\\ &= \lim_{r\to\infty}\int_{\mathcal{C}} F(w)\tilde{\mu}_{r,0}(dw)=\int_E F(w)\mu(dw). \quad \blacksquare\end{aligned}$$

Proof of Theorem 3.1. We recall that the Gibbs measure μ is $\{P_t^{(0)}\}$-reversible. See Lemma 2.9 in Iwata [7]. Hence by virtue of Proposition 3.3 and Lemma 3.5, we easily have

$$\begin{aligned}\int_E P_tF(w)\mu(dw) &= \int_E P_t^{(0)}(Q_tF)(w)\mu(dw)\\ &= \int_E Q_tF(w)\mu(dw)=\int_E F(w)\mu(dw) \end{aligned}\tag{3.25}$$

for any $F\in\mathcal{FC}_b^\infty$. Hence by recalling (3.3), we complete the proof. $\blacksquare$

Proof of Theorem 3.2. Let $\mu,\tilde{\mu}\in\mathcal{S}(b)$. Let X and $\tilde{X}$ be corresponding solutions of the SPDE (1.1) with initial distributions μ and $\tilde{\mu}$, respectively.

By a similar argument to the proof of Lemma 2.1, we have

$$\|X_t-\tilde{X}_t\|_\lambda\le e^{(\frac{-K_3+2\lambda^2}{2}t)}\|X_0-\tilde{X}_0\|_\lambda.$$

for P-almost surely. Then for every $F\in\mathcal{FC}_b^\infty$, we have

$$\begin{aligned}|\mathbb{E}^\mu[F]-\mathbb{E}^{\tilde{\mu}}[F]| &= \big|\mathbb{E}[F(X_t)]-\mathbb{E}[F(\tilde{X}_t)]\big|\\ &\le K(\lambda)e^{(\frac{-K_3+2\lambda^2}{2}t)}\mathbb{E}\big[\|X_0-\tilde{X}_0\|_\lambda\big], \end{aligned}\tag{3.26}$$

where the positive constant $K(\lambda)$ is defined in the proof of Corollary 2.2.

Now we fix $\lambda>0$ such that $-K_3+2\lambda^2<0$. Then by letting $t\to\infty$ on both sides of (3.26), $\mathbb{E}^\mu[F]=\mathbb{E}^{\tilde{\mu}}[F]$ holds for every $F\in\mathcal{FC}_b^\infty$. Hence $\mu=\tilde{\mu}$ holds. This means the uniqueness of $\mathcal{S}(b)$. $\blacksquare$

Remark 3.6 *By Theorem* 3.1*, we have* $\|P_tF\|_{L^1(E;\mu)} \le \|F\|_{L^1(E;\mu)}$ *holds for* $F \in \mathcal{FC}_b^\infty$*. Hence Riesz-Thorin's interpolation theorem implies that* $\{P_t\}$ *can be extended to a strongly continuous contraction semigroup on* $L^p(E,\mathbb{R};\mu)$*,* $1 \le p < \infty$*.*

4 Fundamental Properties of a Suitable Domain for the Infinitesimal Generator

In this section, we formulate a suitable domain for the infinitesimal generator of the semigroup $\{P_t\}$ via a stochastic approach. This approach may be found in Revuz-Yor's book [16]. They called the generator by the extended infinitesimal generator. In this paper, we give a slightly different formulation such that the domain has both the ring property and the stability under $\{P_t\}$. These properties will play fundamental roles in the sequel.

4.1 Definition of the Domain for the Infinitesimal Generator

Let $(L_p, \mathrm{Dom}(L_p))$ be the infinitesimal generator of the strongly continuous contraction semigroup $\{P_t\}$ on $L^p(E;\mu)$, $1 \le p < \infty$ which is defined by

$$\begin{aligned} \mathrm{Dom}(L_p) \;&:=\; \Big\{ F \in L^p(E;\mu) \mid \lim_{t\downarrow 0} \frac{1}{t}\big(P_tF - F\big) \text{ exists } \Big\} = (I-L_p)^{-1}L^p(E;\mu), \\ L_pF \;&:=\; \lim_{t\downarrow 0} \frac{1}{t}\big(P_tF - F\big),\ F \in \mathrm{Dom}(L_p). \end{aligned}$$

We consider the operator $\mathcal{L}$ with a suitable domain $\mathcal{D}(\mathcal{L})$ as follows:

$$\begin{cases} \mathcal{D}(\mathcal{L}) := \bigcap_{p\ge 1} \mathrm{Dom}(L_p), \\ \mathcal{L}F := L_pF,\ F \in \mathcal{D}(\mathcal{L}). \end{cases} \tag{4.1}$$

In this subsection, we give a stochastic representation for $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$. We have

Proposition 4.1 *A function* $F : E \to \mathbb{R}$ *belongs to* $\mathcal{D}(\mathcal{L})$ *if and only if there exist a function* $\Phi^{[F]} : E \to \mathbb{R}$ *with* $\Phi^{[F]} \in \bigcap_{1\le p<\infty} L^p(E;\mu)$ *and a* $\{\mathcal{F}_t\}$*-martingale* $\{M_t^{[F]}\}_{t\ge 0}$ *such that the following identities hold:*

$$\text{(i)}\quad M_t^{[F]} = F(X_t) - F(X_0) - \int_0^t \Phi^{[F]}(X_s)ds \quad \textit{for } P_\mu\textit{-almost surely,} \tag{4.2}$$

$$\text{(ii)}\quad \mathbb{E}^{P_\mu}\big[|M_t^{[F]}|^p\big] < \infty \quad \textit{for any } t \ge 0,\ 1 \le p < \infty, \tag{4.3}$$

where $P_\mu := \int_E P_w\mu(dw)$*. Moreover the function* $\Phi^{[F]}$ *is equal to the generator* $\mathcal{L}F$*.*

Here we have to mention that the martingale $\{M_t^{[F]}\}$ and the function $\Phi^{[F]}$ in (4.2) are determined uniquely.

Proof. We denote by $\mathbb{D}(\mathcal{L})$ the set of functions in $\bigcap_{1\le p<\infty} L^p(E;\mu)$ which satisfies (4.2) and (4.3). Firstly we aim to show $\mathbb{D}(\mathcal{L}) \subset \mathcal{D}(\mathcal{L})$.

Let $F \in \mathbb{D}(\mathcal{L})$. Then by (4.2), the following identity holds for μ-a.e. $w \in E$:

$$P_tF(w) = F(w) + \int_0^t P_s\Phi^{[F]}(w)ds, \quad t \ge 0.$$

Hence for every $p \ge 1$, we have

$$\begin{aligned} \Big\| \frac{1}{t}\big(P_tF - F\big) - \Phi^{[F]} \Big\|_{L^p(E;\mu)} &= \Big\| \frac{1}{t}\int_0^t (P_s\Phi^{[F]} - \Phi^{[F]})ds \Big\|_{L^p(E;\mu)} \\ &\le \frac{1}{t}\int_0^t \|P_s\Phi^{[F]} - \Phi^{[F]}\|_{L^p(E;\mu)}ds \end{aligned} \tag{4.4}$$

Therefore we have $F \in \mathrm{Dom}(L_p)$ and $\Phi^{[F]} = L_pF$ by recalling the right hand side of (4.4) tends to 0 as $t \to 0$. It leads us that we have shown $\mathbb{D}(\mathcal{L}) \subset \mathcal{D}(\mathcal{L})$ and $\Phi^{[F]} = \mathcal{L}F$.

Next we aim to show $\mathcal{D}(\mathcal{L}) \subset \mathbb{D}(\mathcal{L})$. For $F \in \mathcal{D}(\mathcal{L})$, we set

$$\tilde{M}_t^{[F]} := F(X_t) - F(X_0) - \int_0^t \mathcal{L}F(X_s)ds.$$

Since $\mathcal{L}F \in \bigcap_{1\le p<\infty} L^p(E;\mu)$, we want to show that $\{M_t^{[F]}\}_{t\ge 0}$ is a $\{\mathcal{F}_t\}$-martingale with (4.3). Since $\{X_t\}$ is the mild solution of SPDE (1.1), the Markov property

$$\mathbb{E}^{P_\mu}\big[F(X_{s+t})|\mathcal{F}_s\big] = \mathbb{E}^{P_{X_s}}[F(X_t)] \ \text{ for } P_\mu\text{-almost surely} \tag{4.5}$$

holds. See Section 9 in [3] for the details. Therefore by combining (4.5) and

$$(P_tF) - F - \int_0^t P_r(\mathcal{L}F)dr = 0 \tag{4.6}$$

holds, we can easily obtain that $\{\tilde{M}_t^{[F]}\}$ is a $\{\mathcal{F}_t\}$-martingale under P_μ as follows.

$$\begin{aligned} \mathbb{E}^{P_\mu}\big[\tilde{M}_t^{[F]}|\mathcal{F}_s\big] &= \tilde{M}_s^{[F]} + \mathbb{E}^{P_\mu}\Big[F(X_t) - F(X_s) - \int_s^t \mathcal{L}F(X_r)dr|\mathcal{F}_s\Big] \\ &= \tilde{M}_s^{[F]} + \mathbb{E}^{P_{X_s}}\Big[F(X_{t-s}) - F(X_0) - \int_0^{t-s} \mathcal{L}F(X_r)dr\Big] \\ &= \tilde{M}_s^{[F]} + \Big\{P_{t-s}F(X_s) - F(X_s) - \int_0^t P_r(\mathcal{L}F)(X_r)dr\Big\} \\ &= \tilde{M}_s^{[F]}, \quad 0 \le s \le t, \ P_\mu\text{-almost surely.} \end{aligned} \tag{4.7}$$

On the other hand, we have the following identity for every $p \geq 1$ by recalling the $\{P_t\}$-invariance of μ.

$$\mathbb{E}^{P_\mu}[|F(X_t)|^p] = \int_E \mathbb{E}^{P_w}[|F(X_t)|^p]\mu(dw) = \int_E |F(w)|^p \mu(dw) < \infty.$$

We also have the following estimate for every $p \geq 1$:

$$\begin{aligned}
\mathbb{E}^{P_\mu}\Big[|\int_0^t \mathcal{L}F(X_s)ds|^p\Big] &\leq \int_E \mathbb{E}^{P_w}\Big[(\int_0^t ds)^{p-1}(\int_0^t |L_p F(X_s)|^p ds)\Big]\mu(dw) \\
&= t^{p-1}\int_0^t ds \int_E \mathbb{E}^{P_w}\Big[|\mathcal{L}F(X_s)|^p\Big]\mu(dw) \\
&= t^{p-1}\int_0^t ds \int_E |\mathcal{L}F(w)|^p \mu(dw) \\
&= t^p \int_E |\mathcal{L}F(w)|^p \mu(dw) < \infty.
\end{aligned}$$

Therefore we can conclude $\mathbb{E}^{P_\mu}\Big[|\tilde{M}_t^{[F]}|^p\Big] < \infty$ for every $p \geq 1$. Then we have shown $F \in \mathbb{D}(\mathcal{L})$ and $\mathcal{L}F = \Phi^{[F]}$. This completes the proof. ∎

4.2 Preliminary Facts on the Symmetric Diffusion Process $\mathbb{M}^{(0)}$

In this subsection, we discuss the relationship between the solution of the SPDE (1.4) and a certain Dirichlet form. For $F \in \mathcal{FC}_b^\infty$, we also define the Fréchet derivative $DF : E \longrightarrow H$ by

$$DF(w)(x) := \sum_{k=1}^n \frac{\partial f}{\partial \alpha_k}(\langle w, \phi_1\rangle, \cdots, \langle w, \phi_n\rangle)\phi_k(x), \quad x \in \mathbb{R}. \tag{4.8}$$

Now, we consider a symmetric bilinear form $\mathcal{E}$ which is given by

$$\mathcal{E}(F) = \frac{1}{2}\int_E \|DF(w)\|_H^2 \mu(dw), \quad F \in \mathcal{FC}_b^\infty.$$

We also define $\mathcal{E}_1(F) := \mathcal{E}(F) + \|F\|^2_{L^2(E;\mu)}$ and $\mathcal{D}(\mathcal{E})$ by the completion of $\mathcal{FC}_b^\infty$ with respect to $\mathcal{E}_1^{1/2}$-norm. For $F \in \mathcal{D}(\mathcal{E})$, we also denote by DF the closed extension of (4.8). By virtue of the $C_0^\infty(\mathbb{R}, \mathbb{R}^d)$-quasi-invariance and the strictly positive property of the Gibbs measure μ, Theorem 1 and Proposition 3.6 in Kusuoka [12] derive that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form on $L^2(E;\mu)$, i.e., $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a closed Markovian symmetric bilinear form.

Now we can summarize the relationship between this Dirichlet form and our dynamics as the following proposition. The reader is referred to Theroem 2.1 in [5] or Proposition 2.3 in [9] for the proof.

Proposition 4.2 (1) *There exists a diffusion process* $\tilde{\mathbb{M}}^{(0)} := (\tilde{Y}_t, \tilde{P}_w^{(0)})$ *on* E *associated with the Dirichlet form* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.
(2) *If the initial distribution of* $\tilde{Y}_0$ *is the Gibbs measure* μ, *the distribution on* $C([0,\infty), E)$ *of the process* $\tilde{Y}_t$ *coincides with that of* Y_t.

Here we give a remark. Let $\{\tilde{P}_t^{(0)}\}$ be a $L^2(E;\mu)$-strongly contraction semigroup associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Then the assertion (2) implies that

$$\int_E P_t^{(0)} F(w) G(w) \mu(dw) = \int_E \tilde{P}_t^{(0)} F(w) G(w) \mu(dw)$$

holds for any $F, G \in L^2(E;\mu)$. So by Riesz's representation theorem, we have $P_t^{(0)} F = \tilde{P}_t^{(0)} F$ holds for any $F \in L^2(E;\mu)$. This means that $\{\tilde{P}_t^{(0)}\}$ coincides with $\{P_t^{(0)}\}$ as a $L^p(E;\mu)$, $1 \le p < \infty$ -strongly continuous contraction semigroup. We denote by $L_p^{(0)}$ the infinitesimal generator on $L^p(E;\mu)$.

Before closing this subsection, we introduce a relationship of Sobolev norms. Quite recently, Kawabi-Miyokawa [11] showed the Littlewood-Paley-Stein inequality for the symmetric diffusion semigroup under the gradient estimate condition on the semigroup which is slightly weaker than the lower boundedness condition of Bakry-Emery's Γ_2. Moreover, we have already obtained the gradient estimate

$$\|D(P_t^{(0)} F)(w)\|_H \le e^{\frac{K_1 t}{2}} P_t^{(0)} \big(\|DF\|_H\big)(w), \ t \ge 0, \ \mu\text{-a.e. } w \in E$$

for $F \in \mathcal{D}(\mathcal{E})$. See Proposition 2.4 in Kawabi [9] for the detail. Hence we can apply the result in [11] to our dynamics $\mathbb{M}^{(0)}$. Then we have the following proposition as a by-product of the Littlewood-Paley-Stein inequality. See Theorem 1.2 in [11] for the detail.

Proposition 4.3 *For any* $p \ge 2$, $q > 1$ *and* $\alpha > \frac{K_1}{2}$, *the following inequality holds for* $F \in L^p(E;\mu)$:

$$\Big\| D(\sqrt{\alpha - L_p^{(0)}})^{-q} F \Big\|_{L^p(E,H;\mu)} \lesssim \|F\|_{L^p(E;\mu)}. \tag{4.9}$$

In (4.9), *the notation* $\|F\|_{L^p(E;\mu)} \lesssim \|G\|_{L^p(E;\mu)}$ *stands for* $\|F\|_{L^p(E;\mu)} \le C_p \|G\|_{L^p(E;\mu)}$, *where* C_p *is a positive constant depending only on* p. (4.9) *means the following inclusion holds:*

$$\mathrm{Dom}\big((\sqrt{1 - L_p^{(0)}})^q\big) \subset W^{1,p}(E;\mu) := \big\{F \in L^p(E;\mu) \cap \mathcal{D}(\mathcal{E}) \mid DF \in L^p(E,H;\mu)\big\}.$$

4.3 Fundamental Properties of $\mathcal{D}(\mathcal{L})$

In this subsection, we present the following fundamental properties of the domain $\mathcal{D}(\mathcal{L})$ which will play central roles to establish functional inequalities.

Theorem 4.4 (1) $\mathcal{F}\mathcal{C}_b^\infty \subset \mathcal{D}(\mathcal{L})$.
(2) $P_t\big(\mathcal{D}(\mathcal{L})\big) \subset \mathcal{D}(\mathcal{L})$ *holds for* $t \geq 0$.
(3) $\mathcal{D}(\mathcal{L}) \subset \bigcap_{p\geq 1} W^{1,p}(E;\mu) \subset \mathcal{D}(\mathcal{E})$.
(4) *For any* $F \in \mathcal{D}(\mathcal{L})$,

$$M_t^{[F]} = \int_0^t (DF(X_s), dW_s)_H, \quad t \geq 0. \tag{4.10}$$

(5) *For* $F_1, F_2 \in \mathcal{D}(\mathcal{L})$, $F_1F_2 \in \mathcal{D}(\mathcal{L})$ *and the following equality holds.*

$$\mathcal{L}(F_1F_2) = F_1\mathcal{L}F_2 + F_2\mathcal{L}F_1 + (DF_1, DF_2)_H. \tag{4.11}$$

Proof. (1) Let $F \in \mathcal{F}\mathcal{C}_b^\infty$ be given. Then the Itô formula implies the following equality by recalling (1.3).

$$\begin{aligned} F(X_t) &= F(X_0) + \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial \alpha_i}\big(\langle X_s, \phi_1\rangle, \cdots, \langle X_s, \phi_n\rangle\big) d\langle X_s, \phi_i\rangle \\ &\quad + \frac{1}{2}\sum_{i,j=1}^n \int_0^t \frac{\partial^2 f}{\partial \alpha_i \partial \alpha_j}\big(\langle X_s, \phi_1\rangle, \cdots, \langle X_s, \phi_n\rangle\big) d\Big\langle \langle X_\cdot, \phi_i\rangle, \langle X_\cdot, \phi_j\rangle \Big\rangle_s \\ &= F(X_0) + \int_0^t LF(X_s)ds + \int_0^t (DF(X_s), dW_s)_H, \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} LF(w) &:= \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 f}{\partial \alpha_i \partial \alpha_j}(\langle w, \phi_1\rangle, \cdots, \langle w, \phi_n\rangle)\langle \phi_i, \phi_j\rangle \\ &\quad + \sum_{i=1}^n \frac{\partial f}{\partial \alpha_i}(\langle w, \phi_1\rangle, \cdots, \langle w, \phi_n\rangle)\Big(\frac{1}{2}\langle w, \Delta_x \phi_i\rangle + \langle b(w(\cdot)), \phi_i\rangle\Big). \end{aligned} \tag{4.13}$$

Then by recalling (3.3), $LF \in \bigcap_{1\leq p<\infty} L^p(E;\mu)$ holds. By Burkholder's inequality, we also have $M_t^{[F]} := \int_0^t (DF(X_s), dW_s)_H$ such that $\mathbb{E}^{P_\mu}\big[|M_t^{[F]}|^p\big] < \infty$ for any $p \geq 1$. Hence we easily see $F \in \mathcal{D}(\mathcal{L})$ by setting $\mathcal{L}F := LF$.

(2) For $F \in \mathcal{D}(\mathcal{L})$, we consider $\{M_t^{[P_rF]}\}_{t\geq 0}$ defined by

$$M_t^{[P_rF]} := P_rF(X_t) - P_rF(X_0) - \int_0^t P_r(\mathcal{L}F)(X_s)ds, \quad t \geq 0. \tag{4.14}$$

By the Markov property and (4.6), we easily see the martingale property of $\tilde{M}_t$ as follows:

$$\begin{aligned} &\mathbb{E}^{P_\mu}\big[M_t^{[P_rF]} | \mathcal{F}_s\big] \\ &= M_s^{[P_rF]} + \mathbb{E}^{P_\mu}\Big[P_rF(X_t) - P_rF(X_s) - \int_s^t P_r(\mathcal{L}F)(X_\tau)d\tau | \mathcal{F}_s\Big] \end{aligned}$$

$$= M_s^{[P_rF]} + \mathbb{E}^{P_{X_s}}\Big[P_rF(X_{t-s}) - P_rF(X_0) - \int_0^{t-s} P_r(\mathcal{L}F)(X_\tau)d\tau\Big]$$
$$= M_s^{[P_rF]} + \big\{P_{t-s+r}F(X_s) - P_{0+r}F(X_s) - \int_0^{t-s} P_{\tau+r}(\mathcal{L}F)(X_s)ds\big\}$$
$$= \tilde{M}_s^{[P_rF]},\ \ 0 \le s \le t,\ P_\mu\text{-almost surely.} \tag{4.15}$$

On the other hand, we easily obtain $\mathbb{E}^{P_\mu}\big[|M_t^{[P_rF]}|^p\big] < \infty$ and $P_r(\mathcal{L}F) \in L^p(E;\mu)$ for any $p \ge 1$. Hence we obtain our desired assertion by setting $\mathcal{L}(P_tF) := P_t(\mathcal{L}F),\ t \ge 0$.

(3) We take $1 < q < 2$. We aim to show

$$\Big\|\sqrt{1-L_p^{(0)}}^{\,q}(1-L_p)^{-1}F\Big\|_{L^p(E;\mu)} \le C\|F\|_{L^p(E;\mu)},\ F \in L^p(E;\mu). \tag{4.16}$$

In what follow, constants C depend on p and q but not on F. They may differ from lines to lines.

Here by recalling $T_t := e^{-t}P_t^{(0)}$ is an analytic semigroup on $L^p(E;\mu)$, we have

$$\|(1-L_p^{(0)})T_tF\|_{L^p(E;\mu)} \le Ct^{-1}e^{-t}\|F\|_{L^p(E;\mu)},\ F \in L^p(E;\mu).$$

Then we have

$$\begin{aligned}
\Big\|\sqrt{1-L_p^{(0)}}^{\,q}T_tF\Big\|_{L^p(E;\mu)} &= \Big\|\sqrt{1-L_p^{(0)}}^{\,(q-2)}\Big\{(1-L_p^{(0)})T_tF\Big\}\Big\|_{L^p(E;\mu)} \\
&\le \frac{1}{\Gamma(1-\frac{q}{2})}\int_0^\infty s^{-q/2}\big\|(1-L_p^{(0)})T_{s+t}F\big\|_{L^p(E;\mu)}ds \\
&\le \frac{C}{\Gamma(1-\frac{q}{2})}\int_0^\infty s^{-q/2}\big\{(s+t)^{-1}e^{-(s+t)}\|F\|_{L^p(E;\mu)}\big\}ds \\
&\le \frac{C\|F\|_{L^p(E;\mu)}}{\Gamma(1-\frac{q}{2})}t^{-q/2}e^{-t}\int_0^\infty \tau^{-q/2}(1+\tau)^{-1}d\tau \\
&= Ct^{-q/2}e^{-t}\|F\|_{L^p(E;\mu)}.
\end{aligned}$$

Hence we can show (4.16) as follows:

$$\begin{aligned}
\Big\|\sqrt{1-L_p^{(0)}}^{\,q}(1-L_p)^{-1}F\Big\|_{L^p(E;\mu)} &= \Big\|\sqrt{1-L_p^{(0)}}^{\,q}\big(\int_0^\infty e^{-t}P_tF\,dt\big)\Big\|_{L^p(E;\mu)} \\
&= \Big\|\int_0^\infty e^{-t}\sqrt{1-L_p^{(0)}}^{\,q}P_t^{(0)}(Q_tF)dt\Big\|_{L^p(E;\mu)} \\
&\le \int_0^\infty \Big\|\sqrt{1-L_p^{(0)}}^{\,q}T_t(Q_tF)\Big\|_{L^p(E;\mu)}dt \\
&\le \int_0^\infty (Ct^{-q/2}e^{-t})\|Q_tF\|_{L^p(E;\mu)}dt \\
&\le C\big(\int_0^\infty t^{-q/2}e^{-t}dt\big)\|F\|_{L^p(E;\mu)} = C\|F\|_{L^p(E;\mu)}.
\end{aligned}$$

Here we note that (4.16) means $\mathrm{Dom}(L_p) \subset \mathrm{Dom}\big(\sqrt{1-L_p^{(0)}}^{\,q}\big)$. On the other hand, we have seen that $\mathrm{Dom}(\sqrt{1-L_p^{(0)}}^{\,q}) \subset W^{1,p}(E;\mu)$ in Proposition 4.3 and $\mathcal{D}(\mathcal{L}) \subset \mathrm{Dom}(L_p)$ in Proposition 4.1. So we have our assertion.

(4) Let $F \in \mathcal{D}(\mathcal{L})$ and $\{\phi_i\}_{i=1}^{\infty} \subset C_0^{\infty}(\mathbb{R}, \mathbb{R}^d)$ be a C.O.N.S. of H. Since $\{\mathcal{F}_t\}_{t\geq 0}$ is the Brownian filtration, there exists a H-valued progressively measurable process $\{f_t\}_{t\geq 0}$ such that $\mathbb{E}\big[\int_0^T \|f_s(\omega)\|_H^2 ds\big] < \infty$ for all $T > 0$ and the martingale $\{M_t^{[F]}\}$ is represented by

$$M_t^{[F]} = \int_0^t \big(f_s(\omega), dW_s\big)_H = \sum_{i=1}^{\infty} \int_0^t f_s^{(i)}(\omega) d\beta_i(s), \quad t \geq 0, \tag{4.17}$$

where $f_s^{(i)}, i \in \mathbb{N}$ is defined by $\big(f_s, \phi_i\big)_H$.

For $i \in \mathbb{N}$, we consider the function $G_i := \langle \cdot, \phi_i \rangle$. By (1.3), we easily see

$$\beta_i(t) = G_i(X_t) - G_i(X_0) - \int_0^t \big(\frac{1}{2}\langle X_s, \Delta_x \phi_i \rangle - \big\langle b(X_s(\cdot)), \phi_i \big\rangle\big) ds, \quad t \geq 0$$

for P-almost surely. Hence by setting $\mathcal{L}G_i(w) := \frac{1}{2}\langle w, \Delta_x \phi_i \rangle - \big\langle b(w(\cdot)), \phi_i \big\rangle$ and $M_t^{[G_i]} = \beta_i(t)$, we obtain $G_i \in \mathcal{D}(\mathcal{L})$.

Then the quadratic variation $\langle M^{[F]}, M^{[G_i]} \rangle_t$ is given by

$$\langle M^{[F]}, M^{[G_i]} \rangle_t(\cdot) = \int_0^t f_s^{(i)}(\cdot) ds, \quad t > 0. \tag{4.18}$$

Here we regard both sides of (4.18) as $L^2(\Theta; P)$-valued continuous stochastic processes. Then Lebesgue's theorem implies that for a.e. $t > 0$,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big(\langle M^{[F]}, M^{[G_i]} \rangle_{t+\varepsilon} - \langle M^{[F]}, M^{[G_i]} \rangle_t\Big)(\cdot) = f_t^{(i)}(\cdot) \tag{4.19}$$

holds in $L^2(\Theta; P)$.

On the other hand, we remember that

$$\begin{aligned} \langle M^{[F]}, M^{[G_i]} \rangle_t \;\; = \;\; & P\text{-}\lim_{|\Delta| \to 0} \sum_{j=1}^{\infty} \Big(F(X_{t_{j+1}\wedge t}) - F(X_{t_j \wedge t}) - \int_{t_j \wedge t}^{t_{j+1}\wedge t} \mathcal{L}F(X_s) ds\Big) \\ & \times \Big(G_i(X_{t_{j+1}\wedge t}) - G_i(X_{t_j \wedge t}) - \int_{t_j \wedge t}^{t_{j+1}\wedge t} \mathcal{L}G_i(X_s) ds\Big), \end{aligned}$$

where $\Delta : t_0 = 0 < t_1 < t_2 < \cdots < t_j < \cdots \to \infty$, and $|\Delta| := \max_{j \in \mathbb{N}}(t_j - t_{j-1})$. Hence we have the following.

$$\langle M^{[F]}, M^{[G_i]} \rangle_{t+\varepsilon} - \langle M^{[F]}, M^{[G_i]} \rangle_t \in \mathcal{G}_t^{t+\varepsilon} := \sigma(X_u; t \leq u \leq t+\varepsilon), \quad t, \varepsilon \geq 0.$$

Then for $t > 0$, we have

$$\Big\{\omega \in \Omega \,\Big|\, \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big(\langle M^{[F]}, M^{[G_i]}\rangle_{t+\varepsilon}(\omega) - \langle M^{[F]}, M^{[G_i]}\rangle_{t}(\omega)\Big) \text{ exists}\Big\}$$
$$\in \Big(\bigcap_{\varepsilon>0} \mathcal{G}_t^{t+\varepsilon}\Big) \cap \Big(\bigcap_{\varepsilon>0} \mathcal{G}_{t-\varepsilon}^{t}\Big) = \sigma(X_t). \tag{4.20}$$

Here by recalling (4.19) and (4.20), we obtain

$$P\Big(f_t^{(i)}(\omega) \text{ is } \sigma(X_t)\text{-measurable}\Big) = 1 \quad \text{for a.e. } t > 0.$$

Hence for a.e. $t > 0$, there exists a Borel measurable function $\Psi_t^{(i)} : E \to \mathbb{R}$ such that

$$P\Big(f_t^{(i)}(\omega) = \Psi_t^{(i)}(X_t(\omega))\Big) = 1.$$

Then Fubini's theorem implies that

$$f_t^{(i)}(\omega) = \Psi_t^{(i)}(X_t) \quad \text{for } (m \otimes P)\text{-a.e. } (t, \omega) \in [0, \infty) \times \Theta, \tag{4.21}$$

where m is one-dimensional Lebesgue measure. We also have

$$\langle M^{[F]}, M^{[G_i]}\rangle_t = \int_0^t f_s^{(i)}(\omega) ds = \int_0^t \Psi_s^{(i)}(X_s(\omega)) ds, \quad t > 0$$

holds for P-almost surely ω.

By using Itô's formula, we have

$$\begin{aligned}
F(X_t)G_i(X_t) &= F(X_0)G_i(X_0) + \int_0^t \big(F(X_s) dM_s^{[G_i]} + G_i(X_s) dM_s^{[F]}\big) \\
&\quad + \langle M^{[F]}, M^{[G_i]}\rangle_t + \int_0^t \big(F(X_s)\mathcal{L}G_i(X_s) + G_i(X_s)\mathcal{L}F(X_s)\big) ds \\
&= F(X_0)G_i(X_0) + \int_0^t F(X_s) d\beta_i(s) + \int_0^t G(X_s)\big(f_s, dW_s\big)_H \\
&\quad + \int_0^t \big(F(X_s)\mathcal{L}G_i(X_s) + G_i(X_s)\mathcal{L}F(X_s) + \Psi_s^{(i)}(X_s)\big) ds.
\end{aligned} \tag{4.22}$$

By taking the expectation on both sides in (4.22) and remembering that μ is $\{P_t\}$-invariant, we have

$$-\int_E (F(w)\mathcal{L}G_i(w) + G_i(w)\mathcal{L}F(w))\mu(dw) = \int_E \Big(\frac{1}{t}\int_0^t \Psi_s^{(i)}(w) ds\Big)\mu(dw), \quad t > 0. \tag{4.23}$$

By remarking the left hand side of (4.23) does not depend on t, there exists a Borel measurable function $\Psi^{(i)} : E \to \mathbb{R}$ such that

$$\Psi^{(i)} = \frac{1}{t}\int_0^t \Psi_s^{(i)} ds$$

holds for $t > 0$. Then by taking the differential both sides in t, we have

$$\Psi^{(i)} = \Psi_t^{(i)} \quad \text{for a.e. } t > 0. \tag{4.24}$$

Moreover Burkholder's inequality leads us $\Psi^{(i)} \in L^1(E;\mu)$.

By returning to (4.22), we can see $FG_i \in \mathrm{Dom}(L_1)$ and

$$L_1(FG_i)(w) = F(w)\mathcal{L}G_i(w) + \mathcal{L}F(w)G_i(w) + \Psi^{(i)}(w).$$

So we can define the bilinear form $\Gamma : \mathcal{D}(\mathcal{L}) \times \mathcal{D}(\mathcal{L}) \longrightarrow L^1(E;\mu)$ by

$$\Gamma(F, G_i) := \frac{1}{2}\big\{L_1(FG_i) - F\mathcal{L}G_i - G_i\mathcal{L}F\big\}.$$

For $F \in \mathcal{D}(\mathcal{L}) \subset \mathcal{D}(\mathcal{E})$, we take a sequence $\{F_n\}_{n=1}^{\infty} \subset \mathcal{FC}_b^{\infty}$ such that $F_n \to F$ in $\mathcal{D}(\mathcal{E})$ as $n \to \infty$. Then we easily have the following convergence in $L^1(E;\mu)$:

$$2\Gamma(F_n, G_i) = (DF_n, \phi_i)_H \to (DF, \phi_i)_H \quad \text{strongly as } n \to \infty. \tag{4.25}$$

Next we want to show the following convergence in $L^1(E;\mu)$:

$$\Gamma(F_n, G_i) \to \Gamma(F, G_i) \quad \text{weakly as } n \to \infty. \tag{4.26}$$

Since $F_n \to F$ strongly in $L^2(E;\mu)$, we have the following for every $G \in \mathcal{FC}_b^{\infty}$ by using the integration by parts for the Gibbs measure μ:

$$\begin{aligned}
\mathbb{E}^{\mu}\big[\Gamma(F_n, G_i)G\big] &= \frac{1}{2}\mathbb{E}^{\mu}\Big[\{L_1(F_nG_i) - F_n\mathcal{L}G_i - G_iLF_n\}G\Big] \\
&= \frac{1}{2}\mathbb{E}^{\mu}\Big[(F_nG_i)L^*G - F_n(\mathcal{L}G_i)G - F_nL^*(G_iG)\Big] \\
&\to \frac{1}{2}\mathbb{E}^{\mu}\Big[(FG_i)L^*G - F(\mathcal{L}G_i)G - FL^*(G_iG)\Big] \quad \text{as } n \to \infty,
\end{aligned} \tag{4.27}$$

where LF_n is defined as (4.13) and L^*G and $L^*(G_iG)$ are denoted by

$$\begin{aligned}
L^*G(w) &:= \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2 g}{\partial\alpha_j\partial\alpha_k}(\langle w, \phi_1\rangle, \cdots, \langle w, \phi_n\rangle)\langle\phi_j, \phi_k\rangle \\
&\quad + \frac{1}{2}\sum_{j=1}^{n} \frac{\partial g}{\partial\alpha_j}(\langle w, \phi_1\rangle, \cdots, \langle w, \phi_n\rangle) \\
&\qquad \times \big\{\langle w, \Delta_x\phi_j\rangle - \langle\nabla U(w(\cdot)) + 2Bw(\cdot), \phi_j\rangle\big\}, \\
L^*(G_iG) &:= G_i \cdot L^*G + \sum_{j=1}^{n} \frac{\partial g}{\partial\alpha_j}(\langle w, \phi_1\rangle, \cdots, \langle w, \phi_n\rangle)\langle\phi_i, \phi_j\rangle \\
&\quad + \frac{1}{2}G(w) \cdot \big\{\langle w, \Delta_x\phi_i\rangle - \langle\nabla U(w(\cdot)) + 2Bw(\cdot), \phi_i\rangle\big\}.
\end{aligned}$$

On the other hand, we also have

$$\mathbb{E}^\mu\big[\Gamma(F,G_i)G\big] = \frac{1}{2}\mathbb{E}^\mu\Big[(FG_i)L^*G - F(\mathcal{L}G_i)G - FL^*(G_iG)\Big]. \tag{4.28}$$

Hence by combining (4.27) and (4.28), we complete the proof of (4.26). Therefore we have

$$\Psi^{(i)}(w) = 2\Gamma(F,G)(w) = (DF(w),\phi_i)_H \quad \text{for} \quad \mu\text{-a.e. } w \in E.$$

Finally, by combining (4.17), (4.21) and (4.24), we have the desired assertion.

(5) By using Itô's formula and (4.10), we have the following expansion for $F_1, F_2 \in \mathcal{D}(\mathcal{L})$.

$$\begin{aligned} & F_1(X_t)F_2(X_t) \\ = \ & F_1(X_0)F_2(X_0) + \int_0^t \big\{F_1(X_s)\big(DF_2(X_s), dW_s\big)_H + F_2(X_s)\big(DF_1(X_s), dW_s\big)_H\big\} \\ & + \int_0^t \big\{F_1(X_s)\mathcal{L}F_2(X_s) + F_2(X_s)\mathcal{L}F_1(X_s) + \big(DF_1(X_s), DF_2(X_s)\big)_H\big\}ds. \end{aligned}$$

Hence we easily see $F_1F_2 \in \mathcal{D}(\mathcal{L})$ and (4.11) by recalling $F_1, F_2 \in \bigcap_{p\ge 1} W^{1,p}(E;\mu)$. ∎

Remark 4.5 *In infinite dimensional settings, Stannat* [19] *studied the relationship between the generator of a non-symmetric semigroup* $\{P_t\}$ *and a certain symmetric Dirichlet forms* $(\mathcal{E},\mathcal{D}(\mathcal{E}))$*. Moreover Trutnau* [20] *establised the Fukushima decomposition of additive functionals in the framework of generalized Dirichlet forms. In these studies, the generator is of type*

$$LF(w) = L^{(0)}F(w) + \big(\mathbb{B}(w), DF(w)\big)_H,$$

where $L^{(0)}$ *is associated with* $(\mathcal{E},\mathcal{D}(\mathcal{E}))$*. On the other hand, in this paper, we can not regard* $\mathbb{B}(w)$ *as an element of* H *since we treat rotation. Hence we emphasize that Theorem* 4.4 *is not included in* [19] *and* [20]*.*

5 Gradient Estimate for $\{P_t\}$

In this section, we establish a gradient estimate for the transition semigroup $\{P_t\}$ which plays a key role in the proof of Theorem 6.1. We note that this type estimate is studied in Proposition 2.3 in Bakry [2] by using Γ_2-method. Here we note that the existence of a suitable core $\mathcal{A} \subset L^2(E;\mu)$ which has the stability under the operation $\{P_t\}$ is assumed in [2]. In finite dimensional cases, we can easily check this assumption. But in infinite dimensional situations, it is not trivial to find such a core. Needless to say, $\mathcal{FC}_b^\infty$ does not satisfy above property.

In this paper, we adopt another approach to prove this estimate. Here we represent P_tF as the expectation of the functional associated with our dynamics. In this approach, a stochastic flow estimate (2.2) is the key tool when we take the differential in the expectation.

We state the gradient estimate as follows.

Theorem 5.1 (Gradient Estimate for $\{P_t\}$) *For $F \in \mathcal{D}(\mathcal{E})$, the following gradient estimate holds for any $t \in [0,\infty)$ and μ-a.e. $w \in E$.*

$$\|D(P_tF)(w)\|_H \le e^{\frac{K_1 t}{2}} P_t\big(\|DF\|_H\big)(w). \tag{5.1}$$

Proof. We first assume that $F \in \mathcal{F}C_b^\infty$, i.e., $F(w) = f(\langle w,\phi_1\rangle,\cdots,\langle w,\phi_n\rangle)$. Here $\{\phi_i\}_{i=1}^\infty \subset C_0^\infty(\mathbb{R},\mathbb{R}^d)$ denotes a C.O.N.S. of H for simplicity. Here we have to notice that $P_tF \in \mathcal{D}(\mathcal{L}) \subset \mathcal{D}(\mathcal{E})$ by recalling Theorem 4.4.

For $w \in E, h \in H$, we take approximate sequences $\{w_n\}_{n=1}^\infty \subset \mathcal{C}, \{h_n\}_{n=1}^\infty \subset H \cap \mathcal{C}$ such that $\lim_{n\to\infty} w_n = w$ in E and $\lim_{n\to\infty} h_n = h$ in H.

Then by Lemma 2.1, we have

$$\begin{aligned}
|(P_tF)(w+h)-(P_tF)(w)| &\le \liminf_{n\to\infty} \mathbb{E}[|F(X_t^{w_n+h_n})-F(X_t^{w_n})|] \\
&\le \|\nabla f\|_{L^\infty(\mathbb{R}^n)} \cdot \liminf_{n\to\infty} \mathbb{E}[\|X_t^{w_n+h_n}-X_t^{w_n}\|_H] \\
&\le e^{\frac{K_1 t}{2}} \|\nabla f\|_{L^\infty(\mathbb{R}^n)} \cdot \lim_{n\to\infty} \|h_n\|_H \\
&= e^{\frac{K_1 t}{2}} \|\nabla f\|_{L^\infty(\mathbb{R}^n)} \cdot \|h\|_H.
\end{aligned} \tag{5.2}$$

Then by Lemma 1.3 in [12], there exists $\Omega_0 \in \mathcal{B}(E)$ such that $\Omega_0 \subset \mathcal{C}$, $\mu(\Omega_0)=1$ and the following identity holds:

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\big\{(P_tF)(w+\varepsilon h)-(P_tF)(w)\big\} = \big(D(P_tF)(w),h\big)_H \quad \text{for any } w\in\Omega_0,\ h\in H.$$

For $w \in \Omega_0$, $\varepsilon > 0$ and $h \in H \cap \mathcal{C}$, we define $Z_t^{w,\varepsilon,h} := \dfrac{1}{\varepsilon}(X_t^{w+\varepsilon h}-X_t^w)$. By Lemma 2.1, we can easily see that $\|Z_t^{w,\varepsilon,h}\|_H \le e^{\frac{K_1 t}{2}}\|h\|_H$ holds for P-almost surely. Then for any $t>0$, $w\in\Omega_0$ and $h \in H\cap\mathcal{C}$, we have

$$\begin{aligned}
&\frac{1}{\varepsilon}\big\{(P_tF)(w+\varepsilon h)-(P_tF)(w)\big\} \\
&\quad= \mathbb{E}\Big[\frac{1}{\varepsilon}\big(F(X_t^{w+\varepsilon h})-F(X_t^w)\big)\Big] \\
&\quad= \mathbb{E}\left[\sum_{i=1}^n \int_0^1 \frac{\partial f}{\partial\alpha_i}\Big((1-\theta)\langle X_t^w,\phi\rangle+\theta\langle X_t^{w+\varepsilon h},\phi\rangle\Big)\cdot\langle Z_t^{w,\varepsilon,h},\phi_i\rangle d\theta\right] \\
&\quad\le \mathbb{E}\left[\Big(\sum_{i=1}^n\int_0^1 \big|\frac{\partial f}{\partial\alpha_i}\big((1-\theta)\langle X_t^w,\phi\rangle+\theta\langle X_t^{w+\varepsilon h},\phi\rangle\big)\big|^2 d\theta\Big)^{1/2}\cdot\|Z_t^{w,\varepsilon,h}\|_H\right] \\
&\quad\le e^{\frac{K_1 t}{2}}\|h\|_H\cdot\mathbb{E}\left[\Big(\sum_{i=1}^n\int_0^1 \big|\frac{\partial f}{\partial\alpha_i}\big((1-\theta)\langle X_t^w,\phi\rangle+\theta\langle X_t^{w+\varepsilon h},\phi\rangle\big)\big|^2 d\theta\Big)^{1/2}\right].
\end{aligned} \tag{5.3}$$

Then by combining (5.3) and Lebesgue's dominated convergence theorem, we have the following estimate for any $w \in \Omega_0$:

$$\begin{aligned}
&\big(D(P_tF)(w), h\big)_H \\
&\quad \le e^{\frac{K_1 t}{2}} \|h\|_H \cdot \mathbb{E}\Big[\lim_{\varepsilon\to 0}\Big(\sum_{i=1}^n \int_0^1 \big|\frac{\partial f}{\partial \alpha_i}\big((1-\theta)\langle X_t^w, \phi\rangle + \theta\langle X_t^{w+\varepsilon h}, \phi\rangle\big)\big|^2 d\theta\Big)^{1/2}\Big] \\
&\quad \le e^{\frac{K_1 t}{2}} \|h\|_H \cdot \mathbb{E}\Big[\Big(\sum_{i=1}^n \big|\frac{\partial f}{\partial \alpha_i}\big(\langle X_t^w, \phi\rangle\big)\big|^2\Big)^{1/2}\Big] \\
&\quad = e^{\frac{K_1 t}{2}} \|h\|_H \cdot P_t\big(\|DF\|_H\big)(w).
\end{aligned}$$

Therefore we have the following for any $w \in \Omega_0$:

$$\begin{aligned}
\|D(P_tF)(w)\|_H &= \sup\Big\{\big(D(P_tF)(w), h\big)_H \Big| \; h \in \mathcal{C} \cap H, \|h\|_H = 1\Big\} \\
&\le e^{\frac{K_1 t}{2}} P_t(\|DF\|_H)(w).
\end{aligned}$$

Next we consider in the case of $F \in \mathcal{D}(\mathcal{E})$. For $F \in \mathcal{D}(\mathcal{E})$, we can take a sequence $\{F_j\}_{j=1}^\infty \subset \mathcal{FC}_b^\infty$ such that $F_j \to F$ in $\mathcal{D}(\mathcal{E})$ as $j \to \infty$. Since $\{P_t\}$ is a strongly continuous contraction semigroup in $L^2(E;\mu)$, we easily have $P_t\{\|DF_j\|_H\} \to P_t\{\|DF\|_H\}$ in $L^2(E;\mu)$ as $j \to \infty$. Hence we have the convergence of the right hand side of (5.1).

On the other hand, we obtain the following estimate by using (5.1):

$$\sup_{j\in\mathbb{N}} \mathcal{E}(P_tF_j) \le \frac{e^{K_1 t}}{2} \sup_{j\in\mathbb{N}} \Big\{\int_E P_t(\|DF_j\|_H)(w)^2 \mu(dw)\Big\} < \infty.$$

Hence by recalling Lemma 2.12 in Ma-Röckner [14], there exists a subsequence $\{P_tF_{j_k}\}_{k=1}^\infty$ of $\{P_tF_j\}_{j=1}^\infty$ such that its Cesaro mean $f_j := \frac{1}{j}\sum_{k=1}^j P_tF_{j_k} \to P_tF$ in $\mathcal{D}(\mathcal{E})$ as $j \to \infty$. Therefore we also have the convergence of the left hand side of (5.1). This completes the proof. ∎

6 Parabolic Harnack Inequality for $\{P_t\}$

In this section, we present a parabolic Harnack inequality for the transition semigroup $\{P_t\}$. This is an infinite dimensional version of the celebrated Li-Yau's parabolic Harnack inequality. Our inequality is as follows.

Theorem 6.1 (Parabolic Harnack Inequality) *Let $F \in \mathcal{FC}_b^\infty$. Then for any $h \in H$, $\alpha > 1$ and $t > 0$, the following dimension free parabolic Harnack inequality holds for all $w \in E$.*

$$\big|P_tF(w)\big|^\alpha \le P_t\big|F\big|^\alpha(w+h) \cdot \exp\left(\frac{\alpha\|h\|_H^2}{2(\alpha-1)} \cdot \frac{K_1}{1-e^{-K_1 t}}\right). \tag{6.1}$$

Here K_1 is the constant denoted in the condition (U2) *and we set* $\dfrac{K_1}{1-e^{-K_1 t}} := \dfrac{1}{t}$ *if* $K_1 = 0$. *In the case of* $F \in L^\infty(E;\mu)$ *and* $h \in C_0^\infty(\mathbb{R}, \mathbb{R}^d)$, *this inequality also holds for* μ*-a.e.* $w \in E$.

Originally, Wang [21] established this type inequality for the transition semigroup of symmetric diffusion processes on finite dimensional non-compact Riemannian manifolds to give a lower bound of the transition probability. On the other hand, Kusuoka [13] independently proved this inequality for the Ornstein-Uhlenbeck semigroup on an abstract Wiener space. After their works, Aida-Kawabi [1] proved this inequality for a certain symmetric diffusion process on an abstract Wiener space by using Bakry-Emery's Γ_2-method. Recently, Röckner-Wang [17] also proved this inequality for generalized Mehler semigroups.

Contrary to their approaches, we employ a stochastic approach based on the formulation of Section 4 and Kawabi [9]. Especially, we use Itô's formula for semi-martingales when we need to expand the term $(P_t F)^\alpha$. So it is different from the original functional analytic proof as [21], [1] and [17].

To prove Theorem 6.1, we need to prepare a new probability measure which is important to show the differentiability property of functions in $\mathcal{D}(\mathcal{E})$. We fix $h \in C_0^\infty(\mathbb{R}, \mathbb{R}^d)$ and $t > 0$ in this section. We assume supp $h \subset (-T, T)$. We define a cut-off function $\phi \in C_0^\infty(\mathbb{R}, \mathbb{R})$ by $\phi(x) \equiv 1$ for $|x| \le T$ and $\phi(x) \equiv 0$ for $|x| \ge T+1$. For $w \in \mathcal{C}$, we define

$$V(w(x)) := K_3\big(1 + |w(x)|^{p+1}\big)\phi(x), \quad x \in \mathbb{R}. \tag{6.2}$$

Here $K_3 := K_3(p, K_2)$ is a sufficient large constant which will be determined in the proof of Lemma 6.2. p and K_2 are positive constant in the condition (U3). By using this function, we define a weighted Gibbs measure μ^V by

$$\mu^V(dw) := Z_V^{-1} \exp\big(- \int_{\mathbb{R}} V(w(x))dx\big)\mu(dw),$$

where Z_V is the normalization constant. Clearly, this measure is equivalent to the original Gibbs measure μ.

Then we can state the following by recalling the $C_0^\infty(\mathbb{R}, \mathbb{R}^d)$-quasi-invariance of the Gibbs measure μ. See Lemma 3.1 in [9] for the proof.

Lemma 6.2 (1) *For* $F \in B_b(E, \mathbb{R})$ *and* $k \in C_0^\infty(\mathbb{R}, \mathbb{R}^d)$, *the following quasi-invariance of* μ^V *holds:*

$$\begin{aligned}&\int_E F(w+k)\mu^V(dw) \\ &\quad = \int_E F(w)\exp\Big(\Phi(k,w) + \int_{\mathbb{R}} \big(V(w(x)) - V(w(x)-k(x))\big)dx\Big)\mu^V(dw). \end{aligned} \tag{6.3}$$

(2) *Let* $F \in L^2(E;\mu)$ *and* $v(\cdot) \in C([0,t],\mathbb{R})$. *Then there exists a positive constant* $K_4 := K_4(\|h\|_{L^\infty}, K_2, K_3, p, T, \|v\|_{L^\infty})$ *such that*

$$\int_E |F(w+v(s)h)|\mu^V(dw) \leq Z_V^{-1} e^{K_4}\Big(\int_E |F(w)|^2\mu(dw)\Big)^{1/2} \tag{6.4}$$

for any $0 \leq s \leq t$.
(3) *Let* $F \in \mathcal{D}(\mathcal{E})$ *and* $v(\cdot) \in C^1([0,t],\mathbb{R})$ *such that* $v(0) = 0$ *and* $v(t) = 1$. *Then* $F(\cdot + v(s)h) : s \in [0,t] \to L^1(E;\mu^V)$ *is a* C^1*-function. Moreover the following identity holds for* $0 < s < t$*:*

$$\frac{d}{ds}F(\cdot + v(s)h) = \big(DF(\cdot + v(s)h), v'(s)h\big)_H. \tag{6.5}$$

From now, we devote ourselves to give a proof of Theorem 6.1.
Proof of Theorem 6.1. We may assume that $F \in \mathcal{FC}_b^\infty, F(w) > \delta > 0$ since $|P_tF(w)| \leq P_t|F|(w)$ holds generally. For fixed $t > 0$, we define $v(\cdot) \in C^\infty([0,t],\mathbb{R})$ by

$$v(s) := \frac{\int_0^s e^{-K_1 r}dr}{\int_0^t e^{-K_1 r}dr}.$$

For $\alpha > 1$ and $h \in C_0^\infty(\mathbb{R},\mathbb{R}^d)$, we will consider a function $G : [0,t] \to \mathcal{D}(\mathcal{E}) \subset L^1(E;\mu^V)$ by $G(s) := P_s(P_{t-s}F)^\alpha(\cdot + v(s)h)$.

First we study the differentiability of G with respect to s. This is the most important property in this proof. We claim the following lemma:

Lemma 6.3 *The following identity holds in* $L^1(E;\mu^V)$*:*

$$\begin{aligned} G'(s) &= \frac{\alpha(\alpha-1)}{2}P_s\big\{(P_{t-s}F)^{\alpha-2}\|D(P_{t-s}F)\|_H^2\big\}(\cdot + v(s)h) \\ &\quad + \big(D\{P_s(P_{t-s}F)^\alpha\}(\cdot + v(s)h), v'(s)h\big)_H, \quad 0 < s < t. \end{aligned} \tag{6.6}$$

Proof. We consider a function $H(r_1,r_2,r_3) : (0,t)\times(0,t)\times(0,t) \to L^1(E;\mu^V)$ which is defined by $H(r_1,r_2,r_3) := P_{r_1}(P_{t-r_2}F)^\alpha(\cdot + v(r_3)h)$.

To show that $H(r_1,r_2,r_3)$ is a C^1-function, we expand this function. By virtue of the assertions (1) and (2) in Theorem 4.4, we have $P_{t-r_2}F \in \mathcal{D}(\mathcal{L})$ for $F \in \mathcal{FC}_b^\infty$. Hence there exists a continuous $\{\mathcal{F}_{r_1}\}$-martingale $\{M_{r_1}^{[P_{t-r_2}F]}\}_{0\leq r_1 \leq t}$ defined by

$$M_{r_1}^{[P_{t-r_2}F]} = (P_{t-r_2}F)(X_{r_1}) - (P_{t-r_2}F)(X_0) - \int_0^{r_1}\mathcal{L}(P_{t-r_2}F)(X_\tau)d\tau. \tag{6.7}$$

Here the assertion (4) in Theorem 4.4 also leads us that the quadratic variation of $M^{[P_{t-r_2}F]}$ is given by

$$M_{r_1}^{[P_{t-r_2}F]} = \int_0^{r_1}\big(D(P_{t-r_2}F)(X_\tau), dW_\tau\big)_H. \tag{6.8}$$

Now we apply Itô's formula for (6.7). Then we can expand $(P_{t-r_2}F)^\alpha$ as

$$\begin{aligned}
(P_{t-r_2}F)^\alpha(X_{r_1}) &= (P_{t-r_2}F)^\alpha(X_0) + \alpha\int_0^{r_1}(P_{t-r_2}F)^{\alpha-1}(X_\tau)dM_\tau^{[P_{t-r_2}F]} \\
&\quad + \alpha\int_0^{r_1}(P_{t-r_2}F)^{\alpha-1}(X_\tau)\mathcal{L}(P_{t-r_2}F)(X_\tau)d\tau \\
&\quad + \frac{\alpha(\alpha-1)}{2}\int_0^{r_1}(P_{t-r_2}F)^{\alpha-2}(X_\tau)d\langle M^{[P_{t-r_2}F]}\rangle_\tau \\
&= (P_{t-r_2}F)^\alpha(X_0) + \alpha\int_0^{r_1}(P_{t-r_2}F)^{\alpha-1}(X_\tau)\big(D(P_{t-r_2}F)(X_\tau), dW_\tau\big)_H \\
&\quad + \int_0^{r_1}\big\{\alpha(P_{t-r_2}F)^{\alpha-1}(X_\tau)\mathcal{L}(P_{t-r_2}F)(X_\tau) \\
&\qquad + \frac{\alpha(\alpha-1)}{2}(P_{t-r_2}F)^{\alpha-2}(X_\tau)\|D(P_{t-r_2}F)(X_\tau)\|_H^2\big\}d\tau. \qquad (6.9)
\end{aligned}$$

Hence we easily see $(P_{t-r_2}F)^\alpha \in \mathcal{D}(\mathcal{L})$ and

$$\mathcal{L}(P_{t-r_2}F)^\alpha = \alpha(P_{t-r_2}F)^{\alpha-1}\mathcal{L}(P_{t-r_2}F) + \frac{\alpha(\alpha-1)}{2}(P_{t-r_2}F)^{\alpha-2}\|D(P_{t-r_2}F)\|_H^2. \qquad (6.10)$$

Moreover by combining (6.9) and (6.10), we obtain the following expansion for any $r_1, r_2, r_3 \in [0, t]$:

$$\begin{aligned}
&P_{r_1}(P_{t-r_2}F)^\alpha(\cdot + v(r_3)h) \\
&\quad = \mathbb{E}\Big[(P_{t-r_2}F)^\alpha(X_0^{\cdot+v(r_3)h})\Big] + \mathbb{E}\Big[\int_0^{r_1}\mathcal{L}(P_{t-r_2}F)^\alpha(X_\tau^{\cdot+v(r_3)h})d\tau\Big] \\
&\qquad + \alpha\mathbb{E}\Big[\int_0^{r_1}(P_{t-r_2}F)^{\alpha-1}(X_\tau^{\cdot+v(r_3)h})\cdot\big(D(P_{t-r_2}F)(X_\tau^{\cdot+v(r_3)h}), dW_\tau\big)_H\Big] \\
&\quad = (P_{t-r_2}F)^\alpha(\cdot + v(r_3)h) \\
&\qquad + \alpha\int_0^{r_1}P_\tau\big\{(P_{t-r_2}F)^{\alpha-1}P_{t-r_2}(\mathcal{L}F)\big\}(\cdot + v(r_3)h)d\tau \\
&\qquad + \frac{\alpha(\alpha-1)}{2}\int_0^{r_1}P_\tau\big\{(P_{t-r_2}F)^{\alpha-2}\big\|D(P_{t-r_2}F)\big\|_H^2\big\}(\cdot + v(r_3)h)d\tau. \qquad (6.11)
\end{aligned}$$

Hence for any $r_1, r_2, r_3 \in (0, t)$, we have

$$\begin{aligned}
\frac{\partial H}{\partial r_1}(r_1, r_2, r_3) &= \alpha P_{r_1}\big\{(P_{t-r_2}F)^{\alpha-1}P_{t-r_2}(\mathcal{L}F)\big\}(\cdot + v(r_3)h) \\
&\quad + \frac{\alpha(\alpha-1)}{2}P_{r_1}\big\{(P_{t-r_2}F)^{\alpha-2}\|D(P_{t-r_2}F)\|_H^2\big\}(\cdot + v(r_3)h) \\
&=: H_1(r_1, r_2, r_3) + H_2(r_1, r_2, r_3). \qquad (6.12)
\end{aligned}$$

Before discussing the continuity of $\frac{\partial H}{\partial r_1}(r_1, r_2, r_3)$, we show the following identity holds for $G \in \mathcal{D}(\mathcal{L})$.

$$\lim_{\varepsilon\to 0}\mathcal{E}(P_\varepsilon G - G) = 0. \qquad (6.13)$$

In the case of $G := g(\langle \cdot, \phi_1\rangle, \cdots, \langle \cdot, \phi_n\rangle) \in \mathcal{FC}_b^\infty$, the conditions (B) and (U1) implies the expression

$$
\begin{aligned}
L^{(0)} Q_\varepsilon G(w) &= \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 g}{\partial\alpha_i\partial\alpha_j}\big(\langle R_\varepsilon w, \phi_1\rangle, \cdots, \langle R_\varepsilon w, \phi_n\rangle\big)\langle \phi_i, \phi_j\rangle \\
&\quad + \frac{1}{2}\sum_{i=1}^n \frac{\partial g}{\partial \alpha_i}\big(\langle R_\varepsilon w, \phi_1\rangle, \cdots, \langle R_\varepsilon w, \phi_n\rangle\big)\big\{\langle R_\varepsilon w, \Delta_x \phi_i\rangle - \langle \nabla U(w(\cdot)), \phi_i\rangle\big\}.
\end{aligned}
$$

So we have that $\|L^{(0)}(Q_\varepsilon G)\|_{L^2(E;\mu)}$ is dominated by a constant which is independent of ε. Therefore Proposition 3.3 and the strongly continuity of $\{P_t\}$ lead us to

$$
\begin{aligned}
\mathcal{E}(P_\varepsilon G - G) &= -\Big(L^{(0)}(P_\varepsilon^{(0)} Q_\varepsilon G - G), P_\varepsilon G - G\Big)_{L^2(E;\mu)} \\
&\leq \big\|P_\varepsilon^{(0)}(L^{(0)} Q_\varepsilon G) - (L^{(0)} G)\big\|_{L^2(E;\mu)} \cdot \|P_\varepsilon G - G\|_{L^2(E;\mu)} \\
&\leq \big(\|L^{(0)}(Q_\varepsilon G)\|_{L^2(E;\mu)} + \|L^{(0)} G\|_{L^2(E;\mu)}\big)\|P_\varepsilon G - G\|_{L^2(E;\mu)} \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}
\tag{6.14}
$$

In the case of $G \in \mathcal{D}(\mathcal{L})$, we take a sequence $\{G_j\}_{j=1}^\infty \subset \mathcal{FC}_b^\infty$ such that $G_j \to G$ in $\mathcal{D}(\mathcal{E})$ as $j \to \infty$. Then by using Theorem 5.1 and the contraction property of $\{P_t\}$ in $L^2(E;\mu)$, we have

$$
\begin{aligned}
\mathcal{E}(P_\varepsilon G - G) &\leq 3\Big(\mathcal{E}(P_\varepsilon G - P_\varepsilon G_j) + \mathcal{E}(P_\varepsilon G_j - G_j) + \mathcal{E}(G - G_j)\Big) \\
&\leq 3\Big\{(e^{K_1\varepsilon} + 1)\mathcal{E}(G - G_j) + \mathcal{E}(P_\varepsilon F - P_\varepsilon G_j)\Big\}.
\end{aligned}
\tag{6.15}
$$

Hence by letting $\varepsilon \to 0$ and $j \to \infty$ and recalling (6.14), we complete the proof of (6.13).

Then by (6.13) and Theorem 5.1, we can also obtain

$$
\begin{aligned}
\mathcal{E}(P_{t-r_2}F - P_{t-r_2-\varepsilon}F) &\leq \frac{1}{2}\int_E e^{K_1(t-r_2-\varepsilon)}\Big\{P_{t-r_2-\varepsilon}\big(\|D(F - P_\varepsilon F)\|_H\big)(w)\Big\}^2 \mu(dw) \\
&\leq e^{K_1(t-r_2-\varepsilon)}\mathcal{E}(P_\varepsilon F - F) \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}
\tag{6.16}
$$

Now we return to discuss the continuity of $\frac{\partial H}{\partial r_1}(r_1, r_2, r_3)$. By recalling the assertion (2) in Lemma 6.2 and the contraction property of $\{P_t\}$ in $L^2(E;\mu)$, we have the following estimate for sufficient small numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3$:

$$
\begin{aligned}
&\Big\|\frac{\partial H}{\partial r_1}(r_1+\varepsilon_1, r_2+\varepsilon_2, r_3+\varepsilon_3) - \frac{\partial H}{\partial r_1}(r_1, r_2, r_3)\Big\|_{L^1(E;\mu^V)} \\
&\quad \leq \sum_{i=1}^2 \Big\{\|H_i(r_1, r_2, r_3+\varepsilon_3) - H_i(r_1, r_2, r_3)\|_{L^1(E;\mu^V)} \\
&\qquad\qquad + Z_V^{-1} e^{K_4}\|H_i(r_1+\varepsilon_1, r_2+\varepsilon_2, 0) - H_i(r_1, r_2, 0)\|_{L^2(E;\mu)}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&\le \sum_{i=1}^{2} \Big\{ \|H_i(r_1, r_2, r_3+\varepsilon_3) - H_i(r_1, r_2, r_3)\|_{L^1(E;\mu^V)} \\
&\qquad + Z_V^{-1} e^{K_4} \|H_i(r_1+\varepsilon_1, r_2+\varepsilon_2, 0) - H_i(r_1+\varepsilon_1, r_2, 0)\|_{L^2(E;\mu)} \\
&\qquad + Z_V^{-1} e^{K_4} \|H_i(r_1+\varepsilon_1, r_2, 0) - H_i(r_1, r_2, 0)\|_{L^2(E;\mu)} \Big\} \\
&\le \sum_{i=1}^{2} \Big\{ \|H_i(r_1, r_2, r_3+\varepsilon_3) - H_i(r_1, r_2, r_3)\|_{L^1(E;\mu^V)} \\
&\qquad + Z_V^{-1} e^{K_4} \|H_i(0, r_2+\varepsilon_2, 0) - H_i(0, r_2, 0)\|_{L^2(E;\mu)} \\
&\qquad + Z_V^{-1} e^{K_4} \|H_i(r_1+\varepsilon_1, r_2, 0) - H_i(r_1, r_2, 0)\|_{L^2(E;\mu)} \Big\}. \qquad (6.17)
\end{aligned}
$$

By remembering (6.16), Theorem 5.1 and the uniformly boundedness of $\{P_{t-r_2-\varepsilon_2}\}$ with respect to ε_2, we have

$$
\lim_{\varepsilon_2 \to 0} \big\| H_2(0, r_2+\varepsilon_2, 0) - H_2(0, r_2, 0) \big\|_{L^2(E;\mu)} = 0. \qquad (6.18)
$$

Hence by combining the assertion (3) in Lemma 6.2, (6.18) and the strongly continuity of $\{P_t\}$, (6.17) leads us to

$$
\lim_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \to 0} \Big\| \frac{\partial H}{\partial r_1}(r_1+\varepsilon_1, r_2+\varepsilon_2, r_3+\varepsilon_3) - \frac{\partial H}{\partial r_1}(r_1, r_2, r_3) \Big\|_{L^1(E;\mu^{V,h})} = 0.
$$

Next, we discuss the continuity of $\frac{\partial H}{\partial r_2}(r_1, r_2, r_3)$ which is given by the following for $r_1, r_2, r_3 \in (0, t)$:

$$
\frac{\partial H}{\partial r_2}(r_1, r_2, r_3) = -\alpha P_{r_1} \Big\{ (P_{t-r_2}F)^{\alpha-1} P_{t-r_2}(\mathcal{L}F) \Big\} (\cdot + v(r_3)h). \qquad (6.19)
$$

By using the same argument in (6.17) and the strongly continuity of $\{P_t\}$, we can easily have

$$
\lim_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \to 0} \Big\| \frac{\partial H}{\partial r_2}(r_1+\varepsilon_1, r_2+\varepsilon_2, r_3+\varepsilon_3) - \frac{\partial H}{\partial r_2}(r_1, r_2, r_3) \Big\|_{L^1(E;\mu^V)} = 0.
$$

Finally, we consider $\frac{\partial H}{\partial r_3}(r_1, r_2, r_3)$. By virtue of (6.5), we have

$$
\frac{\partial H}{\partial r_3}(r_1, r_2, r_3) = \big(D\{P_{r_1}(P_{t-r_2}F)^\alpha\}(\cdot + v(r_3)h), v'(r_3)h \big)_H. \qquad (6.20)
$$

Here we denote $H_3(r_1, r_2, r_3, r_4) := \big(D\{P_{r_1}(P_{t-r_2}F)^\alpha\}(\cdot + v(r_3)h), v'(r_4)h \big)_H$. By using the similar argument in (6.17) , we have the following estimate for sufficient small numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3$:

$$
\begin{aligned}
&\big\| H_3(r_1+\varepsilon_1, r_2+\varepsilon_2, r_3+\varepsilon_3, r_3+\varepsilon_3) - H_3(r_1, r_2, r_3, r_3) \big\|_{L^1(E;\mu^V)} \\
&\quad \le \|H_3(r_1, r_2, r_3, r_3+\varepsilon_3) - H_3(r_1, r_2, r_3, r_3)\|_{L^1(E;\mu^V)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \|H_3(r_1,r_2,r_3+\varepsilon_3,r_3+\varepsilon_3) - H_3(r_1,r_2,r_3,r_3+\varepsilon_3)\|_{L^1(E;\mu^V)} \\
&+ Z_V^{-1}e^{K_4}\Big\{\|H_3(r_1+\varepsilon_1,r_2+\varepsilon_2,0,r_3+\varepsilon_3) - H_3(r_1+\varepsilon_1,r_2,0,r_3+\varepsilon_3)\|_{L^2(E;\mu)} \\
&\qquad + \|H_3(r_1+\varepsilon_1,r_2,0,r_3+\varepsilon_3) - H_3(r_1,r_2,0,r_3+\varepsilon_3)\|_{L^2(E;\mu)}\Big\}. \qquad (6.21)
\end{aligned}
$$

We treat the third term of the right hand side in (6.21). By Theorem 5.1, (6.16) and the strongly continuity of $\{P_t\}$, we have

$$
\begin{aligned}
&\|H_3(r_1+\varepsilon_1,r_2+\varepsilon_2,0,r_3+\varepsilon_3) - H_3(r_1+\varepsilon_1,r_2,0,r_3+\varepsilon_3)\|^2_{L^2(E;\mu)} \\
&\le \int_E \big\|D\big[P_{r_1+\varepsilon_1}\{(P_{t-r_2-\varepsilon_2}F)^\alpha - (P_{t-r_2}F)^\alpha\}\}\big](w)\big\|_H^2 \mu(dw)\cdot\|v'(r_3+\varepsilon_3)h\|_H^2 \\
&\le e^{K_1(r_1+\varepsilon_1)}\int_E P_{r_1+\varepsilon_1}\Big[\big\|D\big\{(P_{t-r_2-\varepsilon_2}F)^\alpha - (P_{t-r_2}F)^\alpha\big\}\big\|_H\Big](w)^2\mu(dw)\cdot\|v'(r_3+\varepsilon_3)h\|_H^2 \\
&\le e^{K_1(r_1+\varepsilon_1)}\int_E \big\|D\{(P_{t-r_2-\varepsilon_2}F)^\alpha - (P_{t-r_2}F)^\alpha\}(w)\big\|_H^2\mu(dw)\cdot\|v'(r_3+\varepsilon_3)h\|_H^2 \\
&= \int_E \big\|(P_{t-r_2-\varepsilon_2}F)^{\alpha-1}(w)D(P_{t-r_2-\varepsilon_2}F)(w) - (P_{t-r_2}F)^{\alpha-1}(w)D(P_{t-r_2}F)(w)\big\|_H^2\mu(dw) \\
&\quad \times\alpha^2 e^{K_1(r_1+\varepsilon_1)}\cdot\|v'(r_3+\varepsilon_3)h\|_H^2 \longrightarrow 0 \ \text{ as } \ \varepsilon_2\to 0. \qquad (6.22)
\end{aligned}
$$

For the fourth term of the right hand side in (6.21), we also have the following by remarking $(P_{t-r_2-\varepsilon_2}F)^\alpha \in \mathcal{D}(\mathcal{L})$ and the similar argument in (6.22).

$$
\begin{aligned}
&\|H_3(r_1+\varepsilon_1,r_2,0,r_3+\varepsilon_3) - H_3(r_1,r_2,0,r_3+\varepsilon_3)\|^2_{L^2(E;\mu)} \\
&\le \int_E \big\|D\big[P_{r_1}\{P_{\varepsilon_1}(P_{t-r_2-\varepsilon_2}F)^\alpha - (P_{t-r_2}F)^\alpha\}\big](w)\big\|_H^2\mu(dw)\cdot\|v'(r_3+\varepsilon_3)h\|_H^2 \\
&\le e^{K_1 r_1}\int_E \big\|DP_{\varepsilon_1}(P_{t-r_2-\varepsilon_2}F)^\alpha(w) - D(P_{t-r_2-\varepsilon_2}F)^\alpha(w)\big\|_H^2\mu(dw)\cdot\|v'(r_3+\varepsilon_3)h\|_H^2 \\
&\quad \longrightarrow 0 \ \text{ as } \ \varepsilon_1\to 0. \qquad (6.23)
\end{aligned}
$$

Hence we can obtain

$$
\lim_{\varepsilon_1,\varepsilon_2,\varepsilon_3\to 0}\big\|H_3(r_1+\varepsilon_1,r_2+\varepsilon_2,r_3+\varepsilon_3,r_3+\varepsilon_3) - H_3(r_1,r_2,r_3,r_3)\big\|_{L^1(E;\mu^V)} = 0
$$

by using (6.21), (6.22), (6.23) and the continuity of $v'(\cdot)$.

Therefore we can conclude that $H(r_1,r_2,r_3)$ is a C^1-function. Hence we have the following calculation by combining (6.12), (6.19) and (6.20):

$$
\begin{aligned}
G'(s) &= \sum_{i=1}^3 \frac{\partial H}{\partial r_i}(r_1,r_2,r_3)\Big|_{r_1=r_2=r_3=s} \\
&= \frac{\alpha(\alpha-1)}{2}P_s\big\{(P_{t-s}F)^{\alpha-2}\|D(P_{t-s}F)\|_H^2\big\}(\cdot+v(s)h) \\
&\quad +\big(D\{P_s(P_{t-s}F)^\alpha\}(\cdot+v(s)h), v'(s)h\big)_H.
\end{aligned}
$$

This completes the proof of Lemma 6.3. ∎

Continuation of the Proof of Theorem 6.1. The proof is same as [9], however, to make this paper self-contained, we give the proof below. By virtue of Lemma 6.3, we have the following estimate for $0 < s < t$:

$$\begin{aligned} G'(s) &\geq \frac{\alpha(\alpha-1)}{2} P_s\{(P_{t-s}F)^{\alpha-2}\|D(P_{t-s}F)\|_H^2\}(\cdot + v(s)) \\ &\quad - \left\|D\{P_s(P_{t-s}F)^\alpha\}(\cdot + v(s))\right\|_H \cdot \left\|v'(s)h\right\|_H. \end{aligned}$$

Since $P_s(P_{t-s}F)^\alpha \in \mathcal{D}(\mathcal{L})$, we can use Theorem 5.1. Then we can continue as

$$\begin{aligned} G'(s) &\geq \frac{\alpha(\alpha-1)}{2} P_s\{(P_{t-s}F)^{\alpha-2}\|D(P_{t-s}F)\|_H^2\}(\cdot + v(s)h) \\ &\quad - e^{\frac{K_1 s}{2}} P_s\{\|D(P_{t-s}F)^\alpha\|_H\}(\cdot + v(s)h) \cdot \left\|v'(s)h\right\|_H \\ &= \frac{\alpha}{2} P_s\Big\{(\alpha-1)(P_{t-s}F)^{\alpha-2}\|D(P_{t-s}F)\|_H^2 \\ &\quad - 2e^{\frac{K_1 s}{2}}\left\|v'(s)h\right\|_H \cdot (P_{t-s}F)^{\alpha-1}\|D(P_{t-s}F)\|_H\Big\}(\cdot + v(s)h) \\ &\geq -\frac{\alpha}{2(\alpha-1)} P_s\left\{e^{K_1 s}\left\|v'(s)h\right\|_H^2 \cdot (P_{t-s}F)^\alpha\right\}(\cdot + v(s)h). \\ &= -\frac{\alpha e^{K_1 s}}{2(\alpha-1)} \cdot \left\{\frac{K_1^2 e^{-2K_1 s}}{(1-e^{-K_1 t})^2}\|h\|_H^2\right\} \cdot P_s(P_{t-s}F)^\alpha(\cdot + v(s)h). \end{aligned} \tag{6.24}$$

By (6.24), we can get the following estimate for $0 < s < t$:

$$\begin{aligned} \frac{d}{ds}\log G(s) &= \frac{G'(s)}{G(s)} \\ &\geq -\frac{\alpha e^{K_1 s}}{2(\alpha-1)} \cdot \frac{K_1^2 e^{-2K_1 s}}{(1-e^{-K_1 t})^2}\|h\|_H^2. \end{aligned} \tag{6.25}$$

By integrating both sides of (6.25) over s from 0 to t and letting $\delta \downarrow 0$, we obtain the inequality (6.1). If $F \in L^\infty(E;\mu)$, the $C_0^\infty(\mathbb{R},\mathbb{R}^d)$-quasi-invariance of μ also implies our assertion. ∎

Before closing this section, we present an application of Theorem 6.1. The following corollary is an H-smoothing property of the transition semigroup $\{P_t\}$. Since the proof is same as [9], we omit the proof.

Corollary 6.4 (H-Smoothing Property) *Let $F \in L^\infty(E;\mu)$. Then for every $t > 0$, the function $P_tF(w + \cdot) : C_0^\infty(\mathbb{R},\mathbb{R}^d) \subset H \to \mathbb{R}$ is continuous for μ-a.e. $w \in E$.*

7 Application: Certain Lower Estimate on Short Time Asymptotics of the Transition Probability

In this section, we present an application of Theorem 6.1. We give a certain lower estimate of $p_t(A,B)$ in terms of the geometric H-distance, where $p_t(A,B)$ is defined for Borel measurable sets $A, B \subset E$ by

$$p_t(A,B) := \int_A P_t \mathbf{1}_B(w)\mu(dw). \tag{7.1}$$

Here $\mathbf{1}_B$ is the indicator function on B. Briefly speaking, this is the probability of our dynamics $\mathbb{M}$ starting from A and reaching B at time t.

We define the H-distance between two Borel measurable sets in E. This notion is due to [1]. For $u, v \in E$, we define $d_H(u,v)$ by

$$d_H(u,v) := \begin{cases} \|u-v\|_H & \text{if } u-v \in H, \\ \\ +\infty & \text{otherwise.} \end{cases} \tag{7.2}$$

For a Borel measurable set $A \subset E$, we define the distance function $d_H(\cdot, A) : E \longrightarrow [0,\infty]$ by $d_H(u,A) := \inf_{v\in A} d_H(u,v)$. Then $d_H(\cdot, A)$ is Borel measurable. We also define the distance $d_H(A,B)$ between two Borel measurable sets $A, B \subset E$ with $\mu(A), \mu(B) > 0$ as follows:

$$\begin{aligned} d_H(A,B) \;\; := \;\; \sup \Big\{ & \mathrm{essinf}_{u\in A} d_H(u,\tilde{B}), \mathrm{essinf}_{v\in B} d_H(v,\tilde{A}) \;\Big|\; \tilde{A}, \tilde{B} \subset E \text{ are } \sigma\text{-compact sets} \\ & \text{with } \mu\big((A\setminus\tilde{A})\cup(\tilde{A}\setminus A)\big) = \mu\big((B\setminus\tilde{B})\cup(\tilde{B}\setminus B)\big) = 0 \Big\}. \end{aligned} \tag{7.3}$$

We remark that $d_H(A,B) < \infty$ under the condition (U5). For fundamental properties of this distance, the reader is referred to Proposition 4.3 and Remark 4.4 in [9].

Before giving our lower estimate, we recall the notion of H-open set from [1]. We call that a Borel measurable set $A \subset E$ is a H-open set if for any $u \in A$, there exists $\varepsilon > 0$ such that $\{u+h \mid h \in H,\ \|h\|_H < \varepsilon\} \subset A$ holds. This is a weaker notion than a open set.

We present the following lower estimate of $p_t(A,B)$.

Theorem 7.1 *Let $A, B \subset E$ be Borel measurable sets with $\mu(A), \mu(B) > 0$. Assume $d_H(A,B) < \infty$ and A or B is H-open. Then the following asymptotics holds:*

$$\mathrm{liminf}_{t\to 0} 2t \log p_t(A,B) \geq -d_H(A,B)^2. \tag{7.4}$$

Proof. Without loss of generality, we assume that A is H-open. Firstly, we recall Definition 3.4 and Lemma 3.5 in [1]. Since A is H-open and $d_H(A,B) < \infty$, for any $\varepsilon > 0$, there exist a Borel set $D \subset B$ with $\mu(D) > 0$ and $h \in C_0^\infty(\mathbb{R}, \mathbb{R}^d)$ such that $D + h \subset A$ and $\|h\|_H \leq d_H(A,B) + \varepsilon$ hold.

Then by Lemma 5.2 in [1], there exist a Borel measurable set $D' \subset D$, a sequence $\{t_j\}_{j=1}^\infty \downarrow 0$ and $N \in \mathbb{N}$ such that

$$P_{t_j}\mathbf{1}_D(w) \geq \frac{1}{2} \tag{7.5}$$

holds for any $w \in D'$ and $j \geq N$.

By remembering (3.4) and (3.5), we obtain the following estimate for $\alpha > 1$.

$$
\begin{aligned}
p_t(A,B) &\geq \int_E \mathbf{1}_{D+h}(w) P_t \mathbf{1}_D(w)\mu(dw) \\
&= \int_E \mathbf{1}_{D+h}(w+h) P_t \mathbf{1}_D(w+h) \exp(\Phi(-h,w))\mu(dw) \\
&= \int_D P_t \mathbf{1}_D(w+h) \exp\left(\Phi(-h,w)\right)\mu(dw) \\
&\geq \int_{D'} P_t |\mathbf{1}_D|^\alpha (w+h) \exp\left(\Phi(-h,w)\right)\mu(dw). \qquad (7.6)
\end{aligned}
$$

Now we use Theorem 6.1 and (7.5). Then for $j \geq N$, we can continue to estimate as

$$
\begin{aligned}
&p_{t_j}(A,B) \\
&\geq \exp\left(\frac{-\alpha\|h\|_H^2}{2(\alpha-1)} \cdot \frac{K_1}{1-e^{-K_1 t_j}}\right) \cdot \int_{D'} |P_{t_j}\mathbf{1}_D(w)|^\alpha \exp(\Phi(-h,w))\mu(dw) \\
&\geq \exp\left(\frac{-\alpha\|h\|_H^2}{2(\alpha-1)} \cdot \frac{K_1}{1-e^{-K_1 t_j}}\right) \cdot \left(\frac{1}{2}\right)^\alpha \left(\int_E \exp\big(\Phi(-h,w)\big)\mu(dw)\right) \\
&\geq \exp\left\{\frac{-\alpha(d_H(A,B)+\varepsilon)^2}{2(\alpha-1)} \cdot \frac{K_1}{1-e^{-K_1 t_j}}\right\} \cdot \left(\frac{1}{2}\right)^\alpha \left(\int_E \exp\big(\Phi(-h,w)\big)\mu(dw)\right). \qquad (7.7)
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
2t_j \log p_{t_j}(A,B) &\geq \left\{\frac{-\alpha(d_H(A,B)+\varepsilon)^2}{\alpha-1} \cdot \frac{K_1 t_j}{1-e^{-K_1 t_j}}\right\} \\
&\quad +2t_j \log\left\{\left(\frac{1}{2}\right)^\alpha \left(\int_E \exp\big(\Phi(-h,w)\big)\mu(dw)\right)\right\}. \qquad (7.8)
\end{aligned}
$$

Finally, we have our desired estimate (7.4) by letting $j \to \infty$, $\alpha \to \infty$ and $\varepsilon \downarrow 0$. ∎

Remark 7.2 *For symmetric diffusion semigroups, Ramirez* [15] *and Hino-Ramirez* [6] *established the Varadhan type short time asymptotics*

$$\lim_{t\to 0} 2t \log p_t(A,B) = -d(A,B)^2$$

in general state spaces. In [15]*, this type asymptotics was also proved for non-symmetric diffusion processes by using the Girsanov transformation. However, in this paper, we can not apply the Girsanov transformation since we treat rotation. Hence it seems that Theorem* 7.1 *is not included in their results.*

Remark 7.3 *We give a comment on the upper bound of $p_t(A,B)$. Ramirez* [15] *and Hino-Ramirez* [6] *proved the upper bound by using Davies' method which is not effective for non-symmetric cases. In former papers Aida-Kawabi* [1] *and Kawabi* [9]*, we employed Lyons-Zheng's martingale decomposition theorem in the proof. However in our case, there exists a difficulty for the formulation of this decomposition theorem. So it seems that we can not use the method as* [1] *and* [9]*. In what follows, we explain the difficulty.*

For the non-symmetric diffusion process $\mathbb{M}$ and a fixed constant $T>0$, we have the following equality for every $F \in \mathcal{FC}_b^\infty$:

$$\begin{aligned} F(X_t)-F(X_0) &= \frac{1}{2}M_t^{[F]} - \frac{1}{2}\big(\bar{M}_T^{[F]} - \bar{M}_{T-t}^{[F]}\big) + \frac{1}{2}\int_0^t \big(LF(X_s) - L^*F(X_s)\big)ds \\ &= \frac{1}{2}M_t^{[F]} - \frac{1}{2}\big(\bar{M}_T^{[F]} - \bar{M}_{T-t}^{[F]}\big) - \int_0^t {}_E\langle \mathbb{B}(X_s), DF(X_s)\rangle_{E^*} ds, \\ &\qquad \text{for } P_\mu\text{-almost surely,} \end{aligned} \tag{7.9}$$

where $\{M_t^{[F]}\}_{t\geq 0}$ is a continuous $\sigma(X_s;\ 0\leq s\leq t)$ -martingale, $\{\bar{M}_t^{[F]}\}_{t\geq 0}$ is a continuous $\sigma(X_s;\ T-t\leq s\leq T)$-martingale and ${}_E\langle \mathbb{B}(X_s), DF(X_s)\rangle_{E^*}$ is defined by

$${}_E\langle \mathbb{B}(X_s), DF(X_s)\rangle_{E^*} := \int_{\mathbb{R}} \big(BX_s(x), DF(X_s(x))\big)_{\mathbb{R}^d} dx.$$

Hence in the case of $F \in \mathcal{D}(\mathcal{E})$, it is not clear whether the the third term of the right hand side in (7.9) *is well-defined. It will be a challenging problem to give a suitable meaning to* (7.9)*. It is also an important problem to give another approach for the proof of the upper bound. It will be discussed in forthcoming papers.*

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